Willmore two-spheres in S^{n+2} via Loop group theory

Peng Wang (with Josef Dorfmeister)

Tongji University

The 10th Pacific Rim Geometric Conference 2011 Osaka-Fukuoka

Background

- $x : M \to S^{n+2}$ Willmore surface: critical surface of the Willmore functional

$$W(M) = \int_M (H^2 - K + 1) dM$$

- Bryant, R. (1984), $x : M \to S^3$ Willmore,
  1. harmonicity of conformal Gauss map
  $$Gr : M \to Gr_{3,1}(\mathbb{R}^5_1) = S^4_1,$$
  2. Duality theorems.
  3. Willmore $S^2 \Longrightarrow$ conformal to minimal surface in $\mathbb{R}^3$.

Background

- $x : M \to S^{n+2}$ Willmore surface: critical surface of the Willmore functional

$$W(M) = \int_M (H^2 - K + 1) dM$$

- Bryant, R. (1984), $x : M \to S^3$ Willmore,
  1. harmonicity of conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^5_1) = S^4_1,$$

  2. Duality theorems.
  3. Willmore $S^2 \Longrightarrow$ conformal to minimal surface in $\mathbb{R}^3$.

Ejiri (1988) $x : M \to S^{n+2}$ Willmore:

- harmonicity of conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1)$$

- S-Willmore surface: Willmore surface with a dual surface,
- Classification of S-Willmore S^2 in S^{n+2}.
- All Willmore S^2 in S^4 are S-Willmore.

Ejiri (1988) $x : M \to S^{n+2}$ Willmore:

- harmonicity of conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1)$$

- S-Willmore surface: Willmore surface with a dual surface,
- Classification of S-Willmore S^2 in S^{n+2}.
- All Willmore S^2 in S^4 are S-Willmore.

Ejiri (1988) $x : M \to S^{n+2}$ Willmore:

- harmonicity of conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1)$$

- S-Willmore surface: Willmore surface with a dual surface,
- Classification of S-Willmore S^2 in S^{n+2}.
- All Willmore S^2 in S^4 are S-Willmore.

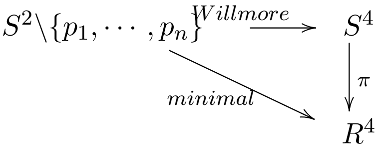
Ejiri (1988) $x : M \to S^{n+2}$ Willmore:

- harmonicity of conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1)$$

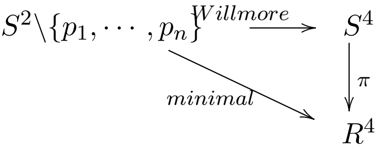
- S-Willmore surface: Willmore surface with a dual surface,
- Classification of S-Willmore S^2 in S^{n+2}.
- All Willmore S^2 in S^4 are S-Willmore.

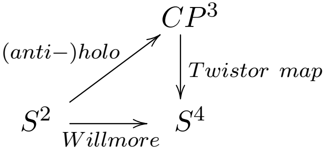
Ejiri (1988), Musso(1990), Montiel (2000): Classification of Willmore S^2 in S^4

- 

-

Ejiri (1988), Musso(1990), Montiel (2000): Classification of Willmore S^2 in S^4

- 

- 

Questions:

- Are there Willmore two spheres in S^5 or S^6 which are non-S-Willmore ?
- Classification of all Willmore S^2 in S^{n+2}.
- How to do with Willmore surfaces by use of the theory on harmonic maps into symmetric spaces?

Questions:

- Are there Willmore two spheres in S^5 or S^6 which are non-S-Willmore ?
- Classification of all Willmore S^2 in S^{n+2}.
- How to do with Willmore surfaces by use of the theory on harmonic maps into symmetric spaces?

Questions:

- Are there Willmore two spheres in S^5 or S^6 which are non-S-Willmore ?
- Classification of all Willmore S^2 in S^{n+2}.
- How to do with Willmore surfaces by use of the theory on harmonic maps into symmetric spaces?

Loop group methods

- Uhlenbeck, K. (1989): All harmonic S^2 in $U(n)$. finite uniton.
- Dorfmeister, J., Pedit, F., Wu, H.Y.(1998): DPW methods for harmonic map $f: M^2 \to G/K$, G, K compact. Normalized potential $\eta = \lambda^{-1}\eta_{-1}$.
- Burstall, F, Guest, M. (1997): All harmonic S^2 in (compact semisimple Lie group) G. ($\Rightarrow$) η_{-1} locates in some nilpotent Lie sub-algebra.

Loop group methods

- Uhlenbeck, K. (1989): All harmonic S^2 in $U(n)$. finite uniton.
- Dorfmeister, J., Pedit, F., Wu, H.Y.(1998): DPW methods for harmonic map $f : M^2 \to G/K$, G, K compact. Normalized potential $\eta = \lambda^{-1}\eta_{-1}$.
- Burstall, F, Guest, M. (1997): All harmonic S^2 in (compact semisimple Lie group) G. $(\Rightarrow)$ η_{-1} locates in some nilpotent Lie sub-algebra.

Loop group methods

- Uhlenbeck, K. (1989): All harmonic S^2 in $U(n)$. finite uniton.
- Dorfmeister, J., Pedit, F., Wu, H.Y.(1998): DPW methods for harmonic map $f : M^2 \to G/K$, G, K compact. Normalized potential $\eta = \lambda^{-1}\eta_{-1}$.
- Burstall, F, Guest, M. (1997): All harmonic S^2 in (compact semisimple Lie group) G. ($\Rightarrow$) η_{-1} locates in some nilpotent Lie sub-algebra.

Our main strategy

- Study Willmore surface by considering the conformal harmonic Gauss map via loop group methods.
- Harmonic map from S^2 into compact Lie group is of finite uniton. For the non-compact case, this property holds too.
- One can describe the conformal harmonic maps of finite uniton explicitly, and as an application, giving all Willmore two-spheres (may have branch points).

Our main strategy

- Study Willmore surface by considering the conformal harmonic Gauss map via loop group methods.
- Harmonic map from S^2 into compact Lie group is of finite uniton. For the non-compact case, this property holds too.
- One can describe the conformal harmonic maps of finite uniton explicitly, and as an application, giving all Willmore two-spheres (may have branch points).

Our main strategy

- Study Willmore surface by considering the conformal harmonic Gauss map via loop group methods.
- Harmonic map from S^2 into compact Lie group is of finite uniton. For the non-compact case, this property holds too.
- One can describe the conformal harmonic maps of finite uniton explicitly, and as an application, giving all Willmore two-spheres (may have branch points).

Basic methods of our work

- Moving frame of Willmore surface in S^{n+2} by Burstall-Pedit-Pinkall.
- DPW methods for harmonic maps in symmetric space, i.e., using Lie-algebra-valued meromorphic 1-form (Normalized potential) to describe harmonic maps.
- Burstall-Guest theory on harmonic map from S^2 into compact Lie group (in term of DPW, the normalized potential is in some nilpotent Lie sub-algebra).

Basic methods of our work

- Moving frame of Willmore surface in S^{n+2} by Burstall-Pedit-Pinkall.
- DPW methods for harmonic maps in symmetric space, i.e., using Lie-algebra-valued meromorphic 1-form (Normalized potential) to describe harmonic maps.
- Burstall-Guest theory on harmonic map from S^2 into compact Lie group (in term of DPW, the normalized potential is in some nilpotent Lie sub-algebra).

Basic methods of our work

- Moving frame of Willmore surface in S^{n+2} by Burstall-Pedit-Pinkall.
- DPW methods for harmonic maps in symmetric space, i.e., using Lie-algebra-valued meromorphic 1-form (Normalized potential) to describe harmonic maps.
- Burstall-Guest theory on harmonic map from S^2 into compact Lie group (in term of DPW, the normalized potential is in some nilpotent Lie sub-algebra).

Main results

- Harmonic maps into compact symmetric space v.s. non compact symmetric spaces.
- From Willmore surface to the conformal Gauss map, and how to go back.
- The finite uniton case:
  classification of nilpotent normalized potential, and going back to the corresponding Willmore surfaces.

Main results

- Harmonic maps into compact symmetric space v.s. non compact symmetric spaces.
- From Willmore surface to the conformal Gauss map, and how to go back.
- The finite uniton case:
  classification of nilpotent normalized potential, and going back to the corresponding Willmore surfaces.

Main results

- Harmonic maps into compact symmetric space v.s. non compact symmetric spaces.
- From Willmore surface to the conformal Gauss map, and how to go back.
- The finite uniton case:
  classification of nilpotent normalized potential, and going back to the corresponding Willmore surfaces.

Strategy of DPW

- G K compact. $f: M^2 \to G/K$ harmonic
  $\dashrightarrow F(z,\bar{z},\lambda): M^2 \to \Lambda G_\sigma$, $\lambda \in S^1$. $\dashrightarrow$
  $F(z,\bar{z},\lambda) = F_-(z,\bar{z},\lambda)F_+(z,\bar{z},\lambda)$ (Birkhoff decomposition)

  $F_- dF_- = \eta = \lambda^{-1}\eta_{-1}dz$.(meromorphic) Normalized potential

- $\eta = \lambda^{-1}\eta_{-1}dz$. $\dashrightarrow F_- dF_- = \eta$
  $\dashrightarrow F_- = F(z,\bar{z},\lambda)F_+(z,\bar{z},\lambda)$ Iwasawa decomposition
  $\dashrightarrow F(z,\bar{z},\lambda): M^2 \to \Lambda G_\sigma \dashrightarrow f: M^2 \to G/K$ harmonic.
- Burstall-Guest: f finite uniton $\Longleftrightarrow \eta_{-1}$ locates in some nilpotent Lie subalgebra.

Strategy of DPW

- G K compact. $f : M^2 \to G/K$ harmonic
  $\dashrightarrow F(z, \bar{z}, \lambda) : M^2 \to \Lambda G_\sigma$, $\lambda \in S^1$. $\dashrightarrow$
  $F(z, \bar{z}, \lambda) = F_-(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda)$ (Birkhoff decomposition)

  $F_- dF_- = \eta = \lambda^{-1} \eta_{-1} dz$.(meromorphic) Normalized potential
- $\eta = \lambda^{-1} \eta_{-1} dz$. $\dashrightarrow F_- dF_- = \eta$
  $\dashrightarrow F_- = F(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda)$ Iwasawa decomposition
  $\dashrightarrow F(z, \bar{z}, \lambda) : M^2 \to \Lambda G_\sigma \dashrightarrow f : M^2 \to G/K$ harmonic.
- Burstall-Guest: f finite uniton $\iff \eta_{-1}$ locates in some nilpotent Lie subalgebra.

Strategy of DPW

- G K compact. $f : M^2 \to G/K$ harmonic
  $\dashrightarrow F(z, \bar{z}, \lambda) : M^2 \to \Lambda G_\sigma$, $\lambda \in S^1$. $\dashrightarrow$
  $F(z, \bar{z}, \lambda) = F_-(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda)$ (Birkhoff decomposition)

  $F_- dF_- = \eta = \lambda^{-1} \eta_{-1} dz$.(meromorphic) Normalized potential

- $\eta = \lambda^{-1} \eta_{-1} dz$. $\dashrightarrow F_- dF_- = \eta$
  $\dashrightarrow F_- = F(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda)$ Iwasawa decomposition
  $\dashrightarrow F(z, \bar{z}, \lambda) : M^2 \to \Lambda G_\sigma \dashrightarrow f : M^2 \to G/K$ harmonic.
- Burstall-Guest: f finite uniton $\Longleftrightarrow \eta_{-1}$ locates in some nilpotent Lie subalgebra.

Non-compact case vs compact case

- G non-compact Lie group, G/K inner symmetric. $\dashrightarrow$ $U \subset G^{\mathbb{C}}$, U compact, and $U^{\mathbb{C}} = G^{\mathbb{C}}, (U \cap K^{\mathbb{C}})^{\mathbb{C}} = K^{\mathbb{C}}$.
- $f : M^2 \to G/K$, f harmonic, $\dashrightarrow$ $(Iwasawa)$ $f_U : M^2 \to U/(U \cap K^{\mathbb{C}})$
  f has the same normalized potential as f_U.
  Especially, f is of finite uniton if and only if f_U is of finite uniton.

Non-compact case vs compact case

- G non-compact Lie group, G/K inner symmetric. $\dashrightarrow$ $U \subset G^{\mathbb{C}}$, U compact, and $U^{\mathbb{C}} = G^{\mathbb{C}}, (U \cap K^{\mathbb{C}})^{\mathbb{C}} = K^{\mathbb{C}}$.
- $f : M^2 \to G/K$, f harmonic, $\dashrightarrow (Iwasawa)$
  $f_U : M^2 \to U/(U \cap K^{\mathbb{C}})$
  f has the same normalized potential as f_U.
  Especially, f is of finite uniton if and only if f_U is of finite uniton.

Willmore surfaces in S^{n+2}

- Let $\mathcal{C}^{n+3}$ be the light cone of Lorentz-Minkowski space $\mathbb{R}^{n+4}_1$, then $S^{n+2} = Q^{n+2} = \{ [x] \in \mathbb{R}P^{n+3} \mid x \in C^{n+3} \setminus \{0\}\}$.
- The conformal group of S^{n+2}: $= SO(1, n+3)$.
- $y : M \to S^{n+2}$ immersion, the conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1) = SO(1, n+3)/SO(1,3) \times SO(n).$$

Gr corresponds to the mean curvature sphere congruence. y is a conformal enveloping surface of Gr.

- y Willmore $\Longleftrightarrow$ Gr harmonic

Willmore surfaces in S^{n+2}

- Let $\mathcal{C}^{n+3}$ be the light cone of Lorentz-Minkowski space $\mathbb{R}^{n+4}_1$, then $S^{n+2} = Q^{n+2} = \{\ [x] \in \mathbb{R}P^{n+3} \mid x \in C^{n+3} \setminus \{0\}\}$.
- The conformal group of S^{n+2}: $= SO(1, n+3)$.
- $y : M \to S^{n+2}$ immersion, the conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1) = SO(1, n+3)/SO(1,3) \times SO(n).$$

  Gr corresponds to the mean curvature sphere congruence. y is a conformal enveloping surface of Gr.
- y Willmore $\Longleftrightarrow$ Gr harmonic

Willmore surfaces in S^{n+2}

- Let $\mathcal{C}^{n+3}$ be the light cone of Lorentz-Minkowski space $\mathbb{R}^{n+4}_1$, then $S^{n+2} = Q^{n+2} = \{\ [x] \in \mathbb{R}P^{n+3} \mid x \in C^{n+3} \setminus \{0\}\}$.
- The conformal group of S^{n+2}: $= SO(1, n+3)$.
- $y : M \to S^{n+2}$ immersion, the conformal Gauss map

$$Gr : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1) = SO(1, n+3)/SO(1,3) \times SO(n).$$

  Gr corresponds to the mean curvature sphere congruence. y is a conformal enveloping surface of Gr.
- y Willmore $\Longleftrightarrow$ Gr harmonic

Willmore surfaces in S^{n+2}

- Let $\mathcal{C}^{n+3}$ be the light cone of Lorentz-Minkowski space $\mathbb{R}^{n+4}_1$, then $S^{n+2} = Q^{n+2} = \{ [x] \in \mathbb{R}P^{n+3} \mid x \in C^{n+3} \setminus \{0\}\}$.
- The conformal group of S^{n+2}: $= SO(1, n+3)$.
- $y : M \rightarrow S^{n+2}$ immersion, the conformal Gauss map

$$Gr : M \rightarrow Gr_{3,1}(\mathbb{R}^{n+4}_1) = SO(1, n+3)/SO(1,3) \times SO(n).$$

  Gr corresponds to the mean curvature sphere congruence. y is a conformal enveloping surface of Gr.
- y Willmore $\Longleftrightarrow$ Gr harmonic

From Willmore surfaces to harmonic maps

- $y : M \to S^{n+2}$ Willmore $\Longrightarrow$,

  $Gr : M \to SO(1, n+3)/SO(1,3) \times SO(n)$ harmonic, the Maurer-Cartan form is of the form

  $$\alpha' = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{pmatrix} dz,$$

  with

  $$B_1^t I_{1,3} B_1 = 0.(\Longrightarrow Rank(B_1) \leqslant 2).$$

- y **S-Willmore** $\Longleftrightarrow$ B_1 is of rank one.

From Willmore surfaces to harmonic maps

- $y: M \to S^{n+2}$ Willmore $\Longrightarrow$,

  $Gr: M \to SO(1,n+3)/SO(1,3) \times SO(n)$ harmonic, the Maurer-Cartan form is of the form

$$\alpha' = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{pmatrix} dz,$$

  with

$$B_1^t I_{1,3} B_1 = 0. (\Longrightarrow Rank(B_1) \leqslant 2).$$

- y **S-Willmore** $\Longleftrightarrow$ B_1 is of rank one.

From harmonic maps going back to Willmore surfaces

Let $f : M \to SO(1,n+3)/SO(1,3) \times SO(n)$ be a harmonic map with its Maurer-Cartan form of f satisfying $B_1^t I_{1,3} B_1 = 0$.

- f envelops a pair of dual Willmore surfaces (hence S-Willmore)
  $\Longleftrightarrow Rank(B_1) = 1$. (One of them may degenerate to a point).
- f envelops a unique surface $y \Longleftrightarrow Rank(B_1) = 2$.(y may degenerate to a point).

From harmonic maps going back to Willmore surfaces

Let $f : M \to SO(1, n+3)/SO(1,3) \times SO(n)$ be a harmonic map with its Maurer-Cartan form of f satisfying $B_1^t I_{1,3} B_1 = 0$.

- f envelops a pair of dual Willmore surfaces (hence S-Willmore)
  $\Longleftrightarrow Rank(B_1) = 1$. (One of them may degenerate to a point).
- f envelops a unique surface $y \Longleftrightarrow Rank(B_1) = 2$.(y may degenerate to a point).

Harmonic maps enveloping a point

- Let $f : M \to SO(1, n+3)/SO(1,3) \times SO(n)$ be a harmonic map with $B_1^t I_{1,3} B_1 = 0$. Then there exists an enveloping surface of f degenerating to a point, if and only if the normalized potential is of the form

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \hat{B}_1 = (v_1, \cdots, v_n),$$

with

$$v_j \hookrightarrow \mathrm{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}, j = 1, \cdots, n.$$

Examples of Willmore surfaces of finite uniton in S^n

- Minimal surfaces in $\mathbb{R}^n$.
- Surfaces in S^4 coming from (anti-)holomorphic curves of the twistor bundle $\mathbb{C}P^3$.

Examples of Willmore surfaces of finite uniton in S^n

- Minimal surfaces in $\mathbb{R}^n$.
- Surfaces in S^4 coming from (anti-)holomorphic curves of the twistor bundle $\mathbb{C}P^3$.

Willmore surfaces of finite uniton in S^6

For a harmonic map $f : M \to SO(1,7)/SO(1,3) \times SO(4)$ of finite uniton, with $B_1^t I_{1,3} B_1 = 0$. Suppose that the normalized potential

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \hat{B}_1 = (v_1, \cdots, v_4).$$

Then up to a conjugation of $SO(1,3) \times SO(4)$, $\hat{B}_1$ must be one of the three cases:

- (1). $v_j \hookrightarrow \text{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}, j = 1, \cdots, 4.$
- (2). $v_2 = iv_1,\ v_3 = iv_4.$
- (3). $v_2 = iv_1,\ v_3, v_4 \hookrightarrow \text{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}.$

Willmore surfaces of finite uniton in S^6

For a harmonic map $f : M \to SO(1,7)/SO(1,3) \times SO(4)$ of finite uniton, with $B_1^t I_{1,3} B_1 = 0$. Suppose that the normalized potential

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \hat{B}_1 = (v_1, \cdots, v_4).$$

Then up to a conjugation of $SO(1,3) \times SO(4)$, $\hat{B}_1$ must be one of the three cases:

- (1). $v_j \hookrightarrow \text{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}, j = 1, \cdots, 4.$
- (2). $v_2 = iv_1$, $v_3 = iv_4$.
- (3). $v_2 = iv_1$, $v_3, v_4 \hookrightarrow \text{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}.$

Willmore surfaces of finite uniton in S^6

For a harmonic map $f : M \to SO(1,7)/SO(1,3)\times SO(4)$ of finite uniton, with $B_1^t I_{1,3} B_1 = 0$. Suppose that the normalized potential

$$\eta = \lambda^{-1}\begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \hat{B}_1 = (v_1, \cdots, v_4)\,.$$

Then up to a conjugation of $SO(1,3)\times SO(4)$, $\hat{B}_1$ must be one of the three cases:

- (1). $v_j \hookrightarrow \mathrm{Span}_{\mathbb{C}}\left\{(1,1,0,0)^t, (0,0,1,i)^t\right\}, j = 1, \cdots, 4.$
- (2). $v_2 = iv_1$, $v_3 = iv_4$.
- (3). $v_2 = iv_1$, $v_3, v_4 \hookrightarrow \mathrm{Span}_{\mathbb{C}}\left\{(1,1,0,0)^t, (0,0,1,i)^t\right\}$.

Going back to Willmore surfaces of finite uniton in S^6

- Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ conformal to minimal surface in } \mathbb{R}^6,$$

$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ degenarates to a point.}$$

- Case (2) $\Rightarrow y$ totally isotropic. For Case (2)\Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ S-Willmore},$$

$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ not S-Willmore.}$$

- Case (3)\Case (2) and Case (1): $\Rightarrow Rank(\hat{B}_1) = 2$

  y having non isotropic Hopf differential, not S-Willmore.
- Case (1) and Case (2) are all the cases such that f is S^1-invariant.

Going back to Willmore surfaces of finite uniton in S^6

- Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ conformal to minimal surface in } \mathbb{R}^6,$$
$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ degenarates to a point.}$$

- Case (2) $\Rightarrow$ y totally isotropic. For Case (2)\Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ S-Willmore},$$
$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ not S-Willmore.}$$

- Case (3)\Case (2) and Case (1): $\Rightarrow Rank(\hat{B}_1) = 2$
  y having non isotropic Hopf differential, not S-Willmore.
- Case (1) and Case (2) are all the cases such that f is S^1-invariant.

Going back to Willmore surfaces of finite uniton in S^6

- Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ conformal to minimal surface in } \mathbb{R}^6,$$
$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ degenarates to a point.}$$

- Case (2) $\Rightarrow$ y totally isotropic. For Case (2)\Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ S-Willmore},$$
$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ not S-Willmore.}$$

- Case (3)\Case (2) and Case (1): $\Rightarrow Rank(\hat{B}_1) = 2$
  y having non isotropic Hopf differential, not S-Willmore.

- Case (1) and Case (2) are all the cases such that f is S^1-invariant.

Going back to Willmore surfaces of finite uniton in S^6

- Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ conformal to minimal surface in } \mathbb{R}^6,$$
$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ degenarates to a point.}$$

- Case (2) $\Rightarrow$ y totally isotropic. For Case (2)\Case (1).

$$Rank(\hat{B}_1) = 1 \Leftrightarrow y \text{ S-Willmore,}$$
$$Rank(\hat{B}_1) = 2 \Leftrightarrow y \text{ not S-Willmore.}$$

- Case (3)\Case (2) and Case (1): $\Rightarrow Rank(\hat{B}_1) = 2$
  y having non isotropic Hopf differential, not S-Willmore.
- Case (1) and Case (2) are all the cases such that f is S^1-invariant.

Examples of Case (2)

- The normalized potential

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz,$$

with

$$\hat{B}_1 = \frac{1}{2} \begin{pmatrix} 2iz & -2z & -i & 1 \\ -2iz & 2z & -i & 1 \\ -2 & -2i & -z & -iz \\ 2i & -2 & -iz & z \end{pmatrix}.$$

- $$Y=\begin{pmatrix} \left(1+r^2+\frac{5r^4}{4}+\frac{4r^6}{9}+\frac{r^8}{36}\right) \\ \left(1-r^2-\frac{3r^4}{4}+\frac{4r^6}{9}-\frac{r^8}{36}\right) \\ -i\left(z-\bar{z})(1+\frac{r^6}{9})\right) \\ \left(z+\bar{z})(1+\frac{r^6}{9})\right) \\ -i\left((\lambda^{-1}z^2-\lambda\bar{z}^2)(1-\frac{r^4}{12})\right) \\ \left((\lambda^{-1}z^2+\lambda\bar{z}^2)(1-\frac{r^4}{12})\right) \\ -i\frac{r^2}{2}(\lambda^{-1}z-\lambda\bar{z})(1+\frac{4r^2}{3}) \\ \frac{r^2}{2}(\lambda^{-1}z+\lambda\bar{z})(1+\frac{4r^2}{3}) \end{pmatrix},\ r=|z|.$$
- $y=[Y]:S^2\to S^6$ is a totally isotropic immersed Willmore sphere which is not S-Willmore .

- $$Y=\begin{pmatrix} \left(1+r^2+\frac{5r^4}{4}+\frac{4r^6}{9}+\frac{r^8}{36}\right) \\ \left(1-r^2-\frac{3r^4}{4}+\frac{4r^6}{9}-\frac{r^8}{36}\right) \\ -i\left(z-\bar{z})(1+\frac{r^6}{9})\right) \\ \left(z+\bar{z})(1+\frac{r^6}{9})\right) \\ -i\left((\lambda^{-1}z^2-\lambda\bar{z}^2)(1-\frac{r^4}{12})\right) \\ \left((\lambda^{-1}z^2+\lambda\bar{z}^2)(1-\frac{r^4}{12})\right) \\ -i\frac{r^2}{2}(\lambda^{-1}z-\lambda\bar{z})(1+\frac{4r^2}{3}) \\ \frac{r^2}{2}(\lambda^{-1}z+\lambda\bar{z})(1+\frac{4r^2}{3}) \end{pmatrix},\ r=|z|.$$
- $y=[Y]:S^2\to S^6$ is a totally isotropic immersed Willmore sphere which is not S-Willmore .

The S^4 case

- Case (3) can not happen.
- Case (1) $\Longrightarrow$ minimal surfaces in R^4.
- For case (2), $rank(B_1) = 1$. The corresponding Willmore surfaces are always S-Willmore having isotropic Hopf differential. $\Rightarrow$ holomorphic or anti-holomorphic curves in CP^3 .

The S^4 case

- Case (3) can not happen.
- Case (1) $\Longrightarrow$ minimal surfaces in R^4.
- For case (2), $rank(B_1) = 1$. The corresponding Willmore surfaces are always S-Willmore having isotropic Hopf differential. $\Rightarrow$ holomorphic or anti-holomorphic curves in CP^3 .

The S^4 case

- Case (3) can not happen.
- Case (1) $\Longrightarrow$ minimal surfaces in R^4.
- For case (2), $rank(B_1) = 1$. The corresponding Willmore surfaces are always S-Willmore having isotropic Hopf differential. $\Rightarrow$ holomorphic or anti-holomorphic curves in CP^3 .

The S^{2m+2} case for Willmore surfaces of finite uniton

- Suppose that $\hat{B}_1 = (v_1, \cdots, v_{2m})$. Then up to a conjugation of $SO(1,3) \times SO(2m)$, $\hat{B}_1$ must be one of the $(m+1)$ cases:

(1).

$$v_j \hookrightarrow \mathrm{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}, j = 1, \cdots, 2m.$$

(2).

$$v_2 = iv_1, \ v_j \hookrightarrow \mathrm{Span}_{\mathbb{C}} \left\{ (1,1,0,0)^t, (0,0,1,i)^t \right\}, j = 3, \cdots, 2m.$$

$$\vdots$$

(m+1).

$$v_2 = iv_1, \ v_4 = iv_3, \cdots, \ v_{2m} = iv_{2m-1}.$$

- Bryant, R. *A duality theorem for Willmore surfaces,* J. Diff.Geom. 20(1984), 23-53.
- Burstall, F.E., Guest, M.A., *Harmonic two-spheres in compact symmetric spaces, revisited,* Math. Ann. 309 (1997), 541-572.
- Burstall, F., Pedit, F., Pinkall, U. *Schwarzian derivatives and flows of surfaces,* Contemporary Mathematics 308, 39-61, Providence, RI: Amer. Math. Soc., 2002
- Dorfmeister, J., Pedit, F., Wu, H., *Weierstrass type representation of harmonic maps into symmetric spaces,* Comm. Anal. Geom. 6 (1998), 633-668.
- Ejiri, N. *Willmore surfaces with a duality in* $S^n(1)$, Proc. London Math.Soc. (3), 57(2)(1988), 383-416.

- Bryant, R. *A duality theorem for Willmore surfaces,* J. Diff.Geom. 20(1984), 23-53.
- Burstall, F.E., Guest, M.A., *Harmonic two-spheres in compact symmetric spaces, revisited,* Math. Ann. 309 (1997), 541-572.
- Burstall, F., Pedit, F., Pinkall, U. *Schwarzian derivatives and flows of surfaces,* Contemporary Mathematics 308, 39-61, Providence, RI: Amer. Math. Soc., 2002
- Dorfmeister, J., Pedit, F., Wu, H., *Weierstrass type representation of harmonic maps into symmetric spaces,* Comm. Anal. Geom. 6 (1998), 633-668.
- Ejiri, N. *Willmore surfaces with a duality in* $S^n(1)$, Proc. London Math.Soc. (3), 57(2)(1988), 383-416.

- Bryant, R. *A duality theorem for Willmore surfaces,* J. Diff.Geom. 20(1984), 23-53.
- Burstall, F.E., Guest, M.A., *Harmonic two-spheres in compact symmetric spaces, revisited,* Math. Ann. 309 (1997), 541-572.
- Burstall, F., Pedit, F., Pinkall, U. *Schwarzian derivatives and flows of surfaces,* Contemporary Mathematics 308, 39-61, Providence, RI: Amer. Math. Soc., 2002
- Dorfmeister, J., Pedit, F., Wu, H., *Weierstrass type representation of harmonic maps into symmetric spaces,* Comm. Anal. Geom. 6 (1998), 633-668.
- Ejiri, N. *Willmore surfaces with a duality in* $S^n(1)$, Proc. London Math.Soc. (3), 57(2)(1988), 383-416.

- Bryant, R. *A duality theorem for Willmore surfaces,* J. Diff.Geom. 20(1984), 23-53.
- Burstall, F.E., Guest, M.A., *Harmonic two-spheres in compact symmetric spaces, revisited,* Math. Ann. 309 (1997), 541-572.
- Burstall, F., Pedit, F., Pinkall, U. *Schwarzian derivatives and flows of surfaces,* Contemporary Mathematics 308, 39-61, Providence, RI: Amer. Math. Soc., 2002
- Dorfmeister, J., Pedit, F., Wu, H., *Weierstrass type representation of harmonic maps into symmetric spaces,* Comm. Anal. Geom. 6 (1998), 633-668.
- Ejiri, N. *Willmore surfaces with a duality in $S^n(1)$,* Proc. London Math.Soc. (3), 57(2)(1988), 383-416.

- Bryant, R. *A duality theorem for Willmore surfaces,* J. Diff.Geom. 20(1984), 23-53.
- Burstall, F.E., Guest, M.A., *Harmonic two-spheres in compact symmetric spaces, revisited,* Math. Ann. 309 (1997), 541-572.
- Burstall, F., Pedit, F., Pinkall, U. *Schwarzian derivatives and flows of surfaces,* Contemporary Mathematics 308, 39-61, Providence, RI: Amer. Math. Soc., 2002
- Dorfmeister, J., Pedit, F., Wu, H., *Weierstrass type representation of harmonic maps into symmetric spaces,* Comm. Anal. Geom. 6 (1998), 633-668.
- Ejiri, N. *Willmore surfaces with a duality in* $S^n(1)$, Proc. London Math.Soc. (3), 57(2)(1988), 383-416.

- Hélein *Willmore immersions and loop groups,* J. Differ. Geom., 50, 1998, 331-385.
- Montiel, S. *Willmore two spheres in the four-sphere,* Trans. Amer.Math. Soc. 2000, 352(10), 4469-4486.
- Musso, E. *Willmore surfaces in the four-sphere,* Ann. Global Anal. Geom. Vol 8, No.1(1990), 21-41.
- Uhlenbeck, K.*Harmonic maps into Lie groups (classical solutions of the chiral model),* J. Diff. Geom. 30 (1989), 1-50.
- Wu, H.Y. *A simple way for determining the normalized potentials for harmonic maps,* Ann. Global Anal. Geom. 17 (1999), 189-199.

- Hélein *Willmore immersions and loop groups,* J. Differ. Geom., 50, 1998, 331-385.
- Montiel, S. *Willmore two spheres in the four-sphere,* Trans. Amer.Math. Soc. 2000, 352(10), 4469-4486.
- Musso, E. *Willmore surfaces in the four-sphere,* Ann. Global Anal. Geom. Vol 8, No.1(1990), 21-41.
- Uhlenbeck, K.*Harmonic maps into Lie groups (classical solutions of the chiral model),* J. Diff. Geom. 30 (1989), 1-50.
- Wu, H.Y. *A simple way for determining the normalized potentials for harmonic maps,* Ann. Global Anal. Geom. 17 (1999), 189-199.

- Hélein *Willmore immersions and loop groups,* J. Differ. Geom., 50, 1998, 331-385.
- Montiel, S. *Willmore two spheres in the four-sphere,* Trans. Amer.Math. Soc. 2000, 352(10), 4469-4486.
- Musso, E. *Willmore surfaces in the four-sphere,* Ann. Global Anal. Geom. Vol 8, No.1(1990), 21-41.
- Uhlenbeck, K.*Harmonic maps into Lie groups (classical solutions of the chiral model),* J. Diff. Geom. 30 (1989), 1-50.
- Wu, H.Y. *A simple way for determining the normalized potentials for harmonic maps,* Ann. Global Anal. Geom. 17 (1999), 189-199.

- Hélein *Willmore immersions and loop groups,* J. Differ. Geom., 50, 1998, 331-385.
- Montiel, S. *Willmore two spheres in the four-sphere,* Trans. Amer.Math. Soc. 2000, 352(10), 4469-4486.
- Musso, E. *Willmore surfaces in the four-sphere,* Ann. Global Anal. Geom. Vol 8, No.1(1990), 21-41.
- Uhlenbeck, K.*Harmonic maps into Lie groups (classical solutions of the chiral model),* J. Diff. Geom. 30 (1989), 1-50.
- Wu, H.Y. *A simple way for determining the normalized potentials for harmonic maps,* Ann. Global Anal. Geom. 17 (1999), 189-199.

- Hélein *Willmore immersions and loop groups,* J. Differ. Geom., 50, 1998, 331-385.
- Montiel, S. *Willmore two spheres in the four-sphere,* Trans. Amer.Math. Soc. 2000, 352(10), 4469-4486.
- Musso, E. *Willmore surfaces in the four-sphere,* Ann. Global Anal. Geom. Vol 8, No.1(1990), 21-41.
- Uhlenbeck, K.*Harmonic maps into Lie groups (classical solutions of the chiral model),* J. Diff. Geom. 30 (1989), 1-50.
- Wu, H.Y. *A simple way for determining the normalized potentials for harmonic maps,* Ann. Global Anal. Geom. 17 (1999), 189-199.

Thank you!