Symplectic nilmanifolds and applications.

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Abstract.

The talk is devoted to the remarkable towers of bundles

$$M^n \to M^{n-1} \to \cdots \to S^1, \quad n \geqslant 2,$$

with fiber the circle S^1.
This towers are defined by the nilpotent groups of the polynomial transformations of the real line.
Each M^n, $n \geqslant 2$, is a smooth nilmanifold with a 2-form which gives a symplectic structure on any M^{2k}.

Such manifolds play an important role in different areas of mathematics.
We will discuss the differential-geometric and algebro-topologic results and unsolved problems, concerning this manifolds.

Groups of polynomial transformations.

Put $L^n = \{p_x(t) = t + \sum_{k=1}^{n} x_k t^{k+1}, x_k \in \mathbb{R}\}$.
We have $L^n \cong \mathbb{R}^n : p_x(t) \Rightarrow x = (x_1, \dots, x_n)$.

We will consider L^n as the n-dim group of polynomial transformations of the real line

$$\mathbb{R} \to \mathbb{R} : \; t \mapsto p_x(t),$$

with the multiplication: $x * y = z$, where

$$(p_x * p_y)(t) = p_z(t) = p_y(p_x(t)) \quad \text{mod } t^{n+2}.$$

Example.

For $n = 4$:

$$p_z(t) = (p_x * p_y)(t) = p_x(t) + \sum_{k=1}^{4} y_k p_x(t)^{k+1} \mod t^6 :$$

$$\begin{aligned}
z_1 &= x_1 + y_1, \\
z_2 &= x_2 + 2x_1y_1 + y_2, \\
z_3 &= x_3 + (2x_2 + x_1^2)y_1 + 3x_1y_2 + y_3 \\
z_4 &= x_4 + 2(x_3 + x_1x_2)y_1 + 3(x_2 + x_1^2)y_2 + 4x_1y_3 + y_4.
\end{aligned}$$

Nilpotent group structure on $\mathbb{R}^n$.

The group $L^n \cong \mathbb{R}^n$ has the structure of *nilpotent* group with the upper central series

$$L^n_n \subset \cdots \subset L^n_q \subset \cdots \subset L^n_0 = L^n,$$

where $L^n_n = \{0 \in \mathbb{R}\}$,

$$\mathbb{R}^{n-q} \cong L^n_q = \{p_x(t) = t + \sum_{k=q+1}^{n} x_k t^{k+1}\}.$$

We have

$$L^n_q = \{x \in L^n \mid \forall y \in L^n : \quad [x, y] \in L^n_{q+1}\}$$

and $L^n_q / L^n_{q-1} \cong \mathbb{R}$ is the center of L^n / L^n_q, $q = 0, \ldots, n-1$.

The canonical matrix representation.

The left multiplication $*$ gives the canonical matrix representation

$$(x : v \to x * v) : \; \rho : L^n \to GT(n+1) : \; \rho(p_x(t)) \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ x * v \end{pmatrix}$$

into the group of lower triangular $(n+1) \times (n+1)$-matrices with ones on the diagonal:

$$\rho(p_x(t)) = X = (x_{ik}), \quad i, k = 0, \ldots, n,$$

where $x_{i,k} = [p_x(t)^{k+1}]_{i+1}$ is the coefficient of t^{i+1} in $p_x(t)^k$.

Example.

For $n = 4$:

$$\rho(p_x(t))\begin{pmatrix}1\\ v\end{pmatrix} = \begin{pmatrix}1 & & & & \\ x_1 & 1 & & & \\ x_2 & 2x_1 & 1 & & \\ x_3 & 2x_2 + x_1^2 & 3x_1 & 1 & \\ x_4 & 2(x_3 + x_1 x_2) & 3(x_2 + x_1^2) & 4x_1 & 1\end{pmatrix}\begin{pmatrix}1\\ v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}.$$

Deformation to the standard group structure.

The multiplication $*$ on $\mathbb{R}^n$ can be written down as

$$x * y = x + y + A(x)y,$$

where $A(x) = (a_{ik}(x))$ is the lower triangular $(n \times n)$-matrix with zeros on the diagonal and

$$a_{ik}(x) = x_{i,k} = [p_x(t)^{k+1}]_{i+1}, \quad i \neq k.$$

Any linear transformation $B : \mathbb{R}^n \to \mathbb{R}^n$ of coordinates in $\mathbb{R}^n$ by $B \in GL(n, \mathbb{R})$ gives a *transformed* multiplication on $\mathbb{R}^n$:

$$x *_B y \stackrel{def}{=} B^{-1}((Bx) * (By)) =$$
$$= B^{-1}(Bx + By + A(Bx)By) =$$
$$= x + y + (B^{-1}A(Bx)B)y.$$

In the case of a scalar matrix τE, we obtain

$$x *_\tau y = x + y + A(\tau x)y.$$

This gives a *deformation of multiplication* $*$ ($\tau = 1$) to the standard addition ($\tau = 0$) on $\mathbb{R}^n$.

Example. For $n = 4$:

$$x * y = x + y + \tau A_1(x)y + \tau^2 A_2(x)y,$$

where

$$A_1(x) = \begin{pmatrix} 0 & & & \\ 2x_1 & 0 & & \\ 2x_2 & 3x_1 & 0 & \\ 2x_3 & 3x_2 & 4x_1 & 0 \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ x_1^2 & 0 & 0 & \\ 2x_1x_2 & 3x_1^2 & 0 & 0 \end{pmatrix}.$$

Cocompact lattices.

The multiplication $*$ gives the free actions of L^n on $\mathbb{R}^n$:

The left shift	$v \to x * v$	gives a linear action ρ,
The right shift	$v \to v * x$	gives a non-linear action.

Let us consider the canonical lattice:

$$\Gamma^n = \{p_x(t) \in L^n : x_i \in \mathbb{Z}\}$$

with the upper central series:

$$\Gamma^n_n \subset \cdots \subset \Gamma^n_q \subset \cdots \subset \Gamma^n_0 = \Gamma^n.$$

This lattice $\Gamma^n \cong \mathbb{Z}^n$ is cocompact (uniform).

Nilmanifolds.

With respect to the right shifts we obtain a smooth *closed* and *compact nilmanifold*

$$M^n = \mathbb{R}^n/\Gamma^n.$$

The tangent bundle of M^n is

$$T(M^n) = \mathbb{R}^n \times_{\Gamma^n} \mathbb{R}^n \to M^n = \mathbb{R}^n/\Gamma^n$$

with respect to the linear action ρ (left shift) on a fiber $\mathbb{R}^n$.

We have the towers of groups

$$L^n \to L^{n-1} \to \cdots \to L^1,$$

$$\Gamma^n \to \Gamma^{n-1} \to \cdots \to \Gamma^1$$

and the induced tower

$$M^n \to M^{n-1} \to \cdots \to M^1 = S^1$$

of bundles $M^n \to M^{n-1}$ with the fiber S^1.

For each n the monomorphism holds

$$i_n : L^1 \to L^n : i_n(x_1) = (x_1, \dots, x_1^k, \dots, x_1^n).$$

Its composition with the projection $L^n \to L^1$ is the identity map. Thus for each n the bundle

$$M^1 \to S^1 \quad \text{with the fiber } L_1^n/\Gamma_1^n$$

has a section.

Left invariant differential operators.

Let us fix the polynomial ring $R[x_1, \dots, x_n]$ as the ring of functions on $L^n \cong \mathbb{R}^n$.
Put for $f(x) \in \mathbb{R}[x_1, \dots, x_n]$

$$R_x^y f(x) \stackrel{def}{=} f(x * y) = \sum_{|I| \geqslant 0} \mathcal{D}_I(f(x)) y^I$$

where R_x^y is the *right shift* operator,
$I = (i_1, \dots, i_n)$ and $y^I = y_1^{i_1} \dots y_n^{i_n}$.

From the associativity equation $R_x^y R_x^z = R_y^z R_x^y$ we have

$$\sum_{|I| \geqslant 0} \sum_{|J| \geqslant 0} \mathcal{D}_I \mathcal{D}_J f(x) y^J z^I = \sum_{|K| \geqslant 0} \mathcal{D}_K f(x) (y * z)^K.$$

Example $n = 3$. We have $\mathcal{D}_0 f(x) = f(x)$,

$$\mathcal{D}_{(1,0,0)} = \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + (2x_2 + x_1^2) \frac{\partial}{\partial x_3},$$
$$\mathcal{D}_{(0,1,0)} = \frac{\partial}{\partial x_2} + 3x_1 \frac{\partial}{\partial x_3},$$
$$\mathcal{D}_{(0,0,1)} = \frac{\partial}{\partial x_3}.$$

$$\mathcal{D}_{(1,0,0)}\mathcal{D}_{(0,1,0)} = \mathcal{D}_{(1,1,0)} + 2\mathcal{D}_{(0,0,1)},$$

$$\mathcal{D}_{(0,1,0)}\mathcal{D}_{(1,0,0)} = \mathcal{D}_{(1,1,0)} + 3\mathcal{D}_{(0,0,1)}.$$

The algebra $\mathcal{A}^n$ generated by the operators $\mathcal{D}_I$ is the algebra of all left invariant differential operators on $\mathbb{R}[x_1, \ldots, x_n]$ for the left shift L_x^z:

$$L_x^z f(x) = f(z * x),$$

that is

$$L_x^z \mathcal{D}_I f(x) = \mathcal{D}_I L_x^z f(x)$$

for z as parameter.

Algebra of the left invariant operators.

The algebra $\mathcal{A}^n$ is multiplicatively generated by the operators

$$\xi_i = \partial_i + \sum x_{i,q}\partial_q,\ i = 1, \dots, n,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and x_{iq} is the coefficient of t^{q+1} in the polynomial $p_x(t)^{i+1}$, as before.

The commutators of this operators are

$$[\xi_i, \xi_j] = (j - i)\xi_{i+j}$$

with $\xi_q = 0$ if $q > n$.

Example. For $n = 3$

$$\mathcal{A}^3 = \mathbb{R}[\xi_1, \xi_2, \xi_3]/([\xi_1, \xi_2] = \xi_3, [\xi_1, \xi_3] = [\xi_2, \xi_3] = 0).$$

The operators $\{\xi_i\}$ constitute a basis in the Lie algebra $\mathcal{L}^n$ of the left invariant vector fields on the group L^n, and the operator ξ_m corresponds to the one-parameter subgroup $\phi_m(s)$ of polynomials

$$\left\{\varphi_m(t;s) = t(1-mst^m)^{-\frac{1}{m}} \mod t^{n+2}\right\}, \quad m = 1, 2, \ldots, n.$$

We have

$$\varphi_m(t;s) = t+st^{m+1}+\sum_{k\geqslant 2}(1+m)(1+2m)\ldots(1+(k-1)m)\frac{s^k}{k!}t^{km+1}.$$

Note $\phi_m(t,1) \notin \Gamma^n$ for $m > 1$, but $\varphi_m(t;m) = \varphi_m(t;1)^m \in \Gamma^n$.

Example. For $n = 4$

$$\varphi_1(t;s) = t + st^2 + s^2t^3 + s^3t^4 + s_4t^5,$$
$$\varphi_2(t;s) = t + st^3 + \frac{3}{2}s^2t^5,$$
$$\varphi_3(t;s) = t + st^4,$$
$$\varphi_4(t;s) = t + st^5.$$

$$\varphi_1(t;1) = e_1 * e_2 * e_3^{-2} * e_4^6,$$

where $e_3^{-1}(e_3(t)) = t$.

Cohomology ring of a differential graded algebra.

A differential graded algebra (d. g. a.) (C, d) is a graded algebra

$$C = \sum_{p \geqslant 0} C^p$$

with a differential $d : C \to C$ of degree 1, i. e. $d(C^p) \subset C^{p+1}$ and $d^2 = 0$, such that $a \cdot b = (-1)^{pq} ba$ for $a \in C^p$, $b \in C^q$, $dab = (da)b + (-1)^p a(db)$ for $a \in C^p$.

Put $\quad Z^p C = ker(d : C^p \to C^{p+1})$ – cocycles group,

$B^p C = Im(d : C^{p-1} \to C^p)$ – coboundaries group,

and $\quad H^p C = Z^p C / B^p C$ – cohomology group.

Then $\quad H^* C = \sum_{p \geqslant 0} H^p C$

is a d.g.a. (with $d = 0$) — *cohomology ring* of C.

Example.

Let X be a smooth n-dimensional compact manifold.
Then we have a d.g.a. of smooth real differential forms

$$C(X) = \sum_{p \geqslant 0} C^p(X).$$

In a coordinate neighbourhood $U \subset X$ we have for $\omega \in C^p(X)$

$$\omega = \sum u_{i_1 \dots i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad x = (x^{i_1}, \dots, x^{i_1}) \in U,$$

$$d\omega = \sum_{i_1 < i_2 < \cdots < i_p} du_{i_1 \dots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} =$$

$$= \sum_{i_1 < i_2 < \cdots < i_p} \frac{\partial}{\partial x^{i_0}} u_{i_1 \dots i_p} dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

and $H^* C(X) = H^*(X; \mathbb{R})$.

Differential graded algebra of the left invariant differential forms on the nilmanifold.

Let $\omega_1, \ldots, \omega_n$ be the basis of the left invariant differential 1-forms on L^n dual to the basis $\{\xi_i\}$.

Let

$$\omega = \sum_{i=1}^{n} \omega_i \xi_i$$

be the Maurer-Cartan form taking values in the Lie algebra $\mathcal{L}^n$ of vector fields ξ_i, $i = 1, \ldots, n$.

The Maurer–Cartan equation

$$d\omega = -\frac{1}{2}[\omega, \omega]$$

in our case takes the form

$$d\omega_q = \sum_{\{(k,l):\ k>l>0,\ k+l=q\}} (k-l)\ \omega_k \wedge \omega_l. \qquad (1)$$

Here $[\omega, \omega](\zeta_1, \zeta_2) = [\omega(\zeta_1), \omega(\zeta_2)]$.

Note that $d\omega_1 = d\omega_2 = 0$ and (1) is independent of n.

Examples: $d\omega_3 = \omega_2 \wedge \omega_1$, $d\omega_4 = 2\omega_3 \wedge \omega_1$, $d\omega_5 = 3\omega_4 \wedge \omega_1 + \omega_3 \wedge \omega_2$.

Bigraded cohomology ring.

We have $H^*(M^n;\mathbb{R}) = H(\Lambda(\omega_1,\dots,\omega_n),d)$
where $\Lambda(\ )$ is the exterior algebra, and d has the form (1).

Set bideg $\omega_{\mathrm{q}} = (1,-2\mathrm{q})$. It follows from (1) that
the differential complex $(\Lambda(\omega_1,\dots,\omega_n),d)$ can be decomposed as
a sum of differential subcomplexes

$$\Lambda^0 + \sum_{q=1}^{n} (\Lambda^{-2q},d),$$

where $\Lambda^0 = \mathbb{R}$ and (Λ^{-2q},d) is generated by the forms
$\omega_{i_1}\wedge\cdots\wedge\omega_{i_s},\ s = 1,\dots,n,\ i_1 > i_2 > \cdots > i_s > 0,\ i_1+\cdots+i_s = q.$

For any $n \geqslant 2$ we have:

$$H^1(M^n;\mathbb{R}) = H^{1,-2}(M^n;\mathbb{R}) + H^{1,-4}(M^n;\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$$

with the generators $[\omega_1]$ and $[\omega_2]$ correspondingly.

Thus $H^1(M^n;\mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$, $n \geqslant 2$.

The ring $H^*(M^n,\mathbb{R})$ has the structure of a bigraded ring

$$\mathbb{R} + \sum_{s=1}^{n} \sum_{2q=s(s+1)}^{s(2n+1-s)} H^{s,-2q}(M^n;\mathbb{R}).$$

Set $n(s,q) = q - \frac{1}{2}(s-1)(s-2)$. For any $k \geqslant n(s,q)$ we have

$$H^{s,-2q}(M^k;\mathbb{R}) = H^{s,-2q}(M^{k+1};\mathbb{R}).$$

Example for $n = 4$.

We have:

$$H^*(M^4;\mathbb{R}) = H^*(\Lambda(\omega_1,\omega_2,\omega_3,\omega_4),d),$$

where $d\omega_1 = d\omega_2 = 0$, $d\omega_3 = \omega_2 \wedge \omega_1$, $d\omega_4 = 2\omega_3 \wedge \omega_1$.

$$H^*(M^4;\mathbb{R}) = \mathbb{R} + \sum_{s=1}^{4} \sum_{2q=s(s+1)}^{s(9-s)} H^{s,-2q}(M^4;\mathbb{R}).$$

10 differential subcomplexes.

$q \backslash s$	0	1		2		3	4
0	1						
1		ω_1					
2		ω_2					
3		ω_3	$\xrightarrow{1}$	$\omega_2 \wedge \omega_1$			
4		ω_4	$\xrightarrow{2}$	$\omega_3 \wedge \omega_1$			
5				$\omega_4 \wedge \omega_1$			
				$\omega_3 \wedge \omega_2$			
6				$\omega_4 \wedge \omega_2$	$\xrightarrow{-2}$	$\omega_3 \wedge \omega_2 \wedge \omega_1$	
7				$\omega_4 \wedge \omega_3$	$\xrightarrow{-1}$	$\omega_4 \wedge \omega_2 \wedge \omega_1$	
8						$\omega_4 \wedge \omega_3 \wedge \omega_1$	
9						$\omega_4 \wedge \omega_3 \wedge \omega_2$	
10							$\omega_4 \wedge \omega_3 \wedge \omega_2 \wedge \omega_1$

Generators of the Poincare duality.

$$
\begin{array}{cccccc}
\text{dim} & 0 & 1 & 2 & 3 & 4 \\
 & & [\omega_1] & \longleftrightarrow & [\omega_4 \wedge \omega_3 \wedge \omega_2] & \\
 & & [\omega_2] & \longleftrightarrow & [\omega_4 \wedge \omega_3 \wedge \omega_1] & \\
 & & & [\omega_1 \wedge \omega_4] & & \\
 & & & \updownarrow & & \\
 & & & [\omega_2 \wedge \omega_3] & & \\
 & 1 & & \longleftrightarrow & & [\omega_4 \wedge \omega_3 \wedge \omega_2 \wedge \omega_1]
\end{array}
$$

Toric bundles.

For each n and $q < \left[\frac{n+1}{2}\right]$ there are exact sequences

$$0 \to \mathbb{R}^{q+1} \to L^{n+1} \to L^{n-q} \to 0,$$
$$0 \to \mathbb{Z}^{q+1} \to \Gamma^{n+1} \to \Gamma^{n-q} \to 0,$$

which give a smooth bundle

$$\pi_n^q : M^{n+1} \to M^{n-q}$$

with fibre torus $\mathbb{T}^{q+1}$.

Symplectic nilmanifolds M^n.

A smooth manifold M is called *symplectic* if it carries a *nondegenerate closed* 2-form Ω which is called a *symplectic* form.

Consider the smooth bundle with fibre circle S^1

$$\pi_n = \pi_n^0 : M^{n+1} \to M^n.$$

The left invariant 1-form ω_{n+1} is a *connection* in the bundle π_n.
The *curvature* form of this bundle is

$$\Omega_n = \sum_{\{(k,l):\, k+l=n+1,\, k>l>0\}} (k-l)\omega_k \wedge \omega_l$$

and we have $\pi^*\Omega_n = d\omega_{n+1}$.
The nilmanifold M^{2n} with the form Ω_{2n} is *symplectic*.

Conjecture. Ω_n is an integer form for any n.

Example. For $n = 3$, $q = 1$ we have the smooth bundle

$$\pi_3^1 : M^4 \to M^2 = \mathbb{T}^2$$

with the fibre $\mathbb{T}^2$ and the symplectic form:

$$\Omega_4 = 3\omega_4 \wedge \omega_1 + \omega_3 \wedge \omega_2.$$

The base is the symplectic manifold with the form $\Omega_2 = \omega_2 \wedge \omega_1$ and $(\pi_3^1)^* \Omega_2 = 0$.

The manifold $M^3 \times S^1$ is symplectic with the form

$$2\omega_3 \wedge \omega_1 + \omega_2 \wedge dt.$$

The manifold M^{2n-1} has the form

$$\Omega_{2n-1} = \sum_{\{(k,l):\ k+l=2n,\ k>l>0\}} (k-l)\omega_k \wedge \omega_l.$$

For $n > 2$ the form ω_n is not closed, thus the 2-form

$$\Omega = \Omega_{2n-1} + \omega_n \wedge dt$$

is not closed on $M^{2n-1} \times S^1$ but Ω^n is closed and gives the fundamental cocycle on this manifold.

Nonformality of nilmanifolds.

A simplicial complex X is called *formal* if its rational homotopy type is a formal consequence of its cohomology ring.

Theorem. (F. E. A. Johnson, E. G. Rees, 1989)
If G is a nilpotent Lie group and $\Gamma \subset G$ is a discrete cocompact subgroup, then G/Γ is *formal* if and only if G is abelian.

Corollary. The symplectic nilmanifolds M^{2m} are *nonformal*, $m \geqslant 2$, and $M^2 = T^2$ is *formal*.

Realizing nilmanifolds as symplectic submanifolds
of complex projective spaces $\mathbb{C}P^N$,
denote by $X_m(N)$ the *symplectic* blow up of $\mathbb{C}P^N$ along M^{2n}.

Theorem. (I.K. Babenko, I.A. Taimanov, 1999)
For $m \geqslant 2$ and $N \geqslant 2m+1$ the symplectic manifolds $X_m(N)$ are *simply connected* and *nonformal*.

The proof of this result makes use of the fact that in the cohomology ring $H^*(M^{2n})$ there are *nontrivial* Massey products.

Universal properties of M^n.

The manifold $M^n = K(\Gamma^n, 1)$ is the *Eilenberg-MacLane space* and thus for any CW-complex X

$$[X, M^n] = H^1(X, \Gamma^n).$$

The manifold M^n is the *classifying space* for the discrete group Γ^n, that is $M^n = B\Gamma^n$ and thus $[X, M^n]$ is the set of isomorphism classes of principal Γ^n-bundles over a CW-complex X; we have

$$[X, M^n] = \mathrm{Hom}(\pi_1(\mathrm{X}), \Gamma^{\mathrm{n}}),$$

$$H_k(M^n; \mathbb{Z}) = H_k(\Gamma^n; \mathbb{Z}), \qquad H^k(M^n; \mathbb{Z}) = H^k(\Gamma^n; \mathbb{Z}).$$

Cellular subdivision of M^n.

Consider the cellular subdivision

$$(pt) = M_0^n \subset M_1^n \subset \cdots \subset M_{n-1}^n \subset M_n^n = M^n,$$

where $M_1^n = \vee_{i=1}^n S_i^1$, $M_{k+1}^n / M_k^n = \vee S^{k+1}$, $M_n^n / M_{n-1}^n = \vee S^n$.

For the $\mathbb{Z}$-homology groups of pair we obtain the exact sequence

$$0 \to H_2(M^n) \to H_2(M^n / M_1^n) \to \oplus_{i=1}^n \mathbb{Z} \to H_1(M^n) \to 0.$$

Using that $M^n = K(\Gamma^n; 1)$ and $M_1^n = K(\vee_{i=1}^n \mathbb{Z}; 1)$
for the homotopy groups of pair we obtain the exact sequence

$$0 \to R_n \to \vee_{i=1}^n \mathbb{Z} \to \Gamma^n \to 0.$$

Here $\vee_{i=1}^n \mathbb{Z}$ is the free product of $\mathbb{Z}$
and $R_n = \pi_2(M^n, M_1^n)$ is its subgroup. It is a free group.

The multiplicative generators of the group $\Gamma^n \subset L^n$ are $e_k(t) = t + t^{k+1}$, $k = 1, \dots, n$. Put $e_0(t) = t$. Note $\varphi_q(t; 1) = e_q(t)$ for $q > [\frac{n}{2}]$.

It is clear that if $e^I = e_1^{i_1} * \dots * e_n^{i_n} = e_0$ where $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$, then $I = 0$.

We have

$$[e_k, e_{k+2}] = e_{2k+2}^2 * e_i * \dots, \qquad i > 2k + 2, \quad k \geqslant 1,$$

$$[e_k, e_{k+1}] = e_{2k+1} * e_j * \dots, \qquad j > 2k + 1, \quad k \geqslant 1.$$

Thus the group $H_1(M^n; \mathbb{Z}) = \Gamma_n / [\Gamma_n, \Gamma_n]$ has only 2-torsion.

Hopf's integral homology formula.

Let $G = F/R$ and F is a free group. Then

$$H_2(G, \mathbb{Z}) \cong (R \cap [F, F])/[F, R].$$

Thus

$$H_2(M^n, \mathbb{Z}) \cong (R_n \cap [F^n, F^n])/[F^n, R_n],$$

where $F^n = \vee_{i=1}^n \mathbb{Z}$ and

$$0 \to R_n \to \vee_{i=1}^n \mathbb{Z} \to \Gamma^n \to 0,$$

and therefore to each element $a \in H_2(M^n, \mathbb{Z})$ corresponds an element

$$g = [a_1, b_1] \cdot ... \cdot [a_g, b_g] \in (R_n \cap [F^n, F^n]).$$

Example $n = 3$.

Γ^3 has the generators e_1, e_2, e_3 and the relations

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_0, \quad [e_2, e_3] = e_0.$$

Thus $H_1(M^3, \mathbb{Z}) = \Gamma^3/[\Gamma^3, \Gamma^3] = \mathbb{Z} \oplus \mathbb{Z}$.

In this case F^3 has the generators c_1, c_2, c_3,
R_3 has the generators r_1, r_2, r_3 and

$$R_3 \to F^3 : \quad r_1 \mapsto [c_1, c_3],\ r_2 \mapsto [c_2, c_3],\ r_3 \mapsto [c_1, c_2]c_3^{-1}.$$

We have $r_3 \notin [F^3, F^3]$.
The generators of

$$H_2(M^3; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

correspond to r_1 and r_2.

Example $n = 4$.

Γ^4 has the generators e_1, e_2, e_3, e_4 and the relations

$$[e_1, e_2] = e_3 * e_4, \quad [e_1, e_3] = e_4^2, \quad [e_1, e_4] = e_0,$$

$$[e_i, e_j] = e_0, \quad i, j = 2, 3, 4.$$

Thus $H_1(M^4, \mathbb{Z}) = \Gamma^4/[\Gamma^4, \Gamma^4] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

Consider an oriented 2-dimentional surface S_g^2 of genus g.

We have $S_g^2 = K(G_g, 1)$,
where $G_g = \pi_1(S_g^2)$ is the group with the generators $a_1, b_1, \ldots, a_g, b_g$
and a single relation $[a_1, b_1] \cdot ... \cdot [a_g, b_g] = 1$,
that is

$$0 \to \mathbb{Z} \to \vee_{i=1}^{2g} \mathbb{Z} \to G_g \to 0.$$

We have

$$[S_g^2, M^n] = Hom[G_g, \Gamma^n].$$

Corollary.
Each element $a \in H_2(M^n, \mathbb{Z})$, $n \geqslant 2$, is realised by a smooth mapping

$$f_a : S^2_g \to M^n, \quad (f_a)_*([S^2_g]) = a$$

for some g.

The form Ω_n is integer if and only if

$$\langle f_a^* \Omega_n,\ [S^2_g] \rangle \in \mathbb{Z}.$$

Let $\pi_n : M^{n+1} \to M^n$ be a smooth bundle with the fibre S^1. Denote by $\xi_{n+1} = \xi_{n+1}(\pi_n)$ the field of vectors tangent to the fiber of this bundle.

Problem. Classify the sequences of smooth manifolds

$$\pi_n : M^{n+1} \to M^n, n \geqslant 0,$$

with the fiber S^1, such that

- for each $n > 1$ there exists an *integer* closed 2-form Ω_n on M^n satisfying the condition

$$\pi_n^* \Omega_n = d\omega_{n+1}, \quad \text{where } \langle \omega_{n+1}, \xi_{n+1} \rangle = ||\xi_{n+1}||,$$

- for each even n the form Ω_n is nondegenerate.

The following problem is closely related to the previous one and has self-contained interest:

Problem. For the towers

$$M^n \to M^{n-1} \to \cdots \to S^1$$

of fibrations described above
calculate the cohomology rings $H^*(M^n; k)$ for $k = \mathbb{Z}$ and $\mathbb{Q}$.

Consider the bundle

$$\widehat{\pi}_n : E \to M^n$$

with the fiber D^2, such that $\partial E = M^{n+1}$.
In the exact sequence of the pair $(E, \partial E)$
the Gyzin homomorphism has the form

$$j_q^n : H^q(M^n) \to H^{q+2}(M^n) : j_q^n a = [\Omega_n]a.$$

Thus we get the exact sequence

$$0 \leftarrow ker j_{q-1}^n \leftarrow H^q(M^{n+1}) \leftarrow coker j_{q-2}^n \leftarrow 0.$$

In the case of rational coefficients we get

$$\dim H^q(M^{n+1}) = (\dim ker j_{q-1}^n) + (\dim coker j_{q-2}^n).$$

Denote the Betti number $\dim H^q(M^n, \mathbb{Q})$ by b_q^n.
Thus we have the estimate

$$b_q^{n+1} \leqslant b_{q-1}^n + b_q^n.$$

D. V. Millionshikov has obtained results on the Betti numbers b_q^n for manifolds M^n defined by the groups L^n.
His approach is based on the calculations by L. Goncharova of infinite dimensional Lie algebras cohomologies.

For such manifolds he proved that

$$b_2^n = 3 \text{ for all } n > 5;$$

$$b_3^n = 5 \text{ for all } n > 11.$$

D. V. Millionshchikov used some combinatorial arguments and the Goncharova theorem to sketch the proof of the statement

$$b_q^n = F_{q+2}$$

for n sufficiently large ($n > 3q + 2$),
where F_{q+2} is the $(q + 2)$-th Fibonacci number.
That is $F_{q+2} = F_{q+1} + F_q$, $q \geqslant 0$, $F_0 = 0$, $F_1 = 1$.
However no detailed proof of this statement appeared till now.
Recently he suggested to consider the last statement as a conjecture.

Using the computer, Millionschikov calculated Betti numbers b_q^n for $n \leqslant 30$.

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Addendum.

Massey products.

Let (C, d) — a d.g.a. For $a \in C^p$ put

$$\bar{a} = (-1)^p a.$$

Then we obtain the involution on C, i.e. $\overline{ab} = \bar{a}\bar{b}$, $\bar{\bar{a}} = a$, and

$$dab = (da)b + \bar{a}(db) \text{ for } a \in C^p.$$

Let $T_k^0 = T_k^0(C)$ — the algebra of upper triangular $(k \times k)$-matrices over C with zeros on the diagonal. For $A = (a_{ij}) \in T_k^0$ put $dA = (da_{ij})$ and $\bar{A} = (\bar{a}_{ij})$.

Let $J^k = (J^k_{ij}) \in T_k^0$, such that $J^k_{ij} = 0$, if $(i, j) \neq (1, k)$, and $J^k_{1k} = 1$.

Lemma. Let $A = (a_{ij}) \in T^0_{n+1}$, such that $a_{i,i+1} \in C^{k_i}$ and

$$dA = \bar{A}A - cJ^{n+1}$$

for some $c \in C$. Then
- $da_{i,i+1} = 0$,
- $c \in C^m$, where $m = k_1 + \cdots + k_n - n + 2$,
- $dc = 0$.

Show that $dc = 0$. We have:

$$d\bar{A} = -\bar{dA} = -A\bar{A} - cJ^{n+1}.$$

Using that $J^{n+1}A = AJ^{n+1} = 0$, we obtain

$$d\bar{A}A = (d\bar{A})A + \bar{\bar{A}}(dA) = -A\bar{A}A + A\bar{A}A = 0.$$

So

$$(dc)J^{n+1} = d(\bar{A}A) - ddA = 0.$$

Definition. Take n homogeneous elements $a_1, \ldots, a_n$ in C, which are cocycles, i.e. $da_i = 0$, $i = 1, \ldots, n$.
Assume that there exists a matrix $A \in T^0_{n+1}$ such that:
- $a_{i,i+1} = a_i$
- A satisfies the equation

$$dA = \bar{A}A - cJ^{n+1}$$

for some $c \in C$.

In this case it is told that the *Massey product* $\langle a_1, \ldots, a_n \rangle$ of the cocycles $a_1, \ldots, a_n$ is defined and equals cocycle c.

Examples $n = 2$.

$$\begin{pmatrix} 0 & \bar{a}_1 & \bar{a}_{13} \\ & 0 & \bar{a}_2 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & a_{13} \\ & 0 & a_2 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

So

$$c \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 \\ & 0 & 0 \\ & & 0 \end{pmatrix} - \begin{pmatrix} 0 & da_1 & da_{13} \\ & 0 & da_2 \\ & & 0 \end{pmatrix}$$

and

$$da_1 = da_2 = 0, \quad c = \langle a_1, a_2 \rangle = \bar{a}_1 a_2 - da_{13}$$

for some a_{13}.

Examples $n = 3$.

$$\begin{pmatrix} 0 & \bar{a}_1 & \bar{a}_{13} & \bar{a}_{14} \\ & 0 & \bar{a}_2 & \bar{a}_{24} \\ & & 0 & \bar{a}_3 \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & a_{13} & a_{14} \\ & 0 & a_2 & a_{24} \\ & & 0 & a_3 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 & \bar{a}_1 a_{24} + \bar{a}_{13} a_3 \\ & 0 & 0 & \bar{a}_2 a_3 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

So

$$c \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 & \bar{a}_1 a_{24} + \bar{a}_{13} a_3 \\ & 0 & 0 & \bar{a}_2 a_3 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & da_1 & da_{13} & da_{14} \\ & 0 & da_2 & da_{24} \\ & & 0 & da_3 \\ & & & 0 \end{pmatrix}.$$

So, $\quad da_i = 0, \quad i = 1, 2, 3, \quad \bar{a}_1 a_2 = da_{13}, \quad \bar{a}_2 a_3 = da_{24},$
and $\langle a_1, a_2, a_3 \rangle = c = \bar{a}_1 a_{24} + \bar{a}_{13} a_3, \quad \deg c = k_1 + k_2 + k_3 - 1.$

Examples $n = 3$.

$H^*(M^3) = H^*(\Lambda(\omega_1, \omega_2, \omega_3), d)$, where $d\omega_1 = d\omega_2 = 0$, $d\omega_3 = \omega_2 \wedge \omega_1$. The generators of $H^*(M^3)$:

$$[\omega_1],\ [\omega_2],\ [\omega_3 \wedge \omega_1],\ [\omega_3 \wedge \omega_2],\ [\omega_3 \wedge \omega_2 \wedge \omega_1].$$

So, for

$$a_1 = \omega_1,\ a_2 = \omega_2,\ a_3 = \omega_1 \Rightarrow a_{13} = \omega_3,\ a_{24} = -\omega_3$$

and $\langle \omega_1, \omega_2, \omega_1 \rangle = -2\omega_3 \wedge \omega_1$,

$$a_1 = \omega_1,\ a_2 = \omega_1,\ a_3 = \omega_2 \Rightarrow a_{13} = 0,\ a_{24} = \omega_3$$

and $\langle \omega_1, \omega_1, \omega_2 \rangle = \omega_3 \wedge \omega_1$,

$$a_1 = \omega_1,\ a_2 = \omega_2,\ a_3 = \omega_2 \Rightarrow a_{13} = \omega_3,\ a_{24} = 0$$

and $\langle \omega_1, \omega_2, \omega_2 \rangle = -\omega_3 \wedge \omega_2$.

The matrix equation

$$dA = \bar{A}A - cJ^{n+1}$$

for $n \geqslant 4$ gives the following relations:

$$da_{i,i+1} = da_i = 0,$$

$$da_{i,k} = \sum_{q=i+1}^{k-1} \bar{a}_{i,q} a_{q,k}, \qquad i+2 \leqslant k \leqslant n$$

and

$$\langle a_1, \ldots, a_n \rangle = c = \sum_{q=2}^{n} \bar{a}_{1,q} a_{q,n+1} - da_{1,n+1},$$

where $dc = 0$. So

$$da_{i,i+2} = \bar{a}_{i,i+1} a_{i+1,i+2} = \bar{a}_i a_{i+1},$$

$$da_{i,i+3} = \bar{a}_{i,i+1} a_{i+1,i+3} + \bar{a}_{i,i+2} a_{i+2,i+3} = \langle a_i, a_{i+1}, a_{i+2} \rangle.$$

Example $n = 4$.

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_{13} & a_{24} & \langle a_1, a_2, a_3 \rangle \\ \omega_1 & \omega_2 & \omega_1 & \omega_3 & -\omega_3 & 2\omega_3 \wedge \omega_1 = d\omega_4 \\ \omega_2 & \omega_1 & \omega_1 & -\omega_3 & 0 & \omega_3 \wedge \omega_1 = \frac{1}{2} d\omega_4 \\ \omega_1 & \omega_2 & \omega_2 & \omega_3 & 0 & \omega_3 \wedge \omega_2 \end{array}$$

We have

$$\langle a_1, a_2, a_3, a_4 \rangle = \bar{a}_{12} a_{25} + \bar{a}_{13} a_{35} + \bar{a}_{14} a_{45}.$$

For $\langle \omega_2, \omega_1, \omega_1, \omega_1 \rangle$:

$$da_{25} = \langle \omega_1, \omega_1, \omega_1 \rangle = 0 \Rightarrow a_{25} = 0$$

$$da_{35} = -\omega_1 \wedge \omega_1 = 0 \Rightarrow a_{35} = 0$$

$$da_{14} = \langle \omega_2, \omega_1, \omega_1 \rangle = \frac{1}{2} d\omega_4 \Rightarrow a_{14} = \frac{1}{2} \omega_4.$$

So, we obtained:

$$\langle \omega_2, \omega_1, \omega_1, \omega_1 \rangle = -\frac{1}{2} \omega_4 \wedge \omega_1 \neq 0.$$

For $\langle \omega_1, \omega_2, \omega_2, \omega_2 \rangle$:

$$a_{25} = 0, \quad a_{35} = 0$$

and

$$da_{14} = \langle \omega_1, \omega_2, \omega_2 \rangle = \omega_3 \wedge \omega_2.$$

We can't find such a_{14} and therefore the Massey product $\langle \omega_1, \omega_2, \omega_2, \omega_2 \rangle$ is not well defined in $H^*(M^4)$.

Infinite-dimensional algebra of vector fields of the line.

Introduce:
the group $L_\infty = \lim_{\leftarrow} L_n$
Lie algebra $\mathcal{L}_\infty = \lim_{\leftarrow} \mathcal{L}_n$
and algebra of operators $\mathcal{A}_\infty = \lim_{\leftarrow} \mathcal{A}_n$.

Let $l_1 = \{x^{k+1}\frac{d}{dx}, k \geqslant 1\}$ be the well known Lie algebra of vector fields on the line.

We have $\mathcal{L}_\infty \cong l_1$.

Theorem.(L. V. Goncharova, 1973)

$$\dim H_k^q(l_1) = \begin{cases} 1, & \text{if } k = \frac{3q^2 \pm q}{2}, \\ 0, & \text{otherwise} \end{cases}$$

Thus, $\dim H^q(l_1) = 2$ for $q \geqslant 1$.

The cohomological product in $H^*(l_1)$ is trivial.
It was V. M. Buchstaber (1978) who raised the problem whether $H^*(l_1)$ is *generated, with respect to Massey products*, by $H^1(l_1)$.

The Heisenberg group.

Let us fix a decomposition $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, $n = k + l$,
and a bilinear mapping $\mathcal{B} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^l$.

We define a multiplication on $\mathbb{R}^n$ by the formula

$$(v_1, w_1) \cdot (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + \mathcal{B}(v_1, v_2))$$

where $v_i \in \mathbb{R}^k$, $w_i \in \mathbb{R}^l$, $i = 1, 2$.

Note the relation

$$\mathcal{B}(v_1, v_2) = \mathcal{A}(v_1)v_2,$$

where $\mathcal{A}$ is the *linear* mapping $\mathbb{R}^k \to Hom(\mathbb{R}^k, \mathbb{R}^l)$.

Thus we obtain a group structure on $\mathbb{R}^n$,
which is noncommutative for nonsymmetric mapping $\mathcal{B}$.

A linear change of coordinates

$$B = (B_1, B_2) \in GL(k, \mathbb{R}) \times GL(l, \mathbb{R}) \subset GL(n, \mathbb{R})$$

gives a new multiplication

$$(v_1, w_1) * (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + B_2^{-1}\mathcal{B}(B_1 v_1, B_1 v_2)).$$

For the scalar matrix τE we get

$$(v_1, w_1) *_\tau (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + \tau\mathcal{B}(v_1, v_2))$$

and this gives a deformation into the standard addition.

Note: this is a bilinear deformation.

To obtain the well-known Heisenberg group take $k = 2$, $l = 1$ and for $v_i = (x_i, y_i)$ put

$$\mathcal{B} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^1 : \mathcal{B}(v_1, v_2) = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \tau x_1 y_2.$$

The Heisenberg multiplication on $\mathbb{R}^3$:

$$(x_1, y_1, w_1) \cdot (x_2, y_2, w_2) = (x_1 + x_2, y_1 + y_2, w_1 + w_2 + \tau x_1 y_2).$$