The geometry of orbits of Hermann actions

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2013

$$
\begin{aligned}
M &= \mathbf{sym}(n) = \{{}^t X = X \in M_n(\mathbb{R})\} \\
&\supset A = D(n) = \{\mathbf{diagonal}\} \cong \mathbb{R}^n
\end{aligned}
$$

$$G = SO(n) \curvearrowright M; \rho(g)X = gX{}^t g$$

$$\mathbf{orbit\ space} = \{x_1 \leq \cdots \leq x_n\}$$

$G \curvearrowright M$; **<u>hyperpolar</u>**

$\Leftrightarrow$ $\exists$**conn. flat closed** $A \subset M$

s.t. $G \cdot x \perp A$; **<u>section</u> (t.g.)**

In this case

orbit space $G \backslash M \cong A / \sim$

$(G, K_1, K_2), \pi_i : G \to M_i := G/K_i$

$K_2 \curvearrowright M_1$; **<u>Hermann action</u>**

$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1 = \mathfrak{k}_2 \oplus \mathfrak{m}_2$

$\mathfrak{a} \subset \mathfrak{m}_1 \cap \mathfrak{m}_2$; **max. abel.**

$\pi_1(\exp \mathfrak{a}) \subset M_1$; **section (H.P.T.T.)**

$\theta_1\theta_2 = \theta_2\theta_1$; **<u>commutative</u>**

$$U(n) \curvearrowright G_k^{\mathbb{R}}(\mathbb{C}^n) \quad (1 \leq k := 2m \leq n)$$

$$\theta = (\theta_1, \cdots, \theta_m) \in \mathbb{R}^m$$

$$V(\theta_j) = \langle e_{2j-1}, \sqrt{-1}\cos\theta_j e_{2j-1} + \sin\theta_j e_{2j}\rangle_{\mathbb{R}}$$

$$V(\theta) = \sum_{j=1}^{m} V(\theta_j), \ \textbf{section} = \{V(\theta)\}$$

H. Tasaki

$$U(n)\backslash G_k^{\mathbb{R}}(\mathbb{C}^n) \cong \{0 \leq \theta_1 \leq \cdots \leq \theta_m \leq \frac{\pi}{2}\}$$

(G, K_1, K_2) : **compact symm. triad**

(A) **or** (B)

(A) G : **simple**, $\theta_1 \not\sim \theta_2$

(B) $\exists(U, K), U$: **simple**, $G = U \times U$,

$K_1 = \Delta G,\ K_2 = K \times K$

For $\alpha \in \mathfrak{a}$

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \{X \in \mathfrak{g}^{\mathbb{C}} \mid$$
$$[H, X] = \sqrt{-1}\langle \alpha, H \rangle X \quad (H \in \mathfrak{a})\}$$

$$\tilde{\Sigma} = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}(\mathfrak{a}, \alpha) \neq \{0\}\}$$

$$\mathfrak{g}(\mathfrak{a}, \alpha, \pm 1) = \{X \in \mathfrak{g}(\mathfrak{a}, \alpha) \mid \theta_1\theta_2 X = \pm X\}$$

$$\Sigma = \{\lambda \in \tilde{\Sigma} \mid \mathfrak{g}(\mathfrak{a}, \lambda, 1) \neq \{0\}\}$$

$$W = \{\alpha \in \tilde{\Sigma} \mid \mathfrak{g}(\mathfrak{a}, \alpha, -1) \neq \{0\}\}$$

$(\tilde{\Sigma}, \Sigma, W)$ **symmetric triad of** $\mathfrak{a}$

(1) $\tilde{\Sigma}$ **: irr. root system of** $\mathfrak{a}$

(2) Σ **is a root system of** $\mathfrak{a}$

(3) $W = -W \neq \emptyset, \quad \tilde{\Sigma} = \Sigma \cup W$

(4) $l := \max\{\|\alpha\| \mid \alpha \in \Sigma \cap W\}$**.**

$\Sigma \cap W = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| \leq l\}$

(5) For $\alpha \in W$ and $\lambda \in \Sigma - W$,

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} : \textbf{odd} \Leftrightarrow s_\alpha \lambda \in W - \Sigma.$$

(6) For $\alpha \in W$ and $\lambda \in W - \Sigma$,

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} : \textbf{odd} \Leftrightarrow s_\alpha \lambda \in \Sigma - W.$$

Theorem

(G, K_1, K_2)**:compact symm. triad**

with (A) or (B)

$\rightsquigarrow (\tilde{\Sigma}, \Sigma, W)$**: symmetric triad**

Conversely every symmetric triad is obtained in this way.

If $(B) \Rightarrow \tilde{\Sigma} = \Sigma = W$

$(\tilde{\Sigma}, \Sigma, W)$: **sym.t.with multiplicities**

$m, n : \tilde{\Sigma} \to \mathbb{Z}_{\geq 0}$

(1) $m(\lambda) = m(-\lambda),\ n(\alpha) = n(-\alpha)$

$$m(\lambda) > 0 \Leftrightarrow \lambda \in \Sigma, \quad n(\alpha) > 0 \Leftrightarrow \alpha \in W$$

(2) When $\lambda \in \Sigma, \alpha \in W, s \in W(\Sigma)$**,**

$$m(\lambda) = m(s\lambda), \quad n(\alpha) = n(s\alpha)$$

(3) When $\sigma \in W(\tilde{\Sigma}), \lambda \in \tilde{\Sigma}$ then

$$n(\lambda) + m(\lambda) = n(\sigma\lambda) + m(\sigma\lambda)$$

(4) Let $\lambda \in \Sigma \cap W$ and $\alpha \in W$.

If $\dfrac{2\langle \alpha, \lambda \rangle}{\|\alpha\|^2}$ is even then $m(\lambda) = m(s_\alpha \lambda)$,

If $\dfrac{2\langle \alpha, \lambda \rangle}{\|\alpha\|^2}$ is odd then $m(\lambda) = n(s_\alpha \lambda)$

Theorem

(G, K_1, K_2) **: compact symmetric triad satisfying (A) or (B).**
$\rightsquigarrow (\tilde{\Sigma}, \Sigma, W)$**: symmetric triad.**

For $\lambda \in \Sigma$ **and** $\alpha \in W$ **set**

$$m(\lambda) = \dim_{\mathbb{C}} \mathfrak{g}(\mathfrak{a}, \lambda, 1), \; n(\alpha) = \dim_{\mathbb{C}} \mathfrak{g}(\mathfrak{a}, \alpha, -1),$$

then $m(\lambda), n(\alpha)$ **are multiplicities of** λ **and** α **respectively.**

$$\mathfrak{a}_r = \bigcap_{\lambda\in\Sigma,\alpha\in W} \left\{ \langle \lambda, H\rangle \notin \pi\mathbb{Z}, \langle \alpha, H\rangle \notin \frac{\pi}{2} + \mathbb{Z} \right\}.$$

$$\tilde{W}(\tilde{\Sigma}, \Sigma, W)\overline{P_0} = \mathfrak{a},$$

<u>**Theorem**</u> $\exists! \tilde{\alpha} \in W^+$ **s.t.**

$$P_0 = \left\{ H \in \mathfrak{a} \mid \langle \tilde{\alpha}, H\rangle < \frac{\pi}{2},\ 0 < \langle \lambda, H\rangle\ (\lambda \in \Pi) \right\}$$

$H \in \mathfrak{a}$:**<u>totally geodesic point</u>**

$\Leftrightarrow \langle \lambda, H \rangle \in \frac{\pi}{2}\mathbb{Z}$ **for any** $\lambda \in \tilde{\Sigma}$**.**

$H \in \mathfrak{a}$:**<u>austere point</u>** $\Leftrightarrow$

$$\{-\lambda \cot\langle \lambda, H \rangle \ \mid \lambda \in \Sigma^{+}, \langle \lambda, H \rangle \notin \frac{\pi}{2}\mathbb{Z}\} \cup$$

$$\{\alpha \tan\langle \alpha, H \rangle \ \mid \alpha \in W^{+}, \langle \alpha, H \rangle \notin \frac{\pi}{2}\mathbb{Z}\}$$

is -1**-invariant with multiplicities.**

mean curvature of $H \in \mathfrak{a}$

$$m_H = - \sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin \frac{\pi}{2}\mathbb{Z}}} m(\lambda) \cot(\langle \lambda, H \rangle) \lambda$$

$$+ \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \notin \frac{\pi}{2}\mathbb{Z}}} n(\alpha) \tan(\langle \alpha, H \rangle) \alpha.$$

H:**minimal point** $\Leftrightarrow m_H = 0$

$H \in \mathfrak{a}$: <u>c**-metric point**</u> $(0 < c \leq 1)$

$\Leftrightarrow |\sin(\langle \lambda, H \rangle)| = |\cos(\langle \alpha, H \rangle)| = c$

$\lambda \in \Sigma$ **with** $\langle \lambda, H \rangle \notin \pi\mathbb{Z}$ **,**

$\alpha \in W$ **with** $\langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z}$**.**

1**-metric pt.**$\Leftrightarrow$ **totally geodesic pt.**

1-metric point		**c-metric point**
		$(0 < c < 1)$
$\Updownarrow$		$\Downarrow$
t.g. point	$\Rightarrow$	**austere point**
$\Downarrow$		$\Downarrow$
vertex of cell	$\Rightarrow$	**minimal point**

Harvey-Lawson (1982)

$N \subset M$, A:**shape operator**

N: **austere**$\Leftrightarrow$ **the set of eigenvalues of** $A_\xi(\xi \in N(M))$ **is** -1**-invariant.**

○ **totally geodesic**$\Rightarrow$ **austere** $\Rightarrow$ **minimal**

$K_2\pi_1(\exp H)$ **is regular** $\Leftrightarrow$ H **is regular**

(singular $\Leftrightarrow$ **singular)**

$K_2\pi_1(\exp H)$ **:t.g.** $\Leftrightarrow$ H **:t.g.**

$\Leftrightarrow$ H**:**1**-metric point**

$K_2\pi_1(\exp H)$ **:austere** $\Leftrightarrow$ H **:austere**

$K_2\pi_1(\exp H)$ **:minimal** $\Leftrightarrow$ H **:minimal**

$$K_2\pi_1(\exp H)(\subset M_1) \quad \cong \quad K_2/N^H$$

$$g_i \qquad\qquad\qquad\qquad g_n$$

$$\mathfrak{k}_2 = \mathfrak{n}^H \oplus (\mathfrak{n}^H)^\perp \rightsquigarrow g_n \ \textbf{(n. h. m.)}$$

$$K_2/\pi_1(\exp H)(\subset M_1) \ \textbf{:t.g.}\Rightarrow g_n//g_i$$

Proposition

Assume $K_2\pi_1(\exp H)$**: not t.g.**

$g_n//g_i \Leftrightarrow$

H : c**-metric pt.** $(0 < \exists c < 1)$ **and**

$\{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [X, \mathfrak{a}] = 0\} = \{0\}$

<u>Theorem</u>

(G, K_1, K_2) **:(A) or (B)**

Assume $K_2\pi_1(\exp H)$**: not t.g.**

$g_n // g_i \Leftrightarrow$

(1)

$SO(1+s) \curvearrowright SU(1+s)/S(U(1)\times U(s))$**.**

$(s \geq 2)$

$H \neq 0$**:end point of segment** $\overline{P_0}$**.**

(2) $SO(4) \curvearrowright SU(4)/Sp(2)$ **and**
$SO(2)^2 \curvearrowright SU(2) \times SU(2)/\Delta SU(2)$**.**

H**: mid point of the segment** $\overline{P_0}$**.**

$K_2\backslash G/K_1 \ni [g]$	$\leftrightarrow$	$[g^{-1}] \in K_1\backslash G/K_2$
codimension	$=$	**codimension**
regular (singular)	$\leftrightarrow$	**regular (singular)**
totally geodesic	$\leftrightarrow$	**totally geodesic**
austere	$\leftrightarrow$	**austere**
minimal	$\leftrightarrow$	**minimal**
$g_i//g_n$	$\not\leftrightarrow$	$g_i//g_n$