Geometry of $R$-spaces canonically embedded in Kähler $C$-spaces as Lagrangian submanifolds

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This article is dedicated to Professor Young Jin Suh on the occasion of his 65th birthday.

Abstract. An $R$-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. It is known that any $R$-space has the canonical embedding into a Kähler $C$-space as a real form and thus it is a compact totally geodesic Lagrangian submanifold. In this article we provide an exposition on such nice properties of $R$-spaces as Lagrangian submanifolds and our recent work on minimal Maslov number of $R$-spaces canonically embedded in Einstein-Kähler $C$-spaces ([20]).

1 Introduction

A smooth immersion (resp. embedding) $\varphi : L \to M$ of a smooth manifold $L$ into a symplectic manifold $(M, \omega)$ is called a Lagrangian immersion (resp. Lagrangian embedding) if $2 \dim L = \dim M$ and $\varphi^* \omega = 0$. For a Lagrangian immersion $\varphi : L \to M$, we have the vector bundle isomorphism $\varphi^{-1}TM/\varphi_* TL \ni v \mapsto \alpha_v := \omega(v, \cdot) \in T^*L$. A smooth family of Lagrangian immersions $\varphi_t : L \to M$ with $\varphi_0 = \varphi$ is called a Lagrangian deformation of $\varphi$, which is characterized by the closedness of the 1-form $\alpha_{V_t} \in \Omega^1(L)$ corresponding to the variational vector field $V_t := \frac{\partial \varphi_t}{\partial t} \in \varphi^{-1}TM$ for each $t$. A Lagrangian deformation $\varphi_t : L \to M$ with

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\( \varphi_0 = \varphi \) is called a Hamiltonian deformation of \( \varphi \) if \( \alpha \in \Omega^1(L) \) is exact for each \( t \). Suppose that \( [\omega] \in H^2(M, \mathbb{R}) \) is an integral class, that is, there is a complex line bundle \( E \) over \( M \) and a \( U(1) \)-connection \( \nabla \) of \( E \) whose curvature form is equal to \( 2\pi \sqrt{-1} \omega \). It is known that a Lagrangian deformation \( \varphi_t : L \to M \) with \( \varphi_0 = \varphi \) is a Hamiltonian deformation if and if a family of flat connections \( \{ \varphi_t^{-1}\nabla \} \) has same holonomy homomorphism \( \rho : \pi_1(L) \to U(1) \).

Two group homomorphisms \( I_{\mu,L} : \pi_2(M, L) \to \mathbb{Z} \) and \( I_{\omega,L} : \pi_2(M, L) \to \mathbb{R} \) are defined for any Lagrangian submanifold of a symplectic manifold in general (see Section 3) so that \( I_{\mu,L} \) is a symplectic invariant and \( I_{\omega,L} \) is not a symplectic invariant but a Hamiltonian invariant. The minimal Maslov number of a Lagrangian submanifold in a symplectic manifold is defined by the condition that \( I_{\mu,L} = \lambda I_{\omega,L} \) (\( \exists \lambda > 0 \)). The Floer homology theory for the intersection of monotone Lagrangian submanifolds was initiated and well-developed by Y.-G. Oh ([15], [16], [17], [18] and so on). It is known that any compact minimal Lagrangian submanifold of an Einstein-Kähler manifold with positive Einstein constant is monotone (Cieliebak-Goldstein [2], Hajime Ono [21]). Moreover he ([21]) gave a nice formula of the minimal Maslov number for a compact monotone Lagrangian submanifold in a simply connected Einstein-Kähler manifold with positive Einstein constant (see the formula (3.1) in Section 3).

An \( R \)-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. Note that an \( R \)-space is not a symmetric space in general and it is called a symmetric \( R \)-space when it is a symmetric space. It is known that each \( R \)-space has the canonical embedding into a Kähler \( C \)-space as a real form. A Kähler \( C \)-space is a simply connected compact homogeneous complex manifold which admits invariant Kähler metrics, and it is also called a generalized flag manifold. A real form means the fixed point subset by an anti-holomorphic isometry of a Kähler \( C \)-space and thus it is a compact embedded totally geodesic Lagrangian submanifold. So \( R \)-spaces canonically embedded in Kähler \( C \)-spaces constitute a nice class of Lagrangian submanifolds. As explained in Section 2 any \( R \)-space can be canonically embedded in an Einstein-Kähler \( C \)-space. In this case it is a compact monotone Lagrangian submanifold and so we can use H. Ono’s formula in order to study the minimal Maslov number for \( R \)-spaces canonically embedded in Einstein-Kähler \( C \)-spaces. In [20] we showed a Lie theoretic formula for the minimal Maslov number of such an \( R \)-space and some examples of the calculation by that formula.

In this article we provide an exposition on such nice properties of \( R \)-spaces as Lagrangian submanifolds and our related recent work ([20]). This article is organized as follows: In Section 2 we review the definitions and elementary properties of \( R \)-spaces and their canonical embeddings into Kähler \( C \)-spaces and the description of the invariant symplectic structures, invariant complex structures, invariant Kähler metrics and invariant Einstein-Kähler metrics on a Kähler \( C \)-space. We also discuss several properties from the viewpoint of geometry of Lagrangian sub-
manifolds such as an anti-symplectic involutive diffeomorphism, the moment maps, Morse theory and related intersection problem. In Section 3 we recall the definitions of two Hamiltonian invariants $I_{u,L}$ and $I_{\omega,L}$ and the monotonicity for Lagrangian submanifolds of general symplectic manifolds. Moreover we refer the monotonicity theorem and minimal Maslov number formula by Cieliebak-Goldstein and H. Ono for Lagrangian submanifolds in Einstein-Kähler manifolds, and mention our applications to the case of the Gauss images of isoparametric hypersurfaces. In Section 4 we describe the construction of the Lie theoretic formula for minimal Maslov number for $R$-spaces canonically embedded in Einstein-Kähler $C$-spaces.

Throughout this article any manifold is smooth and connected.

2 The canonical embeddings of an $R$-space into a Kähler $C$-space

In this section we review the definitions and elementary properties of $R$-spaces and their canonical embeddings into Kähler $C$-spaces from the viewpoint of geometry of Lagrangian submanifolds (cf. [1], [22], [27], [23], [24], [25], [20]).

Let $(G, K, \theta)$ be a Riemannian symmetric pair with an involutive automorphism $\theta$. Suppose that $G$ is a connected compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ and $K$ is a connected compact Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. We choose an $Ad\mathfrak{g}$- and $\theta$-invariant inner product $\langle , \rangle$ of $\mathfrak{g}$ and extend it to the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ by the complex bi-linearity. Let $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ with respect to $(G, K, \theta)$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Then we have $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}, \mathfrak{b} = \mathfrak{t} \cap \mathfrak{t}, \mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ and $\mathfrak{t}$ is invariant by $\theta$. Let $(\ ), ( )$ denote an inner product of $\mathfrak{t}$ defined by a restriction of $(\ ), ( )$ to $\mathfrak{t}$. The root space decomposition of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$ is given as

$$\mathfrak{g}^C = \mathfrak{t}^C + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^\alpha,$$

where

$$\mathfrak{g}^\alpha := \{ X \in \mathfrak{g}^C | (ad\xi)(X) = \sqrt{-1}(\alpha, \xi)X \ (\forall \xi \in \mathfrak{t}) \}$$

and $\Sigma(\mathfrak{g}) \subset \mathfrak{t}$ denotes the set of all roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$. Set $\Sigma_0(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \mathfrak{b}$. Define the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a}) := \{ \gamma = \bar{\alpha} | \alpha \in \Sigma(\mathfrak{g}) \}$, where $\bar{\alpha}$ denotes the $\mathfrak{a}$-component of $\alpha \in \Sigma(\mathfrak{g}) \subset \mathfrak{t} = \mathfrak{b} + \mathfrak{a}$. We choose an involutive orthogonal transformation $\sigma \in O(\mathfrak{t})$ by $\sigma := -\theta|_{\mathfrak{t}}$. We choose a $\sigma$-order $> 0$ on $\mathfrak{t}$ so that if $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$ and $\alpha > 0$, then $\sigma \alpha > 0$ and thus $\theta \alpha = -\sigma \alpha < 0$ ([22]). Set $\Sigma^+(\mathfrak{g}) := \{ \alpha \in \Sigma(\mathfrak{g}) | \alpha > 0 \}, \Sigma^0(\mathfrak{g}) := \Sigma_0(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{ \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}) | \gamma > 0 \} = \{ \bar{\alpha} | \alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g}) \}$.

We choose $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Sigma(\mathfrak{g})$ such that $[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha, \langle E_\alpha, E_{-\alpha} \rangle = 1$, $E_\alpha = E_{-\alpha}$ for each $\alpha \in \Sigma(\mathfrak{g})$ and let $\{ \omega^\alpha | \alpha \in \Sigma(\mathfrak{g}) \}$ be the linear forms on $\mathfrak{g}^C$ dual to $\{ E_\alpha | \alpha \in \Sigma(\mathfrak{g}) \}$ so that $\omega^\alpha(\mathfrak{t}^C) = \{ 0 \}, \omega^\alpha(\mathfrak{p}^C) = 0$ for each $\alpha, \beta \in \Sigma(\mathfrak{g})$. The restricted root space decompositions of $\mathfrak{t}$ and $\mathfrak{p}$ with respect to $\mathfrak{a}$ are given as

$$\mathfrak{t} = \mathfrak{t}_0 + \sum_{\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{t}_\gamma, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{p}_\gamma,$$
where $\mathfrak{k}_0 := \{ X \in \mathfrak{k} \mid (\text{ad}H)X = 0 \ (\forall H \in \mathfrak{a}) \}$ and for each $\gamma \in \Sigma^+(\mathfrak{g},\mathfrak{a})$ set

$$\mathfrak{t}_\gamma := \{ X \in \mathfrak{t} \mid (\text{ad}H)^2X = -\gamma, H)^2X \ (\forall H \in \mathfrak{a}) \},$$

$$\mathfrak{p}_\gamma := \{ X \in \mathfrak{p} \mid (\text{ad}H)^2Y = -\gamma, H)^2Y \ (\forall H \in \mathfrak{a}) \}.$$  

For $\gamma \in \Sigma^+(\mathfrak{g},\mathfrak{a})$, there are an orthonormal basis $\{ S_{\gamma,i} \mid i = 1, \ldots, m(\gamma) \}$ of $\mathfrak{k}_0$, and an orthonormal basis $\{ T_{\gamma,i} \mid i = 1, \ldots, m(\gamma) \}$ of $\mathfrak{p}_\gamma$, where $m(\gamma) := \dim \mathfrak{k}_0 = \dim \mathfrak{p}_\gamma$, such that $[H, S_{\gamma,i}] = (\gamma, H)T_{\gamma,i}$, $[H, T_{\gamma,i}] = -\gamma, H)S_{\gamma,i}$ for each $H \in \mathfrak{a}$.

Now we fix an arbitrary non-zero element $Z$ of $\mathfrak{a}$. Set

$$\Sigma_\mathfrak{g}(Z) := \{ \alpha \in \Sigma(\mathfrak{g}) \mid \langle \alpha, Z \rangle = 0 \} \quad \text{and} \quad \Sigma^+_{\mathfrak{g}}(Z) := \Sigma_\mathfrak{g}(Z) \cap \Sigma^+(\mathfrak{g}).$$

The element $Z$ is called regular if $\Sigma_\mathfrak{g}(Z) = \Sigma_0(\mathfrak{g})$. Define closed subgroups $G_Z$ and $K_Z$ of $G$ by

$$G_Z := \{ a \in G \mid \text{Ad}(a)Z = Z \}$$

and

$$K_Z := \{ a \in K \mid \text{Ad}(a)Z = Z \} = K \cap G_Z.$$  

It is well-known that $G_Z$ is always connected. Denote by $\mathfrak{g}_Z$ and $\mathfrak{k}_Z$ Lie algebras of $G_Z$ and $K_Z$, respectively. Note that $\theta(G_Z) = G_Z$, $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ and thus $(G_Z, K_Z, \theta|_{G_Z})$ is also a compact symmetric pair.

**Definition.** The compact homogeneous space $L := K/K_Z$ is called an $R$-space, and it has the standard imbedding into the vector space $\mathfrak{p}$ defined by

$$\varphi_Z : L = K/K_Z \ni aK_Z \mapsto \text{Ad}(a)Z \in \text{Ad}(K)Z \subset \mathfrak{p}.$$  

If $Z$ is a regular element of $\mathfrak{a}$, then $L = K/K_Z$ is called a regular $R$-space. Another compact homogeneous space $M := G/G_Z$ is called a generalized flag manifold or a Kähler $C$-space, and it also has the standard imbedding into the Lie algebra $\mathfrak{g}$

$$\Phi_Z : M = G/G_Z \ni aG_Z \mapsto \text{Ad}(a)Z \in \text{Ad}(G)Z \subset \mathfrak{g}.$$  

The canonical embedding of $K/K_Z$ into $G/G_Z$ is a map defined by

$$\iota_Z : L = K/K_Z \ni aK_Z \mapsto aG_Z \in G/G_Z = M.$$  

We take the orthogonal direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{k}$ as $\mathfrak{g} = \mathfrak{g}_Z + \mathfrak{m}$, $\mathfrak{m} \cong T_{eG_Z}M$ and $\mathfrak{k} = \mathfrak{t}_Z + \mathfrak{l} \cong T_{eK_Z}L$. Note that $\mathfrak{t}_Z = \mathfrak{k} \cap \mathfrak{g}_Z$. By using the property $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ one can show that $\iota_Z$ is an embedding and $2 \dim L = \dim M$.

The author has heard from Professor Masaru Takeuchi that the “$R$-space” was named first by Jacques Tits ([31]). Here we should notice that an $R$-space is not a symmetric space in general, and however the $R$-space can be considered as a class
Geometry of $R$-spaces Canonically Embedded in Kähler $C$-spaces

of the most important compact homogeneous spaces related to symmetric spaces. An $R$-space $K/K_Z$ is called a symmetric $R$-space if $K/K_Z$ is a symmetric space. It is known that an $R$-space is a symmetric $R$-space if and only if one of the following conditions is satisfied:

1. $(K, K_Z)$ is a symmetric pair.
2. There is an element $Z \in \mathfrak{p}$ satisfying the equation $(\text{ad}Z)^3 + (\text{ad}Z) = 0$ such that $L = K/K_Z$ and $G = G/G_Z$.
3. $(G, G_Z)$ is a Hermitian symmetric pair.
4. The standard imbedding $\varphi_Z$ has the parallel second fundamental form (Dirk Ferus [3]).
5. $\varphi_Z(L)$ is an (extrinsic) symmetric submanifold in Euclidean space $p$ (Dirk Ferus [4]).

For such $Z$, we can define a $G$-invariant symplectic form $\omega_Z$ on $M = G/G_Z$ by

$$\omega_Z(X,Y) := \langle [Z,X], Y \rangle$$

for each $X, Y \in \mathfrak{g}$.

and $\omega_Z$ can be also expressed as

$$\omega_Z = -\sqrt{-1} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} (Z, \alpha) \omega^{-\alpha} \wedge \omega^{-\alpha}.$$

Then the canonical embedding $\iota_Z : L = K/K_Z \to G/G_Z = M$ is a Lagrangian embedding with respect to $\omega_Z$.

The involutive automorphism $\theta$ of $G$ induces an involutive diffeomorphism

$$\hat{\theta}_Z : M = G/G_Z \ni aG_Z \mapsto \theta(a)G_Z \in G/G_Z = M$$

which is equivariant with respect to the Lie group automorphism $\theta : G \to G$ Then $\hat{\theta}_Z : G/G_Z \to G/G_Z$ is anti-symplectic with respect to $\omega_Z$, that is, $\hat{\theta}_Z^* \omega_Z = -\omega_Z$.

Define the fixed point subset of $M$ by $\hat{\theta}_Z$ as

$$\text{Fix}(M, \hat{\theta}_Z) := \{ p \in M \mid \hat{\theta}_Z(p) = p \}.$$  

Then $\iota_Z(K/K_Z) \subset \text{Fix}(M, \hat{\theta}_Z) \subset G/G_Z$.

The natural left action of $G$ on a symplectic manifold $(M = G/G_Z, \omega_Z)$ is a Hamiltonian group action with the moment map

$$\mu_G := \Phi_Z : G/G_Z \to \mathfrak{g}^*.$$  

Moreover the natural left action of $K \subset G$ on $(M = G/G_Z, \omega_Z)$ is also a Hamiltonian group action with the moment map

$$\mu_K := \pi_t \circ \mu_G = \pi_t \circ \Phi_Z : G/G_Z \to \mathfrak{k} \cong \mathfrak{k}^*.$$
Here \( \pi_\kappa : \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \rightarrow \mathfrak{t} \) denotes the orthogonal projection of \( \mathfrak{g} \) onto \( \mathfrak{t} \). Then 
\[ \mu_G \circ \hat{\theta}_Z = -\theta \circ \mu_G \quad \text{and} \quad \mu_K \circ \hat{\theta}_Z = -\mu_K. \]
It follows from these formulas that
\[ \text{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0). \]

Since \( K \) and \( M \) are compact, by a result of Kirwan ([12, p.549, (3.1)]) \( \mu_K^{-1}(0) \) is connected and thus \( \text{Fix}(M, \hat{\theta}_Z) \) is also connected. Therefore we obtain
\[ \iota_Z(K/K) = \text{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0). \]

The Weyl group of \((G, K)\) is defined by \( W(G, K) := N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \). By the action of the Weyl group \( W(G, K) \), we may assume that \( \mathcal{Z} \in \mathfrak{a} \subset \mathfrak{t} \) satisfies \((\alpha, \mathcal{Z}) \geq 0 \) for \( \forall \alpha \in \Sigma^+(\mathfrak{g}) \).

Now we describe an invariant complex structure on \( M = G/G, Z \) corresponding to \( Z \). Note that \( Z \in \mathfrak{c}(\mathfrak{g}) \subset \mathfrak{t} \subset \mathfrak{g} \).

Then
\[ \mathfrak{g}^C_\mathcal{Z} = \mathfrak{t}^C + \sum_{\alpha \in \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^\alpha, \]
\[ T_{eG_\mathcal{Z}}(G/G_\mathcal{Z})^C \cong \mathfrak{m}^C = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^{-\alpha} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^\alpha. \]

Note that \( \mathfrak{g}^\alpha = \hat{\mathfrak{g}}^{-\alpha} \). Thus we can define a \( G \)-invariant complex structure \( J_\mathcal{Z} \) on \( G/G_\mathcal{Z} \) such that
\[ T_{eG_\mathcal{Z}}(G/G_\mathcal{Z})^{1,0} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^{-\alpha}, \quad T_{eG_\mathcal{Z}}(G/G_\mathcal{Z})^{0,1} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^\alpha, \]

Since
\[ \theta \left( \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} \mathfrak{g}^\alpha, \]
the involutive diffeomorphism \( \hat{\theta}_\mathcal{Z} : G/G_\mathcal{Z} \rightarrow G/G_\mathcal{Z} \) is anti-holomorphic with respect to \( J_\mathcal{Z} \), that is, \( J_\mathcal{Z} \circ d\theta_\mathcal{Z} = -d\theta_\mathcal{Z} \circ J_\mathcal{Z} \).

Moreover the corresponding \( G \)-invariant Kähler metric \( g_\mathcal{Z} \) on \( M = G/G_\mathcal{Z} \) is defined by
\[ \omega_\mathcal{Z}(X, Y) = (-1)g_\mathcal{Z}(J_\mathcal{Z}X, Y) \quad \text{for each} \ X, Y \in \mathfrak{m} \]

or
\[ g_\mathcal{Z} = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_\mathcal{Z}(\mathfrak{g})} (Z, \alpha) \omega^{-\alpha} \cdot \overline{\omega^{-\alpha}}. \]

Hence the diffeomorphism \( \theta_\mathcal{Z} : M \rightarrow M \) is an isometry of \( M \) with respect to \( g_\mathcal{Z} \).

Let \( \Pi := \Pi(\mathfrak{g}) = \{ \alpha_1, \ldots, \alpha_\ell \} \) be the fundamental root system of \( \mathfrak{g} \) with respect to the \( \sigma \)-order \( < \) of \( \mathfrak{t} \). Set \( U(\mathfrak{g}) := \Pi(\mathfrak{g}) \cap \mathfrak{b} \). For the above \( Z \), set \( \Pi_Z := \Pi_Z(\mathfrak{g}) := \{ \alpha_i \in \Pi(\mathfrak{g}) \mid (\alpha_i, Z) = 0 \} \). Note that \( \Pi_0(\mathfrak{g}) \subset \Pi_Z(\mathfrak{g}) \) and
thus $\Pi(g) \setminus \Pi_Z(g) \subset \Pi(g) \setminus \Pi_0(g)$. Let $\{\Lambda_1, \cdots, \Lambda_\ell\} \subset \mathfrak{t}$ be the fundamental weight system of $\mathfrak{g}$ corresponding to $\Pi(g)$ defined by
\[
\frac{2\langle \Lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij} \quad (i, j = 1, \cdots, \ell).
\]
Now we set
\[
\lambda = \frac{\chi_Z}{\langle \alpha, \alpha \rangle} \quad \text{for each } \alpha \in \Pi_H.
\]
Note that $\Lambda_i \subset \mathfrak{t}$, and therefore $\Pi(\mathfrak{g}) = \Pi(\mathfrak{g}) \setminus \Pi(\mathfrak{g})$. Thus $\mathfrak{g} = \mathfrak{g}_Z$ and $\mathfrak{g} = \mathfrak{g}_Z$. By the connectedness of $G_\chi$ and $G_Z$, we obtain $G_\chi = G_Z$ and $G/G_\chi = G/G_Z = M$. In particular $\omega_\chi$ is a $G$-invariant symplectic form on $M = G/G_\chi = G/G_Z$. However $\lambda$ and $H$ define the same $G$-invariant complex structure $J_\chi = J_H$ on $M = G/G_H = G/G_Z$.

Since $\theta(g_Z) = g_Z$ and thus $\theta(\mathfrak{c}(g_Z)) = \mathfrak{c}(g_Z)$, there is a direct sum decomposition
\[
\mathfrak{c}(g_Z) = \mathfrak{c}_Z = (\mathfrak{c}_Z \cap \mathfrak{b}) + (\mathfrak{c}_Z \cap \mathfrak{a}).
\]
For each $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$, since $H = H_Z$ and $G/H = G/G_Z$, we have $H = H_Z$ and $H = H_Z$. By the connectedness of $G_\chi$ and $G_Z$, we obtain $G_\chi = G_Z$ and $G/G_\chi = G/G_Z = M$. In particular $\omega_\chi$ is a $G$-invariant symplectic form on $M = G/G_\chi = G/G_Z$. However $\lambda$ and $H$ define the same $G$-invariant complex structure $J_\chi = J_H$ on $M = G/G_H = G/G_Z$.

Let $\mathcal{H}_Z^2(M)$ denote the real vector space of all $G$-invariant closed 2-forms on $M = G/G_Z$. Then the natural linear map $\omega : \mathcal{H}_Z^2(M) \ni \omega \longmapsto [\omega] \in H^2(M, \mathbb{R})$, is a linear isomorphism and there is a linear isomorphism $\omega : \frac{1}{2\pi\sqrt{-1}}\mathcal{H}_Z^2(M) \longrightarrow \mathcal{H}_Z^2(M)$ defined by
\[
\omega \left( \frac{1}{2\pi\sqrt{-1}} \right) (X, Y) := -\frac{1}{2\pi} ([\lambda, X], Y) \quad (X, Y \in \mathfrak{m})
\]
for each $\lambda \in \mathfrak{c}_Z$. Moreover the linear isomorphism $\tau = \mathfrak{w} \circ \omega : \frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_Z \longrightarrow \mathcal{H}_Z^2(M) \longrightarrow H^2(M, \mathbb{R})$ given by the transgression operator is restricted to a $\mathfrak{Z}$-module isomorphism $\tau = \mathfrak{w} \circ \omega : \frac{1}{2\pi\sqrt{-1}}\mathcal{Z}_Z \longrightarrow H^2(M, \mathbb{Z})$. The 2nd cohomology and homology groups of $G/G_Z$ are described as follows:
\[
\mathfrak{c}_Z = \bigoplus_{\alpha \in \mathfrak{H} \setminus \Pi_\mathfrak{Z}} \mathbb{R} \Lambda_\alpha, \quad \mathfrak{c}_Z \triangleright \lambda \longmapsto \left[ \frac{1}{2\pi\sqrt{-1}} \omega(\lambda) \right] \in H^2(G/G_Z, \mathbb{R}),
\]
\[
\mathcal{Z}_Z = \bigoplus_{\alpha \in \mathfrak{H} \setminus \Pi_\mathfrak{Z}} \mathbb{Z} \Lambda_\alpha, \quad \longmapsto H^2(G/G_Z, \mathbb{Z}).
\]
and if we set $\alpha^*_i := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$, \( Z^*_c := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R} \alpha^*_i \uplus \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \sum x_i \alpha^*_i \leftrightarrow \sum x_i [S^2(\alpha^*_i)] \in H_2(G/G_Z, \mathbb{R}) \cup H_2(G/G_Z, \mathbb{Z}) \cong \pi_2(G/G_Z). \)

For each $\lambda \in Z^*_c \cap a$, define a $G$-invariant Kähler metric on a complex manifold $M = G/G_Z, J_Z$ by
\[
g \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right) := \frac{1}{2\pi} \sum_{\alpha \in \Sigma^+(g) \setminus \Sigma_Z(g)} (\lambda, \alpha) \omega^{-\alpha} \cdot \bar{\omega}^{-\alpha}
\]
whose Kähler form coincides with $\omega \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right)$ as
\[
\omega \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right) (X, Y) = g \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right) (J_Z X, Y).
\]
Note that $Z \in Z^*_c \cap a$ and $\omega \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right) = -\omega Z$. Namely, the convex open set $Z^*_c$ of the vector space $Z^*_c$ parametrizes all $G$-invariant Kähler metrics on $M = G/G_Z$ relative to the complex structure $J_Z$. So the parameter spaces of all $G$-inv. Kähler metrics on $G/G_Z$ are given as
\[
Z^+_c := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}^+ \Lambda_{\alpha_i} \uplus \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \{ \text{G-inv. Kähler met. on } G/G_Z \}
\]
\[
\uplus Z^+_c = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}^+ \Lambda_{\alpha_i} \uplus \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \{ \text{G-inv. Hodge met. on } G/G_Z \}.
\]

For each $H \in Z^*_c \cap a$, the diffeomorphism $\hat{\theta}_Z : M = G/G_Z \to M = G/G_Z$ preserves a $G$-invariant Kähler metric $g \left( \frac{1}{2\pi \sqrt{-1}} H \right)$ on $M$, that is, $\hat{\theta}_Z : M = G/G_Z \to M = G/G_Z$ is an isometry with respect to $g \left( \frac{1}{2\pi \sqrt{-1}} H \right)$. Hence the canonically embedded $R$-space $\nu_Z(K/K_Z)$ is a real form, that is, the fixed point subset of a Kähler $C$-space $M = G/G_Z$ by the anti-holomorphic isometry $\hat{\theta}_Z$ with respect to $J_Z$ and a Kähler metric $g \left( \frac{1}{2\pi \sqrt{-1}} H \right)$ for any $H \in Z^*_c \cap a$.

Next we mention the characterization of a $G$-invariant Einstein-Kähler metric on $M = G/G_Z$. Set
\[
\delta_m := \frac{1}{2} \sum_{\alpha \in \Sigma^+(g) \setminus \Sigma_Z(g)} \alpha \in \mathfrak{t}.
\]

Lemma 2.1 ([1]).

\[
(2.11) \quad 2\delta_m = \sum_{\alpha \in \Sigma^+(g) \setminus \Sigma_H(g)} \alpha \in Z^+_c = \bigoplus_{\alpha \in \Pi \setminus \Pi_H} Z^+ \Lambda_{\alpha}.
\]
and it corresponds to the first Chern class of the complex manifold $(M, J_M)$:

$$c_1(M) = \left[ \omega \left( \frac{1}{2\pi \sqrt{-1}} 2 \delta_m \right) \right] = \tau \left( \frac{1}{2\pi \sqrt{-1}} 2 \delta_m \right).$$

**Proposition 2.2** ([24]). The $G$-invariant Kähler metric $g = g \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right)$ on $M$ is Einstein if and only if $\lambda = b \delta_m$ for some $b > 0$.

Then we can show that $2 \delta_m \in a$ ([20]). Therefore we obtain

**Proposition 2.3.** The element $Z^{\text{ein}} := 2 \delta_m \in Z_{c^+} \cap a \subset C^+ \cap a$ corresponds to the canonical embedding $\tau_{Z^{\text{ein}}}$ of the same $R$-space $L = K/K_Z$ into an Einstein-Kähler $C$-space $M = G/G_Z, \omega_{Z^{\text{ein}}}, J_z, g \left( \frac{1}{2\pi \sqrt{-1}} Z^{\text{ein}} \right)$. Moreover, the element $Z^{\text{ein}}$ is such a unique element of $c^+ \cap a$ up to the multiplication by a positive constant.

Here we shall mention about geometry of $R$-spaces as homogeneous spaces of noncompact real semisimple Lie groups. Set $p^z := \sqrt{-1}p$. Then $g^z := \mathfrak{t} + p^z$ is the Cartan decomposition of a noncompact real semisimple Lie algebra $g^z$ with Cartan involution $\tau$. Let $G^C$ be a connected complex Lie group without center with Lie algebra $g^C$ and then $G$ can be regarded as an analytic subgroup of $G^C$. Let $G^l$ be a connected real semisimple Lie subgroup of $G^C$ with Lie algebra $g^l$. The root space decomposition of $g^l$ with respect to $\sqrt{-1}a$ is given as

$$g^l = g^0_l + \bigoplus_{\gamma \in \Sigma(g,a)} g^\gamma_l,$$

where $g^0_l := \{ X \in g^l \mid [\sqrt{-1}H, X] = 0 \ (\forall H \in a) \}$ and for each $\gamma \in \Sigma(g,a)$

$$g^\gamma_l := \{ X \in g^l \mid [\sqrt{-1}H, X] = (\gamma, H)X \ (\forall H \in a) \}.$$

Then

$$u := g^0_l + \bigoplus_{\gamma \in \Sigma(g,a), \gamma(Z) \geq 0} g^\gamma_l.$$

is a parabolic subalgebra of $g^l$. Let $U$ be a parabolic subgroup of $G^l$ with Lie algebra $u$, which is always connected. The complexification of $u$

$$u^C = (g^0_l)^C + \bigoplus_{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}), \gamma(Z) \geq 0} (g^\gamma_l)^C = \mathfrak{t}^C + \bigoplus_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}), \alpha(Z) \geq 0} \mathfrak{g}_\alpha^C.$$

is a complex parabolic subalgebra of $g^C$. Let $U^C$ be a complex parabolic subgroup of $G^l$ with Lie algebra $\mathfrak{u}$, which is always connected. Then we know ([23]) that

$$\begin{align*}
(2.12) & \quad KU = G^l, K \cap U = K_Z, \text{ and thus } L = K/K_Z \cong G^l/U, \\
(2.13) & \quad GU^C = G^C, G \cap U^C = G_Z, \text{ and thus } G/G_Z \cong M = G^C/U^C.
\end{align*}$$
The induced complex structure of $M$ under the identification of $M = G/G_Z$ with the complex homogeneous space $G^C/U^C$ coincides with the $G$-invariant complex structure $J_Z$ of $M$.

Define two subgroups of $K$ and $K_Z$ as

$$N_K(a) := \{ k \in K \mid \text{Ad}(k)a = a \} \subset K, \quad N_{K_Z}(a) := N_K(a) \cap K_Z \subset K_Z.$$  

Note that $N_{K_Z}(a)$ is not a normal subgroup of $N_K(a)$, $C_K(a) \subset N_{K_Z}(a)$, and if $Z \in a$ is regular, then $C_K(a) = N_{K_Z}(a)$:

$$1 \rightarrow N_{K_Z}(a)/C_K(a) \rightarrow W(G,K) = N_K(a)/C_K(a) \rightarrow N_K(a)/N_{K_Z}(a) \rightarrow 1$$

**Theorem 2.4** (Masaru Takeuchi [23]). Let $k_1, \cdots, k_b \in N_K(a)$ be complete representatives of $N_K(a)/N_{K_Z}(a) = \{ [k_1], \cdots, [k_b] \}$. Then the orbits $Nk_1, \cdots, Nk_b$ of $N$ through the origin $o = eU \in G^2/U = L$ provide a cellular decomposition of $L$ as $L = G^2/U = Nk_1 \cup \cdots \cup Nk_b$ and these cells are all cycles mod 2 of $L$. In particular, $\dim H_* (L;\mathbb{Z}_2) = \sum_{i=0}^L \dim H_i (L;\mathbb{Z}_2) = b$.

We briefly discuss related results of Masaru Takeuchi and Shoshichi Kobayashi ([27]) on perfect Morse functions on $R$-spaces. For each $X \in \mathfrak{g}$, we define a linear function $u_X : \mathfrak{g} \rightarrow \mathbb{R}$ defined by $u_X(\xi) := \langle \xi, X \rangle$ for each $\xi \in \mathfrak{g}$. A smooth function $f_X$ on $M = G/G_Z$ is defined by

$$\tilde{f}_X := u_X \circ \Phi_Z = u_X \circ \mu_G = \langle \mu_G, X \rangle : M = G/G_Z \rightarrow \mathfrak{g} \rightarrow \mathbb{R}.$$  

Then by the moment map equation and (2.10) we have

$$d\tilde{f}_X = d(\mu_G, X) = \omega_Z (\tilde{X}, \cdot) = -g_Z (J_Z \tilde{X}, \cdot),$$  

where $\tilde{X}$ denotes a vector field on $M = G/G_Z$ induced by a one-parameter subgroup $\{ \exp(tX) \mid t \in \mathbb{R} \}$ of $G$, which is a Killing vector field on $M$ with respect to a Kähler metric $g_Z$. Hence the gradient vector field $\text{grad}(\tilde{f}_X)$ on $M = G/G_Z$ with respect to the invariant Kähler metric $g_Z$ is equal to $-J_Z \tilde{X}$:

$$\text{grad}(\tilde{f}_X) = J_Z \tilde{X} = (\sqrt{-1}X).$$  

Here $J_Z \tilde{X} = (\sqrt{-1}X)$ is a holomorphic vector field on $M = G^C/U^C$ induced by a one-parameter subgroup $\{ \exp(t\sqrt{-1}X) \mid t \in \mathbb{R} \}$ of $G^C$.

Now assume that $X \in \mathfrak{p}$. A smooth function $f_X$ on $L = K/K_Z$ is defined by

$$f_X = \tilde{f}_X \circ t_Z = u_X \circ \phi_Z = \langle \mu_G \circ t_Z, X \rangle : L = K/K_Z \rightarrow \mathfrak{p} \rightarrow \mathbb{R}.$$  

By pulling back the equation (2.14) by the canonical embedding $t_Z$, we have

$$df_X = t_Z^* d\tilde{f}_X = t_Z^* \omega_Z (\tilde{X}, \cdot) = -g_Z ((\sqrt{-1}X) \circ t_Z, (t_Z)_*(\cdot)).$$
Since $\sqrt{-1}X \in \sqrt{-1}p = p^2 \subset g^2 \subset g^2$, $\exp(t\sqrt{-1}X) \mid t \in \mathbb{R}$ is a one-parameter subgroup of $G^2$ and it induces a vector field $(\sqrt{-1}X)$ on $L = G^2/U$. Since

$$(t_Z)_*(((\sqrt{-1}X)) = (\sqrt{-1}X) \circ t_Z = J_Z \circ t_Z,$$

the equation (2.16) becomes

$$(2.17) \quad df_X = -\langle i_Z g_Z \rangle((\sqrt{-1}X), \cdot)$$

Hence the gradient vector field $\nabla (f_X)$ on $L = K/K_Z$ with respect to the induced Riemannian metric $\langle i_Z g_Z \rangle$ is equal to a vector field $-(\sqrt{-1}X)$ on $L = K/K_Z = G^2/U$ induced by $-\sqrt{-1}X \in \sqrt{-1}p \subset g^2$. In particular, the critical point set of $f_X$ on $L$ coincides with the zero set $\text{Zero}(\sqrt{-1}X) = \text{Zero}(\tilde{X} \circ t_Z)$ of vector fields $(\sqrt{-1}X)$ and $\tilde{X} \circ t_Z$ on $L$. In [27] they showed that for each $X = \text{Ad}(k)H \in p \ (k \in K, H \in a)$, it holds $\ell_k(N_K(a)eK_Z) \subset \text{Zero}(\sqrt{-1}X)$ and if $X$ is regular, then $\ell_k(N_K(a)eK_Z) = \text{Zero}(\sqrt{-1}X)$. Here $\ell_k : K/K_Z \rightarrow K/K_Z$ denotes the left natural action by $k \in K$ on $K/K_Z$. Therefore, for each regular $X \in p$, the number of critical points of $f_X$ is equal to $\sharp(N_K(a)/N_{K_Z}(a))$ and thus $b = \dim H_o(K/K_Z, \mathbb{Z}_2)$. In particular $f_X$ is a connected open dense subset of codimension at least 2 and

$$(2.18) \quad \psi : K/Z_K(a) \times \hat{A} \ni (kZ_K(a), \hat{a}) \longrightarrow k\hat{a} \in G/K$$

is a surjective smooth map. Define the diagram of a compact symmetric pair $(G, K)$ by

$$D(g, a) := \{H \in a \mid (\gamma, H) \in \pi Z \ (\exists r \in \Sigma(g, a))\}$$

and thus we have $a \setminus D(g, a) = \{H \in a \mid (\gamma, H) \not\in \pi Z \ (\forall r \in \Sigma(g, a))\}$. Set $A_s := (\exp D(g, a))eK \subset A$ and $A_r := A \setminus A_s := (\exp a \setminus D(g, a))eK$. Each element of $A_s$ (resp. $A_r$) is called a regular (resp. singular) element of $A$. Then $G/K = (G/K)_s \cup (G/K)_r$ (disjoint union), where $(G/K)_s := \psi(K/Z_K(a) \times A_s)$ is a closed set of codimension at least 2 and

$$(2.19) \quad (G/K)_r := \psi(K/Z_K(a) \times A_r)$$

is a connected open dense subset of $G/K$. Each element of $(G/K)_r$ is called a regular element of $G/K$. The surjective smooth map

$$(2.20) \quad \psi : K/Z_K(a) \times A_r \longrightarrow (G/K)_r$$

is a covering map whose covering transformation group is the right natural action of $W(G, K)$ on $K/Z_K(a) \times A_r$, and thus it induces a diffeomorphism $\tilde{\psi} : (K/Z_K(a) \times A_r)/W(G, K) \longrightarrow (G/K)_r$ which is equivariant with the actions of $K$. Here note that $K/Z_K(a) \times A_r$ is not connected in general.
Using the geometry of a compact symmetric space $G/K$, we discuss the intersection property of $a \iota_Z(L)$ and $\iota_Z(L)$ under the left group action of $a \in G$ on $M = G/G_Z$.

For any $a \in G$, by the surjectivity of $\psi$ we have $aK = \psi(kZ_K(a), \exp(H)eK)$ for some $k \in K$ and some $H \in a$ and thus $ak_1 = kk_0 \exp(H)$ for some $k_0 \in Z_K(a)$ and some $k_1 \in K$. Thus $ak_1G_Z = kk_0 \exp(H)G_Z = kG_Z \subset G/G_Z$ and hence $a \iota_Z(k_1K_Z) = \iota_Z(kK_Z) = a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ gives an intersection point of $a \iota_Z(K/K_Z)$ and $\iota_Z(K/K_Z)$. Moreover, for any $k' \in N_K(a)$, we have $ak_1k' = kk'(k'^{-1}k_0k') \exp(\text{Ad}(k'^{-1})H)$ where note that $k'^{-1}k_0k' \in Z_K(A)$ and $\text{Ad}(k'^{-1})H \in a$. Thus $ak_1k'G_Z = kk'(k'^{-1}k_0k') \exp(\text{Ad}(k'^{-1})H)G_Z = kk'G_Z \subset G/G_Z$ and hence $a \iota_Z(k_1k'K_Z) = \iota_Z(kk'K_Z) = a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ also gives an intersection point of $a \iota_Z(K/K_Z)$ and $\iota_Z(K/K_Z)$. Note that if $kk'K_Z = kk''K_Z$ for $k, k'' \in N_K(a)$, then $kkN_KZ(a) = kk''N_KZ(a)$. Therefore, combining the above argument with Theorem 2.4, we obtain

**Proposition 2.5.** For any $a \in G$, it holds

\[ \dim (a \iota_Z(L) \cap \iota_Z(L)) \geq \dim (N_K(a)/N_KZ(a)) = \dim H_\ast(L, \mathbb{Z}_2). \]

Next we mention about the transversality condition of the intersection $a \iota_Z(L) \cap \iota_Z(L)$. Suppose that $p \in a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$. Then $p = a \iota_Z(k_2K_Z) = \iota_Z(k_1K_Z)$ ($\exists k_1, k_2 \in K$). Since $k_1^{-1}ak_2 \in G_Z$, using the symmetric Lie algebra $\mathfrak{g}_Z = \mathfrak{t}_Z + \mathfrak{p}_Z$ of a compact symmetric pair $(G_Z, K_Z)$ so that $a \subset \mathfrak{p}_Z$, there are $k_Z, k'_Z \in K_Z$ and $H_p \in a$ such that $k_1^{-1}ak_2 = k_Z(\exp H_p)k_Z^{-1}k'_Z$. Thus $ak_2 = k_1k_Z(\exp H_p)k_Z^{-1}k'_Z$. Since

\[
(\Phi_Z)_*(T_p a \iota_Z(K/K_Z)) = \text{Ad}(ak_2)[t, Z] = \text{Ad}(k_1k_Z(\exp H_p)k_Z^{-1}k'_Z)[t, Z] = \text{Ad}(k_1)[\text{Ad}(k_Z)\text{Ad}(\exp H_p)[t, Z], 
\]

the transversality of $a \iota_Z(L) \cap \iota_Z(L)$ at $p$ is equivalent to the transversality of $\text{Ad}(\exp H_p)[t, Z]$ and $[t, Z]: \text{Ad}(\exp H_p)[t, Z] \cap [t, Z] = \{0\}$. Then by a simple computation using the basis $\{S_\gamma, T_{\gamma, t}\}$ we can show

**Lemma 2.6.** $a \iota_Z(L)$ intersects transversally with $\iota_Z(L)$ in $M = G/G_Z$ if and only if at each intersection point $p$ such an $H_p \in a$ satisfies $(\gamma, H) \notin \pi \mathbb{Z}$ for each $\gamma \in \Sigma^+(\mathfrak{g}, a)$ with $(\gamma, Z) \neq 0$.

First we suppose that $a \in G$ satisfies $aK \in (G/K)_\gamma$. Then $a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is transversal at each intersection point. Fix an intersection point $\iota_Z(k_1K_Z) \in a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$. Then

\[
\begin{align*}
  aK &= k_1k_Z(\exp H_1)K \quad (\exists H_1 \in a \setminus \mathfrak{D}(\mathfrak{g}, a), \exists k_z \in K_Z) \\
  &= \psi(k_1k_Z, Z_K(a), \exp(H_1)K)
\end{align*}
\]
For an arbitrary intersection point \( \iota_Z(k_2K_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \),
\[
aK = k_2k_{Z,2} \exp(H_2)K \quad (\exists H_2 \in a \setminus \mathbb{D}(G, K), k_{Z,2} \in K_Z)
\]
\[
= \psi(k_2k_{Z,2}Z_K(a), \exp(H_2)K)
\]
Then there is \( s = [k'] \in W(G, K) = N_K(a)/Z_K(a) \) such that
\[
k_2k_{Z,2}Z_K(a) = k_1k_{Z,1}Z_K(a)s = k_1k_{Z,1}k'Z_K(a)
\]
and
\[
(\exp H_2)K = s^{-1}(\exp H_1)K = (\exp s^{-1}H_1)K = (\exp Ad(k')^{-1}H_1)K.
\]
Thus \( k_2K_Z = k_1k_{Z,1}k'K_Z \) and hence \( \iota_Z(k_2K_Z) = \iota_Z(k_1k_{Z,1}k'K_Z) \). Therefore we obtain
\[
a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) = \{ \iota_Z(k_1k_{Z,1}k'K_Z) \mid [k'] \in N_K(a)/N_{K_Z}(a) \}.
\]
In particular \( \sharp(a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(a)/N_{K_Z}(a)) \).

In general suppose that \( a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \) is transversal at each intersection point. Particularly \( a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \) is a finite set. We may assume that \( aK \in (G/K)_r \). Since \( (G/K)_r \) is open and dense in \( G/K \), we choose a smooth perturbation \( a_t \in G \) of \( a_0 = a \) such that \( a_tK \in (G/K)_r \) for all \( t \). Then \( a_t\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \) is also transversal at each intersection point and for sufficiently small \( t > 0 \) we have
\[
\sharp(a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(a_t\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(a)/N_{K_Z}(a)).
\]
Therefore we obtain

**Proposition 2.7.** For any \( a \in G \) with transversal \( a\iota_Z(L) \cap \iota_Z(L) \), it holds
\[
\sharp(a\iota_Z(L) \cap \iota_Z(L)) = \sharp(N_K(a)/N_{K_Z}(a)).
\]
Combining it with Theorem 2.4, we see

**Corollary 2.8.** For any \( a \in G \) with transversal \( a\iota_Z(L) \cap \iota_Z(L) \), it holds
\[
\sharp(a\iota_Z(L) \cap \iota_Z(L)) = \dim H_*(L, \mathbb{Z}_2).
\]

Such a property is called the **global tightness** for a Lagrangian submanifolds in a Kähler C-space ([14], [9], [5]). It was proved by [28] in the case when \( L \) is a symmetric R-space. It is still an open problem to classify compact globally tight or simply tight Lagrangian submanifolds of Kähler C-spaces. More generally the intersection theory and Floer homology for two real forms in Kähler C-spaces are discussed in [7], [11].
3 Minimal Maslov number and monotonicity of Lagrangian submanifolds in symplectic manifolds and Einstein-Kähler manifolds

Let \( L \) be a Lagrangian submanifold of a symplectic manifold \((M, \omega)\). Define two kinds of group homomorphisms \( I_{\mu,L} : \pi_2(M,L) \to \mathbb{Z} \) and \( I_{\omega,L} : \pi_2(M,L) \to \mathbb{R} \). For a smooth map \( u : (D^2, \partial D^2) \to (M,L) \) with \( A = [u] \in \pi_2(M,L) \), choose a trivialization of the pull-back bundle as a symplectic vector bundle (which is unique up to the homotopy). \( u^{-1}TM \cong D^2 \times \mathbb{C}^n \). This gives a smooth map \( \tilde{u} : S^1 = \partial D^2 \to \Lambda(\mathbb{C}^n) \). Here \( \Lambda(\mathbb{C}^n) \) denotes the Grassmann manifold of Lagrangian vector subspaces of \( \mathbb{C}^n \). Using the Moslov class \( \mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z} \), we define \( I_{\mu,L}(A) = \mu(\tilde{u}) \). Another homomorphism \( I_{\omega,L} : \pi_2(M,L) \to \mathbb{R} \) is defined by \( I_{\omega,L}(A) = \int_{D^2} u^*\omega \). Note that \( I_{\mu,L} \) is invariant under symplectic diffeomorphisms and \( I_{\omega,L} \) is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.

A Lagrangian submanifold \( L \) of \((M, \omega)\) is called monotone ([15]) if \( I_{\mu,L} = \lambda I_{\omega,L} \) for some \( \lambda > 0 \). If \( I_{\mu,L} = 0 \), we define \( \Sigma_L = 0 \). We assume that \( I_{\mu,L} \neq 0 \), and denote by \( \Sigma_L \in \mathbb{Z}^+ \) the positive generator of \( \text{Im}(I_{\mu,L}) \) as an additive subgroup of \( \mathbb{Z} \). We call \( \Sigma_L \) the minimal Maslov number of \( L \).

**Theorem 3.1** ([2], [21]). Suppose that \((M, \omega, J, g)\) is an Einstein-Kähler manifold of positive Einstein constant. If \( L \) is a compact minimal Lagrangian submanifold of \( M \), then \( L \) is monotone.

Suppose that \((M, \omega, J, g)\) is a simply connected Einstein-Kähler manifold with positive Einstein constant and \( L \) is a compact monotone Lagrangian submanifold of \( M \). Then Hajime Ono ([21]) showed the formula for \( \Sigma_L \):

\[
(3.1) \quad n_L \Sigma_L = 2\gamma_{c_1},
\]

where we set

\[
\gamma_{c_1} := \min \{ c_1(M)(A) \mid A \in H_2(M; \mathbb{Z}), c_1(M)(A) > 0 \},
\]

\[
n_L := \min \{ k \in \mathbb{Z}^+ \mid \otimes^k E \text{ trivial} \}.
\]

Here \( \frac{1}{2} \omega = c_1(E, \nabla) \) for some constant \( \gamma > 0 \).

As an application of that formula (3.1), we mention the minimal Maslov number formula for the Gauss images of isoparametric hypersurfaces in the standard sphere.

Let \( N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} \) be an oriented hypersurface of \( S^{n+1}(1) \) and let \( \hat{N}^n := \{(x(p), n(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\} \) be the Legendrian lift of \( N^n \) to \( T^1 S^{n+1} \). Then we have the following diagram:
Then it holds \( \Sigma = \frac{2^n}{g} \), \((19), (13)\). This formula was crucial to the Hamiltonian non-displaceability theorem for Gauss images of isoparametric hypersurfaces \((8)\).

### 4 Minimal Maslov number of \(R\)-spaces canonically embedded in Einstein-Kähler \(C\)-spaces

We take the universal cover \(\tilde{G} \to G\) of \(G\). Let \((\tilde{G}, \tilde{K}, \theta)\) be a Riemannian symmetric pair of compact type with simply connected \(\tilde{G}\) and connected \(\tilde{K}\).

By Proposition 2.3 we can choose \(Z = Z^{e}m = 2\delta_{m}\). Then \(\iota_{Z} : L = K/K_{Z} \to M = G/G_{Z}\) is the canonical embedding of an \(R\)-space into an Einstein-Kähler \(C\)-space. As in \((24)\) we use expression

\[
2\delta_{m} = \sum_{\alpha_{i} \in \Pi \setminus \Pi_{Z}} k_{i}\Lambda_{i} = \kappa(M) \sum_{\alpha_{i} \in \Pi \setminus \Pi_{Z}} \kappa_{i}\Lambda_{i} \quad (k_{i} \in \mathbb{Z}^{+}),
\]
where $\kappa(M)$ denotes the greatest common divisor of $\{k_i \mid \alpha_i \in \Pi \setminus \Pi_Z\}$ and set $\kappa_i := \frac{k_i}{\kappa(M)}$ for each $\alpha_i \in \Pi \setminus \Pi_Z$. Then the invariant $\gamma_c$ in (3.1) is given as $\gamma_c = \kappa(M)$ (cf. [20]).

\[\bar{L} = \tilde{K}/\tilde{K}_Z \quad \longrightarrow \quad \tilde{P} = \tilde{G}/\tilde{G}'_Z \quad \longrightarrow \quad \tilde{\pi} \quad \rho(\pi_1(L)) \quad \pi \quad U(1) \cong S^1 \quad \longrightarrow \quad \bar{M} = \tilde{G}/\tilde{G}_Z \quad \text{Einstein-Kähler C-space.}\]

Here we take the orthogonal direct sum decomposition $g_Z = \mathbb{R} \cdot 2\delta_m \oplus g'_Z$. Denote by $\tilde{G}'_Z$ a connected Lie subgroup of $\tilde{G}_Z$ with Lie algebra $g'_Z$ and set $\tilde{K}_Z := \tilde{K} \cap \tilde{G}'_Z$.

Theorem 4.1 ([20]). The minimal Maslov number $\Sigma_L$ of an $R$-space $L$ canonically embedded in an Einstein-Kähler C-space $M$ is given by the formula

\[(4.1) \quad \Sigma_L = \frac{2\kappa(M)}{\sharp(K_Z/K'_Z)}.\]

Some concrete examples of computations by this formula are given in [20] in the case when (1) $(G, K) = (SU(n+1), SO(n+1))$ and $L$ is $RP^n$ or a regular $R$-space, (2) $L$ is a maximal flag manifold of a compact semisimple Lie group $K$, (3) $L$ is an irreducible symmetric $R$-space. It is an interesting problem to study the minimal Maslov number for all other $R$-spaces by this formula.

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References


