DIFFERENTIAL GEOMETRY OF REAL HYPERSURFACES IN HERMITIAN SYMMETRIC SPACES WITH RANK 2

JÜRGEN BERNDT AND YOUNG JIN SUH

Abstract. In this talk, first we introduce the classification of homogeneous hypersurfaces in some Hermitian symmetric spaces of rank 1 or rank 2. In particular, we give a full expression of the geometric structures for hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ or in complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2})$.

Next by using the isometric Reeb flow we give a complete classification for hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2})$, complex quadric Q^n and its noncompact dual Q^{n*}. Moreover, we give a classification of contact hypersurfaces with constant mean curvature in the complex quadric $Q^n = SO_{n+2}/SO_nSO_2$ and its noncompact dual $Q^{n*} = SO^o_{n,2}/SO_nSO_2$ for $n \geq 3$

1. Compact Hermitian Symmetric Space with rank 2

The study of real hypersurfaces in non-flat complex space forms or quaternionic space forms which belong to HSSP with rank 1 of compact type in section 1 is a classical topic in differential geometry. For instance, there have been many investigations for homogeneous hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$. They are completely classified by Cecil and Ryan [9], Kimura [13] and Takagi [30]. Here, explicitly, we mention that A_1: Geodesic hyperspheres, A_2: a tube around a totally geodesic complex projective spaces $\mathbb{C}P^k$, B: a tube around a complex quadric Q^{m-1} and can be viewed as a tube around a real projective space $\mathbb{R}P^m$, C: a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k \geq 2$, D: a tube around the Plücker embedding into $\mathbb{C}P^9$ of the complex Grassmannian manifold $G_2(\mathbb{C}^5)$ of complex 2-planes in $\mathbb{C}^5$ and E: a tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space $SO(10)/U(5)$.

Now let us study hypersurfaces in complex two-plane Grassmanians $G_2(\mathbb{C}^{m+2})$ which is a kind of HSSP with rank two of compact type. The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J.

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[1] *2000 Mathematics Subject Classification.* Primary 53C40; Secondary 53C15.

[2] *Key words and phrases* : Hermitian symmetric spaces, Real hypersurfaces, Noncompact complex two-plane Grassmannians, Isometric Reeb flow, Totally geodesic, Horosphere at infinity, Contact hypersurfaces.

[3] This work was supported by grant Proj. No. NRF-2012-R1A2A2A-01043023 from National Research Foundation of Korea.

On the other hand, Cecil and Ryan [9] proved that any tube M around a complex submanifold in complex projective space $\mathbb{C}P^m$ are characterized by the invariancy of $A\xi = \alpha\xi$, where the Reeb vector ξ is defined by $\xi = -JN$ for a Kähler structure J and a unit normal N to hypersurfaces M in $P_m(\mathbb{C})$. Moreover, the corresponding geometrical feature for hypersurfaces in $\mathbb{Q}P^m$ is the invariance of the distribution $\mathfrak{D}^\perp = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ by the shape operator, where $\xi_i = -J_iN$, $J_i \in \mathfrak{J}$. In fact every tube around a quaternionic submanifold $\mathbb{Q}P^m$ satisfies such kind of geometrical feature (See [14], [16], [17]).

From such a view point, we considered two natural geometric conditions for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that means that the maximal complex subbundle $\mathcal{C}$ and a maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M, where the maximal complex subbundle $\mathcal{C}$ of the tangent bundle TM of M is defined by $\mathcal{C} = \{X \in TM | JX \in TM\}$, and the maximal quaternionic subbundle $\mathcal{Q}$ of TM is defined by $\mathcal{Q} = \{X \in TM | \mathfrak{J}X \in TM\}$ respectively. By using such conditions and the result in Alekseevskii [1], Berndt and Suh [3] proved the following

Theorem 1.1. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the maximal complex subbundle $\mathcal{C}$ and a maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if*
(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, we say that the Reeb vector field ξ on M is Killing. Moreover, the Reeb vector field ξ is said to be *Hopf* if it is invariant by the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the *Hopf foliation* of M is totally geodesic.

By using Theorem 1, in a paper due to Berndt and Suh [4] we have given a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow as follows:

Theorem 1.2. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

2. Complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$

Now let us consider for the case that the Riemannian manifold $\bar{M}$ becomes a Riemannian symmetric spaces of non compact type with rank 1 or rank 2. As some examples of non compact type with rank 1 we say a real hyperbolic space $\mathbb{R}H^n = SO_0(1,n)/SO(n)$, a complex hyperbolic space $\mathbb{C}H^n = SU(1,n)/S(U(1) \times U(n))$, a quaternionic hyperbolic space $\mathbb{H}H^n = Sp(1,n)/Sp(1) \times Sp(n)$, and a Caley projective plane $\mathbb{O}P^2 = F_4/Spin(9)$. The study of homogeneous hypersurfaces in such a symmetric spaces of noncompact type with rank 1 was investigated in Berndt [5], Berndt and Tamaru [8].

In this section we consider a hypersurface in HSSP of noncompact type with rank 2. Among some examples of noncompact type with rank 2 given in section 1 we focus on a dual complex two-plane Grassmannian $SU(2,m)/S(U(2){\times}U(m))$. The Riemannian symmetric space $SU(2,m)/S(U(2){\times}U(m))$ is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type with rank 2.

Let $G = SU_{2,m}$ and $K = S(U_2{\cdot}U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 and I_m denote the identity (2×2)-matrix and $(m \times m)$-matrix respectively. Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2{\cdot}U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis J_1, J_2, J_3 of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2{\cdot}U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $\mathrm{t}r(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

Then by the argument asserted in section 1, we note that any homogeneous hypersurfaces in $G_2^*(\mathbb{C}^{m+2})$ becomes a tube around one singular orbit. By virtue of this fact and using geometric tools given in Helgason [11], [12], Eberlein [10] , Berndt and Suh [5] proved a characterization of homogeneous hypersurfaces in $G_2^*(\mathbb{C}^{m+2})$ as follows:

Theorem 2.1. *Let M be a connected real hypersurface in the complex hyperbolic two-plane Grassmanniaan $SU(2,m)/S(U(2){\times}U(m))$, $m \geq 2$, with constant principal curvatures. Then the maximal complex subbundle $\mathcal{C}$ and a maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

(A) a tube around a totally geodesic $SU(2,m-1)/S(U(2)\times U(m-1))$ in $SU(2,m)/S(U(2)\times U(m))$,
(B) a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $SU(2,2)/S(U(2)\times U(m))$, $m=2n$,
(C) a horosphere in $SU(2,m)/S(U(2)\times U(m))$ whose center at infinity is singular.

In this section we give a classification of all real hypersurfaces with isometric *Reeb flow* in complex hyperbolic two-plane Grassmann manifold $SU_{2,m}/S(U_2{\cdot}U_m)$ as follows (see Suh [26]):

Theorem 2.2. *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m\geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular.*

A tube around $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ is a principal orbit of the isometric action of the maximal compact subgroup $SU_{1,m+1}$ of SU_{m+2}, and the orbits of the *Reeb flow* corresponding to the orbits of the action of U_1. The action of $SU_{1,m+1}$ has two kinds of singular orbits. One is a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ and the other is a totally geodesic $\mathbb{C}H^m$ in $SU_{2,m}/S(U_2{\cdot}U_m)$.

A remarkable consequence of our Main Theorem is that a connected complete real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m\geq 3$ with isometric *Reeb flow* is homogeneous. This was also true in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which could be identified with symmetric space of compact type $SU_{m+2}/S(U_2{\cdot}U_m)$, as follows from the classification. It would be interesting to understand the actual reason for it (See [3], [4], [17], and [22]).

3. Isometric Reeb Flow in Complex Quadric Q^m

The homogeneous quadratic equation $z_1^2+\ldots+z_{m+2}^2=0$ on $\mathbb{C}^{m+2}$ defines a complex hypersurface Q^m in the $(m+1)$-dimensional complex projective space $\mathbb{C}P^{m+1}=SU_{m+2}/S(U_{m+1}U_1)$. The hypersurface Q^m is known as the m-dimensional complex quadric. The complex structure J on $\mathbb{C}P^{m+1}$ naturally induces a complex structure on Q^m which we will denote by J as well. We equip Q^m with the Riemannian metric g which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The 1-dimensional quadric Q^1 is isometric to the round 2-sphere S^2. For $m\geq 2$ the triple (Q^m,J,g) is a Hermitian symmetric space of rank two and its maximal sectional curvature is equal to 4. The 2-dimensional quadric Q^2 is isometric to the Riemannian product $S^2\times S^2$. We will assume $m\geq 3$ for the main part of this paper.

For a nonzero vector $z\in\mathbb{C}^{m+1}$ we denote by $[z]$ the complex span of z, that is, $[z]=\{\lambda z \mid \lambda\in\mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. As usual, for each $[z]\in\mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2}\ominus[z]$ of $[z]$ in $\mathbb{C}^{m+2}$. For $[z]\in Q^m$ the tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2}\ominus([z]\oplus[\bar{z}])$ of $[z]\oplus[\bar{z}]$

in $\mathbb{C}^{m+2}$. Note that $\bar{z} \in \nu_{[z]}Q^m$ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

We denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to $\bar{z}$. Then we have $A_{\bar{z}}w = \overline{w}$ for all $w \in T_{[z]}Q^m$, that is, $A_{\bar{z}}$ is just complex conjugation restricted to $T_{[z]}Q^m$. The shape operator $A_{\bar{z}}$ is an antilinear involution on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$. Recall that a real structure on a complex vector space V is by definition an antilinear involution $A : V \to V$. Since the normal space $\nu_{[z]}Q^m$ of Q^m in $\mathbb{C}P^{m+1}$ at $[z]$ is a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ of complex dimension one, every normal vector in $\nu_{[z]}Q^m$ can be written as $\lambda\bar{z}$ with some $\lambda \in \mathbb{C}$. The shape operators $A_{\lambda\bar{z}}$ of Q^m define a rank two vector subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$. Since the second fundamental form of the embedding $Q^m \subset \mathbb{C}P^{m+1}$ is parallel (see e.g. [29]), $\mathfrak{A}$ is a parallel subbundle of $\mathrm{End}(TQ^m)$. For $\lambda \in S^1 \subset \mathbb{C}$ we again get a real structure $A_{\lambda\bar{z}}$ on $T_{[z]}Q^m$ and we have $V(A_{\lambda\bar{z}}) = \lambda V(A_{\bar{z}})$. We thus have an S^1-subbundle of $\mathfrak{A}$ consisting of real structures on the tangent spaces of Q^m.

The Gauss equation for the complex hypersurface $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor R of Q^m can be expressed in terms of the Riemannian metric g, the complex structure J and a generic real structure A in $\mathfrak{A}$:

$$\begin{aligned} R(X,Y)Z \;=\;& g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &+ g(AY,Z)AX - g(AX,Z)AY \\ &+ g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned}$$

Note that the complex structure J anti-commutes with each endomorphism $A \in \mathfrak{A}$, that is, $AJ = -JA$.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

Basic complex linear algebra shows that for every unit tangent vector $W \in T_{[z]}Q^m$ there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$.

Let M be a real hypersurface in a Kähler manifold $\bar{M}$. The complex structure J on $\bar{M}$ induces locally an almost contact metric structure (ϕ, ξ, η, g) on M. In the context of contact geometry, the unit vector field ξ is often referred to as the Reeb vector field on M and its flow is known as the Reeb flow. The Reeb flow has been of significant interest in recent years, for example in relation to the Weinstein Conjecture. We are interested in the Reeb flow in the context of Riemannian geometry, namely in the classification of real hypersurfaces with isometric Reeb flow in homogeneous Kähler manifolds.

For the complex projective space $\mathbb{C}P^m$ a full classification was obtained by Okumura in [19]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0,\dots,m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ the classification was obtained by the author in [26]. We have proved that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Finally, related to the isometric Reeb flow, we give a mention for our recent work due to Berndt and Suh [6]. In this lecture we want to investigate this problem for the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. In view of the previous two results a natural expectation is that the classification involves at least the totally geodesic $Q^{m-1} \subset Q^m$. Surprisingly, this is not the case. Our main result states:

Theorem 3.1. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

Every tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$ is a homogeneous hypersurface. In fact, the closed subgroup U_{k+1} of SO_{2k+2} acts on Q^{2k} with cohomogeneity one. The two singular orbits are totally geodesic $\mathbb{C}P^k \subset Q^{2k}$ and the principal orbits are the tubes around any of these two singular orbits. So as a corollary we get:

Corollary 3.1. *Let M be a connected complete real hypersurface in the complex quadric Q^{2k}, $k \geq 2$. If the Reeb flow on M is isometric, then M is a homogeneous hypersurface of Q^{2k}.*

It is remarkable that in this situation the existence of a particular one-parameter group of isometries implies transitivity of the isometry group. As another interesting consequence we get:

Corollary 3.2. *There are no real hypersurfaces with isometric Reeb flow in the odd-dimensional complex quadric Q^{2k+1}, $k \geq 1$.*

To our knowledge the odd-dimensional complex quadrics are the first examples of homogeneous Kähler manifolds which do not admit a real hypersurface with isometric Reeb flow. Recently, as a further characterization of the tube around a totally geodesic $\mathbb{C}P^k$ we want to remark the following two theorems in [27] and [28] respectively.

When the shape operator S of M in $\mathbb{Q}^m$ is parallel, that is, $\nabla_\xi S = 0$ along the direction the structure vector field ξ, we say that the shape operator is *Reeb parallel*. Moreover, we say that the Reeb principal curvature is constant if the function α defined by $\alpha = g(S\xi,\xi)$ is constant. Motivated by theses results, we have given a complete classification of real hypersurfaces in complex quadric Q^m with Reeb parallel shape operator in [27] as follows:

Theorem 3.2. *Let M be a real hypersurface in complex quadric Q^m, $m{\geq}3$ with Reeb parallel shape operator and non-vanishing constant Reeb curvature. Then*

$m = 2k$, and M is locally congruent to a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ in Q^{2k}.

The shape operator S is said to be *Reeb invariant* if the shape operator is Lie parallel along the Reeb direction, that is, $\mathcal{L}_\xi S = 0$. From such a view point, we want to give another characterization of the tube around a totally geodesic $\mathbb{C}P^k$ in [28] as follows:

Theorem 3.3. *Let M be a real hypersurface in complex quadric Q^m, $m \geq 3$ with Reeb parallel shape operator and non-vanishing constant Reeb curvature. Then $m = 2k$, and M is locally congruent to a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ in Q^{2k}.*

4. Contact hypersurfaces in Complex Quadric Q^n

This section is a recent work due to Berndt and Suh [7]. A contact manifold is a smooth $(2n-1)$-dimensional manifold M together with a one-form η satisfying $\eta \wedge (d\eta)^{n-1} \neq 0$, $n \geq 2$. The one-form η on a contact manifold is called a contact form. The kernel of η defines the so-called contact distribution $\mathcal{C}$ in the tangent bundle TM of M. Note that if η is a contact form on a smooth manifold M, then $\rho\eta$ is also a contact form on M for each smooth function ρ on M which is nonzero everywhere. The origin of contact geometry can be traced back to Hamiltonian mechanics and geometric optics. The standard example of a contact manifold is $\mathbb{R}^3$ together with the contact form $\eta = dz - y\,dx$.

Another standard example is a round sphere in an even-dimensional Euclidean space. Consider the sphere $S^{2n-1}(r)$ with radius $r \in \mathbb{R}_+$ in $\mathbb{C}^n$ and denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{C}^n$ given by $\langle z, w \rangle = \mathrm{Re} \sum_{\nu=1}^n z_\nu \bar{w}_\nu$. By defining $\xi_z = -\frac{1}{r} iz$ for $z \in S^{2n-1}(r)$ we obtain a unit tangent vector field ξ on $S^{2n-1}(r)$. We denote by η the dual one-form given by $\eta(X) = \langle X, \xi \rangle$ and by ω the Kähler form on $\mathbb{C}^n$ given by $\omega(X,Y) = \langle iX, Y \rangle$. A straightforward calculation shows that $d\eta(X,Y) = -\frac{2}{r}\omega(X,Y)$. Since the Kähler form ω has rank $2(n-1)$ on the kernel of η it follows that $\eta \wedge (d\eta)^{n-1} \neq 0$. Thus $S^{2n-1}(r)$ is a contact manifold with contact form η. This argument for the sphere motivates a natural generalization to Kähler manifolds.

Let $(\bar{M}, J, g)$ be a Kähler manifold of complex dimension n and let M be a connected oriented real hypersurface of $\bar{M}$. The Kähler structure on $\bar{M}$ induces an almost contact metric structure (ϕ, ξ, η, g) on M. The Riemannian metric on M is the one induced from the Riemannian metric on $\bar{M}$, both denoted by g. The orientation on M determines a unit normal vector field N of M. The so-called Reeb vector field ξ on M is defined by $\xi = -JN$ and η is the dual one form on M, that is, $\eta(X) = g(X, \xi)$. The tensor field ϕ on M is defined by $\phi X = JX - g(JX, N)N = JX - \eta(X)N$, so that ϕX is just the tangential component of JX. The tensor field ϕ determines the fundamental 2-form ω on M by $\omega(X,Y) = g(\phi X, Y)$. M is said to be a contact hypersurface if there exists an everywhere nonzero smooth function ρ on M such that $d\eta = 2\rho\omega$. It is clear that if $d\eta = 2\rho\omega$ holds then $\eta \wedge (d\eta)^{n-1} \neq 0$, that is, every contact hypersurface in a Kähler manifold is a contact manifold.

Contact hypersurfaces in complex space forms of complex dimension $n \geq 3$ have been investigated and classified by Okumura [20] (for the complex Euclidean space

$\mathbb{C}^n$ and the complex projective space $\mathbb{C}P^n$) and Vernon [21] (for the complex hyperbolic space $\mathbb{C}H^n$). In this paper we carry out a systematic study of contact hypersurfaces in Kähler manifolds. We will then apply our results to the complex quadric $Q^n = SO_{n+2}/SO_nSO_2$ and its noncompact dual space $Q^{n*} = SO^o_{n,2}/SO_nSO_2$ to prove the following two classifications:

Theorem 4.1. *Let M be a connected orientable real hypersurface with constant mean curvature in the complex quadric $Q^n = SO^o_{n+2}/SO_nSO_2$ and $n \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around a real form S^n of Q^n.*

Theorem 4.2. *Let M be a connected orientable real hypersurface with constant mean curvature in the noncompact dual $Q^{n*} = SO^o_{n,2}/SO_nSO_2$ of the complex quadric and $n \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following contact hypersurfaces in Q^{n*}:*

(i) *the tube of radius $r \in \mathbb{R}_+$ around the totally geodesic $Q^{(n-1)*}$ in Q^{n*};*
(ii) *a horosphere in Q^{n*} whose center at infinity is determined by an $\mathcal{A}$-principal geodesic in Q^{n*};*
(iii) *the tube of radius $r \in \mathbb{R}_+$ around a real form $\mathbb{R}H^n$ in Q^{n*}.*

The symbol $\mathcal{A}$ refers to a circle bundle of real structures on Q^{n*} and the notion of $\mathcal{A}$-principal will be explained later. Every contact hypersurface in a complex space form has constant mean curvature. Our results on contact hypersurfaces in Kähler manifolds suggest that it is natural to impose the condition of constant mean curvature in the more setting.

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Jürgen Berndt
King's College London,
Department of Mathematics,
London WC2R 2LS, United Kingdom
E-mail address: jurgen.berndt@kcl.ac.uk

Young Jin Suh
Kyungpook National University,
Department of Mathematics,
Taegu 702-701, Korea
E-mail address: yjsuh@knu.ac.kr