Gromov-Lawson-Schoen-Yau theory and isoparametric foliations

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1 Introduction

Definition 1.1. *A Riemannian manifold M is said to carry a metric of positive scalar curvature R_M if*

$$R_M \geq 0 \ \text{ and } \ R_M(p) > 0 \ \text{ at some point } \ p \in M.$$

□

Denote by $R_M > 0$ if M carries a metric of positive scalar curvature (*p.s.c.*).

Question: Which compact manifolds admit Riemannian metrics of *p.s.c.*?

Theorem (A. Lichnerowicz, 1963) ***For a Rie. manifold X^{4k}, which is compact and Spin***

$$R_X > 0 \Longrightarrow \widehat{A}(X) = 0.$$

□

Remark For example: $\mathbb{C}P^{2k}$ is not Spin, but $\widehat{A}(\mathbb{C}P^{2k}) = (-1)^k 2^{-4k} \binom{2k}{k} \neq 0.$

Theorem (N. Hitchin, 1974) ***There is a ring homomorphism***

$$\alpha : \Omega_*^{spin} \longrightarrow KO^{-n}(pt)$$

$\alpha = \widehat{A}$ ***if*** $dim = 4k$***. For*** X ***compact spin,*** $R_X > 0 \ \Rightarrow \ \alpha(X) = 0$. □

For example There exist $8k+1$ and $8k+2$ dimensional exotic spheres with $\alpha \neq 0$. Thus, these exotic spheres admit no metrics of *p.s.c.*

Theorem
(Gromov-Lawson, [Ann. of Math. 1980]; Schoen-Yau, [Manuscripta Math. 1979])

Let M ***be a manifold obtained from a compact Riemannian manifold*** N ***by surgeries of*** $codim \geq 3$***. Then***

$$R_N > 0 \Longrightarrow R_M > 0.$$

□

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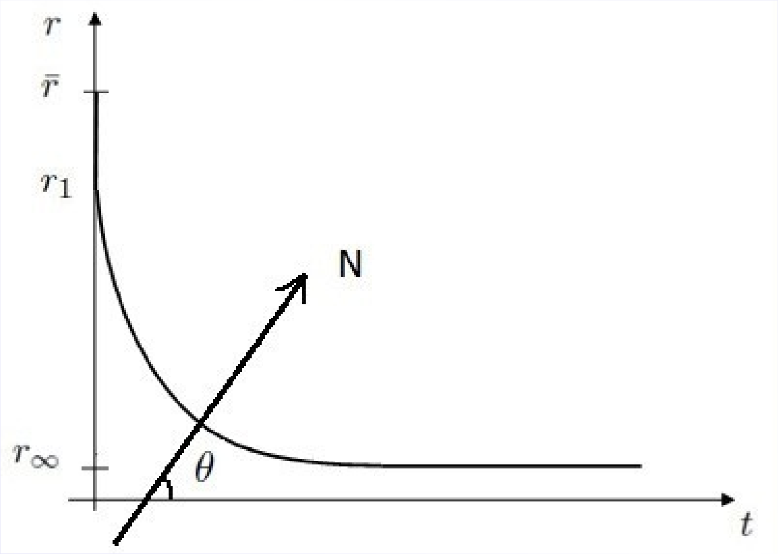
2 Gromov-Lawson theory around a point

Let X be a Rie. manifold of dimension n with $R_X > 0$. Fix $p \in X$ with $R_X(p) > 0$. $D^n := \{x \in X^n : |x| \le \bar{r}\}$: a small normal ball centered at p.

Consider a hypersurface of $D^n \times \mathbb{R}$:

$$M^n := \{(x,t) \in D^n \times \mathbb{R} : \ \ (|x|, t) \in \gamma\}$$

where $|x| = dist(x,p)$, and γ is a curve in the (r,t)-plane as pictured below:



N: the unit exterior normal vector of M. The curve γ begins with a vertical line segment $t = 0$, $r_1 \le r \le \bar{r}$, and ends with a horizontal line segment $r = r_\infty > 0$, with r_∞ small enough.

Fix $q = (x,t) \in M$ corresponding to $(r,t) \in \gamma$.

$$\textit{orthonormal basis on } T_qM \longleftrightarrow \textit{principal curvatures of } M$$
$$e_1, e_2, ..., e_{n-1}, e_n \longleftrightarrow \underbrace{\lambda_1, \lambda_2, ..., \lambda_{n-1}}_{=(-\frac{1}{r}+O(r))sin\theta}, \lambda_n := k.$$

where e_n is the tangent vector to γ, $k \geq 0$ is the curvature of the plane curve γ.

By Gauss equation:

$$K_{ij}^M = K_{ij}^{D\times\mathbb{R}} + \lambda_i\lambda_j,$$

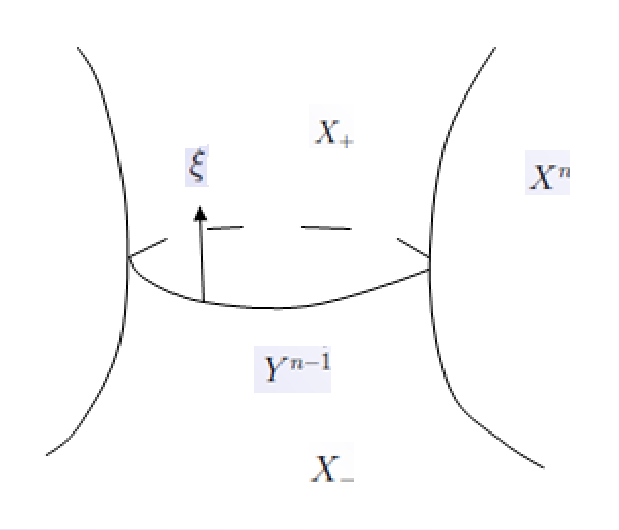
Since $D \times \mathbb{R}$ has the product metric,

$$K_{ij}^{D\times\mathbb{R}} = K_{ij}^D, \qquad 1 \leq i,j \leq n-1$$
$$K_{n,j}^{D\times\mathbb{R}} = K_{\frac{\partial}{\partial r},j}^D cos^2\theta,$$

$$\Longrightarrow R_M = R_D - 2Ric^D(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})sin^2\theta + (n-1)(n-2)(\frac{1}{r^2} + O(1))sin^2\theta$$
$$+\mathbf{2}(n-1)(-\frac{1}{r} + O(r))ksin\theta$$

3 The "double" manifold on isoparametric foliation

Assumptions: X^n $(n \geq 3)$ compact, connected, $\partial X = \emptyset$.
Y^{n-1}: a compact, connected embedding hypersurface in X,
with trivial normal bundle   ($\Rightarrow \exists$ *a unit normal vector field* ξ *on* Y),
and $\pi_0(X - Y) \neq 0$ ($\Rightarrow Y^{n-1}$ *separates* X^n *into two components,* X^n_+, X^n_-).



ξ on Y $\rightsquigarrow$ a unit normal v.f. in a neighborhood of Y, still denoted by ξ.

$D(X_\pm)$:= the double of $X_\pm$, the manifold obtained by gluing $X_\pm$ with itself along the boundary Y.

Define a continuous function $r : X^n \longrightarrow \mathbb{R}$

$$x \mapsto \begin{cases} dist(x,Y) & \text{if } x \in X_+ \\ -dist(x,Y) & \text{if } x \in X_- \end{cases}$$

where $dist(x, Y)$ is the distance from x to the hypersurface Y.

Let $Y_r := \{x \in X | r(x) = r\}, \bar{r} > 0$ small. Consider a manifold

$$M^n := \{(x,t) \in X^n \times \mathbb{R} \mid (|r(x)|, t) \in \gamma, |r(x)| \leq \bar{r}\}$$

where γ is the plane curve as before.

Fix $q = (x, t) \in M \cap (X_+ \times \mathbb{R})$, corresponding to $(r, t) \in \gamma$ $(r > 0)$.
Choose an o.n. basis $e_1, e_2, ..., e_{n-1}$ on T_xY_r such that

$$A_\xi e_i = \mu_i e_i \quad for\ i = 1, ..., n-1,$$

where A_ξ is the shape operator of the hypersurface Y_r in X.
Principal curvatures of M in $X_+ \times \mathbb{R}$:

$$\lambda_i = \mu_i sin\theta\ \ for\ i = 1, ..., n-1, where\ \ sin\theta := \langle N, \xi \rangle$$
$$\lambda_n := k.$$

We obtain:

$$R_M = \sum_{i \neq j}^{k} K_{ij}^{M} = R_X + 2Asin^2\theta + 2kH(r)sin\theta \tag{1}$$

where

$$A := \sum_{i<j\leq n-1} \mu_i\mu_j - Ric^X(\xi, \xi), \quad H(r) = \sum_{i=1}^{n-1} \mu_i(r) : \textit{mean curvature of } Y_r.$$

Gromov and Lawson computed the scalar curvature of M constructed from a submanifold with trivial normal bundle. Their formula is expressed in form of estimate, losing a factor **2** and **one item related to the second fundamental form** of the submanifold. But this mistake would result in the missing of the item $H(r)$ in our formula (1), which is essential for our research.

Rosenberg and Stolz *[Ann. Math. Studies, 2001]* modified Gromov-Lawson's expression, but they also lost the second fundamental form.

From now on, we deal with $X^n = S^n(1)$, and Y^{n-1} is a minimal isoparametric hypersurface in $S^n(1)$, *i.e.*, minimal hypersurface with constant principal curvatures, separating S^n into S^n_+ ($r \geq 0$) and S^n_- ($r \leq 0$).

Gauss equation implies

$$S = (n-1)(n-2) - R_Y$$

where S is norm square of the second fundamental form.

Peng and Terng:(*[Annals of Math. Studies, 1983]*)

If Y is a minimal isoparametric hypersurface in S^n, then

$$S = (g-1)(n-1),$$

where g is the number of distinct principal curvatures of Y.

Therefore, $R_Y \geq 0$, and

$$R_N = 0 \iff (m_+, m_-) = (1, 1).$$

Theorem 3.1 *Let Y^{n-1} be a minimal isoparametric hypersurface in $S^n(1)$, $n \geq 3$. Then each of doubles $D(S^n_+)$ and $D(S^n_-)$ has a metric of positive scalar curvature. Moreover, there is still an isoparametric foliation in $D(S^n_+)$ (or $D(S^n_-)$).* □

Outline of proof. The scalar curvature of M restricted to Y_r is

$$R_M|_{Y_r} = n(n-1)cos^2\theta + (n-g-1)(n-1)sin^2\theta + a(r)sin^2\theta + 2kH(r)sin\theta,$$

where $H(r)$ has the property that

$$H(0) = 0 \quad and \quad H(r) > 0 \; for \; any \; r > 0,$$

and $a(r)$ satisfies

$$\lim_{r \to 0} a(r) = 0$$

In fact, $a(r)$ is identically 0 when $n - 1 - g = 0$.

In each of two cases $n - 1 - g > 0$ and $n - 1 - g = 0$, we can control the "bending angle" of the curve γ, so that $R_M|_{Y_r} > 0$.

Let Y be a compact minimal isoparametric hypersurface in S^n with focal submanifolds M_+ and M_-.

Proposition 3.2 *Let the ring of coefficient $R = \mathbb{Z}$ if M_+ and M_- are both orientable and $R = \mathbb{Z}_2$, otherwise. Then for the cohomology groups, we have isomorphisms:*

$$\begin{cases} H^0(D(S^n_+)) \cong R \\ H^1(D(S^n_+)) \cong H^1(M_+) \\ H^q(D(S^n_+)) \cong H^{q-1}(M_-) \oplus H^q(M_+) \quad \textit{for } 2 \le q \le n-2 \\ H^{n-1}(D(S^n_+)) \cong H^{n-2}(M_-) \\ H^n(D(S^n_+)) \cong R \end{cases}$$

For $D(S^n_-)$, similar identities hold. □

Proposition 3.3 *$D(S^n_+)$ is a π-manifold,* i.e. *stably parallelizable manifold. In particular, $D(S^n_+)$ is an orientable, spin manifold with all the Stiefel-Whitney and Pontrjagin classes vanishing.*

Corollary 3.4 *The KO-numbers* $\alpha(D(S^n_+)) = 0$, $\alpha(D(S^n_-)) = 0$.

Proof of Prop 3.3.

$$\begin{array}{c} B^{m_+ +1} \hookrightarrow S^n_+ = B(\nu_+) \\ \downarrow \pi \\ M_+ \end{array}$$

Since S^n_+ has a metric, we can define

$$B^n_1 \sqcup_{id} B^n_2 \longrightarrow S(\nu_+ \oplus \mathbf{1})$$
$$e \longmapsto \begin{cases} (e, \sqrt{1-|e|^2}) & \text{for } e \in B^n_1 \\ (e, -\sqrt{1-|e|^2}) & \text{for } e \in B^n_2 \end{cases}$$

where B^n_1, B^n_2 are two copies of $S^n_+ = B(\nu_+)$.

Thus $D(S^n_+) \cong S(\nu_+ \oplus \mathbf{1})$, sphere bundle of Whitney sum $\nu_+ \oplus \mathbf{1}$.

$\Longrightarrow T(S(\nu_+ \oplus \mathbf{1})) \oplus \mathbf{1} \cong \pi^* TM_+ \oplus \pi^*(\nu_+ \oplus \mathbf{1}) \cong \pi^* TS^n \oplus \mathbf{1} \cong (\mathbf{n+1})$

$\Longrightarrow D(S^n_+)$ is stably parallelizable, *i.e.*, a π-manifold.

For isoparametric hypersurfaces in $S^n(1)$,

Münzner: g can only be $1, 2, 3, 4$ *or* 6.

g=1, an isoparametric hypersurface must be a hypersphere, $D(S^n_+) = S^n$.

g=2, an isoparametric hypersurface must be $S^k(r) \times S^{n-k-1}(s)$, $r^2 + s^2 = 1$, $D(S^n_+) = S^k \times S^{n-k}$ or $S^{k+1} \times S^{n-k-1}$.

g=3, all the isoparametric hypersurfaces are homogeneous. (***E.Cartan***, 1930's)

g=4, except for the unknown case (m_+, m_-)=$(7, 8)$, all isoparametric hypersurfaces are either of OT-FKM-type or homogeneous.
([***CCJ***, *Ann. Math.2007*], [***Q.S.Chi***, *preprint, 2011*])

g=6, all the isoparametric hypersurfaces are homogeneous.
([***Dorfmeister and Neher***, 1983], [***R.Miyaoka***, *preprint,2009*])

Homogeneous hypersurfaces in $S^n(1)$: principal orbits of the isotropy representation of symmetric spaces of rank two, classified completely by **Hsiang and Lawson** (*[J. Diff. Geom. 1971]*).

G: compact Lie group.
$G \times S^n \to S^n$: cohomogeneity one action. $S^n/G = [-1, 1]$.

$$orbits \;\; Y, M_\pm \;\; \longleftrightarrow \;\; isotropy \; subgroups \;\; K_0, K_\pm.$$

By the group actions

$$\begin{aligned} K_\pm \times (G \times B_\pm^{m_+ +1}) &\longrightarrow G \times B_\pm^{m_+ +1} \\ (k, g, x) &\longmapsto (gk^{-1}, k \bullet x) \end{aligned}$$

we obtain a decomposition

$$S^n = G \times_{K_+} B_+^{m_+ +1} \cup_Y G \times_{K_-} B_-^{m_- +1},$$

where $B_\pm^{m_\pm +1}$ denote the normal disc to the orbit $M_\pm = G/K_\pm$, and $\bullet$ is a slice representation.

Next, by defining a new action of the isotropy subgroup K_+ on $G \times S^{m_++1}$

$$\begin{aligned} K_+ \times (G \times S^{m_++1}) &\longrightarrow G \times S^{m_++1} \\ (k, g, (x,t)) &\longmapsto (gk^{-1}, k \star (x,t) := (k \bullet x, t)) \end{aligned}$$

we have a diffeomorphism

$$D(S^n_+) = G \times_{K_+} B^{m_++1} \cup_Y G \times_{K_+} B^{m_++1} = G \times S^{m_++1}/K_+.$$

g	(m_+, m_-)	(U, K)	K_0	K_+	K_-
1	$n-1$	$(S^1 \times SO(n+1), SO(n))$ $n \geq 2$	$SO(n-1)$	$SO(n)$	$SO(n)$
2	(p, q)	$(SO(p+2) \times SO(q+2),$ $SO(p+1) \times SO(q+1))$ $p, q \geq 1$	$SO(p) \times SO(q)$	$SO(p+1) \times SO(q)$	$SO(p) \times SO(q+1)$
3	(1, 1)	$(SU(3), SO(3))$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$S(O(2) \times O(1))$	$S(O(1) \times O(2))$
3	(2, 2)	$(SU(3) \times SU(3), SU(3))$	T^2	$S(U(2) \times U(1))$	$S(U(1) \times U(2))$
3	(4, 4)	$(SU(6), Sp(3))$	$Sp(1)^3$	$Sp(2) \times Sp(1)$	$Sp(2) \times Sp(1)$
3	(8, 8)	(E_6, F_4)	$Spin(8)$	$Spin(9)$	$Spin(9)$
4	(2, 2)	$(SO(5) \times SO(5), SO(5))$	T^2	$SO(2) \times SO(3)$	$U(2)$
4	(4, 5)	$(SO(10), U(5))$	$SU(2)^2 \times U(1)$	$Sp(2) \times U(1)$	$SU(2) \times U(3)$
4	(6, 9)	$(E_6, T \cdot Spin(10))$	$U(1) \cdot Spin(6)$	$U(1) \cdot Spin(7)$	$S^1 \cdot SU(5)$
4	(1, m-2)	$(SO(m+2), SO(m) \times SO(2))$ $m \geq 3$	$SO(m-2) \times \mathbb{Z}_2$	$SO(m-2) \times SO(2)$	$O(m-1)$
4	(2, 2m-3)	$(SU(m+2), S(U(m) \times U(2)))$ $m \geq 3$	$S(U(m-2) \times T^2)$	$S(U(m-2) \times U(2))$	$S(U(m-1) \times T^2)$
4	(4, 4m-5)	$(Sp(m+2), Sp(m) \times Sp(2))$ $m \geq 2$	$Sp(m-2) \times Sp(1)^2$	$Sp(m-2) \times Sp(2)$	$Sp(m-1) \times Sp(1)^2$
6	(1, 1)	$(G_2, SO(4))$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$O(2)$	$O(2)$
6	(2, 2)	$(G_2 \times G_2, G_2)$	T^2	$U(2)$	$U(2)$

(*cf.* *[H.Ma and H.Ohnita, Math. Z., 2009]*)

Example: $(g, m_+, m_-) = (3, 1, 1)$.

Cartan: the isoparametric hypersurface must be a tube of constant radius over a standard Veronese embedding of $\mathbb{R}P^2$ into S^4.

ν: the normal bundle of $\mathbb{R}P^2 \hookrightarrow S^4$, so $T\mathbb{R}P^2 \oplus \nu = \mathbf{4}$.

η: Hopf line bundle over $\mathbb{R}P^2$.

$$\begin{aligned} & T\mathbb{R}P^2 \oplus \mathbf{1} = 3\eta \\ \Longrightarrow \; & 3\eta \oplus \nu = T\mathbb{R}P^2 \oplus \mathbf{1} \oplus \nu = \mathbf{5} \\ \Longrightarrow \; & 4\eta \oplus \nu = 5 \oplus \eta. \end{aligned}$$

Since $4\eta = \mathbf{4}$, by obstruction theory, we have $\nu \oplus \mathbf{1} = \eta \oplus \mathbf{2}$.

Thus $D(S^4_+) = S(\nu_+ \oplus \mathbf{1}) = S(\eta \oplus \mathbf{2})$, furthermore,

$$D(S^4_+) \cong S^2 \times S^2/(x, y_1, y_2, y_3) \sim (-x, -y_1, y_2, y_3),$$

where $x \in S^2$, $(y_1, y_2, y_3) \in S^2$.

On the other hand, the Grassmannian manifold is represented by

$$G_2(\mathbb{R}^4) \cong S^2 \times S^2/(x, y) \sim (-x, -y).$$

By calculation, we see $G_2(\mathbb{R}^4)$ is not spin, while as mentioned before, $D(S^4_+)$ is spin!

When $g = 4$, the OT-FKM-type isoparametric hypersurfaces are level hypersurfaces of the following isoparametric functions restricted on S^{2l-1}:

$$F : \mathbb{R}^{2l} \to \mathbb{R}$$
$$F(z) = |z|^4 - 2\sum_{k=0}^{m} \langle P_k z, z\rangle^2,$$

where $\{P_0, \cdots, P_m\}$ is a symmetric Clifford system on $\mathbb{R}^{2l}$.

Multiplicities : $(m, l\text{-}m\text{-}1, m, l\text{-}m\text{-}1)$.

Focal submanifolds $M_+ := (F|_{S^{2l-1}})^{-1}(1)$, $M_- := (F|_{S^{2l-1}})^{-1}(-1)$.

Since M_+ has a trivial normal bundle in S^{2l-1}, we just consider M_-.

If $m \not\equiv 0 \ (mod\ 4)$, F is determined by m and l up to a rigid motion of S^{2l-1};

If $m \equiv 0 \ mod\ 4$, there are inequivalent representations of the Clifford algebra on $\mathbb{R}^l$ parameterized by an integer q, the index of the representation. (*cf.* *[**Q.M.Wang**, J. Diff. Geom. 1988]*)

In fact,

$$tr(P_0 P_1 \cdots P_m) = 2q\delta(m),$$

where $\delta(m)$ is the dimension of the irreducible Clifford algebra $\mathcal{C}_{m-1}$-modules.

Denote by $M_-(m, l, q)$ the corresponding focal submanifold.

For the topology on $D(S_-^{2l-1})$, we have:

Theorem 3.5 *Given an odd prime p, for any q_1, q_2, if $q_1 \not\equiv \pm q_2 \ (mod\ p)$, then $D(S_-^n)(m, l, q_1)$ and $D(S_-^n)(m, l, q_2)$ have different homotopy types.*

Outline of proof. By Pontrjagin class, Wu square modular $\mathbb{Z}_p$, Thom isomorphism as well as Gysin sequence.

Thank you!