Gauge-theoretic equations for symmetric spaces and certain minimal submanifolds in moduli spaces

Introduction

A Riemannian manifold with holonomy group contained in $Sp(n)$ is called a \textit{hyperKähler manifold}. It is well-known that such a Riemannian manifold is Ricci flat and has three complex structures $J_1, J_2, J_3$ parallel with respect to the Levi-Civita connection which satisfy the quaternionic relation. A submanifold $f : M \rightarrow N$ immersed in a Kähler manifold $(N, \omega, J)$ is called a \textit{complex submanifold} if $J_f(x)^*(df)_xT_xM \subseteq (df)_xT_xM$ for each $x \in M$, and hence, $M$ itself becomes a Kähler manifold. An immersed submanifold $f : M \rightarrow N$ in a Kähler manifold $(N, \omega, J)$ is called a \textit{totally real submanifold} if, for each $x \in M$, $J_f(x)^*(T_xM)$ is orthogonal to the tangent space $T_xM$ relative to the Riemannian metric of $N$, or equivalently the pull back of $\omega$ to $M$ vanishes. A totally real submanifold with $2 \dim M = \dim N$ is nothing but a \textit{Lagrangian submanifold} in the symplectic manifold $(N, \omega)$. A Lagrangian submanifold $M$ in a Riemannian manifold $N$ with holonomy group contained in $SU(n)$ is called a \textit{special Lagrangian submanifold} if $M$ is calibrated by the real part $\alpha$ of the parallel holomorphic $n$-form $\alpha + \sqrt{-1} \beta$ on $N$ (see [2]).

Hitchin [4] investigated conformally invariant gauge-theoretic equations — the so-called self-duality equations — on a principal bundle $P$ with the structure group $G$, which is a compact Lie group, over a compact Riemann surface $M$ and showed that the moduli space of solutions to the self-duality equations is obtained as a hyperKähler quotient, and the smooth part of the moduli space becomes a hyperKähler manifold, which is possibly noncompact.

In this paper we shall provide a construction of a series of certain minimal submanifolds in the hyperKähler moduli space. Each minimal submanifold obtained here is a complex submanifold relative to $J_1$ and a totally real submanifold relative to $J_2$ and $J_3$ whose dimension is half of the dimension of the ambient space; and in particular, it becomes a special Lagrangian submanifold. Such a special Lagrangian submanifold in a hyperKähler manifold is called a \textit{complex Lagrangian submanifold} (see [8]). Moreover, it is shown that it is a totally geodesic submanifold. For a given compact symmetric pair $(G, K, \sigma)$ and reduction of the structure group $G$ of $P$ to the compact subgroup $K$, the construction is done by using the moduli spaces of solutions to the gauge-theoretic equations defined in [17] and the natural immersions of the moduli spaces into Hitchin’s moduli space. At the same time, we shall also provide a similar con-
struction in the moduli space of solutions to the gauge-theoretic equations harmonic maps into compact symmetric spaces (cf. [17]), which differ from Hitchin's self-duality equations by a sign.

The definitions of complex submanifolds, totally real submanifolds, and complex Lagrangian submanifolds for smooth immersed submanifolds can be extended in a natural way to submanifolds which are not necessarily smooth, but which admit Zariski tangent spaces at each point. Throughout this paper we use such terminology also for immersed submanifolds in the wider sense.

1 Gauge-theoretic equations for compact Lie group G

Let $M$ be a compact Riemann surface and $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $P$ be a principal bundle over a compact Riemann surface $M$ with structure group $G$. Let $\mathcal{A}_P$ be the affine space of all smooth connections in the principal bundle $P$ and $\Omega^1(\mathfrak{g}_P)$ be the vector space of all smooth 1-forms on $M$ with values in the adjoint bundle $\mathfrak{g}_P$. It is known that there are two types of conformally invariant gauge-theoretic equations for a connection $A$ on $P$ and a 1-form $\phi$ on $M$ with values in the adjoint bundle $\mathfrak{g}_P$, which is called a Higgs field,

\[
\begin{aligned}
F(A) \pm \frac{1}{2}[\phi \wedge \phi] &= 0, \\
& (\ast)_g^G \\
d_A\phi = d_A^* \phi &= 0,
\end{aligned}
\]

where $F(A)$ is the curvature form of the connection $A$. The equations $(\ast)_g^G$ are the self-duality equations investigated by Hitchin [5]. The equations $(\ast)_g^G$ are a gauge-theoretic formulation of the harmonic map equations of a Riemann surface $M$ into the Lie group $G$ ([6],[18],[19],[16]). Let $\widetilde{\mathcal{M}}_\pm(M,G,P)$ denote the space of all smooth solutions to $(\ast)_g^G$. We denote by $[A,\phi]$ the gauge equivalence class of an element $(A,\phi)$ in $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ under the action of the group $G_P$ of gauge transformations of $P$. Let $\mathcal{M}_\pm(M,G,P) := \widetilde{\mathcal{M}}_\pm(M,G,P)/G_P$ denote the moduli space of all gauge equivalence classes $[A,\phi]$ of smooth solutions to $(\ast)_g^G$. We equip the moduli space $\mathcal{M}_\pm(M,G,P)$ with a Riemannian metric induced by the $L^2$-inner product. $(A,\phi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ is called regular if

$$\Gamma_{(A,\phi)}(G_P) := \{g \in G_P \mid g^* (A,\phi) = (A,\phi)\} = C(G),$$

where $C(G)$ denotes the center of Lie group $G$ with Lie algebra $\mathfrak{c}(\mathfrak{g})$.

Set

$$\widetilde{\mathcal{M}}_\pm(M,G,K) := \{(A,\phi) \in \widetilde{\mathcal{M}}_\pm(M,G,P) \mid (A,\phi) \text{ is regular}\}$$

and define

$$\mathcal{M}_\pm(M,G/K,Q) := \widetilde{\mathcal{M}}_\pm(M,G/K,Q)/K_Q.$$
We recall the local theory of the moduli spaces of the equations \((\ast)_{\pm}^c\) ([5], [16]).

The linearized equations for \((\ast)^c_{\pm}\) are

\[
\begin{align*}
d_A \alpha \pm [\phi \wedge \eta] &= 0, \\
d_A \eta + [\phi \wedge \alpha] &= 0, \\
d_A \ast \eta + [\ast \phi \wedge \alpha] &= 0,
\end{align*}
\]

for \((\alpha, \eta) \in \Omega^1(g_P) \oplus \Omega^1(g_P)\).

The elliptic cochain complex associated with deformations of solutions to the equations \((\ast)^c_{\pm}\) is

\[0 \rightarrow \Omega^0(g_P) \rightarrow \Omega^1(g_P) \oplus \Omega^1(g_P) \rightarrow \Omega^2(g_P) \oplus \Omega^2(g_P) \oplus \Omega^2(g_P) \rightarrow 0, \quad (C_{(A, \phi)})\]

where the linear differential operators

\[d_0 = d_{0(A, \phi)} : \Omega^0(g_P) \rightarrow \Omega^1(g_P) \oplus \Omega^1(g_P)\]

and

\[d_1 = d_{1(A, \phi)} : \Omega^1(g_P) \oplus \Omega^1(g_P) \rightarrow \Omega^2(g_P) \oplus \Omega^2(g_P) \oplus \Omega^2(g_P)\]

are defined by

\[
\begin{align*}
d_{0(A, \phi)}(\xi) &= (d_A \xi, [\phi, \xi]), \\
d_{1(A, \phi)}(\alpha, \eta) &= (d_A \alpha \pm [\phi \wedge \eta], d_A \eta + [\phi \wedge \alpha], d_A \ast \eta + [\ast \phi \wedge \alpha]).
\end{align*}
\]

The formal adjoint operators \(d_0^*\) and \(d_1^*\) become

\[d_0^*(\alpha, \eta) = d_0^* \alpha - [\phi \wedge \ast \eta] = -*(d_A \ast \alpha + [\phi \wedge \ast \eta]),\]

and

\[d_1^*(u, v, w) = (d_A^* u - [\phi \wedge v] - [\ast \phi \wedge w], \ast [\phi \wedge u] + d_A^* v - d_A \ast w).\]

Set \(P_0 := d_0^* d_0, P_1^\pm := d_0 d_0^* + d_1^* d_1\) and \(P_2^\pm := d_1 d_1^*\). Then the \(i\)-th cohomology group \(H^i_{(A, \phi)}(M, G, P)\) of the cochain complex is isomorphic to \(\text{Ker} P_i\) for each \(i = 0, 1, 2\).

We know ([5], [16]) that for each \((A, \phi) \in \tilde{M}_\pm(M, G, P)\), there exists a real analytic \(\Gamma_{(A, \phi)}\)-equivariant map \(\psi_{(A, \phi)} : U \rightarrow H^2_{(A, \phi)}(M, G, P)\) defined on a neighborhood \(U\) of the origin in \(H^2_{(A, \phi)}(M, G, P)\) with \(\psi_{(A, \phi)}(0) = 0\) and \((d\psi_{(A, \phi)})_0 = 0\) such that a neighborhood of \([A, \phi]\) in \(M_\pm(M, G, P)\) is homeomorphic to \(\psi_{(A, \phi)}^{-1}(0) / \Gamma_{(A, \phi)}\).

Since the vanishing of \(H^0_{(A, \phi)}(M, G, P)\) implies the vanishing of \(H^2_{(A, \phi)}(M, G, P)\) by the Bochner-Weitzenböck formula in the case of \((\ast)^c\), we see that a neighborhood of each point \([A, \phi]\) of \(M^*_\pm(M, G, P)\) becomes a smooth manifold of dimension equal to \(\dim H^1_{(A, \phi)}(M, G, P)\) (see [5], [16]). And we know that the real dimension of \(M_\pm(M, G, P)\) is equal to \(4 \dim G(g - 1)\) if the genus of \(M\) is greater than 1 and \(G\) is semisimple ([5], [6]).

In [17], we showed the weak compactness theorem on sequences of solutions to the equations \((\ast)_\pm\) with bounded energy and a removable theorem for solutions with an isolated singularity, which are analogous to Sacks-Uhlenbeck’s theorems.
2 Gauge-theoretic equations for compact symmetric space $G/K$

In [17], generalizing the equations $(\ast)_\pm^G$, we introduced certain gauge-theoretic equations for each compact symmetric space $G/K$ and discussed local structure of the moduli spaces. Now let $(G, K, \sigma)$ be a compact symmetric pair, more generally $(G, K)$ be a pair of a compact Lie group $G$ and its compact subgroup $K$ associated with an orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k}, \sigma)$ (see [3]). Let $Q$ be a principal $K$-bundle over $M$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ with respect to a symmetric space $G/K$. Define associated vector bundles over $M$ as $\mathfrak{t}_Q := Q \times_K \mathfrak{k}$, $\mathfrak{m}_Q := Q \times_K \mathfrak{m}$, $\mathfrak{g}_Q := \mathfrak{Q} \times_K \mathfrak{g}$. Note that we have $\mathfrak{g}_Q = \mathfrak{t}_Q \oplus \mathfrak{m}_Q$. Note that $\mathfrak{g}_Q \cong \mathfrak{g}_P$ as vector bundles. Let $\mathcal{A}_Q$ denote the space of all smooth connections in $Q$. For each $i = 0, 1, 2$, let $\Omega^i(\mathfrak{t}_Q)$ and $\Omega^i(\mathfrak{m}_Q)$ denote the vector spaces of all smooth $i$-forms with values in $\mathfrak{t}_Q$ and $\mathfrak{m}_Q$ on $M$, respectively. The space $\mathcal{A}_Q$ is an infinite dimensional affine space with the associated vector space $\Omega^1(\mathfrak{t}_Q)$.

For $(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$, consider the gauge-theoretic equations

\[
\begin{align*}
F(A) \pm \frac{1}{2}[\phi \wedge \phi] &= 0, \\
A \phi &= d_A \phi = 0.
\end{align*}
\]

A solution $(A, \phi)$ to the equations $(\ast)_\pm^{G/K}$ with $\phi \neq 0$ is called nontrivial. A connection $A \in \mathcal{A}_Q$ is called irreducible if the holonomy algebra of the connection $A$ is equal to the Lie algebra $\mathfrak{k}$ of the structure group $K$.

Set

\[
\widetilde{\mathcal{M}}_\pm(M, G/K, Q) := \{(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q) \mid (A, \phi) \text{ is a smooth solution to } (\ast)_\pm^{G/K}\}.
\] (5)

Denote by $\mathcal{K}_Q$ the group of gauge transformations of the principal $K$-bundle $Q$. The right action of $\mathcal{K}_Q$ on $\mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$ is defined by

\[k^* (A, \phi) = (k^* A, \text{Ad}(k)^{-1} \phi).\]

The tangent space of the gauge orbit through $(A, \phi)$ is given by

\[
\{(d_A \xi, [\phi, \xi]) \mid \xi \in \Omega^1(\mathfrak{t}_Q)\}.
\]

The moduli spaces of all gauge equivalence classes $[A, \phi]$ of smooth solutions to $(\ast)_\pm^{G/K}$ are defined by

\[
\mathcal{M}_\pm(M, G/K, Q) := \widetilde{\mathcal{M}}_\pm(M, G/K, Q)/\mathcal{K}_Q.
\] (6)

For $(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{t}_Q)$

\[
(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{t}_Q)
\]

and define $\mathcal{M}_\pm(M, G/K, Q)$.

We recall results discussed in [17]. The linear differential equation

\[
0 \to \Omega^0(\mathfrak{t}_Q) \to
\]

for $(\alpha, \eta) \in \Omega^1(\mathfrak{t}_Q) \oplus \Omega^1(\mathfrak{m}_Q)$

The elliptic cochain equations $(\ast)_\pm$ is

where the linear differential

\[
d_1 = d_1(A, \phi)
\]

are defined by

\[
d_0(\eta)(\xi) := d_1(A, \phi)(\eta, \xi).
\]

The formal adjoint of $d_0$ is

\[
d_0^*(\alpha, \eta) = (d_1(A, \phi))^*(\alpha, \eta).
\]

The formal adjoint of $d_1$ is

\[
d_1^*(\alpha, \eta) = (d_0(\phi))^*(\alpha, \eta).
\]

Set $P_0 := d_0^* d_0$, $P_1^2 = P_0^2$

$H^1_{(A, \phi)}(M, G/K, Q)$ of
For \((A, \phi) \in \mathcal{A}_Q \times \Omega^1(m_Q)\), we set
\[
\Gamma_{(A, \phi)}(K_Q) := \{k \in K_Q | k^*(A, \phi) = (A, \phi)\}.
\]
(7)

\((A, \phi) \in \mathcal{A}_Q \times \Omega^1(m_Q)\) is called regular if \(\Gamma_{(A, \phi)}(K_Q) = C(G) \cap K\). Set
\[
\widetilde{\mathcal{M}}_\pm(M, G/K, Q) := \{(A, \phi) \in \widetilde{\mathcal{M}}_\pm(M, G/K, Q) | (A, \phi) \text{ is regular}\}
\]
and define \(\mathcal{M}_\pm(M, G/K, Q) := \widetilde{\mathcal{M}}_\pm(M, G/K, Q)/K_Q\).

We recall results on local structures of the moduli spaces \(\mathcal{M}_\pm(M, G/K, Q)\) discussed in [17]. The linearized equations for \((*)_{G/K}\) are
\[
\begin{cases}
d_A \alpha \pm [\phi \wedge \eta] = 0, \\
d_A \eta + [\phi \wedge \alpha] = 0, \\
d_A * \eta + [\ast \phi \wedge \alpha] = 0,
\end{cases}
\]
(9)±

for \((\alpha, \eta) \in \Omega^1(t_Q) \oplus \Omega^1(m_Q)\).

The elliptic cochain complex associated with deformations for solutions to the equations \((*)_{\pm}\) is
\[
0 \rightarrow \Omega^0(t_Q) \rightarrow \Omega^1(t_Q) \oplus \Omega^1(m_Q) \rightarrow \Omega^2(t_Q) \oplus \Omega^2(m_Q) \rightarrow \Omega^3(t_Q) \oplus \Omega^3(m_Q) \rightarrow 0,
\]
(C\(_{(A, \phi)}\))

where the linear differential operators
\[
d_0 = d_0(A, \phi) : \Omega^0(t_Q) \rightarrow \Omega^1(t_Q) \oplus \Omega^1(m_Q)
\]
and
\[
d_1 = d_1(A, \phi) : \Omega^1(t_Q) \oplus \Omega^1(m_Q) \rightarrow \Omega^2(t_Q) \oplus \Omega^2(m_Q) \oplus \Omega^2(m_Q)
\]
are defined by
\[
d_0(A, \phi)(\xi) := (d_A \xi, \phi, \xi),
\]
\[
d_1(A, \phi)(\alpha, \eta) := (d_A \alpha \pm [\phi \wedge \eta], d_A \eta + [\phi \wedge \alpha], d_A * \eta + [\ast \phi \wedge \alpha]).
\]
(10)±

The formal adjoint operators \(d_0^*\) and \(d_1^*\) become
\[
d_0^*(\alpha, \eta) = d_1^* \alpha - *[\phi \wedge \ast \eta] = -*(d_A * \alpha + [\phi \wedge \ast \eta]),
\]
(12)

and
\[
d_1^*(u, v, w) = (d_A^* u - *[\phi \wedge u] - *[\phi \wedge w], \mp [\phi \wedge u] + d_A^* v - d_A * w).
\]
(13)±

Set \(P_0 := d_0^* d_0, P_1^\pm := d_0 d_0^* + d_1^* d_1\) and \(P_2^\pm := d_1 d_1^*\). Then the i-th cohomology group \(H_i^\pm(A, \phi)(M, G/K, Q)\) of the cochain complex is isomorphic to \(\text{Ker} P_i\) for each \(i = 0, 1, 2\).
In [17], we showed that for each $(A, \phi) \in \tilde{M}_\pm(M, G/K, Q)$, there exists a real analytic $\Gamma_{(A, \phi)}$-equivariant map

$$\psi_{(A, \phi)} : U \to H^2_{(A, \phi)}(M, G/K, Q)$$

defined on a neighborhood $U$ of the origin in $H^1_{(A, \phi)}(M, G/K, Q)$ with $\psi_{(A, \phi)}(0) = 0$ and $(d\psi_{(A, \phi)})_0 = 0$ such that a neighborhood of $[A, \phi]$ in $\mathcal{M}_\pm(M, G/K, Q)$ is homeomorphic to $\psi_{(A, \phi)}^{-1}(0)/\Gamma_{(A, \phi)}$.

We define subsets of $\mathcal{M}_\pm(M, G/K, Q)$ as follows:

$$\mathcal{M}_\pm^+(M, G/K, Q) := \{[A, \phi] \in \mathcal{M}_\pm(M, G/K, Q) \mid H^1_{(A, \phi)}(M, G/K, Q) = (c(\mathfrak{g}) \cap \mathfrak{t}) \oplus (c(\mathfrak{g}) \cap \mathfrak{m}) \},$$

$$\mathcal{M}_\pm^-(M, G/K, Q)^{irr} := \{[A, \phi] \in \mathcal{M}_\pm(M, G/K, Q) \mid A \text{ is irreducible} \},$$

$$\mathcal{M}_\pm^+(M, G/K, Q)^{irr} := \mathcal{M}_\pm^+(M, G/K, Q) \cap \mathcal{M}_\pm^-(M, G/K, Q)^{irr}.$$

Then we see that a neighborhood of each point $[A, \phi]$ of $\mathcal{M}_\pm^+(M, G/K, Q)$ becomes a smooth manifold of dimension equal to dim $H^1_{(A, \phi)}(M, G/K, Q)$.

If we specify the case of the equations $(\ast_{G/K})^s$, then we obtain the following results on the vanishing theorem for the second cohomology group and the manifold structure of the moduli space.

**Proposition 1.** [17] (1) If $G/K$ is a compact Lie group, then we have

$$\mathcal{M}_\pm^+(M, G/K, Q) = \mathcal{M}_\pm^-(M, G/K, Q).$$

(2) If $G/K$ is of compact type, then $\mathcal{M}_\pm^-(M, G/K, Q)^{irr}$ is an open submanifold of $\mathcal{M}_\pm^+(M, G/K, Q)$.

3 Maps between moduli spaces

Let $(G, K, \sigma)$ be a compact symmetric pair. A subbundle $Q$ of $P$ with structure group $K$ which is a compact subgroup of $G$ is called a reduced subbundle to $K$. It is known (cf. [13]) that

$$\{Q \mid \text{reduced subbundles of } P \text{ to the structure group } K\}$$

$$\cong C^\infty(M; P \times_G (G/K)), \quad (14)$$

where $P \times_G (G/K)$ is a fibre bundle associated to $P$ with respect to the left action of $G$ on the compact symmetric space $G/K$. Note that any principal $K$-bundle $Q$ over $M$ becomes a reduced subbundle of a principal $G$-bundle $P$ to $K$. Indeed, the structure group $K$ of the principal bundle $Q$ acts on the Lie group $G$ on the left. We define a principal bundle $P = Q \times_K G$ over $M$ with the structure group $G$ associated with the left action of $K$ on $G$. Then the principal bundle $Q$ with the structure group $K$ is embedded into $P$. Conversely, if $Q$ is to $Q \times_K G$ as $\pi$.

Since $Q$ is a re and

Since $k \in K_Q$ exte homomorphism

These embedd which are equivar between the modi for each compact $s$ $(G/K)$. We shoul the following:

Let $N_G(\mathfrak{t}) = \{\}$ has Lie algebra $\mathfrak{t}$ with $(\mathfrak{g}, \mathfrak{k}, \sigma)$.

**Proposition 2.**

is injective.

**Proof.** Suppose $t \mathfrak{g}((A_1, \phi_1)) = \mathfrak{g}$ is $g \in G_P$ such the $u \in Q$, we have $g$ that $H^1_{(A)}(u) = a$ a connection $A$ w $u \in Q$ we have $\mathfrak{t} =$. Therefore, we obt

The gauge grc Then we have $j_{\beta(\mathfrak{t})}$

Therefore, when $w$ $C^\infty(M; P \times_G (G/K))$
embedded into the principal bundle $P$ with the structure group $G$ as a reduced bundle. Conversely, if $Q$ is a reduced subbundle of $P$ to $K$, then $P$ is canonically isomorphic to $Q \times_K G$ as principal $G$-bundles.

Since $Q$ is a reduced subbundle of $P$ to $K$, we have natural embeddings

$$\mathcal{A}_Q \to \mathcal{A}_P$$

and

$$\Omega^1(\mathfrak{m}_Q) \to \Omega^1(\mathfrak{g}_P).$$

Since $k \in \mathcal{K}_Q$ extends to a gauge transformation of $P \cong Q \times_K G$, we have an injective homomorphism

$$\mathcal{K}_Q \to \mathcal{G}_P.$$

These embeddings induce embeddings between the solution spaces

$$\tilde{j} = j_Q : \widetilde{\mathcal{M}}_{\pm}(M, G/K, Q) \to \widetilde{\mathcal{M}}_{\pm}(M, G, P),$$

which are equivariant under the actions of $\mathcal{K}_Q$ and $\mathcal{G}_P$, and hence, we obtain a map between the moduli spaces

$$j = j_Q : \mathcal{M}_{\pm}(M, G/K, Q) \to \mathcal{M}_{\pm}(M, G, P),$$

(15)

for each compact symmetric pair $(G, K, \sigma)$ with the same $G$ and each $Q \in C^\infty(M; P \times_G (G/K))$. We should note that $j = j_Q$ is not necessarily injective. However, we can show the following:

Let $N_G(\mathfrak{k}) = \{a \in G \mid Ad(a)\mathfrak{k} = \mathfrak{k}\}$ be the normalizer of $\mathfrak{k}$ in $G$. Note that $N_G(\mathfrak{k})$ also has Lie algebra $\mathfrak{k}$ and $N_G(\mathfrak{k})/K$ is finite. The pair $(G, N_G(\mathfrak{k}))$ is also a pair associated with $(\mathfrak{k}, \sigma)$.

**Proposition 2.** Assume that $N_G(\mathfrak{k}) = K$. Then the map

$$j_Q : \mathcal{M}_{\pm}(M, G/K, Q)^{irr} \to \mathcal{M}_{\pm}(M, G, P)$$

is injective.

**Proof.** Suppose that $[A_1, \phi_1], [A_2, \phi_2] \in \mathcal{M}_{\pm}(M, G/K, Q)^{irr}$ and $j_Q([A_1, \phi_1]) = j_Q([A_2, \phi_2])$. We write $(\tilde{A}_1, \tilde{\phi}_1) = j_Q(A_1, \phi_1)$ for each $i = 1, 2$. Since there is $g \in \mathcal{G}_P$ such that $g^*(\tilde{A}_1, \tilde{\phi}_1) = (A_2, \phi_2)$, we have $g^*\tilde{A}_1 = \tilde{A}_2$ and $g^*\tilde{\phi}_1 = \phi_2$. For each $u \in Q$, we have $g(u) = a(u)$ for some $a = a(u) \in G$. The condition $g^*\tilde{A}_1 = \tilde{A}_2$ implies that $H^0_{A_1}(u) = a^{-1}H^0_{A_2}(u)a$. Here $H^0_{A_2}(u)$ denotes the restricted holonomy group of a connection $A$ with reference point $u$. By the irreducibility of $A_1$ and $A_2$, for each $u \in Q$ we have $\mathfrak{k} = a(u)^{-1}\mathfrak{a}(u)$, namely $a(u) \in N_G(\mathfrak{k}) = K$. This means that $g \in \mathcal{K}_Q$.

Therefore, we obtain $[A_1, \phi_1] = [A_2, \phi_2]$. \qed

The gauge group $\mathcal{G}_P$ acts on $C^\infty(M; P \times_G (G/K))$ by transforming $Q$ to $g(Q)$. Then we have $j_Q(g) \circ g = j_Q$ and thus,

$$j_Q(\mathcal{M}_{\pm}(M, G/K, Q)) = j_Q(\mathcal{M}_{\pm}(M, G/K, g(Q))).$$

Therefore, when we discuss the image of the map $j$, we may consider the quotient space $C^\infty(M; P \times_G (G/K))/\mathcal{G}_P$ by the gauge group $\mathcal{G}_P$ rather than $C^\infty(M; P \times_G (G/K))$. 199
4 HyperKähler structure on the moduli space $\mathcal{M}_-(M, G, P)$

Here we recall Hitchin’s construction ([5]) of the hyperKähler structure on the moduli space $\mathcal{M}_-(M, G, P)$. The geometric structure on $\mathcal{M}_+(M, G, P)$ is discussed in [16].

The affine space $\mathcal{A}_P$ and the vector space $\Omega^1(\mathfrak{g}_P)$ have complex structures defined by

$$J(\alpha) := -\ast \alpha$$

for each $\alpha \in \Omega^1(\mathfrak{g}_P)$, where $\ast$ denotes the star operator of the Riemann surface $M$.

The Kähler forms, and hence symplectic forms, on $\mathcal{A}_P$ and $\Omega^1(\mathfrak{g}_P)$ are defined by

$$\omega(\alpha_1, \alpha_2) := \langle J(\alpha_1), \alpha_2 \rangle_{L^2}.$$  \hfill (18)

The moment map for the action $\mathcal{G}_P$ on a symplectic manifold $(\mathcal{A}_P, \omega)$ is a map

$$\nu_1 : \mathcal{A}_P \longrightarrow \Omega^0(\mathfrak{g}_P) \cong \Omega^2(\mathfrak{g}_P),$$

defined by

$$\nu_1(A) = \ast F(A) \equiv F(A).$$ \hfill (19)

The moment map for the action $\mathcal{G}_P$ on a symplectic manifold $(\Omega^1(\mathfrak{g}_P), \omega)$ is a map

$$\nu_2 : \Omega^1(\mathfrak{g}_P) \longrightarrow \Omega^0(\mathfrak{g}_P) \cong \Omega^2(\mathfrak{g}_P),$$

defined by

$$\nu_2(\phi) = \frac{1}{2}[\phi \wedge \phi].$$ \hfill (20)

Three symplectic forms on $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ are defined as

$$\omega_1((\alpha_1, \eta_1), (\alpha_2, \eta_2)) := -\langle \ast \alpha_1, \alpha_2 \rangle_{L^2} + \langle \ast \eta_1, \eta_2 \rangle_{L^2},$$

$$\omega_2((\alpha_1, \eta_1), (\alpha_2, \eta_2)) := -\langle \alpha_1, \eta_2 \rangle_{L^2} + \langle \eta_1, \alpha_2 \rangle_{L^2},$$

$$\omega_3((\alpha_1, \eta_1), (\alpha_2, \eta_2)) := \langle \eta_1, \ast \alpha_2 \rangle_{L^2} - \langle \eta_2, \ast \alpha_1 \rangle_{L^2},$$ \hfill (21)

for each $(\alpha_1, \eta_1), (\alpha_2, \eta_2) \in \Omega^1(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P)$.

The corresponding moment maps for the action of $\mathcal{G}_P$ on $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ relative to $\omega_1, \omega_2$ and $\omega_3$ are

$$\mu_1(A, \phi) := \nu_1(A) - \nu_2(\phi) = F(A) - \frac{1}{2}[\phi \wedge \phi],$$

$$\mu_2(A, \phi) := d_A * \phi,$$

$$\mu_3(A, \phi) := d_A \phi,$$ \hfill (22)

respectively. The moduli space

We equip the spa

We define three $\nu$

Then we have

By the method $c$

with the hyperKähler Hitchin map

Here we observe

hyperKähler metric

invariant polyno-

is defined by $p$

bundle of $M$ an Hitchin ([5],[6]) $\mathcal{M}_-(M, G, P)$, becomes a comp-

a different class
respectively. The solution space $\hat{M}$ can be expressed as

$$\hat{M}_-(M, G, P) = \bigcap_{i=1}^{3} \mu_i^{-1}(0).$$  \(\text{(23)}\)

The moduli space is given by

$$\mathcal{M}_-(M, G, P) = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/G_P.$$  \(\text{(24)}\)

We equip the space $A_P \times \Omega^1(g_P)$ with an $L^2$-Riemannian metric defined by

$$g((\alpha_1, \eta_1), (\alpha_2, \eta_2)) := \langle \alpha_1, \alpha_2 \rangle_{L^2} + \langle \eta_1, \eta_2 \rangle_{L^2}.$$  \(\text{(25)}\)

We define three complex structures on $A_P \times \Omega^1(g_P)$ by

$$J_1(\alpha, \eta) := (- \ast \alpha, \ast \eta),$$

$$J_2(\alpha, \eta) := (\eta, -\alpha),$$

$$J_3(\alpha, \eta) := (- \ast \eta, - \ast \alpha).$$  \(\text{(26)}\)

Then we have

$$\omega_1((\alpha_1, \eta_1), (\alpha_2, \eta_2)) = g(J_1(\alpha_1, \eta_1), (\alpha_2, \eta_2)),$$

$$\omega_2((\alpha_1, \eta_1), (\alpha_2, \eta_2)) = g(J_2(\alpha_1, \eta_1), (\alpha_2, \eta_2)),$$

$$\omega_3((\alpha_1, \eta_1), (\alpha_2, \eta_2)) = g(J_3(\alpha_1, \eta_1), (\alpha_2, \eta_2)).$$  \(\text{(27)}\)

By the method of hyperKähler quotient ([10], [5]), we can equip the moduli space

$$\mathcal{M}_-(M, G, P) = \hat{M}_-(M, G, P)/G_P = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/G_P$$

with the hyperKähler structure.

Here we observe a class of examples of complex Lagrangian submanifolds of the hyperKähler moduli space. We choose a basis $\{p_1, \cdots, p_k\}$ of the ring consisting of invariant polynomials on $g^C$. We denote by $d_i$ the degree of the polynomial $p_i$. The Hitchin map

$$p: \mathcal{M}_-(M, G, P) \longrightarrow \bigoplus_{i=1}^{k} H^0(M; K_M^{d_i})$$

is defined by $p([A, \phi]) = (p_1(\phi'), \cdots, p_k(\phi'))$. Here $K_M$ denotes the canonical line bundle of $M$ and $\phi'$ denotes the $(1,0)$-component of $\phi$. It is an important fact due to Hitchin ([5],[6]) that this map gives an algebraically integrable Hamiltonian system on $\mathcal{M}_-(M, G, P)$. We see that for generic $a \in \bigoplus_{i=1}^{k} H^0(M; K_M^{d_i})$, the level subset $p^{-1}(c)$ becomes a complex Lagrangian torus submanifold $\mathcal{M}_-(M, G, P)$. In Section 6, we give a different class of complex Lagrangian submanifolds in the same moduli space.

201
5  Kähler structure on $\mathcal{M}_-(M, G/K, Q)$

In this section we discuss the geometric structure of the moduli spaces $\mathcal{M}_-(M, G/K, Q)$ of solutions to the equations $(\ast)^G_{G/K}$. In [17], we posed a question asking what kinds of geometric structures the moduli spaces $\mathcal{M}_-(M, G/K, Q)$ have for each compact symmetric space $N = G/K$, and how the geometry of each compact symmetric space $N = G/K$ reflects the geometry of the moduli space $\mathcal{M}_-(M, G/K, Q)$. Here we give a partial answer to the question.

We define a complex structure on $\mathcal{A}_Q \times \Omega^1(m_Q)$ by

$$J(\alpha, \eta) = (- \ast \alpha, \ast \eta)$$

and we define

$$\omega((\alpha_1, \eta_1), (\alpha_2, \eta_2)) = g(J(\alpha_1, \eta_1), (\alpha_2, \eta_2)).$$

The formal tangent vector space $T_{(A, \phi)}\mathcal{M}_\pm(M, G/K, Q)$ can be identified with a vector space of all elements $(\alpha, \eta) \in \Omega^1(\xi_P) \oplus \Omega^1(m_P)$ satisfying

$$\begin{cases}
  d_A \alpha \pm [\phi \wedge \eta] = 0, \\
  d_A \eta + [\phi \wedge \alpha] = 0, \\
  d_A \ast \eta + [\ast \phi \wedge \alpha] = 0, \\
  d_A \ast \alpha + [\phi \wedge \ast \eta] = 0,
\end{cases}$$

Then in the case of the equations $(\ast)_-$ we can observe that $J$, $\omega$ and the $L^2$-metric $g$ induce a closed 2-form and an almost complex structure on the moduli space $\mathcal{M}_-(M, G/K, Q)$. Therefore, we obtain

**Proposition 3.** The smooth part of $\mathcal{M}_-(M, G/K, Q)$ is an almost Kähler manifold.

In the next section, we see that it is really a Kähler manifold.

6  Main results

Let $(A, \phi) \in \mathcal{M}_\pm(M, G/K, Q)$ and set $(\hat{A}, \hat{\phi}) = j(A, \phi)$. The following commutative diagram describes the local structure of the map $j_Q : \mathcal{M}_\pm(M, G/K, Q) \to \mathcal{M}_\pm(M, G, P)$.

We shall clari...
We shall clarify the local property of the maps
\[ j = j_Q : \mathcal{M}_\pm(M, G/K, Q) \to \mathcal{M}_\pm(M, G, P) \]

First we discuss the smoothness of a neighborhood of \( j_Q[A, \phi] \) in \( \mathcal{M}_\pm(M, G, P) \) when a neighborhood of \([A, \phi]\) in \( \mathcal{M}_\pm(M, G/K, Q) \) is smooth.

**Lemma 1.** Let \((A, \phi) \in \mathcal{M}_-(M, G/K, Q)\) be a regular point. Assume that (1) \( G/K \) is a compact Lie group, or (2) \( G/K \) is of compact type and \( A \) is irreducible. Then we have \( H^0_{(A, \phi)}(M, G, P) = c(g) \).

**Proof.** Note that if \((A, \phi) \in \mathcal{M}_-(M, G/K, Q)\) is regular, then we have
\[ H^0_{(A, \phi)}(M, G/K, Q) = c(g) \cap \mathfrak{k}. \]
Let \( \xi \in H^0_{(A, \phi)}(M, G, P) \). Since \( d_A \xi = 0 \) and \([\phi, \xi] = 0\), setting \( \xi = \xi_\xi + \xi_m \) we have
\[ d_A \xi_\xi = 0, \quad [\phi, \xi_\xi] = 0 \quad (31) \]
and
\[ d_A \xi_m = 0, \quad [\phi, \xi_m] = 0. \quad (32) \]

Since the equation (31) means that \( \xi_\xi \in H^0_{(A, \phi)}(M, G/K, Q) \), we obtain \( \xi_\xi = c(g) \cap \mathfrak{k} \).
In the case of (1), via the natural isomorphism \( \xi_Q \cong m_Q \), the section \( \xi_m \) also defines an element of \( H^0_{(A, \phi)}(M, G/K, Q) \). Thus, we obtain \( \xi_m = c(g) \). In the case of (2), since \( \xi_m \) defines a \( \mathfrak{k} \)-fixed element in \( m \), the compactness and semisimplicity of \( g \) (see [3]) imply that \( \xi_m = 0 \). \( \square \)
Proposition 4. Assume that $N_G(\mathfrak{t}) = K$. Then we have

$$j_Q(\mathcal{M}_*^+(M, G/K, Q)^{irr}) \subset \mathcal{M}_*^+(M, G, P)$$

and hence, $j_Q$ induces an embedding of $\mathcal{M}_*^+(M, G/K, Q)^{irr}$ into $\mathcal{M}_*^+(M, G, P)$.

Proof. Let $(A, \phi) \in \widetilde{\mathcal{M}}^+_* (M, G/K, Q)^{irr}$ and set $(\tilde{A}, \tilde{\phi}) = \tilde{j}(A, \phi) \in \widetilde{\mathcal{M}}_+ (M, G, P)$. Suppose that $g \in \Gamma_{(\tilde{A}, \tilde{\phi})}(G_P)$ and $g^*(A, \phi) = (A, \phi)$. For each $u \in Q$, we set $g(u) = uu$ for some $a = a(u) \in G$. The condition $g^*A = A$ and the irreducibility of $A$ imply that $\mathfrak{t} = a^{-1}u\mathfrak{a}$, and thus $a = a(u) \in N_G(\mathfrak{t}) = K$ for each $u \in Q$. Hence, $g \in K_Q$. Therefore, we obtain $g \in \Gamma_{(A, \phi)}(K_Q) = C(G) \cap K$. Therefore, we conclude that $\Gamma_{(A, \phi)}(G_P) = C(G)$.

Concerning to the smoothness of a neighborhood of $[A, \phi]$ in $\mathcal{M}_+^+(M, G/K, Q)$ when a neighborhood of $j_Q[A, \phi]$ in $\mathcal{M}_+^+(M, G, P)$ is smooth, we obtain the following:

Proposition 5. Let $[A, \phi] \in \mathcal{M}_+^+(M, G, P)$. If $j_Q(A, \phi) \in \mathcal{M}_+^+(M, G, P)$, then we have $[A, \phi] \in \mathcal{M}_+^+(M, G/K, Q)$. Hence $j_Q$ is an immersion of a neighborhood of $[A, \phi]$ in $\mathcal{M}_+^+(M, G/K, Q)$.

Proof. This statement follows from the inclusions

$$\Gamma_{(A, \phi)}(K_Q) \hookrightarrow \Gamma_{(\tilde{A}, \tilde{\phi})}(G_P)$$

and

$$H^0_{(A, \phi)}(M, G/K, Q) \hookrightarrow H^0_{(\tilde{A}, \tilde{\phi})}(M, G, P), \quad \text{(33)}$$

$$H^2_{(A, \phi)}(M, G/K, Q) \hookrightarrow H^2_{(\tilde{A}, \tilde{\phi})}(M, G, P). \quad \text{(34)}$$

We must determine the tangent space and the normal space of an immersed submanifold in the wider sense by the map

$$\mathcal{M}_+^+(M, G/K, Q) \hookrightarrow \mathcal{M}_+^+(M, G, P).$$

For each $[A, \phi] \in \mathcal{M}_+^+(M, G/K, Q)$, let

$$T_{[A, \phi]}\mathcal{M}_+^+(M, G/P) = T_{[A, \phi]}\mathcal{M}_+^+(M, G/K, Q) \oplus T_{[A, \phi]}\mathcal{M}_+^+(M, G/K, Q) \quad \text{(35)}$$

be the orthogonal decomposition of the tangent space of $\mathcal{M}_+^+(M, G, P)$ at $[A, \phi]$ into the tangent space and the normal space of the submanifold $\mathcal{M}_+^+(M, G/K, Q)$ at the same point.
Let \([A, \phi] \in \mathcal{M}_\pm(M, G/K, Q)\). The formal tangent vector space \(T_{[A, \phi]}\mathcal{M}_\pm(M, G, P)\) is a vector space of all elements \((\alpha, \eta) \in \Omega^1(g_P) \oplus \Omega^1(g_P)\) satisfying

\[
\begin{align*}
  d_A \alpha &\pm [\phi \wedge \eta] = 0, \\
  d_A \eta + [\phi \wedge \alpha] &= 0, \\
  d_A \ast \eta + [\ast \phi \wedge \alpha] &= 0, \\
  d_A \ast \alpha + [\phi \wedge \ast \eta] &= 0.
\end{align*}
\]

(36)±

If we decompose \(\alpha\) and \(\eta\) as \(\alpha = \alpha_t + \alpha_m\) and \(\eta = \eta_t + \eta_m\) relative to the canonical decomposition \(g = t + m\), then we have

\[
(\alpha, \eta) = (\alpha_t, \eta_m) + (\alpha_m, \eta_t).
\]

(37)

Since

\[
\begin{align*}
  d_A \alpha &\pm [\phi \wedge \eta] = d_A \alpha_t + d_A \alpha_m \pm [\phi \wedge \eta_t] \pm [\phi \wedge \eta_m] = 0, \\
  d_A \eta + [\phi \wedge \alpha] = d_A \eta_t + d_A \eta_m + [\phi \wedge \alpha_t] + [\phi \wedge \alpha_m] = 0, \\
  d_A \ast \eta + [\ast \phi \wedge \alpha] = d_A \ast \eta_t + d_A \ast \eta_m + [\ast \phi \wedge \alpha_t] + [\ast \phi \wedge \alpha_m] = 0, \\
  d_A \ast \alpha + [\phi \wedge \ast \eta] = d_A \ast \alpha_t + d_A \ast \alpha_m + [\phi \wedge \ast \eta_t] + [\phi \wedge \ast \eta_m] = 0,
\end{align*}
\]

(38)

we see that \((\alpha_t, \eta_m)\) satisfies

\[
\begin{align*}
  d_A \alpha_t &\pm [\phi \wedge \eta_m] = 0, \\
  d_A \eta_m + [\phi \wedge \alpha_t] &= 0, \\
  d_A \ast \eta_m - [\phi \wedge \ast \alpha_t] = 0, \\
  d_A \ast \alpha_t + [\phi \wedge \ast \eta_m] &= 0
\end{align*}
\]

(39)±

and \((\alpha_m, \eta_t)\) satisfies

\[
\begin{align*}
  d_A \alpha_m &\pm [\phi \wedge \eta_t] = 0, \\
  d_A \eta_t + [\phi \wedge \alpha_m] = 0, \\
  d_A \ast \eta_t - [\phi \wedge \ast \alpha_m] = 0, \\
  d_A \ast \alpha_m - [\phi \wedge \ast \eta_t] = 0.
\end{align*}
\]

(40)±

In the decomposition (35)±, the tangent vector space \(T_{[A, \phi]}\mathcal{M}_\pm(M, G/K, Q)\) is a vector space of all elements \((\alpha, \eta) \in \Omega^1(g_P) \oplus \Omega^1(g_P)\) satisfying

\[
\begin{align*}
  d_A \alpha &\pm [\phi \wedge \eta] = 0, \\
  d_A \eta + [\phi \wedge \alpha] = 0, \\
  d_A \ast \eta + [\ast \phi \wedge \alpha] = 0, \\
  d_A \ast \alpha + [\phi \wedge \ast \eta] = 0.
\end{align*}
\]

(41)±
and the normal vector space $T_{[\alpha, \varphi]}^K \mathcal{M}_\pm(M, G/K, Q)$ is a vector space of all elements $(\beta, \zeta) \in \Omega^1(m_P) \oplus \Omega^1(f_P)$ satisfying

\[
\begin{align*}
(d_A \beta \pm [\varphi \wedge \zeta] &= 0, \\
(d_A \zeta + [\varphi \wedge \beta] &= 0, \\
(d_A \ast \zeta - [\varphi \wedge \ast \beta] &= 0, \\
(d_A \ast \beta + [\varphi \wedge \ast \zeta] &= 0.
\end{align*}
\]

Theorem 1. Let $G$ be a compact Lie group and let $P$ be a principal $G$-bundle over a compact Riemann surface $M$. For each compact symmetric pair $(G, K, \sigma)$ with the same $G$ and each $Q \in C^\infty(M; P \times_G (G/K))/G_P,$

1. $\mathcal{M}_-(M, G, K, Q)$ is a complex submanifold of $\mathcal{M}_-(M, G, P)$ with respect to $J_1$.
2. $\mathcal{M}_-(M, G, K, Q)$ is a totally real submanifold of $\mathcal{M}_-(M, G, P)$ with respect to $J_2$ and $J_3$.
3. $2 \dim \mathcal{M}_-(M, G, K, Q) = \dim \mathcal{M}_-(M, G, P)$.

In particular, $\mathcal{M}_-(M, G, K, Q)$ is a complex Lagrangian submanifold of a hyperKähler moduli space $\mathcal{M}_-(M, G, P)$.

Proof. (1) For each $(\alpha, \eta) \in T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q) \subset \Omega^1(f_Q) \times \Omega^1(m_Q)$, $J_1(\alpha, \eta) = (- \ast \alpha, \ast \varphi) \ast \eta$ satisfies (41)\pm, and hence $J_1(\alpha, \eta) \in T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, P)$. (2) For each $(\alpha, \eta) \in T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q) \subset \Omega^1(f_Q) \times \Omega^1(m_Q)$, $J_2(\alpha, \eta)$ and $J_3(\alpha, \eta)$ belong to $\Omega^1(m_Q) \times \Omega^1(f_Q)$. Hence $J_2(T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q))$ and $J_3(T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q))$ are orthogonal to $T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q)$. (3) In the case where the equations are $(\ast)_\pm$, we have

\[
(\alpha, \eta) = (\alpha_t, \eta_m) + J_2(-\eta_t, \alpha_m).
\]

The above equations (39)\pm, (40)\pm mean that $(\alpha_t, \eta_m), (-\eta_t, \alpha_m) \in T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q).$ Therefore, we obtain

\[
T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, P) = T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q) \oplus J_2(T_{[\alpha, \varphi]} \mathcal{M}_-(M, G, K, Q))).
\]

\[\square\]

Theorem 2. Let $G$ be a compact Lie group and let $P$ be a principal $G$-bundle over a compact Riemann surface $M$. For each compact symmetric pair $(G, K, \sigma)$ with the same $G$ and each $Q \in C^\infty(M; P \times_G (G/K))/G_P$, the maps $\mathcal{M}_\pm(M, G, K, Q) \rightarrow \mathcal{M}_\pm(M, G, P)$ are totally geodesic.
Proof. First we consider the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{A}_Q \times \Omega^1(m_Q) & \xrightarrow{i} & \mathcal{A}_P \times \Omega^1(g_P) \\
\uparrow & & \uparrow \\
\widetilde{\mathcal{M}}_\pm(M, G/K, Q) & \xrightarrow{j} & \widetilde{\mathcal{M}}_\pm(M, G, P) \\
\downarrow \kappa_Q & & \downarrow \rho_P \\
\mathcal{M}_\pm(M, G/K, Q) & \xrightarrow{j} & \mathcal{M}_\pm(M, G, P)
\end{array}
\]

Let \( \gamma \) be a geodesic in \( \mathcal{M}_\pm(M, G/K, Q) \). Then \( \gamma \) lifts to a horizontal geodesic \( \tilde{\gamma} \) in \( \widetilde{\mathcal{M}}_\pm(M, G/K, Q) \). Since \( \tilde{\gamma} \) is a geodesic in \( \widetilde{\mathcal{M}}_\pm(M, G/K, Q) \subset \mathcal{A}_Q \times \Omega^1(m_Q) \), we have \( \tilde{\gamma}'' \in \Omega^1(t_Q) \oplus \Omega^1(m_Q) \). Set \( \tilde{\delta} = j \circ \tilde{\gamma} \). Then we have \( \tilde{\delta}'' \in \Omega^1(t_Q) \oplus \Omega^1(m_Q) \). Since \( T^*_Q \mathcal{M}_\pm(M, G/K, Q) \subset \Omega^1(m_Q) \oplus \Omega^1(t_Q) \), \( \tilde{\delta}'' \) is orthogonal to \( T^*_Q \mathcal{M}_\pm(M, G/K, Q) \). On the other hand, \( \tilde{\gamma}'' \) is orthogonal to \( T^*_Q \mathcal{M}_\pm(M, G/K, Q) \). Thus, by (6.1) \( \tilde{\delta}'' \) is orthogonal to \( T^*_Q \mathcal{M}_\pm(M, G/K, Q) \). Hence, \( \tilde{\delta} \) is a horizontal geodesic in \( \mathcal{M}_\pm(M, G, P) \) and, therefore, \( \delta = j \circ \gamma \) is a geodesic in \( \mathcal{M}_\pm(M, G, P) \). We conclude that the map \( j \) is totally geodesic.

By the equation of Gauss, we obtain the following:

Corollary 1. The smooth part of \( \mathcal{M}_-(M, G/K, Q) \) is a Kähler manifold whose curvature tensor field is the restriction of the curvature tensor field of \( \mathcal{M}_-(M, G, P) \).

Remark 1. It is an interesting problem to describe the Ricci curvature of the Kähler moduli space \( \mathcal{M}_-(M, G/K, Q) \).

7 Gauge-theoretic equations for \( k \)-symmetric spaces

Assume that \((G, K, \sigma)\) is a compact \( k \)-symmetric pair, where \( \sigma \) is an automorphism of \( G \) of order \( k > 2 \). Let

\[ g^C = \bigoplus_{i \in \mathbb{Z}_k} g_i^C \]

be the decomposition of \( g^C \) into eigenspaces of \( \sigma \). Note that \([g_i^C, g_j^C] \subset g_{i+j}^C\). We have the decomposition \( g = \mathfrak{e} + m \) such that

\[ e^C = e_0^C \quad \text{and} \quad m^C = \bigoplus_{i \in \mathbb{Z}_k \setminus \{0\}} g_i^C. \]

Let \( Q \) be a principal \( K \)-bundle over \( M \). Then we define associated vector bundles

\[ g_Q = e_Q \oplus m_Q, \quad g_Q^C = e_Q^C \oplus m_Q^C \]

207
and
\[ \mathfrak{g}_Q^C(g_0^C)_Q \ (m^C)_Q = \bigoplus_{i \in \mathbb{Z}_k \setminus \{0\}} (g_i^C)_Q. \]

Now we define the gauge-theoretic equations for \((A, \phi') \in A_Q \times \Omega^1,0((g_i^C)_Q),\)
\[
\begin{cases}
F(A) + \frac{1}{2} [\phi \wedge \phi]_k = 0, \\
d_A \phi = 0,
\end{cases}
\tag{(*)}^{G/K}_{\pm}
\]

where we set \(\phi = \phi' + \bar{\phi}'\) and \((\cdot)_k\) denotes the \(k\)-part of \((\cdot)\) with respect to the decomposition \(g = \mathfrak{k} + \mathfrak{m}\). This definition is based on the notion of primitive maps into \(k\)-symmetric spaces ([11]).

Set
\[ \tilde{\mathcal{M}}_{\pm}(M, G/K, Q) = \{(A, \phi') \in A_Q \times \Omega^1,0((g_i^C)_Q) \mid (A, \phi) \text{ is a solution to } (*)^{G/K}_{\pm}\}. \]

The moduli spaces of solutions to \((*)^{G/K}_{\pm}\) are defined by
\[ \mathcal{M}_{\pm}(M, G/K, Q) = \tilde{\mathcal{M}}_{\pm}(M, G/K, Q)/K_Q. \]

**Theorem 3.** Let \(G\) be a compact Lie group and let \(P\) be a principal \(G\)-bundle over a compact Riemann surface \(M\). Then for each compact \(k(>2)\)-symmetric pair \((G, K, \sigma)\) with the same \(G\) and each \(Q \in C^\infty(M; P \times_G (G/K))/G_P,\)

1. \(\mathcal{M}_-(M, G/K, Q)\) is a complex submanifold of \(\mathcal{M}_-(M, G, P)\) with respect to \(J_1\).
2. \(\mathcal{M}_-(M, G/K, Q)\) is a totally real submanifold of \(\mathcal{M}_-(M, G, P)\) with respect to \(J_2\) and \(J_3\).

In particular, \(\mathcal{M}_-(M, G/K, Q)\) is a minimal submanifold of \(\mathcal{M}_-(M, G, P)\).

**Remark 2.** Very recently Professor M. Itoh suggested to the author some papers [11], [14], [15] which have interesting relationships with this work. In a later paper, we will discuss the geometry of submanifolds in moduli spaces constructed here, including relations with their papers.

**Acknowledgements**

The author would like to thank M.A. Guest and M. Mukai for their interest in this work and their valuable suggestions. He would also like to thank R. Miyaoka for a long discussion at Submanifold Theory-Yuzawa 1997, which inspired him to find the construction in this paper.
References


7. , Harmonic maps from a 2-torus to the 3-sphere, *J. Differential Geom.* 31 (1990), 627-710.


19. Department of Mathematics, Graduate School of Science, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

ohnita@comp.metro-u.ac.jp

[11], will ding