Degeneration formulae and its applications to local GW and DT invariants

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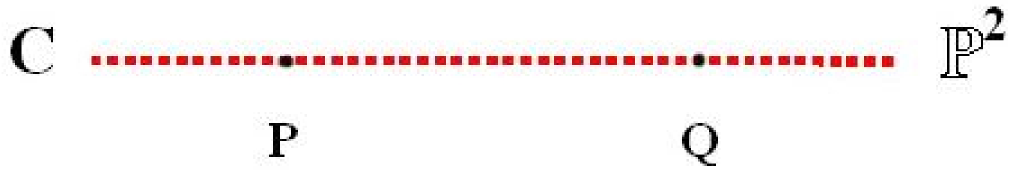
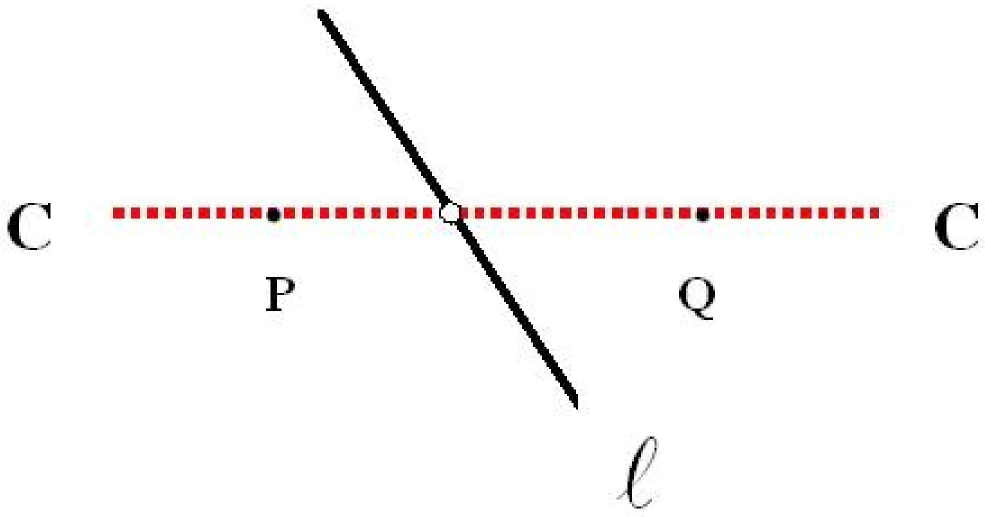
§1 Motivations

Example: Consider the projective plane $\mathbb{P}^2$

• Fix two general points P, Q, there is only one line C passing through P and Q, with the homology class $\beta = [\ell]$.



• Fix two general points P, Q, and a line ℓ, there is only one line C passing through P, Q and intersecting the line ℓ, with the homology class $\beta = [\ell]$.



Question:

Fix a homology class $A \in H_2(X, \mathbb{Z})$ and some cycles Z_i in a projective (or symplectic) manifold X, assuming the Z_i are in general position. The basic question is :

How many curves on X satisfy:

$$C \subset X \text{ of genus } g, \text{ homology class } A, \text{ and } \quad C \cap Z_i \neq \emptyset \text{ for all } i. \tag{1}$$

Naively, Gromov-Witten invariant is defined as the number of curves (1).

Physical origin of Gromov-Witten invariant

The origins in physics of Gromov-Witten invariants is the topological sigma model coupled to gravity. In particular, the genus zero (sometimes called tree level) Gromov-Witten invariants originate from the topological sigma model, which is a topological quantum field theory. In fact, in topological quantum field theory, the Gromov-Witten invariants appear as correlation functions.

§2 Gromov-Witten invariant

Notations:

• (X, ω): a compact symplectic (or projective) manifold of $\dim 2n$, here ω is a nondegenerate closed 2-form, i.e. $\omega^n \neq 0$.

• There exist almost complex structures $J : TX \to TX$ such that $J^2 = -id$.

Fact:The space of all tamed almost complex structures is contractible(This implies that symplectic geometry is much more flexible than complex geometry).

• Example: $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$.

Stable map

An n-pointed stable map consists of a connected marked curve $(C, p_1, \cdots, p_n)$ and a morphism $f : C \longrightarrow X$ satisfying the following conditions:

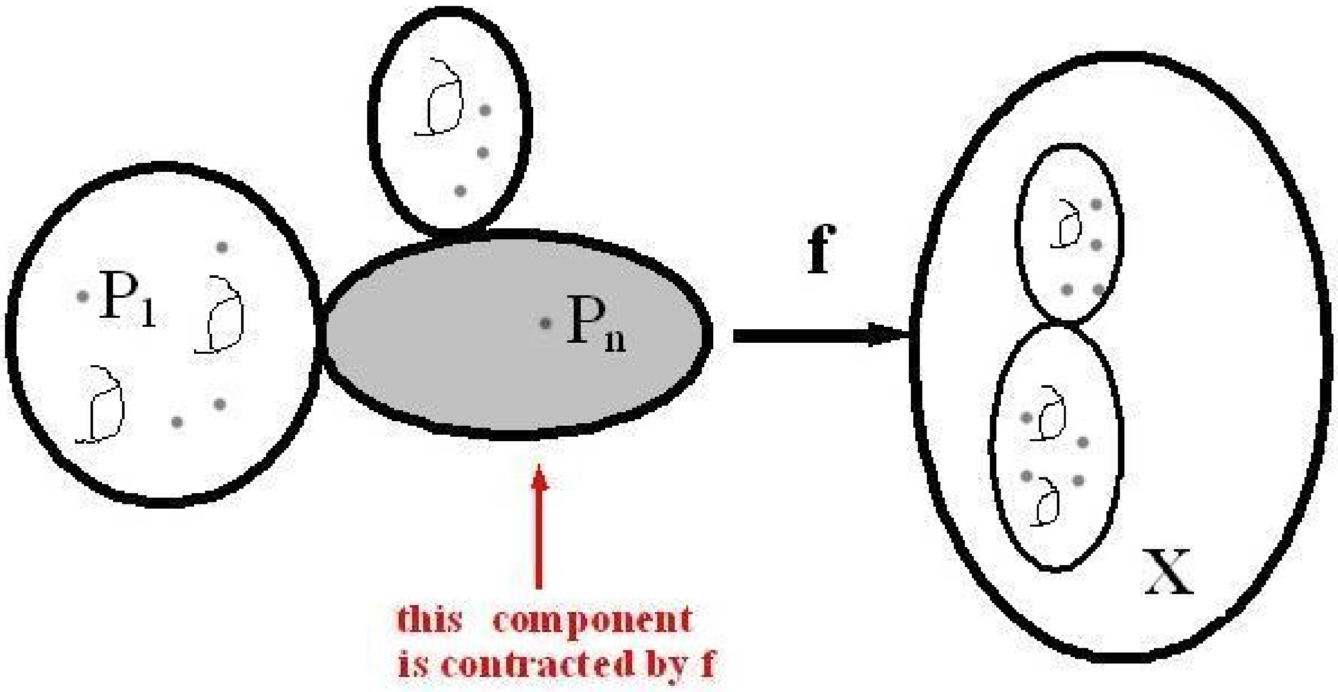
(i) The only singularities of C are ordinary double points(nodal Riemann Surface).

(ii) $p_1, \cdots, p_n$ are distinct ordered smooth points in C.

(iii) If C_i is a component of C such that $C_i \cong \mathbb{P}^1$ and $f\mid_{C_i}$ is constant, then C_i contains at least 3 special points(nodal points and marked points).

(iv) If C has arithmetic genus one and $n = 0$, then f is not constant.

(v) $f\mid_{C_i}: C_i \longrightarrow X$ is holomorphic where C_i is a smooth component of C.



P_1
P_n
f
X
this component is contracted by f

- Equivalence of stable map

$(C, p_1, \cdots, p_n; f)$ is isomorphic to $(C', p'_1, \cdots, p'_n; f')$ if there is an isomorphism $\tau : C \longrightarrow C'$ such that $\tau(p_i) = p'_i$ for all i and $f' \circ \tau = f$.

$$\begin{array}{ccc} (C, p_1, \cdots, p_n) & \stackrel{\exists\tau(\cong)}{\longrightarrow} & (C', p'_1, \cdots, p'_n) \\ f \searrow & & \swarrow f' \\ & X & \end{array}$$

Denote by $[(C, p_1, \cdots, P_n; f)]$ the equivalence class.

Moduli space of stable maps

For $A \in H_2(X, \mathbb{Z})$, define the moduli space of stable maps as follows

$$\overline{\mathcal{M}}_{g,n}(X, A) \quad : \quad =$$

$$\{[(C, p_1, \cdots, p_n; f)] \quad | \quad (C, p_1, \cdots, p_n; f) \text{is a genus } g \text{ stable map and } f_*[C] = A\}.$$

Remark: In general, $\overline{\mathcal{M}}_{g,n}(X, A)$ is very singular. In many case, different component has different dimension. For example, assume that $X = \mathbf{P}^1$, $g > 0$, $A = dL$ with $d > 2$ where L is the homology class of a line in $\mathbf{P}^1$. Then $\overline{\mathcal{M}}_{g,n}(X, A)$ has more than one components. The most interesting one consists (generically) of irreducible genus g curves. Call this one $\overline{\mathcal{M}}_{g,0}(\mathbf{P}^1, A)^o$. The second consists (generically) of two intersecting components, one of genus g and mapping to a point, and the other rational and mapping to $\mathbf{P}^1$ with degree d. The first one has dimension $2d + 2g - 2$, and the second has dimension $2d + 3g - 3$, so the second is not in the closure of the first.

Proposition:[**Ruan1, LT, FO, S**] $\overline{\mathcal{M}}_{g,n}(X,A)$ has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,A)]^{vir}$$

with the expected dimension

$$C_1(A)+(\dim X-3)(1-g)+n.$$

Once we have the moduli space of stable maps, then we may define the following evaluation maps:

$$
\begin{aligned}
ev_i \quad : \quad & \overline{\mathcal{M}}_{g,n}(X,A) \longrightarrow X \\
& [C, p_1, \cdots, p_n, f] \mapsto f(p_i), i = 1, 2, \cdots, n.
\end{aligned}
$$

Definition: Given cohomology classes $\alpha_i \in H^*(X, \mathbf{R})$, roughly define the **(primitive) Gromov-Witten** invariant

$$
\Psi_{(A,g)}(\alpha_1, \cdots, \alpha_n) = \int_{[\overline{\mathcal{M}}_{g,n}(X,A)]^{vir}} \prod_{i=1}^{n} ev_i^* \alpha_i,
$$

if $\sum_{i=1}^n \deg \alpha_i = 2C_1(A) + 2(\dim X - 3)(1-g) + 2n$. Otherwise, we simply define the invariants to be zero.

Remark: (Enumerative meaning) If Z_i is a cycle in X dual to α_i, then the primitive Gromov-Witten invariant $\langle \alpha_1, \cdots, \alpha_n \rangle_{g,A}^X$ should count genus g curves $(C, p_1, \cdots, p_n)$ for which we can find f such that

$$f : (C, p_1, \cdots, p_n) \longrightarrow X \text{ is stable and}$$

$$f_*[C] = A, f(p_i) \in Z_i.$$

Relative Gromov-Witten invariants

Let $Z \subset X$ be a real codimension 2 symplectic submanifold. Suppose that J is an $\omega-$tamed almost complex structure on X preserving TZ, i.e. making Z an almost complex submanifold. The relative GW invariants are defined by counting stable $J-$holomorphic maps intersecting Z at finitely many points with prescribed tangency. More precisely, fix a k-tuple $T_k = (t_1, \cdots, t_k)$ of positive integers, consider a marked pre-stable curve

$$(C, x_1, \cdots, x_l, y_1, \cdots, y_k)$$

and stable $J-$holomorphic maps $f : C \longrightarrow X$ such that the divisor f^*Z is

$$f^*Z = \sum_i t_i y_i.$$

We consider the moduli space of such curves, $\mathcal{M}_{g,T_k}(X, Z, A)$. Unfortunately, this moduli space is not compact. Similar to the case of absolute Gromov-Witten invariant, we may compactify this moduli space by relative stable maps. Denote by $[\overline{\mathcal{M}}_{g,T_k}(X, Z, A)]^{vir}$ the virtual fundamental class. Then use the virtual technique to define the relative Gromov-Witten invariant.

Evaluation maps:

$$\begin{array}{rcl} ev_i: & \overline{\mathcal{M}}_{g,T_k}(X,Z,A) & \longrightarrow X \\ & (C,x_1,\cdots,x_l;y_1,\cdots,y_k;f) & \mapsto f(x_i), \quad 1\leq i\leq l. \end{array}$$

$$\begin{array}{rcl} ev_j^Z: & \overline{\mathcal{M}}_{g,T_k}(X,Z,A) & \longrightarrow Z \\ & (C,x_1,\cdots,x_l;y_1,\cdots,y_k;f) & \mapsto f(y_j), \quad 1\leq j\leq k. \end{array}$$

Definition: (relative Gromov-Witten invariant) Let $\alpha_i \in H^*(X,\mathbb{R})$, $1\leq i\leq l$, $\beta_j \in H^*(Z,\mathbb{R})$, $1\leq j\leq k$. Define the relative Gromov-Witten invariant

$$\langle \Pi_i \tau_{d_i}\alpha_i \mid \Pi_j \beta_j \rangle^{X,Z}_{g,A,T_k} = \frac{1}{|Aut(T_k)|}\int_{[\overline{\mathcal{M}}_{g,T_k}(X,Z,A)]^{vir}} \Pi_i \psi^{d_i}\wedge ev_i^*\alpha_i \wedge (ev_j^Z)^*\beta_j.$$

§3 Degeneration formula for symplectic cutting

- Symplectic cutting

Suppose that $X_0 \subset X$ is an open subset with a hamiltonian S^1-action such that $H : X_0 \longrightarrow \mathbb{R}$ is a Hamiltonian function with 0 as a regular value and $H^{-1}(0)$ is a separating hypersurface in X.

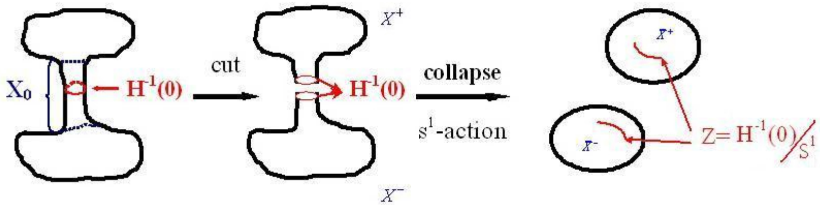
Cut X along $H^{-1}(0)$, we obtain two connected manifolds $X^{\pm}$ with boundary $\partial X^{\pm} = H^{-1}(0)$.

Denote by $Z = H^{-1}(0)/S^1$ the symplectic reduction.

Collapsing the S^1-action on $\partial X^{\pm} = H^{-1}(0)$, we obtain closed smooth manifolds $\bar{X}^{\pm}$.

Definition: Two symplectic manifolds $(\bar{X}^{\pm}, \omega^{\pm})$ are called the symplectic cuts of X along $H^{-1}(0)$.

Here is the geometric description of symplectic cut:

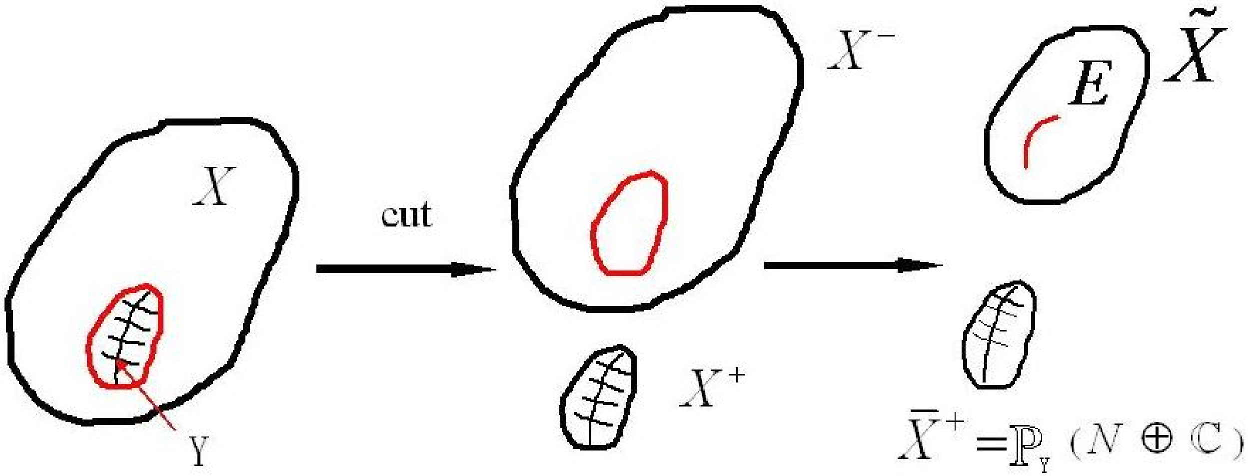


- Symplectic bow-up

Let $Y \subset X$ be a symplectic submanifold of X of codimension $2k$, $N_{Y|X}$ the normal bundle of Y in X. Perform the symplectic cut along the sphere bundle of $N_{Y|X}$, we obtain two symplectic cuts $\bar{X}^{\pm}$:

$$\begin{aligned} \bar{X}^+ &:= \mathbb{P}_Y(N_{Y|X} \oplus \mathbb{C}) \\ \bar{X}^- &:= \tilde{X}, \text{ symplectic blowup of } X \text{ along } Y. \end{aligned}$$

- Example: $Y = pt$. Then $\bar{X}^+ = \mathbb{P}^n$, $\bar{X}^- = \tilde{X}$.



Denote by $p : \tilde{X} \longrightarrow X$ the natural projection of the blow-up. $E = \mathbb{P}_Y(N_{Y|X})$ the exceptional divisor.

- Symplectic blow-down: the opposite operation from $\tilde{X}$ to X.

Degeneration formula

Denote the reduction map by

$$\pi : X \longrightarrow \bar{X}^{+} \cup_Z \bar{X}^{-}.$$

So we have a map

$$\pi_* : H_2(X, \mathbb{Z}) \longrightarrow H_2(\bar{X}^{+} \cup_Z \bar{X}^{-}, \mathbb{Z}).$$

For $A \in H_2(X, \mathbb{Z})$, define $[A] = A + \ker \pi_*$ and define

$$\langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,[A]} := \sum_{B \in [A]} \langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,B}.$$

Degeneration Formula: (gluing formula)

$$\langle \Pi_i \tau_{d_i} \alpha_i \rangle^{X}_{g,[A]} = \sum \langle \Pi_{i\in I_1} \tau_{d_i} \alpha_i^{+} \mid \beta_j \rangle^{\bar{X}^{+},Z}_{g_1,A_1,T_k} \Delta(T_k) \langle \Pi_{i\in I_2} \tau_{d_i} \alpha_i^{-} \mid \check{\beta}_j \rangle^{\bar{X}^{-},Z}_{g_2,A_2,T_k},$$

where the summation runs over all the splittings of g and A, all distribution of the insertion $\alpha_i^{\pm}$, all intermediate cohomology weighted partitions (T_j, β_j) and all configurations of connected components yielding a connected total domain,

$$\Delta(T_k) := \Pi_j t_j |Aut(T_k)|, \quad I_1 \cup I_2 = \{1, 2, \cdots, l\},$$

and $\check{\beta}_j$ is dual to β_j.

Local Gromov-Witten invariants

Let S be a Fano surface and K_S its canonical bundle. For $\beta \in H_2(S, \mathbb{Z})$, denote by $\overline{\mathcal{M}}_{g,k}(S,\beta)$ the moduli space of k-pointed stable maps of degree β to S. Then the following diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,1}(S,\beta) & \xrightarrow{ev} & S \\ \rho \downarrow & & \\ \overline{\mathcal{M}}_{g,0}(S,\beta) & & \end{array}$$

defines the obstruction bundle $R^1\rho_* ev^* K_S$ whose fiber over a stable map $f : C \longrightarrow S$ is given by $H^1(C, f^* K_S)$.

Chiang-Klemm-Yau-Zaslow defined the local Gromov-Witten invariants of K_S as follows

$$K^S_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}} e(R^1\rho_* ev^* K_S). \tag{2}$$

• Yang-Zhou(2009) generalize this definition to the case of toric non Fano surfaces.

Observation:

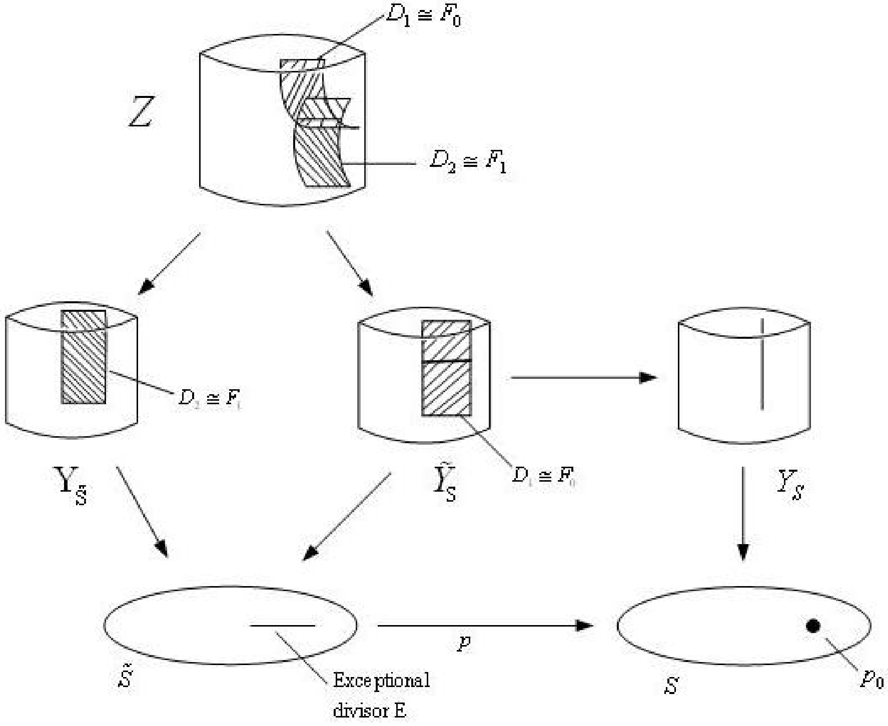
$$[\overline{\mathcal{M}}_{g,0}(Y_S,\beta)]^{vir} = [\overline{\mathcal{M}}_{g,0}(S:\beta)]^{vir} \cap e(R^1\rho_* ev^* K_S).$$

• This implies that the local Gromov-Witten invariant of K_S of degree $\beta \in H_2(S,\mathbb{Z})$ equals the corresponding Gromov-Witten invariant of Y_S, i. e.,

$$K^S_{g,\beta} = n^{Y_S}_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(Y_S,\beta)]^{vir}} 1.$$

Projective completion of K_S

$$
\begin{aligned}
Y_S &= \mathbb{P}(K_S \oplus \mathcal{O}) \\
p &: \tilde{S} \longrightarrow S \\
\tilde{Y_S} &= p^* Y_S \\
D_1 &\cong \mathbb{F}_0 \\
D_1 &= \mathbb{F}_1.
\end{aligned}
$$


$D_1 \cong F_0$
Z
$D_2 \cong F_1$
$D_2 \cong F_1$
$Y_{\tilde{S}}$
$\tilde{Y}_S$
$D_1 \cong F_0$
Y_S
p
$\tilde{S}$
Exceptional
divisor E
S
p_0

Results on local Gromov-Witten invariants

Lemma 1: Suppose that S and its blowup $\tilde{S}$ are Fano surfaces. Let $\tilde{Y}_S$ be the blowup of Y_S along the fiber over $p_0 \in S$. Then for any $\beta \in H_2(S, \mathbb{Z})$, we have

$$n_{g,\beta}^{Y_S} = \langle 1 \mid \emptyset \rangle_{g,p!(\beta)}^{\tilde{Y}_S, D_1}$$

where $D_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional divisor in $\tilde{Y}_S$, $p!(\beta) = PDp^*PD(\beta)$ and $p : \tilde{S} \longrightarrow S$ is the natural projection of the blowup.

Lemma 2: For any $\beta \in H_2(S, \mathbb{Z})$, we have

$$n_{g,p!(\beta)}^{\tilde{Y}_S} = \langle 1 \mid \emptyset \rangle_{g,p!(\beta)}^{\tilde{Y}_S, D_1}.$$

Summarizing Lemma 1 and Lemma 2, we have

Theorem 3:

$$n_{g,\beta}^{Y_S} = n_{g,p!(\beta)}^{\tilde{Y}_S}.$$

Next, we want to compare the Gromov-Witten invariants $n_{g,p!(\beta)}^{\tilde{Y}_S}$ of $\tilde{Y}_S$ to the Gromov-Witten invariants of Z. In fact, we have

Theorem 4:

$$n_{g,p!(\beta)}^{\tilde{Y}_S} = n_{g,p!(\beta)}^{Z}.$$

Donaldson-Thomas invariants

Let X be a smooth projective 3-fold and $\mathcal{I}$ be an ideal sheaf of rank 1 on X.

Fact: $\mathcal{I}$ determines a sub-scheme Y of dimension ≤ 1.

Fact: There is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

• Fix $\beta \in H_2(X, \mathbb{Z})$. Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $\mathcal{I}$ of rank 1 satisfying

$$\chi(\mathcal{O}_Y) = n, \quad [Y] = \beta.$$

Fact: $I_n(X,\beta)$ is projective and a fine moduli space.

Fact: The virtual dimension of $I_n(X,\beta)$ equals $\int_\beta c_1(T_X)$.

For $\gamma \in H^l(X,\mathbb{Z})$, one can introduce some descendent field $(-1)^{k+1}ch_{k+2}(\gamma)$ on the moduli space $I_n(X,\beta)$ by the Chern classes of the universal ideal sheaf $\mathcal{J} \longrightarrow I_n(X,\beta)\times X$.

Definition: Suppose that X is a nonsingular,projective, Calabi-Yau 3-fold. Then for $\gamma_i \in H^*(X,\mathbb{R})$, $1 \le i \le r$, and integers $k_1,\cdots,k_r$, the Donaldson-Thomas invariant is defined via integration against the virtual fundamental class,

$$\langle \tilde{\tau}_{k_1}(\gamma_1),\cdots,\tilde{\tau}_{k_r}(\gamma_r)\rangle_{n,\beta} := \int_{[I_n(X,\beta)]^{vir}} \prod_{i=1}^{r} (-1)^{k_i+1} ch_{k_i+2}(\gamma_i).$$

Donaldson-Thomas partition function

DT partition function:

$$Z_{DT}(X, q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_\beta := \sum_{n\in\mathbb{Z}} \langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \rangle_{n,\beta} q^n.$$

Reduced DT partition function:

$$Z'_{DT}(X, q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_\beta := \frac{Z_{DT}(X, q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_\beta}{Z_{DT}(X; q)_0}.$$

Relative DT invariants and its partition function

Similar to Gromov-Witten invariants, If S is a smooth surface in X, then for a partition $\eta = (\eta_1, \cdots, \eta_s)$ of $[S$one can define the relative Donaldson-Thomas invariant

$$\langle \tilde{\tau}_{k_1}(\gamma_1), \cdots, \tilde{\tau}_{k_r}(\gamma_r) \mid \eta \rangle_{n,\beta}$$

Relative DT partition function:

$$Z_{DT}(X/S, q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_{\beta,\eta} := \sum_{n \in \mathbb{Z}} \langle \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \mid \eta \rangle_{n,\beta} q^n.$$

Reduced relative DT partition function:

$$Z'_{DT}(X/S, q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_{\beta,\eta} := \frac{Z'_{DT}(X, q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_{\beta,\eta}}{Z'_{DT}(X/S; q)_0}.$$

Degeneration formula for DT invariants

Let $\pi : \mathcal{X} \longrightarrow \mathbb{C}$ be a semistable degeneration such that $\mathcal{X}_t = \pi^{-1}(t) \cong X$ for $t \neq 0$ and $\mathcal{X}_0$ is a union of two smooth e-folds X_1 and X_2 intersecting transversely along a smooth surface S, Write

$$i_t : X = \mathcal{X}_t \to \mathcal{X},\ i_0 : \mathcal{X}_0 \to \mathcal{X},\ j_1 : X_1 \to \mathcal{X}_0,\ j_2 : X_2 \longrightarrow \mathcal{X}_0.$$

Then the degeneration formula take the following form

$$\begin{aligned} & Z'_{DT}(\mathcal{X}_t; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_i(t)))_\beta \\ = \quad & \sum Z'_{DT}(X_1/S; q \mid \prod \tilde{\tau}_0(j_1^*\gamma_i(0)))_{\beta_1,\eta} \frac{(-1)^{|\eta|-\ell(\eta)} \triangle(\eta)}{q^{|\eta|}} \\ & \times Z'_{DT}(X_2/S; q \mid \prod \tilde{\tau}_0(j_2^*\gamma_i(0)))_{\beta_2,\eta^\vee}, \end{aligned}$$

where the sum runs over the splittings $\beta_1 + \beta_2 = \beta$ and cohomology weighted partitions η.

Donaldson-Thomas invariants of local surfaces

Let $Y_S = \mathbb{P}(K_S \oplus \mathcal{O}_S)$ be the projective bundle over the surface S. Since Y_S has an anticanonical section, the Donaldson-Thomas theory of Y_S is well-defined in every rank.

Main result: Suppose that $\tilde{S}$ is the blowup of S and $p : \tilde{S} \longrightarrow S$ is the projection. For $\beta \in H_2(S, \mathbb{Z})$, we have

$$Z'_{DT}(Y_S; q)_\beta = Z'_{DT}(Y_{\tilde{S}}; q)_{p!(\beta)},$$

where $p!(\beta) = PDp^* PD(\beta)$.

Thank You!