A new condition of real hypersurfaces in complex two-plane Grassmannians

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- $G_2(\mathbb{C}^{m+2})$: the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$

- $G_2(\mathbb{C}^{m+2}) = G/K$
  - $G = SU(m+2)$, $K = S(U(2) \times U(m))$
  - $dim(G_2(\mathbb{C}^{m+2})) = 4m$

$G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Riemannian symmetric space equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathcal{J}$, not containing J.

- (geometric structure)

$$JJ_\nu = J_\nu J, \ Tr(JJ_\nu) = 0 \quad \nu = 1, 2, 3$$

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- g : the induced Riemannian metric on M
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$J \longrightarrow (\phi, \xi, \eta, g)$: an almost contact metric structure,
where $JX = \phi X + \eta(X)N$, $-JN = \xi$

- $\eta(\xi) = 1$, $\phi\xi = 0$, $\eta(\phi X) = 0$, $\phi^2 X = -X + \eta(X)\xi$

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- $[\xi]$ is invariant under the shape operator A of M, that is, $A\xi = \alpha\xi$.
- $\mathfrak{D}^\perp$ is invariant under the shape operator A of M, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

Theorem A [Berndt and Suh, Monatshefte für Math., 1999]

Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

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Proposition A. [Berndt and Suh Monatshefte für Math., 1999]

Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathcal{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three(if $r = \pi/2\sqrt{8}$) or four(otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8}cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}cot(\sqrt{2}r)\ ,\ \lambda = -\sqrt{2}tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\lambda) = 2m-2 = m(\mu)$$

and for the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,\ T_\beta = \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$

$$T_\lambda = \{X | X \perp \mathbb{H}\xi, JX = J_1X\},\ T_\mu = \{X | X \perp \mathbb{H}\xi, JX = -J_1X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Proposition B. [Berndt and Suh Monatshefte für Math., 1999]

Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$.
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$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathfrak{J}J\xi\ ,\ T_\gamma = \mathfrak{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^{\perp}\ ,\ \mathfrak{J}T_\lambda = T_\lambda\ ,\ \mathfrak{J}T_\mu = T_\mu\ ,\ JT_\lambda = T_\mu.$$

Berndt and Suh have some equivalent condition of the properties as follows

Theorem B [Berndt and Suh, Monatshefte für Math., 2002]

Let M be a connected orentable real hypersurface in a Kaehler manifold $\widetilde{M}$. The following statements are equivalent:

(1) The Reeb flow on M is isometric,
(2) The shape operator A and the structure tensor field ϕ commute with each other,
(3) The Reeb vector field ξ is a principal curvature vector of M everywhere and the principal curvature spaces contained in the maximal complex subbundle $\mathfrak{D}$ of TM are complex subspaces.

Also, Berndt and Suh gave a charaterization of Type(A) in Theorem A.

Theorem C [Berndt and Suh, Monatshefte für Math., 2002]

Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

$A\phi = \phi A \Longleftrightarrow$ The Reeb flow on M is isometric $\Longleftrightarrow M \approx$ Type (A)

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In 2003, Suh considered the condition that the almost contact 3-structure tensor $\{\phi_1, \phi_2, \phi_3\}$ commute with the shape opertor A of real hypersurface M in $G_2(\mathbb{C}^{m+2})$. So he proved that there does not exist any real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $A\phi_\nu X = \phi_\nu AX,\ \nu = 1, 2, 3$ for any tangent vector field X on M.

In addition, he gave a characterization of real hypersurface of Type (B).

Theorem D [Suh, Bull. Austral. Math. Soc., 2003]

Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying

$$A\phi_\nu X = \phi_\nu AX,\ \nu = 1, 2, 3 \qquad X \in [\xi]^\perp$$

where the distribution $[\xi]^\perp$ is the orthogonal complement of the one-dimensional distribution $[\xi]$. Then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

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Problem

$$M \hookrightarrow G_2(\mathbb{C}^{m+2}) \quad \Rightarrow \quad M \approx (?)$$

Hopf

$$\phi\phi_i AX = A\phi_i\phi X \quad \text{for some } i = 1, 2, 3 \qquad (*)$$

- M : Hopf $\rightleftharpoons$ any integral curve of the Reeb vector field ξ are geodesic ($A\xi = \alpha\xi$, where $\alpha = g(A\xi, \xi)$).

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Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &- g(JX,Z)JY - 2g(JX,Y)JZ + \sum_{\nu=1}^{3} g(J_\nu Y,Z)J_\nu X \\ &- \sum_{\nu=1}^{3}\{g(J_\nu X,Z)J_\nu Y + 2g(J_\nu X,Y)J_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathcal{J}$.

- $\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$: the Gauss formula

  $\bar{\nabla}_X N = -AX$: the Weingarten formula

$$\begin{aligned}\bar{R}(X,Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\ &= R(X,Y)Z - g(AY,Z)AX + g(AX,Z)AY \\ &\quad + g((\nabla_X A)Y,Z)N - g((\nabla_Y A)X,Z)N\end{aligned}$$

- Codazzi equation

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = \ & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ & + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu\end{aligned}$$

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The following identities can be proved in a straightforward method

$$\begin{aligned} &\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu,\ \eta_\nu(\xi_\nu) = 1,\ \phi_\nu\xi_\nu = 0, \\ &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2},\ \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{aligned} \tag{1}$$

Moreover, from the commuting property of $J_\nu J = JJ_\nu,\ \nu = 1, 2, 3$, it can be given by

$$\begin{aligned} &\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ &\eta_\nu(\phi X) = \eta(\phi_\nu X),\ \phi\xi_\nu = \phi_\nu\xi. \end{aligned} \tag{2}$$

The following identities can be proved in a straightforward method

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&\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu,\\
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\end{aligned} \tag{2}$$

From the Kaehler structure J and the quaternionic Kaehler structure $\mathcal{J}$, together with Gauss and Weingarten formulas it follows that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \ \nabla_X \xi = \phi AX, \tag{3}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{4}$$

$$\begin{aligned}(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ - g(AX, Y)\xi_\nu.\end{aligned} \tag{5}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu \xi) &= \nabla_X(\phi \xi_\nu) \\ &= (\nabla_X \phi)\xi_\nu + \phi(\nabla_X \xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu \phi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{6}$$

From the Kaehler structure J and the quaternionic Kaehler structure $\mathcal{J}$, together with Gauss and Weingarten formulas it follows that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \ \nabla_X \xi = \phi AX, \tag{3}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{4}$$

$$\begin{aligned}(\nabla_X \phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX, Y)\xi_\nu.\end{aligned} \tag{5}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu \xi) &= \nabla_X(\phi \xi_\nu) \\ &= (\nabla_X \phi)\xi_\nu + \phi(\nabla_X \xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu \phi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{6}$$

Proposition A. [Berndt and Suh Monatshefte für Math., 1999]

Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathcal{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three(if $r = \pi/2\sqrt{8}$) or four(otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8}cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}cot(\sqrt{2}r)\ ,\ \lambda = -\sqrt{2}tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\lambda) = 2m - 2 = m(\mu)$$

and for the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,\ T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$

$$T_\lambda = \{X | X \perp \mathbb{H}\xi, JX = J_1 X\},\ T_\mu = \{X | X \perp \mathbb{H}\xi, JX = -J_1 X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Key Lemma

Lemma 1

Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M has the commuting shape operator $\phi\phi_1 AX = A\phi_1\phi X$, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

proof)
Let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.
From the assumption $\phi\phi_1 AX = A\phi_1\phi X$ for $X = \xi$, we have

$$\phi_1 A\xi = \eta(\phi_1 A\xi)\xi. \tag{7}$$

Since M is Hopf, we see that

$$A\xi = \alpha\xi = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1. \tag{8}$$

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Combining with above two formulas, we have

$$\alpha\eta(X_0)\phi_1 X_0 = 0. \tag{9}$$

But we see that $\phi_1 X_0$ is non-vanishing at all point of M. In fact, we obtain $\|\phi_1 X_0\|^2 = 1$. Then it gives

$$\alpha\eta(X_0) = 0. \tag{10}$$

Case 1. $\alpha = 0$, that is, $A\xi = 0$

This case is trivial by Lemma 3.1 due to Pérez and Suh.

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Lemma 2

Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $\xi \in \mathfrak{D}^{\perp}$. If M satisfies the following conditions

$$(\phi\phi_1)AX = A(\phi\phi_1)X, \quad X \in \mathfrak{D}^{\perp}, \tag{**}$$

then the distribution $\mathfrak{D}^{\perp}$ is invariant under the shape operator A of M, that is, $g(A\mathfrak{D}^{\perp}, \mathfrak{D}) = 0$.

proof)

Since $\xi \in \mathfrak{D}^{\perp}$, let us put $\xi = \xi_1$. Taking the covariant derivative along any direction $Y \in TM$, we have

$$\nabla_Y \xi = \nabla_Y \xi_1$$

$$\phi AY = q_3(Y)\xi_2 - q_2(Y)\xi_3 + \phi_1 AY. \tag{11}$$

From this, taking an inner product with ξ_2, ξ_3, we have

$$q_3(Y) = 2g(AY, \xi_3), \; q_2(Y) = 2g(AY, \xi_2), \tag{12}$$

respectively.

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Applying the structure tensor ϕ in (11), this equation can be written by

$$AY = \alpha\eta(Y)\xi + 2g(AY, \xi_2)\xi_2 + 2g(AY, \xi_3)\xi_3 - \phi\phi_1 AY, \quad Y \in TM, \tag{13}$$

where we have used that M is Hopf. Putting $Y = \xi_2$ in (13), we get

$$A\xi_2 = 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - A\xi_2.$$

From this together with the condition (**) we have $\phi\phi_1 A\xi_2 = A\phi\phi_1\xi_2 = A\xi_2$, because of $\xi_2 \in \mathfrak{D}^\perp$. Then, it implies

$$A\xi_2 = g(A\xi_2, \xi_2)\xi_2 + g(A\xi_2, \xi_3)\xi_3. \tag{14}$$

Similarly, if we put $Y = \xi_3$ in (13), it follows

$$A\xi_3 = g(A\xi_3, \xi_2)\xi_2 + g(A\xi_3, \xi_3)\xi_3. \tag{15}$$

From above two equtions and the assumption $A\xi_1 = A\xi = \alpha\xi = \alpha\xi_1$, we have $A\xi_\nu \in \mathfrak{D}^\perp$ for any $\nu = 1, 2, 3$.

So we assert that the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator A of M which gives a complete proof of our lemma.

Using Lemma 2, we have the following

Lemma 3

Let M be a connected orientable real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $\xi \in \mathfrak{D}^\perp$. If M has the commuting shape operator, that is, the shape operator A on M satisfies the condition (*), then the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator A on M, that is, $g(A\mathfrak{D}^\perp, \mathfrak{D}) = 0$.

proof. Substituting $X = \xi$ in $\phi\phi_1 AX = A\phi_1\phi X$, we have

$$\phi\phi_1 A\xi = 0.$$

Applying ϕ in above equation gives

$$\phi_1 A\xi = \eta(\phi_1 A\xi)\xi.$$

Taking an inner product with ξ_1, we obtain

$$\eta(\phi_1 A\xi)\eta(\xi_1) = 0.$$

Since $\xi = \xi_1$, it means that $\eta(\phi_1 A\xi) = 0$. So, we have

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Moreover, from (2), we have

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Then the condition (*) can be written by

$$\phi\phi_1 AX = A\phi_1\phi X = A\phi\phi_1 X, \;\; X \in TM. \tag{16}$$

Now putting $X = \xi_\nu$, $\nu = 2, 3$ in (16), we have

$$\phi\phi_1 A\xi_\nu = A\phi\phi_1\xi_\nu, \quad \nu = 2, 3. \tag{17}$$

By Lemma 2, we have $A\xi_\nu \in \mathfrak{D}^\perp$, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, $\nu = 2, 3$ under our assumption.

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$\phi\phi_1 AX = A\phi_1\phi X$, $\xi \in \mathfrak{D}^\perp$

$\Rightarrow A\xi = \alpha\xi$, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$ ($\because$ Lemma 3)

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Conversely,

- Check!! $M \approx$ type (A) $\Rightarrow \phi\phi_1 AX = A\phi_1\phi X$

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[subcase 1] $X \in T_\alpha$, (i.e. $X = \xi = \xi_1$)
It can be easily checked that two sides are equal to each other.

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We put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Then by putting $X = \xi_2$ in (*) we have

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$$\text{Left Side} = \phi\phi_1 AX = \lambda\phi\phi_1 X = \lambda\phi^2 X = -\lambda X \qquad \text{and}$$
$$\text{Right Side} = A\phi_1\phi X = A\phi_1^2 X = -AX = -\lambda X.$$

From this we know that they are equal to $-\lambda X$.

[subcase 1] $X \in T_\alpha$, (i.e. $X = \xi = \xi_1$)
It can be easily checked that two sides are equal to each other.

[subcase 2] $X \in T_\beta$, (i.e. $X = \xi_2$ or $X = \xi_3$)
We put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Then by putting $X = \xi_2$ in (*) we have

$$\text{Left Side} = \phi\phi_1 A\xi_2 = \beta\phi\phi_1\xi_2 = \beta\phi\xi_3 = \beta\xi_2 \qquad \text{and}$$
$$\text{Right Side} = A\phi_1\phi\xi_2 = A\phi_1\phi_2\xi = A\phi_1\phi_2\xi_1 = A\xi_2 = \beta\xi_2.$$

Similarly, by putting $X = \xi_3$ in (*) we know that they are equal to $\beta\xi_3$.

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[subcase 4] $X \in T_\mu = \{X \mid X \perp \mathbb{H}\xi,\ \phi X = -\phi_1 X\}$

We have that

$$\text{left side} = \phi\phi_1 AX = \mu X \qquad \text{and}$$
$$\text{right side} = A\phi_1\phi X = \mu X.$$

So we know that they are equal to $\mu X = 0$, because of $\mu = 0$.

Remark 1. The real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ satisfy the condition $(*)$.

Theorem 1

Let M be a connected Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with satisfying commuting condition $(*)$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

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To consider this case, we introduce one theorem as follows

Theorem E [Lee and Suh, Bull. Korean Math. Soc., 2010]

Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

From Theorem E,

- $\xi \in \mathfrak{D} \quad \Rightarrow \quad M \approx$ type (B)

Conversely,

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Proposition A. [Berndt and Suh Monatshefte für Math., 1999]

Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$.
The corresponding multiplicities are

$$m(\alpha) = 1\ ,\quad m(\beta) = 3 = m(\gamma)\,,\quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathcal{J}J\xi\ ,\ T_\gamma = \mathcal{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp\ ,\ \mathcal{J}T_\lambda = T_\lambda\ ,\ \mathcal{J}T_\mu = T_\mu\ ,\ JT_\lambda = T_\mu.$$

If $X = \xi$, where $\xi \in T_\alpha$, then we have

$$\begin{aligned} \phi\phi_1 A\xi = A\phi_1\phi\xi \Leftrightarrow & \ \phi\phi_1 A\xi - A\phi_1\phi\xi = 0 \\ \Leftrightarrow & \ \phi\phi_1 A\xi = 0 \\ \Leftrightarrow & \ \alpha\phi\phi_1\xi = 0 \\ \Leftrightarrow & \ \alpha\phi^2\xi_1 = 0 \\ \Leftrightarrow & \ -\alpha\xi_1 = 0 \\ \Leftrightarrow & \ \alpha = 0. \end{aligned}$$

But this case can not occur for some $r \in (0, \pi/4)$. In fact, $\alpha = -2tan(2r)$ is non-vanishing for $r \in (0, \pi/4)$.
This gives a contradiction.

Remark 2. The real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ do not satisfy the commuting condition (*).

Theorem 2

There does not exist a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with the commuting shape operator $\phi\phi_1 A = A\phi_1\phi$ if the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.

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Theorem 3

Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the shape operator A satisfies the commuting condition $\phi\phi_1 A = A\phi_1\phi$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

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Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the shape operator A satisfies the commuting condition $\phi\phi_i A = A\phi_i\phi$ for some $i = 1, 2, 3$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

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Thank you for your attention!!