Mean curvature flows and isotopy problems

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Columbia University

10th Pacific Rim Geometry Conference
Osaka City University, Osaka
December 5, 2011

Plan of the talk

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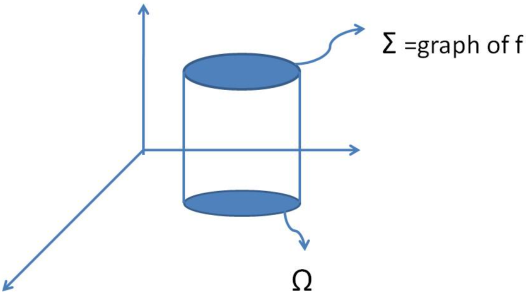
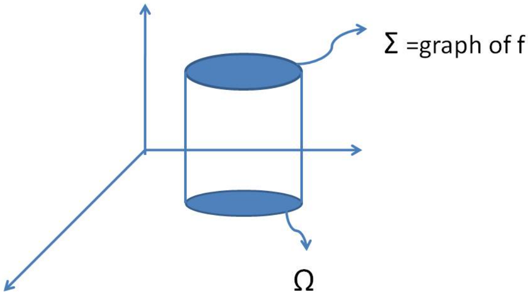
- Joint works with I. Medos, K. Smoczyk, and M.-P. Tsui.



Minimal surface equation



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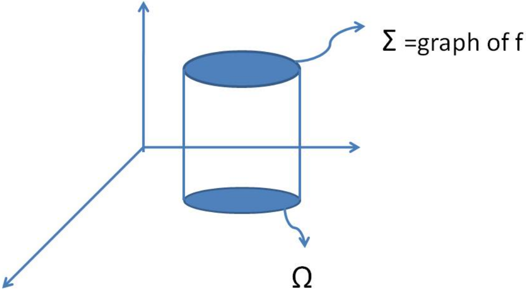

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Euler-Lagrange equation is $div(\frac{\nabla f}{\sqrt{1+|\nabla f|^2|}}) = 0.$

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Any Lipschitz solution is smooth and analytic. (J.Moser, C. B. Morrey, etc.)

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Note that the equation is of non-divergence form.

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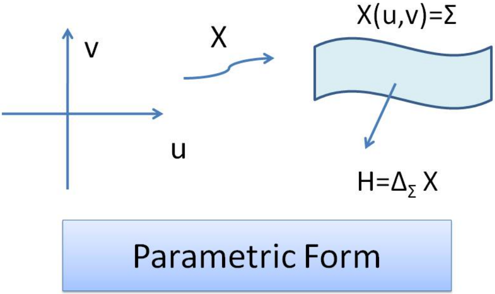

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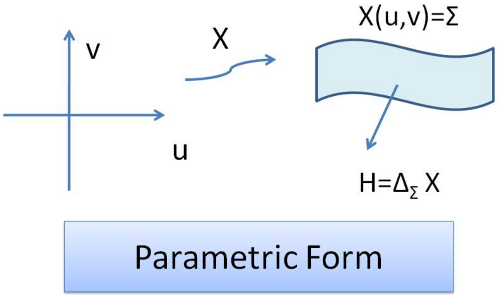

The minimal surface equation is equivalent to

$$\vec{H} = \Delta_\Sigma \overrightarrow{X} = (\Delta_\Sigma X_1, \Delta_\Sigma X_2, \Delta_\Sigma X_3) = (0,0,0)$$

where Σ is the image surface of $\overrightarrow{X}$.

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"Heat equation" for submanifolds.

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Recent results in higher-dimensional parametric case, by Andrews-Baker and Liu-Xu-Ye-Zhao for submanifolds with pinched second fundamental forms.

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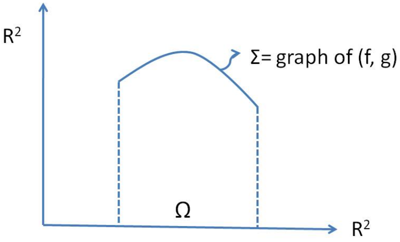

2-dimensional surface in R^4

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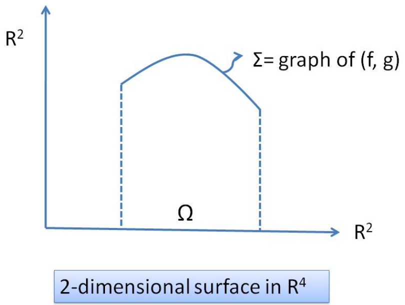

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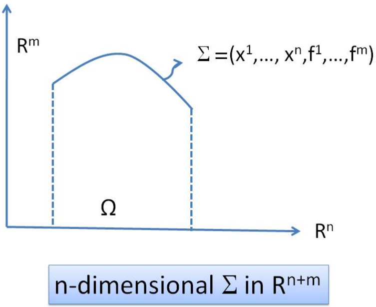
Euler-Lagrange equation is a non-linear elliptic system for f and g.

In general, consider $\vec{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and Σ is the graph of $\vec{f}$ in $\mathbb{R}^{n+m}$.



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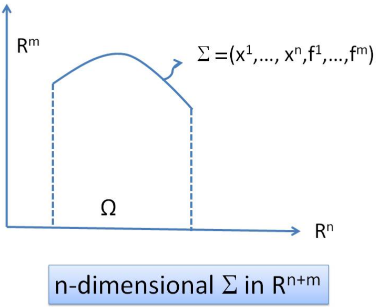

Denote

$$g_{ij} = \delta_{ij} + \sum_{\alpha=1}^{m} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \text{ and } g^{ij} = (g_{ij})^{-1}.$$

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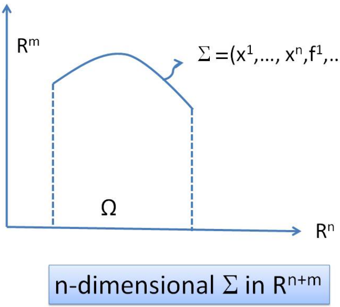

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We can also consider the more general situation when $\mathbf{f} : M_1 \to M_2$ is a differentiable map between Riemannian manifolds, and Σ is the graph of $\mathbf{f}$ in $M_1 \times M_2$.

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Shall discuss estimates and global existence theorems for higher-codimensional mean curvature flows with appropriate initial data.

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$m = 2, n = 2$, $J_2 = \frac{1}{\sqrt{1+|\nabla f|^2+|\nabla g|^2+(f_x g_y - f_y g_x)^2}}$ satisfies

$$\frac{d}{dt} J_2 = \Delta_\Sigma J_2 + R_2(\nabla f, \nabla g, \nabla^2 f, \nabla^2 g)$$

R_2 is quadratic in $\nabla^2 f$ and $\nabla^2 g$ and is positive if $|f_x g_y - f_y g_x| \leq 1$.

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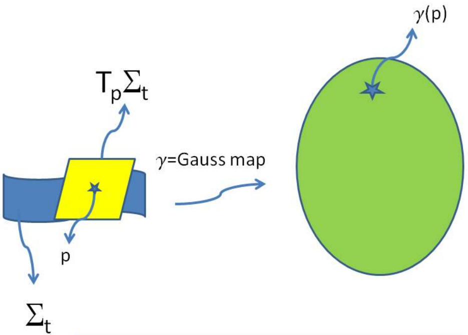
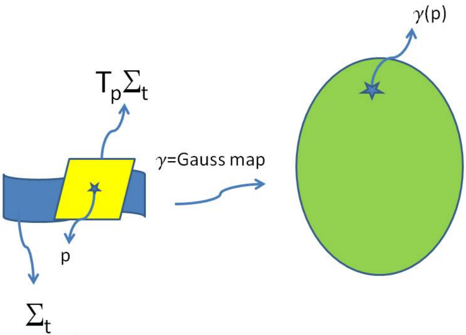
Combining with the evolution equation of J_2, this gives C^1 estimates, and shows that the graphical condition is preserved.



Gauss map of mean curvature flows



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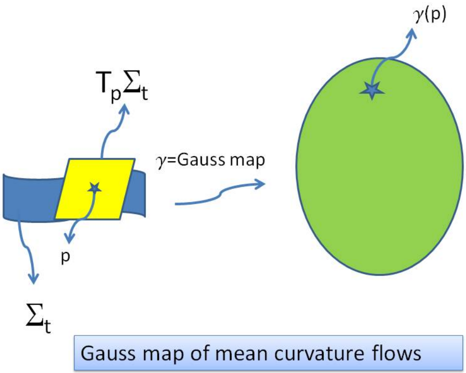

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Underlying fact: the Gauss map of the mean curvature flow is a (nonlinear) harmonic map heat flow.

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Exclusion of self-similar "area-preserving" or "area-decreasing" singularity profiles and ϵ regularity theorems give the desired C^2 estimates.

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Statements of current theorems are cleanest when M_1 and M_2 are closed Riemannian manifolds with suitable curvature conditions.

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- A oriented area-preserving map is a symplectomorphism, i.e. $f^*\omega_2 = \omega_1$ where ω_1 and ω_2 are the area forms of g_1 and g_2, respectively.
- The area A of the graph is a symmetric function on the symplectomorphism group, i.e. $A(f) = A(f^{-1})$ and the mean curvature flow gives a deformation retract.

- (W, 2001, 2004) The mean curvature flow Σ_t exists for all t and converges smoothly to a minimal submanifold as $t \to \infty$. Σ_t is the graph of a symplectic isotopy f_t from f_0 to a canonical minimal map f_∞.

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- For area-decreasing maps, the flow exists for all time and converges to the graph of a constant map.

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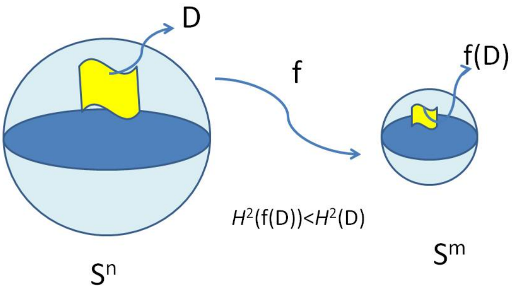
- Area decreasing condition is preserved along the mean curvature flow for $f : S^n \to S^m$ between spheres of constant curvature 1.



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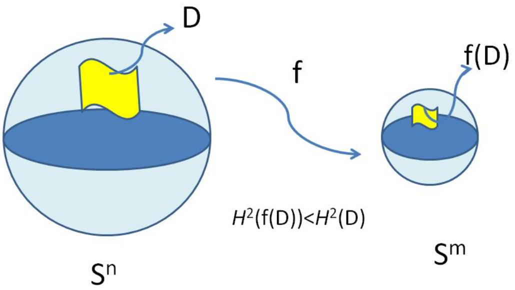
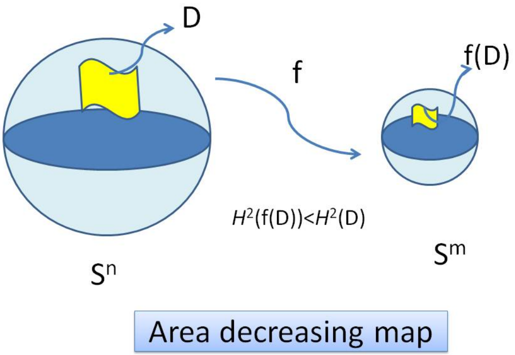


Area decreasing map

(Tsui-W, 2004) For $n, m \geq 2$. If $f : S^n \to S^m$ is an area-deceasing Lipschitz map, the mean curvature flow of the graph of f exists for all time, remains a graph, and converges smoothly to a constant map as $t \to \infty$.



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We express the area-decreasing condition as two-positivity of a Lorentzian metric of signature (n, m) and compute the evolution equation of the Lorentzian metric.

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- L. Guth constructed homotopically non-trivial maps from S^n to S^m with arbitrarily small 3-dilation.

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- Thus $f_t : M_1 \to M_2$ being a symplectomorphism is preserved along the mean curvature flow if both M_1 and M_2 are Kähler manifolds equipped with Kähler-Einstein metric of the same Ricci curvature.
- Take $M_1 = M_2 = \mathbb{CP}^n$, $g_1 = g_2 = g$ Fubini-Study metric.

- (Medos-W, 2011 JDG) There exists $\Lambda > 1$ depending only on n (explicitly computable), such that any symplectomorphism $f : \mathbb{CP}^n \to \mathbb{CP}^n$ with

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- (M. Gromov) : when $n = 2$, the statement holds without any pinching condition by the method of pseudoholomorphic curves. For $n \geq 3$, this seems to be the first known result.

- Unlike previous theorems, Grassmannian geometry does not help here, as the subset that corresponds to biholomorphic isometries does not have any convex neighborhood in the Grassmannian. The integrability condition (Gauss-Codazzi equations) is used in an essential way.

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- $*\Omega > 0 \Leftrightarrow \Sigma$ graphical.
- Along the MCF, $*\Omega$ evolves by

$$\frac{\partial}{\partial t} * \Omega = \Delta * \Omega + *\Omega(Q + A)$$

where Q involves the 2nd fundamental form of Σ and A involves the ambient curvature of $\mathbb{CP}^n$.

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- (1) a_i $\tilde{a}_i$ both unitary. $a_{2k} = J_1 a_{2k-1}$ and $\tilde{a}_{2k} = J_2 \tilde{a}_{2k-1}$, $k = 1 \cdots n$.

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- (3) $\lambda_i \lambda_{i+1} = 1$ and $\lambda_i > 0$ for i odd.

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$$\frac{\partial}{\partial t} * \Omega = \Delta * \Omega + *\Omega \left[Q(\lambda_i, h_{ijk}) + \sum_{i \text{ odd}} \frac{(1-\lambda_i^2)^2}{(1+\lambda_i^2)^2} \right].$$

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- Decompose $h_{ijk} \in \bigodot^3 T_q\Sigma$ into irreducible representations of symmetric groups and estimate the eigenvalue of the restriction of Q on each sub-space.
- We prove $Q(1, \cdots, 1, h_{ijk}) \geq (3 - \sqrt{5}) \sum h_{ijk}^2$.

- By continuity, there exists a Λ such that $\frac{1}{\Lambda} < \lambda_i < \Lambda$ for all i implies $Q(\lambda_i, h_{ijk}) \geq \delta \sum h_{ijk}^2$ for $\delta > 0$.

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- Comparison with ODE gives λ_i approaches 1 as $t \to \infty$.
- The limit f_∞ satisfies $\lambda_i = 1$ for all i and $df_\infty(J_1 X) = J_2 df_\infty(X)$ and f_∞ is holomorphic.

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- Smoczyk-W. defined a generalized Lagrangian mean curvature flow when the ambient space is a cotangent bundle. Short-time existence and preservation of "exactness" and "zero Maslov class" have been established.
- Long time existence that converges to the zero section with applications to the nearby Lagrangian conjecture.

Thank you!