On the moduli spaces of left-invariant geometric structures

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Intro (1/4)

- I moved from Hiroshima University to Osaka City University on Oct/2018.
- This talk is based on several joint works with my young collaborators.

Announcement (conference)

- Symmetry and shape
  - Celebrating the 60th birthday of Prof. J. Berndt,
  28–31 October 2019, Santiago de Compostela, Spain.

Intro (2/4)

Motivation

- For some distinguish geometric structures, left-invariant ones on Lie groups provide several nice examples.
- It is important to examine whether given Lie groups admit some distinguished left-invariant geometric structures or not.

Intro (3/4)

Our Approach

- In this talk, we consider an approach from the "moduli spaces" of some left-invariant geometric structures.
- As applications, we determine the existence and nonexistence of distinguished ones, for some very particular Lie groups.

Key Tool

- Group actions on (Riemannian or pseudo-Riemannian) symmetric spaces.

Intro (4/4)

Contents

Topic 1: Riemannian metrics (1/7)

Setting

- G : a (simply-connected) Lie group, $\mathfrak{g} := \mathrm{Lie}(G)$,
- $\widetilde{\mathfrak{M}}(\mathfrak{g}) := \{$positive definite inner product on $\mathfrak{g}\}$,
  $\cong \{$left-inv. Riem. metrics on $G\}$.

Def.

- The orbit space $\mathfrak{PM}(\mathfrak{g}) := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}(\mathfrak{g})$ is called the **moduli space** of left-invariant Riem. metrics on G.

Note

- $\mathbb{R}^\times \curvearrowright \widetilde{\mathfrak{M}}(\mathfrak{g})$ gives scaling; $\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}(\mathfrak{g})$ gives isometry.
- So it is enough to work on $\mathfrak{PM}(\mathfrak{g})$.

Topic 1: Riemannian metrics (2/7)

Note

- $\widetilde{\mathfrak{M}}(\mathfrak{g}) \cong \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ if $n := \dim \mathfrak{g}$;
- so $\widetilde{\mathfrak{M}}(\mathfrak{g})$ is a Riemannian symmetric space.

Note

- $$\begin{aligned} \mathfrak{PM}(\mathfrak{g}) &:= \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}(\mathfrak{g}) \\ &\cong \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n) \text{ (double coset space).} \end{aligned}$$

We are interested in the case that $\mathfrak{PM}(\mathfrak{g})$ is small.

Topic 1: Riemannian metrics (3/7)

Thm. (Lauret 2003)

- $\mathfrak{PM}(\mathfrak{g}) = \{\mathrm{pt}\}$ iff $\mathfrak{g} = \mathbb{R}^n$, $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.

Notation

- $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ is the solvable part of the Iwasawa dec. of $\mathfrak{so}(n,1)$.
  ($\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} = \mathrm{span}\{e_1, \dots e_n\}$ with $[e_1, e_j] = e_j$ $(j \in \{2, \dots, n\})$)
- $\mathfrak{h}^3$ is the 3-dim. Heisenberg Lie algebra.

Note

- Recall: $\mathfrak{PM}(\mathfrak{g}) := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}(\mathfrak{g})$.
- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is parabolic in $\mathrm{GL}(n, \mathbb{R})$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.

Topic 1: Riemannian metrics (4/7)

Prop. (Hashinaga-T. 2017)

- If $\mathfrak{g}$ is 3-dim. solvable, then $\dim \mathfrak{PM}(\mathfrak{g}) \leq 1$.

Ex. (Hashinaga-T. 2017)

Consider $\mathfrak{g} := \mathfrak{r}_{3,a}$ $(a \in [-1, 1))$ spanned by $\{e_1, e_2, e_3\}$ with

- $[e_1, e_2] = e_2, \ \ [e_1, e_3] = ae_3, \ \ [e_2, e_3] = 0.$

Then we have

- $\mathfrak{PM}(\mathfrak{g}) \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \mid \lambda \geq 0 \right\} \cong [0, +\infty).$

Topic 1: Riemannian metrics (5/7)

$\mathfrak{PM}(\mathfrak{g})$ derives a generalization of "Milnor frames".

Thm. (Milnor 1976)

- $\mathfrak{g}$: 3-dim. unimodular Lie algebra;
- $\langle , \rangle$: (positive definite) inner product on $\mathfrak{g}$.

Then $\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $\exists \{x_1, x_2, x_3\}$ onb of $\mathfrak{g}$ wrt $\langle , \rangle$ such that

- $[x_1, x_2] = \lambda_3 x_3, \ \ [x_2, x_3] = \lambda_1 x_1, \ \ [x_3, x_1] = \lambda_2 x_2.$

Note

- If $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ then $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$;
- But $\mathfrak{M}(\mathfrak{g}) \cong \mathrm{GL}(3, \mathbb{R})/\mathrm{O}(3)$ has dimension 6;
- So λ_1, λ_2, λ_3 parametrize the orbit space $\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}(\mathfrak{g})$.

Topic 1: Riemannian metrics (6/7)

Rem.

- "varying inner products" $\leftrightarrow$ "varying bracket products".

Prop. (Hashinaga-T. 2017)

Consider $\mathfrak{g} := \mathfrak{r}_{3,a}$ $(a \in [-1, 1))$. Then $\forall\langle , \rangle$ on $\mathfrak{g}$,

- $\exists \lambda \in [0, +\infty)$, $\exists k > 0$, $\exists \{x_1, x_2, x_3\}$: ONB wrt $k\langle , \rangle$ such that $[x_1, x_2] = x_2 + \lambda(a-1)x_3$, $[x_1, x_3] = ax_3$, $[x_2, x_3] = 0$.

Cor.

Let G be a Lie groups with $\mathrm{Lie}(G) = \mathfrak{r}_{3,a}$. Then

- $\nexists$ left-inv. Einstein metrics on G;
- $\exists$ left-inv. Ricci soliton metric on G (corresponding to $\lambda = 0$).

Topic 1: Riemannian metrics (7/7)

Comments

- One can obtain a generalization of Milnor's theorem, theoretically for any Lie algebras.

However

- If $\mathfrak{PM}(\mathfrak{g})$ is large (i.e., $\mathrm{Aut}(\mathfrak{g})$ is small), then the resulting bracket still contains lots of parameters.

Topic 2: Pseudo-Riemannian metrics (1/7)

We can apply the same strategy for pseudo-Riemannian cases.

Setting

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\dim G = p + q$,
- $\widetilde{\mathfrak{M}}_{(p,q)}(\mathfrak{g}) :=$ {inner product on $\mathfrak{g}$ with signature (p, q)}

  $\cong$ {left-inv. metrics on G with signature (p, q)}.

Def.

- $\mathfrak{PM}_{(p,q)}(\mathfrak{g}) := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{(p,q)}(\mathfrak{g})$ is called the **moduli space** of left-invariant metrics on G with signature (p, q).

Topic 2: Pseudo-Riemannian metrics (2/7)

Note

- $\widetilde{\mathfrak{M}}_{(p,q)}(\mathfrak{g}) \cong \mathrm{GL}(p+q,\mathbb{R})/\mathrm{O}(p,q)$;
- which is a pseudo-Riemannian symmetric space.

We are interested in the case that $\mathfrak{PM}_{(p,q)}(\mathfrak{g})$ is small.

Topic 2: Pseudo-Riemannian metrics (3/7)

Nice Fact (Wolf 1974)

- Let G/H be a reductive symmetric space, and $G \supset Q$ a parabolic subgroup. Then $\#(Q\backslash G/H)$ is finite.

Cor.

- $\#\mathfrak{PM}_{(p,q)}(\mathfrak{g}) < \infty$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}$ or $\mathfrak{h}^3 \oplus \mathbb{R}^{p+q-3}$.

Known Results

- $\# \mathfrak{PM}_{(n-1,1)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}) = 3$ (Nomizu 1979).
- $\# \mathfrak{PM}_{(2,1)}(\mathfrak{h}^3) = 3$ (Rahmani 1992).

Topic 2: Pseudo-Riemannian metrics (4/7)

Recall

- $\# \mathfrak{PM}_{(n-1,1)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}) = 3$ (Nomizu 1979).
- $\# \mathfrak{PM}_{(2,1)}(\mathfrak{h}^3) = 3$ (Rahmani 1992).

Thm. (Kubo-Onda-Taketomi-T. 2016)

- $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}) = 3$ for $\forall p, q \in \mathbb{Z}_{\geq 1}$.

Thm. (Kondo-T.)

- $\# \mathfrak{PM}_{(n,1)}(\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}) = 6$ for $\forall n \geq 4$.

Topic 2: Pseudo-Riemannian metrics (5/7)

One can give a generalization of Milnor's theorem for these cases, and can calculate the curvatures directly.

Thm. (Kubo-Onda-Taketomi-T. 2016)

Among $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}) = 3$ for $\forall p, q \in \mathbb{Z}_{\geq 1}$,

- all of them have constant sectional curvatures;
- one is positive, one is negative, and one is flat.

Thm. (Kondo-T.)

Among $\# \mathfrak{PM}_{(n,1)}(\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}) = 6$ for $\forall n \geq 4$,

- only one is flat;
- other five are (algebraic) Ricci soliton, but not Einstein.

Topic 2: Pseudo-Riemannian metrics (6/7)

Why $\# \mathfrak{PM}_{(p,q)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}) = 3$?

- Recall: $\mathfrak{PM}_{(p,q)}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}) \cong Q_{1,p+q-1} \backslash \mathrm{GL}(p+q,\mathbb{R}) / \mathrm{O}(p,q)$;
- Consider: $\mathrm{O}(p,q) \curvearrowright \mathrm{GL}(p+q,\mathbb{R}) / Q_{1,p+q-1} \cong \mathbb{R}\mathrm{P}^{p+q-1}$;
- Since $\mathrm{O}(p,q)$ preserves $\langle , \rangle_{p,q}$, it also preserves

$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{spacelike}\},$$
$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{lightlike}\},$$
$$\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1} \mid v : \text{timelike}\}.$$

Similar for $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$

- $\mathrm{O}(n-1,1) \curvearrowright \mathrm{GL}(n,\mathbb{R}) / Q_{1,n-3,2} \cong F_{1,n-3}(\mathbb{R}^n)$;
- it is a (partial) flag manifold.

Topic 2: Pseudo-Riemannian metrics (7/7)

Comments

- For $H^3 \times \mathbb{R}^{n-3}$ with $n \geq 4$, there exist exactly 6 left-inv. Lorentzian metrics up to "automorphism" and scaling.
- At the moment we do not know whether these 6 metrics are "nonisometric" up to scaling or not.

Topic 3: Nondegenerate 2-forms (1/4)

We apply same strategy for studying left-inv. symplectic structures.

Setting

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\dim G = 2n$,
- $\widetilde{\Omega}(\mathfrak{g}) := \{\text{nondegeneate 2-forms on } \mathfrak{g}\}$
  $\cong \{\text{left-inv. nondegeneate 2-forms on } G\}$.

Def.

- $\mathfrak{P}\Omega(\mathfrak{g}) := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\Omega}(\mathfrak{g})$ is called the **moduli space** of left-invariant nondegenerate 2-forms on G.

Topic 3: Nondegenerate 2-forms (2/4)

Note

- $\widetilde{\Omega}(\mathfrak{g}) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(n, \mathbb{R})$,
- which is a pseudo-Riemannian symmetric space.

We are interested in the case that $\mathfrak{P}\Omega(\mathfrak{g})$ is small.

Cor. (of Theorem by Wolf)

- $\#\mathfrak{P}\Omega(\mathfrak{g}) < \infty$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^{2n}}$ or $\mathfrak{h}^3 \oplus \mathbb{R}^{2n-3}$.

Topic 3: Nondegenerate 2-forms (3/4)

Thm. (CastellanosMoscoso-T.)

- $\#\mathfrak{P}\Omega(\mathfrak{g}_{\mathbb{R}\mathrm{H}^{2n}}) = 1$;
- $\#\mathfrak{P}\Omega(\mathfrak{h}^3 \oplus \mathbb{R}^{2n-3}) \leq 5$.

We can construct a kind of Milnor frames of symplectic basis.

Cor.

- $\exists 1$ left-inv. symplectic structure on $G_{\mathbb{R}\mathrm{H}^2}$;
- $\nexists$ left-inv. symplectic structures on $G_{\mathbb{R}\mathrm{H}^{2n}}$ if $n \geq 2$;
- $\exists 1$ left-inv. symplectic structure on $H^3 \times \mathbb{R}^{2n-3}$.

Topic 3: Nondegenerate 2-forms (4/4)

Note

Classification of left-invariant symplectic structures are known for

- $\dim = 2$ (easy)
- $\dim = 4$ (Ovando 2006)
- $\dim = 6$ & nilpotent (Goze-Khakimdjanov-Medina 2004)

Plan

- We will try to apply our method to some other Lie groups (whose automorphism groups are not small...).

Summary (1/2)

Summary

- Left-invariant Riem. metrics can be studied by isometric actions on some noncompact Riem. symmetric spaces.

- Left-invariant **pseudo-Riem**. metrics can be studied by isometric actions on some **pseudo-Riem**. symmetric spaces.

- Left-invariant **symplectic structures** can also be studied by isometric actions on some **other pseudo-Riem**. symmetric spaces.

Summary (2/2)

Plan

- Our method could be applied to other geometric structures.
- It would be practical if $\mathrm{Aut}(\mathfrak{g})$ is large.

Thank you very much!