In this article we shall provide an exposition on harmonic maps of Riemann surfaces and pluriharmonic maps of complex manifolds into compact symmetric spaces from the viewpoint of integrable system theory, and we shall mention recent results in a joint work with S. Udagawa.

1. Harmonic maps of Riemann surfaces and pluriharmonic maps of complex manifolds

Let $M$ be a compact Riemann surface and $N$ be a Riemannian manifold, more generally a smooth manifold with a torsion-free affine connection $\nabla^N$. Let $f : M \rightarrow N$ be a smooth map and $f^{-1}TN$ be the pull-back bundle over $M$ by $f$ from the tangent bundle $TN$ of $N$. Denote by $\nabla^f$ the induced connection in $f^{-1}TN$ by $f$ from $\nabla^N$. The harmonic map equation for $f$ is

$$\nabla^f_{\frac{\partial}{\partial \bar{z}}} df \left( \frac{\partial}{\partial z} \right) dz \wedge d\bar{z} = 0, \quad (1.1)$$

where $\{z\}$ denotes a local holomorphic coordinate of $M$.

The notion of pluriharmonic maps of complex manifolds is a natural generalization of harmonic maps of Riemann surfaces. A smooth map $f : M \rightarrow N$ of a complex manifold $M$ is called a pluriharmonic map if $f$ satisfies the equation

$$d\sigma f = \nabla^f_{\frac{\partial}{\partial z^i}} df \left( \frac{\partial}{\partial \bar{z}^j} \right) dz^i \wedge d\bar{z}^j = 0, \quad (1.2)$$

The following property is fundamental for pluriharmonic maps.

**Proposition 1.1.** A smooth map $f : M \rightarrow N$ of a complex manifold $M$ is pluriharmonic if and only if for every holomorphic map $i : S \rightarrow M$ of a Riemann surface the composition $f \circ i : S \rightarrow N$ is harmonic.

See [15], [16], [17], [20] for some basic results on pluriharmonic maps of Kähler manifolds.

2. Pluriharmonic maps into homogeneous spaces

2.1. Zero curvature representation of the pluriharmonic map equation. Suppose that $N = G/H$ is a reductive homogeneous space
of a Lie group $G$ by a closed subgroup $H$. By definition we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ as vector spaces. Here $\mathfrak{p}$ is a vector subspace of $\mathfrak{g}$ such that $\text{Ad}H(\mathfrak{p}) = \mathfrak{p}$. Let $\pi: G \rightarrow G/H$ be the natural projection. For a smooth map $f: M \rightarrow G/H$, a smooth map $F: M \rightarrow G$ such that $f = \pi \circ F$ is called a framing of $f$. If $M$ is contractible, then there always exists a framing of any smooth map into $G/H$. Set

$\alpha = F^*\theta_G = F^{-1}dF$,

where $\theta_G$ denotes the left invariant Maurer-Cartan form of $G$. Then $\alpha$ satisfies the Maurer-Cartan equation

$\frac{1}{2}d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$.

We decompose a 1-form $\alpha$ on $M$ with values in $\mathfrak{g}$ into the $\mathfrak{h}$-part and the $\mathfrak{p}$-part along $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ as

$\alpha = \alpha_h + \alpha_p$.

Suppose that $M$ is a complex manifold. Moreover we decompose $\alpha_h$ and $\alpha_p$ into $(1,0)$- and $(0,1)$-parts as

$\alpha_h = \alpha_h' + \alpha_h''$, and $\alpha_p = \alpha_p' + \alpha_p''$.

Choose an arbitrary $G$-invariant Riemannian metric $g$ on $G/H$ and denote by $\langle , \rangle$ the corresponding $\text{Ad}(H)$-invariant inner product of $\mathfrak{p}$. The equation (2.2) is equivalent to the system of the equations

$\partial\alpha_h' + \frac{1}{2}[\alpha_h' \wedge \alpha_h']_h + \frac{1}{2}[\alpha_p' \wedge \alpha_p']_h = 0$,

$\bar{\partial}\alpha_p' + \partial\alpha_p'' + [\alpha_h' \wedge \alpha_h'']_h + [\alpha_p' \wedge \alpha_p'']_h = 0$,

$\partial\alpha_h'' + [\alpha_h'' \wedge \alpha_p']_h + \frac{1}{2}[\alpha_p' \wedge \alpha_p']_h = 0$,

$\bar{\partial}\alpha_p'' + \partial\alpha_p' + [\alpha_h'' \wedge \alpha_p']_h + [\alpha_h'' \wedge \alpha_p']_h + [\alpha_p'' \wedge \alpha_p']_h = 0$.

Note that the first and third equations are always satisfied in the case when $M$ is a Riemann surface.

The pluriharmonic map equation for $f$ is given as follows:

$\bar{\partial}\alpha_p' + [\alpha_h'' \wedge \alpha_p'] = -\frac{1}{2}[\alpha_p' \wedge \alpha_p'']_p + U(\alpha_p' \wedge \alpha_p'')$.

Here $U: \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ is a symmetric bilinear form defined by

$2\langle U(X,Y),Z \rangle = \langle X, [Z,Y]_p \rangle + \langle Y, [Z,X]_p \rangle$.

We can obtain the pluriharmonic map equation (2.9) by using the harmonic map equation described in [8].

For each $\lambda \in S^1$, we define a 1-form $\alpha^\lambda$ on $M$ with values in $\mathfrak{g}$ by

$\alpha^\lambda := \alpha_h + \lambda^{-1}\alpha_p' + \lambda\alpha_p''$.
We assume that \( f : M \to G/K \) is a pluriharmonic map satisfying the condition (\( \ast \)):

\[
\begin{align*}
[\alpha'_p \wedge \alpha''_p]_p &= 0, \\
U(\alpha'_p \wedge \alpha''_p) &= 0, \\
[\alpha'_p \wedge \alpha'_p] &= 0.
\end{align*}
\]

Note that the third equation here is always satisfied in the case when \( M \) is a Riemann surface. Then the Maurer-Cartan equation

\[
d\alpha^\lambda + \frac{1}{2} [\alpha^\lambda \wedge \alpha^\lambda] = 0
\]

holds for each \( \lambda \in S^1 \). The converse also holds. Thus if \( M \) is simply connected, then there is an \( S^1 \)-family of framings \( F_\lambda : M \to G \ (\lambda \in S^1) \) such that \( F_\lambda^*\theta_G = \alpha^\lambda \) for each \( \lambda \in S^1 \). The \( S^1 \)-family of framings is called an extended framing. Thus we obtain the \( S^1 \)-associated family of pluriharmonic maps \( f_\lambda = \pi \circ F_\lambda : M \to G/H \). The following result is a well-known fact.

**Theorem 2.1.** If \( G/H \) is a symmetric space, then every pluriharmonic map \( f : M \to G/H \) always satisfies the first equation and the second equation in the condition (\( \ast \)). Furthermore, if \( G/H \) is a Riemannian symmetric space without noncompact factor or without compact factor, then every pluriharmonic map \( f : M \to G/H \) always satisfies the third equation in the condition (\( \ast \)).

The second statement of this theorem was proved in [20].

**Problem 2.1.** Find new examples of harmonic maps or pluriharmonic maps satisfying the condition (\( \ast \)) other than super-horizontal holomorphic maps and primitive maps which will be discussed in the later subsections.

Using these results we want to point out the gauge-theoretic formulation for pluriharmonic maps into symmetric spaces. Consider the gauge-theoretic equation in a principal \( G \)-bundle over a complex manifold \( M \) : For \( (A, \phi) \in \mathcal{A}_P \times \Omega^1(g_P) \),

\[
(\ast)_{\hat{G}}^G
\]

\[
\begin{align*}
F(A) &\pm \frac{1}{2} [\phi \wedge \phi] = 0, \\
d'_{A'} \phi' &= d''_{A'} \phi' = 0, \\
[\phi' \wedge \phi'] &= 0,
\end{align*}
\]

where \( \phi' \) denotes the (1,0)-component of \( \phi \). The equation (\( \ast \))_{\hat{G}}^G is a gauge-theoretic equation for pluriharmonic maps into \( G \).

Suppose that \( G/K \) is a Riemannian symmetric space with an involutive automorphism \( \sigma \). Then we can consider a reduction of the gauge-theoretic equation (\( \ast \))_{\hat{G}}^G to \( G/K \). Let \( g = \mathfrak{k} + \mathfrak{m} \) be the standard decomposition of the symmetric Lie algebra for \( G/K \). Let \( Q \) be an
arbitrary subbundle with structure group $K$ reduced from a principal $G$-bundle $P$. We denote by

$$m_Q := Q \times_K m$$

the vector bundle associated with the adjoint action of $K$ on $m$. Consider the gauge-theoretic equation with respect to $(A, \phi) \in A_Q \times \Omega^1(m_Q)$,

$$F(A) \pm \frac{1}{2} [\phi \wedge \phi] = 0,$$

$$d' \phi' = \alpha^\prime = 0,$$

$$[\phi' \wedge \phi'] = 0.$$

The equation $(*)_G$ is a gauge-theoretic equation for pluriharmonic maps into $G/K$.

The gauge-theoretic equations for harmonic maps of Riemann surfaces and the moduli spaces of their solutions were discussed in detail in [14].

2.2. Loop group actions for pluriharmonic maps. Let $G/K$ be a symmetric space with an involutive automorphism $\sigma$ of $G$. Suppose that $G/K$ has no noncompact factor or no compact factor. The associated twisted loop group and the twisted loop algebra are defined as

$$\Lambda G^C_\sigma := \{ \gamma \in \Lambda G^C \mid \sigma(\gamma(\lambda)) = \gamma(-\lambda) \}$$

and

$$\Lambda g^C_\sigma := \{ \xi \in \Lambda g^C \mid \sigma(\xi(\lambda)) = \xi(-\lambda) \}.$$

Decomposition theorem for loops. Let $K^C = KB$ be the Iwasawa decomposition of the complexification $K^C$. Set $D_0 := \{ \lambda \in C \mid |\lambda| < 1 \}$. Define the subgroups and the subalgebras of $\Lambda G^C_\sigma$ and $\Lambda g^C_\sigma$ as follows:

$$\Lambda^+ G^C_\sigma := \{ \gamma \in \Lambda G^C_\sigma \mid \gamma \text{ extends continuously to a holomorphic map } \tilde{\gamma} : D_0 \longrightarrow G^C \},$$

$$\Lambda^+ g^C_\sigma := \{ \xi \in \Lambda g^C_\sigma \mid \xi(\lambda) = \sum_{i=0}^{\infty} \lambda^i \xi_i \}$$

and

$$\Lambda^+ B G^C_\sigma := \{ \gamma \in \Lambda^+ G^C_\sigma \mid \tilde{\gamma}(0) \in B \},$$

$$\Lambda^+ B g^C_\sigma := \{ \xi \in \Lambda^+ g^C_\sigma \mid \xi_0 \in b \},$$

where $b$ denotes the Lie algebra of $B$. Note that we have a decomposition $\Lambda^+ G^C_\sigma = K \cdot \Lambda^+ B G^C_\sigma$.

Since

$$\Lambda g^C_\sigma = \Lambda g_\sigma + \Lambda^+ B g^C_\sigma,$$
is a direct sum decomposition as vector spaces, by the inverse function theorem the multiplication map

\[(2.20)\quad \Lambda G_\sigma \times \Lambda^+_B G^C_\sigma \ni (\gamma, h) \mapsto \gamma h \in \Lambda G_\sigma \cdot \Lambda^+_B G^C_\sigma \subset \Lambda G^C_\sigma\]

becomes a local diffeomorphism around a neighborhood of the identity. In the case when $G$ is compact, the multiplication map is a diffeomorphism and we obtain a global decomposition

\[(2.21)\quad \Lambda G^C_\sigma = \Lambda G_\sigma \cdot \Lambda^+_B G^C_\sigma\]

([21], [5]). Recently global decompositions in the case when $G$ is noncompact have been investigated by J. Dorfmeister, J. Inoguchi and others.

Let $f : M \rightarrow G/K$ be a pluriharmonic map. The starting point to study such harmonic maps via integrable system theory is to regard an extended framing $F_\lambda : M \rightarrow G$ ($\lambda \in S^1$) of $f$ as a map into the loop group $\tilde{F} : M \rightarrow \Lambda G_\sigma$.

As the first application of the decomposition theorem of loops, we shall observe the existence of the group action of $\Lambda G^C_\sigma$ on extended framings of pluriharmonic maps. For each $\gamma \in \Lambda G^C_\sigma$, we define

\[(2.22)\quad \gamma \tilde{F} := (\gamma \tilde{F})_{\Lambda G_\sigma},\]

where $(\cdot)_{\Lambda G_\sigma}$ denotes the $\Lambda G_\sigma$-component in the decomposition (2.21). Then we can show that $\gamma \tilde{F} : M \rightarrow \Lambda G_\sigma$ is an extended framing of a pluriharmonic map $\gamma \tilde{f} : M \rightarrow G/K$. For an extended framing $\tilde{F} : M \rightarrow \Lambda G_\sigma$, let $[\tilde{F}]$ denote an equivalent class of extended framings containing $\tilde{F}$ modulo right multiplications by smooth maps $M \rightarrow K$.

### 2.3. The DPW formula for pluriharmonic maps

A Weierstrass type formula for harmonic maps of a Riemann surface into a compact symmetric space is well-known as the so-called DPW formula due to Dorfmeister, Pedit and Wu [5]. It is a very powerful formula for explicit constructions of harmonic maps and several kinds of integrable surfaces such as CMC surfaces and pseudo-spherical surfaces.

In this section we shall discuss a generalization of such a formula for pluriharmonic maps.

Set

\[(2.23)\quad \Lambda_{-1,\infty} := \{\xi \in \Lambda g^C_\sigma \mid \xi = \sum_{i=-1}^{\infty} \lambda_i \xi_i\}.
\]

Denote by $\Omega^{1,0}(\Lambda_{-1,\infty})$ the vector space of all smooth $(1,0)$-forms defined on $M$ with values in $\Lambda_{-1,\infty}$. Define the space $\mathcal{P}$ as

\[(2.24)\quad \mathcal{P} := \{\mu \in \Omega^{1,0}(\Lambda_{-1,\infty}) \mid \mu \text{ satisfies the following equations}\}.\]
Here

\[
\begin{aligned}
\partial \mu + \frac{1}{2}[\mu \land \mu] &= 0, \\
\partial \mu &= 0.
\end{aligned}
\]  

(2.25)

If \( \mu \in \mathcal{P} \), then we have

\[
\mu = \sum_{i=-1}^{\infty} \lambda_i \mu_i.
\]  

(2.26)

Here, if \( k \) is even, then \( \mu_k \in \Omega^{1,0}(\mathfrak{t}^C) \), and if \( k \) is odd, then \( \mu_k \in \Omega^{1,0}(\mathfrak{p}^C) \).

For \( \mu \in \mathcal{P} \), the condition

\[
d\mu + \frac{1}{2}[\mu \land \mu] = \bar{\partial} \mu + (\partial \mu + \frac{1}{2}[\mu \land \mu]) = 0
\]  

(2.27)

is equivalent to the condition (2.25).

Assume that \( g_\mu : M \to \Lambda G^C_{\sigma} \) satisfies

\[
\begin{aligned}
g_\mu^{-1}d\mu &= \mu, \\
g_\mu(z_0) &= e.
\end{aligned}
\]  

(2.28)

where \( z_0 \) is a base point of \( M \). If \( M \) is simply connected, then such a \( g_\mu \) always exists on the whole \( M \). By virtue of the decomposition theorem of loops, we can decompose \( g_\mu \) as follows :

\[
\begin{aligned}
\Phi_\mu^{-1}d\Phi_\mu &= b_\mu g_\mu^{-1}d(g_\mu b_\mu^{-1}) \\
&= b_\mu g_\mu^{-1}(dg_\mu b_\mu^{-1} - g_\mu b_\mu^{-1}db_\mu b_\mu^{-1}) \\
&= (\text{Ad}(b_\mu))\mu - db_\mu \cdot b_\mu^{-1}.
\end{aligned}
\]  

(2.30)

Thus we have

\[
\Phi_\mu^{-1}d\Phi_\mu = [(\text{Ad}(b_\mu))\mu]_{\lambda_0} \\
= (\text{Ad}(b_\mu(0))\mu_{-1}\lambda^{-1} + \alpha_0 + (\text{Ad}(b_\mu(0))\mu_{-1}\lambda),
\]  

(2.31)

where \( \alpha_0 \) denotes the \( \mathfrak{t} \)-component of the \( \lambda^0 \)-term for \( (\text{Ad}(b_\mu(\lambda)))\mu(\lambda) \). Therefore \( \Phi_\mu \) is an extended framing of a pluriharmonic map \( M \to G/K \).

Define the holomorphic gauge transformation group as

\[
\mathcal{G} := \{ h : M \to \Lambda^+ G^C_{\sigma} | \bar{\partial}h = 0 \}.
\]  

(2.32)

For each \( h \in \mathcal{G} \) and each \( \mu \in \mathcal{P} \), we define the action of \( h \) on \( \mu \) as

\[
h \cdot \mu := (\text{Ad}h)\mu - (dh)h^{-1}.
\]  

(2.33)
Then we have $h \cdot \mu \in \mathcal{P}$. Thus we obtain the action of the group $\mathcal{G}$ on the space $\mathcal{P}$. Now if we define
\begin{equation}
(2.34) \quad g_{h,\mu} := h(z_0)g_\mu h^{-1} : M \rightarrow \Lambda G^C_{\sigma},
\end{equation}
then $g_{h,\mu}(z_0) = e$ and
\begin{equation}
(2.35) \quad g_{h,\mu}^{-1}dg_{h,\mu} = h \cdot \mu
\end{equation}
is satisfied. Then we obtain
\begin{equation}
(2.36) \quad \Phi_h \mu = (g_{h,\mu})_\Lambda G_{\sigma} = h(z_0)^\sharp(\Phi_\mu \tilde{h})
\end{equation}
for some smooth map $\tilde{h} : M \rightarrow K$, that is,
\begin{equation}
(2.37) \quad [\Phi_h \mu] = h(z_0)^\sharp[\Phi_\mu].
\end{equation}
Assume that $M$ is contractible and Stein. Let $f : M \rightarrow G/K$ be a pluriharmonic map and $\tilde{F} : M \rightarrow \Lambda G_{\sigma}$ be an extended framing of $f$ with $\tilde{F}(z_0) = e$. Then we shall show that there exists $\mu \in \mathcal{P}$ such that
\begin{equation}
(2.38) \quad [\Phi_\mu] = [\tilde{F}].
\end{equation}
In order to show it, we have only to prove that there exists $h : M \rightarrow \Lambda^+ G_{\sigma}^C$ with $h(z_0) = e$ such that $g = \tilde{F}h$ and $\partial h = 0$.

The $(0,1)$-component of
\begin{equation}
(2.39) \quad g^{-1}dg = \text{Ad}(h^{-1})(\lambda^{-1}\alpha_m'' + \alpha_t + \lambda\alpha_m'') + h^{-1}dh
\end{equation}
is equal to
\begin{equation}
(2.40) \quad \text{Ad}(h^{-1})(\alpha_t'' + \lambda\alpha_m'') + h^{-1}\partial h.
\end{equation}
As the holomorphicity of $g$ is equivalent to the vanishing of the term (2.40), the problem reduces to the $\partial$-problem of solving the complex equations
\begin{equation}
(2.41) \quad \begin{cases}
(\partial h)h^{-1} = -(\alpha_t'' + \lambda\alpha_m''), \\
h(z_0) = e.
\end{cases}
\end{equation}
Here we should note that
\begin{equation}
(2.42) \quad \partial(\alpha_t'' + \lambda\alpha_m'') + \frac{1}{2}[(\alpha_t'' + \lambda\alpha_m'') \wedge (\alpha_t'' + \lambda\alpha_m'')]
\end{equation}
\begin{equation}
= (\partial\alpha_t'' + \frac{1}{2}[\alpha_t'' \wedge \alpha_t'']) + \lambda(\partial\alpha_m'' + [\alpha_t'' \wedge \alpha_m'']) + \frac{\lambda^2}{2}[\alpha_m'' \wedge \alpha_m''] = 0.
\end{equation}
Since the argument in Appendix of [5] still works also for the higher dimensional case, we can solve locally the $\partial$-problem.

The relationship between Frobenius manifolds with special geometry and pluriharmonic maps into $GL(n; \mathbb{R})/O(n)$ was discovered by Dubrovin [7] (see also [6]). We want to discuss also them elsewhere.
2.4. Super-horizontal holomorphic maps into generalized flag manifolds. First we shall briefly review geometry of flag manifolds. Let $G$ be a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\langle \ , \ \rangle$ denote an Ad-invariant inner product of $\mathfrak{g}$. Choose a canonical element $\xi \in \mathfrak{g}$ corresponding to a subset $I \subset \{1, 2, \cdots, \ell\}$ (cf. [4]), where $\ell = \text{rank} \mathfrak{g}$. Note that the eigenvalues of $\text{ad}\xi$ belongs to $\sqrt{-1}\mathbb{Z}$.

We take the eigenspace decomposition of $\mathfrak{g}^\mathbb{C}$ as
\begin{equation}
\mathfrak{g}^\mathbb{C} = \sum_r \mathfrak{g}_r(\xi),
\end{equation}
where $\sqrt{-1}r$ runs through all eigenvalues of $\text{ad}\xi$ and
\begin{equation}
\mathfrak{g}_r(\xi) := \{X \in \mathfrak{g}^\mathbb{C} \mid (\text{ad}\xi)X = \sqrt{-1}rX\}.
\end{equation}
Define the centralizer in $G$ of $\xi$ as
\begin{equation}
H = C_G(\xi) = \{a \in G \mid \text{Ad}(a)\xi = \xi\}.
\end{equation}
The compact homogeneous space $Z = G/H = \text{Ad}(G)$ is called a generalized flag manifold. Denote by $\mathfrak{h}$ the Lie algebra of $H$. Let $\mathfrak{p}$ be the orthogonally complementary vector subspace of $\mathfrak{g}$ to $\mathfrak{k}$ with respect to $\langle \ , \ \rangle$ so that
\begin{equation}
\mathfrak{p}^\mathbb{C} = \sum_{r<0} \mathfrak{g}_r(\xi).
\end{equation}
Then we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and the vector subspace $\mathfrak{p}$ can be identified with the tangent vector space $T_oZ$. The space $Z$ has the following rich geometric structures.

1. $Z$ has a $G$-invariant complex structure defined by
\begin{equation}
T_oZ^{1,0} \cong \bigoplus_{r>0} \mathfrak{g}_r(\xi)
\end{equation}
and $Z = G/H = G^\mathbb{C}/U$ is a simply connected compact complex homogeneous space. Here $U$ is the parabolic subgroup of $G^\mathbb{C}$ with parabolic subalgebra $\mathfrak{u} = \sum_{i\geq 0} \mathfrak{g}_{-i}$.

2. $Z$ has a $G$-invariant Kähler structure defined by
\begin{equation}
\omega^\mathbb{C}(X, Y) := -\langle \xi, [X, Y]\rangle
\end{equation}
for each $X, Y \in \mathfrak{p}$.

3. There is a compact symmetric spaces of inner type $N = G/K$ defined by an involutive inner automorphism $\sigma = \text{Ad}(\exp(\pi\xi))$ and
\begin{equation}
K = \{a \in G \mid \text{Ad}(a)\exp(\pi\xi) = \exp(\pi\xi)\}.
\end{equation}
Its symmetric Lie algebra
\begin{equation}
\mathfrak{g} = \mathfrak{k} + \mathfrak{m}
\end{equation}
defined by
\[ \mathfrak{t}^C = \mathfrak{t}^C + \sum_{r \in \mathbb{Z}} \mathfrak{g}_r(\xi), \]
(2.49)
\[ \mathfrak{m}^C = \sum_{r \in \mathbb{Z} + 1} \mathfrak{g}_r(\xi). \]

(4) There is a twistor fibration \( \pi_{\xi} : Z = G/H \longrightarrow N = G/K \) over a compact inner symmetric space.

(5) The generalized flag manifold has a \( k \)-symmetric structure on \( Z = G/H \) defined by \( \tau = \text{Ad}(\exp(2\pi\xi/k)) \). Here if we express the highest weight as \( \tilde{\alpha} = \sum_{i=1}^{\ell} m_i \alpha_i \), set \( k := n_I(\tilde{\alpha}) = \sum_{i \in I} m_i + 1. \)
\( \tau \) defines a \( k \)-symmetric space structure on \( G/H \). We call it the canonical \( k \)-symmetric space structure.

Let \( G/H \) be a generalized flag manifold. A smooth map \( f : M \longrightarrow G/H \) is called a super-horizontal holomorphic map if \( \alpha'_p \) has values in \( \mathfrak{g}_1(\xi) \). Then a super-horizontal holomorphic map \( f : M \longrightarrow G/H \) is a pluriharmonic map satisfying the condition (\( \star \)).

Under the twistor fibration \( \pi_{\xi} : Z = G/H \longrightarrow N = G/K \) over a compact inner symmetric space, each horizontal holomorphic map \( f : M \longrightarrow G/H \) projects a weakly conformal harmonic map \( \varphi = \pi_{\xi} \circ f : M \longrightarrow G/H \). Such harmonic maps were called pseudo-holomorphic curves, superminimal or isotropic harmonic maps.

2.5. Primitive maps into \( k \)-symmetric spaces. Let \( G \) be a compact connected semisimple Lie group with Lie algebra \( \mathfrak{g} \) and \( G/H \) be a \( k(\geq 2) \)-symmetric space defined by the automorphism \( \tau \) of order \( k \). Denote by \( \omega := e^{2\pi\sqrt{-1}/k} \) the primitive \( k \)-root of 1. We take the eigenspace decomposition of \( \mathfrak{g} \) by \( \tau \):

\[ \mathfrak{g}^C = \sum_{i \in \mathbb{Z}_k} \mathfrak{g}_i, \]
(2.50)
where \( \mathfrak{g}_i \) denotes the \( \omega^i \)-eigenspace of \( \tau \):

\[ \mathfrak{g}_k := \{ X \in \mathfrak{g}^C \mid \tau(X) = \omega^k X \}. \]
(2.51)
Then the Lie algebra \( \mathfrak{h} \) of \( H \) satisfies \( \mathfrak{t}^C = \mathfrak{g}_0 \) and there is a subspace \( \mathfrak{p} \) of \( \mathfrak{g} \) such that

\[ \mathfrak{g} = \mathfrak{h} + \mathfrak{p} \]
(2.52)
is a direct sum as vector spaces and

\[ \mathfrak{p}^C = \sum_{i \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_i. \]
(2.53)
We can identify the subspace \( \mathfrak{p} \) with the tangent vector space \( T_o(G/H) \) of \( G/H \) at the origin \( o = eH \). A smooth map \( f : M \longrightarrow G/H \) is called a primitive map if \( \alpha'_p \) has values in \( \mathfrak{g}_{-1} \). Then a primitive map \( f : M \longrightarrow G/H \) is a pluriharmonic map satisfying the condition (\( \star \)) provided that \( k > 2. \)
3. Harmonic maps of finite type

Harmonic maps of finite type are a class of harmonic maps coming from integrable systems on twisted loop algebras.

Let \( G/H \) be a \( k(\geq 2) \)-symmetric space defined by the automorphism \( \tau \) of a compact connected Lie group \( G \) of order \( k \).

The twisted loop algebra associated with \( \tau \) is defined as

\[
\Lambda^C_g := \{ \xi : S^1 \to g^C | \xi(\omega \lambda) = \tau(\xi(\lambda)) \text{ for each } \lambda \in S^1 \}
\]

and its real form is defined as

\[
\Lambda^r_g := \{ \xi \in \Lambda^C_g | \xi : S^1 \to g \}.
\]

For a positive integer \( d \equiv 1 \mod k \), consider the finite dimensional real vector subspace :

\[
\Lambda_d := \{ \xi = \sum_j \xi_j \lambda^{-j} \in \Lambda^r_g | \xi_j = 0 \text{ for } |j| > d \}.
\]

We take the differential equation of Lax type for unknown function \( \xi : \mathbb{R}^2 \to \Lambda_d : \)

\[
d\xi = [\xi, \lambda^{-1}\xi_d dz + (\xi_{d-1} dz)_h + \lambda \xi_{-d} d\bar{z}].
\]

The first fundamental fact is that

**Proposition 3.1.** The partial differential equation of the first order (3.4) is completely integrable, i.e. involutive, in the sense of Frobenius.

Hence for an arbitrary point \( \xi_o \in \Lambda_d \) there exists a unique solution \( \xi : \mathbb{R}^2 \to \Lambda_d \) to the equation (3.4) satisfying the initial condition \( \xi(0) = \xi_o \). Using the solution \( \xi \), we define

\[
\alpha_\lambda := \lambda^{-1}\xi_d dz + (\xi_{d-1} dz)_h + \lambda \xi_{-d} d\bar{z}
\]

for each \( \lambda \in S^1 \). The second fundamental fact is that

**Proposition 3.2.** Each \( \alpha_\lambda \) satisfies the Maurer-Cartan equation

\[
d\alpha_\lambda + \frac{1}{2} [\alpha_\lambda \wedge \alpha_\lambda] = 0.
\]

Thus there is a framing \( F_\lambda : \mathbb{R}^2 \to G \) such that \( F_\lambda(0) = \text{Id} \) and \( \alpha_\lambda = F_\lambda^{-1} dF_\lambda \). Note that we have \( \xi = (\text{Ad}F_\lambda^{-1})\xi_0 \). Using the natural projection \( \pi : G \to G/H \), we obtain primitive harmonic maps \( f = \pi \circ F_\lambda : \mathbb{R}^2 \to G/H \) for \( \lambda \in S^1 \). We call each \( f_\lambda : \mathbb{R}^2 \to G/H \) obtained in this way a primitive harmonic map of finite type.

These results were proved via the Adler-Kostant-Symes theory by Burstall, Ferus, Pedit, Pinkall [2],[3]. In [19] we have generalized them to a primitive pluriharmonic map of finite type.

The above Lax equation also comes from an algebraically completely integrable Hamiltonian system (ACIHS) a so called polynomial matrix system ([9]).
I. McIntosh established the correspondence between harmonic maps of finite type into complex projective spaces and algebraic geometric data consisting of spectral curves ([11], [12], [22]). We refer to [13] for the relationship of superconformal minimal surfaces in spheres with Toda field equations and its application to Lawson’s problem.

4. CLASSIFICATION PROBLEM OF HARMONIC TORI IN SYMMETRIC SPACES

We refer to [18] for the classification problem of harmonic 2-spheres (i.e., harmonic maps of compact Riemann surfaces of genus 0) in compact symmetric spaces.

We shall briefly mention the classification problem of harmonic tori (i.e., harmonic maps of compact Riemann surfaces of genus 1) in compact symmetric spaces.

Theorem 4.1 ([2], [3]). If \( f : \mathbb{C}/\Gamma \rightarrow G/H \) is a harmonic map of a torus into a \( k \)-symmetric space \( G/H \) and \( f \) satisfies the semisimple-adapted condition, then \( f \) is a harmonic map of finite type.

Theorem 4.2 ([1], [2]). If \( \varphi : \mathbb{C}/\Gamma \rightarrow G/K \) is a non-isotropic harmonic map of a torus into \( G/K = S^n \) or \( \mathbb{C}P^n \), then there is a twistor fibration \( \pi : G/H \rightarrow G/K \) of a \( k \)-symmetric space \( G/H \) onto \( G/K \) and a semisimple-adapted primitive harmonic map \( f : M \rightarrow G/H \) of finite type such that \( \varphi = \pi \circ f \).

The cases when \( G/K \) is a quaternionic projective space or complex Grassmann manifold are studied by [23], [24].

5. HARMONIC MAPS OF FINITE TYPE INTO GENERALIZED FLAG MANIFOLDS AND Twistor Fibrations

Let \( G/H \) be a generalized flag manifold with the canonical \( k(>2) \)-symmetric structure \( \tau \) and \( p : G/H \rightarrow G/K \) be the natural twistor projection. We assume that

1. \( H \subset K \) and \( G/K \) is a compact symmetric space of inner type with an involutive automorphism \( \sigma \) of \( G \),
2. the canonical decomposition \( g = \mathfrak{k} + \mathfrak{m} \) with respect to \( \sigma \) is \( \tau \)-stable and orthogonal,
3. \( \mathfrak{g}_{-1} \cap \mathfrak{n} \subset \mathfrak{m}^C \).

Theorem 5.1 ([19]). Suppose that \( G/K \) is a Hermitian symmetric space. If \( \psi : M \rightarrow G/H \) is a primitive harmonic map of finite type, then \( \varphi = p \circ \psi : M \rightarrow G/K \) is a harmonic map of finite type.

It is an interesting problem to extend this result to more general cases. Also in the case when \( G/K \) is the standard sphere \( S^n \) and \( G/H \) is a \( 2r+2 \)-symmetric space \( SO(n+1)/SO(2) \times \cdots \times SO(2) \times SO(n-2r) \), we can show the same statement directly ([19]). Combining Theorem
4.2 with Theorem 5.1, we obtain that all harmonic tori in $G/K = S^n$ and $G/K = CP^n$ are of finite type. Moreover pluriharmonic maps of higher dimensional complex tori into $G/K = S^n$ and $G/K = CP^n$ were discussed by [19].

The essential point in the proof of Theorem 5.1 is to prove the existence of a Lie algebra isomorphism between two different twisted loop algebras $\Lambda g^C_\tau$ and $\Lambda g^C_\sigma$ which transforms an integrable system to another integrable system.

6. Harmonic maps of finite type from compact Riemann surfaces

Let $M_g$ be a compact Riemann surface of genus $g (> 1)$ and $G/H$ a $k$-symmetric space. Let $V_1, \cdots, V_n : g \rightarrow g$ be $\tau$-compatible $AdG^C$-equivariant polynimial maps. Let $C = \{C_1, \cdots, C_n\}$ be holomorphic 1-forms on $M$. For a positive integer $d \equiv 1 \mod k$, consider the differential equation of Lax type for $\xi : M \rightarrow \Lambda d$

(6.1)

$$d\xi = [\xi, \sum_{j=1}^{n} (\lambda^{-1}V_j(\xi_d)C_j + (V_j(\xi_d + \xi_{d-1}))_1)C_j]_t + \lambda V_j(\xi_d)C_j].$$

Suppose that $\xi : M \rightarrow \Lambda d$ is a solution of the above differential equation (6.2) on $M$. Using $\xi : M \rightarrow \Lambda d$, we define a 1-form with values in $\Lambda d$ on $M$ by

(6.2)

$$\alpha(C)\lambda := \sum_{j=1}^{n} \left( \lambda^{-1}V_j(\xi_d)C_j + (V_j(\xi_d + \xi_{d-1}))_1C_j \right)_{\lambda} + \lambda V_j(\xi_d)C_j \right)$$

for $\lambda \in S^1$. Then we have

**Theorem 6.1** ([19]). For each $\lambda \in S^1$, $\alpha(C)\lambda$ satisfies the Maurer-Cartan equation.

Let $\tilde{M}$ be the universal covering space of $M$ and $\pi_M : \tilde{M} \rightarrow M$ a holomorphic universal covering map.

**Definition 6.1.** $f : M \rightarrow G/H$ is called a primitive harmonic map of generalized finite type if there is a solution $\xi : M \rightarrow \Lambda d$ such that the map $f \circ \pi_M : \tilde{M} \rightarrow G/H$ has an extended framing $F_\lambda : \tilde{M} \rightarrow G (\lambda \in S^1)$ which satisfies $f \circ \pi_M = \pi \circ F_1$ and

$$\pi_M^* \alpha(C)\lambda = F_\lambda^* \theta_G = F_\lambda^{-1} dF_\lambda.$$

**Theorem 6.2** ([19]). Assume that $M$ is a compact Riemann surface of genus $g \geq 2$. Let $f : M \rightarrow G/H$ be a primitive harmonic map of generalized finite type. Let $J(M) = C^d/\Gamma$ be the Jacobian torus of $M$. Then there is a primitive pluriharmonic map of finite type $\tilde{f} : M \rightarrow \tilde{M}$.
\[ C^9/T \rightarrow G/H \] such that \( f = \hat{f} \circ j \), where \( j \) denotes the Abel map for the Riemann surface \( M \). Conversely, such a map is a primitive harmonic map of generalized finite type.

At the Differential Geometry meeting at Fukuoka University Seminar House in Dec. 2000, Professor A. T. Huckleberry suggested to the author that this result has an interesting similarity to the result in [10].

**References**


Department of Mathematics, Graduate School of Science, Tokyo Metropolitan University

Current address: Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

E-mail address: ohnita@comp.metro-u.ac.jp