The Global Geometry of Stationary Surfaces in 4-dimensional Lorentz space

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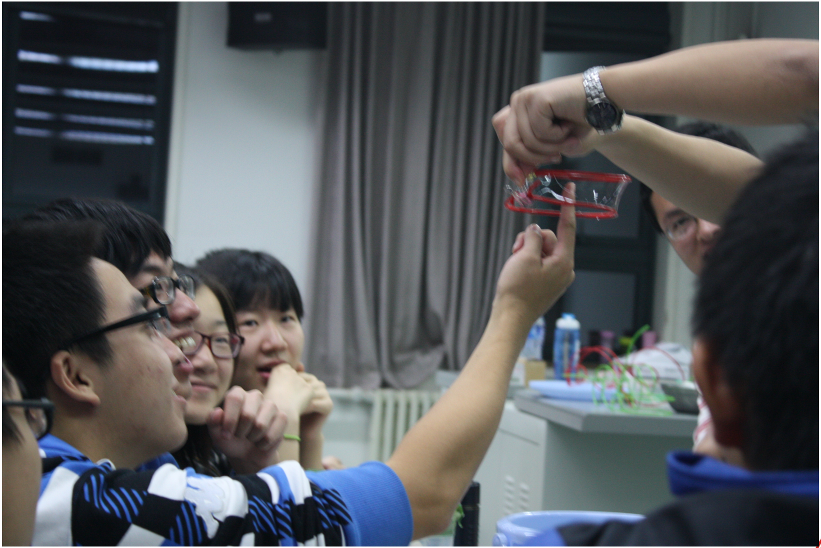
the 10th Pacific Rim Geometry Conference

3 December, 2011, Osaka



Freshman attempting to break a soap film

Sharing soap films with kids

Stationary surfaces = spacelike surfaces with $H = 0$

In $\mathbb{R}^4_1$: $\langle X, X\rangle := X_1^2 + X_2^2 + X_3^2 - X_4^2$.

$H = 0 \Leftrightarrow X : M \to \mathbb{R}^4_1$ is harmonic (for induced metric).

Special cases:

- In $\mathbb{R}^3$: Minimizer of the surface area.
- In $\mathbb{R}^3_1$: Maximizer of the surface area.

In $\mathbb{R}^4_1$: Not local minimizer or maximizer.

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Motivation

Stationary surfaces in $\mathbb{R}^4_1$ are:

- special examples of Willmore surfaces (critical points for $\int (H^2 - K)\mathrm{d}M$).
- corresponding to Laguerre minimal surfaces (critical points for $\int \frac{H^2-K}{K}\mathrm{d}M$).
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Main Results

- Osserman's theorem fails.

  We construct examples with $\int |K| < \infty$ whose Gauss maps could not extend to the ends.

- Singular ends.

  We divide them into two types; define index for good type.

- Gauss-Bonnet type result:

$$\int_M K \mathrm{d}M = 2\pi(2 - 2g - m - \sum \widetilde{d}_j).$$

- We construct many embedded examples (in contrast to uniqueness results in $\mathbb{R}^3$).

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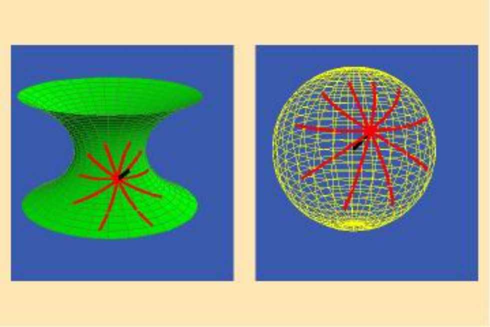
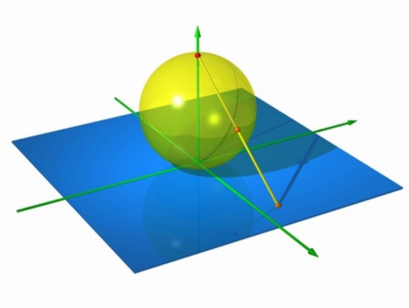
The Gauss Map in $\mathbb{R}^3$





$$\text{Minimal} \Leftrightarrow N : M \to S^2 \text{ anti-conformal.}$$

$$\Leftrightarrow G = p \circ N \text{ meromorphic.}$$

The Gauss Maps in $\mathbb{R}^4_1$

Space-like $X : M^2 \to \mathbb{R}^4_1$:

normal plane $(TM)^\perp$ is a Lorentz plane;

splits into light-like lines $(TM)^\perp = \mathrm{Span}\{Y, Y^*\}$.

$$(M^2, z) \xrightarrow{[Y],[Y^*]} Q^2 \cong S^2, \qquad p: Q^2 \to \mathbb{C}, \qquad \phi, \overline{\psi}: (M^2, z) \to \mathbb{C}$$

$$\langle Y, Y \rangle = \langle Y^*, Y^* \rangle = 0,$$
$$\langle Y, Y^* \rangle = 1.$$
$$Q^2 = \{[v] \in \mathbb{R}P^3 | \langle v, v \rangle = 0\}.$$

Stationary $\Leftrightarrow$ $[Y]$ conformal, $[Y^*]$ anti-conformal.

$\Leftrightarrow$ $\phi, \psi : M \to \mathbb{C}$ meromorphic.

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The W-representation for Minimal $X : M^2 \hookrightarrow \mathbb{R}^3$

$X_z \mathrm{d}z = (\omega_1, \omega_2, \omega_3)$ is a vector-valued holomorphic 1-form with $(\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2 = 0$.

$$X = \mathrm{Re} \int_{z_0}^{z} \left[G - \frac{1}{G}, -\mathrm{i} \left(G + \frac{1}{G} \right), 2 \right] \mathrm{d}h \ .$$

- M: a Riemann surface (non-compact).
- G: the Gauss map; meromorphic function on M;
- $\mathrm{d}h$: height differential; holomorphic on M.

The W-representation in $\mathbb{R}^4_1$

For stationary $X : M^2 \to \mathbb{R}^4_1$ with $X_z \mathrm{d}z = (\omega_1, \omega_2, \omega_3, \omega_4)$ one has: $(\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2 - (\omega_4)^2 = 0$.

$$X = \mathrm{Re} \int_{z_0}^{z} \left[\phi + \psi, -\mathrm{i}(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi\right] \mathrm{d}h \,.$$

$\phi, \psi, \mathrm{d}h$ are Gauss maps and height differential, respectively.

$$\text{Special cases} \left\{ \begin{array}{lll} \psi = -1/\phi & \Rightarrow & M \to \mathbb{R}^3 \\ \psi = 1/\phi & \Rightarrow & M \to \mathbb{R}^3_1 \\ \psi = 0 & \Rightarrow & M \to \mathbb{R}^3_0 \end{array} \right\} \text{Unified in } \mathbb{R}^4_1.$$

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Induced metric $\mathrm{d}s^2 = |\phi - \overline{\psi}|^2 |\mathrm{d}h|^2$.

- **Regularity**: $\phi \neq \overline{\psi}$ on M (because $[Y] \neq [Y^*]$);

  poles of ϕ or $\psi \leftrightarrow$ zeros of $\mathrm{d}h$.

- **Period Condition**: meromorphic differentials ω_j have no real periods along any closed path.

$$(-K + \mathrm{i}K^{\perp})\mathrm{d}M = 2\mathrm{i}\frac{\phi_z \overline{\psi}_{\bar{z}}}{(\phi - \overline{\psi})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

$$= 2\mathrm{i}\left[\log(\phi - \bar{\psi})\right]_{z\bar{z}}\mathrm{d}z \wedge \mathrm{d}\bar{z}\ .$$

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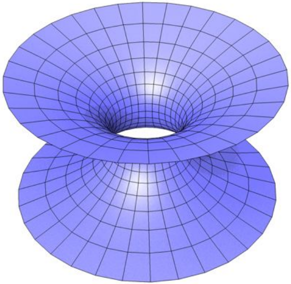
Minimal Surfaces of Finite Toal Curvature

Thm [Osserman, Jorge-Meeks]
Complete minimal $X : M \to \mathbb{R}^3$,
$\int_M -K\mathrm{d}M < \infty$. $\Rightarrow$

- $M \cong \overline{M} - \{p_1, \cdots, p_m\}$.
  conformal equivalence [Huber].
  $\overline{M}$ compact. p_j **Ends**.
- $G, \mathrm{d}h$ extends analytically to p_j;
  be meromorphic objects on $\overline{M}$.
- $\int K\mathrm{d}M = -4\pi\deg(G)$
  $= 2\pi(2-2g-m-\sum_{j=1}^m d_j)$.

  g: genus of $\overline{M}$;

  d_j: multiplicity of the j-th end.



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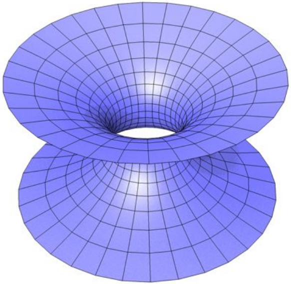

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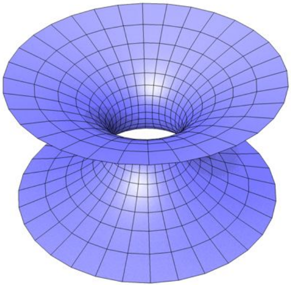

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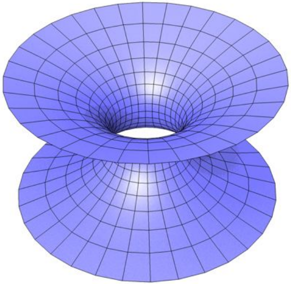

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Basic Difficulties for $X : M \longrightarrow \mathbb{R}^4_1$

$$(-K + \mathrm{i}K^{\perp})\mathrm{d}M = 2\mathrm{i}\frac{\phi_z \overline{\psi}_{\bar{z}}}{(\phi - \overline{\psi})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

- There might be $\phi = \overline{\psi}$ at one end. Called a **singular end**.
- The sign of K is not fixed in general.
  (Compare to $K \leq 0$ in $\mathbb{R}^3$, $K \geq 0$ in $\mathbb{R}^3_1$, $K \equiv 0$ in $\mathbb{R}^3_0$.)
- The integral of Gauss curvature losses the old geometric meaning as the area of Gauss map image.
- **Essential singularities** of ϕ, ψ on $\overline{M}$. **EXIST OR NOT**?
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Osserman's Theorem NOT True in $\mathbb{R}^4_1$

Counter-example X_k $(k \geq 2)$:

$$M = \mathbb{C} - \{0\},\ \phi(z) = \frac{-1}{z^k} e^z,\ \psi(z) = z^k e^z,\ \mathrm{d}h = e^{-z}\mathrm{d}z\ .$$

- No singular points/ends. $\phi \neq \overline{\psi}$ on $\mathbb{C} \cup \{\infty\}$.
- X_k is complete with two end $z = 0, \infty$; no periods.
- The absolute total curvature of X_k is finite:

$$\int_M |-K + \mathrm{i}K^{\perp}|\mathrm{d}M < \infty.$$

(Indeed $\int_M K\mathrm{d}M = -4k\pi$, $\int_M K^{\perp}\mathrm{d}M = 0$.)

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Singular Ends — Good or Bad

Let $X : D - \{0\} \to \mathbb{R}^4_1$ be one end at $z = 0$. Recall that

$$(-K + \mathrm{i}K^{\perp})\mathrm{d}M = 2\mathrm{i}\frac{\phi_z \overline{\psi}_{\bar{z}}}{(\phi - \overline{\psi})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Definition $z = 0$ is called a singular end if $\phi(0) = \overline{\psi}(0)$.

Definition It is called a BAD singular end if both ϕ and ψ have the same multiplicity at 0, or a GOOD singular end otherwise.

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Index of a Good Singular End

Definition The index of a good singular end p is

$$\mathrm{ind}(\phi - \overline{\psi}) := \lim_{D_p \to \{p\}} \frac{1}{2\pi \mathrm{i}} \oint_{\partial D_p} \mathrm{d} \ln(\phi - \overline{\psi}).$$

Lemma

$$\lim_{D \to \{0\}} \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \mathrm{d} \ln(z^m - \bar{z}^n) = \begin{cases} m, & \text{if } m < n, \\ -n, & \text{if } m > n. \end{cases}$$

When $m = n$, $\oint \frac{\phi_z}{\phi - \overline{\psi}} \mathrm{d}z$ and $\oint \frac{\bar{\psi}_{\bar{z}}}{\phi - \overline{\psi}} \mathrm{d}\bar{z}$ won't converge!

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Index of a Good Singular End

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$$\mathrm{ind}(\phi - \overline{\psi}) := \lim_{D_p \to \{p\}} \frac{1}{2\pi \mathrm{i}} \oint_{\partial D_p} \mathrm{d} \ln(\phi - \overline{\psi}).$$

Lemma

$$\lim_{D \to \{0\}} \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \mathrm{d} \ln(z^m - \bar{z}^n) = \begin{cases} m, & \text{if } m < n, \\ -n, & \text{if } m > n. \end{cases}$$

When $m = n$, $\oint \frac{\phi_z}{\phi - \bar{\psi}} \mathrm{d}z$ and $\oint \frac{\bar{\psi}_{\bar{z}}}{\phi - \bar{\psi}} \mathrm{d}\bar{z}$ won't converge!

G-B Theorem for Algebraic Minimal Surfaces

Theorem Let complete stationary surface $X : M \to \mathbb{R}^4_1$ satisfy:

1) $M \cong \overline{M} - \{p_1, \cdots, p_m\}$;
2) $\phi, \psi, \mathrm{d}h$ extends analytically to $\overline{M}$;
3) There are NO bad singular ends.

Then

$$\begin{aligned} \int_M K \mathrm{d}M &= -2\pi \left[\deg(\phi) + \deg(\psi) - \sum |\mathrm{ind}|\right] \\ &= 2\pi(2 - 2g - m - \sum \widetilde{d_j}), \\ \int_M K^\perp \mathrm{d}M &= 0\,. \end{aligned}$$

Remark Here we modify $\widetilde{d_j} := d_j - |\mathrm{ind}|$ at p_j.

Remark $\deg(\phi) - \deg(\psi) = \sum_{p_j} \mathrm{ind}(\phi - \overline{\psi})$.

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Sketch of the Proof

1) Cut out small neighborhood D_j for each end p_j.

2) Using Stokes theorem on $\overline{M} - \cup_{j=1}^{m} D_j$, we get

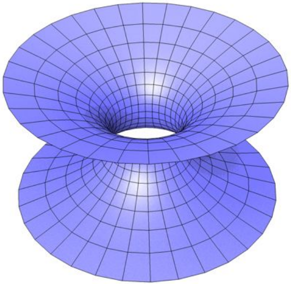
$$\begin{aligned}\int_M (-K + \mathrm{i}K^{\perp})\mathrm{d}M &= 2\mathrm{i} \lim \int_{\overline{M}-\cup D_j} \frac{\phi_z \overline{\psi}_{\bar{z}}}{(\phi - \overline{\psi})^2} \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= 2\mathrm{i} \sum_j \lim_{D_j \to \{p_j\}} \int_{\partial D_j} \frac{\phi_z}{\phi - \overline{\psi}} \mathrm{d}z \\ &= 2\mathrm{i} \cdot 2\pi\mathrm{i} \left[-\sum \mathrm{poles}(\phi) + \sum_{\mathrm{ind}>0} \mathrm{ind} \right] \\ &= 4\pi \deg(\phi) - 2\pi \left(\sum |\mathrm{ind}| + \sum \mathrm{ind} \right).\end{aligned}$$

3) Similarly, $LHS = 4\pi \deg(\psi) - 2\pi \left(\sum |\mathrm{ind}| - \sum \mathrm{ind} \right)$.

Generalized Catenoid

Classical catenoid:

$M = \mathbb{C} - \{0\}, \phi = -\frac{1}{\psi} = z, \mathrm{d}h = \frac{\mathrm{d}z}{z}$.



Lopez-Ros theorem:

A complete, genus zero, finite total curvature, embedded minimal surface in $\mathbb{R}^3$ is a plane or a catenoid.

Generalized to $\mathbb{R}^4_1$:

$$M = \mathbb{C} - \{0\},\ \phi = z + a,\ \psi = \frac{-1}{z-a},\ \mathrm{d}h = \frac{z-a}{z^2}\mathrm{d}z\ .$$

It has no real periods and no singular points/ends for $a \in (-1, 1)$.

This surface is embedded.

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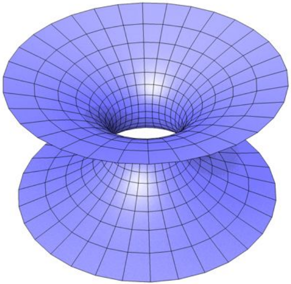

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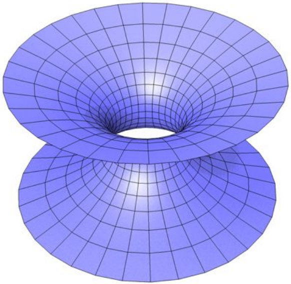

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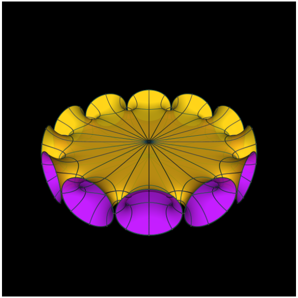
Generalized k-noids

The Jorge-Meeks k-noids ($k \geq 3$) in $\mathbb{R}^3$:

$$M = \mathbb{C}P^1 \backslash \{\epsilon^j | \epsilon^k = 1\},$$

$$G = z^{k-1}, \quad dh = \frac{z^{k-1}}{(z^k - 1)^2} dz \ .$$



One can deform it to an embedded stationary surface in $\mathbb{R}^4_1$:

$$X = \mathrm{Re} \int_{z_0}^{z} \left[G - \frac{1}{G}, -\mathrm{i}\left(G + \frac{1}{G} \right), \sqrt{3}, i \right] \mathrm{d}h \ .$$

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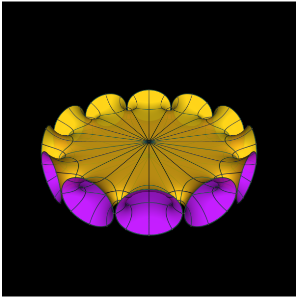

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Classical helicoid:

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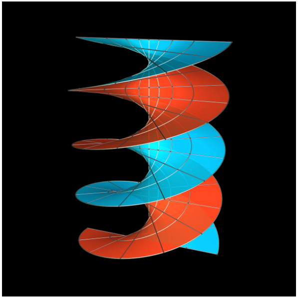
A complete, simply connected, embedded minimal surface in $\mathbb{R}^3$ is a plane or a helicoid.



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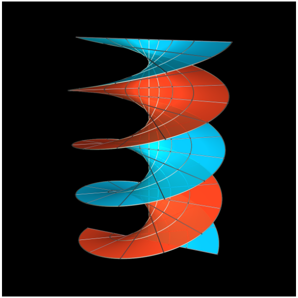

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Stationary Graph

In $\mathbb{R}^3$, a complete graph is a plane (Bernstein theorem).

In $\mathbb{R}^3$, an embedded end must have multiplicity 1, and be either a catenoid end or a planar end.

In $\mathbb{R}^4_1$, stationary surfaces as graph over a 2-plane (hence embedded) could has one planar end of arbitrary multiplicity n:

$$X_z = \left[\left(\frac{1}{z^n} - \frac{z^n}{2}\right), i\left(\frac{1}{z^n} + \frac{z^n}{2}\right), 1, i\right].$$

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Generalized Enneper Surfaces

Classical Enneper surface:
$M = \mathbb{C}, \phi = -\frac{1}{\psi} = z, \mathrm{d}h = z\mathrm{d}z.$

- Simply connected.
- Total curvature -4π.
- One end of multiplicity 3; with self intersection.

Generalized to $\mathbb{R}^4_1$:

$$M = \mathbb{C},\ \phi = z + 1,\ \psi = \frac{c}{z},\ \mathrm{d}h = s \cdot z\mathrm{d}z.$$

This deformation preserves completeness, regularity, period condition... (choose $c, s \in \mathbb{C} \setminus \{0\}$ appropriately).

It could be EMBEDDED in $\mathbb{R}^4_1$ (when $c < -\frac{1}{4}, s \notin \mathbb{R}$).

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Other Results

- Classification of algebraic minimal surfaces in $\mathbb{R}^4_1$ with total curvature -4π.

  (We have to show that $\bar{z}(\bar{z}+\bar{a}) = \frac{z^2}{z+b}$ has only trivial solutions $z = 0, \infty$ for any parameters $a, b \in \mathbb{C}$ satisfying $a + b = 1$.)

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Open Problems

- For essential singularities with finite total curvature, define indices and establish G-B type theorem. In particular we conjecture that
$$\int_M K\mathrm{d}M = -4\pi n$$
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THANK YOU !

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