Moment maps and isoparametric hypersurfaces in spheres

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Abstract (1/2)

Story

(起) Isoparametric hypersurfaces in spheres with $g = 4$ (four distinct principal curvatures) are interesting.

(承) $\exists$ both homogeneous and inhomogeneous examples, which we would like to understand more nicely.

(転) Our expectation: $g = 4$ cases would be related to moment maps (of linear Hamiltonian actions).

(結) Our results: for most of the homogeneous examples, our expectation is true.

Abstract (2/2)

Note

Based on joint works with Shinobu Fujii:

- [F2010] Fujii: Homogeneous isoparametric hypersurfaces in spheres with four distinct principal curvatures and moment maps, Tohoku Math. J. (2010)
- [FT2015] Fujii & T.: Moment maps and isoparametric hypersurfaces in spheres — Hermitian cases, Transf. Groups (2015)
- [F] Fujii: Moment maps and isoparametric hypersurfaces in spheres — Grassmannian cases, (almost) preprint

(We will comment on [Miyaoka2013] later.)

Sec. 1 - Isoparametric Hypersurfaces in Spheres (1/6)

Recall isoparametric hypersurfaces (probably everyone knows...)

Def.

$S^{n+1} \supset M^n$: **isoparametric** if

- the principal curvatures are constant.

Thm. (Münzner)

The number g of distinct principal curvatures satisfies

- $g \in \{1, 2, 3, 4, 6\}$.

Note

- Any homogeneous hypersurfaces in S^{n+1} is isoparametric.
- Only $g = 4$ case, $\exists$ inhomogeneous examples (OT-FKM type).

Sec. 1 - Isoparametric Hypersurfaces in Spheres (2/6)

We think S^{n+1} is the unit sphere of $\mathbb{R}^{n+2}$.

Thm. (Münzner)

For an (complete) isoparametric hypersurface $M^n \subset S^{n+1}$,

- Let $\lambda_1 > \cdots > \lambda_g$ the principal curvatures, and m_i the multiplicity of λ_i. Then $m_i = m_{i+2}$ (subscription $\mathrm{mod}\, g$);
- M is a regular level set of $f : \mathbb{R}^{n+2} \to \mathbb{R}$ satisfying
  - f is a homogeneous polynimial of degree g;
  - $||\mathrm{grad}\, f(P)||^2 = g^2 ||P||^{2g-2}$;
  - $2\triangle f(P) = (m_2 - m_1) g^2 ||P||^{g-2}$.

The above f is called the **Cartan-Münzner polynomial** of M.

Sec. 1 - Isoparametric Hypersurfaces in Spheres (3/6)

Our Expectation (rough version)

- Every Cartan-Münzner polynomial f with $g = 4$ are obtained by some linear Hamiltonian action $H \curvearrowright (\mathbb{R}^{n+2}, \omega)$.

Note

- $n = \dim M = m_1 + m_2 + m_3 + m_4 = 2(m_1 + m_2)$ is even.

Sec. 1 - Isoparametric Hypersurfaces in Spheres (4/6)

Thm (Hsiang-Lawson, Takagi-Takahashi)

Isoparametric hypersurfaces in spheres with $g = 4$ are precisely obtained by the isotropy representations of

(1) SO_{n+2}/SO_nSO_2;

(2) $SU_{n+2}/S(U_nU_2)$;

(3) SO_{10}/U_5;

(4) E_6/U_1Spin_{10};

(5) Sp_{n+2}/Sp_nSp_2;

(6) $SO_5 \times SO_5/SO_5$.

Sec. 1 - Isoparametric Hypersurfaces in Spheres (5/6)

Note (isotropy rep.)

For a Riemannian symmetric space G/K,

- the isotropy representation is $K \curvearrowright T_o(G/K) = \mathbb{R}^{n+2}$;
- K preserves the inner product on $T_o(G/K)$;
- hence K acts on the unit sphere S^{n+1} in $T_o(G/K)$;
- if $\mathrm{rank}\,(G/K) = 2$, then it is cohomogeneity one on S^{n+1}.

Sec. 1 - Isoparametric Hypersurfaces in Spheres (6/6)

Note (invariant polynomials)

For homogeneous cases; an orbit of $K \curvearrowright T_o(G/K) = \mathbb{R}^{n+2}$,

- the Cartan-Münzner polynomial f is K-invariant.
- not so many K-inv. homogeneous polynomials on $T_o(G/K)$...

So, inv. polynomials have many chances to be Cartan-Münzner.

Note (symplectic structures)

- if G/K is Hermitian ((1)–(4)), $\exists$ canonical ω on $T_o(G/K)$.
- otherwise, the choice of ω is already a problem.

Sec. 2 - Moment maps (1/4)

Let $(\mathfrak{p}, \omega)$ be a symplectic manifold.

Def.

$H \curvearrowright (\mathfrak{p}, \omega)$ is **Hamiltonian** if

- H preserves ω;
- $\exists \mu : \mathfrak{p} \to \mathfrak{h}^*$: moment map.

Def.

$\mu : \mathfrak{p} \to \mathfrak{h}^*$ is a **moment map** for $H \curvearrowright (\mathfrak{p}, \omega)$ if

- μ is H-equivariant;
- $d\mu$ is "compatible" with ω.

The moment maps give a procedure to obtain equivariant maps.

Sec. 2 - Moment maps (2/4)

Note

Let us take

- $\mu : \mathfrak{p} \to \mathfrak{h}^*$ be a moment map for $H \curvearrowright (\mathfrak{p}, \omega)$,
- $|| \cdot ||$: an H-invariant norm on $\mathfrak{h}^*$.

Then $||\mu||^2 : \mathfrak{p} \to \mathbb{R}$ is H-invariant.

Note (cf. Ohnita)

Let us consider

- G/K : Hermitian symm. space with cplx str $J = \mathrm{ad}_Z$ $(Z \in \mathfrak{k})$,
- $\langle , \rangle := -$[Killing form of $\mathfrak{g}$]. (and $\omega(X, Y) = \langle JX, Y \rangle$)

Then the moment map μ of $K \curvearrowright T_o(G/K)$ is

- $\mu(P) = (1/2)\langle [P, [P, Z]], \cdot \rangle$,
- hence $||\mu||^2$ is K-inv. homogeneous polynomial of degree 4.

Sec. 2 - Moment maps (3/4)

Our Expectation (exact version)

Every Cartan-Münzner polynomial f with $g = 4$ can be written as $f = ||\mu||^2$, where μ is a moment map of some linear Hamiltonian action $K \curvearrowright \mathbb{R}^{n+2}$, and $||\cdot||$ is a K-inv. norm on $\mathfrak{k}^*$.

Note

If this is true, then isoparametric hypersurfaces in S^{n+1} could be understood in terms of group actions (even for inhomogeneous cases)...

Sec. 2 - Moment maps (4/4)

$\exists$ related but different study:

Thm. (Miyaoka 2013)

Every Cartan-Münzner polynomial f with $g = 4$ are written by

- $||\mu(\cdot, Y_\cdot)||^2$,

where μ is a moment map of some Hamiltonian $H \curvearrowright T\mathbb{R}^{n+2}$, and some mysterious $Y \in \mathfrak{X}(\mathbb{R}^{n+2})$.

The advantage of this is $T\mathbb{R}^{n+2}$ has a natural symplectic structure, but we would like to consider $H \curvearrowright \mathbb{R}^{n+2}$, not on $T\mathbb{R}^{n+2}$.

Sec. 3 - Results (1/7)

Recall

Isoparametric hypersurfaces in spheres with $g = 4$ are precisely obtained by the isotropy representations of

(1) SO_{n+2}/SO_nSO_2;

(2) $SU_{n+2}/S(U_nU_2)$;

(3) SO_{10}/U_5;

(4) E_6/U_1Spin_{10};

(5) Sp_{n+2}/Sp_nSp_2;

(6) $SO_5 \times SO_5/SO_5$.

Main Thm. (F2010, FT2015, F)

- Our expectation is true for (1)–(5).

Sec. 3 - Results (2/7)

Note

In order to show our expectation for $M^n \subset S^{n+1}$, we need to

- find ω on $\mathbb{R}^{n+2}$;
- find a Hamiltonian action $H \curvearrowright (\mathbb{R}^{n+2}, \omega)$;
- find an H-invariant norm on $\mathfrak{h}^*$;
- and check that it is Cartan-Münzner polynomial.

Sec. 3 - Results (3/7)

Proof for (1)–(4):

Since they are Hermitian,

- $\exists$ canonical symplectic form ω on $T_o(G/K)$;
- $K \curvearrowright (T_o(G/K), \omega)$ is Hamiltonian;
- we know μ (recall $2\mu(P) = \langle [P, [P, Z]], \cdot \rangle$);
- since $\mathfrak{k} = u_1 \oplus \mathfrak{k}'$ (with $\mathfrak{k}'$ semisimple part), we have two-parameter family of K-invariant norms, say $|| \cdot ||_{a,b}$;
- by calculating grad and $\triangle$, we find a, b to be Cartan-Mňzner.

Note

- [F2010] proved the classical cases (1)–(3) by matrix.
- [FT2015] proved (1)–(4) by Lie algebra theory (roots).

Sec. 3 - Results (4/7)

Observation

In order to attack Case (5): Sp_{n+2}/Sp_nSp_2:

- this is Grassmannian: $G_2(\mathbb{H}^{n+2}) = Sp_{n+2}/Sp_nSp_2$;
- this is not Hermitian, and the isotropy representation does not look like Hamiltonian;
- one can identify $T_o(G_2(\mathbb{H}^{n+2})) = \mathbb{H}^n \oplus \mathbb{H}^n$;
- this is same for other Grassmannians:

$$T_o(G_2(\mathbb{R}^{n+2})) = \mathbb{R}^n \oplus \mathbb{R}^n, \quad T_o(G_2(\mathbb{C}^{n+2})) = \mathbb{C}^n \oplus \mathbb{C}^n.$$

Hence

Let us consider

- $V := \mathbb{K}^n \oplus \mathbb{K}^n$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.
- $\mathbb{K}^n$ has $\langle x, y \rangle := \mathrm{Re}({}^t\overline{x}y)$. Then also on V.
- V is symplectic by $\omega((x_1, x_2), (y_1, y_2)) := \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle$.

Sec. 3 - Results (5/7)

Prop.

$H \curvearrowright (V = \mathbb{K}^n \oplus \mathbb{K}^n, \omega)$ is Hamiltonian, where

- $SO_2 \curvearrowright V$ as "rotation" $(V \cong \mathbb{K}^n \otimes \mathbb{R}^2)$.
- $U := SO_n, U_n, SP_n Sp_1$ acts on $\mathbb{K}^n$, and also on V diagonally.
- $H := U \cdot SO_2$.

Proof for (1), (2), (5)

- Calculate a moment map μ (one can get explicit forms);
- $\mathfrak{h}$ has two or three components, so $\exists$ inv. norms $|| \cdot ||_{a,b,c}$.
- Let $f := ||\mu||^2_{a,b,c}$. Calculate $||\mathrm{grad}\, f||^2$ and $\triangle f$.
- Find a, b, c so that f is Cartan-Münzner.

Sec. 3 - Results (6/7)

Note

For $\mathbb{K} = \mathbb{H}$,

- the isoparametric hypersurface is an orbit of $Sp_n Sp_2$,
- but the Hamiltonian action is by $Sp_n Sp_1 SO_2$.

Remark 1

In order to prove our expectation,

- the Hamitonian action could be given by a smaller group.

  (smaller than the full normalizer of the given isoparametric hypersurface)

Sec. 3 - Results (7/7)

Note

For $\mathbb{K} = \mathbb{C}$,

- the isoparametric hypersurface is an orbit of $S(U_n U_2)$,
- since this is Hermitian, the Cartan-Mñzner polynomial f is written by the moment map of this action.
- since this is Grassmannian, f can also be written by the moment map of $U_n SO_2$-action.

Remark 2

In order to prove our expectation, remind that

- some different Hamitonian actions could give the same Cartan-Mñzner polynomial.

Sec. 4 - Summary (1/3)

Our Expectation

Every Cartan-Münzner polynomial f with $g = 4$ can be written as $f = ||\mu||^2$, where μ is a moment map of some linear Hamiltonian action $H \curvearrowright \mathbb{R}^{n+2}$, and $|| \cdot ||$ is an H-inv. norm on $\mathfrak{h}^*$.

Results

- Our expectation is true for most of homogeneous cases.

Sec. 4 - Summary (2/3)

Problem 1 (ongoing):

- Our expectation is also true for other remaining cases ?
  - homogeneous one obtained by SO_5 (adjoint rep.) ?
  - inhomogeneous ones (i.e., OT-FKM type) ?

Problem 2 (dreaming):

- Prove our expectation without using classification.

Sec. 4 - Summary (3/3)

Problem 3 (just thinking):

- Our method can be applied to other ambient spaces?
  - particulary, hypersurfaces in Hermitian symmetric spaces
  (e.g., $\mathbb{C}\mathrm{P}^n$, $\mathbb{C}\mathrm{H}^n$, higher rank cases, ...)

- Thank you very much!