

# FINITE GAP SOLUTIONS FOR HORIZONTAL MINIMAL SURFACES OF FINITE TYPE IN 5-SPHERE

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Abstract. We consider a horizontal minimal surface in 5-sphere  $s_0 : M \longrightarrow S^5 \subset \mathbf{C}^3$ . The Gauss, Codazzi and Ricci equations for  $s_0$  becomes a unified equation called “Tzitéica equation” of elliptic type as follows.

$$(1) \quad \partial_{\bar{z}}\partial_z u = e^{-2u} - e^u,$$

where  $z = x + \sqrt{-1}y$  a local coordinate system for a Riemann surface  $M$  and  $u = u(z, \bar{z})$  is a real valued function on  $M$ . The induced metric  $g$  on  $M$  is given by  $g = 2e^u dz d\bar{z}$ . Let  $\langle, \rangle$  be a Hermitian fibre metric on  $M \times \mathbf{C}^3$  compatible with  $g$ . Since  $s_0$  is a minimal immersion, we have

$$(2) \quad \langle s_0, s_0 \rangle = 1, \quad \partial_{\bar{z}}\partial_z s_0 = -e^u s_0.$$

The horizontality of  $s_0$  with respect to the Hopf fibration  $S^5 \longrightarrow \mathbf{C}P^2$  means that

$$(3) \quad \langle \partial_z s_0, s_0 \rangle = 0, \quad \langle \partial_{\bar{z}} s_0, s_0 \rangle = 0$$

We also have  $\langle s_0, \partial_{\bar{z}} s_0 \rangle = 0, \langle s_0, \partial_z s_0 \rangle = 0$ . The conformality of  $s_0$  means that

$$(4) \quad \langle \partial_z s_0, \partial_{\bar{z}} s_0 \rangle = 0, \quad \langle \partial_{\bar{z}} s_0, \partial_z s_0 \rangle = 0$$

We set

$$(5) \quad s_1 = e^{-\frac{u}{2}} \partial_z s_0, \quad s_2 = e^{-\frac{u}{2}} \partial_{\bar{z}} s_0, \quad \phi = e^{\frac{u}{2}} \langle \partial_z \partial_z s_0, s_2 \rangle.$$

It then follows from (2), (3), (4) and (5) that  $F = (s_0 \ s_1 \ s_2)$  is a unitary frame on  $M \times \mathbf{C}^3$ . We have

$$(6) \quad F^{-1}\partial_z F = \begin{pmatrix} 0 & 0 & -e^{\frac{u}{2}} \\ e^{\frac{u}{2}} & \frac{u_z}{2} & 0 \\ 0 & \phi e^{-u} & -\frac{u_{\bar{z}}}{2} \end{pmatrix}, \quad F^{-1}\partial_{\bar{z}} F = \begin{pmatrix} 0 & -e^{\frac{u}{2}} & 0 \\ 0 & -\frac{u_{\bar{z}}}{2} & -\bar{\phi} e^{-u} \\ e^{\frac{u}{2}} & 0 & \frac{u_z}{2} \end{pmatrix},$$

where we have set  $u_z = \partial_z u, u_{\bar{z}} = \partial_{\bar{z}} u$ . If we set  $U = F^{-1}\partial_z F, V = F^{-1}\partial_{\bar{z}} F$ , then the compatibility condition for (6) is given by  $\partial_{\bar{z}} U - \partial_z V = [U, V]$ , which is equivalent to the following equations.

$$\begin{cases} \partial_{\bar{z}}\partial_z u &= |\phi|^2 e^{-2u} - e^u, \\ \partial_{\bar{z}}\phi &= 0. \end{cases}$$

Changing the local complex coordinate, we may assume that  $\phi = -1$ . We then obtain the Tzitzéica equation stated in (1). This is a special case of the famous Toda

equation in the theory of integrable systems. We first give an explicit solution of the Tzitzéica equation in terms of the Jacobi elliptic function. Secondly, we express the solution in terms of the Riemann theta function, which is so-called a finite gap solution. Moreover, some examples of horizontal minimal surfaces in 5-sphere can be described in terms of the Jacobi elliptic functions, which are also described explicitly in terms of the Baker-Akhiezer function. In this work, we give a spectral curve explicitly, which is a hyperelliptic curve of genus 2 and given in the affine coordinate by  $\hat{\mathcal{C}} : \tilde{\nu}^2 = \prod_{j=1}^3 (\mu - \mu_j)(\mu + \mu_j)$ . We also give Abelian differentials of second kind explicitly. Our Baker-Akhiezer function is given by

$$\hat{\Psi}(z, \bar{z}, \hat{P}, \mathbf{e}) = \frac{\theta(\mathcal{B}(\hat{P}) - (z + \bar{z})\mathbf{U}^0 - \mathbf{e}) \theta(\mathbf{e})}{\theta(\mathcal{B}(\hat{P}) - \mathbf{e}) \theta((z + \bar{z})\mathbf{U}^0 + \mathbf{e})} \Phi_e(z, \bar{z}, \hat{P}),$$

where  $\mathbf{e} = \pi\sqrt{-1}$ ,  $\mathbf{U}^0 = \frac{\pi\sqrt{-1}}{2\omega^0}$  and  $\hat{P} \in \hat{\mathcal{C}}$ . Moreover,  $\Phi_e(z, \bar{z}, \hat{P})$  is given by

$$\Phi_e(z, \bar{z}, \hat{P}) = \exp \left( z \left( \int_{\hat{P}_1}^{\hat{P}} \hat{\Omega}_{\infty} - \frac{\sqrt{-1}}{2} \mu_1 \right) - \bar{z} \left( \int_{\hat{P}_1}^{\hat{P}} \hat{\Omega}_0 - \frac{\sqrt{-1}}{2} \mu_1 \right) \right),$$

where  $\hat{P}_1 = (\mu_1, 0) \in \hat{\mathcal{C}}$ .

Finally, remark that a generalization of the reconstruction from some spectral curve of higher genus is also possible.

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