Lecture Note for

Introduction to Geometry of $K3$ Surfaces

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Abstract

What is a $K3$ surface? I'd try to answer you about this question. In the end, you shall find that $K3$ surfaces popping up everywhere, and that they have many characters and aspects in geometry. I hope that you'd get acknowledged and familier with, and interested in $K3$'s so as to discover something in common with your own interests and to find what we can do. One may consult [2] as to results for surfaces.

First Talk : I define $K3$ surface as a 2-dimensional version of elliptic curve that is also regarded as Riemannian surface of genus one. Then we explore several areas (differential and algebraic geometry, topology, differential equation, math.physics) in which $K3$ surfaces play important roles. Lastly, I introduce *Torelli-type theorem* that is fundamental and important for study of $K3$ surfaces because it interprets the geometry of $K3$ into the study of *lattices*.

Second Talk : It is necessary to study algebraic and transcendental parts of $K3$ surfaces in complex algebraic geometry. I introduce the Picard lattices as algebraic part, and the Hodge decomposition as transcendental. Finally, I relate them to the Torelli-type theorem.

Third Talk : In the third and last talk of the series, I introduce an example of study of $K3$ surfaces. Elliptic curves have projective model as the smooth cubic curve in $\mathbb{P}^2$, whilst $K3$ surfaces are realized as smooth quartic surfaces in $\mathbb{P}^3$. I often deal with its generalization: *hypersurfaces as anticanonical divisors in Fano* 3*-folds*. I discuss such $K3$ surfaces together with the Picard lattices.

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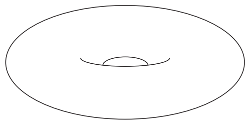
1 Definition of a $K3$ surface

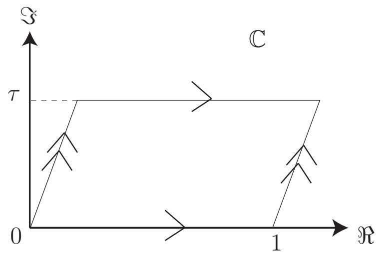
Introductory example



Torus = Riemannian surface of genus 1

$$\sim \qquad \mathbb{C}\,/(\mathbb{Z}\oplus\tau\mathbb{Z}) \qquad (\tau\in\mathbb{H})$$

$$\Gamma_\tau := \mathbb{Z}\oplus\tau\mathbb{Z}$$



Here $\quad \mathbb{H} := \{z\in\mathbb{C}\,|\,\Im z \geq 0\} \quad$ is the upper half plane.

As is well-known, a torus is topologically isomorphic to $\mathbb{C}/\Gamma_\tau$. Define the *Weierstrass* $\wp$*-function* $\wp(z)$ on the *lattice* $\mathbb{C}/\Gamma_\tau$ by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Gamma_\tau\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

$\wp(z)$ has poles of degree two on Γ_τ, and is regular at other points. (It is easy to verify that this function is well-defined on $\mathbb{C}/\Gamma_\tau$.)

Define a function φ on $\mathbb{C}/\Gamma_\tau$ by

$$\varphi:\ \mathbb{C}/\Gamma_\tau \ \rightarrow\ \mathbb{C}^3 \ \ ;\ \ z\ \mapsto\ (1,\,\wp(z),\,\wp'(z)).$$

$$\Rightarrow\ \ \mathrm{Im}\,\varphi = \left\{ (x,\,y)\in\mathbb{C}^2 \ \middle|\ \begin{array}{l} y^2 = 4x^3 - g_2x - g_3,\ \text{ where} \\ g_2 = 60 \displaystyle\sum_{\omega\in\Gamma_\tau\setminus\{0\}} \frac{1}{\omega^4},\ g_3 = 140 \displaystyle\sum_{\omega\in\Gamma_\tau\setminus\{0\}} \frac{1}{\omega^6} \end{array} \right\}.$$

The *n-dimensional projective space* $\mathbb{P}^n$ is defined to be a quotient space $\mathbb{P}^n := \mathbb{C}^{n+1}\setminus\{0\}\big/\sim$, where $(x_0,\,x_1,\,\ldots,x_n)\sim(y_0,\,y_1,\,\ldots,y_n)\in\mathbb{C}^{n+1}\setminus\{0\}$ if there exists $\lambda\in\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ such that $(y_0,\,y_1,\,\ldots,y_n) = \lambda(x_0,\,x_1,\,\ldots,x_n)$.

A point in $\mathbb{P}^n$ is denoted by $(x_0 : x_1 : \ldots : x_n)$. The n-dimensional projective space is covered by $n+1$ affine spaces : $\mathbb{P}^n = \bigcup_{i=0}^n U_i$, where

$$U_i = \{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\} = \left\{ \left(\frac{x_0}{x_i} : \cdots : 1 : \cdots : \frac{x_n}{x_i} \right) \in \mathbb{P}^n \right\} \simeq \mathbb{C}^n.$$

Homogenise the equation $y^2 = 4x^3 - g_2 x - g_3$ by setting $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$, and we get a homogeneous equation

$$E_\tau : Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$$

whose set $E_\tau$of zero-points are defined on $\mathbb{P}^2$. Now the set E_τ is

- an algebraic curve on $\mathbb{P}^2$ (= algebraic plane curve), because it has a defining *polynomial.*
- smooth, because the defining polynomial $F_\tau := Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3$ satisfies $\left(F_\tau, \frac{\partial F_\tau}{\partial X}, \frac{\partial F_\tau}{\partial Y}, \frac{\partial F_\tau}{\partial Z}\right) \neq 0$.
- cubic, because F_τ is homogeneously of degree three.

Definition 1.1 *A smooth algebraic plane cubic curve is called an* elliptic curve.

Alternatively, an elliptic curve is defined as a smooth algebraic curve of genus one.

The complex number $\tau \in \mathbb{H}$ is called the *period* of an elliptic curve E_τ. Via the period $\tau \in \mathbb{H}$, there is a one-to-one correspondence

$$\{\text{ellipric curves } E_\tau\}/\text{isom} \leftrightarrow SL(2,\mathbb{Z})\backslash\mathbb{H}\,(\leftrightarrow \{\Gamma_\tau\})\,.$$

Remark 1.1 More precisely, define the *j-invariant* $j(E)$ for an elliptic curve $E : y^2 = x^3 - g_2x - g_3$ by

$$j(E) = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

Since g_2 and g_3 only depend on the period, so does the j-invariant.

The j-invariant is a holomorphic invariant of elliptic curves. For instance

$j = 0 : E = (Y^2Z = X^3 - Z^3)$, and $\mathrm{Aut}(E) = \mathbb{Z}/6\mathbb{Z}$.

$j = 1728 : E = (Y^2Z = X^3 - XZ^2)$, and $\mathrm{Aut}(E) = \mathbb{Z}/4\mathbb{Z}$.

$j \neq 0,\, 1728 : \mathrm{Aut}(E_j) = \mathbb{Z}/2\mathbb{Z}$.

These examples are defined over an algebraically-closed field of any characteristic save 2 and 3.

An algebraic variety is a pair $V = (X, \mathcal{O}_X)$ of a topological space X in Zariski topology, and a sheaf $\mathcal{O}_X$ of holomorphic functions on X. For a nonsingular algebraic variety V of dimension n, a line bundle $\mathcal{E} \to X$ with a section $s : X \to \mathcal{E}$ defines a *(Weil) divisor* D on V by $D = (s = 0)$. Two divisors D and D' are *linearly equivalent* denoting $D \sim D'$ if there exists a nonzero holomorphic function f on V such that $D' = D + (f)$. The *canonical divisor* K_V of V is the linear equivalence class of a divisor associated to a line bundle $\bigwedge^n T_V^\wedge$, where T_V is the holomorphic tangent bundle on V. Divisors form an Abelian group with zero element 0. A divisor D is described as a formal sum : $D = \sum a_i D_i$, where $a_i \in \mathbb{Z}$ and D_i is an irreducible subvariety of codimension one in V.

Definition 1.2 *A nonzero divisor D is* effective *denoted by $D \geqslant 0$, if the coefficients $a_i \geq 0$ for all i. For a divisor E, define the* complete linear system $|E|$ *to be a set of effective divisors that are linearly equivalent to E.*

⟨ Property of elliptic curves ⟩
The canonical divisor K_{E_τ} of an elliptic curve E_τ is trivial : $K_{E_\tau} \sim 0$.

If the algebraic variety V has at most Gorenstein singularities, then, V admits the canonical divisor. For the *anticanonical divisor* $-K_V$ of V, the linear system $|-K_V|$ is called the *anticanonical linear system*, and its elements *anticanonical members*.

Fact ([6] §II-5 and 6)

(1) The *principal divisor* (f) of a function f on V is defined to be the sum of zero- $(f)_0$ and polar- $(f)_\infty$ loci of f.
(2) An algebraic variety being covered by affine varieties U_i, a *Cartier divisor* D on V is locally a Weil divisor on each U_i: $D|_{U_i} = (f_i = 0)$ with a function $f_i \in \mathcal{O}_X(U_i)$ on U_i for each i. If V is nonsingular, Cartier and Weil divisor coincide.
(3) An $\mathcal{O}_X$-module $\mathcal{L}(D)$ such that $\mathcal{L}(D)(U_i) = \mathcal{O}_X|_{U_i} f_i^{-1}$ is an invertible sheaf associated to D.
(4) In general, a rank-n vector bundle $\pi : \mathcal{E} \to X$ on V defines a set $\mathcal{S}(\mathcal{E}) := \{s : X \to \mathcal{E}\}$ of sections, which in fact turns to one-to-one correspond to a rank-n locally free sheaf by $\mathcal{E}^\wedge \xrightarrow{\sim} \mathcal{S}(\mathcal{E})$. In particular, a line bundle is associated to an invertible sheaf.
(5) Adding up (3) and (4), we occasionally identify Cartier divisors, invertible sheaves, and line bundles.

We define $K3$ surface as a 2-dimensional analogy of elliptic curve.

Definition 1.3 *Let S be a compact complex connected 2-dimensional algebraic variety. S is called a $K3$* surface *if S is smooth, the canonical divisor is trivial : $K_S \sim 0$, and irregularity is zero : $h^1(\mathcal{O}_S) = 0$.*

$\langle$ Properties of $K3$ surfaces $\rangle$ Let S be a $K3$ surface.
0°) (i) S is simply-connected.
(ii) There exists a nowhere-vanishing holomorphic 2-form ω_S on S such that $H^{2,0}(S) = \mathbb{C}\omega_S$.
(iii) There is no obstruction in deforming $K3$ surfaces.
1°) Any $K3$ surface is diffeomorphic to a smooth quartic hypersurface in $\mathbb{P}^3$. Every $K3$ surface admits a Kähler form [15].
2°) Denote by $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}(S)$ the dimension of $H^{p,q}(S)$.
The Hodge diamond of a $K3$ surface is given as

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

3°) Introduction of *Gorenstein $K3$ surfaces*

Definition 1.4 *Let $F = (f, 0)$ be a germ of singularity, i.e., $(f = 0)$ defines a singularity at 0 in $\mathbb{C}^3$.*
(1) *F is of type A_n $(n \geq 1)$ if*

$$f = x^2 + y^2 + z^{n+1},$$

(2) *F is of type D_n $(n \geq 4)$ if*

$$f = x^2 + y^2 z + z^{n-1},$$

(3) *F is of type*

$$\begin{array}{lll} E_6 & \text{if} & f = x^2 + y^3 + z^4, \\ E_7 & \text{if} & f = x^2 + y^3 + yz^3, \\ E_8 & \text{if} & f = x^2 + y^3 + z^5, \end{array}$$

after an appropriate transformation.

Remark 1.2 This is the full list of *rational double points* on a surface, and we have identifications
rational double point (RDP) $\Leftrightarrow$ canonical $\Leftrightarrow$ rational Gorenstein $\Leftrightarrow$ ADE.

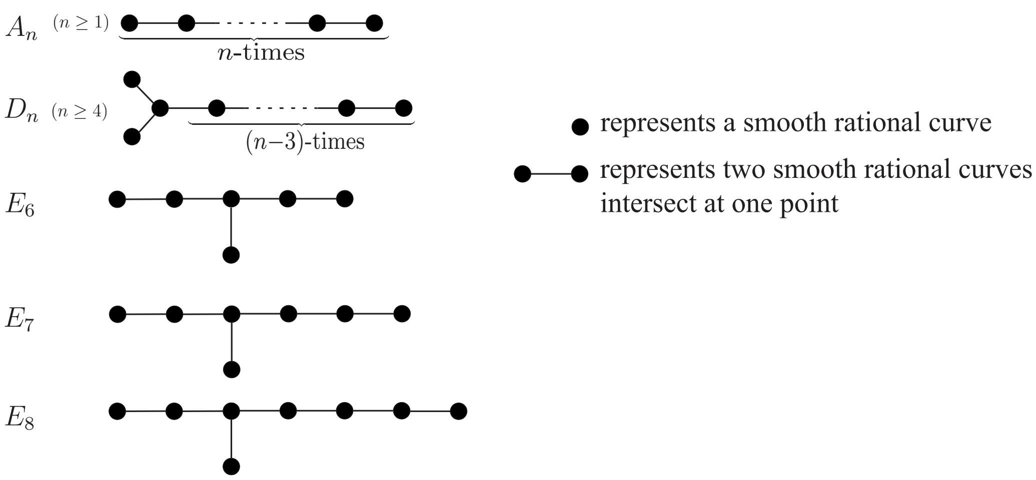


Figure 1: Dynkin diagrams of desingularisation of ADE singularities

Let S be a compact complex connected 2-dimensional algebraic variety with at most ADE singularities. A *resolution of singularities* (= desingularisation) is a birational morphism

$$\phi : \tilde{S} \to S$$

with $\tilde{S}$ being a smooth surface. By adjunction formula, we get

$$K_{\tilde{S}} = \phi^* K_S + \sum a_i E_i$$

with $a_i \geq 0$ since ADE are canonical, and E_i's are exceptional curves. The resolution ϕ is called *crepant* if the coefficients a_i are zero for all i.

Theorem 1.1 *There exists a crepant resolution of ADE singularities of S.*

If S satisfies $K_S \sim 0$, and $h^1(\mathcal{O}_S) = 0$, then, since properties of cohomology groups are birational-invariant, we have $h^1(\mathcal{O}_{\tilde{S}}) = h^1(\mathcal{O}_S) = 0$, and by crepant-ness, we have $K_{\tilde{S}} = \phi^* K_S \sim \phi^* 0 = 0$. Thus, the smooth model $\tilde{S}$ of S, which is unique when it exists, is a $K3$ surface.

Definition 1.5 *The surface S is called a* Gorenstein $K3$ surface *if S has at most ADE singularities, $K_S \sim 0$, and $h^1(\mathcal{O}_S) = 0$.*

4°) Let S be a $K3$ surface and $\gamma_1, \gamma_2, \ldots, \gamma_{22}$ be a generator of $H_3(S, \mathbb{Z})$, and $\omega_S \in H^{2,0}(S)$ be a nowhere-vanishing holomorphic 2-form on S so that we can consider the *period point*

$$p(S) := \left(\int_{\gamma_1} \omega_S : \int_{\gamma_2} \omega_S : \cdots : \int_{\gamma_{22}} \omega_S \right) \in \mathbb{P}^{21}.$$

The point $p = p(S)$ satisfies $(p, p) = 0$, $(p, \bar{p}) > 0$. In fact,

$$\Omega := \left\{ p \in \mathbb{P}^{21} \mid (p, p) = 0,\ (p, \bar{p}) > 0 \right\}$$

is the moduli space of $K3$ surfaces.

Now let us consider a one-parameter family $\{S_z\}_{z \in \mathbb{C}}$ of $K3$ surfaces with

$$p(z) := p(S_z) = \left(\int_{\gamma_1(z)} \omega(z) : \int_{\gamma_2(z)} \omega(z) : \cdots : \int_{\gamma_{22}(z)} \omega(z) \right) \in \mathbb{P}^{21}.$$

The period points of the family $\{S_z\}_{z \in \mathbb{C}}$ satisfy the Picard-Fuchs differential equation as is explained below following [12].

Let $v_j(z)$ be a 22-dimensional vector defined as

$$v_j(z) := {}^t \left(\frac{d^j}{dz^j} \int_{\gamma_1(z)} \omega(z),\ \frac{d^j}{dz^j} \int_{\gamma_2(z)} \omega(z),\ \cdots,\ \frac{d^j}{dz^j} \int_{\gamma_{22}(z)} \omega(z) \right)$$

and

$$d_j(z) := \dim_{\mathbb{C}} \left(\mathrm{Span}\{v_0(z),\ v_1(z), \ldots, v_j(z)\} \right) \leq 22.$$

Hence for $j > 21$, vectors $v_0(z), v_1(z), \ldots, v_j(z)$ are linearly dependent. Therefore there exists a number s such that $v_s(z) \in \mathrm{Span}\{v_0(z),\ v_1(z), \ldots, v_{s-1}(z)\}$, more precisely, there exist functions $C_j(z)$ in z such that

$$v_s(z) = \sum_{j=0}^{s-1} -C_j(z) v_j(z).$$

This means the period point $p(z)$ satisfies a differential equation

$$\frac{d^s}{dz^s} \left({}^t p(z) \right) + \sum_{j=0}^{s-1} C_j(z) \frac{d^j}{dz^j} \left({}^t p(z) \right) = 0.$$

Thus $p(z)$ is a solution of

$$\left(\frac{d^s}{dz^s} + \sum_{j=0}^{s-1} C_j(z) \frac{d^j}{dz^j} \right) \mathbf{F}(z) = 0,$$

which is called the *Picard-Fuchs differential equation.* Determining the coefficient functions $C_j(z)$ is a chief problem (*e.g.* for toric hypersurfaces [4]).

5°) Study of automorphism groups of $K3$ surfaces is now applied to *dynamic systems* (*e.g.* [1]), as well as moduli problem.

6°) *Mirror symmetry* from mathematical sciences requires an interchange of invariants of families $\{(S_z,\ \kappa(z))\}_z$ of $K3$'s together with Kähler forms and of $\{(S_t,\ \omega(t))\}_t$ of $K3$'s with complex structures.

2 Fundamental theorem for $K3$ surfaces: Torelli-type theorem

The aim of this section is to explain the statement of following theorem.

Theorem 2.1 (Pjateckiĭ-Šapiro & Šafarevič [13]) *There exists an isomorphism $f : S \to S'$ of $K3$ surfaces if and only if there exists an effective Hodge isometry $\phi : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$. Moreover, we have $f^* = \phi$.*

2.1 H^2-, Picard, and Transcendental lattices

Definition 2.1 (1) *A* lattice *is a pair $(L, \langle\, ,\, \rangle)$ of a finitely-generated free $\mathbb{Z}$-module L and a symmetric bilinear form $\langle\, ,\, \rangle : L \times L \to \mathbb{Z}$ called* pairing*.*
(2) *Two lattices $(L, \langle\, ,\, \rangle_L)$ and $(L', \langle\, ,\, \rangle_{L'})$ are* isometric *if there exists an isomorphism $\phi : L \to L'$ of $\mathbb{Z}$-modules which preserves the pairings, that is, $\langle \phi(x),\, \phi(y)\rangle_{L'} = \langle x,\, y\rangle_L$ for all $x, y \in L$.*

For a $K3$ surface S, there exists an *intersection pairing*

$$\langle\, ,\, \rangle : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to \mathbb{Z}$$

on the second cohomology group, which installs a pair $(H^2(S, \mathbb{Z}), \langle\, ,\, \rangle)$ a structure of lattice. It is known that $(H^2(S, \mathbb{Z}), \langle\, ,\, \rangle)$ is an even unimodular lattice of rank 22 with signature $(3, 19)$, thus by a general theory of $\mathbb{Z}$-modules, this is isometric to a lattice $\Lambda := U^3 \oplus E_8^2$, called the $K3$ *lattice*,where U is the hyperbolic lattice, and E_8 is the negative-definite even unimodular lattice of rank 8 whose intersection matrix is associated to the Dynkin diagram of type E_8. We call the lattice $(H^2(S, \mathbb{Z}), \langle\, ,\, \rangle)$ the H^2*-lattice* of S.

By a standard exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

we get an exact sequence of cohomology groups as

$$\dashrightarrow H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbb{Z}) \dashrightarrow .$$

By definition, $H^1(S, \mathcal{O}_S) = 0$ thus we get an inclusion mapping

$$c_1 : H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbb{Z}).$$

Definition 2.2 *The group $H^1(S, \mathcal{O}_S^*)$ of* linear equivalence classes of invertible sheaves on *S is called the* Picard group*, and the lattice* $\mathrm{NS}(S) := H^1(S, \mathcal{O}_S^*)/\ker(c_1)$ *of* algebraically-equivalent classes of invertible sheaves on *S is called the* Néron-Severi lattice*.*

By the inclusion c_1, we can install a structure of lattice into $H^1(S, \mathcal{O}_S^*)$ induced from that of $H^2(S, \mathbb{Z})$. Moreover, in case of $K3$ surfaces, the fact $\ker(c_1) = 0$ leads that the lattices $\mathrm{Pic}(S) := (H^1(S, \mathcal{O}_S^*), \langle\, , \rangle)$ and $\mathrm{NS}(S)$ coincide. Note also that more precise description of $\mathrm{NS}(S)$ as a sublattice of $H^2(S, \mathbb{Z})$ is given as follows.

Theorem 2.2 (Lefschetz's Theorem on $(1,1)$-classes) *For a compact surface V, the image of the Picard group by c_1 is equal to $c_1(H^{1,1}(V)) \cap H^2(V, \mathbb{Z})$. In other words, $c_1(H^1(V, \mathcal{O}_V^*))$ consists of classes represented by real closed $(1,1)$-forms of algebraic coefficient.*

Therefore, the Néron-Severi lattice of a $K3$ surface is also presented as a sublattice $NS(S) = c_1(H^{1,1}(S)) \cap H^2(S, \mathbb{Z})$ of $H^2(S, \mathbb{Z})$.

Definition 2.3 *The lattice* $\mathrm{Pic}(S)$ *is called the* Picard lattice *of S, and the rank of the Picard lattice the* Picard number *and is denoted by $\rho(S)$.*

Remark 2.1 1) The Picard lattice $\mathrm{Pic}(S)$ of a $K3$ surface S is primitively embedded into $H^2(S, \mathbb{Z})$. The signature is $\mathrm{sgn}\,\mathrm{Pic}(S) = (1, \rho(S) - 1)$ since the first Betti number $b_1(S) = 0$ is even, and by using signature theorem.
2) A compact complex surface V is projective iff there exists a line bundle D on V such that $c_1(D)^2 > 0$.
3) If a $K3$ surface S is algebraic, we have $1 \leq \rho(S) \leq 22$. Moreover if S is complex, we have $1 \leq \rho(S) \leq 20$. In case S is defined over an algebraically-closed field of positive characteristic, we may have $\rho(S) = 21$, 22.
4) It is a very delicate problem to tell the difference between cohomology groups $H^1(\mathcal{O}^*)$ and $H^{1,1}$ in general. We once again strongly remark that in case of $K3$ surfaces, we have the identity

$$(H^1(S, \mathcal{O}_S^*), \langle\, , \rangle) \simeq c_1\left(H^{1,1}(S)\right) \cap H^2(S, \mathbb{Z})$$

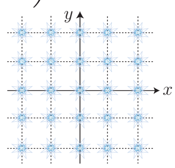
of lattices (see Figure 2).

Definition 2.4 *The orthogonal complement of the Picard lattice of a $K3$ surface in the H^2-lattice is called the* transcendental lattice*:*

$$T(S) := \mathrm{Pic}(S)^{\perp} \subset H^2(S, \mathbb{Z}).$$

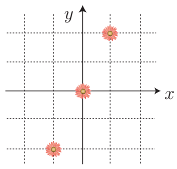
Roughly speaking, $\mathrm{Pic}(S)$ shows *algebraic* side of S, whilst $T(S)$ does *transcendental* part of S, so that $\mathrm{Pic}(S)$ and $T(S)$ together give the whole geometry of S.

Let $L := \mathbb{Z}\begin{pmatrix}1\\0\end{pmatrix} + \mathbb{Z}\begin{pmatrix}0\\1\end{pmatrix}$ be a rank-2 lattice. $L = \{\ \}$



Let $V := \mathbb{R}\begin{pmatrix}p\\q\end{pmatrix}$ be a vector space which is embedded into $\mathbb{R}^2 = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\1\end{pmatrix}$.

(1) If $p, q \in \mathbb{Q}$, then, $V \cap L$ is of rank 1.
The case of $p = 1$, $q = 2$, $V \cap L = \{\ \}$.



(2) If p or $q \notin \mathbb{Q}$, then, $V \cap L = \{0\}$ is of rank 0.
The case of $p = 1$, $q = \sqrt{2}$, the line $y = \sqrt{2}x$ intersects at the only lattice point $(0, 0) =$.

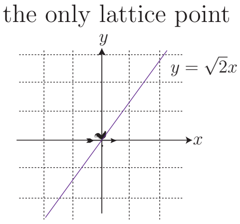


Figure 2: A toy model of the embedding $c_1 : H^1(S, \mathcal{O}_S^*) \hookrightarrow H^2(S, \mathbb{Z})$

2.2 Hodge decomposition

Definition 2.5 *Let S be a $K3$ surface. A subcone $\mathcal{C}_S^+$ of the cone*

$$\mathcal{C}_S := \left\{ x \in H^{1,1}(S) \,|\, \langle x, x \rangle > 0 \right\}$$

in $H^{1,1}(S)$ is called the positive cone *of S if $\mathcal{C}_S^+$ contains Kähler classes .*

Let S be a $K3$ surface. Owing to the fact that S is complex, there exists a *Hodge decomposition*

$$H^2(S, \mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S),$$

where $H^{0,2}(S) = \overline{H^{2,0}(S)}$.

Definition 2.6 *Let S and S' be $K3$ surfaces.*
(1) *An isometry $\phi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ is called* Hodge isometry *if the $\mathbb{C}$-extension $\phi_{\mathbb{C}} : H^2(S, \mathbb{C}) \to H^2(S', \mathbb{C})$ preserves the Hodge decompositions.*
(2) *An isometry $\phi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ is called* effective *if ϕ preserves effective classes.*

2.3 Surjectivity of the period mapping

With a fixed *marking* $H^2(S, \mathbb{Z}) \xrightarrow{\sim} \Lambda$, let us call a map $p : S \mapsto p(S) \in \Omega$ the *period mapping.*

Theorem 2.3 *The period mapping is surjective.*

It is known that there exists a universal family of marked $K3$ surfaces that are parametrised by a non-Hausdorff space of dimension 20.

For a sublattice $L \subset \Lambda$ of signature $(1, t)$, a *L-polarised $K3$ surface* is defined to be a $K3$ surface S with a marking $\phi : H^2(S, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that $\phi^{-1}(L)$ consists of divisors on S. The period domain D_L of L-polarised $K3$ surfaces is described as $D_L = \Omega_L^{\mathrm{pol}}/O(\Lambda, L)$, where

$$O(\Lambda, L) := \{g \in O(\Lambda) \mid g|_L = id\}, \qquad \Omega_L^{\mathrm{pol}} := \Omega_L \setminus \bigcup_{d \in \Delta_L} H_d \cap \Omega_L,$$
$$\Delta_L := \{d \in L \mid d^2 = -2\}, \qquad H_d := d^{\perp},$$
$$\Omega_L := \{[\omega] \in \Omega \mid \langle [\omega], l \rangle = 0 \, \forall l \in L\}.$$

Summary

$$\left.\begin{array}{c} 0 \neq \omega \in H^{2,0}(S) \subset H^{1,1}(S)^{\perp} \\ + \\ \gamma_1, \gamma_2, \ldots, \gamma_{22} \in H_3(S, \mathbb{Z}) \end{array}\right\} \rightsquigarrow \text{period} \in \Omega$$

$$\left.\begin{array}{c} \text{Surjectivity of the period mapping} \\ + \\ \text{Torelli-type theorem} \end{array}\right\} \rightsquigarrow \boxed{\begin{array}{c} \textbf{Slogan} \\ \text{Study of } K3 \text{ surfaces is reduced to} \\ \text{a study of Lattices !} \end{array}}$$

3 How to study $K3$ surfaces – an example

Definition 3.1 *Let X be an 3-dimensional algebraic variety with at most Gorenstein singularities. X is called a* Fano 3-fold *if the anticanonical divisor $-K_X$ is ample, that is, $-K_X.C > 0$ and $(-K_X)^2 > 0$ for all effective divisors C on X.*

Fact Let X be a smooth Fano 3-fold, and $S \in |-K_X|$ be general. Then,

(i)	$h^1(\mathcal{O}_S)$	$=$	$h^1(-K_X + K_X) = 0$	by Kodaira vanishing, $-K_X$ being ample.
(ii)	K_S	$=$	$(K_X + S)\vert_S$	by adjunction formula
		$=$	$(K_X + (-K_X))\vert_S = 0$	since $S \sim -K_X$.
(iii)	S	is	smooth	by Šokurov [16].

Therefore general anticanonical member of X is a $K3$ surface.

Examples

1°) $X = \mathbb{P}^3$, S: smooth quartic surface in $X \Rightarrow S$ is $K3$.

2°) $X =$ smooth Fano 3-fold, $S \in |-K_X|$ is general $\Rightarrow S$ is $K3$.

c.f. Smooth Fano 3-folds are classified by Mori and Mukai [10][11] if the second Betti number $B_2 \geq 2$, and by Iskovskih [8][9] in case $B_2 = 1$.

3°) A quadruple (a_0, a_1, a_2, a_3) of positive integers is called *well-posed* if

(i) $1 \leq a_0 \leq a_1 \leq a_2 \leq a_3$, and

(ii) $\gcd(a_i,\, a_j,\, a_k) = 1\, (0 \leq i, j, k \leq 3)$.

For a well-posed quadruple $a = (a_0,\, a_1,\, a_2,\, a_3)$, set $d := a_0 + a_1 + a_2 + a_3$, and define the *weighted projective space* $\mathbb{P}(a) = \mathbb{P}(a_0,\, a_1,\, a_2,\, a_3)$ of weight $(a_0,\, a_1,\, a_2,\, a_3)$ as follows:
$\mathbb{P}(a) := \mathbb{C}^4 \backslash \{0\} \Big/ \sim_W$, where

$$(x_0, x_1, x_2, x_3) \sim_W (y_0, y_1, y_2, y_3) \quad \Leftrightarrow \quad \begin{array}{l} \text{there exists } \lambda \in \mathbb{C}^* \text{ such that} \\ (y_0, y_1, y_2, y_3) = (\lambda^{a_0} x_0, \lambda^{a_1} x_1, \lambda^{a_2} x_2, \lambda^{a_3} x_3). \end{array}$$

Let $(x_0 : x_1 : x_2 : x_3)$ be a coordinate of $\mathbb{P}(a)$, then it means that the weight of x_i is $a_i\, (i = 0, 1, 2, 3)$.

Any anticanonical member in a weighted projective space $X = \mathbb{P}(a)$ of weight $a = (a_0, a_1, a_2, a_3)$ is a hypersurface of weighted degree d.

Theorem 3.1 *General anticanonical member in $\mathbb{P}(a)$ is a Gorenstein $K3$ surface if and only if the weight a is one of those in a list of 95 weights classified by Reid and Iano-Fletcher, and Yonemura.*

Remark 3.1 Yonemura [17] is in a relation with *simple $K3$ singularities* which is an analogue of simple elliptic singularities that are identified with $T_{p,q,r}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Reid ((4.1), (4.5) [14]) and Iano-Fletcher [7] are by combinatorically interpreting conditions that a hypersurface to have canonical singularities.

We call the list in this theorem the *list of 95*, and *weighted $K3$ surfaces* for (Gorenstein) $K3$ surfaces in the weighted projective spaces. Using the fact that all the weighted projective spaces are toric, one may study families of weighted $K3$ surfaces.

Definition 3.2 *Let $a = (a_0,\, a_1,\, a_2,\, a_3)$ be a weight out of the list of 95. Define the* full Newton polytope of degree d in $\mathbb{P}(a)$ *as*

$$\Delta_{(a;d)} := \mathrm{Conv} \left\{ (m_0,\, m_1,\, m_2,\, m_3) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} \sum_{i=0}^{3} a_i m_i = 0 \text{ and} \\ m_i \geq -1\, (i = 0, 1, 2, 3) \end{array} \right\} \subset \mathbb{R}^3.$$

Remark 3.2 Let $(x_0 : x_1 : x_2 : x_3)$ be a global coordinate system of $\mathbb{P}(a)$, thus a monomial $x_0^{m'_0} x_1^{m'_1} x_2^{m'_2} x_3^{m'_3}$ of weighted degree d satisfies

$$a_0 m'_0 + a_1 m'_1 + a_2 m'_2 + a_3 m'_3 = d, \text{ and } m'_i \geq 0 \text{ for all } i = 0, 1, 2, 3.$$

Since $d = a_0 + a_1 + a_2 + a_3$, we have

$$a_0(m'_0 - 1) + a_1(m'_1 - 1) + a_2(m'_2 - 1) + a_3(m'_3 - 1) = 0.$$

Thus $(m'_0 - 1,\, m'_1 - 1,\, m'_2 - 1,\, m'_3 - 1)$ is a lattice point in $\Delta_{(a;d)}$.

The full Newton polytope of degree d in $\mathbb{P}(a)$ of weight a in the list of 95 is characterised to be reflexive.

Definition 3.3 [3] *Let $M \simeq \mathbb{Z}^3$ be a lattice of rank 3, and $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be its dual lattice with respect to a natural pairing*

$$\langle\, ,\, \rangle : M \times N \to \mathbb{Z}.$$

(1) *Let $\Delta \subset M \otimes \mathbb{R}$ be an 3-dimensional integral convex polytope such that the origin 0 is in the interior of Δ. Define the* polar dual polytope *of Δ as*

$$\Delta^* := \{y \in N \otimes \mathbb{R} \,|\, \langle x, y\rangle \geq -1 \,\text{for all}\, x \in \Delta\}.$$

(2) *Let Δ be a polytope as in* (1). *Δ is called* reflexive *if the origin 0 is the only lattice point in the interior of Δ.*

Theorem 3.2 [3] *Polar duality preserves the reflexivity.*

The dual of an edge Γ of Δ is an edge Γ^* of Δ^*, and the dual of a face F of Δ is a vertex $v := F^*$ of Δ^*.

Denote by $\mathcal{F}_a$ the family of weighted $K3$ surfaces in the weighted projective space $\mathbb{P}(a)$. We call a member $S \in \mathcal{F}_a$ *generic* if the Picard number $\rho(\tilde{S})$ of the smooth model $\tilde{S}$ of S is equal to that of the smooth model $\widetilde{\mathbb{P}(a)}$ of the projective space $\mathbb{P}(a)$. Denote by $\mathrm{Pic}(\mathcal{F}_a) := \mathrm{Pic}(\tilde{S})$ the Picard lattice of the family $\mathcal{F}_a$, and the Picard number $\rho(a) := \rho(\tilde{S})$.

Facts (1) The Picard number $\rho(a)$ is computed in two ways:

$$\begin{aligned} \rho(a) &= 22 - \sharp\left\{\text{lattice points on edges of } \Delta_{(a;d)}\right\} + 1 \\ &= \sum_{\Gamma:\,\text{edge of}\,\Delta_{(a;d)}} l^*(\Gamma)\, l^*(\Gamma^*) + \left(\sum_{\Gamma:\,\text{edge of}\,\Delta_{(a;d)}} l(\Gamma^*) - 3\right), \quad \text{where} \\ l^*(\Gamma) &= \sharp\{\text{lattice points in the interior of } \Gamma\}, \\ l^*(\Gamma^*) &= \sharp\{\text{lattice points in the interior of } \Gamma^*\}, \\ l(\Gamma^*) &= \sharp\{\text{lattice points in } \Gamma\}. \end{aligned}$$

(2) A vertex v of $\Delta^*_{(a;d)}$ defines a *toric divisor* $D_v := \overline{\mathrm{orb}\,(\mathbb{R}_{\geq 0} v)}$, which is a smooth curve in X. The dual v^* of v is a face F in $\Delta_{(a;d)}$, and the genus of

D_v is given as $g(D_v) = l^*(F)$, and $D_v^2|_{-K_X} = 2l^*(F) - 2$.
(3) (see also [5]) Suppose for an edge Γ of $\Delta_{(a;d)}$, we have $n = l^*(\Gamma^*)$, and $m = l^*(\Gamma)$. Then there is a singularity of type A_n of multiplicity $m+1$. More precisely, if $\Gamma = F \cap F'$ with faces F and F', which is always true in our case, then there exists a singularity of type A_n of multiplicity $m+1$ on the intersection of D_v and $D_{v'}$. Thus the dual graph of resolution of singularity is shown in Figure 3.

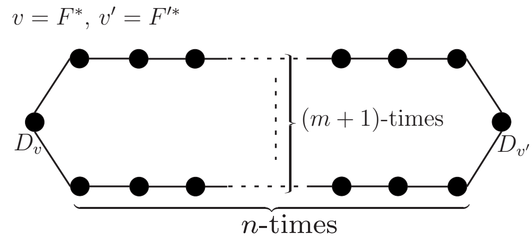
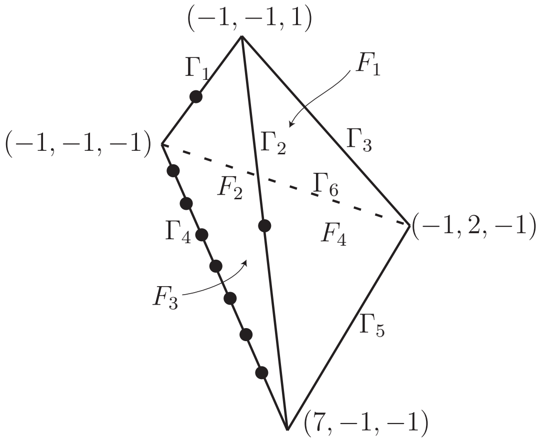


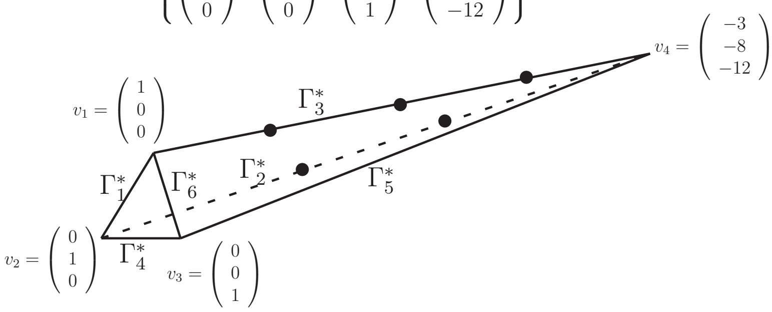
Figure 3: The dual graph of resolution of singularity

EXAMPLE.

$$\begin{aligned}\Delta_{(1,3,8,12;24)} &= \mathrm{Conv}\left\{(m_0,m_1,m_2,m_3)\in\mathbb{Z}^4 \;\middle|\; \begin{array}{l} m_0+3m_1+8m_2+12m_3=0 \\ m_i\geq -1\,(i=0,1,2,3)\end{array}\right\} \\ &= \mathrm{Conv}\left\{(-1,-1,-1,1),\ (23,-1,-1,-1),\ (-1,7,-1,-1),\ (-1,-1,2,-1)\right\} \\ &= \mathrm{Conv}\left\{(-1,-1,1),\ (-1,-1,-1),\ (7,-1,-1),\ (-1,2,-1)\right\}\end{aligned}$$



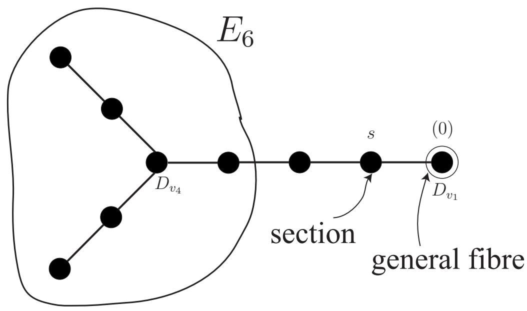
$$\begin{aligned}\Delta^*_{(1,3,8,12;24)} &= \mathrm{Conv}\left\{m^*={}^t(m_0^*,m_1^*,m_2^*,m_3^*)\in\mathbb{Z}^4\,|\,\langle x,m^*\rangle\geq -1 \text{ for all } x\in\Delta_{(1,3,8,12;24)}\right\} \\ &= \mathrm{Conv}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}-3\\-8\\-12\end{pmatrix}\right\}\end{aligned}$$



$$\begin{aligned}\rho(1,3,8,12) &= \sum_{i=1}^{6} l^*(\Gamma_i)\,l^*(\Gamma_i^*) + \left(\sum_{i=1}^{6} l(\Gamma_i^*)-3\right) \\ &= 1\cdot 0+1\cdot 2+0\cdot 3+7\cdot 0+0\cdot 0+2\cdot 0+(9-3) \\ &= 2+6=8.\end{aligned}$$

$$\begin{array}{lll} l^*(\Gamma_3^*)=3 \quad \& \quad l^*(\Gamma_3)=0 & : & A_3\text{-sing of mult 1 on } D_{v_1}\cap D_{v_4}\cap(-K_{\mathbb{P}(1,3,8,12)}) \\ l^*(\Gamma_2^*)=2 \quad \& \quad l^*(\Gamma_2)=1 & : & A_2\text{-sing of mult 2 on } D_{v_2}\cap D_{v_4}\cap(-K_{\mathbb{P}(1,3,8,12)}) \\ l^*(F_1)=1 & & \therefore D_{v_1}^2|_{-K_{\mathbb{P}(1,3,8,12)}}=2\cdot 1-2=0 \\ & & \rightsquigarrow \text{there exists an elliptic fibration.}\end{array}$$

The dual graph is as follows:



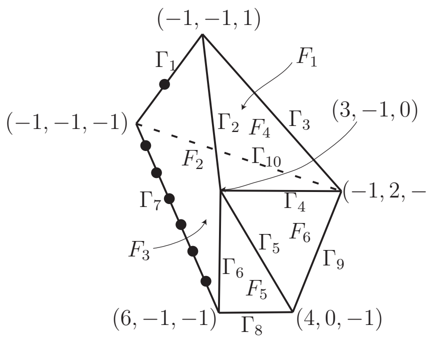
Therefore, $\mathrm{Pic}(\mathcal{F}_{(1,3,8,12)}) = E_6 \oplus U$.

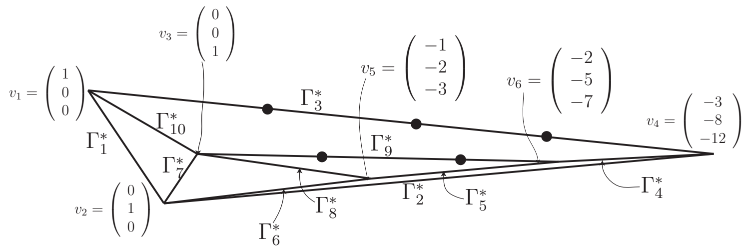
EXAMPLE.

$$\Delta_{(1,2,5,7;15)} = \mathrm{Conv}\left\{(m_0,m_1,m_2,m_3)\in\mathbb{Z}^4 \,\middle|\, \begin{array}{l} m_0+2m_1+5m_2+7m_3=0 \\ m_i \geq -1\,(i=0,1,2,3)\end{array}\right\} = \mathrm{Conv}\left\{\begin{array}{l}(0,-1,-1,1),\,(14,-1,-1,-1),\,(-1,3,-1,0),\\ (0,6,-1,-1),\,(-1,4,0,-1),\,(-1,-1,2,-1)\end{array}\right\}$$

$$= \mathrm{Conv}\left\{(-1,-1,1),\,(-1,-1,-1),\,(3,-1,0),\,(6,-1,-1),\,(4,0,-1),\,(-1,2,-1)\right\}$$

$$\Delta^*_{(1,2,5,7;15)} = \mathrm{Conv}\left\{m^*\in\mathbb{Z}^4\,|\langle x,m^*\rangle \geq -1 \forall\, x\in\Delta_{(1,2,5,7;15)}\right\} = \mathrm{Conv}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}-3\\-8\\-12\end{pmatrix},\begin{pmatrix}-1\\-2\\-3\end{pmatrix},\begin{pmatrix}-2\\-5\\-7\end{pmatrix}\right\}$$

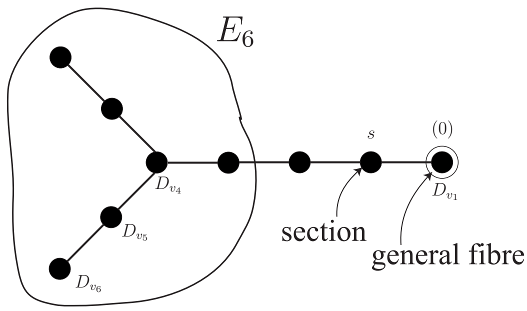




$$\begin{aligned}\rho(1,2,5,7) &= \sum_{i=1}^{10} l^*(\Gamma_i)\,l^*(\Gamma_i^*) + \left(\sum_{i=1}^{10} l(\Gamma_i^*) - 3\right)\\ &= 1\cdot 0+0\cdot 2+0\cdot 3+0\cdot 0+0\cdot 0+0\cdot 0+6\cdot 0+0\cdot 0+0\cdot 0+(11-3)\\ &= 0+8=8.\end{aligned}$$

$$\begin{array}{llll} l^*(\Gamma_2^*)=2 & \& & l^*(\Gamma_2)=0 & :\ A_2\text{-sing of mult 1 on } D_{v_2}\cap D_{v_4}\cap(-K_{\mathbb{P}(1,2,5,7)}) \\ l^*(\Gamma_3^*)=3 & \& & l^*(\Gamma_3)=0 & :\ A_3\text{-sing of mult 1 on } D_{v_1}\cap D_{v_4}\cap(-K_{\mathbb{P}(1,2,5,7)}) \\ & & l^*(F_1)=1 & \therefore D_{v_1}^2|_{-K_{\mathbb{P}(1,2,5,7)}} = 2\cdot 1-2=0 \\ & & & \rightsquigarrow \text{there exists an elliptic fibration.}\end{array}$$

The dual graph is as follows:



Therefore, $\mathrm{Pic}(\mathcal{F}_{(1,2,5,7)}) = E_6 \oplus U$.

⟨ Observations ⟩
(1) $\mathrm{Pic}(\mathcal{F}_{(1,3,8,12)}) \simeq \mathrm{Pic}(\mathcal{F}_{(1,2,5,7)}) = U \oplus E_6$.
(2) Polytopes $\Delta_{(1,3,8,12;24)}$ and $\Delta_{(1,2,5,7;15)}$ have several vertices in common.

Final Problem Is there any correspondence between general members in $\mathcal{F}_{(1,3,8,12)}$ and $\mathcal{F}_{(1,2,5,7)}$?

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