Lagrangian submanifolds in complex projective space with parallel second fundamental form

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Contents

Kähler manifold

Let $\bar{M}^n$ be an Kähler n-manifold, that is, an 2n-dimensional manifold with a almost complex structure $J : T\bar{M}^n \to T\bar{M}^n$ satisfying that

$$\begin{cases} J^2 = -I, \\ < Jv, Jw >=< v, w >, \\ DJ = 0, \end{cases}$$

where $v, w \in T\bar{M}^n$ and D is the Levi-Civita connection on $\bar{M}^n$.

Complex space forms are the simplest Kähler-Einstein manifold.

Let $\bar{M}^n(4c)$ denote an n-dimensional complex space form with constant holomorphic sectional curvature $4c$.

When $c > 0$, $\bar{M}^n(4c) = CP^n(4c)$,
When $c = 0$, $\bar{M}^n(4c) = C^n$,
When $c < 0$, $\bar{M}^n(4c) = CH^n(4c)$.

Lagrangian submanifolds

Let $\phi : M \to \bar{M}^n$ be an isometric immersion from an n-dimensional Riemannian manifold M into a Kähler n-manifold $\bar{M}^n$.

Then M is called a **Lagrangian submanifold** if the almost complex structure J of $\bar{M}^n$ carries each tangent space of M into its corresponding normal space.

Example 1-3: $\mathbb{R}^n \to \mathbb{C}^n$, $\mathbb{RP}^n \to \mathbb{CP}^n$, $\mathbb{RH}^n \to \mathbb{CH}^n$.
Example 4: Whitney sphere in $\mathbb{C}^n$. It is defined as the Lagrangian immersion of the unit sphere $\mathbb{S}^n$, centered at the origin of $\mathbb{R}^{n+1}$, in $\mathbb{C}^n$, given by

$$\phi : \mathbb{S}^n \to \mathbb{C}^n : \phi(x_1, x_2, \dots, x_n, x_{n+1}) = \frac{1 + i x_{n+1}}{1 + x_{n+1}^2}(x_1, \dots, x_n). \qquad (1)$$

Example 5: Whitney spheres in $\mathbb{CP}^n$. They are a one-parameter family of Lagrangian spheres in $\mathbb{CP}^n$, given by

$$\bar{\phi}_\theta : \mathbb{S}^n \to \mathbb{CP}^n(4) :$$

$$\bar{\phi}_\theta(x_1, x_2, \dots, x_n, x_{n+1}) = \pi \circ \left(\frac{(x_1, \dots, x_n)}{c_\theta + i s_\theta x_{n+1}}; \frac{s_\theta c_\theta (1 + x_{n+1}^2) + i x_{n+1}}{c_\theta^2 + s_\theta^2 x_{n+1}^2} \right), \quad (2)$$

where $\theta > 0$, $c_\theta = \cosh\theta$, $s_\theta = \sinh\theta$, $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{CP}^n(4)$ is the Hopf fibration.

Example 6: Whitney spheres in $\mathbb{CH}^n$. They are a one-parameter family of Lagrangian spheres in $\mathbb{CH}^n$, given by

$$\bar{\phi}_\theta : \mathbb{S}^n \to \mathbb{CH}^n(-4) :$$

$$\bar{\phi}_\theta(x_1, x_2, \dots, x_n, x_{n+1}) = \pi \circ \left(\frac{(x_1, \dots, x_n)}{s_\theta + i c_\theta x_{n+1}}; \frac{s_\theta c_\theta (1 + x_{n+1}^2) - i x_{n+1}}{s_\theta^2 + c_\theta^2 x_{n+1}^2} \right), \quad (3)$$

where $\theta > 0$, $c_\theta = \cosh\theta$, $s_\theta = \sinh\theta$, $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(4)$ is the Hopf fibration.

A general method for constructing Lagrangian submanifolds in complex projective space

In view of Reckziegel's results, we have

Let $\phi : M \to \mathbb{CP}^n(4c)$ be a Lagrangian isometric immersion.

We consider the Hopf fibration: $\pi : \mathbb{S}^{2n+1}(c) \to \mathbb{CP}^n(4c)$.

Then there exists an isometric covering map $\tau : \hat{M} \to M$ and a $\mathbb{C}$-totally real isometric immersion $\tilde{\phi} : \hat{M} \to \mathbb{S}^{2n+1}(c)$ such that $\phi \circ \tau = \pi \circ \tilde{\phi}$.

Conversely, let $\tilde{\phi} : \hat{M} \to \mathbb{S}^{2n+1}(c)$ be a $\mathbb{C}$-totally real isometric immersion. Then $\phi = \pi \circ \tilde{\phi} : M \to \mathbb{CP}^n(4c)$ is an Lagrangian isometric immersoin.

Under this correspondence, the second fundamental form h^{ϕ} and $h^{\tilde{\phi}}$ satisfy $h^{\phi} = \pi_* h^{\tilde{\phi}}$. We shall denote h^{ϕ} and $h^{\tilde{\phi}}$ simply by h.

Parallel submanifolds

Let $\phi : M \to \bar{M}$ be an isometric immersion. If at each point p of M, the first derivative of the second fundamental form ∇h vanishes, i.e., $\nabla h \equiv 0$, we call M a submanifold with parallel second fundamental form, i.e, a parallel submanifold.

Examples: straight lines, circles, planes, round spheres, round cylinders in $\mathbb{R}^3$; circles, round spheres, a product of two circles in $\mathbb{S}^3$, Veronese surface in $\mathbb{S}^4$.

Some examples of Lagrangian parallel submanifolds:

- $\mathbb{R}^n \to \mathbb{C}^n$, $\mathbb{RP}^n \to \mathbb{CP}^n$, $\mathbb{RH}^n \to \mathbb{CH}^n$.
- $\mathbf{SU}(k)/\mathbf{SO}(k)\,(k \geq 3) \to \mathbb{CP}^{\frac{1}{2}k(k+1)-1}$.
- $\mathbf{SU}(k)\,(k \geq 3) \to \mathbb{CP}^{k^2-1}$.
- $\mathbf{SU}(2k)/\mathbf{Sp}(k)\,(k \geq 3) \to \mathbb{CP}^{2k^2-k-1}$.
- $\mathbf{E}_6/\mathbf{F}_4 \to \mathbb{CP}^{26}$.

Motivation of our research work

From the Riemannian geometric point of view, one of the most fundamental problems in the study of Lagrangian submanifolds is:

the classification of Lagrangian submanifolds in complex space forms with parallel second fundamental form.

In 1980s, H. Naitoh classified the Lagrangian submanifolds with parallel second fundamental form in complex projective space.

Prof. Naitoh's method is based on the theory of Lie groups and symmetric spaces.

In the irreducible case, the classification is clear, Naitoh completely classified the Lagrangian submanifolds with parallel second fundamental form and without Euclidean factor in complex projective space. He showed that such a submanifold is always locally symmetric and is one of the symmetric spaces:

- $\mathbf{SO}(k+1)/\mathbf{SO}(k)\,(k \geq 2)$.
- $\mathbf{SU}(k)/\mathbf{SO}(k)\,(k \geq 3)$.
- $\mathbf{SU}(k)\,(k \geq 3)$.
- $\mathbf{SU}(2k)/\mathbf{Sp}(k)\,(k \geq 3)$.
- $\mathbf{E}_6/\mathbf{F}_4$.

However, little information is given on how to construct all reducible examples.

Question

How to determine all reducible parallel Lagrangian submanifolds of complex projective space?

Our main result

We obtain a complete and explicit classification of all (irreducible and reducible) parallel Lagrangian submanifolds of complex projective space by an **elementary** geometric method.

Calabi product Lagrangian immersion

Definition

Let $\psi_1 : (M_1, g_1) \to \mathbb{CP}^{n_1}(4)$ and $\psi_2 : (M_2, g_2) \to \mathbb{CP}^{n_2}(4)$ be two Lagrangian immersions.

$\pi : \mathbb{S}^{2n+1}(c) \to \mathbb{CP}^n(4c)$ is the Hopf fibration.

We denote by $\tilde{\psi}_i : M_i \to \mathbb{S}^{2n_i+1}(1)$ the horizontal lifts of ψ_i, $i = 1, 2$, respectively.

Let $\tilde{\gamma}(t) = (r_1 e^{i(\frac{r_2}{r_1} t)}, r_2 e^{i(-\frac{r_1}{r_2} t)})$, be a special Legendre curve, where r_1 and r_2 are positive constants with $r_1^2 + r_2^2 = 1$,

Then $\psi = \pi(\tilde{\gamma}_1 \tilde{\psi}_1; \tilde{\gamma}_2 \tilde{\psi}_2) : I \times M_1 \times M_2 \to \mathbb{CP}^n(4)$ is a Lagrangian immersion, where $n = n_1 + n_2 + 1$.

We call ψ a *Calabi product Lagrangian immersion* of ψ_1 and ψ_2.

When n_1 (or n_2) is zero, we call ψ a *Calabi product Lagrangian immersion* of ψ_2 (or ψ_1) and a point.

Characterizations of the Calabi products

Theorem (1.6, Li-Wang, Results Math, 2011)

Let $\psi : M \to \mathbb{CP}^n(4)$ be a Lagrangian immersion. If M admits two orthogonal distributions $\mathcal{T}_1$ (of dimension 1, spanned by a unit vector E_1) and $\mathcal{T}_2$ (of dimension $n-1$, spanned by $\{E_2, \ldots, E_n\}$), and there exist local functions $\lambda_1,\ \lambda_2$ such that

$$\begin{cases} h(E_1, E_1) = \lambda_1 JE_1,\ h(E_1, E_i) = \lambda_2 JE_i, \\ \lambda_1 \neq 2\lambda_2,\ i = 2, \ldots, n. \end{cases} \tag{4}$$

If M has parallel second fundamental form, then ψ is locally a Calabi product Lagrangian immersion of a point and an $(n-1)$-dimensional Lagrangian immersion $\psi_1 : M_1 \to \mathbb{CP}^{n-1}(4)$ which has parallel second fundamental form.

Conversely, if ψ is locally a Calabi product Lagrangian immersion of a point and an $(n-1)$-dimensional Lagrangian immersion $\psi_1 : M_1 \to \mathbb{CP}^{n-1}(4)$ which has parallel second fundamental form, then M has parallel second fundamental form.

Characterizations of the Calabi products

Theorem (4.6, Li-Wang, Results Math, 2011)

Let $\psi : M \to \mathbb{CP}^n(4)$ be a Lagrangian immersion. If M admits three mutually orthogonal distributions $\mathcal{T}_1$ (spanned by a unit vector E_1), $\mathcal{T}_2$, and $\mathcal{T}_3$ of dimension 1, n_1 and n_2 respectively, with $1 + n_1 + n_2 = n$, and there three real functions λ_1, λ_2 and λ_3 ($2\lambda_3 \neq \lambda_1 \neq 2\lambda_2 \neq 2\lambda_3$) such that for all $E_i \in \mathcal{T}_2,\ E_\alpha \in \mathcal{T}_3$,

$$\begin{cases} h(E_1, E_1) = \lambda_1 JE_1,\ h(E_1, E_i) = \lambda_2 JE_i, \\ h(E_1, E_\alpha) = \lambda_3 JE_\alpha. \end{cases} \tag{5}$$

If M has parallel second fundamental form, then ψ is locally a Calabi product Lagrangian immersion of two lower dimensional Lagrangian submanifolds $\psi_i (i = 1, 2)$ with parallel second fundamental form.

Conversely, if ψ is locally a Calabi product Lagrangian immersion of two lower dimensional Lagrangian submanifolds $\psi_i (i = 1, 2)$ with parallel second fundamental form, then M has parallel second fundamental form.

Theorem (Dillen-Li-Vrancken-Wang, 2011—Main Theorem)

Let M be a Lagrangian submanifold in $\mathbb{CP}^n(4)$ with constant holomorphic sectional curvature 4, suppose that M has parallel second fundamental form, then either M is totally geodesic, or

(i) *M is locally the Calabi product of a point with a lower dimensional Lagrangian submanifold with parallel second fundamental form, or*

(ii) *M is locally the Calabi product of two lower dimensional Lagrangian submanifolds with parallel second fundamental form, or*

(iii) $n = \frac{1}{2}k(k+1) - 1,\ k \geq 3$, *and M is congruent with* $\mathbf{SU}(k)/\mathbf{SO}(k)$, *or*

(iv) $n = k^2 - 1, k \geq 3$, *and M is congruent with* $\mathbf{SU}(k)$, *or*

(v) $n = 2k^2 - k - 1,\ k \geq 3$, *and M is congruent with* $\mathbf{SU}(2k)/\mathbf{Sp}(k)$, *or*

(vi) $n = 26$ *and M is congruent with* $\mathbf{E}_6/\mathbf{F}_4$.

We notice that here we don't assume the minimal condition.

Remark

According to our main theorem, we get a list of all parallel Lagrangian submanifolds in complex projective space. For example,

- When n=1, M^1 is locally isometric to $\mathbb{RP}^1$.
- When n=2, M^2 is locally isometric to $\mathbb{RP}^2$, or $\mathbb{S}^1 \times \mathbb{S}^1$(the latter immersion is a Calabi product Lagrangian immersion).
- When n=3, M^3 is locally isometric to $\mathbb{RP}^3$, or $\mathbb{S}^1 \times \mathbb{S}^2$, or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. (The latter two immersions are Calabi product Lagrangian immersions).
- When n=4, M^4 is locally isometric to $\mathbb{RP}^4$, or $\mathbb{S}^1 \times \mathbb{S}^3$, or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$, or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. (The last three immersions are Calabi product Lagrangian immersions).
- When n=5, M^5 is locally isometric to $\mathbb{RP}^5$, or $\mathbf{SU}(3)/\mathbf{SO}(3)$, or $\mathbb{S}^1 \times \mathbb{S}^4$, or $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{S}^2$, or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^3$, or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$, or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. (The last five immersions are Calabi product Lagrangian immersions).

We notice that here Calabi procuct plays a very important role.

Main techniques

We use the techniques developped in [Hu-Li-Simon-Vrancken, Differential Geom. Appl., 2009] and [Hu-Li-Vrancken, J. Differential Geom., 2011], in which they give a complete classification of locally strongly convex affine hypersurfaces of $\mathbb{R}^{n+1}$ with parallel cubic form, and also the characterizations for Calabi product Lagrangian immersions in [Li-Wang, Results Math., 2011].

There exist many similarities between the study of minimal Lagrangian submanifolds of complex space forms and the study of affine hypersurfaces in affine differential geometry (eg. there exist totally symmetric cubic form in both two cases).

The difference tensor **K** $\longleftrightarrow$ The second fundamental form **h**.

K satisfies the **apolarity** condition, namely $\mathrm{tr}K_X = 0$ for all X
$\longleftrightarrow$
h satisfies the **minimal** condition, namely $\mathrm{tr}h=0$.

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Thanks for your attention!