MOMENT MAPS AND SYMMETRIC LAGRANGIAN SUBMANIFOLDS

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Abstract. In this talk we have discussed about the moment map construction of symmetric Lagrangian submanifolds in complex Euclidean spaces, complex projective spaces, Hermitian symmetric spaces, and so on. The explicit formulae of the moment maps for Hamiltonian group actions induced by the isotropy representaions of Hermitian symmetric spaces of compact type were given in terms of Lie algebras.

There are two kinds of well-known constructions for Lagrangian submanifolds in symplectic manifolds. One is a method of obtaining Lagrangian submanifolds as fixed point subsets by antisymplectic involutions of symplectic manifolds. Another is a method of obtaining Lagrangian submanifolds as inverse images of the moment maps under suitable conditions. In this talk we have discussed about the moment map construction to obtain *nice* Lagrangian submanifolds such as symmetric Lagrangian submanifolds, in complex Euclidean spaces, complex projective spaces, Hermitian symmetric spaces, and so on.

Let (M, ω) be a symplectic manifold of dimension $2n$. Assume that a Lie group G acts symplectically on M. Let $\mathfrak{g}$ donote the Lie algebra of Lie group G. Let $\mu : M \longrightarrow \mathfrak{g}^*$ be the moment map for the Hamiltonian group action G on M. By the definition the moment map μ satisfies the following conditions :

(1) $d\langle \mu, \xi \rangle = \omega(\tilde{\xi}, \cdot)$ for all $\xi \in \mathfrak{g}$.
(2) $\mu(a \cdot x) = \mathrm{Ad}^*(a^{-1})\mu(x)$ for all $x \in M$ and all $a \in G$.

Here $\tilde{\xi}$ denotes the vector field on M induced by the action of the one-parameter subgroup $\exp(t\xi)$. Set

$$\mathfrak{z}(\mathfrak{g}^*) := \{\alpha \in \mathfrak{g}^* \mid \mathrm{Ad}^*(a)\alpha = \alpha \text{ for all } a \in G\}.$$

Lemma 1. *For each $\alpha \in \mathfrak{z}(\mathfrak{g})$, choose an arbitrary point $x \in \mu^{-1}(\alpha)$. Then*

(1) *The orbit $G \cdot x$ of G through x is contained in $\mu^{-1}(\alpha)$.*
(2) *The orbit $G \cdot x$ is an isotropic submanifold of M.*

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(3) *$G \cdot x$ is a Lagrangian submanifold if and only if $G \cdot x$ is open in $\mu^{-1}(\alpha)$, that is $T_y(G \cdot x) = \mathrm{Ker}(d\mu)_y$ for each $y \in G \cdot x$.*

Proof. (1) For each $a \in G$, we have

$$\begin{aligned}\mu(a \cdot x) &= \mathrm{Ad}^*(a)(\mu(x)) \\ &= \mathrm{Ad}^*(a)(\alpha) \\ &= \alpha.\end{aligned}$$

(2) For each $\xi, \eta \in \mathfrak{g}$, we have

$$\begin{aligned}&\omega((\tilde{\xi})_{a \cdot x}, (\tilde{\eta})_{a \cdot x}) \\ =&\langle (d\mu)_{a \cdot x}((\tilde{\xi})_{a \cdot x}), \eta \rangle = 0.\end{aligned}$$

(3) Suppose that $G \cdot x$ is a Lagrangian submanifold of M. For each $X \in \mathrm{Ker}(d\mu)_{a \cdot x}$, we have

$$\omega(X, (\tilde{\eta})_{a \cdot x}) = \langle (d\mu)_{a \cdot x}(X), \eta \rangle = 0.$$

Thus $T_{a \cdot x}(G \cdot x) \oplus \mathbf{R}X$ is an isotropic subspace of $T_{a \cdot x}M$ and hence $X \in T_{a \cdot x}(G \cdot x)$, because $T_{a \cdot x}(G \cdot x)$ is a Lagrangian subspace of $T_{a \cdot x}M$. Conversely, if $X \in T_{a \cdot x}(G \cdot x)$ satisfies

$$\omega(X, (\tilde{\eta})_{a \cdot x}) = 0$$

for each $\eta \in \mathfrak{g}$, then we have

$$\langle (d\mu)_{a \cdot x}(X), \eta \rangle = \omega(X, (\tilde{\eta})_{a \cdot x}) = 0$$

for each $\eta \in \mathfrak{g}$, and hence we obtain $(d\mu)_{a \cdot x}(X) = 0$, that is $X \in \mathrm{Ker}(d\mu)_{a \cdot x}$. □

Next we shall discuss the moment map construction of symmetric Lagrangian submanifolds in complex Euclidean spaces and complex projective spaces.

Let (U, G) be an Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$. Set $\dim(U/G) = 2(n+1)$. Let $\langle \ , \ \rangle$ denote the $\mathrm{Ad}(U)$-invariant inner product of $\mathfrak{u}$ defined by (-1)-times the Killing-Cartan form of the Lie algebra $\mathfrak{u}$. Let $\mathfrak{c}(\mathfrak{g})$ denote the center of $\mathfrak{g}$, which is nontrivial. The endmorphism $\mathrm{ad}|_{\mathfrak{p}}(Z)$ defined by an element Z in $\mathfrak{c}(\mathfrak{g})$ defines the standard complex structure J on $\mathfrak{p}$. Relative to the complex structure, the subspace $\mathfrak{p}$ can be identified with a complex Euclidean space $\mathbf{C}^{n+1}$. The standard symplectic form ω of $\mathfrak{p}$ is defined by

$$\omega(X, Y) = \langle JX, Y \rangle$$

for each $X, Y \in \mathfrak{p}$. We take the decomposition of (U, G) into irreducible Hermitian symmetric pairs of compact type :

$$(U, G) = (U_1, G_1) \oplus \cdots \oplus (U_s, G_s). \tag{0.1}$$

Set $\dim(U_i/G_i) = 2(n_i + 1)$ for $i = 1, \cdots, s$. Let $\mathfrak{u}_i = \mathfrak{g}_i + \mathfrak{p}_i$ be the canonical decomposition of (U_i, G_i) for each $i = 1, 2, \cdots, s$. We denote by $\mathbf{C}P(\mathfrak{p})$ the n-dimensional complex prjective space of 1-dimensioanl complex vector subspaces of $\mathfrak{p}$. The adjoint action of G on $\mathfrak{p} \cong \mathbf{C}^{n+1}$ induces a group action of G on $\mathbf{C}P(\mathfrak{p}) \cong \mathbf{C}P^n$ in the natural way. Let $\hat{\mu} : \mathfrak{p} \cong \mathbf{C}^{n+1} \longrightarrow \mathfrak{g}^* \cong \mathfrak{g}$ be the moment map with respect to the adjoint action of G on $\mathfrak{p}$ and $\mu : \mathbf{C}P(\mathfrak{p}) \cong \mathbf{C}P^n \longrightarrow \mathfrak{g}^* \cong \mathfrak{g}$ be the moment map with respect to the induced action of G on $\mathbf{C}P(\mathfrak{p}) \cong \mathbf{C}P^n$. Then the moment map $\tilde{\mu}$ is explicitly given as follows :

Theorem 1. *For each $p \in \mathfrak{p}$,*

$$\tilde{\mu}(p) - \tilde{\mu}(0) = (\mathrm{ad}(p))^2(Z).$$

Note that $\tilde{\mu}(0) \in \mathfrak{c}(\mathfrak{g})$, because $0 \in \mathfrak{p}$ is a fixed point set by the adjoint action of G on $\mathfrak{p}$.

Proof. At each $p \in \mathfrak{p}$ and for each $\xi \in \mathfrak{g}$, we compute

$$\begin{aligned}(d\langle \tilde{\mu}, \xi \rangle)_p &= \omega_p(\tilde{\xi}, \cdot) \\ &= \langle \tilde{J}(\tilde{\xi})_p, \cdot \rangle \\ &= \langle [Z, [\xi, p]], \cdot \rangle \\ &= -\langle [p, [Z, \cdot]], \xi \rangle.\end{aligned}$$

Hence we obtain

$$(d\mu)_p = -[p, [Z, \cdot]].$$

By integrating it along the line segment $p(t) := tp$ $(0 \leqq t \leqq 1)$, we have

$$\begin{aligned}\tilde{\mu}(p) - \tilde{\mu}(0) &= \int_0^1 (d\tilde{\mu})_{p(t)}(\dot{p}(t))dt \\ &= -\int_0^1 [p, [Z, \dot{p}(t)]]dt \\ &= -\int_0^1 [p, [Z, p]]dt \\ &= -[p, [Z, p]] \\ &= [p, [p, Z]].\end{aligned}$$

□

We remark that the square norm of the moment map is equal to the holomorphic sectional curvatures of U/G.

Let $\zeta \in \mathfrak{c}(\mathfrak{g})$ be an arbitary element. For each $\eta \in \mu^{-1}(\zeta)$, the orbit $\hat{L} = \mathrm{Ad}(G)\eta \subset \tilde{\mu}^{-1}(\zeta)$ of G through η a Lagrangian submanifold fully embedded in a complex Eucledean subspace of $\mathfrak{p} \cong \mathbf{C}^{n+1}$

and the orbit $L = \mathrm{Ad}(G)[\eta] \subset \mu^{-1}(\zeta)$ of G through $[\eta] \in \mathbf{C}P^n$ is a Lagrangian submanifold fully embedded in a complex projective subspace of $\mathbf{C}P(\mathfrak{p}) \cong \mathbf{C}P^n$.

We take a decomposition of η as

$$\eta = \eta_1 + \cdots + \eta_s, \quad \eta_i \in \mathfrak{p}_i (i = 1, \cdots, s). \tag{0.2}$$

Set $\hat{L}_i = \mathrm{Ad}(G_i)(\eta_i)$ which is an irreducible symmetric R-space embedded in the complex Euclidean space $\mathfrak{p}_i$. The following is a list of all irreducible symmetric R-spaces of type $U(r)$:

$$Q_{2,q}(\mathbf{R}), U(p), U(p)/O(p), U(2p)/Sp(p), T \cdot E_6/F_4.$$

Set $\hat{L} = \mathrm{Ad}(G)(\eta) \subset \mathfrak{p}$, which is a symmetric R-space standardly embedded in the complex Euclidean space $\mathfrak{p}$. Then we have

$$\hat{L} = \hat{L}_1 \times \cdots \times \hat{L}_s.$$

In case $s = 1$, L is a minimal Lagrangian submanifold embedded in $\mathbf{C}P^n$. Then the following is a complete list of such L with $s = 1$:

(1) $\mathbf{R}P^n$.
(2) $SU(p)/\mathbf{Z}_p$, $n = p^2 - 1$.
(3) $SU(p)/SO(p)\mathbf{Z}_p$, $n = \frac{(p-1)(p+2)}{2}$.
(4) $SU(2p)/Sp(p)\mathbf{Z}_{2p}$, $n = (p - 1)(2p + 1)$.
(5) $E_6/F_4\mathbf{Z}_3$, $n = 26$.

In [1], we showed that L is a compact Hamiltonian stable minimal Lagrangian submanifold embedded in $\mathbf{C}P^n$. We should remark that any compact Hamiltonian stable minimal Lagrangian submanifold immersed in $\mathbf{C}P^n$ is not simply connected ([1]).

Moreover, in [2], we showed that $\hat{L}$ is a compact Hamiltonian stable H-minimal Lagrangian submanifold embedded in $\mathbf{C}^{n+1}$.

It is known that $\hat{L}$ and L are Lagrangian submanifolds in complex Euclidean spaces and complex projective spaces, which has parallel second fundamental form and hence are extrinsic symmetric.

Real forms in Kähler manifolds, that is the fixed point subset of anti-holomorphic involutive isometry, are Lagrangian submanifolds which are totally geodesic and hence H-minimal. The real forms in Hermitian symmetric spaces of compact type were classified by M. Takeuchi and they are symmetric R-spaces canonically embedded in Hermitian symmetric spaces.

We have determine the Hamiltonian stability of all real forms embedded in compact irredeucible Hermitian symmetric spaces, using

M. Takeuchiś results ([9]). The real forms are Hamiltonian stable except for

$$\begin{aligned}(M, L) = &(SO(4m)/U(2m), U(2m)/Sp(m)) \ (m \geqq 3),\\ &(Q_{p+q-2}(\mathbf{C}), Q_{p,q}(\mathbf{R})) \ (3 \leqq q-p, p \geqq 2),\\ &(E_7/T \cdot E_6, T \cdot E_6/F_4).\end{aligned}$$

Moreover we shall discuss the moment map for the isotropy action on compact Hermitian symmetric spaces U/G. The moment map μ of the action of the isotropy subgroup G on U/G is given as follows:

Theorem 2. *For $\xi \in \mathfrak{g}$, we have the formula*

$$\langle \mu(aG), \xi \rangle - \langle \mu(eG), \xi \rangle = \langle (\mathrm{Ad}(a) - 1)Z, \xi \rangle,$$

at each point $aG \in U/G$.

Note that $\mu(eG) \in \mathfrak{c}(\mathfrak{g})$, because eG is a fixed point by the action of G on U/G.

Moreover from the viewpoint of the moment maps it is interesting to describe symmetric Lagrangian submanifolds in compact Hermitian symmetric spaces.

Furthermore it is also an interesting question to discuss the momnet map constructions for the isotropoy representaion of *generalized flag manifolds* and for adjoint orbits of compact Lie groups.

References

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