Isoparametric submanifolds in Hilbert space

(Joint work with Ernst Heintze)

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- **Conjecture:** M must be standard.
- **Our contribution:** we introduce a (canonical) homogeneous structure Γ on M, and we use it to prove a Rigidity Theorem, which makes such submanifolds accessible to classification.

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  - Conversely, if M is homogeneous isoparametric then M is a principal orbit of an s-representation (Dadok, 1985).
  - Further, if M is isoparametric of codimension $\neq 2$ in V, then M is homogeneous (Thorbergsson, 1991).

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- There is no standard theory of Hilbert-Lie groups and their representations that we can apply. It is not even known what kind of differentiable structure $\mathrm{Iso}(V, M)$ possesses. Instead, we use GEOMETRY!

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- Since $[\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is an ideal of $\mathfrak{g}$ whose generated group is transitive on M, M is determined as well.

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- In other words, $\bar{\Gamma}_X Y = \underbrace{(\bar{\Gamma}_X Y)^\top}_{:=\Gamma_X Y} + \alpha(X, Y)$
- Hence $\bar{\Gamma}_p$ contains the same information as the pair (α_p, Γ_p).
- We call Γ_p the homogeneous structure on M at p associated to the group G and the reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$.

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- **Definition:**

$$\boxed{(\Gamma_p)_X Y := \left[\frac{d}{dt}\Big|_{t=0} (F_X^t)_* Y\right]^\top}$$

  (*Canonical homogeneous structure of* M *at* p),
  where $\mathrm{dom}(\Gamma_p)_X \subset T_pM$ is a dense subspace.

Main results

- **Theorem A.** Each Γ_X is continuous and thus extends to T_pM. Equivalently, the one-parameter groups of H-L are smooth in $\mathrm{Iso}(V)$. Hence their infinitesimal generators are Killing fields on V. (Moreover, Γ is continuous as a bilinear map.)

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- In the other cases, we can also describe the image of Γ, but it is more complicated.

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- **Gauss equation:** For any $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}^*$ and $\mathbf{l} \in \mathbf{I}$ with $\mathbf{k} \neq \mathbf{l}$, and $X_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$, $Z_\mathbf{k} \in E_\mathbf{k}$, $W_\mathbf{l} \in E_\mathbf{l}$,

$$\left\langle \left([\Gamma_{X_\mathbf{i}}, \Gamma_{Y_\mathbf{j}}] - \Gamma_{\Gamma_{X_\mathbf{i}} Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}} X_\mathbf{i}} \right) Z_\mathbf{k}, W_\mathbf{l} \right\rangle = -\langle X_\mathbf{i} \wedge Y_\mathbf{j}, Z_\mathbf{k} \wedge W_\mathbf{l} \rangle \langle v_\mathbf{i}, v_\mathbf{j} \rangle.$$

- Notion of slice: component L containing p of $W \cap M$ for suitable closed affine subspaces $W \subset V$ through p.
  - L is totally geodesic in M and isoparametric in W.
  - Can have $\dim L < \infty$ or $= \infty$.
  - L is determined by some curvature normals.
  - Γ can be computed along L: $\Gamma_{E_\mathbf{i}} E_\mathbf{j}$ where $v_\mathbf{i}$, $v_\mathbf{j}$ can be LD or LI.
  - Most importante are:
    - infinite dimensional rank 1 slices (all curvature normals parallel);
    - finite dimensional rank 2 slices (generated by $v_\mathbf{i}$, $v_\mathbf{j}$ LI).
- **Gauss equation:** For any $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}^*$ and $\mathbf{l} \in \mathbf{I}$ with $\mathbf{k} \neq \mathbf{l}$, and $X_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$, $Z_\mathbf{k} \in E_\mathbf{k}$, $W_\mathbf{l} \in E_\mathbf{l}$,

$$\left\langle \left([\Gamma_{X_\mathbf{i}}, \Gamma_{Y_\mathbf{j}}] - \Gamma_{\Gamma_{X_\mathbf{i}} Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}} X_\mathbf{i}} \right) Z_\mathbf{k}, W_\mathbf{l} \right\rangle = -\langle X_\mathbf{i} \wedge Y_\mathbf{j}, Z_\mathbf{k} \wedge W_\mathbf{l} \rangle \langle v_\mathbf{i}, v_\mathbf{j} \rangle.$$

- **Codazzi equation:**

$$(\Gamma_{Y_\mathbf{j}} X_\mathbf{i})_{E_\mathbf{k}} = \frac{v_\mathbf{j} - v_\mathbf{k}}{v_\mathbf{i} - v_\mathbf{k}} (\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{k}}$$

for all $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$, $\mathbf{k} \in \mathbf{I}$ with $\mathbf{k} \neq \mathbf{i}$ and $X_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$.

- $\Gamma_{X_{\mathbf{i}}}$ is continuous on $E_{\mathbf{0}}$ by closed graph theorem.

- $\Gamma_{X_{\mathbf{i}}}$ is continuous on $E_{\mathbf{0}}$ by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_{\mathbf{i}}} Y|| \leq C||Y||$ for all $Y \in \sum_{\mathbf{j}\neq\mathbf{0}} E_{\mathbf{j}}(p)$.

Some ideas about the proof of Theorem A

- $\Gamma_{X_{\mathbf{i}}}$ is continuous on $E_{\mathbf{0}}$ by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_{\mathbf{i}}} Y|| \leq C||Y||$ for all $Y \in \sum_{\mathbf{j} \neq \mathbf{0}} E_{\mathbf{j}}(p)$.
- If $X_{\mathbf{i}}$, Y tangent to finite dimensional slice: uniform bound; otherwise:

Some ideas about the proof of Theorem A

- $\Gamma_{X_\mathbf{i}}$ is continuous on $E_\mathbf{0}$ by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_\mathbf{i}} Y|| \leq C||Y||$ for all $Y \in \sum_{\mathbf{j}\neq\mathbf{0}} E_\mathbf{j}(p)$.
- If $X_\mathbf{i}$, Y tangent to finite dimensional slice: uniform bound; otherwise:
- Y is tangent to the infinite dimensional slice containing $E_\mathbf{i}$; then:

- $\Gamma_{X_{\mathbf{i}}}$ is continuous on $E_{\mathbf{0}}$ by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_{\mathbf{i}}} Y|| \leq C||Y||$ for all $Y \in \sum_{\mathbf{j} \neq \mathbf{0}} E_{\mathbf{j}}(p)$.
- If $X_{\mathbf{i}}$, Y tangent to finite dimensional slice: uniform bound; otherwise:
- Y is tangent to the infinite dimensional slice containing $E_{\mathbf{i}}$; then:
- A lemma reduces proof to case $Y = Y_{\mathbf{j}} \in E_{\mathbf{j}}(p)$ where $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ are parallel. Then:

Some ideas about the proof of Theorem A

- Γ_{X_i} is continuous on E_0 by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_i} Y|| \leq C||Y||$ for all $Y \in \sum_{j \neq 0} E_j(p)$.
- If X_i, Y tangent to finite dimensional slice: uniform bound; otherwise:
- Y is tangent to the infinite dimensional slice containing E_i; then:
- A lemma reduces proof to case $Y = Y_j \in E_j(p)$ where v_i, v_j are parallel. Then:

$$\frac{1}{2}\langle v_i, v_j\rangle ||X_i||^2||Y_j||^2 \quad = \quad \langle \Gamma_{X_i} Y_j, \Gamma_{Y_j} X_i\rangle \quad \text{(Gauss eqn)}$$

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$$\begin{aligned} \frac{1}{2}\langle v_i, v_j\rangle ||X_i||^2||Y_j||^2 &= \langle \Gamma_{X_i} Y_j, \Gamma_{Y_j} X_i \rangle \quad \text{(Gauss eqn)} \\ &= \sum_{k \in I} c_{ijk} ||(\Gamma_{X_i} Y_j)_{E_k}||^2 \quad \text{(Codazzi eqn)} \end{aligned}$$

Some ideas about the proof of Theorem A

- Γ_{X_i} is continuous on E_0 by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_i} Y|| \leq C||Y||$ for all $Y \in \sum_{j \neq 0} E_j(p)$.
- If X_i, Y tangent to finite dimensional slice: uniform bound; otherwise:
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\frac{1}{2}\langle v_i, v_j\rangle ||X_i||^2 ||Y_j||^2 &= \langle \Gamma_{X_i} Y_j, \Gamma_{Y_j} X_i \rangle \quad \text{(Gauss eqn)} \\
&= \sum_{k \in I} c_{ijk} ||(\Gamma_{X_i} Y_j)_{E_k}||^2 \quad \text{(Codazzi eqn)} \\
&= c_{ij0} ||(\Gamma_{X_i} Y_j)_{E_0}||^2 \quad \text{(by Thm C, if type is not } \tilde{B}_n,\ \tilde{C}_n)
\end{aligned}$$

Some ideas about the proof of Theorem A

- Γ_{X_i} is continuous on E_0 by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_i} Y|| \leq C||Y||$ for all $Y \in \sum_{j \neq 0} E_j(p)$.
- If X_i, Y tangent to finite dimensional slice: uniform bound; otherwise:
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&= c_{ij0} ||\Gamma_{X_i} Y_j||^2
\end{aligned}$$

proving continuity.

Some ideas about the proof of Theorem A

- $\Gamma_{X_{\mathbf{i}}}$ is continuous on $E_{\mathbf{0}}$ by closed graph theorem. Reduced to finding $C > 0$ such that $||\Gamma_{X_{\mathbf{i}}} Y|| \leq C||Y||$ for all $Y \in \sum_{\mathbf{j}\neq\mathbf{0}} E_{\mathbf{j}}(p)$.
- If $X_{\mathbf{i}}$, Y tangent to finite dimensional slice: uniform bound; otherwise:
- Y is tangent to the infinite dimensional slice containing $E_{\mathbf{i}}$; then:
- A lemma reduces proof to case $Y = Y_{\mathbf{j}} \in E_{\mathbf{j}}(p)$ where $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ are parallel. Then:

$$\begin{aligned}
\frac{1}{2}\langle v_{\mathbf{i}}, v_{\mathbf{j}}\rangle ||X_{\mathbf{i}}||^2 ||Y_{\mathbf{j}}||^2 &= \langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{Y_{\mathbf{j}}} X_{\mathbf{i}}\rangle \quad \text{(Gauss eqn)} \\
&= \sum_{\mathbf{k}\in\mathbf{I}} c_{\mathbf{ijk}} ||(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}||^2 \quad \text{(Codazzi eqn)} \\
&= c_{\mathbf{ij0}} ||(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{0}}}||^2 \quad \text{(by Thm C, if type is not } \tilde{B}_n,\ \tilde{C}_n) \\
&= c_{\mathbf{ij0}} ||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}||^2
\end{aligned}$$

proving continuity.

- For $\tilde{B}_n$, $\tilde{C}_n$, need Thms D and E, and affine root systems.

ありがとうございます