CR Li-Yau Gradient Estimate and Perelman Entropy Formulae

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Contents

- Motivations
- Pseudohermitian 3-Manifold
- The CR Li-Yau Gradient Estimate
- The CR Li-Yau-Hamilton and Li-Yau-Perelman Harnack Estimate
- Perelman Entropy Formulas and Li-Yau-Perelman Reduce Distance
- The Proofs
Motivations

Problem

the geometrization problem of contact 3-manifolds via CR curvature flows

- The Cartan Flow: Spherical CR structure
- The torsion flow: the CR analogue of the Ricci flow
The torsion flow

\[
\begin{align*}
\partial_t J(t) &= 2A_{J,\theta} \\
\partial_t \theta(t) &= -2W\theta(t).
\end{align*}
\]

Here \( J = i\theta^1 \otimes Z_1 - i\theta^{1} \otimes Z_{\bar{1}} \) and \( A_{J,\theta} = A_{11}\theta^1 \otimes Z_{\bar{1}} + A_{\bar{1}1}\theta^{1} \otimes Z_1. \)
In particular, we start from the initial data with vanishing torsion:

\[
\begin{align*}
\partial_t J(t) &= 0 \\
\partial_t \theta(t) &= -2 W \theta(t).
\end{align*}
\]


\[
\partial_t \theta(t) = -2 W \theta(t).
\]
Poincare Conjecture and Thurston Geometrization
Conjecture via Ricci Flow

- Sphere and Torus decomposition
- Singularity formation
  1. Li-Yau gradient estimate for heat equation (1986)
- Geometric surgery by Hamilton and Perelman
Geometrization problem of contact 3-manifolds

- Contact Decomposition theorem and Classification
- CR Geometric and Analytic aspects:
  1. Existence of a "best possible geometric CR structure" on closed contact 3-manifolds—spherical CR structure with vanishing torsion.
  2. $R_{ij}$: Ricci curvature tensor $\leftrightarrow A_{11}$: pseudohermitian torsion

Problem

Sub-Laplacian $\Delta_b$ is degenerated along the missing direction $T$ by comparing the Riemannian Laplacian $\Delta$. 
The geometrization problem of contact 3-manifolds

Problem

We proposed to deform any fixed CR structure under the torsion on a contact three dimensional space which shall break up due to the contact topological decomposition.

Problem

The asymptotic state of the torsion flow is expected to be broken up into pieces which satisfy the spherical CR structure with vanishing torsion.

Problem

The deformation will encounter singularities. The major question is to find a way to describe all possible singularities.
Let \((M, J, \theta)\) be the pseudohermitian 3-manifold.

1. \((M, \theta)\) is a contact 3-manifold with \(\theta \wedge d\theta \neq 0\). \(\xi = \ker \theta\) is called the contact structure on \(M\).

2. A CR-structure compatible with \(\xi\) is a smooth endomorphism \(J: \xi \to \xi\) such that \(J^2 = -\text{identity}\).

3. The CR structure \(J\) can extend to \(\mathbb{C} \otimes \xi\) and decomposes \(\mathbb{C} \otimes \xi\) into the direct sum of \(T_{1,0}\) and \(T_{0,1}\) which are eigenspaces of \(J\) with respect to \(i\) and \(-i\), respectively.
Given a pseudohermitian structure \((J, \theta)\):

1. The Levi form \(\langle , \rangle_{L_\theta}\) is the Hermitian form on \(T_{1,0}\) defined by
   \[
   \langle Z, W \rangle_{L_\theta} = -i \langle d\theta, Z \wedge \overline{W} \rangle.
   \]

2. The characteristic vector field of \(\theta\) is the unique vector field \(T\) such that \(\theta(T) = 1\) and \(\mathcal{L}_T \theta = 0\) or \(d\theta(T, \cdot) = 0\).

3. Then \(\{ T, Z_1, Z_{\overline{1}} \}\) is the frame field for \(TM\) and \(\{ \theta, \theta^1, \theta_{\overline{1}} \}\) is the coframe.
The pseudohermitian connection of \((J, \theta)\) is the connection \(\nabla^{\psi.h.}\) on \(TM \otimes C\) (and extended to tensors) given by

\[
\nabla^{\psi.h.} Z_1 = \omega_1^1 \otimes Z_1, \quad \nabla^{\psi.h.} \bar{Z}_1 = \omega_{\bar{1}}^1 \otimes \bar{Z}_1, \quad \nabla^{\psi.h.} T = 0
\]

with

\[
d\theta^1 = \theta^1 \wedge \omega_1^1 + A_{\bar{1}}^1 \theta \wedge \theta^{\bar{1}}
\]

\[
\omega_1^1 + \omega_{\bar{1}}^1 = 0.
\]

Differentiating \(\omega_1^1\) gives

\[
d\omega_1^1 = W \theta^1 \wedge \theta^{\bar{1}} \pmod{\theta}
\]

where \(W\) is the Tanaka-Webster curvature.
Pseudohermitian 3-manifold

- We can define the covariant differentiations with respect to the pseudohermitian connection.

$$f,_{\bar{1}} = Z_{\bar{1}} f ; \quad f,_{\bar{1}\bar{1}} = Z_{\bar{1}} Z_{\bar{1}} f - \omega_{11} (Z_{\bar{1}}) Z_{\bar{1}} f.$$ 

- We define the subgradient operator $\nabla_b$ and the sublaplacian operator $\Delta_b$

$$\nabla_b f = f,_{\bar{1}} Z_{\bar{1}} + f,_{\bar{1}1} Z_{\bar{1}} ,$$

and

$$\Delta_b f = f,_{\bar{1}\bar{1}} + f,_{\bar{1}1} .$$
Example

$D$ is the strictly pseudoconvex domain

$$D \subset \mathbb{C}^2 \quad \text{and} \quad M = \partial D$$

with

$$D = \{ r < 0 \} \quad \text{and} \quad M = \{ r = 0 \}.$$ 

Choose

$$\zeta = TM \cap J_{\mathbb{C}^2} TM \quad \text{and} \quad \theta = -i \partial r |_M$$

with

$$J = J_{\mathbb{C}^2} |_{\zeta}.$$
Theorem

(Li-Yau, 1986) The Li-Yau Harnack estimate

\[ \frac{\partial (\ln u)}{\partial t} - |\nabla \ln u|^2 + \frac{m}{2t} \geq 0 \]

for the positive solution \( u(x, t) \) of the time-independent heat equation

\[ \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) \]

in a complete Riemannian manifold with non-negative Ricci curvature.
Li-Yau-Hamilton Inequality

**Theorem**

(Hamilton, 1993) Hamilton Harnack estimate (trace version)

\[
\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla R \cdot V + 2\text{Ric}(V, V) \geq 0
\]

for the Ricci flow

\[
\frac{\partial g_{ij}}{\partial t} = -2R_{ij}
\]

on Riemannian manifolds with positive curvature operator.
Consider the heat equation

\[(L - \frac{\partial}{\partial t})u(x, t) = 0\]

in a closed \(m\)-manifold with a positive measure and an operator with respect to the sum of squares of vector fields

\[L = \sum_{i=1}^{l} X_i^2, \quad l \leq m,\]

where \(X_1, X_2, \ldots, X_l\) are smooth vector fields which satisfy Hörmander’s condition: the vector fields together with their commutators up to finite order span the tangent space at every point of \(M\).
Subelliptic Li-Yau gradient estimate

Theorem

(Cao-Yau, 1994) Suppose that \([X_i, [X_j, X_k]]\) can be expressed as linear combinations of \(X_1, X_2, ..., X_l\) and their brackets \([X_1, X_2], ..., [X_{l-1}, X_l]\). Then, for the positive solution \(u(x, t)\) of heat flow on \(M \times [0, \infty)\), there exist constants \(C', C'', C'''\) and \(\frac{1}{2} < \lambda < \frac{2}{3}\), such that for any \(\delta > 1\), \(f(x, t) = \ln u(x, t)\) satisfies the following gradient estimate

\[
\sum_i |X_i f|^2 - \delta f_t + \sum_\alpha (1 + |Y_\alpha f|^2)^\lambda \leq \frac{C'}{t} + C'' + C''' t^{\frac{\lambda}{\lambda-1}}
\]

with \(\{Y_\alpha\} = \{[X_i, X_j]\}\).
By choosing a frame \( \{ \mathbf{T}, Z_1, Z_{\bar{1}} \} \) of \( TM \otimes \mathbb{C} \) with respect to the Levi form and \( \{ X_1, X_2 \} \) such that

\[
J(Z_1) = iZ_1 \quad \text{and} \quad J(Z_{\bar{1}}) = -iZ_{\bar{1}}
\]

and

\[
Z_1 = \frac{1}{2}(X_1 - iX_2) \quad \text{and} \quad Z_{\bar{1}} = \frac{1}{2}(X_1 + iX_2),
\]

it follows that

\[
[X_1, X_2] = -2\mathbf{T} \quad \text{and} \quad \Delta_b = \frac{1}{2}(X_1^2 + X_2^2) = \frac{1}{2}L.
\]

Note that

\[
W(Z, Z) = Wx^1x^{\bar{1}} \quad \text{and} \quad Tor(Z, Z) = 2\text{Re} (iA_{1\bar{1}}x^{\bar{1}}x^1)
\]

for all \( Z = x^1Z_1 \in T_{1,0} \).
Definition

(Graham-Lee, 1988) Let \((M^{2n+1}, J, \theta)\) be a complete pseudohermitian manifold. Define

\[
P \varphi = \sum_{\alpha=1}^{n} (\varphi_{\bar{\alpha}} \beta + inA_{\beta\alpha} \varphi^\alpha) \theta^\beta = (P_\beta \varphi) \theta^\beta, \quad \beta = 1, 2, \ldots, n
\]

which is an operator that characterizes CR-pluriharmonic functions. Here

\[
P_\beta \varphi = \sum_{\alpha=1}^{n} (\varphi_{\bar{\alpha}} \beta + inA_{\beta\alpha} \varphi^\alpha)
\]
Theorem

(Chang-Tie-Wu, 2009) Let \((M, J, \theta)\) be a closed pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. If \(u(x, t)\) is the positive solution of CR heat flow on \(M \times [0, \infty)\) such that \(u\) is the CR-pluriharmonic function

\[
P u = 0
\]

at \(t = 0\). Then

\[
|\nabla_b f|^2 + 3f_t \leq \frac{9}{t}
\]

on \(M \times [0, \infty)\).
Theorem

(Chang-Kuo-Lai, 2011) Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)-\text{manifold}\). Suppose that

\[
2\text{Ric} (X, X) - (n - 2) \text{Tor} (X, X) \geq 0
\]

for all \(X \in T_{1,0} \oplus T_{0,1}\). If \(u(x, t)\) is the positive solution of

\[
\left( \Delta_b - \frac{\partial}{\partial t} \right) u(x, t) = 0
\]

with \([\Delta_b, T] u = 0\) on \(M \times [0, \infty)\). Then \(f(x, t) = \ln u(x, t)\) satisfies the following subgradient estimate

\[
\left| \nabla_b f \right|^2 - (1 + \frac{3}{n}) f_t + \frac{n}{3} t (f_0)^2 < \frac{(\frac{9}{n} + 6 + n)}{t}.
\]
CR Li-Yau gradient estimate

- subgradient estimate of the logarithm of the positive solution to heat flow:

**Theorem**

Let $(M, J, \theta)$ be a closed pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. If $u(x, t)$ is the positive solution of CR heat flow on $M \times [0, \infty)$ such that $u$ is the CR-pluriharmonic function

$$Pu = 0$$

at $t = 0$. Then there exists a constant $C_1$ such that $u$ satisfies the subgradient estimate

$$\frac{\left| \nabla_b u \right|^2}{u^2} \leq \frac{C_1}{t}$$

on $M \times (0, \infty)$. 
We consider the heat equation

$$\frac{\partial u(x, t)}{\partial t} = Lu(x, t)$$

in a closed pseudohermitian \((2n + 1)\)-manifold \((M, J, \theta, d\mu)\) with

$$Lu(x, t) := \Delta_b u(x, t) - \nabla_b \phi(x) \cdot \nabla_b u(x, t).$$

Here \(d\mu = e^{-\phi(x)} \theta \wedge (d\theta)^n\).
Bakry-Emery pseudohermitian Ricci curvature

- The $\infty$-dimensional Bakry-Emery pseudohermitian Ricci curvature

$$Ric(L)(W, W) := R_{\alpha\bar{\beta}} W_\alpha W_\beta + \text{Re}[\phi_{\alpha\bar{\beta}} W_\alpha W_\beta]$$

- The $m$-dimensional Bakry-Emery pseudohermitian Ricci curvature

$$Ric_{m,n}(L) := Ric(L) - \frac{\nabla_b \phi \otimes \nabla_b \phi}{2(m - 2n)}, \quad m > 2n$$

$$Tor(L)(W, W) := 2 \text{Re}[\sum_{\alpha, \beta=1}^n (i(n - 2) A_{\alpha\bar{\beta}} - \phi_{\alpha\bar{\beta}}) W_\alpha W_\beta].$$
Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. Suppose that

\[2\text{Ric}_{m,n}(L)(X, X) - \text{Tor}(L)(X, X) \geq 0\]

for all \(X \in T_{1,0} \oplus T_{0,1}\). If \(u(x, t)\) is the positive solution of

\[
\left( L - \frac{\partial}{\partial t} \right) u(x, t) = 0 \text{ with } [L, T] u = 0
\]

on \(M \times [0, \infty)\). Then \(f(x, t) = \ln u(x, t)\) satisfies the following Li-Yau type subgradient estimate

\[
\left[ |\nabla_b f|^2 \right. - \left( 1 + \frac{3}{n} \right) f_t + \frac{n}{3} t(f_0)^2 \right] < \frac{m}{2nt} \left[ \frac{9}{n} + 6 + n \right].
\]
Theorem

Let \((M, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. Suppose that

\[
2 \text{Ric}_{m,n}(L) (X, X) - \text{Tor}(L) (X, X) \geq 0
\]

for all \(X \in T_{1,0} \oplus T_{0,1}\). If \(u(x, t)\) is the positive solution of

\[
\left( L - \frac{\partial}{\partial t} \right) u(x, t) = 0 \quad \text{with}
\]

\[
[L, T] u = 0
\]

on \(M \times [0, \infty)\). Then for any \(x_1, x_2\) in \(M\) and \(0 < t_1 < t_2 < \infty\), we have the Harnack inequality

\[
\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left( \frac{t_2}{t_1} \right)^{-\left[ \frac{m(\frac{9}{n} + 6 + n)}{2n(1 + \frac{3}{n})} \right]} \exp\left\{ -\frac{(1 + \frac{3}{n})}{4} \left[ \frac{d_{cc}(x_1, x_2)^2}{(t_2 - t_1)} \right] \right\}.
\]
Note that

\[ [L, \quad T] u = 2 \text{Im} \quad Qu - 4 \text{Re}(\phi_\alpha u_\beta A_{\bar{\alpha}\bar{\beta}}) + \langle \nabla_b \phi_0, \nabla_b u \rangle \]

and

\[ [\Delta_b, \quad T] u = 2 \text{Im} \quad Qu. \]

Here \( Q \) is the purely holomorphic second-order operator defined by

\[ Qu = 2i(A_{\bar{\alpha}\bar{\beta}} u_\alpha)_\beta. \]
Theorem

(Chang-Kuo, 2011) Let \((M, J, \hat{\theta})\) be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then
\[
4 \frac{\left| \nabla_b W \right|^2}{W^2} - \frac{W_t}{W} - \frac{1}{t} \leq 0
\]
under the CR Yamabe flow
\[
\frac{\partial}{\partial t} \theta (t) = -2W (t) \theta (t), \quad \theta (0) = \hat{\theta}.
\]
Furthermore, we get a subgradient estimate of logarithm of the positive Tanaka-Webster curvature
\[
\frac{\left| \nabla_b W \right|^2}{W^2} \leq \frac{1}{4t}
\]
for all \(t \in (0, T)\).
Theorem

(Chang-Kuo, 2011) Let \((M, J, \theta)\) be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Then

\[
4 \frac{|\nabla_b u|^2}{u^2} - \frac{u_t}{u} - \frac{1}{t} \leq 0
\]

under the time-dependent CR heat equations with potentials evolving by the CR Yamabe flow

\[
\left\{
\begin{array}{l}
\frac{\partial}{\partial t} \theta (t) = -2 W (t) \theta (t), \\
\frac{\partial}{\partial t} u = 4 \Delta_b u + 2 Wu, \quad u_0 (x, 0) = 0.
\end{array}
\right.
\]
Theorem

*Its monotonicity property of the Perelman entropy functional together with Li-Yau-Perelman reduced distance imply the no local collapsing theorem under the Ricci flow.*
G. Perelman proved that the $\mathcal{F}$-functional

\[ \mathcal{F}(g_{ij}, \varphi) = \int_M \left( R + |\nabla \varphi|^2 \right) e^{-\varphi} d\mu \]

is nondecreasing under the following coupled Ricci flow

\[
\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= -2R_{ij}, \\
\frac{\partial \varphi}{\partial t} &= -\Delta \varphi + |\nabla \varphi|^2 - R,
\end{aligned}
\]

in a closed Riemannian $m$-manifold $(M, g_{ij})$. 
Theorem

G. Perelman showed that the $\mathcal{W}$-functional

$$\mathcal{W}(g_{ij}, \varphi, \tau) = \int_M \left[ \tau (R + |\nabla \varphi|^2) + \varphi - m \right] (4\pi\tau)^{-\frac{m}{2}} e^{-\varphi} \, d\mu, \quad \tau > 0$$

is nondecreasing as well under the following coupled Ricci flow

$$\begin{cases} 
\frac{\partial g_{ij}}{\partial t} &= -2R_{ij}, \\
\frac{\partial \varphi}{\partial t} &= -\Delta \varphi + |\nabla \varphi|^2 - R + \frac{m}{2\tau}, \\
\frac{d\tau}{dt} &= -1.
\end{cases}$$
Perelman Entropy Formulae

The Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

coupled with the conjugate heat equation

$$\frac{\partial u}{\partial t} = -\nabla^2 u + Ru.$$ 

1. For $u = e^{-\varphi}$.
2. For $u = (4\pi\tau)^{-\frac{m}{2}} e^{-\varphi}$, $\tau = T - t$. 
Theorem

Let \((M, J, \hat{\theta})\) be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Under

\[
\begin{align*}
\frac{\partial}{\partial t} \theta (t) &= -2 W(t) \theta (t), \\
\frac{\partial u}{\partial t} &= -4 \Delta_b u + 4 W u, \quad u_0 (x, 0) = 0,
\end{align*}
\]

we have

\[
\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W - \frac{1}{\tau} \leq 0
\]

on \(M \times [0, T)\) with \(u = e^{-f}\) and \(\tau = T - t\).
We define the CR Perelman $\mathcal{F}$-functional by

$$\mathcal{F}(\theta(t), f(t)) = 4 \int_M \left[ (\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W) \right] e^{-f} d\mu$$

$$= \int_M (2W + |\nabla_b f|^2) e^{-f} d\mu.$$ 

with the constraint

$$\int_M e^{-f} d\mu = 1.$$
We derive the following monotonicity property of CR $\mathcal{F}$-functional.

**Theorem**

Let $(M, J, \theta)$ be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Then

$$
\frac{d}{dt} \mathcal{F}(\theta(t), f(t)) = 8 \int_M \left| \left( \nabla^H \right)^2 f + \frac{W}{2} L_\theta \right|^2 ud\mu + 2 \int_M W |\nabla_b f|^2 ud\mu \\
\geq 0
$$

under

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \theta(t) = -2W(t) \theta(t), \\
\frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x, 0) = 0.
\end{array} \right.
$$
Theorem

Let \( (M, J, \hat{\theta}) \) be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Under

\[
\begin{align*}
\frac{\partial}{\partial t} \theta (t) &= -2 W (t) \theta (t), \\
\frac{\partial u}{\partial t} &= -4 \Delta_b u + 4 W u, \quad u_0 (x, 0) = 0,
\end{align*}
\]

we have

\[
\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} W + \frac{f}{8 \tau} - \frac{1}{2 \tau} \leq 0
\]

on \( M \times [0, T) \) with \( u = (4\pi \tau)^{-2} e^{-f} \) and \( \tau = T - t \).
Define the CR Perelman $\mathcal{W}$-functional by

$$
\mathcal{W}(\theta(t), f(t), \tau) = 4 \int_M \tau [\Delta_b f - \frac{3}{4} |\nabla_b f|^2 + \frac{1}{2} \mathcal{W} + \frac{f}{8 \tau} - \frac{1}{2 \tau}] \frac{e^{-f}}{(4\pi \tau)^{\frac{3}{2}}} d\mu \\
= \int_M \tau (2 \mathcal{W} + |\nabla_b f|^2) + \frac{f}{2} - 2 (4\pi \tau)^{-2} e^{-f} d\mu,
$$

with the constraint

$$
\int_M (4\pi \tau)^{-2} e^{-f} d\mu = 1.
$$
Theorem

Let \((M, J, \theta)\) be a closed spherical pseudohermitian 3-manifold with nonnegative Tanaka-Webster curvature and vanishing torsion. Then

\[
\frac{d}{dt} \mathcal{W}(\theta(t), f(t), \tau(t)) = 8\tau \int_M \left( \left( \nabla^H \right)^2 f + \frac{W}{2} L_\theta - \frac{1}{4\tau} L_\theta \right)^2 u d\mu \\
+ \tau \int_M \left[ 2W |\nabla_b f|^2 + \frac{|\nabla_b f|^2}{\tau} \right] u d\mu \\
\geq 0
\]

\[ u_{\theta} = \frac{\partial}{\partial t} \theta(t) = -2W(t) \theta(t), \]

\[ \frac{\partial u}{\partial t} = -4\Delta_b u + 4Wu, \quad u_0(x, 0) = 0, \]

for \( u = (4\pi\tau)^{-2} e^{-f} \) and \( \tau = T - t \).
Let $p, q$ be two points in $M$ and $\gamma(\tau), \tau \in [0, \bar{\tau}]$, be a Legendrian curve joining $p$ and $q$ with $\gamma(0) = p$ and $\gamma(\bar{\tau}) = q$.

**Theorem**

Let $(M, J, \theta)$ be a closed spherical pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing torsion. Under the evolution equations

$$\frac{\partial}{\partial t} \theta (t) = -2W (t) \theta (t),$$

$$\frac{\partial u}{\partial t} = -4\Delta_b u + 4W u, \quad u_0 (x, 0) = 0,$$

We have

$$f(q, \bar{\tau}) \leq \frac{2}{\sqrt{\bar{\tau}}} \int_0^{\bar{\tau}} \sqrt{\tau} (W + \frac{1}{8} \langle \dot{\gamma}(\tau), \dot{\gamma}(\tau) \rangle_{L_\theta}) \, d\tau.$$
For
\[ \mathcal{L}(\gamma) = \int_0^\tau \sqrt{\tau}(W + \frac{1}{8} \langle \dot{\gamma}(\tau), \dot{\gamma}(\tau) \rangle_{L_\theta})d\tau, \]
one can define the CR Perelman reduced distance by
\[ l_{cc}(q, \tau) \equiv \inf_{\gamma} \frac{2}{\sqrt{\tau}} \mathcal{L}(\gamma) \]
and CR Perelman reduced volume by
\[ V_{cc}(\tau) \equiv \int_M (4\pi \tau)^{-l_0} \exp\{-\frac{2}{\sqrt{\tau}}L(x, \tau)\} d\mu \]
where \( \inf f \) is taken over all Legendrian curves \( \gamma(\tau) \) joining \( p, q \) and \( L(x, \tau) \) is the corresponding minimum for \( \mathcal{L}(\gamma) \).
**The CR Yamabe Shrinking Soliton**

**Theorem**

There is no nontrivial closed shrinking CR Yamabe soliton on a closed pseudohermitian 3-manifold with positive Tanaka-Webster curvature and vanishing pseudohermitian torsion.

**Theorem**

If \((M, J, \theta)\) is a closed spherical CR 3-manifold with vanishing torsion and positive CR Yamabe constant, then solutions of the CR (normalized) Yamabe flow converge smoothly to, up to the CR automorphism, a unique limit contact form of constant Webster scalar curvature as \(t \to \infty\).
Theorem

(Greenleaf, 1986) Let $\left(M^{2n+1}, J, \theta\right)$ be a complete pseudohermitian manifold. For a real smooth function $u$ on $\left(M, J, \theta\right)$,

$$\frac{1}{2} \Delta_b |\nabla_b u|^2 = |(\nabla^H u)|^2 + <\nabla_b u, \nabla_b \Delta_b u>_{L^\theta}$$

$$+ (2 \text{Ric} - n \text{Tor})(\nabla_b u)_c, (\nabla_b u)_c$$

$$- 2i \sum_{\alpha=1}^n (u_\alpha u_{\bar{\alpha}0} - u_{\bar{\alpha}} u_{\alpha0}).$$
The Proofs: The CR Bochner Formulae

Theorem

(Greenleaf, 1986; Chang-Chiu, 2009) Let \((M^{2n+1}, J, \theta)\) be a complete pseudohermitian manifold. For a real smooth function \(u\) on \((M, J, \theta)\),

\[
\frac{1}{2} \Delta_b |\nabla_b u|^2 = |(\nabla^H)^2 u|^2 + \left(1 + \frac{2}{n}\right) \langle \nabla_b u, \nabla_b \Delta_b u \rangle_{L_\theta} \\
+ [2 \text{Ric} + (n - 4) \text{Tor}](\nabla_b u)_c, (\nabla_b u)_c \\
- \frac{4}{n} \langle Pu + \overline{P}u, d_b u \rangle_{L_\theta^*}.
\]
The Proofs: The CR Bochner Formulae for Witten Sublaplacian

Theorem

(Chang-Kuo-Lai, 2011) Let \((M, J, \theta)\) be a pseudohermitian \((2n + 1)\)-manifold. For a (smooth) real function \(f\) on \(M\) and \(m > 2n\), we have

\[
\frac{1}{2} L |\nabla_b f|^2 \geq 2 \left( \sum_{\alpha, \beta=1}^{n} |f_{\alpha \beta}|^2 + \sum_{\alpha, \beta=1}^{n} |\tilde{f}_{\alpha \beta}|^2 \right) + \frac{1}{m} |Lf|^2 + \frac{n}{2} f_0^2 \\
+ [2Ric_{m,n}(L) - Tor(L)] \langle \nabla_b f, \nabla_b f \rangle \\
+ \langle \nabla_b f, \nabla_b Lf \rangle + 2 \langle J \nabla_b f, \nabla_b f_0 \rangle
\]
Define
\[ F(x, t, a, c) = t \left( |\nabla_b f|^2(x) + af_t + ctf_0^2(x) \right). \]

**Theorem**

Let \((M^3, J, \theta)\) be a pseudohermitian 3-manifold. Suppose that
\[ (2W + Tor) (Z, Z) \geq -2k |Z|^2 \]
for all \(Z \in T_{1,0}\), where \(k\) is an nonnegative constant. If \(u(x, t)\) is the positive solution on \(M \times [0, \infty)\). Then
\[
\left( \Delta_b - \frac{\partial}{\partial t} \right) F \geq \frac{1}{a^2 t} F^2 - \frac{1}{t} F - 2 \langle \nabla_b f, \nabla_b F \rangle + t \left[ + \left( 1 - c - \frac{2c}{a^2} F \right) f_0^2 \right. \\
\left. + \left( - \frac{2(a+1)}{a^2 t} F - 2k - \frac{2}{ct} \right) |\nabla_b f|^2 + 4ctf_0 V(f) \right].
\]
The Proofs

**Theorem**

Let \((M^3, J, \theta)\) be a pseudohermitian 3-manifold. Suppose that

\[
(2W + \text{Tor}) (Z, Z) \geq -2k |Z|^2
\]

for all \(Z \in T_{1,0}\), where \(k\) is an nonnegative constant. Let \(a, c, T < \infty\) be fixed. For each \(t \in [0, T]\), let \((p(t), s(t)) \in M \times [0, t]\) be the maximal point of \(F\) on \(M \times [0, t]\). Then at \((p(t), s(t))\), we have

\[
0 \geq \frac{1}{a^2 t} F (F - a^2) + t \left[ 4 |f_{11}|^2 + (1 - c - \frac{2c}{a^2} F) f_0^2 
+ \left( -\frac{2(a+1)}{a^2 t} F - 2k - \frac{2}{ct} \right) |\nabla_b f|^2 + 4 c tf_0 V(f) \right].
\]
Define

\[ V(\varphi) = (A_{11} \varphi_\bar{1})_{\bar{1}} + (A_{\bar{1}\bar{1}} \varphi_1)_{1} + A_{11} \varphi_\bar{1} \varphi_\bar{1} + A_{\bar{1}\bar{1}} \varphi_1 \varphi_1. \]

**Theorem**

Let \((M^3, J, \theta)\) be a pseudohermitian 3-manifold. Suppose that

\[ [\Delta_b, T] u = 0. \]

Then \(f(x, t) = \ln u(x, t)\) satisfies

\[ V(f) = 0. \]
We claim that for each fixed $T < \infty$, 

$$F (p(T), s(T), -4, c) < \frac{16}{3c},$$

where we choose $a = -4$ and $0 < c < \frac{1}{3}$. Here $(P(T), s(T)) \in M \times [0, T]$ is the maximal point of $F$ on $M \times [0, T]$. We prove by contradiction. Suppose not, that is 

$$F(p(T), s(T), -4, c) \geq \frac{16}{3c}.$$

Due to Proposition ??, $(p(t), s(t)) \in M \times [0, t]$ is the maximal point of $F$ on $M \times [0, t]$ for each $t \in [0, T]$. Since $F (p(t), s(t))$ is continuous in the variable $t$ when $a, c$ are fixed and $F (p(0), s(0)) = 0$, by Intermediate-value theorem there exists a $t_0 \in (0, T]$ such that 

$$F (p(t_0), s(t_0), -4, c) = \frac{16}{3c}.$$
Hence

\[
\left( -\frac{2(a + 1)}{a^2 t_0} F(p(t_0), s(t_0), -4, c) - \frac{2}{ct_0} \right) = 0
\]

and

\[
0 \geq \frac{1}{16 s(t_0)} \frac{16}{3c} \left( \frac{16}{3c} - 16 \right) + \left( 1 - c - \frac{2c}{16} \frac{16}{3c} \right) s(t_0) f_0^2 \\
= \frac{16}{s(t_0)} \frac{1}{3c} \left( \frac{1}{3c} - 1 \right) + \left( \frac{1}{3} - c \right) s(t_0) f_0^2.
\]

Since \(0 < c < \frac{1}{3}\), this leads to a contradiction.

Hence

\[
F(P(T), s(T), -4, c) < \frac{16}{3c}.
\]
This implies that

\[ \max_{(x, t) \in M \times [0, T]} t \left[ |\nabla_b f|^2 (x) - 4f_t + ct f_0^2 (x) \right] < \frac{16}{3c}. \]

When we fix on the set \( \{ T \} \times M \), we have

\[ T \left[ |\nabla_b f|^2 (x) - 4f_t + cT f_0^2 (x) \right] < \frac{16}{3c}. \]

Since \( T \) is arbitrary, we obtain

\[ \frac{|\nabla_b u|^2}{u^2} - 4 \frac{u_t}{u} + ct \frac{u_0^2}{u^2} < \frac{16}{3ct}. \]

Finally let \( c \to \frac{1}{3} \), then we are done. This completes the proof.
Thank you very much!