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Geometry of Lagrangian submanifolds and isoparametric hypersurfaces

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Abstract. In this expository article we shall focus on a relationship between Lagrangian submanifold theory in symplectic geometry and isoparametric hypersurface theory in Riemannian geometry. First we explain some basic notions, invariants and results of Lagrangian submanifolds in symplectic manifolds and in Kähler manifolds. Secondly we discuss theory of isoparametric hypersurfaces in spheres and some necessary results on them. We give attention to Lagrangian submanifolds in complex hyperquadrics obtained from isoparametric hypersurfaces in spheres. The "Gauss map"of an oriented hypersurfaces in a unit sphere provides Lagrangian submanifolds in complex hyperquadrics. The Gauss images of isoparametric hypersurfaces in spheres are the main subject of the author's recent joint work with Hui Ma (Tsingua University, Beijing). We obtain that the Gauss image of a compact oriented isoparametric hypersurface with g distinct constant principal curvatures in a unit sphere $S^{n+1}(1)$ is a compact minimal Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ and a monotone, cyclic Lagrangian submanifold with minimal Maslov number equal to $2n/g$. Moreover we mention further related topics and problems.

Introduction

In this expository article we shall focus on a relationship between Lagrangian submanifold theory in symplectic geometry and isoparametric hypersurface theory in Riemannian geometry.

In symplectic geometry in these twenty years there has been remarkable much

Key words and phrases: Lagrangian submanifold, Isoparameric hypersurface, Minimal Maslov number.

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progress such as Floer cohomology theory for intersections of Lagrangian submanifolds due to K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, (FOOO [9], etc.). Inspired by such progress in symplectic geometry, recently more several developments are made by differential geometers in the study of Lagrangian submanifolds in specific Kähler manifolds, such as complex space forms, Hermitian symmetric spaces, generalized flag manifolds, toric manifolds and so on. Submanifold theory is an area of the longest history in Riemannian geometry, and it can provide various techniques and so many examples to such a study of Lagrangian submanifolds.

The *isoparametric hypersurface* is one of most fundamental and attractive subjects in submanifold theory. An isoparametric hypersurface of a Riemannian manifold M is a level hypersurface for a regular value of a smooth function f on M satisfying the partial differential equations:

$$\begin{cases} \bar{\Delta} f = S(f), \\ \|\bar{\nabla} f\|^2 = T(f). \end{cases} \tag{0.1}$$

Such a function f is called an *isoparametric function*. It is well-known that an isoparametric hypersurface of a real space form M, i.e. a Riemannian manifold of constant sectional curvatures, is nothing but a hypersurface with constant principal curvatures. Nowadays the isoparametric hypersurface theory is a well-developed area as an very interesting class of nice smooth manifolds. The excellent survey articles on isoparametric hypersurfaces are G. Thorbergsson [39], and T. E. Ceci [4].

In this article, first we recall some basic notions, invariants and results of Lagrangian submanifolds in general symplectic manifolds and Lagrangian subamifolds in Kähler manifolds. Secondly we review theory of isoparametric hypersurfaces in spheres and we prepare some fundamental results which are applied to our research.

Now we should notice the construction of Lagrangian submanifolds related to isoparametric hypersurfaces. Especially we shall give attention to Lagrangian submanifolds in complex hyperquadrics obtained from isoparametric hypersurfaces in spheres.

The n-dimensional complex hyperquadrics $Q_n(\mathbf{C})$ is a compact algebraic hypersurface of a complex projective space $\mathbf{C}P^{n+1}$ defined by the homogeneous quadratic equation $(z_0)^2 + (z_1)^2 + \cdots + (z_{n+1})^2 = 0$. It can be identified with the real Grassmann manifold $\tilde{G}r_2(\mathbf{R}^{n+2})$ of oriented 2-dimensional vector subspaces of Euclidean vector space $\mathbf{R}^{n+2}$ in the natural way. It has the homogeneous space expression $Q_n(\mathbf{C}) \cong \tilde{G}r_2(\mathbf{R}^{n+2}) \cong SO(n+2)/(SO(2) \times SO(n))$ and it is a compact rank 2 Hermitian symmetric space equipped with the standard Einstein-Kähler manifold of positive Einstein constant. $Q_1(\mathbf{C}) \cong S^2$ and $Q_2(\mathbf{C}) \cong S^2 \times S^2$ and $Q_n(\mathbf{C})$ is irreducible if $n \geq 3$. Note that the complex hyperquadrics $Q_n(\mathbf{C})$ is considered as a compactification of the tangent vector bundle $TS^n(1)$ of the unit standard sphere $S^n(1)$.

The "Gauss map" $\mathcal{G}$ of oriented hypersurfaces in spheres provides Lagrangian submanifolds in complex hyperquadrics (see Subsection 4.2). The Gauss images of isoparametric hypersurfaces in spheres are the main subject of my recent joint work with Hui Ma (Tsingua University, Beijing). We obtain that the Gauss image $\mathcal{G}(N^n)$ of a compact oriented isoparametric hypersurface N^n with g distinct con-

stant principal curvatures in a unit sphere $S^{n+1}(1)$ is a compact minimal Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ covered by N^n with Deck transformation group $\mathbf{Z}_g$ (see Theorem 4.1). Moreover, $\mathcal{G}(N^n)$ is a compact monotone, cyclic Lagrangian submanifold with minimal Maslov index equal to a natural number $2n/g$ (see Theorem 4.3), and $2n/g$ is even if and only if $\mathcal{G}(N^n)$ is orientable (see Theorem 4.2). Furthermore we will mention related topics and open problems.

This article is organized as follows: In Section 1, we recall the notions and elementary results of the Hamiltonian deformation of a Lagrangian submanifold in a symplectic manifolds, the moment map of Hamiltonian group action and Lagrangian orbits, the Maslov index and minimal Maslov number, the monotone and cyclic properties for Lagrangian submanifolds. In Section 2, we treat Lagrangian submanifolds in a Kähler manifold, especially in an Einstein-Kähler manifold, and describe the definitions and results of the mean curvature form, the Hamiltonian minimality, the Hamiltonian stability, and the integral formula for Maslov index. In Section 3, we review the fundamental structures and known results on isoparametric hypersurfaces in spheres. In Section 4, we shall discuss the basic structures and properties of compact Lagrangian submanifolds in complex hyperquadrics obtained as the Gauss images of isoparametric hypersurfaces in spheres. In Section 5, we shall mention about a construction of special Lagrangian submanifolds in the tangent vector bundle of the standard sphere from isoparametric hypersurfaces in the spheres.

1 Lagrangian submanifolds in a symplectic manifold

1.1 Hamiltonian deformations of Lagrangian submanifolds

Let (M, ω) be a connected symplectic manifold of dimension $2n$ with a symplectic form ω. A smooth immersion (resp. embedding) $\varphi : L \to M$ of a connected smooth manifold L into M is called a *Lagrangian immersion* (resp. *Lagrangian embedding*) if $\dim L = n$ and $\varphi^*\omega = 0$. A submanifold of M satisfying only the condition $\varphi^*\omega = 0$ is called an *isotropic submanifold* of M. We say that L is a immersed (resp. embedded) Lagrangian submanifold in M. If $\varphi : L \to M$ is a Lagrangian immersion, then the vector bundle homomorphism between the normal bundle of φ and the cotangent bundle of L defined by

$$\varphi^{-1}TM/\varphi_*TL \ni v \longmapsto \alpha_v := \omega(v, \cdot) \in T^*L \tag{1.1}$$

is an isomorphism.

The *Lagrangian deformation* of a Lagrangian immersion φ is by definition a one-parameter smooth family of Lagrangian immersions $\varphi_t : L \to M$ $(|t| < \varepsilon)$ with $\varphi_0 = \varphi$. $\{\varphi_t\}$ is a Lagrangian deformation if and only if a 1-form α_{V_t} on L is closed for each t, where $V_t := \frac{\partial}{\partial t}\varphi_t \in C^\infty(\varphi^{-1}TM)$ is the variational vector field of $\{\varphi_t\}$. Moreover, if a 1-form α_{V_t} on L is exact for each t, then such a Lagrangian immersion $\{\varphi_t\}$ is called a *Hamiltonian deformation* of $\varphi = \varphi_0$. Note that if $H^1(L; \mathbf{R}) = \{0\}$, then a Lagrangian deformation and a Hamiltonian deformation are same.

1.2 Moment maps and Lagrangian submanifolds

Suppose that a connected Lie group K has a Hamiltonian group action on (M, ω) with the moment map $\mu_K : M \to \mathfrak{k}^*$. Let $\widetilde{\xi}$ denotes the fundamental vector field on M corresponding to $\xi \in \mathfrak{k}$ defined by

$$(\widetilde{\xi})_x := \frac{d}{dt} \exp(t\xi) \cdot x|_{t=0}$$

for each $x \in M$. The following results are standard and well-known :

Lemma 1.1.

(1) *Let L be a connected isotropic submanifold of (M, ω). If L is invariant under the action of K, then there is an element $\eta \in \mathfrak{z}(\mathfrak{k}^*)$ such that $L \subset \mu_K^{-1}(\eta)$. Here*

$$\mathfrak{z}(\mathfrak{k}^*) := \{\eta \in \mathfrak{k}^* \mid (\mathrm{Ad}^* a)(\eta) = \eta \, (\forall a \in K)\}.$$

When K is compact, we can identify $\mathfrak{z}(\mathfrak{k}^*)$ with the center $\mathfrak{c}(\mathfrak{k})$ of Lie algebra $\mathfrak{k}$.

(2) *An orbit $L = K \cdot p \subset M$ of K is a Lagrangian orbit if and only if $L = K \cdot p \subset M$ is a connected component of $\mu_K^{-1}(\eta)$ for some $\eta \in \mathfrak{z}(\mathfrak{k}^*) \cong \mathfrak{c}(\mathfrak{k})$.*

(3) *Assume that K and M are compact and connected. Then an orbit $L = K \cdot p \subset M$ of K is a Lagrangian orbit if and only if $L = \mu_K^{-1}(\eta)$ for some $\eta \in \mathfrak{z}(\mathfrak{k}^*) \cong \mathfrak{c}(\mathfrak{k})$.*

1.3 Maslov index of monotone and cyclic Lagrangian submanifolds

In this subsection we recall some basic notions and invariants of Lagrangian submanifolds in a symplectic manifold (cf. [26], [28], [31], [32]). The Floer chomology theory of Lagrangian Intersection was investigated in [26], [27], [29] and see FOOO ([9]) for further great developments. In Subsection 2.3 we shall mention some useful results on those invariants for Lagrangian submanifolds in Einstein-Kähler manifolds.

Let (M, ω) be a symplectic manifold and L be a Lagrangian submanifold of M. Let $w : (D^2, \partial D^2) \to (M, L)$ be a smooth map of pairs, where D^2 and ∂D^2 denote a unit open disk of $\mathbf{R}^2$ and a unit circle as the boundary of D^2, respectively. We take an identification $w^{-1}(TM) \cong D^2 \times \mathbf{R}^{2n}$. Set $\mathrm{Lagr}(\mathbf{R}^{2n}) := \{\text{Lagrangian vector subspaces of } \mathbf{R}^{2n}\} = O(2n)/U(n)$. By using the *Maslov class* $\mu \in H^1(O(2n)/U(n); \mathbf{R})$, we define a loop $\tilde{w}$ of $\mathrm{Lagr}(\mathbf{R}^{2n})$ by

$$\tilde{w} : \partial D^2 \ni p \longmapsto T_{w(p)}L \in \Lambda \mathbf{R}^2. \tag{1.2}$$

The *Maslov index* of L is a homomorphism $I_{\mu,L} : \pi_2(M, L) \to \mathbf{Z}$ defined by

$$I_{\mu,L}([w]) := \mu(\tilde{w}) \in \mathbf{Z}. \tag{1.3}$$

$I_{\mu,L}$ is invariant under Lagrangian deformations.

We define the *minimal Maslov number* Σ_L of L by

$$\Sigma_L := \min\{I_{\mu,L}(A) \mid A \in \pi_2(M,L), I_{\mu,L}(A) > 0\} \tag{1.4}$$

Another homomorphism $I_\omega : \pi_2(M,L) \to \mathbf{R}$ is defined by

$$I_\omega([w]) := \int_{D^2} w^*\omega \in \mathbf{R}. \tag{1.5}$$

for any smooth map $w : (D^2, \partial D^2) \to (M,L)$. I_ω is invariant under Hamiltonian deformations, but not invariant under Lagrangian deformations in general.

A Lagrangian submanifold L in a symplectic manifold (M,ω) is called *monotone* if there is a positive constant $\lambda > 0$ such that

$$I_{\mu,L} = \lambda I_\omega. \tag{1.6}$$

Let (M,ω) be a symplectic manifold. The *period group* of (M,ω) is defined by the additive subgroup

$$\Gamma_\omega := \{[\omega](A) \mid A \in H_2(M;\mathbf{Z})\} \subset \mathbf{R}. \tag{1.7}$$

Note that if M is simply connected, then

$$\Gamma_\omega = \{[\omega](u) \mid u : S^2 \longrightarrow M \text{ smooth}\} \subset \mathbf{R}. \tag{1.8}$$

A symplectic manifold (M,ω) is *prequantizable* if Γ_ω is discrete in $\mathbf{R}$, or equivalently there is a non-zero constant γ such that $\left[\frac{\omega}{\gamma}\right]$ is an integral class, i.e. $\left[\frac{\omega}{\gamma}\right] \in i(H^2(M;\mathbf{Z}))$, where i denotes the natural homomorphism $i : H^2(M;\mathbf{Z}) \to H^2(M;\mathbf{R})$ induced by the inclusion $\mathbf{Z} \subset \mathbf{R}$. It is well-known that $\left[\frac{\omega}{\gamma}\right]$ is an integral class if and only if there is a complex line bundle E over M with a $U(1)$-connection ∇ whose curvature form is $2\pi\sqrt{-1}\,\frac{\omega}{\gamma}$. If M is prequantizable, then we can choose a non-negative real number $\gamma = \gamma_\omega$ such that

$$\Gamma_\omega = \gamma_\omega \mathbf{Z}. \tag{1.9}$$

Suppose that a symplectic manifold (M,ω) is prequantizable. A Lagrangian submanifold L of M is called *cyclic* if the additive subgroup

$$\Gamma_{\omega,L} := \{[\omega](B) \mid B \in H_2(M,L;\mathbf{Z})\} \subset \mathbf{R} \tag{1.10}$$

is discrete. If L is cyclic, then we can choose a non-negative real number $\gamma_{\omega,L}$ such that

$$\Gamma_{\omega,L} = \gamma_{\omega,L}\,\mathbf{Z}. \tag{1.11}$$

and there is an integer k such that

$$\gamma_\omega = k\,\gamma_{\omega,L}. \tag{1.12}$$

and moreover, there is an positive integer k such that $\otimes^k(\varphi^{-1}E, \varphi^{-1}\nabla)$ is trivial.

Assume that M is simply connected. Then L is cyclic if and only if there is an integer k such that $\otimes^k(\varphi^{-1}E, \varphi^{-1}\nabla)$ is trivial. Define

$$n_L := \min\{k \in \mathbf{Z} \mid k \geq 1, \otimes^k(\varphi^{-1}E, \varphi^{-1}\nabla) \text{ is trivial}\} = \frac{\gamma_\omega}{\gamma_{\omega,L}} \in \mathbf{Z}. \tag{1.13}$$

2 Lagrangian submanifolds in Kähler manifolds

2.1 Mean curvature form of Lagrangian submanifolds

Suppose that (M, ω, J, g) is a complex n-dimensional Kähler manifold with Kähler form ω, complex structure tensor J and Kähler metric g. Let Ric^M denote the Ricci tensor field of (M, g) and ρ^M denotes the Ricci form of a Kähler manifold (M, ω, J, g) defined by $\rho^M(X, Y) := \mathrm{Ric}^M(JX, Y)$. The first Chern class $c_1(M)$ of (M, J) is an integral class of $H^2(M; \mathbf{R})$ represented by $\dfrac{1}{2\pi}\,\rho^M$.

Let $\varphi : L \to M$ be a Lagrangian immersion into a Kähler manifold. Denote by B and H the second fundamental form and the mean curvature vector field of the immersion φ, respectively. Define a symmetric tensor field S of degree 3 on L by

$$S(X, Y, Z) := \omega(B(X, Y), Z) \quad \text{for each } X, Y, Z \in TL$$

and the *mean curvature form* α_H of φ by

$$\alpha_H(X) := \omega(H, X) \quad \text{for each } X \in TL. \tag{2.1}$$

The mean curvature form α_H satisfies the identity ([6]):

$$d\,\alpha_H(X) = \varphi^*\rho^M. \tag{2.2}$$

The condition $H = 0$, equivalently $\alpha_H = 0$, is the usual minimal submanifold condition. In that case, we call it a *minimal Lagrangian submanifold* in a Kähler manifold.

2.2 Hamiltonian minimality and Hamiltonian stability of Lagrangian submanifolds

The notion of Hamiltonian minimality and Hamiltonian stability for Lagrangian submanifolds in a Kähler manifold was introduced and investigated first by Y.-G. Oh ([23], [24], [25], [28]). For the simplicity we assume that L is compact without boundary. A Lagrangian immersion $\varphi : L \to (M, \omega, J, g)$ is called *Hamiltonian minimal* if for any Hamiltonian deformation $\varphi_t : L \to M$ with $\varphi = \varphi_0$

$$\frac{d}{dt}\mathrm{Vol}(L, \varphi_t^* g)|_{t=0} = 0.$$

The Hamiltonian minimal equation is given by

$$\delta\,\alpha_H = 0, \tag{2.3}$$

where δ denotes the codifferential operator to the exterior differentiation d on L with respect to $\varphi^* g$.

A Hamiltonian minimal Lagrangian immersion $\varphi_t : L \to (M, \omega, J, g)$ is called *Hamiltonian stable* if for any Hamiltonian deformation $\varphi : L \to M$ with $\varphi = \varphi_0$

$$\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi_t^* g)|_{t=0} \geq 0. \tag{2.4}$$

The Lagrangian version of the second variational formula was given in [25]. The null space of the Morse index form of the second variational formula for a compact Hamiltonian minimal Lagrangian submanifold in a Kähler manifold is the vector space of all solutions to the linearized Hamiltonian minimal equation. We denote its dimension by $n(\varphi)$ and call it the *nullity* of a Hamiltonian minimal Lagrangian immersion φ.

For each holomorphic Killing vector field X on the ambient Kähler manifold M, the corresponding 1-form $\alpha_X := \omega(X, \cdot)$ on M is closed. If we assume that $\dim H^1(M; \mathbf{R}) = 0$ or M is simply connected, then α_X is exact, that is, $\alpha_X = d\,f$ for some $f \in C^\infty(M)$. Thus a holomorphic Killing vector field X is a Hamitonian vector field and X generates a Hamiltonian deformation preserving the metric and thus the volume of any Lagrangian submanifold $\varphi : L \to M$. Hence $\varphi^*\alpha_X = d(f \circ \varphi)$ is a solution to the linearized Hamiltonian minimal equation. Denote by

$$n_{\mathrm{hk}}(\varphi) := \dim\{\, \varphi^*\alpha_X \mid X \text{ is a holomorphic Killing vector field on } M \,\},$$

which is called the *holomorphic Killing nullity* of a Hamiltonian minimal Lagrangian immersion φ In general $n_{\mathrm{hk}}(\varphi) \leq n(\varphi)$ and a Hamiltonain minimal Lagrangian immersion φ is said to be *Hamiltonain rigid* if $n_{\mathrm{hk}}(\varphi) = n(\varphi)$. A Hamiltonian minimal Lagrangian immersion is called *strictly Hamiltonian stable* if it is Hamiltonian stable and $n_{\mathrm{hk}}(\varphi) = n(\varphi)$.

In the case when (M, ω, J, g) is an Einstein-Kähler manifold with Einstein constant κ, from the second variational formula we know that a compact minimal Lagrangian Lagrangian submanifold immersed in M if and only if the first (positive) eigenvalue λ_1 of the Laplacian on functions is greater than or equal to κ. When a Lagrangian submanifold embedded in a Kähler manifold M is obtained as a Lagrangian orbit of a compact Lie subgroup of the automorphism group $\mathrm{Aut}(M, \omega, J, g)$, we call it a compact *homogeneous Lagrangian submanifold* in M.

Proposition 2.1 ([18]). *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is Hamiltonian minimal.*

The classification of compact homogeneous Lagrangian submanifolds in a specific Kähler manifold is an interesting and important problem in the sense of Riemannian and Symplectic geometry:

Problem. Classify all compact homogeneous Lagrangian submanifolds in in a specific Kähler manifold such as complex projective spaces, complex Euclidean spaces, complex space forms, Hermitian symmetric spaces and generalized flag manifolds with an invariant symplectic form.

2.3 Minimal Maslov number of cyclic Lagrangian submanifolds in Einstein-Kähler manifolds

Theorem 2.1 ([28]). *Let L be a Lagrangian submanifold in an Einstein-Kähler manifold (M, ω, J, g). Then the real cohomology class $[\alpha_H]$ of the mean curvature form of L in $H^1(L; \mathbf{R})$ is globally invariant under every Hamiltonian deformation of L.*

Hajime Ono showed the following integral formula of Maslov index $I_{\mu,L}$ of a Lagrangian submanifold in a Kähler manifold and it enables us to improve some results of Y. G. Oh ([28]).

Theorem 2.2 ([31]). *Let L be a Lagrangian submanifold in a Kähler manifold (M, ω, J, g). For each smooth map of pairs $w : (D^2, \partial D^2) \to (M, L)$, it holds*

$$I_{\mu,L}([w]) = \frac{1}{\pi}\int_{D^2} w^*\rho_M + \frac{1}{\pi}\int_{\partial D^2} (w|_{\partial D^2})^*\alpha_H. \tag{2.5}$$

The following two theorems are applications of his integral formula.

Theorem 2.3 ([31]). *Suppose that (M, ω, J, g) is a simply connected Einstein-Kähler manifold with positive Einstein constant and L is a compact Lagrangian submanifold in M. Then L is monotone if and only if $[\alpha_H] = 0$ in $H^1(L; \mathbf{R})$.*

We see that if L is a compact monotone Hamiltonian minimal Lagrangian submanifold in M, then L must be minimal, and any compact minimal Lagrangian submanifold in a simply connected Einstein-Kähler manifold with positive Einstein constant is monotone.

Define

$$\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M; \mathbf{Z}), c_1(M)(A) > 0\} \in \mathbf{Z},$$
$$\gamma_{c_1,L} := \min\{c_1(M)(B) \mid B \in H_2(M, L; \mathbf{Z}), c_1(M)(B) > 0\} \in \mathbf{Z}.$$

The following formula (2.6) will be essentially used in the proof of Theorem 4.3.

Theorem 2.4 ([31]). *Suppose that (M, ω, J, g) is a simply connected Einstein-Kähler manifold with positive Einstein constant. If L is a compact monotone Lagrangian submanifold in M, then L is cyclic and it holds the formula*

$$n_L \, \Sigma_L \; = \; 2 \, \gamma_{c_1}. \tag{2.6}$$

Remark. The number γ_{c_1} for each Hermitian symmetric space M of compact type is given as follows ([2, p.521]): If $M = SU(p+q)/S(U(p) \times U(q))$, then $\gamma_{c_1} = p+q$. If $M = SO(2p)/U(p)$, then $\gamma_{c_1} = 2p-2$. If $M = Sp(p)/U(p)$, then $\gamma_{c_1} = p+1$. If $M = SO(p+2)/(SO(2) \times SO(p))\,(p \geq 2)$, then $\gamma_{c_1} = p$. If $M = E_6/T^1 \cdot Spin(10)$, then $\gamma_{c_1} = 12$. If $M = E_7/T^1 \cdot E_6$, then $\gamma_{c_1} = 18$.

3 Isoparametric hypersurfaces in spheres

3.1 Structure theory

In this subsection we shall explain briefly the fundamental structure of isoparametric hypersurfaces in the unit standard sphere due to Elie Cartan and H. F. Münzner ([21], [22], cf. G. Thorbergsson [39], T. E. Cecil [4]).

Let N^n be a connected oriented hypersurface embedded in the unit standard sphere $S^{n+1}(1)$ with g distinct constant principal curvatures $k_1 > k_2 > \cdots > k_g$ and the corresponding multiplicities $m_\alpha \, (\alpha = 1, \cdots, g)$. Let $\mathbf{x}(p)$ and $\mathbf{n}(p)$ denote the position vector from the origin O and the unit normal vector in $S^{n+1}(1)$ at $p \in N^n$. Let $\mathbf{x}$ denote the position vector of N^n and $\mathbf{n}$ the unit normal vector field to N^n in $S^{n+1}(1)$ compatible with the orientation.

Theorem 3.1 ([21]). *Set $k_\alpha = \cot\theta_\alpha \, (\alpha = 1, \cdots, g)$ with $0 < \theta_1 < \cdots < \theta_g < \pi$. Then the following properties hold:*

$$\theta_\alpha = \theta_1 + (\alpha - 1)\frac{\pi}{g} \quad (\alpha = 1, \cdots, g), \tag{3.1}$$

$$m_\alpha = m_{\alpha+2} \qquad \textit{indices modulo } g, \tag{3.2}$$

$$n = \begin{cases} \dfrac{(m_1 + m_2)\, g}{2} & (g \geq 2), \\ m_1 & (g = 1). \end{cases} \tag{3.3}$$

Thus if g is odd, then $m_1 = m_2 = \cdots = m_g$. Since $\theta_1 + (\alpha - 1)\dfrac{\pi}{g} < \pi = \dfrac{g\pi}{g}$, we have $0 < \theta_1 < \dfrac{(g - \alpha + 1)\pi}{g}$, and particularly $0 < \theta_1 < \dfrac{\pi}{g}$.

We define a smooth function V in a tubular neighborhood A of N^n by

$$V(q) := \cos(g\, t(q))$$

for each $q \in A$. Here $\theta_1 - t(q)$ is equal to the distance from a point q to N^n in $S^{n+1}(1)$. Define

$$\bar{F}(q) := r^g \cos(g\, t(q)) = r^g\, V(q)$$

for each $r > 0$ and each $q \in A$. The function $\bar{F}$ on an open cone $\bigcup_{r>0} rA \subset \mathbf{R}^{n+2}$ extends to a homogeneous polynomial $F : \mathbf{R}^{n+2} \to \mathbf{R}$ of degree g, the so called *Cartan-Münzner polynomial*, satisfying the differential equations:

$$\begin{cases} \Delta F = c\, r^{g-2}, \\ \|\mathrm{grad} F\|^2 = g^2 r^{2g-2}, \end{cases} \tag{3.4}$$

where $c := g^2 \dfrac{m_2 - m_1}{2}$ and $r = \|\mathbf{x}\|^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 + \cdots + (x_{n+2})^2$. Moreover $V = F|_{S^{n+1}(1)}$ satisfies the isoparametric function equation in $S^{n+1}(1)$:

$$\begin{cases} \bar{\Delta}\, V \;=\; -g(g+n)\, V \;+ c \;=\; S(V), \\ \|\bar{\nabla} V\|^2 \;=\; g^2(1 - V^2) \;=\; T(V), \end{cases} \tag{3.5}$$

where $\bar{\nabla}$ and $\bar{\Delta}$ denote the covariant differentiation of the Levi-Civita connection and the Laplace-Beltrami operator of $S^{n+1}(1)$. In particular, $V = F|_{S^{n+1}(1)}$ is an *isoparametric function* on $S^{n+1}(1)$. As $0 < \theta_1 < \frac{\pi}{g}$, we have $\cos(g\theta_1) \neq \pm 1$ and thus $\cos(g\theta_1)$ is a regular value of the function V on $S^{n+1}(1)$. The level hypersurface $V^{-1}(\cos(g\theta_1))$ is a compact *connected* orientable *isoparametric hypersurface* embedded in $S^{n+1}(1)$ and N^n is an open subset of $V^{-1}(\cos(g\theta_1))$. Each $N_\pm := V^{-1}(\pm 1)$ is a compact *connected* minimal submanifold embedded in $S^{n+1}(1)$ of codimension at least 2, which is called a *focal submanifold* of an isoparametric hypersurface N^n.

Suppose that N^n is a compact connected oriented isoparametric hypersurface embedded in $S^{n+1}(1)$. From the above argument we can assume that $N^n = V^{-1}(\cos(g\theta_1))$ and $\mathbf{n}(p) = \dfrac{(\mathrm{grad} V)_{\mathbf{x}(p)}}{\|(\mathrm{grad} V)_{\mathbf{x}(p)}\|}$ for each $p \in N^n$. Then we have

Lemma 3.1. *For each $p \in N^n$,*

$$\cos\theta\, \mathbf{x}(p) + \sin\theta\, \mathbf{n}(p) \;\in\; V^{-1}(\cos(g\theta_1)) = N^n$$

if and only if

$$\theta \;=\; \frac{2\pi(\alpha - 1)}{g} \quad \textit{or} \quad 2\theta_1 + \frac{2\pi(\alpha - 1)}{g} \qquad \textit{for some } \alpha = 1, \cdots, g.$$

The following results are famous and important results of Münzner and Abresch.

Theorem 3.2 ([22]).

(1) *g must be $1, 2, 3, 4$ or 6.*

(2) *If $g = 6$, then $m_1 = m_2$.*

Theorem 3.3 ([1]). *If $g = 6$, then $m_1 = m_2 = 1$ or 2.*

3.2 Minimal isoparametric hypersurfaces in spheres

It is well-known that there exists only one *minimal* isoparametric hypersurface N^n in each isoparametric family of $S^{n+1}(1)$. From [21] we easily compute its principal curvatures as follows (cf. [30, p.265]):

Proposition 3.1.

(1) *If $g = 1$, then $k_1 = 0$.*

(2) *If $g = 2$, then $k_1 = \sqrt{\dfrac{m_2}{m_1}}$ and $k_2 = \sqrt{\dfrac{m_1}{m_2}}$.*

(3) *If $g = 3$, then $k_1 = \sqrt{3}$, $k_2 = 0$ and $k_3 = -\sqrt{3}$.*

(4) *If $g = 4$, then*

$$\kappa_1 = \frac{\sqrt{m_1 + m_2} + \sqrt{m_2}}{\sqrt{m_1}},\ \kappa_2 = \frac{\sqrt{m_1 + m_2} - \sqrt{m_1}}{\sqrt{m_2}},$$
$$\kappa_3 = -\frac{\sqrt{m_1 + m_2} - \sqrt{m_2}}{\sqrt{m_1}},\ \kappa_4 = -\frac{\sqrt{m_1 + m_2} + \sqrt{m_1}}{\sqrt{m_2}}.$$

(5) *If $g = 6$, then*

$$\kappa_1 = 2 + \sqrt{3},\ \kappa_2 = 1,\ \kappa_3 = 2 - \sqrt{3},$$
$$\kappa_4 = -(2 - \sqrt{3}),\ \kappa_5 = -1,\ \kappa_6 = -(2 + \sqrt{3}).$$

3.3 Homogeneous isoparametic hypersurfaces in spheres

Due to W.-Y. Hsiang and H. B. Lawson, Jr. ([14]) and R. Takagi and T. Takahashi ([38]), any homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of the isotropy representation of a Riemannian symmetric pair (U, K) of rank 2

Table 1: Homogeneous isoparametric hypersurfaces in spheres

g	*Type*	(U,K)	$\dim N^n$	$m_1,\ m_2$	$N^n=K/K_0$
1	$S^1\times$ BDII	$(S^1\times SO(n+2),SO(n+1))$ $n\geq 1,\ [\mathbf{R}\oplus A_1]$	n	n	S^n
2	BDII$\times$ BDII	$(SO(p+2)\times SO(n+2-p),$ $SO(p+1)\times SO(n+1-p))$ $1\leq p\leq n-1, [A_1\oplus A_1]$	n	$p, n-p$	$S^p\times S^{n-p}$
3	AI_2	$(SU(3),SO(3))\,[A_2]$	3	1,1	$\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3)\times SU(3),SU(3))\,[A_2]$	6	2,2	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6),Sp(3))\,[A_2]$	12	4,4	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	$(E_6,F_4)\,[A_2]$	24	8,8	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5)\times SO(5),SO(5))\,[B_2]$	8	2,2	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2),S(U(2)\times U(m)))$ $m\geq 2, [BC_2](m\geq 3),[B_2](m=2)$	$4m-2$	2, $2m-3$	$\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$
4	BDI_2	$(SO(m+2),SO(2)\times SO(m))$ $m\geq 3, [B_2]$	$2m-2$	1, $m-2$	$\frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$
4	CII_2	$(Sp(m+2),Sp(2)\times Sp(m))$ $m\geq 2, [BC_2](m\geq 3),[B_2](m=2)$	$8m-2$	4, $4m-5$	$\frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$
4	DIII_2	$(SO(10),U(5))\,[BC_2]$	18	4,5	$\frac{U(5)}{SU(2)\times SU(2)\times U(1)}$
4	EIII	$(E_6,U(1)\cdot Spin(10))\,[BC_2]$	30	6,9	$\frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$
6	$\mathfrak{g}_2$	$(G_2\times G_2,G_2)\,[G_2]$	12	2,2	$\frac{G_2}{T^2}$
6	G	$(G_2,SO(4))\,[G_2]$	6	1,1	$\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$

3.4 Isoparametric hypersurfaces of OT-FKM type

The Clifford construction of non-homogeneous isoparametric hypersurfaces in spheres was discovered first by [33], [34] and generalized by [8]. It is another important class of isoparametric hypersurfaces in the unit standard sphere, which are called "isoparametric hypersurfaces of OT-FKM type".

Let $Cl(\mathbf{R}^{m-1})$ be the Clifford algebra over Euclidean space $(\mathbf{R}^{m-1}, \langle\ ,\ \rangle)$. A representation of $Cl(\mathbf{R}^{m-1})$ on $\mathbf{R}^l$ of degree l is an algebra homomorphism

$$Cl(\mathbf{R}^{m-1}) \longrightarrow M(l;\mathbf{R}).$$

Note that

$$Cl(\mathbf{R}^{m-1}) \cong Cl_0(\mathbf{R}^m) \supset Spin(m).$$

Then we can choose $E_1, \cdots, E_{m-1} \in O(l)$ such that

$$E_i^2 = -\mathrm{I},\ E_iE_j = -E_jE_i,\ i \neq j\,.$$

Denote by $\mathfrak{h}(\mathbf{R}^{2l})$ the vector space of symmetric endomorphisms on $\mathbf{R}^{2l}$. Define $P_0, P_1, \cdots, P_m \in \mathfrak{h}(\mathbf{R}^{2l})$ by

$$P_0(u,v) := (u,-v),\ P_1(u,v) := (v,u),\ P_{1+i}(u,v) := (E_iv, -E_iu)$$

$Cl(\mathbf{R}^{m-1})$ has an irreducible representation of degree l if and only if $l = \delta(m)$ as in the table:

m	$Cl(\mathbf{R}^{m-1})$	$\delta(m)$
1	$\mathbf{R}$	1
2	$\mathbf{C}$	2
3	$\mathbf{H}$	4
4	$\mathbf{H} \oplus \mathbf{H}$	4
5	$\mathbf{H}(2) = M(2,\mathbf{H})$	8
6	$\mathbf{C}(4) = M(4,\mathbf{C})$	8
7	$\mathbf{R}(8) = M(8,\mathbf{R})$	8
8	$\mathbf{R}(8) \oplus \mathbf{R}(8) = M(8,\mathbf{R}) \oplus M(8,\mathbf{R})$	8
$k+8$	$M(Cl(\mathbf{R}^{k-1}), 16)$	$16\delta(k)$

Any reducible representation of $Cl(\mathbf{R}^{m-1})$ of degree l for $l = k\,\delta(m)$ with $k > 1$ is a direct sum of k irreducible representations of $Cl(\mathbf{R}^{m-1})$ on $\mathbf{R}^{\delta(m)}$.

The system $(P_1, \cdots, P_m)$ is called a *Clifford system* of $\mathbf{R}^{2l}$. Set

$$m_1 := m, \quad m_2 := l - m - 1 = k\,\delta(m) - m - 1. \tag{3.6}$$

Then the polynomial function $F : \mathbf{R}^{2l} \to \mathbf{R}$ defined by

$$F(x) := \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_ix, x\rangle^2 \tag{3.7}$$

Table 2: Multiplicities of principal curvatures of isoparametric hypersurfaces of OT-FKM type

$k\backslash\delta(m)$	1	2	4	4	8	8	8	8	16	32	$\cdots$
1	–	–	–	–	(5,2)	(6,1)	–	–	(9,6)	(10,21)	$\cdots$
2	–	(2,1)	(3,4)	(4,3)	(5,10)	(6,9)	(7,8)	(8,7)	(9,22)	(10,53)	$\cdots$
3	(1,1)	(2,3)	(3,8)	(4,7)	(5,18)	(6,17)	(7,16)	(8,15)	(9,38)	(10,85)	$\cdots$
4	(1,2)	(2,5)	(3,12)	(4,11)	(5,26)	(6,25)	(7,24)	(8,23)	(9,54)	$\cdot$	$\cdots$
5	(1,3)	(2,7)	(3,16)	(4,15)	(5,34)	(6,33)	(7,32)	(8,31)	$\cdot$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

is a Cartan-Münzner polynomial, that is, F satisfies the Cartan-Münzner differential equations (3.4) for $g = 4$ and thus F provides a family of isoparametric hypersurfaces with 4 distinct principal curvatures and multiplicities $(m_1, m_2) = (m, l - m - 1)$ in the $(2l - 1)$-dimensional unit standard sphere $S^{2l-1}(1) \subset \mathbf{R}^{2l} = \mathbf{R}^l \oplus \mathbf{R}^l$, which is called an *isoparametric hypersurface of OT-FKM type*.

Remark. An isoparametric hypersurface of OT-FKM type in the sphere is homogeneous if and only if it is of type AIII_2, BDI_2, CII_2 or EIII.

3.5 Classification problem of isoparametic hypersurfaces in spheres

Now all known examples of isoparametic hypersurfaces in spheres are homogeneous isoparametric hypersurfaces and isoparametric hypersurfaces of OT-FKM type. It is conjectured that they are ALL isoparametic hypersurfaces in spheres.

Let $N^n \subset S^{n+1}(1)$ be an isoparametric hypersurface in the sphere. If $g = 1, 2$ or 3, then N^n is homogeneous (E. Cartan). If $g = 6$ and $m = 1$, then N^n it is homogeneous (Dorfmeister and Neher [7], R. Miyaoka [20]). The case $g = 6$ and $m = 2$ is investigated by R. Miyaoka. If $g = 4$, then the multiplicities (m_1, m_2) must be the same as those in examples of homogeneous one and of OT-FKM type (Stolz [37]), and N^n must be homogeneous or of OT-FKM type except for the cases $(m_1, m_2) = (4, 5), (3, 4), (6, 9), (7, 8)$ (Cecil, Chi and Jensen [5], Immervoll [15]). The research of the remaining cases is still in progress by Q.-S. Chi.

4 Lagrangian submanifolds in complex hyperquadrics obtained from isoparametric hypersurfaces

4.1 Complex hyperquadrics

The *complex hyperquadric*

$$Q_n(\mathbf{C}) \cong \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong SO(n+2)/(SO(2) \times SO(n))$$

is a compact Hermitian symmetric space of rank 2, where

$$Q_n(\mathbf{C}) := \{[\mathbf{z}] \in \mathbf{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\},$$

$$\widetilde{Gr}_2(\mathbf{R}^{n+2}) := \{W \mid \text{oriented 2-dimensional vector subspace of } \mathbf{R}^{n+2}\}.$$

The identification between $Q_n(\mathbf{C})$ and $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ is given by

$$\mathbf{C}P^{n+1} \supset Q_n(\mathbf{C}) \ni [\mathbf{a} + \sqrt{-1}\,\mathbf{b}] \longleftrightarrow W = \mathbf{a} \wedge \mathbf{b} \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \subset \bigwedge^2 \mathbf{R}^{n+2}.$$

Here $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis of W compatible with its orientation. If $n = 2$, then $Q_2(\mathbf{C}) \cong S^2 \times S^2$. If $n \geq 3$, then $Q_n(\mathbf{C})$ is irreducible.

Note that the standard Kähler metric $g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ on $Q_n(\mathbf{C}) \cong \widetilde{Gr}_2(\mathbf{R}^{n+2})$ induced from the standard inner product of $\mathbf{R}^{n+2}$ is Hermitian symmetric and an Einstein-Kähler metric with the Einstein constant κ equal to n.

4.2 Gauss map of an oriented hypersurface in a sphere

Let $N^n \hookrightarrow S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an oriented hypersurface immersed or embedded in the $(n+1)$-dimensional unit standard sphere.

Let $\mathbf{x}$ and $\mathbf{n}$ denote the position vector of points of N^n and the unit normal vector field of N^n in $S^{n+1}(1)$, respectively. The *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto [\mathbf{x}(p) + \sqrt{-1}\,\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p) \in Q_n(\mathbf{C}) \cong \widetilde{Gr}_2(\mathbf{R}^{n+2})$$

is always a Lagrangian immersion.

Proposition 4.1 ([17], [18])**.**

(1) *Let $N_1, N_2 \subset S^{n+1}(1)$ be oriented hypersurfaces in the unit sphere. Then N_1 is parallel to N_2 if and only if $\mathcal{G}(N_1) = \mathcal{G}(N_2)$.*

(2) *The Gauss maps for any deformation of an oriented hypersurface N^n in $S^{n+1}(1)$ gives a Hamiltonian deformation of the Gauss map $\mathcal{G}$. Conversely, a small Hamiltonian deformation of the Gauss map $\mathcal{G}$ corresponds to a deformation of the oriented hypersurface N^n in $S^{n+1}(1)$.*

Let κ_i $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$. Set $\kappa_i = \cot\theta_i$ $(i = 1, \cdots, n)$, where $0 < \theta_i < \pi$. Choose an orthonormal frame $\{e_i\}$ on $N^n \subset S^{n+1}(1)$ such that the second fundamental form h of N^n in $S^{n+1}(1)$ with respect to $\mathbf{n}$ is diagonalized as $h(e_i, e_j) = \kappa_i\,\delta_{ij}$ and let $\{\theta^i\}$ be its dual coframe. Then the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ on N^n by the Gauss map $\mathcal{G}$ is expressed as

$$\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})} = \sum_{i=1}^{n} (1 + \kappa_i^2)\,\theta^i \otimes \theta^i. \tag{4.1}$$

Let H denote the mean curvature vector field of $\mathcal{G}$. Then B. Palmer showed that the mean curvature form of the Gauss map $\mathcal{G}$ is expressed in terms of the principal curvatures of N^n as follows:

Lemma 4.1 ([35])**.**

$$\alpha_{\mathrm{H}} = d\left(\mathrm{Im}\left(\log \prod_{i=1}^{n} (1 + \sqrt{-1}\kappa_i)\right)\right) = -d\left(\sum_{i=1}^{n} \theta_i\right).$$

Remark. The last equality was pointed by Jianquan Ge (Beijing Normal University).

Particularly, if $N^n \subset S^{n+1}(1)$ is an oriented hypersurface in $S^{n+1}(1)$ with constant principal curvatures, then the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.

4.3 Gauss images of isoparametric hypersurfaces

Suppose that $N^n \hookrightarrow S^{n+1}(1) \subset \mathbf{R}^{n+2}$ is a compact connected oriented isoparametric hypersurface embedded in the $(n+1)$-dimensional unit standard sphere. In this subsection we use the same notations as in Subsection 3.1.

By Lemma 3.1, for each $p \in N^n$ a normal geodesic (a great circle) $\gamma = \gamma(\theta)$ defined by

$$\mathbf{x}_\theta(p) := \cos\theta\,\mathbf{x}(p) + \sin\theta\,\mathbf{n}(p)$$

has intersection with N^n at $2g$ points as

$$\gamma \cap N^n = \{\mathbf{x}_\theta(p) \mid \theta = \frac{2\pi(\alpha-1)}{g} \text{ or } 2\theta_1 + \frac{2\pi(\alpha-1)}{g} \text{ for some } \alpha = 1, \cdots, g\}$$

For each $\mathbf{x}_\theta(p) \in \gamma \cap N^n$, let $p_\theta \in N^n$ be a point of N^n with the position vector $\mathbf{x}_\theta(p) = \mathbf{x}(p_\theta)$. If $\theta = \dfrac{2\pi(\alpha-1)}{g}$ $(\alpha = 1, \cdots, g)$, then

$$\mathcal{G}(p_\theta) = \mathbf{x}(p_\theta) \wedge \mathbf{n}(p_\theta) = \mathbf{x}(p) \wedge \mathbf{n}(p) = \mathcal{G}(p).$$

If $\theta = 2\theta_1 + \dfrac{2\pi(\alpha-1)}{g}$ $(\alpha = 1, \cdots, g)$, then

$$\mathcal{G}(p_\theta) = \mathbf{x}(p_\theta) \wedge \mathbf{n}(p_\theta) = \mathbf{x}(p) \wedge (-\mathbf{n}(p)) = -\mathbf{x}(p) \wedge \mathbf{n}(p) \neq \mathcal{G}(p).$$

Conversely, if $\mathcal{G}(p) = \mathcal{G}(q)$ for $p, q \in N^n$, then $q = p_\theta$ for some $\theta = \dfrac{2\pi(\alpha-1)}{g}$ $(\alpha = 1, \cdots, g)$. Indeed, since $\mathbf{x}(p) \wedge \mathbf{n}(p) = \mathbf{x}(q) \wedge \mathbf{n}(q)$, we can express as

$$\begin{aligned} \mathbf{x}(q) &= \cos\psi\, \mathbf{x}(p) + \sin\psi\, \mathbf{n}(p), \\ \mathbf{n}(q) &= -\sin\psi\, \mathbf{x}(p) + \cos\psi\, \mathbf{n}(p) \end{aligned}$$

for some $0 \leq \psi < 2\pi$, it follows that $\psi = \dfrac{2\pi(\alpha-1)}{g}$ or $2\theta_1 + \dfrac{2\pi(\alpha-1)}{g}$ for some $\alpha = 1, \cdots, g$. If $\psi = 2\theta_1 + \dfrac{2\pi(\alpha-1)}{g}$, then $\mathcal{G}(p) \neq \mathcal{G}(q)$, a contradiction. Hence $\psi = \dfrac{2\pi(\alpha-1)}{g}$ for some $\alpha = 1, \cdots, g$.

From these observations we can define the free action of a finite cyclic group $\mathbf{Z}_g$ of order g on N^n. A map

$$\nu : N^n \ni p \longmapsto \cos\left(\frac{2\pi}{g}\right) \mathbf{x}(p) + \sin\left(\frac{2\pi}{g}\right) \mathbf{n}(p) \in N^n \tag{4.2}$$

is a diffeomorphism of N^n onto itself of order g. $\{\mathrm{Id}_M = \nu^0, \nu, \cdots, \nu^{g-1}\}$ is a finite cyclic group of order g which acts freely on N^n. Set $\mathbf{Z}_g := \{\mathrm{Id}_M = \nu^0, \nu, \cdots, \nu^{g-1}\}$.

Proposition 4.2. *Let $p, q \in N^n$. Then $\mathcal{G}(p) = \mathcal{G}(q)$ if and only if $q = \upsilon(p)$ for some $\upsilon \in \mathbf{Z}_g$.*

Therefore we obtain

Theorem 4.1 ([17])**.** *The image $\mathcal{G}(N^n)$ of the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is diffeomorphic to the quotient smooth manifold $N^n/\mathbf{Z}_g$ by the free action of $\mathbf{Z}_g$, that is, $\mathcal{G}(N^n) \cong N^n/\mathbf{Z}_g$. Hence $\mathcal{G}(N^n)$ is a compact connected minimal Lagrangian submanifold embedded in $Q_n(\mathbf{C})$*

Problem. Investigate properties of compact minimal Lagrangian submanifolds embedded in complex hyperquadrics obtained as Gauss images of isoparametric hypersurfaces in spheres.

We shall compute the differential of a diffeomorphism $\nu \in \mathbf{Z}_g$ at $p \in N^n$. Let $\{e_i \mid i = 1, \cdots, n\}$ be an orthonormal basis of T_pN^n such that $A(e_i) = k_\alpha\, e_i$ and set $k_\alpha = \cot\theta_\alpha$ with $0 < \theta_\alpha < \frac{\pi}{g}$. Then we have

$$(d\nu)_p(e_i) = \partial_{e_i} \mathbf{x}_{\frac{2\pi}{g}} = (\cos\left(\frac{2\pi}{g}\right) - \sin\left(\frac{2\pi}{g}\right) \cot\theta_\alpha)\, e_i.$$

If $g = 1$, then $(d\nu)_p(e_i) = e_i\, (i = 1, \cdots, n)$.

If $g = 2$, then $(d\nu)_p(e_i) = -e_i \, (i = 1, \cdots, n)$.
Suppose that $g \geq 3$. Then

$$\begin{aligned} & \cos(\frac{2\pi}{g}) - \sin(\frac{2\pi}{g}) \cot\theta_\alpha < 0 \\ \Longleftrightarrow \quad & \cot(\frac{2\pi}{g}) < \cot\theta_\alpha \\ \Longleftrightarrow \quad & \frac{2\pi}{g} > \theta_\alpha = \theta_1 + (\alpha - 1)\frac{\pi}{g} \quad (\alpha = 1, \cdots, g) \\ \Longleftrightarrow \quad & \frac{(3-\alpha)\pi}{g} > \theta_1 \quad (\alpha = 1, \cdots, g) \end{aligned}$$

and

$$\frac{2\pi}{g} \, (\alpha = 1) > \frac{\pi}{g} \, (\alpha = 2) > \theta_1 > 0 \, (\alpha = 3) \geq \cdots \geq -\frac{(g-3)\pi}{g} \, (\alpha = g).$$

Hence the number of negative eigenvalues of $(d\nu)_p : T_{\mathbf{x}}N \to T_{\mathbf{x}_\theta}N$ is equal to $m_1 + m_2$.

Lemma 4.2. *Assume that $g \geq 2$. Then the diffeomorphism $\nu : N \to N$ preserves the orientation if and only if $m_1 + m_2$ is even.*

Remark. The $\mathbf{Z}_g$-action on $N^n \subset S^{n+1}(1)$ does not preserve the induced metric from $S^{n+1}(1)$ if $g \geq 3$. The metric (4.1) on N^n induced from $Q_n(\mathbf{C})$ by $\mathcal{G}$ is preserved by the $\mathbf{Z}_g$-action. It is possible to check it by a direct computation too.

From Lemma 3.1 we know

$$\frac{2n}{g} = \begin{cases} m_1 + m_2 & \text{if } g \text{ is even,} \\ 2\, m_1 & \text{if } g \text{ is odd.} \end{cases}$$

Hence by Lemma 4.2 the orientability of $\mathcal{G}(N^n)$ is determined as follows:

Theorem 4.2.

(1) *If $\dfrac{2n}{g}$ is even, then $L = \mathcal{G}(N^n) \cong N^n/\mathbf{Z}_g$ is orientable.*

(2) *If $\dfrac{2n}{g}$ is odd, then $L = \mathcal{G}(N^n) \cong N^n/\mathbf{Z}_g$ is non-orientable.*

Now the minimal Maslov number of the Gauss image of an isoparametric hypersurface in a sphere can be determined as follows :

Theorem 4.3 ([19]). *$L = \mathcal{G}(N^n)$ is a compact monotone and cyclic Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ and its minimal Maslov number Σ_L is given by*

$$\Sigma_L = \frac{2n}{g} = \begin{cases} m_1 + m_2 & \text{if } g \text{ is even,} \\ 2\,m_1 & \text{if } g \text{ is odd.} \end{cases}$$

Proof. By Theorem 2.3 and minimality of $\mathcal{G}(N^n)$, $\mathcal{G}(N^n)$ is a monotone Lagrangian submanifold in $Q_n(\mathbf{C})$. Moreover, by Theorem 2.4 $\mathcal{G}(N^n)$ is a cyclic Lagrangian submanifold in $Q_n(\mathbf{C})$ and the formula (2.6). In case $M = Q_n(\mathbf{C})$ we know that $\gamma_{c_1} = n$ if $n \geq 2$ and $\gamma_{c_1} = 2$ if $n = 1$. We shall determine the number n_L.

Let $\tilde{N}^n$ be the Legendrian lift of N^n to the unit tangent sphere bundle $US^{n+1}(1) = V_2(\mathbf{R}^{n+2})$:

$$N^n \longrightarrow \mathcal{G}(N^n) = N^n/\mathbf{Z}_g.$$

$$\begin{tikzcd}
\tilde{N}^n \arrow[r] \arrow[d, "\mathbf{Z}_g"] & V_2(\mathbf{R}^{n+2})|_{L^n} \arrow[r] \arrow[d, "\pi"', "SO(2)"] & V_2(\mathbf{R}^{n+2}) \arrow[d, "\pi"', "SO(2) \cong U(1) \cong S^1"] \\
\mathcal{G}(N^n) \arrow[r, "="] & \mathcal{G}(N^n) = L^n \subset Q_n(\mathbf{C}) & \\
 & \text{Lagr.} &
\end{tikzcd}$$

Then

$$\pi : V_2(\mathbf{R}^{n+2})|_L \longrightarrow L = \mathcal{G}(N^n)$$

is a flat principal fiber bundle with structure group $SO(2)$ and the covering map

$$\pi : \tilde{N}^n \longrightarrow \mathcal{G}(N^n)$$

with Deck transformation group $\mathbf{Z}_g$ coincides with its holonomy subbundle with the holonomy group $\mathbf{Z}_g$:

$$\begin{aligned} \mathbf{Z}_g = & \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} t = 0, 2\pi\frac{1}{g}, \cdots, 2\pi\frac{g-1}{g} \right\} \\ = & \{ e^{\sqrt{-1}t} \,|\, t = 0, 2\pi\frac{1}{g}, \cdots, 2\pi\frac{g-1}{g} \} \end{aligned}$$

Let E denote the flat complex line bundle over $\mathcal{G}(N^n)$ associated with the principal fiber bundle

$$\pi : V_2(\mathbf{R}^{n+2})|_L \longrightarrow \mathcal{G}(N^n)$$

by the standard action of $SO(2) \cong U(1)$ on $\mathbf{C}$.

The tautological complex line bundle $\mathcal{W}$ over $Q_n(\mathbf{C}) = \widetilde{Gr}_2(\mathbf{R}^{n+2})$ is defined by

$$\mathcal{W}_x := \mathbf{C}(\mathbf{a} + \sqrt{-1}\mathbf{b})$$

for each $x = [\mathbf{a} + \sqrt{-1}\mathbf{b}] \in Q_n(\mathbf{C})$ Then $E = \mathcal{W}$ if $n \geq 2$ and $\otimes^2 E = \mathcal{W}$ if $n = 1$. Indeed, $c_1(\mathcal{W})(\mathbf{C}P^1) = 1$ if $n \geq 2$. Here $\mathbf{C}P^1 := \{W \subset U \mid \text{1-dimensional complex vector subspaces}\}$ and U is a complex 2-dimensional isotropic vector subspace of $\mathbf{C}^{n+2}$, namely, $U \perp \bar{U}$.

For $k = 1, 2, \cdots, g$, the generator $e^{\sqrt{-1}\frac{2\pi}{g}}$ of the holonomy group $\mathbf{Z}_g$ on $E|_L$ induces the multiplication by $e^{\sqrt{-1}\frac{2\pi k}{g}}$ on $\otimes^k E|_L$. Thus the holonomy group of $\otimes^k E|_L$ is generated by $e^{\sqrt{-1}\frac{2\pi k}{g}}$ of $\mathbf{Z}_g$. Hence $\otimes^g E|_L$ has trivial holonomy and for $k = 1, 2, \cdots, g-1$, $\otimes^k E|_L$ has non-trivial holonomy. Therefore we obtain $n_L = g$ if $n \geq 2$ and $n_L = 2$ if $n = 1$. Moreover,

$$\Sigma_L = \frac{2n}{g} = \begin{cases} \dfrac{2}{g}\,\dfrac{(m_1+m_2)g}{2} = m_1 + m_2 & \text{if } g \text{ is even,} \\[2ex] \dfrac{2}{g}\, g\, m_1 = 2m_1 & \text{if } g \text{ is odd} \end{cases}$$

□

Remark. Theorems 4.1 and 4.3 are described in [17], [19] without detail of the proof.

Proposition 4.3 ([17])**.** *An isoparametric hypersurface N^n in $S^{n+1}(1)$ is homogeneous if and only if its Gauss image $\mathcal{G}(N^n)$ is a compact homogeneous Lagrangian submanifold in $Q_n(\mathbf{C})$.*

In [17], we classified ALL compact homogeneous Lagrangian submanifolds in complex hyperquadrics $Q_n(\mathbf{C})$.

Palmer ([35]) showed that the Gauss map $\mathcal{G} : N^n \to Q_n(\mathcal{C})$ is Hamiltonian stable if and only if $g = 1$. In [17], [19], we determined completely the (strict) Hamiltonian stability of the Gauss images $\mathcal{G}(N^n)$ of ALL compact *homogeneous* isoparametric hypersurfaces N^n in spheres $S^{n+1}(1)$.

Problem. Investigate the similar theory in the case of oriented hypersurfaces in a real hyperbolic space form and semi-Riemannian space forms. J. Hahn's work ([10], [11]) will be useful to provide homogeneous and non-homogeneous examples.

5 Cohomogeneity 1 Lagrangian submanifolds in the cotangent bundle of $S^{n+1}(1)$

Let $TS^{n+1}(1)$ and $T^*S^{n+1}(1)$ be the tangent and cotangent vector bundles over the $(n+1)$-dimensional unit standard sphere $S^{n+1}(1)$. The special orthogonal group $SO(n+2)$ acts effectively and transitively on $S^{n+1}(1) \subset \mathbf{R}^{n+2}$ as the identity component of its isometry group and induces the group actions on $TS^{n+1}(1)$ and $T^*S^{n+1}(1)$ in the natural way. Relative to the standard metric of $S^{n+1}(1)$, we identify $TS^{n+1}(1)$ with $T^*S^{n+1}(1)$.

More generally, suppose that N^m is an m-dimensional submanifold immersed in the unit standard sphere $S^{n+1}(1)$. Let ν_N^* denote the conormal bundle of N^m in $S^{n+1}(1)$ and the unit conormal bundle $U(\nu_N^*) := \{\xi \in \nu_N^* \mid \|\xi\| = 1\}$ of N^m in $S^{n+1}(1)$. It is a classical fact that the conormal bundle ν_N^* is a Lagrangian submanifold in the cotangent vector bundle $T^*S^{n+1}(1)$ of the unit standard sphere $S^{n+1}(1)$. Notice that the unit cotangent bundle $U(T^*S^{n+1}(1))$ is diffeomorphic to the Stiefel manifold

(5.1)
$$V_2(\mathbf{R}^{n+2}) := \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbf{R}^{n+2}, \|\mathbf{a}\| = \|\mathbf{b}\| = 1, \langle \mathbf{a}, \mathbf{b} \rangle = 0\} \cong SO(n+2)/SO(n).$$

Furthermore, the unit cotangent bundle $U(T^*S^{n+1}(1))$ is a circle bundle over $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$. Hence $U(T^*S^{n+1}(1))$ carries the canonical contact structure and then the unit conormal bundle $U(\nu_N^*)$ of N is a Legendrian submanifold of $U(T^*S^{n+1}(1))$. Then the projection of $U(\nu_N^*)$ via p_2 gives a Lagrangian submanifold immersed in $Q_n(\mathbf{C})$. In the case when $N^m = N^{n+1}$ is an oriented hypersurface in $S^{n+1}(1)$, (if we take a connected component of $U(\nu_N^*)$,) this construction coincides with the above Gauss map construction. We have the following diagram :

$$\begin{tikzcd}
\nu_N^* \arrow[r, "\text{Lag.}"'] \arrow[d] & T^*S^{n+1}(1) \arrow[d] & \\
U(\nu_N^*) \arrow[r, "\text{Leg.}"'] \arrow[d] & U(T^*S^{n+1}(1)) \arrow[d, "p_2"', "S^1"] \arrow[r, phantom, "\cong"] & V_2(\mathbf{R}^{n+2}) \arrow[d, "p_1"', "S^n"] \\
p_2(U(\nu_N^*)) \arrow[r, "\text{Lag.}"'] & Q_n(\mathbf{C}) & S^{n+1}(1) \supset N^m \\
 & & \text{imm. submfd.}
\end{tikzcd}$$

The unit cotangent bundle $U(T^*S^{n+1}(1)) \cong V_2(\mathbf{R}^{n+2})$ has the standard homogeneous ($SO(n+2)$-invariant) Einstein-Sasakian manifold structure over $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$ and the corresponding cone metric on $(0,\infty) \times V_2(\mathbf{R}^{n+2}) = CV_2(\mathbf{R}^{n+2}) \times CU(T^*S^{n+1}(1)) \cong TS^{n+1}(1) \setminus \{0\}$ is an Ricci-flat Kähler metric (cf. [3]). Moreover, the cone $CU(T^*S^{n+1}(1))$ over $U(T^*S^{n+1}(1))$ is a Lagrangian submanifold in $CV_2(\mathbf{R}^{n+2})$ relative to the Kähler cone metric. Then it is a well-known fact that the following three conditions are equivalent each other:

(1) $p_2(U(\nu_N^*))$ is a minimal Lagrangian submanifold in $Q_n(\mathbf{C})$ with respect to the standard Hermitian symmetric Einstein-Kähler metric.

(2) $U(\nu_N^*)$ is a minimal Legendrian submanifold in $U(T^*S^{n+1}(1)) \cong V_2(\mathbf{R}^{n+2})$ with respect to the standard homogeneous Einstein-Sasakian metric.

(3) $CU(\nu_N^*)$ is a minimal Lagrangian submanifold, i.e. a special Lagrangian submanifold of some phase, in $CV_2(\mathbf{R}^{n+2})$ with respect to the Ricci-flat Kähler cone metric.

Therefore each isoparametric hypersurface N^n in $S^{n+1}(1)$ gives a special Lagrangian submanifold in $CV_2(\mathbf{R}^{n+2})$ equipped with the Ricci-flat Kähler cone metric. In particular we see that if N^n is a homogeneous isoparametric hypersurface in $S^{n+1}(1)$, then the cone $CU(\nu_N^*)$ is a cohomogeneity 1 special Lagrangian submanifold in $CV_2(\mathbf{R}^{n+2})$ with the Ricci-flat Kähler cone metric.

Stenzel ([36]) constructed a complete Ricci-flat Kähler metric on the (co)tangent vector bundle of a compact rank one symmetric space G/K invariant under the induced group action of G, and thus of cohomogeneity 1. The metric is called the *Stenzel meteric*. The Stenzel metric on $T^*S^{n+1}(1) \cong TS^{n+1}(1)$ is a complete Ricci-flat Kähler metric invariant under the standard group action of $SO(n+2)$.

S. Karigiannis and M. Min-Oo ([16]) showed that N^m is an austere submanifold in $S^{n+1}(1)$ if and only if its conormal bundle ν_N^* is a special Lagrangian submanifold in $T^*S^{n+1}(1) \cong TS^{n+1}(1)$ with respect to the Stenzel metric. Here a minimal submanifold of a Riemannian manifold is called *austere* if for each normal vector ν, the eigenvalues of the shape operator A_ν are invariant under multiplication by -1 ([12, p.102]). Note that a minimal isoparametric hypersurface N^n in $S^{n+1}(1)$ is an austere submanifold in $S^{n+1}(1)$ if and only if $m_1 = m_2$ (see Proposition 3.1).

On the other hand by Lemma 4.1 we see that if an oriented minimal hypersurface N^n in $S^{n+1}(1)$ is austere, then the Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. Hence $CU(\nu_{N^n}^*)$ is a special Lagrangian submanifold in $CV_2(\mathbf{R}^{n+2})$ with the Ricci-flat Kähler cone metric and at the same time its conormal bundle $\nu_{N^n}^*$ is a special Lagrangian submanifold in $T^*S^{n+1}(1) \cong TS^{n+1}(1)$ with the Stenzel metric. However we should notice that if N^n is an isoparametric hypersurface in $S^{n+1}(1)$ but not austere, then $CU(\nu_{N^n}^*)$ is a special Lagrangian submanifold in $CV_2(\mathbf{R}^{n+2})$ with the Ricci-flat Kähler cone metric and but by the result of Karigiannis and Min-Oo its conormal bundle $\nu_{N^n}^*$ is not always a special Lagrangian submanifold in $T^*S^{n+1}(1) \cong TS^{n+1}(1)$ with the Stenzel metric. And also we see that if N^n is an austere homogeneous isoparametric hypersurface in $S^{n+1}(1)$, then its conormal bundle $\nu_{N^n}^*$ is a cohomogeneity 1 special Lagrangian submanifold in $T^*S^{n+1}(1) \cong TS^{n+1}(1)$ with the Stenzel metric.

Recently Kaname Hashimoto (Osaka City University, Ph.D. student) and Takashi Sakai (Tokyo Metropolitan University/OCAMI) ([13]) investigated a classification of cohomogeneity 1 special Lagrangian submanifolds in $T^*S^{n+1}(1) \cong TS^{n+1}(1)$ with the Stenzel metric deformed from special Lagrangian cones $CU(\nu^*_{N^n})$ in $CV_2(\mathbf{R}^{n+2})$ with the Ricci-flat Kähler cone metric in the case when N^n is an isoparametric hypersurface in $S^{n+1}(1)$ with $g = 1, 2$. Their research is in progress at present.

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