On the moduli space of left-invariant metrics on a Lie group

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Submanifold Theory in Symmetric Spaces and Lie Theory in Finite and Infinite Dimensions

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1 Introduction

1.1 Abstract

- **Theme:**
  - **G : a given Lie group.**
  - **Examine whether G admits a "distinguished" left-inv. metric. (e.g., Einstein, Ricci soliton)**
- **Successful case:**
  - **Well-studied for 3-dim. unimodular Lie groups.**
  - **By Milnor (1976), using so-called "Milnor frame".**
- **Our study:**
  - **The "moduli space" of left-invariant metric on a Lie group.**
  - **This generalizes Milnor frame to "any" Lie groups.**

1.2 Contents

- **This talk is organized as follows:**

2 Preliminaries: Milnor frame

2.1 Milnor frame (1/3)

- **Thm (Milnor, 1976):**
  - $\mathfrak{g}$: 3-dim. unimodular
  - $\langle , \rangle$: any inner product on $\mathfrak{g}$

    $\Rightarrow\ \exists \{x_1, x_2, x_3\}$: o.n.b. wrt $\langle , \rangle$, $\exists\, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$:

    $$[x_1, x_2] = \lambda_3 x_3,\ [x_2, x_3] = \lambda_1 x_1,\ [x_3, x_1] = \lambda_2 x_2.$$

- **Remark:**
  - For each $\mathfrak{g}$, possible values of $\lambda_1, \lambda_2, \lambda_3$ are determined.
  - One can examine all inner products on $\mathfrak{g}$.

2.2 Milnor frame (2/3) - application

- **Example:**
  - $\langle , \rangle$: any inner product on $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{R})$

    $\Rightarrow$ ∘ $\exists \{x_1, x_2, x_3\}$: o.n.b. wrt $\langle , \rangle$, $\exists \lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$:

    $[x_1, x_2] = \lambda_3 x_3, \ [x_2, x_3] = \lambda_1 x_1, \ [x_3, x_1] = \lambda_2 x_2.$

    ∘ The eigenvalues of $\mathrm{Ric}_{\langle , \rangle}$ satisfy either $(+, -, -)$ or $(0, 0, -)$

- **Cor.:**
  - $SL(2, \mathbb{R})$ does not admit left-invariant Einstein metrics.

- **Comment:**
  - Thanks to Milnor's theorem,

    curvature properties are completely understood

    for 3-dim. unimodular Lie groups.

2.3 Milnor frame (3/3) - comments

- **Recall:**
  - $\forall \langle , \rangle$ on $\mathfrak{g}$, $\exists \{x_1, x_2, x_3\}$: o.n.b., $\exists \lambda_1, \lambda_2, \lambda_3$: ...

- **Comment:**
  - Milnor's thm is quite useful.
  - But, the proof strongly depends on dim $= 3$.

- **Another view:**
  - Milnor's thm expresses

    $$\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{inner products on } \mathfrak{g}\}.$$

  - $\widetilde{\mathfrak{M}} \cong GL_3(\mathbb{R})/O(3)$: dim $= 6$.
  - $(\lambda_1, \lambda_2, \lambda_3)$ represents

    $\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}$, the "moduli space".

3 The moduli space of left-invariant metrics

3.1 The moduli space (1/6) - the space of left-inv. metrics

- **Fact:**
  - G : n-dim. Lie group with Lie algebra $\mathfrak{g}$

    $$\Rightarrow \widetilde{\mathfrak{M}} := \{g : \text{a left-inv. Riemannian metric on } G\}$$
    $$\cong \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\}$$
    $$\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).$$

- **Note that:**
  - We identify $\mathfrak{g} \cong \mathbb{R}^n$, $\mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}_n(\mathbb{R})$.
  - $\mathrm{GL}_n(\mathbb{R})$ acts on $\widetilde{\mathfrak{M}}$ by $g.\langle \cdot, \cdot \rangle := \langle g^{-1} \cdot, g^{-1} \cdot \rangle$.

3.2 The moduli space (2/6) - def

- **Def.:**
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) := \{c\varphi \in GL_n(\mathbb{R}) \mid c \in \mathbb{R}^\times, \ \varphi \in \mathrm{Aut}(\mathfrak{g})\}$.
  - $\mathfrak{PM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$ (the orbit space)

    is called the moduli space of left-invariant metrics.

- **Remark:**
  - $\exists \varphi \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) : \varphi.\langle , \rangle_1 = \langle , \rangle_2$

    $\Rightarrow$ the corresponding left-invariant metrics are isometric up to scaling

    $\Rightarrow$ all curvature properties are preserved.
  - By considering $\mathfrak{PM}$, one can study all left-invariant metrics.

3.3 The moduli space (3/6) - trivial cases

- **Thm. (Lauret, 2003):**
  - **A Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{PM} = \{\mathrm{pt}\}$**

    $\Leftrightarrow$ (1) $\mathbb{R}^n$: abelian,

    (2) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j \geq 2)$,

    (3) $\mathfrak{h}_3 \oplus \mathbb{R}^{n-3} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_2] = e_3$.

- **Comment:**
  - **Milnor (1976): $\forall \langle , \rangle$ on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ has a constant curvature $c < 0$.**
  - **This can be checked easily from $\mathfrak{PM} = \{\mathrm{pt}\}$.**

- **Remark:**
  - **Lauret's proof uses a degeneration of Lie brackets.**
  - **Kodama-Takahara-T. (2011) proved ($\Leftarrow$) by checking $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is transitive.**

3.4 The moduli space (4/6) - an expression

- **Prop.:**
  - $\mathfrak{PM} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}} \cong \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}_n(\mathbb{R}) / \mathrm{O}(n)$.

- **Remark:**
  - In order to express $\mathfrak{PM}$, one needs to express $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$.
    - In general, it is not easy.
    - $\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))$, relatively easier to determine.
  - If one knows $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$, then the remainings are linear algebra...

- **Comment:**
  - In general, $\mathfrak{PM}$ is a complicated space.
  - If $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is nice (e.g., low cohomogeneity, (hyper-)polar) then $\mathfrak{PM}$ is nice.

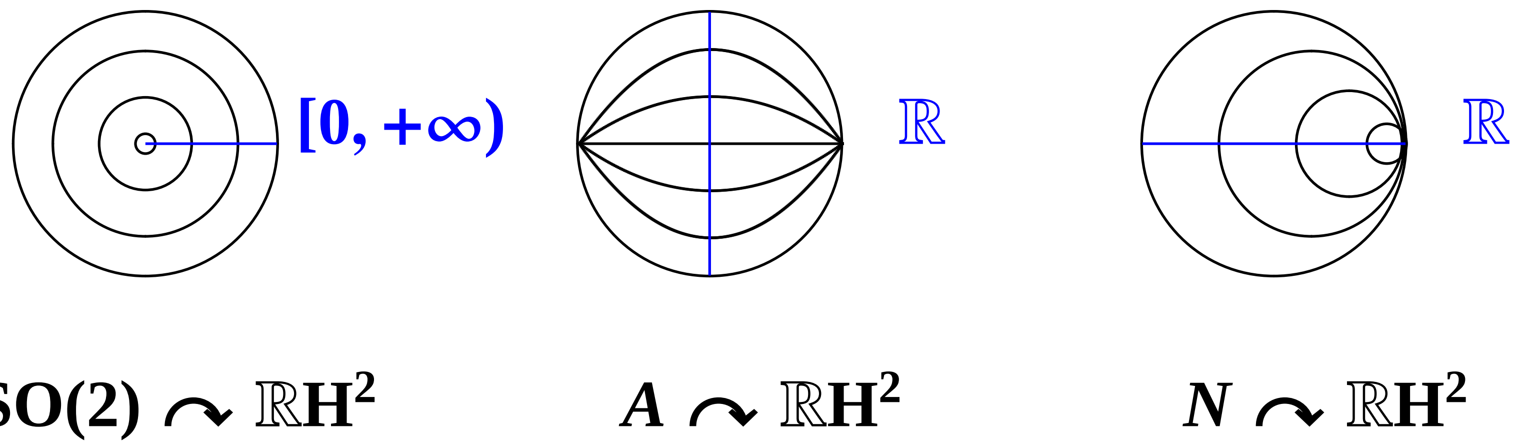
3.5 The moduli space (5/6) - a remark on orbit spaces

- **Recall:**
  - $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$: a noncompact symmetric space.

- **Thm. (Berndt-Brück 2001):**
  - M : a noncompact symmetric space
  - $H \curvearrowright M$: cohomogeneity one (with H connected)

    $\Rightarrow$ $H \backslash M \cong \mathbb{R}$ or $[0, +\infty)$.

- **Example (cohomogeneity one actions on $\mathbb{R}\mathrm{H}^2$):**



3.6 The moduli space (6/6) - an example

- **Example (Hashinaga-T.-Terada, preprint):**
  - $\mathfrak{g} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with $[e_1, e_2] = e_2$
  - $\langle , \rangle_0$: inner product s.t. the above basis is orthonormal

$$\Rightarrow \circ\ \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \left[\begin{array}{c|cc|c} * & 0 & 0 & 0 \\ \hline * & * & * & 0 \\ * & * & * & 0 \\ \hline * & 0 & 0 & * \end{array} \right] \right\},$$

  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: cohomogeneity one.
  - $\mathfrak{P}\mathfrak{M} = \{\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle , \rangle_0) \mid g_\lambda \in \mathfrak{U}\}$,

$$\text{where } \mathfrak{U} = \left\{ \left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & \lambda & 0 & 1 \end{array} \right],\ \lambda \geq 0 \right\} \cong [0, +\infty).$$

4 Milnor-type theorems

4.1 Milnor-type theorems (1/4) - a key theorem

- **Thm. (Hashinaga-T.-Terada, preprint):**
  - $\mathfrak{g}$: Lie algebra
  - $\{e_1, \ldots, e_n\}$: o.n.b. of $\mathfrak{g}$ wrt $\langle , \rangle_0$
  - $GL_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$

    (i.e., $\mathfrak{PM} = \{\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g.\langle , \rangle_0) \mid g \in \mathfrak{U}\}$)
  - $\langle , \rangle$: an arbitrary inner product

  $\Rightarrow \ \exists k > 0, \ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}), \ \exists g \in \mathfrak{U}$:

  $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is orthonormal wrt $k\langle , \rangle$.

- **A sketch of Proof:**
  - By assumption, $\exists g \in \mathfrak{U} : \langle , \rangle \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g.\langle , \rangle_0)$.
  - By definition, $\exists c\varphi \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) : \langle , \rangle = (c\varphi g).\langle , \rangle_0$.

4.2 Milnor-type theorems (2/4) - a comment

- **Recall:**
  - $\mathrm{GL}_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$
  - $\langle , \rangle$: an arbitrary inner product

    $\Rightarrow\ \exists k > 0,\ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}),\ \exists g \in \mathfrak{U}$:

    $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is orthonormal wrt $k\langle , \rangle$.

- **Comment:**
  - By checking the bracket relations among $\{\varphi g e_1, \ldots, \varphi g e_n\}$, one can get a Milnor-type theorem.
  - Note: φ preserves a bracket product.

4.3 Milnor-type theorems (3/4) - an example

▼ **Example (Milnor-type theorem):**

- $\mathfrak{g} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with $[e_1, e_2] = e_2$

  (Recall: $\mathfrak{U} \cong [0, +\infty)$)

- $\langle , \rangle$: an arbitrary inner product

  $\Rightarrow$ $\exists k > 0$, $\exists \lambda \geq 0$, $\exists\{x_1, x_2, x_3, x_4\}$: o.n.b. of $\mathfrak{g}$ wrt $k\langle , \rangle$

  s.t. $[x_1, x_2] = x_2 + \lambda x_4$.

▼ **Cor.:**

- $\mathfrak{g} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with $[e_1, e_2] = e_2$

  $\Rightarrow$
  - eigenvalues of Ric satisfy either $(+, 0, -, -)$ or $(0, 0, -, -)$.
  - $\nexists$ left-invariant Einstein metric.

4.4 Milnor-type theorems (4/4) - a remark

- **Recall:**
  - $\mathfrak{U}$ **: a set of representatives of** $\mathfrak{PM}$

    $\Rightarrow$ **one can get a Milnor-type theorem.**

- **Comment:**
  - **One can get Milnor-type theorems for ANY Lie algebras.**
  - **If one has a nice** $\mathfrak{U}$

    (e.g., $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ **: cohomogeneity one, (hyper-)polar,...)**

    $\Rightarrow$ **one gets a useful Milnor-type theorem as above.**
  - **If not (e.g.,** $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})$ **is very small)**

    $\Rightarrow$ **One can get a Milnor-type theorem, but it is not useful...**

5 A pseudo-Riemannian version

5.1 A pseudo version (1/5) - the moduli space

- **Def:**
  - $\mathfrak{g}$: $(p+q)$-dim. Lie algebra
  - $\widetilde{\mathfrak{M}}_{(p,q)} := \{\langle , \rangle$: an inner product on $\mathfrak{g}$ with signature $(p,q)\}$
    $\cong \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p,q)$.
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{p+q}$: the Moduli space.
- **Thm. (Kubo-Onda-Taketomi-T., in preparation):**
  - A set of representatives of $\mathfrak{PM}_{(p,q)}$
    $\Rightarrow$ a pseudo-Riemannian version Milnor-type theorem
    (by the same procedure)

5.2 A pseudo version (2/5) - a known result

- **Thm. (Nomizu, 1979):**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^n} = \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j \geq 2)$

    $\Rightarrow$ $\forall \langle , \rangle$: Lorentz, it has a constant curvature c.

    (c can take any signature, $c > 0$, $c = 0$, $c < 0$)

- **Comment:**
  - His proof is very direct.
  - Using $\mathfrak{PM}_{(p,q)}$, we can generalize this, and simplify the proof.

5.3 A pseudo version (3/5) - a moduli space

- **Prop. (KOTT):**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}$

    $\Rightarrow \#\mathfrak{PM}_{(p,q)} = 3$, given by $\mathfrak{U} = \{I_n + \lambda E_{1,n} \mid \lambda = 0, 1, 2\}$.

- **Remark:**
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{p+q} \cong \mathrm{O}(p,q)\backslash(\mathrm{GL}_{p+q}(\mathbb{R})/\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}))$
  - If $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, then

    $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})$ is parabolic so that $\mathrm{GL}_{p+q}(\mathbb{R})/\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \cong \mathbb{R}\mathrm{P}^{p+q-1}$.
  - The above 3 orbits correspond to

    $\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1}\}$ with v : timelike, lightlike, spacelike.

5.4 A pseudo version (4/5) - a Milnor-type theorem

- **Recall:**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}$

    $\Rightarrow \; \#\mathfrak{PM}_{(p,q)} = 3$, given by $\mathfrak{U} = \{I_n + \lambda E_{1,n} \mid \lambda = 0, 1, 2\}$.

- **Prop. (Milnor-type theorem):**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^{n}}$
  - $\langle , \rangle$: an arbitrary inner product with signature (p, q)

    $\Rightarrow \; \exists k > 0, \; \exists \lambda \in \{0, 1, 2\}$,

    $\exists \{x_1, \ldots, x_n\}$: pseudo-o.n.b. wrt $k\langle , \rangle$:

    $[x_1, x_j] = x_j, \; [x_1, x_n] = -\lambda x_1 + x_n, \; [x_j, x_n] = -\lambda x_j$

    (for $j \geq 2$)

5.5 A pseudo version (5/5) - Our theorem

- **Thm. (KOTT):**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$

    $\Rightarrow$ $\forall \langle , \rangle$: pseudo-Riemannian, it has a constant curvature c.

    (c can take any signature, $c > 0$, $c = 0$, $c < 0$)

- **Comment:**
  - To get more examples, we need to know:

    isometric actions on pseudo-Riemannian symmetric spaces...

    (the orbit spaces, cohomogeneity one, (hyper-)polar actions, ...)

6 Summary and Problems

6.1 Summary

- **Story:**
  - **The moduli space of left-invariant metrics (both Riemannian and pseudo-Riemannian settings)**
    ⇒ Milnor-type theorems
    ⇒ one can examine ALL left-invariant metrics
  - **This can be applied to the existence and nonexistence problem of distinguished (e.g., Einstein, Ricci soliton) metrics**
- **Point:**
  - **Actions on symmetric spaces play important roles.**

6.2 Problems

- **Problem 1:**
  - **A continuation of this study, i.e.,**
    - **Get more Milnor-type theorems,**
    - **Study isometric actions on symmetric spaces (both Riemannian and pseudo-Riemannian cases).**
- **Problem 2:**
  - **Apply our method to other geometric structures, e.g.,**
    - **left-invariant complex structures,**
    - **left-invariant symplectic structures, ...**