Geometry of $R$-spaces canonically embedded in Kähler $C$-spaces as Lagrangian submanifolds

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This article is dedicated to Professor Young Jin Suh on the occasion of his 65th birthday.

Abstract. An $R$-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. It is known that any $R$-space has the canonical embedding into a Kähler $C$-space as a real form and thus it is a compact totally geodesic Lagrangian submanifold. In this article we provide an exposition on such nice properties of $R$-spaces as Lagrangian submanifolds and our recent work on minimal Maslov number of $R$-spaces canonically embedded in Einstein-Kähler $C$-spaces ([20]).

1 Introduction

A smooth immersion (resp. embedding) $\varphi : L \to M$ of a smooth manifold $L$ into a symplectic manifold $(M, \omega)$ is called a Lagrangian immersion (resp. Lagrangian embedding) if $2 \dim L = \dim M$ and $\varphi^* \omega = 0$. For a Lagrangian immersion $\varphi : L \to M$, we have the vector bundle isomorphism $\varphi^{-1}TM/\varphi^*TL \ni v \leftrightarrow \alpha_v := \omega(v, \cdot) \in T^*L$. A smooth family of Lagrangian immersions $\varphi_t : L \to M$ with $\varphi_0 = \varphi$ is called a Lagrangian deformation of $\varphi$, which is characterized by the closedness of the 1-form $\alpha_{V_t} \in \Omega^1(L)$ corresponding to the variational vector field $V_t := \frac{\partial \varphi_t}{\partial t} \in \varphi^{-1}TM$ for each $t$. A Lagrangian deformation $\varphi_t : L \to M$ with

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ϕ₀ = ϕ is called a Hamiltonian deformation of ϕ if α_{V_t} ∈ Ω¹(L) is exact for each t. Suppose that [ω] ∈ H²(M, ℝ) is an integral class, that is, there is a complex line bundle E over M and a U(1)-connection ∇ of E whose curvature form is equal to 2π√−1ω. It is known that a Lagrangian deformation ϕ_t : L → M with ϕ₀ = ϕ is a Hamiltonian deformation if and if a family of flat connections {ϕ_t⁻¹∇} has same holonomy homomorphism ρ : π₁(L) → U(1).

Two group homomorphisms I_{µ,L} : π₂(M, L) → Z and I_{ω,L} : π₂(M, L) → R are defined for any Lagrangian submanifold of a symplectic manifold in general (see Section 3) so that I_{µ,L} is a symplectic invariant and I_{ω,L} is not a symplectic invariant but a Hamiltonian invariant. The minimal Maslov number of a Lagrangian submanifold in a symplectic manifold is defined by the condition that I_{µ,L} = λI_{ω,L} (∃λ > 0). The Floer homology theory for the intersection of Lagrangian submanifolds was initiated and well-developed by Y.-G. Oh ([15], [16], [17], [18] and so on). It is known that any compact minimal Lagrangian submanifold of an Einstein-Kähler manifold with positive Einstein constant is monotone (Cieliebak-Goldstein [2], Hajime Ono [21]). Moreover he ([21]) gave a nice formula of the minimal Maslov number for a compact monotone Lagrangian submanifold in a simply connected Einstein-Kähler manifold with positive Einstein constant (see the formula (3.1) in Section 3).

An R-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. Note that an R-space is not a symmetric space in general and it is called a symmetric R-space when it is a symmetric space. It is known that each R-space has the canonical embedding into a Kähler C-space as a real form. A Kähler C-space is a simply connected compact homogeneous complex manifold which admits invariant Kähler metrics, and it is also called a generalized flag manifold. A real form means the fixed point subset by an anti-holomorphic isometry of a Kähler C-space and thus it is a compact embedded totally geodesic Lagrangian submanifold. So R-spaces canonically embedded in Kähler C-spaces constitute a nice class of Lagrangian submanifolds. As explained in Section 2 any R-space can be canonically embedded in an Einstein-Kähler C-space. In this case it is a compact monotone Lagrangian submanifold and so we can use H. Ono’s formula in order to study the minimal Maslov number for R-spaces canonically embedded in Einstein-Kähler C-spaces. In [20] we showed a Lie theoretic formula for the minimal Maslov number of such an R-space and some examples of the calculation by that formula.

In this article we provide an exposition on such nice properties of R-spaces as Lagrangian submanifolds and our related recent work ([20]). This article is organized as follows: In Section 2 we review the definitions and elementary properties of R-spaces and their canonical embeddings into Kähler C-spaces and the description of the invariant symplectic structures, invariant complex structures, invariant Kähler metrics and invariant Einstein-Kähler metrics on a Kähler C-space. We also discuss several properties from the viewpoint of geometry of Lagrangian sub-
manifolds such as an anti-symplectic involutive diffeomorphism, the moment maps, Morse theory and related intersection problem. In Section 3 we recall the definitions of two Hamiltonian invariants $I_{u,L}$ and $I_{v,L}$ and the monotonicity for Lagrangian submanifolds of general symplectic manifolds. Moreover we refer the monotonicity theorem and minimal Maslov number formula by Cieliebak-Goldstein and H. Ono for Lagrangian submanifolds in Einstein-Kähler manifolds, and mention our applications to the case of the Gauss images of isoparametric hypersurfaces. In Section 4 we describe the construction of the Lie theoretic formula for minimal Maslov number for $R$-spaces canonically embedded in Einstein-Kähler $C$-spaces.

Throughout this article any manifold is smooth and connected.

2 The canonical embeddings of an $R$-space into a Kähler $C$-space

In this section we review the definitions and elementary properties of $R$-spaces and their canonical embeddings into Kähler $C$-spaces from the viewpoint of geometry of Lagrangian submanifolds (cf. [1], [22], [27], [23], [24], [25], [20]).

Let $(G, K, \theta)$ be a Riemannian symmetric pair with an involutive automorphism $\theta$. Suppose that $G$ is a connected compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ and $K$ is a connected compact Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. We choose an $AdG$- and $\theta$-invariant inner product $(\ , \ )$ of $\mathfrak{g}$ and extend it to the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ by the complex bi-linearity. Let $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ with respect to $(G, K, \theta)$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Then we have $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}, \mathfrak{b} = \mathfrak{t} \cap \mathfrak{t}, \mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ and $\mathfrak{t}$ is invariant by $\theta$. Let $(\ , \ )$ denote an inner product of $\mathfrak{t}$ defined by a restriction of $(\ , \ )$ to $\mathfrak{t}$. The root space decomposition of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$ is given as

$$\mathfrak{g}^C = \mathfrak{t}^C + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^\alpha,$$

where

$$\mathfrak{g}^\alpha := \{ X \in \mathfrak{g}^C \mid (ad_\xi)(X) = \sqrt{-1}(\alpha, \xi)X \ (\forall \xi \in \mathfrak{t}) \}$$

and $\Sigma(\mathfrak{g}) \subset \mathfrak{t}$ denotes the set of all roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$. Set $\Sigma_0(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \mathfrak{b}$. Define the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a}) := \{ \gamma = \alpha \mid \alpha \in \Sigma(\mathfrak{g}) \}$, where $\alpha$ denotes the $\mathfrak{a}$-component of $\alpha \in \Sigma(\mathfrak{g}) \subset \mathfrak{t} = \mathfrak{b} + \mathfrak{a}$. We choose an involutive orthogonal transformation $\sigma \in O(\mathfrak{t})$ by $\sigma := -\theta|_\mathfrak{t}$. We choose a $\sigma$-order $> \gamma$ on $\mathfrak{t}$ so that if $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$ and $\alpha > 0$, then $\sigma\alpha > 0$ and thus $\theta\alpha = -\sigma\alpha < 0$ ([22]). Set $\Sigma^+(\mathfrak{g}) := \{ \alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0 \}$, $\Sigma_0^+(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{ \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \gamma > 0 \} = \{ \alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g}) \}$.

We choose $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Sigma(\mathfrak{g})$ such that $[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha, \langle E_\alpha, E_{-\alpha} \rangle = 1, E_{-\alpha} = E_{-\alpha}$ for each $\alpha \in \Sigma(\mathfrak{g})$ and let $\{ \omega^\alpha \mid \alpha \in \Sigma(\mathfrak{g}) \}$ be the linear forms on $\mathfrak{g}^C$ dual to $\{ E_\alpha \mid \alpha \in \Sigma(\mathfrak{g}) \}$ so that $\omega^\alpha(\mathfrak{t}^C) = \{ 0 \}, \omega^\alpha(\mathfrak{p}^C) = \delta_{\alpha,\beta}$ for each $\alpha, \beta \in \Sigma(\mathfrak{g})$.

The restricted root space decompositions of $\mathfrak{t}$ and $\mathfrak{p}$ with respect to $\mathfrak{a}$ are given as

$$\mathfrak{t} = \mathfrak{t}_0 + \sum_{\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{t}_\gamma, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{p}_\gamma,$$

where

$$\mathfrak{t}_\gamma := \{ X \in \mathfrak{t} \mid (ad_{\mathfrak{a}})(X) = \sqrt{-1}(\gamma, \mathfrak{a})X \ (\forall \mathfrak{a} \in \mathfrak{a}) \}$$

and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ denotes the set of all positive roots of $\mathfrak{g}^C$ with respect to $\mathfrak{a}$. Set $\Sigma_0^+(\mathfrak{g}, \mathfrak{a}) := \Sigma^+(\mathfrak{g}) \cap \mathfrak{b} + \mathfrak{a}$. Define the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a}) := \{ \alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0 \}$, $\Sigma_0^+(\mathfrak{g}, \mathfrak{a}) := \Sigma(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{ \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \gamma > 0 \} = \{ \alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_0^+(\mathfrak{g}) \}$. We choose a $\sigma$-order $> \gamma$ on $\mathfrak{t}$ so that if $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0^+(\mathfrak{g})$ and $\alpha > 0$, then $\sigma\alpha > 0$ and thus $\theta\alpha = -\sigma\alpha < 0$ ([22]). Set $\Sigma^+(\mathfrak{g}) := \{ \alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0 \}$, $\Sigma_0^+(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{ \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \gamma > 0 \} = \{ \alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_0^+(\mathfrak{g}) \}$. We choose $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Sigma(\mathfrak{g})$ such that $[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha, \langle E_\alpha, E_{-\alpha} \rangle = 1, E_{-\alpha} = E_{-\alpha}$ for each $\alpha \in \Sigma(\mathfrak{g})$ and let $\{ \omega^\alpha \mid \alpha \in \Sigma(\mathfrak{g}) \}$ be the linear forms on $\mathfrak{g}^C$ dual to $\{ E_\alpha \mid \alpha \in \Sigma(\mathfrak{g}) \}$ so that $\omega^\alpha(\mathfrak{t}^C) = \{ 0 \}, \omega^\alpha(\mathfrak{p}^C) = \delta_{\alpha,\beta}$ for each $\alpha, \beta \in \Sigma(\mathfrak{g})$.
where $t_0 := \{X \in \mathfrak{t} \mid (\text{ad}H)X = 0 \ (\forall H \in \mathfrak{a})\}$ and for each $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ set

$$t_\gamma := \{X \in \mathfrak{t} \mid (\text{ad}H)^2X = - (\gamma, H)^2X \ (\forall H \in \mathfrak{a})\},$$

$$p_\gamma := \{X \in p \mid (\text{ad}H)^2Y = - (\gamma, H)^2Y \ (\forall H \in \mathfrak{a})\}.$$  

For $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, there are an orthonormal basis $\{S_{i,\gamma} \mid i = 1, \ldots, m(\gamma)\}$ of $t_\gamma$ and an orthonormal basis $\{T_{i,\gamma} \mid i = 1, \ldots, m(\gamma)\}$ of $p_\gamma$, where $m(\gamma) := \dim \mathfrak{t}_\gamma = \dim \mathfrak{p}_\gamma^\perp$, such that $[H, S_{i,\gamma}] = (\gamma, H)T_{i,\gamma}$, $[H, T_{i,\gamma}] = - (\gamma, H)S_{i,\gamma}$ for each $H \in \mathfrak{a}$.

Now we fix an arbitrary non-zero element $Z$ of $\mathfrak{a}$. Set

$$\Sigma_Z(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid (\alpha, Z) = 0\} \text{ and } \Sigma_Z^+(\mathfrak{g}) := \Sigma_Z(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g}).$$

The element $Z$ is called regular if $\Sigma_Z(\mathfrak{g}) = \Sigma_0(\mathfrak{g})$. Define closed subgroups $G_Z$ and $K_Z$ of $G$ by

$$G_Z := \{a \in G \mid \text{Ad}(a)Z = Z\}$$

and

$$K_Z := \{a \in K \mid \text{Ad}(a)Z = Z\} = K \cap G_Z.$$  

It is well-known that $G_Z$ is always connected. Denote by $\mathfrak{g}_Z$ and $\mathfrak{k}_Z$ Lie algebras of $G_Z$ and $K_Z$, respectively. Note that $\theta(G_Z) = G_Z$, $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ and thus $(G_Z, K_Z, \theta|_{G_Z})$ is also a compact symmetric pair.

**Definition.** The compact homogeneous space $L := K/K_Z$ is called an $R$-space, and it has the standard imbedding into the vector space $\mathfrak{p}$ defined by

$$\varphi_Z : L = K/K_Z \ni aK_Z \mapsto \text{Ad}(a)Z \in \text{Ad}(K)Z \subset \mathfrak{p}.$$  

If $Z$ is a regular element of $\mathfrak{a}$, then $L = K/K_Z$ is called a regular $R$-space. Another compact homogeneous space $M := G/G_Z$ is called a generalized flag manifold or a Kähler $C$-space, and it also has the standard imbedding into the Lie algebra $\mathfrak{g}$

$$\Phi_Z : M = G/G_Z \ni aG_Z \mapsto \text{Ad}(a)Z \in \text{Ad}(G)Z \subset \mathfrak{g}.$$  

The canonical embedding of $K/K_Z$ into $G/G_Z$ is a map defined by

$$\iota_Z : L = K/K_Z \ni aK_Z \mapsto aG_Z \in G/G_Z = M.$$  

We take the orthogonal direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{t}$ as $\mathfrak{g} = \mathfrak{g}_Z + \mathfrak{m}, \mathfrak{m} \cong T_eG_Z \mathfrak{m}$ and $\mathfrak{t} = \mathfrak{t}_Z + \mathfrak{l}, \mathfrak{l} \cong T_eK_Z \mathfrak{l}$. Note that $\mathfrak{t}_Z = \mathfrak{t} \cap \mathfrak{g}_Z$. By using the property $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ one can show that $\iota_Z$ is an embedding and $2 \dim L = \dim M$.

The author has heard from Professor Masaru Takeuchi that the “$R$-space” was named first by Jacques Tits ([31]). Here we should notice that an $R$-space is not a symmetric space in general, and however the $R$-space can be considered as a class.
of the most important compact homogeneous spaces related to symmetric spaces. An $R$-space $K/K_Z$ is called a symmetric $R$-space if $K/K_Z$ is a symmetric space. It is known that an $R$-space is a symmetric $R$-space if and only if one of the following conditions is satisfied:

1. $(K, K_Z)$ is a symmetric pair.

2. There is an element $Z \in \mathfrak{p}$ satisfying the equation $(\text{ad} Z)^3 + (\text{ad} Z) = 0$ such that $L = K/K_Z$ and $G = G/G_Z$.

3. $(G, G_Z)$ is a Hermitian symmetric pair.

4. The standard imbedding $\varphi_Z$ has the parallel second fundamental form (Dirk Ferus [3]).

5. $\varphi_Z(L)$ is an (extrinsic) symmetric submanifold in Euclidean space $\mathfrak{p}$ (Dirk Ferus [4]).

For such $Z$, we can define a $G$-invariant symplectic form $\omega_Z$ on $M = G/G_Z$ by

$$\omega_Z(X, Y) := \langle [Z, X], Y \rangle$$

for each $X, Y \in \mathfrak{g}$.

and $\omega_Z$ can be also expressed as

$$\omega_Z = -\sqrt{-1} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} (Z, \alpha) \omega^{-\alpha} \wedge \overline{\omega^{-\alpha}}.$$

Then the canonical embedding $\iota_Z : L = K/K_Z \to G/G_Z = M$ is a Lagrangian embedding with respect to $\omega_Z$.

The involutive automorphism $\theta$ of $G$ induces an involutive diffeomorphism

$$\hat{\theta}_Z : M = G/G_Z \ni aG_Z \mapsto \theta(a)G_Z \in G/G_Z = M$$

which is equivariant with respect to the Lie group automorphism $\theta : G \to G$. Then

$$\hat{\theta}_Z : G/G_Z \to G/G_Z$$

is anti-symplectic with respect to $\omega_Z$, that is, $\hat{\theta}_Z^* \omega_Z = -\omega_Z$.

Define the fixed point subset of $M$ by $\hat{\theta}_Z$ as

$$\text{Fix}(M, \hat{\theta}_Z) := \{p \in M \mid \hat{\theta}_Z(p) = p\}.$$

Then $\iota_Z(K/K_Z) \subset \text{Fix}(M, \hat{\theta}_Z) \subset G/G_Z$.

The natural left action of $G$ on a symplectic manifold $(M = G/G_Z, \omega_Z)$ is a Hamiltonian group action with the moment map

$$\mu_G := \Phi_Z : G/G_Z \to \mathfrak{g} \cong \mathfrak{g}^*.$$

Moreover the natural left action of $K \subset G$ on $(M = G/G_Z, \omega_Z)$ is also a Hamiltonian group action with the moment map

$$\mu_K := \pi_t \circ \mu_G = \pi_t \circ \Phi_Z : G/G_Z \to \mathfrak{k} \cong \mathfrak{k}^*.$$
Here $\pi_t : g = \mathfrak{k} \oplus \mathfrak{p} \rightarrow \mathfrak{k}$ denotes the orthogonal projection of $g$ onto $\mathfrak{k}$. Then $\mu_G \circ \hat{\theta}_Z = -\theta \circ \mu_G$ and $\mu_K \circ \hat{\theta}_Z = -\mu_K$. It follows from these formulas that

$$\text{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0).$$

Since $K$ and $M$ are compact, by a result of Kirwan ([12, p.549, (3.1)]) $\mu_K^{-1}(0)$ is connected and thus $\text{Fix}(M, \hat{\theta}_Z)$ is also connected. Therefore we obtain

$$\iota_Z(K/K) = \text{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0).$$

The Weyl group of $(G, K)$ is defined by $W(G, K) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. By the action of the Weyl group $W(G, K)$, we may assume that $Z \in \mathfrak{a} \subset \mathfrak{t}$ satisfies $(\alpha, Z) \geq 0$ for $\forall \alpha \in \Sigma^+(\mathfrak{g})$.

Now we describe an invariant complex structure on $M = G/G_Z$ corresponding to $Z$. Note that $Z \in \mathfrak{c}(g_Z) \subset \mathfrak{t} \subset g_Z$.

Then

$$g^C_Z = \mathfrak{t}^C + \sum_{\alpha \in \Sigma_Z(\mathfrak{g})} \mathfrak{g}^\alpha,$$

$$T_{cG_Z}(G/G_Z)^C \cong \mathfrak{m}^C = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^\alpha,$$

Note that $\mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$. Thus we can define a $G$-invariant complex structure $J_Z$ on $G/G_Z$ such that

$$T_{cG_Z}(G/G_Z)^{1,0} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha}, \quad T_{cG_Z}(G/G_Z)^{0,1} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^\alpha,$$

Since

$$\hat{\theta} \left( \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^\alpha,$$

the involutive diffeomorphism $\hat{\theta}_Z : G/G_Z \rightarrow G/G_Z$ is anti-holomorphic with respect to $J_Z$, that is, $J_Z \circ d\theta_Z = -d\theta_Z \circ J_Z$.

Moreover the corresponding $G$-invariant Kähler metric $g_Z$ on $M = G/G_Z$ is defined by

$$\omega_Z(X, Y) = (-1)g_Z(J_ZX, Y) \quad \text{for each } X, Y \in \mathfrak{m}$$

or

$$g_Z = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} (Z, \alpha) \omega^{-\alpha} \cdot \omega^{-\alpha}.$$ 

Hence the diffeomorphism $\hat{\theta}_Z : M \rightarrow M$ is an isometry of $M$ with respect to $g_Z$.

Let $\Pi := \Pi(\mathfrak{g}) = \{\alpha_1, \cdots, \alpha_t\}$ be the fundamental root system of $\mathfrak{g}$ with respect to the $\sigma$-order $< \mathfrak{t}$. Set $\Pi(\mathfrak{g})_0 := \Pi(\mathfrak{g}) \cap \mathfrak{b}$. For the above $Z$, set $\Pi_Z := \Pi_Z(\mathfrak{g}) := \{\alpha_i \in \Pi(\mathfrak{g}) \mid (\alpha_i, Z) = 0\}$. Note that $\Pi_0(\mathfrak{g}) \subset \Pi_Z(\mathfrak{g})$ and
thus $\Pi(g) \setminus \Pi_Z(g) \subset \Pi(g) \setminus \Pi_0(g)$. Let $\{\Lambda_1, \cdots, \Lambda_\ell\} \subset t$ be the fundamental weight system of $g$ corresponding to $\Pi(g)$ defined by
\[
\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (i, j = 1, \cdots, \ell).
\]
Now we set $g_\lambda$ for each $\lambda \in c_Z$, since $\Pi(g) = \Pi(g_\lambda) = \Pi(g_Z)$, we have $g_\lambda = g_Z$. By the connectedness of $G_\xi$ and $G_Z$, we obtain $G_\lambda = G_Z$ and $G_\lambda/G_\xi = G/G_Z = M$. In particular $\omega_\lambda$ is a $G$-invariant symplectic form on $M = G/G_\lambda$. However $\lambda$ and $H$ define the same $G$-invariant complex structure $J_\lambda = J_H$ on $M = G/G_H = G/G_\lambda$.

Since $\theta(g_Z) = g_Z$ and thus $\theta(\xi(g_Z)) = \xi(g_Z)$, there is a direct sum decomposition
\[
\xi(g_Z) = c_Z = (c_Z \cap b) + (c_Z \cap a).
\]
For each $H \in c_Z^+ \cap a$, since $G_H = G_Z$ and $G/H = G/G_Z$, we have $K_H = K \cap G_H = K \cap G_Z = K_Z$ and thus $K/K_H = K/K_Z = L$. Hence all $H \in c_Z^+ \cap a$ correspond to the same $R$-space $L = K/K_Z$ and the convex set $c_Z^+ \cap a$ parametrizes orbits of the same type $K_Z$.

Let $\mathfrak{T}_G^2(M)$ denote the real vector space of all $G$-invariant closed 2-forms on $M = G/G_Z$. Then the natural linear map $\omega : \mathfrak{T}_G^2(M) \ni \omega \mapsto \omega \in H^2(M, \mathbb{R})$, is a linear isomorphism and there is a linear isomorphism $\omega : \frac{1}{2\pi \sqrt{-1}} c_Z \rightarrow \mathfrak{T}_G^2(M)$ defined by
\[
\omega \left( \frac{1}{2\pi \sqrt{-1}} \right) (X, Y) := -\frac{1}{2\pi} \langle [\lambda, X], Y \rangle \quad (X, Y \in \mathfrak{m})
\]
for each $\lambda \in c_Z$. Moreover the linear isomorphism $\tau = \omega \circ \omega : \frac{1}{2\pi \sqrt{-1}} c_Z \rightarrow \mathfrak{T}_G^2(M) \rightarrow H^2(M, \mathbb{R})$ given by the transgression operator is restricted to a $\mathbb{Z}$-module isomorphism $\tau = \omega : \frac{1}{2\pi \sqrt{-1}} Z_{c_Z} \rightarrow H^2(M, \mathbb{Z})$. The 2nd cohomology and homology groups of $G/G_Z$ are described as follows:
\[
c_Z = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R} A_{\alpha_i} \quad \overleftarrow{\lambda} \mapsto \left[ \frac{1}{2\pi \sqrt{-1}} \omega(\lambda) \right] \in H^2(G/G_Z, \mathbb{R})
\]
\[
Z_{c_Z} = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z} A_{\alpha_i} \quad \overleftarrow{\longrightarrow} \quad \bigcup \quad H^2(G/G_Z, \mathbb{Z})
\]
and if we set \( \alpha^*_i := \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \),
\[
\mathfrak{c}^+_Z := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}\alpha^*_i \quad \ni \quad \sum x_i\alpha^*_i \iff \sum x_i[S^2(\alpha^*_i)] \in H_2(G/G_Z, \mathbb{R}) \\
\bigcup_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}\alpha^*_i \quad \iff \quad H_2(G/G_Z, \mathbb{Z}) \cong \pi_2(G/G_Z).
\]

For each \( \lambda \in \mathfrak{c}^+_Z \), define a \( G \)-invariant Kähler metric on a complex manifold \( (M = G/G_Z, J_Z) \) by
\[
g\left(\frac{1}{2\pi \sqrt{-1}} \lambda\right) := \frac{1}{2\pi} \sum_{\alpha \in \Sigma^+(g) \setminus \Sigma_Z(g)} (\lambda, \alpha)\omega^{-\alpha} \cdot \bar{\omega}^{-\alpha}
\]
whose Kähler form coincides with \( \omega\left(\frac{1}{2\pi \sqrt{-1}} \lambda\right) \) as
\[
\omega\left(\frac{1}{2\pi \sqrt{-1}} \lambda\right)(X, Y) = g\left(\frac{1}{2\pi \sqrt{-1}} \lambda\right)(J_Z X, Y).
\]

Note that \( Z \in \mathfrak{c}^+_Z \cap a \) and \( \omega(\frac{1}{2\pi \sqrt{-1}} 2\pi Z) = -\omega_Z \). Namely, the convex open set \( \mathfrak{c}^+_Z \) of the vector space \( \mathfrak{c}_Z \) parametrizes all \( G \)-invariant Kähler metrics on \( M = G/G_Z \) relative to the complex structure \( J_Z \). So the parameter spaces of all \( G \)-inv. Kähler metrics on \( G/G_Z \) are given as
\[
\mathfrak{c}^+_Z = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{R}^+\Lambda_{\alpha_i} \quad \ni \lambda \iff \omega\left(\frac{1}{2\pi \sqrt{-1}} \lambda\right) \in \{ G \text{-inv. Kähler met. on } G/G_Z \} \\
\bigcup_{\alpha_i \in \Pi \setminus \Pi_Z} \mathbb{Z}^+\Lambda_{\alpha_i} \quad \iff \quad \{ G \text{-inv. Hodge met. on } G/G_Z \}.
\]

For each \( H \in \mathfrak{c}^+_Z \cap a \), the diffeomorphism \( \hat{\theta}_Z : M = G/G_Z \to M = G/G_Z \) preserves a \( G \)-invariant Kähler metric \( g\left(\frac{1}{2\pi \sqrt{-1}} H\right) \) on \( M \), that is, \( \hat{\theta}_Z : M = G/G_Z \to M = G/G_Z \) is an isometry with respect to \( g\left(\frac{1}{2\pi \sqrt{-1}} H\right) \). Hence the canonically embedded \( R \)-space \( \nu_Z(K/K_Z) \) is a real form, that is, the fixed point subset of a Kähler \( C \)-space \( M = G/G_Z \) by the anti-holomorphic isometry \( \hat{\theta}_Z \) with respect to \( J_Z \) and a Kähler metric \( g\left(\frac{1}{2\pi \sqrt{-1}} H\right) \) for any \( H \in \mathfrak{c}^+_Z \cap a \).

Next we mention the characterization of a \( G \)-invariant Einstein-Kähler metric on \( M = G/G_Z \). Set
\[
\delta_m := \frac{1}{2} \sum_{\alpha \in \Sigma^+(g) \setminus \Sigma_Z(g)} \alpha \in t.
\]

Lemma 2.1 ([1]).

\[
2\delta_m = \sum_{\alpha \in \Sigma^+(g) \setminus \Sigma_H(g)} \alpha \in \mathbb{Z}^+_{\mathfrak{c}_Z} = \bigoplus_{\alpha \in \Pi \setminus \Pi_H} \mathbb{Z}^+\Lambda_{\alpha}.
\]
and it corresponds to the first Chern class of the complex manifold \((M, J_H)\):

\[
c_1(M) = \left[ \omega \left( \frac{1}{2\pi \sqrt{-1}} 2\delta_m \right) \right] = \tau \left( \frac{1}{2\pi \sqrt{-1}} 2\delta_m \right).
\]

**Proposition 2.2 ([24]).** The \(G\)-invariant Kähler metric \(g = g \left( \frac{1}{2\pi \sqrt{-1}} \lambda \right)\) on \(M\) is Einstein if and only if \(\lambda = b \delta_m\) for some \(b > 0\).

Then we can show that \(2\delta_m \in a\) ([20]). Therefore we obtain

**Proposition 2.3.** The element \(Z^\text{ein} := 2\delta_m \in Z^+_p \cap a \subset c^+_Z \cap a\) corresponds to the canonical embedding \(\tau_{Z^\text{ein}}\) of the same \(R\)-space \(L = K/K_Z\) into an Einstein-Kähler \(C\)-space \((M = G/G_Z, \omega, Z^\text{ein}, J_Z, g \left( \frac{1}{2\pi \sqrt{-1}} \right)\). Moreover, the element \(Z^\text{ein}\) is such a unique element of \(c^+_Z \cap a\) up to the multiplication by a positive constant.

Here we shall mention about geometry of \(R\)-spaces as homogeneous spaces of noncompact real semisimple Lie groups. Set \(p^\sharp := \sqrt{-1}p\). Then \(g^\sharp := \mathfrak{t} + p^\sharp\) is the Cartan decomposition of a noncompact real semisimple Lie algebra \(g^\sharp\) with Cartan involution \(\tau\). Let \(G^C\) be a connected complex Lie group without center with Lie algebra \(g^C\) and then \(G\) can be regarded as an analytic subgroup of \(G^C\). Let \(G^\sharp\) be a connected real semisimple Lie subgroup of \(G^C\) with Lie algebra \(g^\sharp\). The root space decomposition of \(g^\sharp\) with respect to \(\sqrt{-1}a\) is given as

\[
g^\sharp = g^\sharp_0 + \bigoplus_{\gamma \in \Sigma(g, a)} g^\sharp_{\gamma},
\]

where \(g^\sharp_0 := \{X \in g^\sharp \mid [\sqrt{-1}H, X] = 0 \ (\forall H \in a)\}\) and for each \(\gamma \in \Sigma(g, a)\)

\[
g^\sharp_{\gamma} := \{X \in g^\sharp \mid [\sqrt{-1}H, X] = (\gamma, H)X \ (\forall H \in a)\}.
\]

Then

\[
u := g^\sharp_0 + \bigoplus_{\gamma \in \Sigma(g, a), \gamma(Z) \geq 0} g^\sharp_{\gamma},
\]

is a parabolic subalgebra of \(g^\sharp\). Let \(U\) be a parabolic subgroup of \(G^\sharp\) with Lie algebra \(u\), which is always connected. The complexification of \(u\)

\[
u^C = (g^\sharp_0)^C + \bigoplus_{\gamma \in \Sigma(g, a), \gamma(Z) \geq 0} (g^\sharp_{\gamma})^C = \mathfrak{t}^C + \bigoplus_{\alpha \in \Sigma(g, a), \alpha(Z) \geq 0} g^\alpha_C.
\]

is a complex parabolic subalgebra of \(g^C\). Let \(U^C\) be a complex parabolic subgroup of \(G^\sharp\) with Lie algebra \(u^C\), which is always connected. Then we know ([23]) that

\[
K U = G^\sharp, \ K \cap U = K_Z, \text{ and thus } L = K/K_Z \cong G^\sharp/U,
\]

\[
G U^C = G^C, \ G \cap U^C = G_Z, \text{ and thus } G/G_Z \cong M = G^C/U^C.
\]
The induced complex structure of $M$ under the identification of $M = G/G_Z$ with the complex homogeneous space $G^C/U^C$ coincides with the $G$-invariant complex structure $J_Z$ of $M$.

Define two subgroups of $K$ and $K_Z$ as

$$N_K(a) := \{ k \in K \mid \text{Ad}(k)a = a \} \subset K, \quad N_{K_Z}(a) := N_K(a) \cap K_Z \subset K_Z.$$  

Note that $N_{K_Z}(a)$ is not a normal subgroup of $N_K(a)$, $C_K(a) \subset N_{K_Z}(a)$, and if $Z \in a$ is regular, then $C_K(a) = N_{K_Z}(a)$:

$$1 \longrightarrow N_{K_Z}(a)/C_K(a) \longrightarrow W(G; K) = N_K(a)/C_K(a) \longrightarrow N_K(a)/N_{K_Z}(a) \longrightarrow 1$$

**Theorem 2.4** (Masaru Takeuchi [23]). Let $k_1, \cdots, k_b \in N_K(a)$ be complete representatives of $N_K(a)/N_{K_Z}(a) = \{[k_1], \cdots, [k_b]\}$. Then the orbits $N_{k_1}o, \cdots, N_{k_b}o$ of $N$ through the origin $o = eU \in G^2/U = L$ provide a cellular decomposition of $L$ as $L = G^2/U = N_{k_1}U \cup \cdots \cup N_{k_b}U$ and these cells are all cycles mod 2 of $L$. In particular, $\dim H_* (L; \mathbb{Z}_2) = \sum_{i=0}^L \dim H_i (L; \mathbb{Z}_2) = b$.

We briefly discuss related results of Masaru Takeuchi and Shosichi Kobayashi ([27]) on perfect Morse functions on $R$-spaces. For each $X \in \mathfrak{g}$, we define a linear function $u_X : \mathfrak{g} \rightarrow \mathbb{R}$ defined by $u_X(\xi) := \langle \xi, X \rangle$ for each $\xi \in \mathfrak{g}$. A smooth function $f_X$ on $M = G/G_Z$ is defined by

$$f_X := u_X \circ \Phi \gamma = u_X \circ \mu_G = \langle \mu_G, X \rangle : M = G/G_Z \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R}.$$  

Then by the moment map equation and (2.10) we have

$$d\tilde{f}_X = d\langle \mu_G, X \rangle = \omega_Z(\tilde{X}, \cdot) = -g_Z (J_Z \tilde{X}, \cdot),$$

where $\tilde{X}$ denotes a vector field on $M = G/G_Z$ induced by a one-parameter subgroup $\{ \exp(tX) \mid t \in \mathbb{R} \}$ of $G$, which is a Killing vector field on $M$ with respect to a Kähler metric $g_Z$. Hence the gradient vector field $\text{grad}(\tilde{f}_X)$ of the function $\tilde{f}_X$ on $M = G/G_Z$ with respect to the invariant Kähler metric $g_Z$ is equal to $-J_Z \tilde{X}$:

$$-\text{grad}(\tilde{f}_X) = J_Z \tilde{X} = (\sqrt{-1}X).$$

Here $J_Z \tilde{X} = (\sqrt{-1}X)$ is a holomorphic vector field on $M = G^{C}/U^C$ induced by a one-parameter subgroup $\{ \exp(t\sqrt{-1}X) \mid t \in \mathbb{R} \}$ of $G^C$.

Now assume that $X \in \mathfrak{p}$. A smooth function $f_X$ on $L = K/K_Z$ is defined by

$$f_X = f_X \circ \iota_Z = u_X \circ \varphi_Z = \langle \mu_G \circ \iota_Z, X \rangle : L = K/K_Z \longrightarrow \mathfrak{p} \longrightarrow \mathbb{R}.$$  

By pulling back the equation (2.14) by the canonical embedding $\iota_Z$, we have

$$df_X = \iota_Z^* df_X = \iota_Z^* \omega_Z(\tilde{X}, \cdot) = -g_Z ((\sqrt{-1}X) \circ \iota_Z)(\cdot, \iota_Z(\cdot)).$$
that $K/Z$ is a perfect Morse function on the symmetric space $G/K$. Since $\psi(2.20)$ is a connected open dense subset of $G/K$ and coincides with the zero set $Z = \{v \in \mathfrak{g}^2 : \sqrt{-1}X \cdot v = 0\}$ of vector fields $\sqrt{-1}X$ on $G/K$ induced by $-\sqrt{-1}X \in \mathfrak{g}^2$. In particular, the critical point set of $f_X$ on $G/K$ coincides with the zero set $\text{Zero}(\sqrt{-1}X) = \text{Zero}(\sqrt{-1}X) \cap \hat{X} \circ \iota_Z)$ of vector fields $\sqrt{-1}X$ and $\hat{X} \circ \iota_Z$ on $L$. In [27] they showed that for each $\gamma \in \mathfrak{g}/\mathfrak{k}$, the number of critical points of $\sqrt{-1}X$ is equal to $\#(N_{(a)}/\mathcal{N}_{\mathcal{K}}(a)) = \dim H_{\mathfrak{a}}(K/K_\mathcal{Z})$. In particular $f_X$ is a Morse function on the $R$-space $L$ for each regular $\gamma \in \mathfrak{g}$. We can also observe that for each regular $\gamma \in \mathfrak{g}$ the equality $\#(\exp(t\gamma)L) \cap \iota_Z(L)) = \#(N_{(a)}/\mathcal{N}_{\mathcal{K}}(a))$ holds for any sufficiently small $t \neq 0$.

Here we recall some fundamental results from the structure theory of a compact symmetric space $G/K$ (cf. [6], [26]). They are necessary to discuss the geometry of $R$-spaces canonically embedded in Kähler $C$-spaces. Set $A := \exp \mathfrak{a} \subset G$ and $\hat{A} := A(a) \subset G/K$. Then the $K$-equivariant map

\begin{equation}
\psi : K/Z_K(a) \times \hat{A} \ni (kZ_K(a), \hat{a}) \mapsto k\hat{a} \in G/K
\end{equation}

is a surjective smooth map. Define the diagram of a compact symmetric pair $(G, K)$ by

\begin{equation}
D(\mathfrak{g}, \mathfrak{a}) := \{H \in \mathfrak{a} \mid (\gamma, H) \in \pi \mathcal{Z} (\exists \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}))\}
\end{equation}

and thus we have $\mathfrak{a} \setminus D(\mathfrak{g}, \mathfrak{a}) = \{H \in \mathfrak{a} \mid (\gamma, H) \notin \pi \mathcal{Z} (\forall \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}))\}$. Set $\hat{A}_s := (\exp D(\mathfrak{g}, \mathfrak{a}))eK \subset \hat{A}$ and $\check{A}_r := \hat{A} \setminus \hat{A}_s := (\exp(\mathfrak{a} \setminus D(\mathfrak{g}, \mathfrak{a}))eK$. Each element of $\hat{A}_s$ (resp. $\check{A}_r$) is called a regular (resp. singular) element of $A$. Then $G/K = (G/K)_r \cup (G/K)_s$ (disjoint union), where $(G/K)_s := \psi(K/Z_K(a) \times \hat{A}_s)$ is a closed set of codimension at least 2 and

\begin{equation}
(G/K)_r := \psi(K/Z_K(a) \times \hat{A}_r)
\end{equation}

is a connected open dense subset of $G/K$. Each element of $(G/K)_r$ is called a regular element of $G/K$. The surjective smooth map

\begin{equation}
\psi : K/Z_K(a) \times \hat{A}_r \mapsto (G/K)_r
\end{equation}

is a covering map whose covering transformation group is the right natural action of $W(G, K)$ on $K/Z_K(a) \times \hat{A}_r$, and thus it induces a diffeomorphism $\hat{\psi} : (K/Z_K(a) \times \hat{A}_r)/W(G, K) \mapsto (G/K)_r$, which is equivariant with the actions of $K$. Here note that $K/Z_K(a) \times \hat{A}_r$ is not connected in general.
Using the geometry of a compact symmetric space $G/K$, we discuss the intersection property of $a_ι Z(L)$ and $ι Z(L)$ under the left group action of $a ∈ G$ on $M = G/G_Z$.

For any $a ∈ G$, by the surjectivity of $ψ$ we have $aK = ψ(kZ_K(a), \exp(H)eK)$ for some $k ∈ K$ and some $H ∈ a$ and thus $ak_1 = kk_0 \exp(H)$ for some $k_0 ∈ Z_K(a)$ and some $k_1 ∈ K$. Thus $ak_1G_Z = kk_0 \exp(H)G_Z = kG_Z ∈ G/G_Z$ and hence $a_ι Z(k_ι kZ) = _ι Z(kZ) ∈ a_ι Z(K/K) _ι Z(K/K)$ gives an intersection point of $a_ι Z(K/K)$ and $ι Z(K/K)$. Moreover, for any $k' ∈ N_K(a)$, we have $ak_1k' = kk'(k'^{-1}k_0k') \exp(Ad(k'^{-1})H)$ where note that $k'^{-1}k_0k' ∈ Z_K(A)$ and $Ad(k'^{-1})H ∈ a$. Thus $ak_1k'G_Z = kk'(k'^{-1}k_0k') \exp(Ad(k'^{-1})H)G_Z = kk'G_Z ∈ G/G_Ż$ and hence $a_ι Z(k_ι k'Z) = _ι Z(kk'Z) ∈ a_ι Z(K/K) _ι Z(K/K)$ also gives an intersection point of $a_ι Z(K/K)$ and $ι Z(K/K)$. Note that if $kk'Z = kk''Z$ for $k, k'' ∈ N_K(a)$, then $k''N_K(a) = k''N_K(a)$. Therefore, combining the above argument with Theorem 2.4, we obtain

**Proposition 2.5.** For any $a ∈ G$, it holds

\[ \sharp(a_ι Z(L) ∩ ι Z(L)) ≥ ζ(N_K(a)/N_K(a)) = \dim H_*(L, Z_2). \]

Next we mention about the transversality condition of the intersection $a_ι Z(L) ∩ ι Z(L)$. Suppose that $p ∈ a_ι Z(K/K) ∩ ι Z(K/K)$. Then $p = a_ι Z(k_ι kZ) = ι Z(k_ι kZ)$ for some $k_ι kZ ∈ G_Z$, using the symmetric Lie algebra $g_Ż = g_Z + p_Z$ of a compact symmetric pair $(G_Ż, K/Z)$ so that $a ⊂ p_Z$, there are $k_Z, k_Z' ∈ K_Ż$ and $H_p ∈ a$ such that $k_ι^{-1}k_2Z = k_Z(\exp H_p)k_Z^{-1}k_Z'$. Thus $ak_2 = k_ι k_Z(\exp H_p)k_Z^{-1}k_2'$. Since

\[
(Φ_Z)_*(T_pJ_a_ι Z(K/K)) = Ad(ak_2)[t, Z]
= Ad(k_ι k_Z(\exp H_p)k_Z^{-1}k_Z')[t, Z]
= Ad(k_1)Ad(k_Z)Ad(\exp H_p)[t, Z],
\]

\[
(Φ_Z)_*(T_pJ_ι Z(K/K)) = Ad(k_1)[t, Z] = Ad(k_1)Ad(k_Z)[t, Z],
\]

the transversality of $a_ι Z(L) ∩ ι Z(L)$ at $p$ is equivalent to the transversality of $Ad(\exp H_p)[t, Z]$ and $[t, Z]$: $Ad(\exp H_p)[t, Z] ∩ [t, Z] = \{0\}$. Then by a simple computation using the basis \{ $S_γs, T_γs, \}$ we can show

**Lemma 2.6.** $a_ι Z(L)$ intersects transversally with $ι Z(L)$ in $M = G/G_Ż$ if and only if at each intersection point $p$ such an $H_p ∈ a$ satisfies $(γ, H) ∉ πZ$ for each $γ ∈ \Sigma^+(g, a)$ with $(γ, Z) ≠ 0$.

First we suppose that $a ∈ G$ satisfies $aK ∈ (G/K)_γ$. Then $a_ι Z(K/K) ∩ ι Z(K/K)$ is transversal at each intersection point. Fix an intersection point $ι Z(k_ι kZ) ∈ a_ι Z(K/K) ∩ ι Z(K/K)$. Then

\[ aK = k_ι k_Z, \exp(H_1)K \quad (∃H_1 ∈ a \setminus D(g, a), ∃k_Z ∈ K_Z) \]
\[ = ψ(k_ι k_Z, Z_K(a), \exp(H_1)K) \]
For an arbitrary intersection point \( \iota_Z(k_2 K_Z) \in a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \),
\[
aK = k_2 k_{Z,2} \exp(H_2) K = (\exists H_2 \in a \setminus \mathbb{D}(G, K), k_{Z,2} \in K_Z)
\]
\[
= \psi(k_2 k_{Z,2} Z_{K}(a), \exp(H_2) K)
\]
Then there is \( s = [k'] \in W(G, K) = N_K(a)/Z_K(a) \) such that
\[
k_2 k_{Z,2} Z_{K}(a) = k_1 k_{Z,1} Z_{K}(a) s = k_1 k_{Z,1} k' Z_{K}(a)
\]
and
\[
(\exp H_2) K = s^{-1}(\exp H_1) K = (\exp s^{-1} H_1) K = (\exp \text{Ad}(k')^{-1} H_1) K.
\]
Thus \( k_2 K_Z = k_1 k_{Z,1} k' K_Z \) and hence \( \iota_Z(k_2 K_Z) = \iota_Z(k_1 k_{Z,1} k' K_Z) \). Therefore we obtain
\[
a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) = \{ \iota_Z(k_1 k_{Z,1} k' K_Z) | [k'] \in N_K(a)/N_{K_Z}(a) \}.
\]
In particular, \( \sharp(a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(a)/N_{K_Z}(a)) \).

In general suppose that \( a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \) is transversal at each intersection point. Particularly \( a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \) is a finite set. We may assume that \( aK \in (G/K)_r \). Since \( (G/K)_r \) is open and dense in \( G/K \), we choose a smooth perturbation \( a_t \in G \) of \( a_0 = a \) such that \( a_t K \in (G/K)_r \) (\( 0 < \forall t < \varepsilon \)). Then \( a_t \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) \) is also transversal at each intersection point and for sufficiently small \( t > 0 \) we have
\[
\sharp(a \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(a_t \iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(a)/N_{K_0}(a)).
\]
Therefore we obtain

**Proposition 2.7.** For any \( a \in G \) with transversal \( a \iota_Z(L) \cap \iota_Z(L) \), it holds
\[
\sharp(a \iota_Z(L) \cap \iota_Z(L)) = \sharp(N_K(a)/N_{K_Z}(a)).
\]

Combining it with Theorem 2.4, we see

**Corollary 2.8.** For any \( a \in G \) with transversal \( a \iota_Z(L) \cap \iota_Z(L) \), it holds
\[
\sharp(a \iota_Z(L) \cap \iota_Z(L)) = \dim H_*(L, Z_2).
\]
Such a property is called the global tightness for a Lagrangian submanifolds in a Kähler C-space ([14], [9], [5]). It was proved by [28] in the case when \( L \) is a symmetric \( R \)-space. It is still an open problem to classify compact globally tight or simply tight Lagrangian submanifolds of Kähler C-spaces. More generally the intersection theory and Floer homology for two real forms in Kähler C-spaces are discussed in [7], [11].
Let \( L \) be a Lagrangian submanifold of a symplectic manifold \((M,\omega)\). Define two kinds of group homomorphisms \( I_{\mu,L} : \pi_2(M,L) \to \mathbb{Z} \) and \( I_{\omega,L} : \pi_2(M,L) \to \mathbb{R} \). For a smooth map \( u : (D^2, \partial D^2) \to (M,L) \) with \( A = [u] \in \pi_2(M,L) \), choose a trivialization of the pull-back bundle as a symplectic vector bundle (which is unique up to the homotopy). \( u^{-1}TM \cong D^2 \times \mathbb{C}^n \). This gives a smooth map \( \tilde{u} : S^1 = \partial D^2 \to \Lambda(\mathbb{C}^n) \). Here \( \Lambda(\mathbb{C}^n) \) denotes the Grassmann manifold of Lagrangian vector subspaces of \( \mathbb{C}^n \). Using the Moslov class \( \mu \in H^1(\Lambda(\mathbb{C}^n),\mathbb{Z}) \cong \mathbb{Z} \), we define \( I_{\mu,L}(A) := \mu(\tilde{u}) \). Another homomorphism \( I_{\omega,L} : \pi_2(M,L) \to \mathbb{R} \) is defined by \( I_{\omega,L}(A) := \int_{D^2} u^*\omega \). Note that \( I_{\mu,L} \) is invariant under symplectic diffeomorphisms and \( I_{\omega,L} \) is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.

A Lagrangian submanifold \( L \) of \((M,\omega)\) is called \textit{monotone} ([15]) if \( I_{\mu,L} = \lambda I_{\omega,L} \) (\( \exists \lambda > 0 \)). If \( I_{\mu,L} = 0 \), we define \( \Sigma_L = 0 \). We assume that \( I_{\mu,L} \neq 0 \), Denote by \( \Sigma_L \in \mathbb{Z}_+ \) the positive generator of \( \text{Im}(I_{\mu,L}) \) as an additive subgroup of \( \mathbb{Z} \). We call \( \Sigma_L \) is called the \textit{minimal Maslov number} of \( L \).

\[ \text{Theorem 3.1 ([2], [21]). Suppose that } (M,\omega,J,g) \text{ is an Einstein-Kähler manifold of positive Einstein constant. If } L \text{ is a compact minimal Lagrangian submanifold of } M, \text{ then } L \text{ is monotone.} \]

Suppose that \((M,\omega,J,g)\) is a \textit{simply connected} Einstein-Kähler manifold with positive Einstein constant and \( L \) is a compact monotone Lagrangian submanifold of \( M \). Then Hajime Ono ([21]) showed the formula for \( \Sigma_L \):

\[ n_L \Sigma_L = 2\gamma_{c_1}, \]

where we set

\[ \gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M;\mathbb{Z}), c_1(M)(A) > 0\}, \]

\[ n_L := \min\{k \in \mathbb{Z}^+ \mid \otimes^k E \text{ trivial}\}. \]

Here \( \frac{1}{2} \omega = c_1(E,\nabla) \) for some constant \( \gamma > 0 \).

As an application of that formula (3.1), we mention the minimal Maslov number formula for the Gauss images of isoparametric hypersurfaces in the standard sphere.

Let \( N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} \) be an oriented hypersurface of \( S^{n+1}(1) \) and let \( \tilde{N}^n := \{(x(p),n(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\} \) be the Legendrian lift of \( N^n \) to \( T^1 S^{n+1} \). Then we have the following diagram:

\[ \begin{array}{ccc}
E_{|L} & \longrightarrow & E \\
\pi_L \text{ flat} & \pi_E \text{ U(1)-connection } \nabla & \\
L \text{ Lag.} & \longrightarrow & M \text{ Einstein-Kähler mfd.}
\end{array} \]

Here \( \frac{1}{2} \omega = c_1(E,\nabla) \) for some constant \( \gamma > 0 \).
The **Gauss Map** is defined by

\[ \mathcal{G} : N \rightarrow \hat{N} \rightarrow \mathbb{C} Q_n(\mathbb{C}) = \mathcal{G} \mathcal{R}_2(\mathbb{R}^{n+2}) \]

\[ p \rightarrow (x(p), n(p)) \rightarrow [x(p) + \sqrt{-1}n(p)] = x(p) \wedge n(p) \]

Suppose that \( N^n \subset S^{n+1}(1) \) is an isoparametric hypersurface with \( g \) distinct principal curvatures. Let \( \hat{N}^n \coloneqq \{(x(p), n(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\} \) be the Legendrian lift of \( N^n \) to \( T^1S^{n+1} \). The following diagram becomes as follows:

\[ \mathcal{G} : N \rightarrow \hat{N} \rightarrow \mathbb{C} Q_n(\mathbb{C}) = \mathcal{G} \mathcal{R}_2(\mathbb{R}^{n+2}) \]

\[ p \rightarrow (x(p), n(p)) \rightarrow [x(p) + \sqrt{-1}n(p)] = x(p) \wedge n(p) \]

Then it holds \( \Sigma_L = \frac{2n}{g} \) ([19], [13]). This formula was crucial to the Hamiltonian non-displaceability theorem for Gauss images of isoparametric hypersurfaces ([8]).

## 4 Minimal Maslov number of R-spaces canonically embedded in Einstein-Kähler C-spaces

We take the universal cover \( \tilde{G} \rightarrow G \) of \( G \). Let \( (\tilde{G}, \tilde{K}, \theta) \) be a Riemannian symmetric pair of compact type with simply connected \( \tilde{G} \) and connected \( \tilde{K} \).

By Proposition 2.3 we can choose \( Z = Z^{cin} = 2\delta_m \). Then \( \iota_Z : L = K/K_Z \rightarrow M = G/G_Z \) is the canonical embedding of an \( R \)-space into an Einstein-Kähler C-space. As in [24] we use expression

\[ 2\delta_m = \sum_{\alpha_i \in \Pi \setminus \Pi_Z} k_i \Lambda_i = \kappa(M) \sum_{\alpha_i \in \Pi \setminus \Pi_Z} \kappa_i \Lambda_i \quad (k_i \in \mathbb{Z}^+) , \]
where \( \kappa(M) \) denotes the greatest common divisor of \( \{ k_i \mid \alpha_i \in \Pi \setminus \Pi_Z \} \) and set \( \kappa_i := \frac{k_i}{\kappa(M)} \) for each \( \alpha_i \in \Pi \setminus \Pi_Z \). Then the invariant \( \gamma_{c_1} \) in (3.1) is given as \( \gamma_{c_1} = \kappa(M) \) (cf. [20]).

Here we take the orthogonal direct sum decomposition \( g_Z = \mathbb{R} \cdot 2\delta_m \oplus g'_Z \). Denote by \( \tilde{G} \) a connected Lie subgroup of \( \tilde{G} \) with Lie algebra \( g'_Z \) and set \( \tilde{K}_Z := \tilde{K} \cap \tilde{G} \). Then \( n_L = \sharp(\tilde{K}_Z/\tilde{K}'_Z) \) ([20]). Therefore by the formula (3.1) we obtain

\[
\Sigma_L = 2\kappa(M) \frac{\sharp(K_H/K'_H)}{\sharp(K_H/K'_H)}.
\]

Some concrete examples of computations by this formula are given in [20] in the case when (1) \((G,K) = (SU(n+1),SO(n+1))\) and \(L\) is \(\mathbb{R}P^n\) or a regular \(R\)-space, (2) \(L\) is a maximal flag manifold of a compact semisimple Lie group \(K\), (3) \(L\) is an irreducible symmetric \(R\)-space. It is an interesting problem to study the minimal Maslov number for all other \(R\)-spaces by this formula.

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References


