Intersections of quadrics and H-minimal Lagrangian submanifolds

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(M,ω) a symplectic Riemannian $2n$-manifold.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

A vector field ξ on M is Hamiltonian if the 1-form $\omega(\,\cdot\,,\xi)$ is exact.

A Lagrangian immersion $i\colon N \looparrowright M$ is Hamiltonian minimal (H-minimal) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, i.e.

$$\frac{d}{dt}\mathrm{vol}(i_t(N))\Big|_{t=0} = 0,$$

where $i_t(N)$ is a Hamiltonian deformation of $i(N) = i_0(N)$, and $\mathrm{vol}(i_t(N))$ is the volume of the deformed part of $i_t(N)$.

Explicit examples of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{C}P^m$ were constructed in the work of Yong-Geun Oh, Castro–Urbano, Hélein–Romon, Amarzaya–Ohnita, among others.

In 2003 A. Mironov suggested a universal construction providing an H-minimal Lagrangian immersion in $\mathbb{C}^m$ from an intersection of special real quadrics.

The same intersections of real quadrics are known to toric geometers and topologists as (real) moment-angle manifolds. They appear, for instance, as level sets of the moment map in the symplectic reduction construction of Hamiltonian toric manifolds.

Here we combine Mironov's construction with the methods of toric topology to produce new examples of H-minimal Lagrangian *embeddings* with interesting and complicated topology.

A convex polyhedron in $\mathbb{R}^n$ obtained by intersecting m halfspaces:

$$P = \Big\{ \boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m \Big\}.$$

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \big(\langle \boldsymbol{a}_1, \boldsymbol{x} \rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x} \rangle + b_m \big).$$

If P has a vertex, then i_P is monomorphic, and $i_P(P)$ is the intersection of an n-plane with $\mathbb{R}^m_{\geqslant} = \{ \boldsymbol{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0 \}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1, \ldots, z_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P / \mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

Proposition 1. *If P is a simple polytope (more generally, if the presentation of P by inequalities is* generic*), then $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$.*

Proof. Write $i_P(\mathbb{R}^n)$ by $m - n$ linear equations in $(y_1, \ldots, y_m) \in \mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by $m - n$ quadrics. $\square$

$\mathcal{Z}_P$: polytopal moment-angle manifold corresponding to P.

Similarly, by considering the projection $\mu \colon \mathbb{R}^m \to \mathbb{R}^m_\geqslant$ instead of $\mu \colon \mathbb{C}^m \to \mathbb{R}^m_\geqslant$ we obtain the real moment-angle manifold $\mathcal{R}_P \subset \mathbb{R}^m$.

Example 1. $P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -\gamma_1 x_1 - \gamma_2 x_2 + 1 \geqslant 0\}$, $\gamma_1, \gamma_2 > 0$ (a 2-simplex). Then

$\mathcal{Z}_P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon \gamma_1 |z_1|^2 + \gamma_2 |z_2|^2 + |z_3|^2 = 1\}$ (a 5-sphere),

$\mathcal{R}_P = \{(u_1, u_2, u_3) \in \mathbb{R}^3 \colon \gamma_1 |u_1|^2 + \gamma_2 |u_2|^2 + |u_3|^2 = 1\}$ (a 2-sphere).

$$\mathcal{Z}_P = \Big\{ \boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \ \sum_{k=1}^m \gamma_{jk} |z_k|^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m-n \Big\}$$

$$\mathcal{R}_P = \Big\{ \boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \ \sum_{k=1}^m \gamma_{jk} u_k^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m-n \Big\}.$$

Set $\gamma_k = (\gamma_{1k}, \ldots, \gamma_{m-n,k}) \in \mathbb{R}^{m-n}$ for $1 \leqslant k \leqslant m$.

Assume that the polytope P is rational. Then have two lattices:

$$\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle \subset \mathbb{R}^n \quad \text{and} \quad L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}.$$

Consider the $(m-n)$-torus

$$T_P = \Big\{ \Big(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle} \Big) \in \mathbb{T}^m \Big\},$$

i.e. $T_P = \mathbb{R}^{m-n}/L^*$, and set

$$D_P = \frac{1}{2} L^*/L^* \cong (\mathbb{Z}_2)^{m-n}.$$

Proposition 2. *The $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ almost freely.*

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u}, \varphi) \mapsto \boldsymbol{u} \cdot \varphi = (u_1 e^{2\pi i \langle \gamma_1, \varphi \rangle}, \dots, u_m e^{2\pi i \langle \gamma_m, \varphi \rangle}).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.

Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Lemma 1. *$f\colon \mathcal{R}_P \times T_P \to \mathbb{C}^m$ induces an immersion $j\colon N_P \looparrowright \mathbb{C}^m$.*

Theorem 1 (Mironov)**.** *The immersion $i_\Gamma\colon N_\Gamma \looparrowright \mathbb{C}^m$ is H-minimal Lagrangian.*

When it is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \dots \cap F_{i_n}.$$

Theorem 2. *The following conditions are equivalent:*

1) $j\colon N_P \to \mathbb{C}^m$ *is an* embedding *of an H-minimal Lagrangian submanifold;*
2) *the* $(m-n)$*-torus* T_P *acts on* $\mathcal{Z}_P$ freely*.*
3) P *is a* Delzant *polytope.*

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:
- simplices and cubes in all dimensions;
- products and face cuts;
- associahedra (Stasheff ptopes), permutahedra, and generalisations.

Example 2 (one quadric). Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$ and $\mathcal{R}_{\Delta^{m-1}}$ is given by a single quadric

$$\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c \qquad (1)$$

with $\gamma_i > 0$, i.e. $\mathcal{R}_{\Delta^{m-1}} \cong S^{m-1}$. Then

$$N \cong S^{m-1} \times_{\mathbb{Z}_2} S^1 \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orient. of } S^{m-1}, \\ \mathcal{K}^m & \text{if } \tau \text{ reverses the orient. of } S^{m-1}, \end{cases}$$

where τ is the involution and $\mathcal{K}^m$ is an m-dimensional Klein bottle.

Proposition 3. *We obtain an H-minimal Lagrangian embedding of $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}_2} S^1$ in $\mathbb{C}^m$ if and only if $\gamma_1 = \cdots = \gamma_m$ in (1). The topological type of $N_{\Delta^{m-1}} = N(m)$ depends only on the parity of m:*

$$N(m) \cong S^{m-1} \times S^1 \qquad \textit{if } m \textit{ is even},$$
$$N(m) \cong \mathcal{K}^m \qquad \textit{if } m \textit{ is odd.}$$

The Klein bottle $\mathcal{K}^m$ with even m does *not* admit Lagrangian embeddings in $\mathbb{C}^m$ [Nemirovsky, Shevchishin].

Example 3 (two quadrics).

Theorem 3. *Let* $m-n=2$, *i.e.* $P \simeq \Delta^{p-1}\times\Delta^{q-1}$.
(a) $\mathcal{R}_P$ *is diffeomorphic to* $\mathcal{R}(p,q)\cong S^{p-1}\times S^{q-1}$ *given by*

$$\begin{aligned}
u_1^2+\ldots+u_k^2+u_{k+1}^2+\cdots+u_p^2 & &= 1,\\
u_1^2+\ldots+u_k^2 & +u_{p+1}^2+\cdots+u_m^2 &= 2,
\end{aligned}$$

where $p+q=m$, $0<p<m$ *and* $0\leqslant k\leqslant p$.
(b) If $N_P\to\mathbb{C}^m$ *is an embedding, then* N_P *is diffeomorphic to*

$$N_k(p,q)=\mathcal{R}(p,q)\times_{\mathbb{Z}_2\times\mathbb{Z}_2}(S^1\times S^1),$$

where the two involutions act on $\mathcal{R}(p,q)$ *by*

$$\begin{aligned}
\psi_1\colon (u_1,\ldots,u_m)&\mapsto(-u_1,\ldots,-u_k,-u_{k+1},\ldots,-u_p,u_{p+1},\ldots,u_m),\\
\psi_2\colon (u_1,\ldots,u_m)&\mapsto(-u_1,\ldots,-u_k,u_{k+1},\ldots,u_p,-u_{p+1},\ldots,-u_m).
\end{aligned} \tag{2}$$

There is a fibration $N_k(p,q)\to S^{q-1}\times_{\mathbb{Z}_2}S^1=N(q)$ with fibre $N(p)$ (the manifold from the previous example), which is trivial for $k=0$.

Example 4 (three quadrics).

In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest P with $m - n = 3$ is a (Delzant) pentagon, e.g.

$$P = \Big\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -x_1+2 \geqslant 0,\ -x_2+2 \geqslant 0,\ -x_1-x_2+3 \geqslant 0\Big\}.$$

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3 \times S^4$.

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Proposition 4. *Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.*

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^m$ which is the total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters into the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}_P$ and $\mathcal{Z}_P$ is even more complicated.

Generalisation to toric varieties:

Consider 2 sets of quadrics:

$$\mathcal{Z}_\Gamma = \left\{ \boldsymbol{z} \in \mathbb{C}^m \colon \textstyle\sum_{k=1}^m \gamma_k |z_k|^2 = \boldsymbol{c} \right\}, \quad \gamma_k, \boldsymbol{c} \in \mathbb{R}^{m-n};$$
$$\mathcal{Z}_\Delta = \left\{ \boldsymbol{z} \in \mathbb{C}^m \colon \textstyle\sum_{k=1}^m \delta_k |z_k|^2 = \boldsymbol{d} \right\}, \quad \delta_k, \boldsymbol{d} \in \mathbb{R}^{m-\ell};$$

s. t. $n + \ell \geqslant m$, and $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ *and* $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ satisfy the conditions above.

Define $\mathcal{R}_\Gamma$, $T_\Gamma \cong \mathbb{T}^{m-n}$, $D_\Gamma \cong \mathbb{Z}_2^{m-n}$, $\mathcal{R}_\Delta$, $T_\Delta \cong \mathbb{T}^{m-\ell}$, $D_\Delta \cong \mathbb{Z}_2^{m-\ell}$ as before.

The idea is to use the first set of quadrics to produce a toric variety M via symplectic reduction, and then use the second set of quadrics to define an H-minimal Lagrangian submanifold in M.

$M := \mathbb{C}^m /\!\!/ T_\Gamma = \mathcal{Z}_\Gamma / T_\Gamma$ toric variety, $\dim M = 2n$.

It contains $(\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma =: R$ as a subset of real points, $\dim R = n + \ell - m$.

Define $N := R \times_{D_\Delta} T_\Delta \subset M$, $\dim N = n$.

Theorem 4. *N is an H-minimal Lagrangian submanifold in M.*

Idea of proof. Consider $\widetilde{M} := M /\!\!/ T_\Delta = (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta)$.

Then

$$\widetilde{N} := N/T_\Delta = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta) = \widetilde{M}$$

is a minimal (totally geodesic) submanifold. Therefore, $N \subset M$ is H-minimal by a result of Y. Dong. □

Example 5.

1. If $m - \ell = 0$, i.e. $\mathcal{Z}_\Delta = \emptyset$, then $M = \mathbb{C}^m$ and we get the original construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$.

2. If $m - n = 0$, i.e. $\mathcal{Z}_\Gamma = \emptyset$, then N is set of real points of M. It is minimal (totally geodesic).

3. $m - \ell = 1$, i.e. $\mathcal{Z}_\Delta \cong S^{2m-1}$, then we get H-minimal Lagrangian submanifolds in $M = \mathbb{C}P^{m-1}$.

Reference:

Andrey Mironov and Taras Panov. *Intersections of quadrics, moment-angle manifolds, and Hamiltonian-minimal Lagrangian embeddings*. Preprint (2011); arXiv:1103.4970.