Symmetry Characterization Theorems for
Homogeneous Siegel Domains

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Siegel domains (history)

— Introduced by Piatetski-Shapiro (1957) for his study of automorphic forms
— Needed an unbounded realization of the unit ball $B^{2N} \subset \mathbb{C}^N$
— Just a moment’s consideration on Shilov boundary convinces us that it cannot be of the half-space type $\mathbb{R}^N + i\Omega \subset \mathbb{C}^N$ if $N > 1$:
  
  The Shilov boundary of $B^{2N}$ is the sphere $S^{2N-1}$,
  whereas the Shilov boundary of $\mathbb{R}^N + i\Omega$ is $\mathbb{R}^N$.

— Piatetski-Shapiro gave an example of non-symmetric homogeneous Siegel domains in $\mathbb{C}^4$ and $\mathbb{C}^5$ (1959).
— This solved a problem posed by E. Cartan (1935), since Siegel domains are holomorphically equivalent to bounded domains.
— By E. Cartan: in $\mathbb{C}^2$ and $\mathbb{C}^3$, every homogeneous bounded domain is symmetric.
— Recall that a domain $\mathcal{D} \subset \mathbb{C}^N$ is symmetric

\[ \begin{align*}
\overset{\text{def}}{\iff} \forall z_0 \in \mathcal{D}, \ \exists \varphi_0 \in \text{Hol}(\mathcal{D}) \text{ s.t.} \begin{cases}
(1) \ z_0 \text{ is an isolated fixed point of } \varphi_0, \\
(2) \ \varphi_0 \circ \varphi_0 = \text{Id}_{\mathcal{D}}.
\end{cases}
\end{align*} \]
E. Cartan’s conjecture

The discovery of non-symmetric bounded domains would have to be based on a fresh idea. — turned out to be correct.

— Note that Cartan never wrote that every homogeneous bounded domains was symmetric, the false conjecture spread by someone who did not read or did not understand what Cartan wrote in French subjonctif.

Now we know a lot of non-symmetric homogeneous Siegel domains.

Basic Question  How do we characterize symmetric domains among homogeneous Siegel (or bounded) domains?
There are already many works. Just list some . . .

— **A. Borel, J.-L. Koszul, J. Hano** (50’s):
  - if the domain is homogeneous under a (semisimple or unimodular) Lie group

— **I. Satake, J. Dorfmeister** (late 70’s)
  - in terms of the defining data of Siegel domains

— **J. E. D’Atri and Dotti Miatello** (1983)
  - by the non-positivity of the sectional curvature of the Bergman metric

— **K. Azukawa** (1985)
  - by the number of distinct eigenvalues of the curvature operator

— **J. Vey** (1970)
  - by the existence of a discrete subgroup \( \Gamma \subset \text{Hol}(\mathcal{D}) \) acting on \( \mathcal{D} \) properly
    s.t. \( \Gamma K = D \) for a compact subset \( K \subset \mathcal{D} \).

— **N** (2001)
  - by the commutativity of the Berezin transform with the Laplace–Beltrami op.

— **Xu Yichao** (1979), **N** (2003)
  - by the harmonicity of the Poisson kernel defined a là Hua.

**Today’s talk**
Symmetry characterization related to Cayley transforms
Siegel domains (definition)
— $V$: a finite-dimensional real vector space (with a norm),
— $\Omega$: an open convex cone in $V$,
   We assume that $\Omega$ is regular (contains no entire line).
— $W := V_\mathbb{C}$: the complexification of $V$,
— $w \mapsto w^*$: the conjugation in $W$ w.r.t. the real form $V$,
— $U$: another finite-dimensional complex vector space,
— $Q$: a Hermitian sesqui-linear map $U \times U \rightarrow W$
   (complex linear in the first variable, conjugate linear in the second),
   We assume that $Q$ is $\Omega$-positive. Thus we have
   $$Q(u', u) = Q(u, u')^*, \quad Q(u, u) \in \mathrm{Cl}(\Omega) \setminus \{0\} \quad (0 \neq \forall u \in U).$$

\textbf{Definition 1}
$$D(\Omega, Q) := \{(u, w) \in U \times W \mid w + w^* - Q(u, u) \in \Omega\}.$$
• $D = D(\Omega, Q)$ is said to be homogeneous if $\text{Hol}(D)$ acts on $D$ transitively.

**Remark** Since $D$ is holomorphically equivalent to a bounded domain, $\text{Hol}(D)$ is a finite-dimensional Lie group (H. Cartan).

• We always assume that our Siegel domain is homogeneous.

**Examples**

(1) $V = \mathbb{R}$, $\Omega = \mathbb{R}_{>0}$, $U = \{0\}$, $W = V_{\mathbb{C}} = \mathbb{C}$.
In this case $D = \mathbb{R}_{>0} + i\mathbb{R}$ is the right half-space in $\mathbb{C}$. The Cayley transform $w \mapsto z := \frac{w - 1}{w + 1}$ maps $D$ onto the unit disk $\{z \in \mathbb{C} ; \ |z| < 1\}$.

(2) $V = \text{Sym}(n, \mathbb{R})$ : the real vector space of $n \times n$ real symmetric matrices,
$\Omega = \mathcal{P}(n, \mathbb{R})$ : the positive-definite matrices in $\text{Sym}(n, \mathbb{R})$,
$U = \{0\}$, $W = V_{\mathbb{C}} = \text{Sym}(n, \mathbb{C})$.
In this case $D = \mathcal{P}(n, \mathbb{R}) + \text{Sym}(n, \mathbb{R})$ is the Siegel right half-space. The Cayley transform $w \mapsto z := (w - e)(w + e)^{-1}$ ($e$ is the identity matrix) maps the Siegel right half-space onto the Siegel disk $\mathcal{D} := \{z \in \text{Sym}(n, \mathbb{C}) ; \ e - z^*z \gg 0\}$.
(3) $V = \mathbb{R}$, $\Omega = \mathbb{R}_{>0}$, $U = \mathbb{C}^m$, $W = V_{\mathbb{C}} = \mathbb{C}$, $Q(u_1, u_2) := 2u_1 \cdot \overline{u}_2$. In this case

$$D(\Omega, Q) = \{(u, w) \in \mathbb{C}^m \times \mathbb{C} ; \ Re w - ||u||^2 > 0\}.$$ 

The Cayley transform

$$(u, w) \mapsto \left( \frac{2u}{w + 1}, \frac{w - 1}{w + 1} \right)$$

maps $D(\Omega, Q)$ onto the unit ball in $\mathbb{C}^{m+1} = \mathbb{C}^m \times \mathbb{C}$. 
**Piatetski-Shapiro algebra** (normal $j$-algebra)

We know \{homogeneous Siegel domains\} $\Leftrightarrow$ \{Piatetski-Shapiro algebras\}

$\mathfrak{g}$ : a split solvable Lie algebra (ad is triangularizable over $\mathbb{R}$)

$J$ : a linear operator on $\mathfrak{g}$ with $J^2 = -\mathrm{Id}_\mathfrak{g}$.

$\omega$ : a linear form on $\mathfrak{g}$.

Then $(\mathfrak{g}, J, \omega)$ (or simply $\mathfrak{g}$) is a **Piatetski-Shapiro algebra** if

1. $[x, y] + J[Jx, y] + J[x, Jy] = [Jx, Jy],$
2. $\langle x | y \rangle_\omega := \langle \omega, [Jx, y] \rangle$ defines a $J$-invariant inner product on $\mathfrak{g}$.

- Linear forms $\omega$ that satisfy (2) are said to be **admissible**.
- The linear form $\beta$ on $\mathfrak{g}$ defined by
  $$\langle \beta, x \rangle := \mathrm{tr}(\mathrm{ad}(Jx) - J \mathrm{ad}(x))$$
  is admissible called the **Koszul form**. This corresponds to the real part of the Hermitian inner product (up to a positive number multiple) coming from the Bergman metric on $D$. 
Structure of a Piatetski-Shapiro algebra

\((g, J, \omega)\): a Piatetski-Shapiro algebra, \(\langle x | y \rangle_\omega : J\)-invariant inner product.

Let \(n := [g, g] \) ; the derived ideal of \(g\). and put \(a := n^\perp\) w.r.t. \(\langle \cdot | \cdot \rangle_\omega\).

Then, \(g = a + n\) with \([a, n] \subset n\).

For \(\alpha \in a^*\), we put

\[n_\alpha := \{x \in n ; [a, x] = \langle \alpha, a \rangle x \ (\forall a \in a)\} .\]

Then, \(\exists \Delta \subset a^* \setminus \{0\} (\#\Delta < +\infty)\) s.t. \(n_\alpha \neq \{0\} (\forall \alpha \in \Delta)\) and \(g = a + \sum_{n \in \Delta} n_\alpha\).

We can choose a basis \(H_1, \ldots, H_r\) of \(a\) so that with \(E_j := -JH_j (\in n)\) we have \([H_j, E_k] = \delta_{jk}E_k\). Let \(\alpha_1, \ldots, \alpha_r\) be the basis of \(a^*\) dual to \(H_1, \ldots, H_r\).

Then, the elements of \(\Delta\) are (not all possibilities occur)

\[
\frac{1}{2}(\alpha_k \pm \alpha_j) \ (j < k), \quad \alpha_k \ (k = 1, \ldots, r), \quad \frac{1}{2}\alpha_k \ (k = 1, \ldots, r).
\]

Moreover, \(n_{\alpha_k} = \mathbb{R}E_k \ (\forall k = 1, \ldots, r)\).

Define \(E_k^* \in g^*\) by requiring

\[
\langle E_k^*, E_k \rangle = 1, \quad E_k^* = 0 \text{ on } a \text{ and on } n_\alpha (\alpha \neq \alpha_k).
\]

We set for \(s = (s_1, \ldots, s_r) \in \mathbb{R}^r\)

\[
E_s^* := s_1E_1^* + \cdots + s_rE_r^*.
\]
We write $s > 0$ if $s_1 > 0, \ldots, s_r > 0$.

**Proposition 2**

The set of the admissible linear forms on $g$ coincides with

$$a^* + \{E_s^* ; s > 0\}.$$  

Thus we only have to consider $E_s^*$ $(s > 0)$ for the inner product on $g$, and we put

$$\langle x \mid y \rangle_s := \langle E_s^*, [Jx, y] \rangle.$$  

Let

$$g(0) := a \oplus \sum_{m>k} n(\alpha_m-\alpha_k)/2, \quad g(1/2) := \sum_{k=1}^{r} n\alpha_k/2,$$

$$g(1) := \sum_{j=1}^{r} n\alpha_j \oplus \sum_{m>k} n(\alpha_m+\alpha_k)/2.$$  

Then, $[g(i), g(j)] \subset g(i + j)$. In particular, $g(0)$ is a Lie subalgebra.

Moreover, $Jg(1) = g(0), \ Jg(1/2) = g(1/2)$. In fact,

$$JE_j = H_j, \ Jn(\alpha_k+\alpha_j)/2 = n(\alpha_k-\alpha_j)/2 \ (k > j), \ Jn\alpha_k/2 = n\alpha_k/2.$$
Siegel domains defined by a Piatetski-Shapiro algebra

$$(g, J, \omega) :$$ our Piatetski-Shapiro algebra,

$$G(0) =: \exp g(0), \quad E := E_1 + \cdots + E_r \in \mathfrak{n}_{\alpha_1} \oplus \cdots \oplus \mathfrak{n}_{\alpha_r} \subset g(1).$$

$G(0)$ acts on $V := g(1)$, and the orbit $\Omega := G(0)E$ through $E$ is a regular open convex cone, and $G(0)$ acts on $\Omega$ simply transitively.

$$W := V_\mathbb{C}, \quad w \mapsto w^*:$$ the conjugation in $W$ relative to $V$.

$$U := (g(1/2), -J):$$ the vector space $g(1/2)$ made complex by $-J$.

$$Q(u, u') := \frac{1}{2}([Ju, u'] - i[u, u']) \quad (u, u' \in U)$$

turns out to be complex sesqui-linear ($\mathbb{C}$-linear in the first variable, conjugate linear in the second) Hermitian map $U \times U \to W$, and $\Omega$-positive.

- We assume that our Siegel domain $D = D(\Omega, Q)$ is defined by these data $\Omega, Q$.
- $\mathfrak{n}_D := g(1) + g(1/2)$ is a 2-step nilpotent Lie algebra.

$$N_D := \exp \mathfrak{n}_D$$ acts on $D$ as follows:

writing elements of $N_D$ by $n(a, b)$ ($a \in g(1)$, $b \in g(1/2)$), we have

$$n(a, b) \cdot (u, w) = (u + b, w + ia + \frac{1}{2}Q(b, b) + Q(u, b))$$

$G(0)$ acts on $V$, and hence on $W = V_\mathbb{C}$. Also $G(0)$ acts on $U$ complex linearly.
- Thus we have the action of $G := \exp g = G(0) \rtimes N_D$ on $D$ by $\mathbb{C}$-affine auto.
Compound power functions

$G(0) = \exp g(0)$ is a semidirect product $A \ltimes N(0)$, where

$$A = \exp a, \quad N(0) = \exp n(0), \quad n(0) := \sum_{m > k} n(\alpha_m - \alpha_k)/2.$$  

For each $s \in \mathbb{R}^r$, let $\chi_s$ be the one-dimensional representation of $A$ defined by

$$\chi_s(\exp \sum_{j=1}^r t_j H_j) = \exp \left( \sum_{j=1}^r s_j t_j \right),$$

and extend it to a one-dimensional representation of $G(0)$ by setting identically equal to 1 on $N(0)$.

Fix a base point $e := (0, E) \in D \subset U \times W$, we have diffeomorphisms

$$G \ni g \mapsto g \cdot e \in D, \quad G(0) \ni h \mapsto hE \in \Omega.$$  

For every $s \in \mathbb{R}^r$, define a function $\Delta_s$ on $\Omega$ by $\Delta_s(hE) := \chi_s(h) \ (h \in G(0))$.

- $\Delta_s$ is relatively invariant: $\Delta_s(hx) = \chi_s(h) \Delta_s(x) \ (h \in G(0), \ x \in \Omega)$.

Theorem 3 (Gindikin 1975, Ishi 2000)

The functions $\Delta_s$ are analytically continued to holomorphic functions on $\Omega + iV$.  

(1) Bergman kernel $\kappa(z_1, z_2)$ of $D$ is written as (up to a positive number multiple)

$$\kappa(z_1, z_2) = \Delta_{-2d-b}(w_1 + w_2^* - Q(u_1, u_2)) \quad (z_j = (u_j, w_j) \in D),$$

where

$$d_k := \text{tr} \text{ad}_{g(1)}(H_k) = \text{tr} [H_k, g(1)], \quad b_k := \text{tr} \text{ad}_{g(1/2)}(H_k).$$

and $d = (d_1, \ldots, d_r)$, $b = (b_1, \ldots, b_r)$.

(2) The characteristic function $\phi$ of $\Omega$:

$$\phi(x) := \int_{\Omega^*} e^{-\langle \lambda, x \rangle} d\lambda \quad (x \in \Omega).$$

$\phi(x) = \Delta_{-d}(x)$ up to a positive number multiple.

(3) Szegö kernel $S(z_1, z_2)$ (the reproducing kernel of the Hardy space $H^2(D)$ over $D$) is $\Delta_{-d-b}(w_1 + w_2^* - Q(u_1, u_2))$ up to a positive number multiple.

$H^2(D)$ is the Hilbert space of holomorphic functions $F$ on $D$ s.t.

$$\|F\|^2 = \sup_{t \in \Omega} \int_U dm(u) \int_V \left| F(u, t + \frac{1}{2}Q(u, u) + ix) \right|^2 dx < +\infty.$$

**Remark** (1) $D$ : symmetric $\implies d_1 = \cdots = d_r$ and $b_1 = \cdots = b_r$.

(2) (D’Atri–Dotti 1983) Irreducible $D$ is quasi-symmetric

$\iff$ (a) $\dim \Pi_{(\alpha_k+\alpha_j)/2}$ is indep. of $k, j$ and (b) $\dim \Pi_{\alpha_m/2}$ is indep. of $m$. 
Cayley transforms

Recall that for the case of right half-plane \( \rightarrow \) the unit disk, we have \( z = \frac{w - 1}{w + 1} \).

Observe that \( \frac{w - 1}{w + 1} = 1 - \frac{2}{w + 1} \). What we need is the denominator \( (w + 1)^{-1} \).

- Recall the following formula for \( x, v \in \mbox{Sym}(n, \mathbb{R}) \), and \( x \gg 0 \):

\[
\frac{d}{dt} \log \det (x + tv)^{-1} \bigg|_{t=0} = \mbox{tr}(x^{-1}v).
\]

**Definition 4**

For \( x \in \Omega \) and \( s > 0 \), we define \( \mathcal{I}_s(x) \in V^* \) by

\[
\langle \mathcal{I}_s(x), v \rangle = -\frac{d}{dt} \log \Delta_s(x + tv) \quad (v \in V).
\]

- \( \mathcal{I}_s(hx) = h \cdot \mathcal{I}_s(x) \) \( (h \in G(0), x \in \Omega) \).
  
  In particular, \( \mathcal{I}_s(\lambda x) = \lambda^{-1} \mathcal{I}_s(x) \) \( (\lambda > 0) \).

- Let \( \Omega^* := \{ \xi \in V^* \ ; \ \langle \xi, v \rangle > 0 \ \forall x \in \mbox{Cl}(\Omega) \setminus \{0\} \} \).

  \( \Omega^* \) is called the **dual cone** of \( \Omega \).
Proposition 5 (N 2003DGA)

Suppose $s > 0$.

(1) For any $x \in \Omega$, we have $\mathcal{I}_s(x) \in \Omega^*$, and $\mathcal{I}_s$ gives a bijection of $\Omega$ onto $\Omega^*$.
(2) $\mathcal{I}_s(E) = E^*_s$.
(3) $\mathcal{I}_s$ is analytically continued to a rational map $W \to W^*$.
(4) $\mathcal{I}_s : \Omega + iV \to \mathcal{I}_s(\Omega + iV)$ is a biholomorphic bijection.

Theorem 6 (Kai–N 2005)

$\mathcal{I}_s(\Omega + iV) = \Omega^* + iV^* \iff s$ is a positive multiple of $d$ and $\Omega$ is selfdual.
In this case we have $s_1 = \cdots = s_r$.

- $\Omega$ is said to be **selfdual** if $\Omega^*$ is transferred to $V$ by means of an appropriate inner product, then $\Omega^* = \Omega$.
- $C_s(w) := \mathcal{I}_s(E) - 2\mathcal{I}_s(w + E)$ ($w \in W$). Recall that $\mathcal{I}_s(E) = E^*_s$.
- $C_s$ is holomorphic on a domain containing $\text{Cl}(\Omega + iV)$. 
Definition 7

For $z = (u, w) \in D$, we define

$$C_s(z) = (2\langle I_s(w + E), Q(u, \cdot) \rangle, C_s(w))$$

Note $U \ni u' \mapsto \langle I_s(w + E), Q(u, u') \rangle \in \mathbb{C}$ is a conjugate linear form on $U$.

Remark (1) For symmetric domains, our Cayley transform with the parameter $s_1 = \cdots = s_r$ is essentially the inverse transform defined by Korány–Wolf (1965).

(2) The parameter $s = d$ corresponds to Penney’s Cayley transform (1996).

Theorem 8 (N 2003DGA)

(1) $C_s$ is a birational and biholomorphic bijection of $D$ onto $C_s(D)$.

(2) $C_s(D)$ is bounded in $U^\dagger + W^*$, where $U^\dagger$ denotes the complex space of all conjugate linear forms on $U$.

Theorem 9 (Kai 2007)

$C_s(D)$ is convex if and only if $D$ is symmetric and $s$ satisfies $s_1 = \cdots = s_r$. 
Convex realization of a homogeneous bounded domain

Harish-Chandra (1956):
- Every non-cpt Hermitian symm. space is realized as a bdd domain in $\mathbb{C}^N$.
- This is given without using classification.
- In particular, Harish-Chandra gave two exceptional symmetric bounded domains which E. Cartan did not treat.

- Harish-Chandra’s realization turned out to be a convex set.
- This is proved first by Hermann (1963), and it is in fact an open unit ball w.r.t some (Banach) norm.
- Nowadays we have a more elementary description in terms of Hermitian Jordan triple system (JTS), and the Harish-Chandra realization is the open unit ball of the spectral norm of the JTS. For domains equivalent to tube domains (Jordan algebra case), see the book of Faraut–Korányi.
Mok-Tsai’s Theorem (1992)
Let $\mathcal{D}$ be an irreducible symmetric bounded domain of rank $\geq 2$. If $\mathcal{D}$ is a convex set, then $\mathcal{D}$ is affinely equivalent to the Harish-Chandra realization.

Gindikin’s conjecture
If a bounded homogeneous domain is a convex set, it is symmetric.

Ishi–Kai’s representative domain of a homogeneous bounded domain (2010)
In the spirit of Bergman (1929), Ishi and Kai introduced the representative domain $\mathcal{D}_0$ of a given homogeneous bounded domain $\mathcal{D}$ by using the Bergman kernel of $\mathcal{D}$. The domain can be considered as a generalization to non-symmetric case of Harish-Chandra realization. They showed that $\mathcal{D}_0$ coincides, up to a positive number multiple, with my Cayley transform image $\mathcal{C}_{2d+b}(\mathcal{D})$ of the corresponding Siegel domain realization $\mathcal{D}$ of $\mathcal{D}$.

Restatement of Kai’s 2007 result
Let $\mathcal{D}$ be a homogeneous bounded domain. Then, its representative domain $\mathcal{D}_0$ is convex if and only if $\mathcal{D}$ is symmetric.
Other symmetry characterization theorems related to Cayley transforms

Recall the $J$-invariant inner product $\langle \cdot | \cdot \rangle_s$ on $g$. By the differential of the diffeomorphphic orbit map $G \ni g \mapsto g \cdot e \in D$, this norm is transferred to a Hermite inner product on the tangent space $T_e(D) = U + W$, and further transferred naturally to a Hermitian inner product on $U^\dagger + W^*$ which we denote by $(\cdot | \cdot)_s$. The corresponding norm is denoted by $\| \cdot \|_s$.

**Theorem 10 (N 2001TG)**

Suppose $D$ is irreducible. Then,

$$\| C_{2d+b}(g \cdot e) \|_s = \| C_{2d+b}(g^{-1} \cdot e) \|_s \quad (\forall g \in G)$$

if and only if $D$ is symmetric and $s > 0$ is positive number multiple of $2s + b$.

- This theorem is used to prove that the Berezin transform on $D$ commutes with the Laplace–Beltrami operator $\mathcal{L}_s$ on $D$ defined by using the metric $\langle \cdot | \cdot \rangle_s$ if and only if $D$ is symmetric and $s$ is a positive number multiple of $2s + b$ (N 2001DGA)

Theorem 10 can be rephrased as follows.
Theorem  Suppose $D$ is irreducible. Then,\
$$\|h \cdot 0\|_s = \|h^{-1} \cdot 0\|_s \quad \text{for } \forall h \in \mathcal{G}_{2d+b} \circ G \circ \mathcal{G}^{-1}_{2d+b}$$
if and only if $\mathcal{D}_{2d+b} := \mathcal{G}_{2d+b}(D)$ is symmetric and $s$ is a positive number multiple of $2d + b$.

$SU(1, 1)$ acts on $\mathcal{D} := \{ |z| < 1 \}$ by $\left( \frac{\alpha}{\beta} \frac{\beta}{\alpha} \right) \cdot z := \frac{\alpha z + \beta}{\beta z + \alpha}$.

$a(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, \quad $A := \{ a(t) \ ; \ t \in \mathbb{R} \}$,

$n(\xi) := \begin{pmatrix} 1 - i\xi & i\xi \\ -i\xi & 1 + i\xi \end{pmatrix}$, \quad $N := \{ n(\xi) \ ; \ \xi \in \mathbb{R} \}$.

$G := NA$ is the Iwasawa subgroup of $SU(1, 1)$.

$P : n(\xi)a(t) \cdot 0 = n(\xi) \cdot \tanh t \in N \cdot r \ (r := \tanh t) : \text{green circle}$.

$Q : (n(\xi)a(t))^{-1} \cdot 0 = n(-e^{-2t}\xi)a(-t) \cdot 0 \in N \cdot (-r) : \text{green circle}$.

$|g \cdot 0| = |g^{-1} \cdot 0| \ (\forall g \in S) \iff \mathcal{D}$ is symmetric.

But this is trivial for $g = \left( \frac{\alpha}{\beta} \frac{\beta}{\alpha} \right) \in SU(1, 1) : \ g \cdot 0 = \frac{\beta}{\alpha}, \ g^{-1} \cdot 0 = -\frac{\beta}{\alpha}$. 
In the following theorem \( \Sigma \) denotes the Silov boundary of \( D \). It is known that
\[
\Sigma = \{(u, w) \in U \times W ; \ 2 \Re w = Q(u, u)\}.
\]
If \( D = \Omega + iV \ (U = \{0\}) \), then \( \Sigma = iV \).

Take \( \Psi_s \in \mathfrak{g} \) s.t. \( \text{tr} \text{ad}(x) = \langle x | \Psi_s \rangle_s \ (\forall x \in \mathfrak{g}) \). We know that \( \Psi_s \in \mathfrak{a} \).

Next, we put \( \alpha_s := \sum_{j=1}^{r} s_j \alpha_j \in \mathfrak{a}^* \).

**Theorem 11 (N 2003JFA)**

Suppose \( D \) is irreducible. Then,
\[
\| \mathcal{C}_{d+b}(\zeta) \|^2_s = \langle \alpha_{d+b}, \Psi_s \rangle \quad (\forall \zeta \in \Sigma)
\]
if and only if \( D \) is symmetric and \( s \) is a positive number multiple of \( d + b \).

The equality in Theorem 11 represents that the image of \( \Sigma \) under \( \mathcal{C}_{d+b} \) lies on a sphere centered at the origin.

Theorem 11 is used to prove that the Poisson-Hua kernel \( P(z, \zeta) := \frac{|S(z, \zeta)|^2}{S(z, z)} \)
\((z \in D, \ \zeta \in \Sigma) \) satisfies \( \mathcal{L}_s P(\cdot, \zeta) = 0 \ (\forall \zeta \in \Sigma) \) if and only if \( D \) is symmetric and \( s \) is a positive number multiple of \( d + b \).
References cited (not all)


— Papers not in the above list are included in References of my article
Unfortunately this is not yet of open access, and the preprint version is available at my webpage in Kyushu University (may not be permanent),
http://www2.math.kyushu-u.ac.jp/~tnomura/nomurappt.pdf