# Vertices on (Other) Riemannian Surfaces

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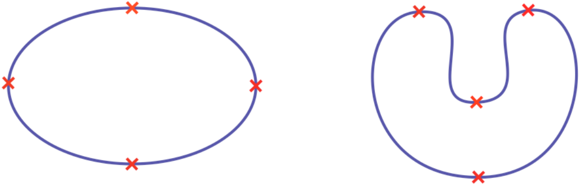
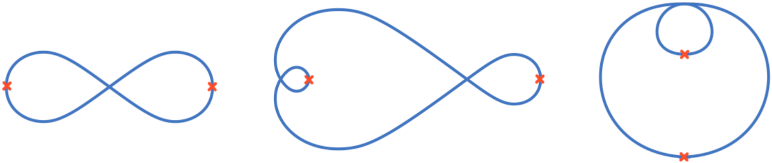
## Theorem (Kneser,1912)

*Any simple closed curve in* $\mathbf{R}^2$ *has (at least) four vertices (local extrema of curvature)*



How about nonsimple curves?

In general they do not have 4 vertices:



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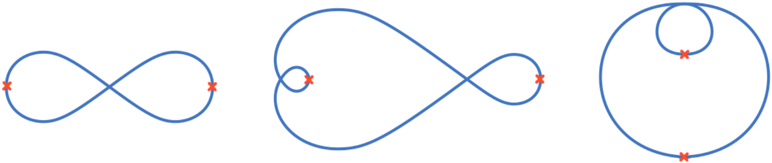

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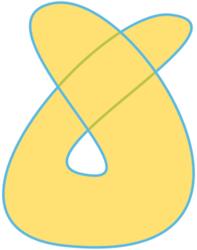

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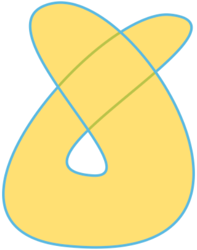

The same result also holds in $\mathbf{S}^2$ and $\mathbf{H}^2$, because the stereographic projection $\pi\colon \mathbf{S}^2 - \{0,0,1\} \to \mathbf{R}^2$ and the inclusion map $i\colon \mathbf{H}^2 \to \mathbf{R}^2$ preserve vertices.

Could Pinkall's theorem be a hint of a purely intrinsic or Riemannian version of the four vertex theorem?

More precisely:

### Question

Let $M$ be a compact surface with boundary and constant curvature. Must the boundary of $M$ have 4-vertices (in terms of geodesic curvature)?

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# A Riemannian 4 -Vertex Theorem for Surfaces with Boundary

**Theorem (MG)**

*Let $M$ be a compact surface with boundary $\partial M$. Then every metric of constant curvature induces four vertices on $\partial M$ if and only if $M$ is simply connected.*

*Indeed, when $M$ is not simply connected, there are elliptic, parabolic and hyperbolic metrics of constant curvature on $M$ which induce only two vertices on $\partial M$.*

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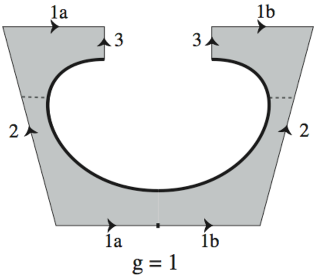
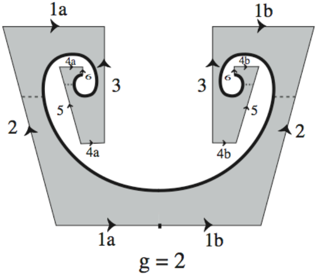

1a
1b
3
3
2
2
1a
1b
g = 1


1a
1b
4a
6
3
3
4b
6
5
5
2
2
4a
4b
1a
1b
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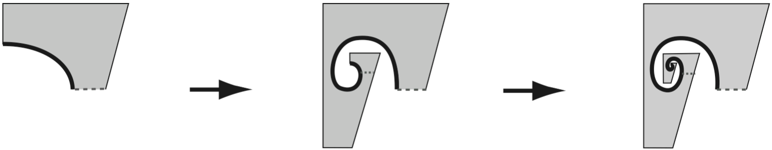
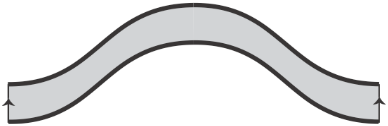


# Flat metrics with fewest vertices

First we show that if $M$ is not simply connected, it admits a flat metric with only two vertices on each boundary component.

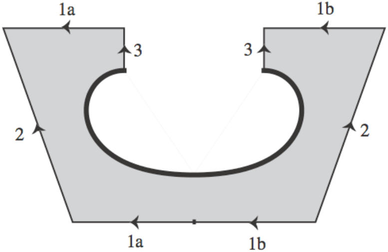
Recall that $M$ is homeomorphic to a closed surface $\overline{M}$ minus $k$-disks. There are three special cases that we consider first:

I. $\overline{M} = \mathbf{S}^2$ & $k = 2$



II. $\overline{M} = \mathbf{RP}^2$ & $k = 1$

## III. $g(\overline{M}) = 1$ & $k = 1$




$$\kappa(t) = 1 - \frac{3}{4}\cos(t),$$

where $-\pi \leq t \leq \pi$. More explicitly, $\gamma(t) := \int_0^t e^{i\theta(s)} ds$, where $e^{i\theta} := (\cos(\theta), \sin(\theta))$, and $\theta(t) := \int_0^t \kappa(s) ds$.

In all the remaining cases we will show that $\overline{M}$ admits a flat metric with exactly $k$ conical singularities.

Then we remove these singularities by cutting $\overline{M}$ along simple closed curves which have only two critical points of geodesic curvature each.

If $\overline{M}$ has $k$ singularities of angles $\theta_i$, then by Gauss-Bonnet theorem,

$$\sum_{i=1}^{k}(2\pi - \theta_i) = 2\pi\chi(\overline{M}).$$

Troyanov has shown that the above condition is also sufficient for the existence of flat metrics with conical singularities of prescribed angles. This quickly yields

### Lemma

*Suppose* $k(\overline{M}) \geq 3$, $2$, $2$, $1$, *according to whether* $\overline{M} = \mathbf{S}^2$, $\overline{M} = \mathbf{RP}^2$, $g(\overline{M}) = 1$, *or* $g(\overline{M}) \geq 2$ *respectively. Then there exists a flat metric on* $\overline{M}$ *with exactly* $k$ *conical singularities.*

## Lemma

*Let $C$ be a cone with angle $\phi \neq 2\pi$ and $\Gamma$ be a circle centered at the vertex of $C$. Then there exists a $C^\infty$ perturbation of $\Gamma$ which has only two critical points of curvature.*

## Proof.

If $\phi = 2n\pi$ (where $n \geq 2$), let

$$r_\lambda(\theta) := 1 - \lambda \cos\left(\frac{\theta}{n}\right).$$

If $\phi \neq 2n\pi$, we cut a segment of theses curves. □

# Perturbations of Flat Metrics

## Proposition

*Let $M$ be a compact surface with boundary and flat metric $g_0$. Then there exists a family $g_\lambda$ of Riemannian metrics on $M$, $\lambda \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, such that $g_\lambda$ has constant curvature $\lambda$, and $\lambda \mapsto g_\lambda$ is continuous with respect to the $C^\infty$ topology.*

This is easy when $M$ is simply connected, for then it isometrically immersed into the plane and we may perturb the whole plane

$$(g_\lambda)_{ij}(x) := \frac{\delta_{ij}}{\left(1 + \frac{\lambda}{4}\|x\|^2\right)^2}.$$

But it requires more work in the general case:

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But it requires more work in the general case:

# A four-vertex theorem for complete surfaces

By Pinkall's theorem, and its extension to $\mathbf{H}^2$ and $\mathbf{S}^2$, *any closed curve bounding a compact surface in a simply connected space form has four vertices*.

Question

Are there any other complete Riemannian surfaces where Pinkall's theorem holds?

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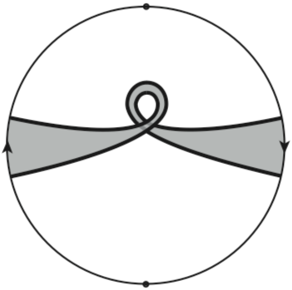
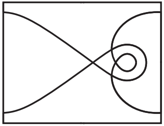


II: The parabolic case ($K = 0$)

So how does one construct a closed curve with only two vertices which bounds a compact immersed surface on a cylinder?

It is not so hard to construct one on a torus:



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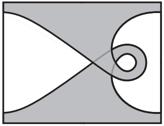

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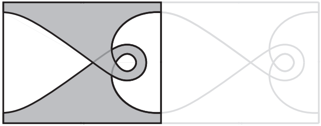

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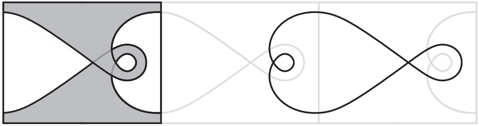

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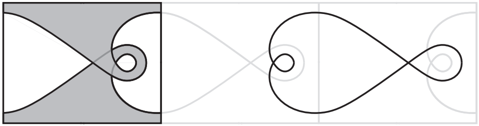
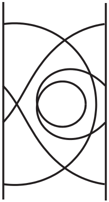
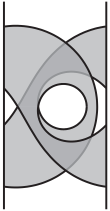
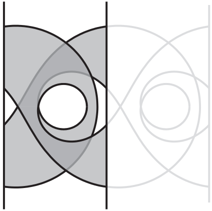
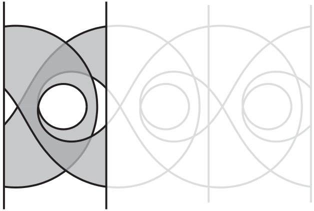


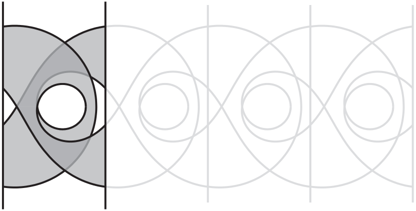
But for a cylinder this will be more complicated:

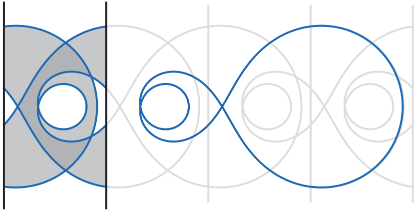


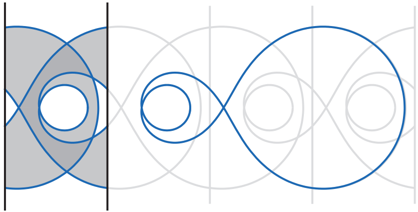






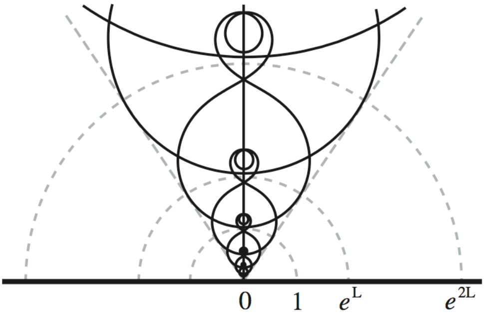


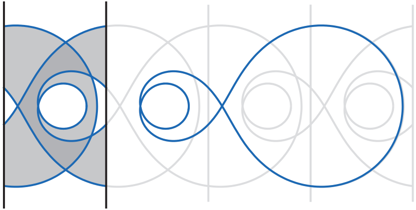




$$\frac{1}{a^2+2a\cos\left(\frac{t}{5}\right)\cos(t)+\cos\left(\frac{t}{5}\right)^2}\left(a+\cos\left(\frac{t}{5}\right)\cos(t),\ \cos\left(\frac{t}{5}\right)\sin(t)\right)$$

III: The hyperbolic case ($K = -1$)





Thanks!