Positivity of singular Hermitian metrics on vector bundles

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1 Introduction

2 The first case

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**Setting**

- $X$: complex manifold (dim $X = n$)
- $E \to X$: holomorphic vector bundle of rank $r$
- $h$: singular Hermitian metric (sHm) on $E$
- $\Theta_h$: Chern curvature current of $h$
### Setting

- $X$: complex manifold ($\dim X = n$)
- $E \to X$: holomorphic vector bundle of rank $r$
- $h$: singular Hermitian metric (sHm) on $E$
- $\Theta_h$: Chern curvature current of $h$

(Def.)

$h$ is a measurable map from $X$ to the space of non-negative Hermitian forms on the fibers, i.e.

- $|u|^2_h$: measurable ($\forall u \in H^0(U, E), \forall U \subset X$ open),
- $0 < \text{det} h < +\infty$ a.e. on each fiber.

$\Theta_h := \overline{\partial}(h^{-1} \partial h)$ locally.
Example (Raufi ’15)

Let $\Delta := \{ z \in \mathbb{C} \mid |z| < 1 \}$, $E := \Delta \times \mathbb{C}^2$, and

\[
h = \begin{pmatrix}
1 + |z|^2 & z \\
\frac{1}{z} & |z|^2
\end{pmatrix}.
\]
Example (Raufi ’15)

Let $\Delta := \{ z \in \mathbb{C} \mid \|z\| < 1 \}$, $E := \Delta \times \mathbb{C}^2$, and

$$h = \begin{pmatrix} 1 + |z|^2 & z \\ \bar{z} & |z|^2 \end{pmatrix}.$$ 

Then,

- $h$ is a sHm on $E$, and has a singularity at the origin. In fact, at $z = 0$,
  $$h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

- $h$ is Griffiths semi-negative (\iff\ \log |u|^2_h$ is psh for $\forall u \in \mathcal{O}(E)$).
- $\Theta_h$ is not a current with measure coefficients.
For this reason, we cannot generally define the positivity or negativity of a sHm on a vector bundle by using the curvature current. For example, we do not know the definition of the Nakano positivity in the singular setting. There are two ways to study positivity notions of a sHm.
Motivation

· For this reason, we cannot generally define the positivity or negativity of a sHm on a vector bundle by using the curvature current.

· For example, we do not know the definition of the Nakano positivity in the singular setting.

· There are two ways to study positivity notions of a sHm.

1. Find some sufficient conditions that curvature currents can be defined with measure coefficients.

2. Seek new positivity notions without using curvature currents.
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The first one

**Theorem (Raufi ’15)**

Let $h$ be a Griffiths semi-negative sHm. If $\det h > \epsilon$ ($\epsilon > 0$), then

1. $\theta_h = h^{-1} \partial h \in L^2_{loc}(X)$, and
2. $\Theta_h$ has measure coefficients.
Theorem (Raufi '15)

Let $h$ be a Griffiths semi-negative sHm. If $\det h > \epsilon$ ($\epsilon > 0$), then

1. $\theta_h = h^{-1} \partial h \in L^2_{loc}(X)$, and
2. $\Theta_h$ has measure coefficients.

- In other words, if a Griffiths semi-negative sHm $h$ is non-degenerate, $\Theta_h$ has measure coefficients.

- In general, a Griffiths semi-negative sHm possibly degenerates, that is, $\{\det h = 0\} \neq \emptyset$.

- We find the condition that the curvature current can be defined with measure coefficients over the degeneracy set $\{\det h = 0\}$. 
Main Theorem 1

**Theorem** (I. ’19)[I]

Let $h$ be a Griffiths semi-negative sHm. Suppose that

(i) $\nu(\log \det h, x) < 1 - \epsilon$ for $x \in X$, and

(ii) $\sqrt{-1}\partial\bar{\partial} \log \det h \in L^{1+\delta}_{loc}$

for $0 \leq \epsilon < 1, \delta > 0$. Then we have

1. $\theta_h \in L^{\frac{2}{2-\epsilon}}_{loc}$, and

2. if $(\epsilon + 1)(\delta + 1) \geq 1$, $\Theta_h$ has measure coefficients.
Main Theorem 1

**Theorem (I. '19)[I]**

Let \( h \) be a Griffiths semi-negative sHm. Suppose that
(i) \( \nu(\log \det h, x) < 1 - \epsilon \) for \( x \in X \), and
(ii) \( \sqrt{-1}\partial\bar{\partial} \log \det h \in L_{loc}^{1+\delta} \)
for \( 0 \leq \epsilon < 1, \delta > 0 \). Then we have

1. \( \theta_h \in L_{loc}^{\frac{2}{2-\epsilon}} \), and
2. if \( (\epsilon + 1)(\delta + 1) \geq 1 \), \( \Theta_h \) has measure coefficients.

We can also prove a version of the above theorem in the case that \( h \) is Griffiths semi-positive.

Namely, if the singularity of \( h \) is ”mild”, the curvature current has measure coefficients over the degeneracy set \( \{ \det h = 0 \} \).

Further applications

We introduce a notion of the sHm with minimal singularities.

(Line bundle cases)

· For every pseudo-effective line bundle $L \to X$, there exist sHms $h_{\text{min}}$ with minimal singularities such that $\sqrt{-1}\Theta h_{\text{min}} \geq 0$ from the results of [Demailly-Peternell-Schneider ’01].

· Let $\varphi_{\text{min}}$ be the local weight of $h_{\text{min}} = e^{-\varphi_{\text{min}}}$. It is known that $\varphi_{\text{min}}$ satisfies like the conditions described in our theorem.
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- Let $\varphi_{min}$ be the local weight of $h_{min} = e^{-\varphi_{min}}$. It is known that $\varphi_{min}$ satisfies like the conditions described in our theorem.

For example, If $X$ is a projective manifold and $L \to X$ is a nef and big line bundle, $\varphi_{min}$ has zero Lelong numbers everywhere [DPS ’01].

- $\partial \bar{\partial}$-Laplacian conditions of $\varphi_{min}$ are obtained by many people (cf. [Berman ’18], [Chu-Tosatti-Weinkove ’18]).
(Vector bundle cases)

- If the sHm with minimal singularities $h_{min}$ on vector bundles is constructed, we can expect that $h_{min}$ satisfies the above regularity properties and $\Theta h_{min}$ has measure coefficients from our theorem.

- However, we do not know anything about $h_{min}$ (existence, construction, property, etc...).

- There are various problems in this field.
  - Construct $h_{min}$ on a Griffiths semi-positive vector bundle.
  - Is det $h_{min}$ a sHm with minimal singularities on det $E$?
  - Does $\Theta h_{min}$ has measure coefficients?
  - …
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The second case

**Theorem (Deng-Wang-Zhang-Zhou '18)**

Let \( X := \Omega \) be a bounded domain in \( \mathbb{C}^n \). Assume that for any \( z \in \Omega \), any \( a \in E_z \setminus \{0\} \) with \( |a|_h < +\infty \), and any \( m \in \mathbb{N} \), there is a \( f_m \in H^0(\Omega, E^\otimes m) \) such that \( f_m(z) = a^\otimes m \) and satisfies the following condition:

\[
\int_{\Omega} |f_m|_{h^\otimes m}^2 \leq C_m |a^\otimes m|_{h^\otimes m}^2,
\]

where \( C_m \) are constants independent of \( z \in \Omega \) and satisfy the growth condition \( \lim_{m \to \infty} \frac{1}{m} \log C_m = 0 \).

Then if \( |\xi|_{h^*} \) is u.s.c. for any \( \xi \in \mathcal{O}(E^*) \), \((E, h)\) is Griffiths semi-positive.
The second case

**Theorem (Deng-Wang-Zhang-Zhou '18)**

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Then if $|\xi|_{h^*}$ is u.s.c. for any $\xi \in \mathcal{O}(E^*)$, $(E, h)$ is Griffiths semi-positive.

- The above theorem implies that an Ohsawa-Takegoshi type condition is a new positivity notion which is stronger than or equivalent to the Griffiths semi-positivity.
Main Theorem 2

Theorem (Hosono-I. ’19)[HI]
This Ohsawa-Takegoshi type positivity is weaker than Nakano semi-positivity in smooth settings.
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**Theorem** (Hosono-I. ’19)[HI]
This Ohsawa-Takegoshi type positivity is weaker than Nakano semi-positivity in smooth settings.

- To be precise, we define the Ohsawa-Takegoshi type positivity in global settings, and show the existence of \((E, h)\) such that \((E, h)\) is positively curved in the Ohsawa-Takegoshi sense, whereas \((E, h)\) is **not** Nakano semi-positive.

- We have the following inclusion relations:
  \[
  \{ \text{Nakano semi-positivity} \} 
  \subset \subset \{ \text{Ohsawa-Takegoshi type positivity} \} 
  \subset \subset \{ \text{Griffiths semi-positivity} \}
  \]

Thank you for your kind attention!