CR Li-Yau Gradient Estimate and Perelman Entropy Formulae

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Contents

Motivations

Problem

geometrization problem of contact 3-manifolds via CR curvature flows

- The Cartan Flow : Spherical CR structure
- The torsion flow : the CR analogue of the Ricci flow

The Torsion Flow

- The torsion flow

$$\begin{cases} \partial_t J_{(t)} = 2A_{J,\theta} \\ \partial_t \theta_{(t)} = -2W\theta_{(t)} \end{cases}.$$

Here $J = i\theta^1 \otimes Z_1 - i\theta^{\bar{1}} \otimes Z_{\bar{1}}$ and $A_{J,\theta} = A_{11}\theta^1 \otimes Z_{\bar{1}} + A_{\bar{1}\bar{1}}\theta^{\bar{1}} \otimes Z_1$.

The CR Yamabe Flow

- In particular, we start from the initial data with vanishing torsion :

$$\begin{cases} \partial_t J_{(t)} = 0 \\ \partial_t \theta_{(t)} = -2W\theta_{(t)} \end{cases}.$$

- The CR Yamabe Flow (Chang-Chiu-Wu, 2010, Chang-Kuo, 2011)

$$\partial_t \theta_{(t)} = -2W\theta_{(t)}.$$

Poincare Conjecture and Thurston Geometrization Conjecture via Ricci Flow

- Sphere and Torus decomposition
- Singularity formation
  1. Li-Yau gradient estimate for heat equation (1986)
  2. Hamilton-Ivy curvature pinching estimate (1982, 1995)
  3. Hamilton Harnack inequality (1982, 1988, 1993, etc)
  4. Perelman entropy formulae and reduce distance (2002, 2003)
- Geometric surgery by Hamilton and Perelman

Geometrization problem of contact 3-manifolds

- Contact Decomposition theorem and Classification
- CR Geometric and Analytic aspects :
  1. *Existence of a " best possible geometric CR structure" on closed contact 3-manifolds- spherical CR structure with vanishing torsion.*
  2.

$$R_{ij} \text{ : Ricci curvature tensor} \leftrightarrow A_{11} \text{ : pseudohermitian torsion}$$

Problem

Sub-Laplacian Δ_b is degenerated along the missing dirction T by comparing the Riemannian Laplacian Δ.

geometrization problem of contact 3-manifolds

Problem

We proposed to deform any fixed CR structure under the torsion on a contact three dimensional space which shall break up due to the contact topological decomposition.

Problem

The asymptoic state of the torsion flow is expected to be broken up into pieces which satisfy the spherical CR structure with vanishing torsion.

Problem

The deformation will encounter singularities. The major question is to find a way to describe all possible singularities.

Pseudohermitian 3-manifold

- Let (M, J, θ) be the pseudohermitian 3-manifold.
  1. (M, θ) is a contact 3-manifold with $\theta \wedge d\theta \neq 0$. $\xi = \ker \theta$ is called the contact structure on M.
  2. A CR-structure compatible with ξ is a smooth endomorphism $J : \xi \to \xi$ such that $J^2 = -identity$.
  3. The CR structure J can extend to $\mathbf{C} \otimes \boldsymbol{\xi}$ and decomposes $\mathbf{C} \otimes \boldsymbol{\xi}$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively.

Pseudohermitian 3-manifold

- Given a pseudohermitian structure (J, θ) :
  1. The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle$.
  2. The characteristic vector field of θ is the unique vector field T such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$.
  3. Then $\{T, Z_1, Z_{\bar{1}}\}$ is the frame field for TM and $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is the coframe.

Pseudohermitian 3-manifold

- The pseudohermitian connection of (J, θ) is the connection $\nabla^{\psi.h.}$ on $TM \otimes C$ (and extended to tensors) given by

$$\nabla^{\psi.h.} Z_1 = {\omega_1}^1 \otimes Z_1, \nabla^{\psi.h.} Z_{\bar{1}} = {\omega_{\bar{1}}}^{\bar{1}} \otimes Z_{\bar{1}}, \nabla^{\psi.h.} T = 0$$

with

$$d\theta^1 = \theta^1 \wedge {\omega_1}^1 + {A^1}_{\bar{1}} \theta \wedge \theta^{\bar{1}}$$
$${\omega_1}^1 + {\omega_{\bar{1}}}^{\bar{1}} = 0.$$

- Differentiating ${\omega_1}^1$ gives

$$d{\omega_1}^1 = W \theta^1 \wedge \theta^{\bar{1}} \pmod{\theta}$$

where W is the Tanaka-Webster curvature.

- We can define the covariant differentiations with respect to the pseudohermitian connection.
  1. $$f_{,1} = Z_1 f \ ; \quad f_{1\bar{1}} = Z_{\bar{1}} Z_1 f - \omega_1{}^1 (Z_{\bar{1}}) Z_1 f.$$
  2. We define the subgradient operator ∇_b and the sublaplacian operator Δ_b
     $$\nabla_b f = f_{,\bar{1}} Z_1 + f_{,1} Z_{\bar{1}},$$
     and
     $$\Delta_b f = f_{,1\bar{1}} + f_{,\bar{1}1}.$$

Pseudohermitian 3-manifold

Example

D is the strictly pseudoconvex domain

$$D \subset \mathbf{C}^2 \quad \text{and} \quad M = \partial D$$

with

$$D = \{r < 0\} \quad \text{and} \quad M = \{r = 0\}.$$

Choose

$$\xi = TM \cap J_{\mathbf{C}^2} TM \quad \text{and} \quad \theta = -i\partial r|_M$$

with

$$J = J_{\mathbf{C}^2}|_\xi.$$

Li-Yau Harnack Estimate

Theorem

(Li-Yau, 1986) The Li-Yau Harnack estimate

$$\frac{\partial(\ln u)}{\partial t} - |\nabla \ln u|^2 + \frac{m}{2t} \geq 0$$

for the positive solution $u(x,t)$ *of the time-independent heat equation*

$$\frac{\partial u\,(x,t)}{\partial t} = \Delta u\,(x,t)$$

in a complete Riemannian m-manifold with nonnegative Ricci curvature.

Li-Yau-Hamilton Inequality

Theorem

(Hamilton, 1993) Hamilton Harnack estimate (trace version)

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla R \cdot V + 2Ric(V, V) \geq 0$$

for the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

on Riemannian manifolds with positive curvature operator.

Subelliptic Li-Yau gradient estimate

Consider the heat equation

$$(L - \frac{\partial}{\partial t}) u(x, t) = 0$$

in a closed m-manifold with a positive measure and an operator with respect to the sum of squares of vector fields

$$L = \sum_{i=1}^{l} X_i^2, \quad l \leq m,$$

where $X_1, X_2, \ldots, X_l$ are smooth vector fields which satisfy Hörmander's condition : the vector fields together with their commutators up to finite order span the tangent space at every point of M.

Subelliptic Li-Yau gradient estimate

Theorem

(Cao-Yau, 1994) Suppose that $[X_i, [X_j, X_k]]$ *can be expressed as linear combinations of* $X_1, X_2, \ldots, X_l$ *and their brackets* $[X_1, X_2], \ldots, [X_{l-1}, X_l]$. *Then, for the positive solution* $u(x,t)$ *of heat flow on* $M \times [0, \infty)$, *there exist constants* C', C'', C''' *and* $\frac{1}{2} < \lambda < \frac{2}{3}$, *such that for any* $\delta > 1$, $f(x,t) = \ln u(x,t)$ *satisfies the following gradient estimate*

$$\sum_i |X_i f|^2 - \delta f_t + \sum_\alpha (1 + |Y_\alpha f|^2)^\lambda \leq \frac{C'}{t} + C'' + C''' t^{\frac{\lambda}{\lambda - 1}}$$

with $\{Y_\alpha\} = \{[X_i, X_j]\}$.

By choosing a frame $\{\mathbf{T}, Z_1, Z_{\bar{1}}\}$ of $TM \otimes \mathbf{C}$ with respect to the Levi form and $\{X_1, X_2\}$ such that

$$J(Z_1) = iZ_1 \text{ and } J(Z_{\bar{1}}) = -iZ_{\bar{1}}$$

and

$$Z_1 = \frac{1}{2}(X_1 - iX_2) \text{ and } Z_{\bar{1}} = \frac{1}{2}(X_1 + iX_2),$$

it follows that

$$[X_1, X_2] = -2\mathbf{T} \quad \text{and} \quad \Delta_b = \frac{1}{2}(X_1^2 + X_2^2) = \frac{1}{2}L.$$

Note that

$$W(Z, Z) = Wx^1 x^{\bar{1}} \quad \text{and} \quad Tor(Z, Z) = 2Re\ (iA_{\bar{1}\bar{1}} x^{\bar{1}} x^{\bar{1}})$$

for all $Z = x^1 Z_1 \in T_{1,0}$.

The CR-pluriharmonic Operator

Definition

(Graham-Lee, 1988) Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold. Define

$$P\varphi = \sum_{\alpha=1}^{n} (\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})\theta^{\beta} = (P_{\beta}\varphi)\theta^{\beta}, \quad \beta = 1, 2, \cdots, n$$

which is an operator that characterizes CR-pluriharmonic functions. Here

$$P_{\beta}\varphi = \sum_{\alpha=1}^{n} (\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})$$

CR Li-Yau gradient estimate

Theorem

(Chang-Tie-Wu, 2009) Let (M, J, θ) *be a closed pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. If* $u(x,t)$ *is the positive solution of CR heat flow on* $M \times [0, \infty)$ *such that u is the CR-pluriharmonic function*

$$Pu = 0$$

at $t = 0$*. Then*

$$|\nabla_b f|^2 + 3f_t \leq \frac{9}{t}$$

on $M \times [0, \infty)$.

CR Li-Yau Gradient Estimate

Theorem

(Chang-Kuo-Lai, 2011) Let $(M,\ J,\ \theta)$ *be a closed pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric\,(X,X)-(n-2)\,Tor\,(X,\ X)\geq 0$$

for all $X\in T_{1,0}\oplus T_{0,1}$. *If* $u\,(x,t)$ *is the positive solution of*

$$\left(\Delta_b-\frac{\partial}{\partial t}\right)u\,(x,t)=0$$

with $[\Delta_b,\ \mathbf{T}]\,u=0$ *on* $M\times[0,\ \infty)$. *Then* $f\,(x,t)=\ln u\,(x,t)$ *satisfies the following subgradient estimate*

$$\left[|\nabla_b f|^2-(1+\frac{3}{n})f_t+\frac{n}{3}t(f_0)^2\right]<\frac{(\frac{9}{n}+6+n)}{t}.$$

CR Li-Yau gradient estimate

- subgradient estimate of the logarithm of the positive solution to heat flow :

Theorem

Let (M, J, θ) be a closed pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. If $u(x,t)$ is the positive solution of CR heat flow on $M \times [0, \infty)$ such that u is the CR-pluriharmonic function

$$Pu = 0$$

at $t = 0$. Then there exists a constant C_1 such that u satisfies the subgradient estimate

$$\frac{|\nabla_b u|^2}{u^2} \leq \frac{C_1}{t}$$

on $M \times (0, \infty)$.

CR Li-Yau Gradient Estimate for Witten Sublaplacian

We consider the heat equation

$$\frac{\partial u\,(x,t)}{\partial t} = Lu(x,t)$$

in a closed pseudohermitian $(2n+1)$-manifold $(M, J, \theta,\ d\mu)$ with

$$Lu\,(x,t) := \Delta_b u\,(x,t) - \nabla_b \phi(x) \cdot \nabla_b u\,(x,t)\,.$$

Here $d\mu = e^{-\phi(x)}\theta \wedge (d\theta)^n$.

Bakry-Emery pseudohermitian Ricci curvature

- The ∞-dimensional Bakry-Emery pseudohermitian Ricci curvature

$$Ric(L)(W,W) := R_{\alpha\bar{\beta}} W_{\bar{\alpha}} W_{\beta} + \mathrm{Re}[\phi_{\alpha\bar{\beta}} W_{\bar{\alpha}} W_{\beta}]$$

- The m-dimensional Bakry-Emery pseudohermitian Ricci curvature

$$Ric_{m,n}(L) := Ric(L) - \frac{\nabla_b \phi \otimes \nabla_b \phi}{2(m-2n)}, \quad m > 2n$$

$$Tor(L)(W,W) := 2\,\mathrm{Re}[\sum_{\alpha,\beta=1}^{n} (i(n-2)A_{\bar{\alpha}\bar{\beta}} - \phi_{\bar{\alpha}\bar{\beta}}) W_{\alpha} W_{\beta}].$$

CR Li-Yau Gradient Estimate for Witten Sublaplacian

Theorem

Let $(M,\ J,\ \theta)$ *be a closed pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric_{m,n}(L)(X,\ X) - Tor(L)(X,\ X) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. *If* $u(x,t)$ *is the positive solution of* $\left(L - \frac{\partial}{\partial t}\right) u(x,t) = 0$ *with*

$$[L,\ \mathbf{T}]\, u = 0$$

on $M \times [0,\ \infty)$. *Then* $f(x,t) = \ln u(x,t)$ *satisfies the following Li-Yau type subgradient estimate*

$$\left[|\nabla_b f|^2 - (1 + \frac{3}{n}) f_t + \frac{n}{3} t (f_0)^2 \right] < \frac{m}{2nt} \left[\frac{9}{n} + 6 + n \right].$$

CR Harnack inequality

Theorem

Let $(M,\ J,\ \theta)$ *be a closed pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric_{m,n}(L)\left(X,\ X\right) - Tor(L)\left(X,\ X\right) \geq 0$$

for all $X \in T_{1,0} \oplus T_{0,1}$. *If* $u\left(x,t\right)$ *is the positive solution of* $\left(L - \frac{\partial}{\partial t}\right) u\left(x,t\right) = 0$ *with*

$$\left[L,\ \mathbf{T}\right] u = 0$$

on $M \times [0,\ \infty)$. *Then for any* x_1, x_2 *in* M *and* $0 < t_1 < t_2 < \infty$, *we have the Harnack inequality*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-\left[\frac{m(\frac{9}{n}+6+n)}{2n(1+\frac{3}{n})}\right]} \exp\{-\frac{(1+\frac{3}{n})}{4}\left[\frac{d_{cc}(x_1,x_2)^2}{(t_2-t_1)}\right]\}.$$

CR Li-Yau Gradient Estimate

Note that

$$[L,\ \mathbf{T}]\,u = 2\,\mathrm{Im}\,Qu - 4\,\mathrm{Re}(\phi_\alpha u_\beta A_{\bar\alpha\bar\beta}) + \langle \nabla_b \phi_0, \nabla_b u\rangle$$

and

$$[\Delta_b,\ \mathbf{T}]\,u = 2\,\mathrm{Im}\,Qu.$$

Here Q is the purely holomorphic second-order operator defined by

$$Qu = 2i(A_{\overline{\alpha}\overline{\beta}}u_\alpha)_\beta.$$

CR Li-Yau-Hamilton Inequality

Theorem

(Chang-Kuo, 2011) Let $(M, J, \mathring{\theta})$ *be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then*

$$4\frac{|\nabla_b W|^2}{W^2} - \frac{W_t}{W} - \frac{1}{t} \leq 0$$

under the CR Yamabe flow

$$\frac{\partial}{\partial t}\theta(t) = -2W(t)\theta(t), \quad \theta(0) = \mathring{\theta}.$$

Furthermore, we get a subgradient estimate of logarithm of the positive Tanaka-Webster curvature

$$\frac{|\nabla_b W|^2}{W^2} \leq \frac{1}{4t}$$

for all $t \in (0, T)$.

CR Li-Yau-Hamilton Inequality

Theorem

(Chang-Kuo, 2011) Let $(M, J, \mathring{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then

$$4\frac{|\nabla_b u|^2}{u^2} - \frac{u_t}{u} - \frac{1}{t} \leq 0$$

under the time-dependent CR heat equations with potentials evolving by the CR Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t}\theta(t) = -2W(t)\theta(t), \\ \frac{\partial u}{\partial t} = 4\Delta_b u + 2Wu, \quad u_0(x,0) = 0. \end{cases}$$

Perelman Entropy Formulae

Theorem

Its monotonicity property of the Perelman entropy functional together with Li-Yau-Perelman reduced distance imply the no local collapsing theorem under the Ricci flow.

Perelman Entropy Formulae

Theorem

G. Perelman proved that the $\mathcal{F}$-functional

$$\mathcal{F}(g_{ij},\varphi)=\int_M (R+|\nabla\varphi|^2)e^{-\varphi}d\mu$$

is nondecreasing under the following coupled Ricci flow

$$\begin{cases} \frac{\partial g_{ij}}{\partial t}=-2R_{ij}, \\ \frac{\partial\varphi}{\partial t}=-\triangle\varphi+|\nabla\varphi|^2-R, \end{cases}$$

in a closed Riemannian m-manifold (M,g_{ij}).

Theorem

G. Perelman showed that the $\mathcal{W}$-functional

$$\mathcal{W}(g_{ij},\varphi,\tau)=\int_M[\tau(R+|\nabla\varphi|^2)+\varphi-m](4\pi\tau)^{-\frac{m}{2}}e^{-\varphi}d\mu,\ \ \tau>0$$

is nondecreasing as well under the following coupled Ricci flow

$$\begin{cases}\frac{\partial g_{ij}}{\partial t}=-2R_{ij},\\ \frac{\partial\varphi}{\partial t}=-\triangle\varphi+|\nabla\varphi|^2-R+\frac{m}{2\tau},\\ \frac{d\tau}{dt}=-1.\end{cases}$$

Perelman Entropy Formulae

- The Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

coupled with the conjugate heat equation

$$\frac{\partial u}{\partial t} = -\triangle u + Ru.$$

1. For $u = e^{-\varphi}$.
2. For $u = (4\pi\tau)^{-\frac{m}{2}} e^{-\varphi}, \ \tau = T - t$.

CR Li-Yau-Perelman Harnack Estimate

Theorem

Let $(M, J, \mathring{\theta})$ *be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Under*

$$\begin{cases} \frac{\partial}{\partial t}\theta(t) = -2W(t)\theta(t), \\ \frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x,0) = 0, \end{cases}$$

we have

$$\Delta_b f - \frac{3}{4}|\nabla_b f|^2 + \frac{1}{2}W - \frac{1}{\tau} \leq 0$$

on $M \times [0, T)$ *with* $u = e^{-f}$ *and* $\tau = T - t$.

CR Perelman Entropy Formulae

We define the CR Perelman $\mathcal{F}$-functional by

$$\begin{aligned}\mathcal{F}(\theta(t), f(t)) &= 4\int_M [(\Delta_b f - \tfrac{3}{4}|\nabla_b f|^2 + \tfrac{1}{2}W)]e^{-f}\,d\mu \\ &= \int_M (2W + |\nabla_b f|^2)e^{-f}\,d\mu.\end{aligned}$$

with the constraint

$$\int_M e^{-f}\,d\mu = 1.$$

Monotonicity Property of CR Perelman Entropy

We derive the following monotonicity property of CR $\mathcal{F}$-functional.

Theorem

Let $(M, J, \mathring{\theta})$ be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Then

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\theta(t), f(t)) &= 8\int_M \left|\left(\nabla^H\right)^2 f + \frac{W}{2}L_\theta\right|^2 ud\mu + 2\int_M W\left|\nabla_b f\right|^2 ud\mu \\ &\geq 0\end{aligned}$$

under

$$\begin{cases} \frac{\partial}{\partial t}\theta(t) = -2W(t)\,\theta(t), \\ \frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x,0) = 0. \end{cases}$$

CR Li-Yau-Perelman Harnack Estimate

Theorem

Let $(M, J, \mathring{\theta})$ *be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Under*

$$\begin{cases} \frac{\partial}{\partial t}\theta(t) = -2W(t)\theta(t), \\ \frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x,0) = 0, \end{cases}$$

we have

$$\Delta_b f - \frac{3}{4}|\nabla_b f|^2 + \frac{1}{2}W + \frac{f}{8\tau} - \frac{1}{2\tau} \leq 0$$

on $M \times [0, T)$ *with* $u = (4\pi\tau)^{-2}e^{-f}$ *and* $\tau = T - t$.

CR Perelman Entropy Formulae

Define the CR Perelman $\mathcal{W}$-functional by

$$\begin{aligned}\mathcal{W}(\theta(t), f(t), \tau) &= 4\int_M \tau[\Delta_b f - \tfrac{3}{4}|\nabla_b f|^2 + \tfrac{1}{2}W + \tfrac{f}{8\tau} - \tfrac{1}{2\tau}]\tfrac{e^{-f}}{(4\pi\tau)^2}d\mu \\ &= \int_M [\tau(2W + |\nabla_b f|^2) + \tfrac{f}{2} - 2](4\pi\tau)^{-2}e^{-f}d\mu,\end{aligned}$$

with the constraint

$$\int_M (4\pi\tau)^{-2}e^{-f}d\mu = 1.$$

Monotonicity Property of CR Perelman Entropy

Theorem

Let $(M, J, \mathring{\theta})$ *be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Then*

$$\begin{aligned}&\tfrac{d}{dt}\mathcal{W}(\theta(t), f(t), \tau(t))\\&= 8\tau \int_M \left| \left(\nabla^H\right)^2 f + \tfrac{W}{2} L_\theta - \tfrac{1}{4\tau} L_\theta \right|^2 u d\mu\\&\quad + \tau \int_M [2W \left|\nabla_b f\right|^2 + \tfrac{|\nabla_b f|^2}{\tau}] u d\mu\\&\geq 0\end{aligned}$$

uner

$$\begin{cases} \frac{\partial}{\partial t}\theta(t) = -2W(t)\theta(t), \\ \frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x, 0) = 0, \end{cases}$$

for $u = (4\pi\tau)^{-2} e^{-f}$ *and* $\tau = T - t$.

CR Li-Yau-Perelman Reduced Distance

Let p, q be two point in M and $\gamma(\tau)$, $\tau \in [0, \overline{\tau}]$, be a Legendrian curve joining p and q with $\gamma(0) = p$ and $\gamma(\overline{\tau}) = q$.

Theorem

Let $(M, J, \mathring{\theta})$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Under under

$$\begin{cases} \frac{\partial}{\partial t}\theta(t) = -2W(t)\theta(t), \\ \frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x, 0) = 0, \end{cases}$$

We have

$$f(q, \overline{\tau}) \leq \frac{2}{\sqrt{\overline{\tau}}} \int_0^{\overline{\tau}} \sqrt{\tau}\left(W + \frac{1}{8}\langle \dot{\gamma}(\tau), \dot{\gamma}(\tau) \rangle_{L_\theta}\right) d\tau.$$

CR Li-Yau-Perelman Reduced Distance

For

$$\mathcal{L}(\gamma) = \int_0^{\overline{\tau}} \sqrt{\tau}\big(W + \frac{1}{8}\langle\, \dot{\gamma}(\tau),\ \dot{\gamma}(\tau)\,\rangle_{L_\theta}\big)d\tau,$$

one can define the CR Perelman reduced distance by

$$l_{cc}(q, \overline{\tau}) \equiv \inf_{\gamma} \frac{2}{\sqrt{\overline{\tau}}}\mathcal{L}(\gamma)$$

and CR Perelman reduced volume by

$$V_{cc}(\overline{\tau}) \equiv \int_M (4\pi\overline{\tau})^{-l_0} \exp\{-\frac{2}{\sqrt{\overline{\tau}}}L(x, \overline{\tau})\}d\mu$$

where $\inf f$ is taken over all Legendrian curves $\gamma(\tau)$ joining p, q and $L(x, \overline{\tau})$ is the corresponding minimum for $\mathcal{L}(\gamma)$.

The CR Yamabe Shrinking Soliton

Theorem

There is no nontrivial closed shrinking CR Yamabe soliton on a closed pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing pseudohermitian torsion.

Theorem

If $(M, J, \mathring{\theta})$ is a closed spherical CR 3-manifold with vanishing torsion and positive CR Yamabe constant, then solutions of the CR (normalized) Yamabe flow converge smoothly to, up to the CR automorphism, a unique limit contact form of constant Webster scalar curvature as $t \longrightarrow \infty$.

The Proofs: The CR Bochner Formulae for Sublaplacian

Theorem

(Greenleaf, 1986) Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold. For a real smooth function u on (M, J, θ),

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b u|^2 &= |(\nabla^H)^2 u|^2 + <\nabla_b u, \nabla_b \Delta_b u>_{L_\theta} \\ &+ (2Ric - nTor)((\nabla_b u)_{\mathbf{C}}, (\nabla_b u)_{\mathbf{C}}) \\ &- 2i\sum_{\alpha=1}^{n}(u_\alpha u_{\bar{\alpha}0} - u_{\bar{\alpha}} u_{\alpha 0}).\end{aligned}$$

The Proofs: The CR Bochner Formulae

Theorem

(Greenleaf, 1986; Chang-Chiu, 2009) Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold. For a real smooth function u on (M, J, θ),

$$\begin{aligned} \frac{1}{2}\Delta_b|\nabla_b u|^2 &= |(\nabla^H)^2 u|^2 + (1+\frac{2}{n}) < \nabla_b u, \nabla_b \Delta_b u >_{L_\theta} \\ &+ [2Ric + (n-4)Tor]((\nabla_b u)_{\mathbf{C}}, (\nabla_b u)_{\mathbf{C}}) \\ &- \frac{4}{n} < Pu + \overline{P}u, d_b u >_{L_\theta^*} . \end{aligned}$$

The Proofs: The CR Bochner Formulae for Witten Sublaplacian

Theorem

(Chang-Kuo-Lai, 2011) Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold. For a (smooth) real function f on M and $m > 2n$, we have

$$
\begin{aligned}
\frac{1}{2}L|\nabla_b f|^2 \quad \geq \quad & 2\Big(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n}|f_{\alpha\bar{\beta}}|^2\Big) + \frac{1}{m}|Lf|^2 + \frac{n}{2}f_0^2 \\
& + [2Ric_{m,n}(L) - Tor(L)](\nabla_b f, \nabla_b f) \\
& + \langle \nabla_b f, \nabla_b Lf \rangle + 2\langle J\nabla_b f, \nabla_b f_0 \rangle
\end{aligned}
$$

The Proofs

Define

$$F(x, t, a, c) = t\left(|\nabla_b f|^2(x) + af_t + ctf_0^2(x)\right).$$

Theorem

Let (M^3, J, θ) be a pseudohermitian 3-manifold. Suppose that

$$(2W + Tor)(Z, Z) \geq -2k|Z|^2$$

for all $Z \in T_{1,0}$, where k is an nonnegative constant. If $u(x,t)$ is the positive solution on $M \times [0, \infty)$. Then

$$\begin{aligned}\left(\Delta_b - \frac{\partial}{\partial t}\right) F \quad \geq \quad & \frac{1}{a^2 t}F^2 - \frac{1}{t}F - 2\langle \nabla_b f, \nabla_b F\rangle + t\left[+\left(1 - c - \frac{2c}{a^2}F\right)f_0^2\right. \\ & \left. + \left(-\frac{2(a+1)}{a^2 t}F - 2k - \frac{2}{ct}\right)|\nabla_b f|^2 + 4ctf_0 V(f)\right].\end{aligned}$$

Theorem

Let $\left(M^3,\ J,\ \theta\right)$ *be a pseudohermitian 3-manifold. Suppose that*

$$(2W + Tor)\,(Z,\ Z) \geq -2k\,|Z|^2$$

for all $Z \in T_{1,0}$, *where* k *is an nonnegative constant. Let* a, c, $T < \infty$ *be fixed. For each* $t \in [0, T]$, *let* $(p(t), s(t)) \in M \times [0, t]$ *be the maximal point of* F *on* $M \times [0, t]$. *Then at* $(p(t), s(t))$, *we have*

$$\begin{aligned} 0 \quad \geq \quad & \tfrac{1}{a^2 t} F\left(F - a^2\right) + t\left[4\left|f_{11}\right|^2 + \left(1 - c - \tfrac{2c}{a^2} F\right) f_0^2 \right. \\ & \left. + \left(-\frac{2(a+1)}{a^2 t} F - 2k - \frac{2}{ct}\right)\left|\nabla_b f\right|^2 + 4ctf_0 V(f)\right]. \end{aligned}$$

The Proofs

Define

$$V(\varphi) = (A_{11}\varphi_{\bar{1}})_{\bar{1}} + (A_{\bar{1}\bar{1}}\varphi_1)_1 + A_{11}\varphi_{\bar{1}}\varphi_{\bar{1}} + A_{\bar{1}\bar{1}}\varphi_1\varphi_1.$$

Theorem

Let (M^3, J, θ) *be a pseudohermitian* 3*-manifold. Suppose that*

$$[\Delta_b, \mathbf{T}]\, u = 0.$$

Then $f(x,t) = \ln u(x,t)$ *satisfies*

$$V(f) = 0.$$

The Proofs

We claim that for each fixed $T < \infty$,

$$F\left(p(T), s\left(T\right), \ -4, \ c\right) < \frac{16}{3c},$$

where we choose $a = -4$ and $0 < c < \frac{1}{3}$. Here $\left(P(T), s\left(T\right)\right) \in M \times [0, T]$ is the maximal point of F on $M \times [0, T]$. We prove by contradiction. Suppose not, that is

$$F(p(T), s\left(T\right), \ -4, \ c) \geq \frac{16}{3c}.$$

Due to Proposition **??**, $\left(p(t), s\left(t\right)\right) \in M \times [0, t]$ is the maximal point of F on $M \times [0, t]$ for each $t \in [0, T]$. Since $F\left(p(t), s\left(t\right)\right)$ is continuous in the variable t when a, c are fixed and $F\left(p(0), s\left(0\right)\right) = 0$, by Intermediate-value theorem there exists a $t_0 \in (0, \ T]$ such that

$$F\left(p(t_0), s\left(t_0\right), \ -4, \ c\right) = \frac{16}{3c}.$$

The Proofs

Hence

$$\left(-\frac{2\left(a+1\right)}{a^{2}t_{0}}F\left(p(t_{0}),s\left(t_{0}\right),\ \ -4,\ c\right)-\frac{2}{ct_{0}}\right)=0$$

and

$$\begin{aligned} 0 &\geq \tfrac{1}{16s(t_0)}\tfrac{16}{3c}\left(\tfrac{16}{3c}-16\right)+\left(1-c-\tfrac{2c}{16}\tfrac{16}{3c}\right)s\left(t_0\right)f_0^2 \\ &= \tfrac{16}{s(t_0)}\tfrac{1}{3c}\left(\tfrac{1}{3c}-1\right)+\left(\tfrac{1}{3}-c\right)s\left(t_0\right)f_0^2. \end{aligned}$$

Since $0<c<\frac{1}{3}$, this leads to a contradiction.
Hence

$$F\left(P(T),s\left(T\right),\ \ -4,\ c\right)<\tfrac{16}{3c}.$$

The Proofs

This implies that

$$\max_{(x,\ \mathrm{t})\in M\times[0,\ T]} t\left[\left|\nabla_b f\right|^2(x) - 4f_t + ctf_0^2(x)\right] \ < \ \tfrac{16}{3c}.$$

When we fix on the set $\{T\}\times M$, we have

$$T\left[\left|\nabla_b f\right|^2(x) - 4f_t + cTf_0^2(x)\right] \ < \ \tfrac{16}{3c}.$$

Since T is arbitrary, we obtain

$$\frac{\left|\nabla_b u\right|^2}{u^2} - 4\frac{u_t}{u} + ct\frac{u_0^2}{u^2} < \frac{16}{3ct}.$$

Finally let $c \to \frac{1}{3}$, then we are done. This completes the proof.

Thank you very much!