Special Lagrangian submanifolds
in the complex sphere and the complex cone

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Plan of this talk

Purpose
Study the geometry of special Lagrangian submanifolds, especially their singularities, in a non-flat Calabi-Yau manifold.

1. Calabi-Yau manifolds and special Lagrangian geometry
2. Calabi-Yau metrics on the complex sphere and the complex cone
3. Special Lagrangian conormal bundles
4. Cohomogeneity one special Lagrangian submanifolds
Motivation and history

1982  Harvey-Lawson, Calibrated geometries
1996  Strominger-Yau-Zaslov, a conjecture on the mirror symmetry
2001- Joyce, Construction of examples of special Lagrangian submanifolds in $\mathbb{C}^n$
2003- Joyce, Conical singularities on a special Lagrangian submanifolds
2004- Haskins, Special Lagrangian cones in $\mathbb{C}^n$
2005- Ionel-MinOo, Karigiannis-MinOo, Anciaux
      Special Lagrangian submanifolds in $T^*S^n$
Calabi-Yau manifolds

**Definition**

\[ (M, J, \omega, \Omega) : \text{Calabi-Yau manifold} \]
\[ \overset{\text{def}}{\iff} (M, J, \omega) : \text{Kähler manifold of complex dimension } n \]
\[ \Omega : \text{non-vanishing holomorphic } (n, 0)\text{-form on } M \]
\[ \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \bar{\Omega} \]

Calabi-Yau manifold \iff Ricci-flat Kähler
\[ \iff \text{Hol}_0(M, g) \subset SU(n) \]
Special Lagrangian submanifolds

\((M, J, \omega, \Omega)\) : Calabi-Yau manifold

**Definition**

\(L \subset M\) : **special Lagrangian submanifold** of phase \(\theta\)

\(\overset{\text{def}}{\iff} L \text{ is calibrated by } \Re(e^{i\theta \Omega}) \text{ for some } \theta \in \mathbb{R}.\)

\(\text{i.e. } \Re(e^{i\theta \Omega})|_L = \text{vol}|_L\)

\(\iff\) \begin{align*}
\dim_{\mathbb{R}} L &= n \\
\omega|_L &\equiv 0 \quad \text{(Lagrangian)} \\
\Im(e^{i\theta \Omega})|_L &\equiv 0
\end{align*}

**Theorem (Harvey-Lawson)**

A special Lagrangian submanifold is a calibrated submanifold, so is volume minimizing in its homology class.
Stenzel metric on $T^*S^n$

$S^n = \text{SO}(n + 1)/\text{SO}(n) =: G/K$

$T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, \xi \rangle = 0\}$

$Q^n = \left\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \left| \sum_{i=1}^{n+1} z_i^2 = 1 \right. \right\} \cong G^{\mathbb{C}}/K^{\mathbb{C}}$

$\Phi : T^*S^n \longrightarrow Q^n \subset \mathbb{C}^{n+1} \text{ diffeo}$

$(x, \xi) \longmapsto x \cosh(\|\xi\|) + \sqrt{-1} \frac{\xi}{\|\xi\|} \sinh(\|\xi\|)$

- $G = \text{SO}(n + 1)$ acts on $T^*S^n$ and $Q^n$ with cohomogeneity one.
- $\varphi$ is $G$-equivariant.
Theorem (Stenzel)

\[ \omega_{Stz} = \sqrt{-1} \partial \bar{\partial} u(r^2) = \sqrt{-1} \sum_{i,j=1}^{n+1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(r^2) dz_i \wedge d\bar{z}_j \]

gives a complete Ricci-flat Kähler metric on \( Q^n \cong T^* S^n \), where \( r^2 = \|z\|^2 \) and \( u : \mathbb{R} \to \mathbb{R} \) is defined by \( U(t) = u(\cosh t) \) where \( U \) satisfies

\[ \frac{d}{dt}(U'(t))^n = cn(\sinh t)^{n-1} \quad (c > 0) \]

- When \( n = 2 \), the Stenzel metric coincides with the Eguchi-Hanson metric on \( T^* S^2 \).
Stenzel metric on $T^* S^n$

$\Omega_{Stz} : \text{holomorphic } (n, 0)\text{-form on } Q^n$

$$\Omega_{Stz} \wedge d(z_1^2 + z_2^2 + \cdots + z_{n+1}^2) = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1}$$

Then, $\exists \lambda \in \mathbb{C}$ s.t.

$$\omega^n_{Stz} = \lambda \Omega_{Stz} \wedge \overline{\Omega}_{Stz}$$

Moreover, $\omega_{Stz}$ and $\Omega_{Stz}$ are $SO(n + 1)$-invariant.

Hence, $(T^* S^n, J, \omega_{Stz}, \Omega_{Stz})$ is a cohomogeneity one Calabi-Yau manifold.
Calabi-Yau metric on the complex cone

\[ T^\circ S^n = T^* S^n \setminus S^n \] excluding the zero-section

\[ Q^n_0 = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum z_i^2 = 0\} \]

\[ \Psi : T^\circ S^n \longrightarrow Q^n_0 \setminus \{0\} \text{ diffeo} \]

\[ (x, \xi) \longmapsto \|\xi\| x + \sqrt{-1} \xi \]

\[ Q^n \xrightarrow{\|z\| \to \infty} Q^n_0 \]

\[ \frac{d}{dt} (U'(t))^n = c (\sinh t)^{n-1} \]

\[ \frac{d}{dt} (F'(t))^n = c e^{t(n-1)} \]

\[ F(t) = c e^{\frac{n-1}{n} t} \]

is a solution, and we define \( f \) by \( F(\tau) = f(\frac{1}{2} e^\tau) \).

**Proposition (Hashimoto-S.)**

Let \( f(r^2) = c r^{\frac{2(n-1)}{n}} \) \((c > 0)\), and define \( \omega_{cone} = \sqrt{-1} \partial \bar{\partial} f(r^2) \).

Then \( \omega_{cone} \) gives a Ricci-flat Kähler metric on \( Q^n_0 \).
Austere submanifold

**Definition (Harvey-Lawson)**

\[ X \subset M : \text{austere submanifold} \]

\[ \iff \] for all \( \xi \in N_x X \), the set of eigenvalues with their multiplicities of the shape operator \( A_\xi \) of \( X \) is invariant under the multiplication by \(-1\).

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

**Theorem (Harvey-Lawson)**

\[ X \subset \mathbb{R}^n : \text{austere} \iff N^* X \subset T^* \mathbb{R}^n \cong \mathbb{C}^n : \text{special Lagrangian} \]
Special Lagrangian conormal bundles

\[ X \subset S^n : \text{submanifold} \]

\[ \Phi : N^1 X \times S^1 \longrightarrow S^{2n+1} \subset \mathbb{R}^{2n+2} \]

\[ (x, \xi, e^{i\theta}) \mapsto (\cos \theta x, \sin \theta \xi) \]

**Theorem (Harvey-Lawson)**

\[ X \subset S^n : \text{austere} \iff \Phi : \text{minimal Legendrian} \]

Borrelli-Gorodski defined a map \( \tilde{\Phi} \) modifying \( \Phi \) and showed that

\[ A_\xi \text{ does not have 0-eigenvalue} \implies \tilde{\Phi} : \text{Legendrian immersion} \]
$X \subset S^n$ : submanifold

Then $L = N^* X$ is a Lagrangian submanifold of $T^* S^n$ with respect to the canonical symplectic structure $\omega_0$.

**Theorem (Karigiannis-Min-Oo)**

$L = N^* X$ is a Lagrangian submanifold of $T^* S^n$ with respect to $\omega_{Stz}$. Moreover, $L$ is a special Lagrangian submanifold of $T^* S^n$ if and only if $X$ is an austere submanifold in $S^n$. 
Weakly reflective submanifold

Definition (Ikawa-Tasaki-S.)

\( X \subset M \) : weakly reflective submanifold (WRS)
\[\xrightarrow{\text{def}} \quad \text{for each } x \in X \text{ and each } \xi \in N_x X, \]
there exists an isometry \( \sigma_\xi \) of \( M \) which satisfies

\[
\sigma_\xi(x) = x, \quad (d\sigma_\xi)_x \xi = -\xi, \quad \sigma_\xi(X) = X.
\]

We call \( \sigma_\xi \) a reflection of \( X \) with respect to \( \xi \).
Weakly reflective submanifolds

Example

\[ S^n(1) \times S^n(1) \subset S^{2n+1}(\sqrt{2}) \] is a weakly reflective submanifold.

Proposition

\[ \text{reflective} \subset \text{WRS} \subset \text{austere} \subset \text{minimal} \]

Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeniety one action on a Riemannian manifold is a weakly reflective submanifold.
An orbit $\text{Ad}(K)H$ of an irreducible $s$-representation which is an austere submanifold in the hypersphere $S$ is one of the following list:

1. An orbit through a restricted root

2. $R = A_2; \quad H = 2e_1 - e_2 - e_3, \quad e_1 + e_2 - 2e_3$

3. $R = A_3; \quad H = e_1 + e_2 - e_3 - e_4$

4. $R = D_n; \quad H = e_1$

5. $R = D_4; \quad H = e_1 + e_2 + e_3 \pm e_4$

6. $R = B_2$ with constant multiplicities; \quad $H = e_1 + \frac{e_1 + e_2}{\sqrt{2}}$

7. $R = G_2; \quad H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$

Moreover, in the cases (1)~(5), these austere orbits are weakly reflective submanifolds in $S$. 

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Special Lagrangian submanifolds in the complex sphere
Case of type $B_2$
Moment map and Lagrangian submanifolds

\((M, \omega) : \text{symplectic manifold}\)

\(G \acts M : \text{Hamiltonian action}\)

\[\mu : M \rightarrow g^* \quad \text{moment map}\]

1. \(\omega(\tilde{X}_x, Y) = \langle X, d\mu(Y) \rangle \quad (\forall X \in g, \forall Y \in T_x M)\)

2. \(\mu(g \cdot x) = \text{Ad}^*(g)\mu(x) \quad (\forall g \in G, \forall x \in M)\)

Proposition

\(L \subset (M, \omega) : G\text{-invariant connected submanifold}\)

Suppose that \(G\) acts on \(L\) with cohomogeneity one. Then

\(L\) is isotropic \(\iff\) \(\exists c \in Z(g^*) \text{ s.t. } L \subset \mu^{-1}(c)\)

\((i.e. \ \omega|_L \equiv 0)\)
Theorem (Harvey-Lawson, Joyce)

\( M \) : Calabi-Yau manifold of complex dimension \( n \)

\( P \subset M \) : isotropic \( C^\infty \)-submanifold of dimension \((n - 1)\)

\( \implies \) locally \( \exists ! \) \( L \) : special Lagrangian submanifold

s.t. \( P \subset L \subset M \)

\( G \actson M, \quad \mu : M \rightarrow \mathfrak{g}^* \)

\( c \in Z(\mathfrak{g}^*) \quad \text{s.t.} \quad G \actson \mu^{-1}(c) \) has \((n - 1)\)-dim. principal orbit

\( \pi : \mu^{-1}(c) \rightarrow \mu^{-1}(c)/G \)

Then, for a curve \( \sigma(s) \subset \mu^{-1}(c)/G \)

\( L := \pi^{-1}(\sigma) \subset M \) is a \( G \)-invariant Lagrangian submanifold.

Moreover, the condition for \( L \) to be special Lagrangian can be described as a first order ODE on the orbit space \( \mu^{-1}(c)/G \).
Moment map of $SO(p) \times SO(q)$-action on $T^* S^n$

\[ G = \left( \frac{SO(p)}{SO(q)} \right) \subset SO(n + 1) \]

$G \ltimes Q^n \subset \mathbb{C}^{n+1}$ : Hamiltonian action

We give a basis of $\mathfrak{g}$ by

\[ \{ X_{ij} \mid 1 \leq i < j \leq p \} \cup \{ X_{ij} \mid p + 1 \leq i < j \leq n + 1 \}, \]

where $X_{ij} = E_{ji} - E_{ij} \in \mathfrak{so}(n + 1)$.

By the dual basis of the above, the moment map $\mu : Q^n \to \mathfrak{g}^*$ of $G \ltimes Q^n$ can be represented as

\[ \mu(z) = u'(r^2) \left( \text{Im}(z_i \bar{z}_j)_{1 \leq i < j \leq p}, \text{Im}(z_i \bar{z}_j)_{p+1 \leq i < j \leq n+1} \right) \]
Here we demonstrate in a generic case $3 \leq p \leq q$.
In this case $Z(g^*) = \{0\}$. Define

$$\Sigma = \left\{ \left( \frac{1}{\cos \tau}, 0, \ldots, 0, \frac{1}{\sin \tau}, 0, \ldots, 0 \right) \in \mathbb{C}^{n+1} \middle| \begin{array}{c} \tau = t + \sqrt{-1} \xi_1, \\
0 \leq t \leq \pi/2, \xi_1 \in \mathbb{R} \end{array} \right\}$$

Then

$$G \cdot \Sigma = \mu^{-1}(0) \subset Q^n.$$ 

Therefore $\Sigma$ can be identified with the orbit space $\mu^{-1}(0)/G$ of $G \sim \mu^{-1}(0)$. 
Let $\tau(s) \subset (t, \xi_1)$ be a regular curve, and define a curve $\sigma \subset \Sigma$

$$\sigma(s) = (\cos \tau(s), 0, \ldots, 0, \sin \tau(s), 0, \ldots, 0)$$

**Theorem (Hashimoto-S.)**

$L = G \cdot \sigma \subset Q^n$ is a Lagrangian submanifold with respect to $\omega_{Stz}$. Moreover, $L = G \cdot \sigma \subset Q^n$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau(s)$ satisfies

$$\text{Im} \left( e^{\sqrt{-1}\theta} \tau'(\cos \tau)^{p-1}(\sin \tau)^{q-1} \right) = 0. \quad (\ast)$$
Case of $n = 6, \ p = 3, \ q = 4, \ \theta = 0$
Case of $n = 6$, $p = 3$, $q = 4$, $\theta = \pi/4$
Case of $n = 6$, $p = 3$, $q = 4$, $\theta = \pi/2$