Division Theorems for Exact Sequences

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Skoda's Division Theorem

Skoda's division theorem is an analogue of Hilbert's Nullstellensatz, but the remarkable feature of effectiveness makes it very powerful.

This theorem has many important applications in complex differential geometry and algebraic geometry, including deformation invariance of plurigenera and effective versions of the Nullstellensatz.

The statement of Skoda's theorem is the following:

Let Ω be a pseudoconvex domain in $\mathbb{C}^n, \psi \in \mathrm{PSH}(\Omega)$, $g_1, \cdots, g_r \in \mathcal{O}(\Omega)$, then for every $f \in \mathcal{O}(\Omega)$ with

$$\int_\Omega |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV < +\infty,$$

there exist holomorphic functions $h_1, \cdots, h_r \in \mathcal{O}(\Omega)$ such that

$$f = \sum g_i h_i \text{ on } \Omega$$

and

$$\int_\Omega |h|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi} dV \leq \frac{1+\varepsilon}{\varepsilon} \int_\Omega |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV$$

where$|g|^2 = \sum_i |g_i|^2, |h|^2 = \sum_i |h_i|^2, q = \min\{n, r-1\}$ and $\varepsilon > 0$ is a constant.

This theorem was generalized by Skoda and Demailly to (generic) surjective homomorphisms between holomorphic vector bundles by solving $\overline{\partial}$-equations.

We will talk about how to establish division theorem for general holomorphic homomorphisms.

We establish division theorems for the homomorphisms in an exact sequence of holomorphic vector bundles (among which the last one is surjective).

We consider a complex of holomorphic vector bundles over M,

$$E \xrightarrow{\Phi} E^{'} \xrightarrow{\Psi} E^{''} \ (*)$$

i.e. $\Phi \in \Gamma(M, \mathrm{Hom}(E, E^{'})), \Psi \in \Gamma(M, \mathrm{Hom}(E^{'}, E^{''}))$ such that $\Psi \circ \Phi = 0$. $E, E^{'}, E^{''}$ are assumed to be endowed with Hermitian structures.

We define for any $x \in M$

$$\mathcal{E}(x) = \min\{((\Psi^*\Psi + \Phi\Phi^*)\xi, \xi) | \xi \in E_x^{'}, |\xi| = 1\}$$

where Φ^*, Ψ^* are the adjoint of Φ and Ψ respectively w.r.t. the given Hermitian structures.

It is easy to see that the above complex is exact at $x \in M$ if and only if $\mathcal{E}(x) > 0$.

When the complex (*) is exact, $\Phi^*(\Psi^*\Psi + \Phi\Phi^*)^{-1}|_{\mathrm{Ker}\Psi}$ is a smooth lifting of Φ, So it is possible to establish division theorems by solving a coupled system consisting of

$$\overline{\partial} g = \overline{\partial}[\Phi^*(\Psi^*\Psi + \Phi\Phi^*)^{-1} f] \text{ and } \Phi g = 0$$

where $f \in \Gamma(E^{'})$ satisfying $\Psi f = 0$.

If g is a solution of this system, then
$h \stackrel{def}{=} \Phi^*(\Psi^*\Psi + \Phi\Phi^*)^{-1} f - g \in \Gamma(E)$ and $\Phi h = f$.

In the special case where Φ is surjective and $E^{'}$ is equipped with the quotient Hermitian structure then $\Psi = 0$, $\Phi\Phi^* = Id_{E'}$, and the above system reduces to

$$\overline{\partial} g = \overline{\partial}(\Phi^* f)$$

on the subbundle $\mathrm{Ker}\Phi$.

The difficulty of this method for our case is that $\mathrm{Ker}\Phi$ is no longer a subbundle of E, so it amounts to solving $\overline{\partial}$-equations for solutions valued in a subsheaf, it seems that it is not easy to give sufficient conditions for the solvability of this system.

Main Results

Theorem 1. Let (M, ω) be a Kähler manifold and let $E, E^{'}, E^{''}$ be Hermitian holomorphic vector bundles over M, L a Hermitian line bundle over M. All the Hermitian structures may have singularities in a subvariety $Z \subsetneqq M$ and $\Phi^{-1}(0) \subseteq Z$. Suppose that (*) is generically exact over M, $M \setminus Z$ is weakly pseudoconvex and that the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0, m \geq \min\{n-k+1, r\}, 1 \leq k \leq n$;
2. the curvature of $\mathrm{Hom}(E, E^{'})$ satisfies
$$(F_{X\overline{X}}^{\mathrm{Hom}(E,E^{'})}\Phi, \Phi) \leq 0 \text{ for every } X \in T^{1,0}M;$$
3. the curvature of L satisfies
$$\sqrt{-1}(\varsigma c(L) - \partial\overline{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \overline{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma+\delta)\partial\overline{\partial}\varphi.$$

Then for every $\overline{\partial}$-closed $(n, k-1)$-form f which is valued in $L \otimes E^{'}$ with $\Psi f = 0$ and $\|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}} < +\infty$, there exists a $\overline{\partial}$-closed $(n, k-1)$-form h valued in $L \otimes E$ such that $\Phi h = f$ and

$$\|h\|_{\frac{1}{\varsigma+\tau}} \leq \|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}}.$$

In the above statement,
$q = \max_{M\setminus Z} \mathrm{rank} B_\Phi, \varphi = \log \|\Phi\|$,$0 < \varsigma, \tau \in C^\infty(M)$ and δ is a measurable function on M satisfying $\mathcal{E}(\varsigma + \delta) \geq ||\Phi||^2 \varsigma$.
B_Φ is the second fundamental form of the holomorphic line bundle $\mathrm{Span}_{\mathbb{C}}\{\Phi\}$ in $\mathrm{Hom}(E, E^{'})$.

A Hermitian holomorphic vector bundle (E, h) is said to be m-tensor semi-positive(semi-negative) if the curvature F (of Chern connection) satisfies $\sqrt{-1}F(\eta, \eta) \geq 0 (\leq 0)$ for every $\eta = \eta_{\alpha i} \frac{\partial}{\partial z_\alpha} \otimes e_i \in T^{1,0}M \otimes E$ with $\mathrm{rank}(\eta_{\alpha i}) \leq m$ where $z_1, \cdots, z_n$ are holomorphic coordinates of M, $\{e_1, \cdots, e_r\}$ is a holomorphic frame of E and m is a positive integer. In this case, we write $E \geq_m 0 (E \leq_m 0)$.

Let E be a holomorphic vector bundle over M, $Z \subsetneqq M$ be a subvariety, and h be a Hermitian structure on $E|_{M \setminus Z}$. If for each $z \in Z$, there exist a neighborhood U of z, a smooth frame $\{e_1, \cdots, e_r\}$ over U and some constant $\kappa > 0$ such that the matrix $\left[h_{i\bar{j}}(w) - \kappa\delta_{ij}\right]$ is semi-positive for every $w \in U \setminus Z$ where $h_{i\bar{j}} := h(e_i, e_j)$ and δ_{ij} is the Kronecker delta, then we call h a singular Hermitian structure on E which has singularities in Z.

The curvature of the Chern connection of a Hermitian holomorphic vector bundle is said to be semi-negative in the sense of Griffiths(Nakano) if and only if it is 1-tensor($\min\{n, r\}$-tensor) semi-negative.

Hence a sufficient condition for $(F_{X\overline{X}}^{\mathrm{Hom}(E,E')}\Phi, \Phi) \leq 0$ is given by(since we always assume $E \geq_m 0$ for some positive integer m): E' is semi-negative in the sense of Griffiths.

Theorem 1 applied to

$$\varsigma = 1, \tau = \text{constant} > 0, \text{ and } \delta = |\Phi|^2\mathcal{E}^{-1},$$

we obtain the following corollary
Corollary1. If the condition 3 in theorem 2 is replaced by

$$\sqrt{-1}c(L) \geq \sqrt{-1}q(|\Phi|^2\mathcal{E}^{-1} + 1)\partial\overline{\partial}\varphi,$$

then for every $\overline{\partial}$-closed $(n, k-1)$-form f which is valued in $L \otimes E'$ with

$$\Psi f = 0 \text{ and } \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}} < +\infty$$

there is a $\overline{\partial}$-closed $(n, k-1)$-form h valued in $L \otimes E$ such that $\Phi h = f$ and the following estimate holds

$$\|h\| \leq \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}}.$$

Let M be a complex manifold and E be a holomorphic vector bundle of rank r over M. The Koszul complex associated to a section $s \in \Gamma(E^*)$ is defined as follows

$$0 \to \det E \xrightarrow{d_r} \wedge^{r-1} E \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_1} \mathcal{O}_M \to 0$$

where the boundary operators are given by the interior product

$$d_p = s \lrcorner, 1 \leq p \leq r.$$

It gives a complex since we have $d_{p-1} \circ d_p = 0$ for $1 \leq p \leq r$.

We will apply theorem 1 to

$$\Phi = s \lrcorner \in \Gamma(M, \mathrm{Hom}(\wedge^p E, \wedge^{p-1} E).$$

We can show by direct computation that

$$(F^{\mathrm{Hom}(\wedge^p E, \wedge^{p-1} E)}_{X\overline{X}} \Phi, \Phi) = \binom{r}{p-1} (F^{E^*}_{X\overline{X}} s, s)$$

where $X \in T^{1,0}_x M, x \in M$, which implies that the condition 2 in theorem 1 holds as soon as E is assumed to be semi-positive in the sense of Griffiths.

In the case of Koszul complex, we have the following division theorem:

Theorem 2. Let (M,ω) be a Käler manifold and let E be a Hermitian holomorphic vector bundle over M, L a line bundle over M, $s \in \Gamma(E^*)$. All the Hermitian structures may have singularities in a subvariety $Z \subsetneqq M$. Assume that $s^{-1}(0) \subseteq Z$, and that $M \setminus Z$ is weakly pseudoconvex and that the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0, m \geq \min\{n-k+1, r-p+1\}$;
2. the curvature of L satisfies

$$\sqrt{-1}(\varsigma c(L) - \partial\overline{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \overline{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma+\delta)\partial\overline{\partial}\varphi.$$

Then for any $\overline{\partial}$-closed $(n,k-1)$-form f which is valued in $L \otimes \wedge^{p-1}E$, if $d_{p-1}f = 0$ and $\|f\|_{\frac{\varsigma+\delta}{\varsigma\delta|s|^2}} < +\infty$ there is at least one $\overline{\partial}$-closed $(n,k-1)$-form h valued in $L \otimes \wedge^p E$ such that $d_p h = f$ and the following estimate holds

$$\|h\|_{\frac{1}{\varsigma+\tau}} \leq \|f\|_{\frac{\varsigma+\delta}{\varsigma\delta|s|^2}}.$$

In the above statement, $1 \leq p \leq r, \varphi = \log|s|, 1 \leq k \leq n, 1 \leq p \leq n, q = \min\{n, r-1\}, n = \dim_{\mathbb{C}} M, r = \mathrm{rank}_{\mathbb{C}} E$,
$0 < \varsigma, \tau \in C^\infty(M)$ and $\delta \geq 0$ is a measurable function on M.

Similar to corollary 1, we have the following result

Corollary 2. Let (M, ω) be a Kähler manifold and let E be a Hermitian holomorphic vector bundle over M, L a line bundle over M, $s \in \Gamma(E^*)$. All the Hermitian structures may have singularities in a subvariety $Z \subsetneqq M$. Assume that $s^{-1}(0) \subseteq Z$, and that $M \setminus Z$ is weakly pseudoconvex and the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0, m \geq \min\{n-k+1, r-p+1\}$;
2. the curvature of L satisfies $\sqrt{-1}c(L) \geq \sqrt{-1}q(1+\varepsilon)\partial\overline{\partial}\varphi$.

Then for any $\overline{\partial}$-closed $(n, k-1)$-form f valued in $L \otimes \wedge^{p-1}E$, if $d_{p-1}f = 0$ and $\|f\|_{|s|^{-2}} < +\infty$ there is at least one $\overline{\partial}$-closed $(n, k-1)$-form h valued in $L \otimes \wedge^p E$ such that $d_p h = f$ and the following estimate holds

$$\|h\|^2 \leq \frac{1+\varepsilon}{\varepsilon} \|f\|^2_{|s|^{-2}} ,$$

where $1 \leq p \leq r, 1 \leq k \leq n, \varphi = \log|s|^2, q = \min\{n, r-1\}, n = \dim_{\mathbb{C}} M, r = \operatorname{rank}_{\mathbb{C}} E$ and ε is a positive constant.

Now we discuss the special case of Koszul complex over a domain $\Omega \subseteq \mathbb{C}^n$.
Let $g_1 \cdots, g_r \in \mathcal{O}(\Omega)$, the Koszul complex associated to $g = (g_1 \cdots, g_r)$ is given by

$$0 \to \wedge^r \mathcal{O}^{\oplus r} \xrightarrow{d_r} \wedge^{r-1} \mathcal{O}^{\oplus r} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_2} \wedge \mathcal{O}^{\oplus r} \xrightarrow{d_1} \mathcal{O} \to 0$$

where the boundary operators are defined by $d_p = g\lrcorner, 1 \leq p \leq r$.
It is easy to see that for every
$h = (h_{i_1 \cdots i_p})_{i_1 \cdots i_p = 1}^r \in \Gamma(\Omega, \wedge^p \mathcal{O}^{\oplus r})$(i.e. $h_{i_1 \cdots i_p} \in \mathcal{O}(\Omega)$ and $h_{i_1 \cdots i_p}$ is skew symmetric in $i_1, \cdots, i_p$),we have

$$d_p h = (f_{i_1 \cdots i_{p-1}})_{i_1 \cdots i_{p-1} = 1}^r \in \Gamma(\Omega, \wedge^{p-1} \mathcal{O}^{\oplus r}) \text{ with}$$

$$f_{i_1 \cdots i_{p-1}} = \sum_{1 \leq \nu \leq r} g_\nu h_{\nu i_1 \cdots i_{p-1}}.$$

By introducing the singular Hermitian structure

$$\frac{1}{(\sum_i |g_i|^2)^{q(1+\varepsilon)} e^{\psi}}$$

on the trivial line bundle, we get the following division theorem:

Corollary3. Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain,$g_1 \cdots, g_r \in \mathcal{O}(\Omega), \psi \in \mathrm{PSH}(\Omega)$ and $\varepsilon > 0$ a constant, then for every global section $(f_{i_1 \cdots i_{\ell-1}})^r_{i_1 \cdots i_{\ell-1}=1} \in \Gamma(\Omega, \wedge^{\ell-1} \mathcal{O}_\Omega^{\oplus r})$ $(1 \leq \ell \leq r$) satisfying $\sum\limits_{1 \leq \nu \leq r} g_\nu f_{\nu i_1 \cdots i_{\ell-2}} = 0$ and

$$\int_\Omega |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV < +\infty$$

there exists at least one $(h_{i_1\cdots i_\ell})_{i_1\cdots i_\ell=1}^r \in \Gamma(\Omega, \wedge^\ell \mathcal{O}_\Omega^{\oplus r})$ such that

$$f_{i_1\cdots i_{\ell-1}} = \sum_{1\le\nu\le r} g_\nu h_{\nu i_1\cdots i_{\ell-1}},$$

and

$$\int_\Omega |h|^2|g|^{-2q(1+\varepsilon)}e^{-\psi}dV \le \frac{1+\varepsilon}{\varepsilon}\int_\Omega |f|^2|g|^{-2(q+q\varepsilon+1)}e^{-\psi}dV,$$

where $|g|^2 = \sum_i |g_i|^2$, $|h|^2 = \sum_{i_1<\cdots<i_\ell} |h_{i_1\cdots i_\ell}|^2$,
$|f|^2 = \sum_{i_1<\cdots<i_{\ell-1}} \left|f_{i_1\cdots i_{\ell-1}}\right|^2$, $q = \min\{n, r-1\}$.
Particularly, if $|g| \neq 0$ holds on Ω then the Koszul complex induces an exact sequence on global sections.

The special case of $p = 1$ recovers Skoda's division theorem.

Let Ω be a domain in $\mathbb{C}^n$, and Φ be a $q \times p$ matrix of holomorphic functions on $\Omega, p \geq q$. We denote by $\delta_{i_1 \cdots i_q}$ the $q \times q$ minors of Φ, i.e.

$$\delta_{i_1 \cdots i_q} = det \begin{pmatrix} \Phi_{1i_1} & \cdots & \Phi_{1i_q} \\ \vdots & \ddots & \vdots \\ \Phi_{qi_1} & \cdots & \Phi_{qi_q} \end{pmatrix},$$

where $1 \leq i_1 < i_2 < \cdots < i_q \leq p$. There are $\binom{p}{q}$ distinct minors of order q.

In complex Euclidean spaces, we also have the following division theorem.

Corollary 4. Let $\psi \in \mathrm{PSH}(\Omega)$, $f \in \mathcal{O}^q(\Omega)$, if $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex and there exists a constant $\alpha > 1$ such that

$$\int_\Omega \frac{|f|^2}{(\sum\limits_{i_1<\cdots<i_q} |\delta_{i_1\cdots i_q}|^2)^\beta} e^{-\psi} dV < +\infty,$$

where $\beta = \min\{n, \binom{p}{q} - 1\} \cdot \alpha + 1$. Then there is at least one $h \in \mathcal{O}^p(\Omega)$ which solves the equations $\Phi h = f$.

The Case $\varepsilon = 0$

The technique of Skoda triple which was introduced by Varolin.

Definition A Skoda triple (φ, F, q) consists of a positive integer q and C^2 functions $\varphi : (1, \infty) \to \mathbb{R}, F : (1, \infty) \to \mathbb{R}$ such that

$$x + F(x) > 0, [x + F(x)]\varphi^{'}(x) + F^{'}(x) + 1 > 0$$

and

$$[x + F(x)]\varphi^{''}(x) + F^{''}(x) < 0$$

hold for every $x > 1$.

It is easy to see that $(\varepsilon \log x, 0, q)$ is a Skoda triple where ε is a positive constant and q is a positive integer.

The notion of Skoda triple is quite useful to produce examples of division theorems.

Theorem 3 Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain, $g_i \in \mathcal{O}(\Omega)(1 \leq i \leq p)$, $\psi \in \mathrm{PSH}(\Omega)$. We assume that

$$\|g\| < 1 \text{ holds on } \Omega.$$

For every $f \in \bigwedge^{\ell-1}\mathcal{O}(\Omega)^{\oplus p}$, if

$$g \lrcorner f = 0$$

and

$$\int_\Omega \|f\|^2 \frac{b}{a(b-1)} \|g\|^{-2(q\ell+1)} e^{\varphi\circ\xi-\psi} dV < \infty,$$

then there exists an $u \in \bigwedge^{\ell}\mathcal{O}(\Omega)^{\oplus p}$ such that $\iota_g u = f$ and

$$\int_\Omega \|u\|^2 \frac{1}{(a+\lambda)} \|g\|^{-2q\ell} e^{\varphi\circ\xi-\psi} \leq \int_\Omega \|f\|^2 \frac{b}{a(b-1)} \|g\|^{-2(q\ell+1)} e^{\varphi\circ\xi-\psi}.$$

In the above statement,

$$p \in \mathbb{N}, 1 \le \ell \le p, \xi = 1 - \log \|g\|^2,$$

$$a = \xi + F \circ \xi,$$

$$b = \frac{a\varphi' \circ \xi + F' \circ \xi + 1}{qa\ell} + 1, \lambda = \Lambda \circ \xi,$$

$$\Lambda(x) = \frac{-(1 + F'(x))^2}{F''(x) + (x + F(x))\varphi''(x)},$$

(φ, F, q) is a Skoda triple and

$$q = \begin{cases} \min\{p-1, n\}, & \ell = 1; \\ \min\{p-\ell+1, n\}, & \ell \ge 2. \end{cases}$$

For the Skoda triple $(\varepsilon \log x, 0, q)$, we have

Corollary 5 Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain, $g_i \in \mathcal{O}(\Omega)(1 \leq i \leq p)$, $\psi \in \mathrm{PSH}(\Omega)$. We assume that

$$\|g\| < 1 \text{ holds on } \Omega.$$

For every $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$, if $\iota_g f = 0$ and

$$\int_\Omega \|f\|^2 \frac{(1 - \log \|g\|^2)^\varepsilon}{\|g\|^{2(q\ell+1)}} e^{-\psi} dV < \infty,$$

then there exists some $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$ such that $\iota_g u = f$ and

$$\int_\Omega \|u\|^2 \frac{(1 - \log \|g\|^2)^{\varepsilon-1}}{\|g\|^{2q\ell}} e^{-\psi} \leq \frac{q\ell + \varepsilon + 1}{\varepsilon} \int_\Omega \|f\|^2 \frac{(1 - \log \|g\|^2)^\varepsilon}{\|g\|^{2(q\ell+1)}} e^{-\psi}$$

where $p \in \mathbb{N}, 1 \leq \ell \leq p$, $\varepsilon > o$ is a constant and q is the constant in the previous theorem.

In the case $\ell = 1$, we see that under the assumption that $\|g\| < 1$ on Ω, the integrability condition in corollary 5 is weaker than that in Skoda's division theorem.

We know by definition that $(0, -\frac{1}{2}e^{-\varepsilon(x-1)}, q)$ is another example of Skoda triples where ε is a positive constant and q is the constant as above. Our previous theorem applied to the Skoda triple $(0, -\frac{1}{2}e^{\varepsilon(x-1)}, q)$ gives the following result.

Corollary 6 Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain, $g_i \in \mathcal{O}(\Omega)(1 \leq i \leq p)$, $\psi \in \mathrm{PSH}(\Omega)$. We assume that

$$\|g\| < 1 \text{ holds on } \Omega.$$

For every $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$, if $g \lrcorner f = 0$ and

$$\int_\Omega \|f\|^2 \|g\|^{-2(q\ell+1)} e^{-\psi} dV < \infty,$$

then there exists some $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$ such that $\iota_g u = f$ and

$$\int_\Omega \|u\|^2 \|g\|^{2(-q\ell+\varepsilon)} e^{-\psi} \leq C_\varepsilon \int_\Omega \|f\|^2 \|g\|^{-2(q\ell+1)} e^{-\psi}$$

where $p \in \mathbb{N}, 1 \leq \ell \leq p$, ε and C_ε are both positive constants(C_ε is determined by ε) and q is the constant as above.

Basic Estimates

The Basic Estimate 1 Let (M,ω) be a Kähler manifold, and let E be a Hermitian holomorphic vector bundle over M, L a Hermitian holomorphic line bundle over M. The Hermitian structures of these bundles may have singularity along $\Phi^{-1}(0)$ and $\Omega \Subset M \setminus \Phi^{-1}(0)$ is a pseudoconvex domain with smooth boundary. Assume that the following conditions hold on Ω :

1. $E \geq_m 0, m \geq \min\{n-k+1, r\}, 1 \leq k \leq n$;
2. the curvature of $\mathrm{Hom}(E, E^{'})$ satisfies

$$(F_{X\overline{X}}^{\mathrm{Hom}(E,E^{'})}\Phi, \Phi) \leq 0 \text{ for every } X \in T^{1,0}M;$$

3. the curvature of L satisfies

$$\sqrt{-1}(\varsigma c(L) - \partial\overline{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \overline{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma+\delta)\partial\overline{\partial}\varphi.$$

Then the following estimate

$$\left\| |\Phi|^{-2}\Phi^{*}u + \overline{\partial}^{*}v \right\|_{\Omega,\varsigma+\tau}^{2} + \left\|\overline{\partial}v\right\|_{\Omega,\varsigma}^{2} \geq \|u\|_{\Omega,\frac{\varsigma(\lambda\delta+\lambda\varsigma-\varsigma)}{(\varsigma+\delta)|\Phi|^{2}}}^{2}$$

holds for every $\overline{\partial}$-closed $u \in A^{n,k-1}(\overline{\Omega}, L\otimes E)$ satisfying $|\Phi^{*}u|^{2} \geq \lambda|\Phi|^{2}|u|^{2}$ a.e.(w.r.t.dV_{ω}) on Ω

and every $v \in A^{n,k}(\overline{\Omega}, L \otimes E) \cap \mathrm{Dom}(\overline{\partial}^*)$, where $c(L)$ denotes the Chern form, $q = \max_{\Omega} \mathrm{rank} B_\Phi, \varphi = \log |\Phi|^2$, $0 < \varsigma \in C^\infty(\overline{\Omega})$ and λ, δ, τ are measurable functions on Ω satisfying $\lambda, \tau > 0, \varsigma + \delta \geq 0$.

The Basic Estimate 2 Let Ω be a bounded pseudoconvex domain with smooth boundary and $g_i \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})(1 \leq i \leq p)$ without common zeros on $\bar{\Omega}$.Let $\varphi_1, \varphi_2 \in C^2(\bar{\Omega})$, $0 < a \in C^2(\bar{\Omega})$ and $1 < b, 0 < \lambda$ be measurable functions on Ω. Assume that

$$\varphi_2 = \varphi_1 + log\|g\|^2,$$

$$a\partial_\alpha\partial_{\bar{\beta}}\varphi_1 - \partial_\alpha\partial_{\bar{\beta}}a - \lambda^{-1}\partial_\alpha a\partial_{\bar{\beta}}a \geq q\ell ab\partial_\alpha\partial_{\bar{\beta}}\log\|g\|^2.$$

Then for any $h \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ satisfying

$$\sum_{1 \le \nu \le r} g_\nu h_{\nu i_1 \cdots i_{p-1}} = 0$$

and any $v \in \mathrm{Dom} \bar{\partial}^*_{\varphi_1} \subseteq \bigwedge^{\ell} L^2_{0,1}(\Omega, \varphi_1)^{\oplus p}$ satisfying $\bar{\partial} v = 0$, we have

$$\| \sqrt{a+\lambda} \frac{\bar{g}}{||g||^2} \wedge h + \sqrt{a+\lambda} \bar{\partial}^*_{\varphi_1} v \|^2_{\varphi_1} \ge \int_\Omega \frac{(b-1)a}{b} \|h\|^2 e^{-\varphi_2} dV.$$

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Thank You!