A CHARACTERIZATION OF DIFFERENTIABILITY FOR THE BEST TRACE SOBOLEV CONSTANT FUNCTION

KAZUYA AKAYAMA\textsuperscript{1} AND FUTOSHI TAKAHASHI\textsuperscript{2}

Abstract. Let $1 < p < N$ and let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$. In this paper we show some regularity results for the best constant $S_q$ of the trace Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$, considering that $S_q$ is a function of $q$. We prove that $S_q$ is absolutely continuous, thus $S'_q = \frac{d}{dq}S_q$ exists a.e. $q \in [1, p_*]$, $p_* = \frac{p(N-1)}{N-p}$. We give a characterization on a set where $S'_q$ exists. These are natural extensions of the recent work by Ercole for the best constant of the Sobolev embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, p^*]$, $p^* = \frac{Np}{N-p}$.

Key words: Best trace Sobolev constant, Absolute continuity, Differentiability

2010 Mathematics Subject Classification: 46E35, 35J20, 35J25

1. Introduction

Let $1 < p < N$ be fixed and let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial \Omega$. The well-known trace Sobolev embedding theorem claims that the continuous inclusion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$ holds true for $1 \leq q \leq p_*$, where $p_* = \frac{p(N-1)}{N-p}$ denotes the trace Sobolev critical exponent. Hence the following trace Sobolev inequality holds true for any $u \in W^{1,p}(\Omega)$:

$$C \left( \int_{\partial \Omega} |u|^q \, dH^{N-1} \right)^{\frac{1}{q}} \leq \int_{\Omega} (|\nabla u|^p + |u|^p) \, dx \quad (1 \leq q \leq p_*) \quad (1.1)$$

\textsuperscript{1}Department of Mathematics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan.
\textsuperscript{2}Department of Mathematics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan.

e-mail:kazuya4876@gmail.com
e-mail:futoshi@sci.osaka-cu.ac.jp

Date: November 12, 2019.
where \( \mathcal{H}^{N-1} \) denotes the \((N-1)\)-dimensional Hausdorff measure on the hypersurface \( \partial \Omega \). The best constant of the trace Sobolev inequality (1.1) (i.e., the largest \( C \) such that the above inequality holds for any \( u \in W^{1,p}(\Omega \setminus W^{1,p}_0(\Omega)) \)) is defined as

\[
S_q = S_q(\Omega) := \inf_{u \in W^{1,p}(\Omega \setminus W^{1,p}_0(\Omega)) \atop \|u\|_{L^q(\partial \Omega)} = 1} \frac{\int_{\Omega} (|\nabla u|^p + |u|^p) \, dx}{\left( \int_{\partial \Omega} |u|^q \, d\mathcal{H}^{N-1} \right)^{\frac{p}{q}}}
\]

(1.2)

It is known that the continuous embedding \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \) for \( 1 \leq q \leq p_* \) is actually compact when \( 1 \leq q < p_* \), thus a minimizer for \( S_q \) exists for \( 1 \leq q < p_* \). A minimizer \( u_q \) for \( S_q \) with the property \( \|u_q\|_{L^q(\partial \Omega)} = 1 \) is a weak solution of the Euler-Lagrange equation

\[
\begin{cases}
\Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = S_q |u|^{q-2}u & \text{on } \partial \Omega,
\end{cases}
\]

(1.3)

where \( \nu \) is the outer unit normal of \( \partial \Omega \). Note that by the strong maximum principle [18], a solution \( u \) of (1.3) has a constant sign on \( \Omega \), and we may assume \( u > 0 \) on \( \Omega \). Also regularity results (see e.g., [15], [17]) imply that \( u \in C^{1,\alpha}(\Omega) \cap C^\alpha(\overline{\Omega}) \) for some \( \alpha \in (0,1) \).

For the case \( q = p_* \), the existence of a minimizer becomes a subtle problem because of the lack of compactness. Recently it is proved in [14] that \( S_{p_*} \) is attained on any smooth bounded domain when \( p \in (1, \frac{N+1}{2} + \beta) \), where \( \beta = \beta(\Omega) > 0 \). See [1], [11], [6], [7] for earlier results on the existence of extremals for \( S_{p_*}(\Omega) \) on bounded domains.

This is a striking difference between the best constant for the Sobolev inequality

\[
\tilde{S}_q = \tilde{S}_q(\Omega) := \inf_{u \in \tilde{W}^{1,p}(\Omega) \atop \|u\|_{\tilde{W}^{1,p}(\Omega)} = 1} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\left( \int_{\Omega} |u|^q \, dx \right)^{\frac{p}{q}}}
\]

(1.4)

for \( 1 \leq q \leq p = \frac{Np}{N-p} \). Indeed, \( \tilde{S}_{p_*}(\Omega) \) is never attained on any domain \( \Omega \) other than \( \mathbb{R}^N \) and \( \tilde{S}_{p_*}(\Omega) \) does not depend on the domain \( \Omega \) but depends only on \( N \). More precisely, \( \tilde{S}_{p_*}(\Omega) = \tilde{S}_{p_*}(\mathbb{R}^N) \) and the explicit value of \( \tilde{S}_{p_*} \) is known, see [16].

Also, the behaviors of both the constants \( S_q(\Omega) \) and \( \tilde{S}_q(\Omega) \) under the dilations of the domain are different from each other. That is, if we define \( \mu \Omega = \{ \mu x \mid x \in \Omega \} \) for \( \mu > 0 \), we have \( \tilde{S}_q(\mu \Omega) = \mu^{N-p-\frac{p^*}{q}} \tilde{S}_q(\Omega) \).
On the other hand, it is easy to see by using $u_\mu(x) = u(\mu x)$ that

$$S_q(\mu\Omega) = \mu^{N-p(N-1)/q} \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega (\mu^{-p} |\nabla u_\mu|^p + |u_\mu|^p) \, dx}{(\int_{\partial\Omega} |u_\mu|^q \, dH^{N-1})^{\frac{p}{q}}}.$$  

Recently, several regularity properties of $\tilde{S}_q$ as a function of $q \in [1, p^*) = [\frac{Np}{N-p}, \infty)$ are proved by G. Ercole [3], [4]; see also [8] and [2]. In fact, in [3] it is proved that the function $q \mapsto \tilde{S}_q$ is Lipschitz continuous on the interval $[1, p^* - \varepsilon]$ for any $\varepsilon > 0$ small. Also $\tilde{S}_q$ is absolutely continuous on the whole closed interval $[1, p^*)$ and thus its derivative $\frac{d}{dq}\tilde{S}_q = \tilde{S}_q'$ exists almost all $q \in [1, p^*)$. In [4], the author characterizes the point $q \in [1, p^*)$ where $\tilde{S}_q$ is differentiable; $\tilde{S}_q'$ exists if and only if the functional

$$\tilde{I}_q(u) = \int_\Omega |u|^q \log |u| \, dx$$

takes a constant value on the set $\tilde{E}_q$ of the $L^q$-normalized extremal functions corresponding to $\tilde{S}_q$:

$$\tilde{E}_q = \{ u \in W_0^{1,p}(\Omega) \mid \|u\|_{L^q(\Omega)} = 1, \text{ and } \int_\Omega |\nabla u|^p \, dx = \tilde{S}_q \}.$$  

We say that $\tilde{S}_q(\Omega)$ is simple if the extremal functions associated with $\tilde{S}_q$ are scalar multiple one of the other. This is equivalent to say that $\tilde{E}_q = \{ \pm u_q \}$ for an $L^q$-normalized extremal $u_q \in W_0^{1,p}(\Omega)$. Recall that there is a long-standing conjecture that $\tilde{S}_q(\Omega)$ is simple if $\Omega$ is a bounded smooth convex domain in $\mathbb{R}^N$ and $1 < q < p^*$. Up to now, only several partial results are available for this conjecture, however, the complete solution has not been obtained. Ercole’s result is interesting since we can disprove the conjecture if we find $q$ such that $\tilde{S}_q'$ does not exist.

Main purpose of this paper is, in spite of the differences between $\tilde{S}_q$ and $\tilde{S}_q'$, to obtain similar regularity results and a characterization of differentiability of the function $[1, p^*) \ni q \mapsto S_q$. In what follows, $|A|$ stands for both the $N$-dimensional Lebesgue measure $\mathcal{L}^N(A)$ when $A \subseteq \Omega$ and the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}(A)$ when $A \subseteq \partial\Omega$. We hope that this abbreviation causes no ambiguity. $\|u\|_{L^q(\Omega)}$ and $\|u\|_{L^q(\partial\Omega)}$ denotes the $L^q$-norm of a function $u : \Omega \to \mathbb{R}$ and $u : \partial\Omega \to \mathbb{R}$ respectively. $\chi_A$ denotes a characteristic function of a set $A$.

2. Monotonicity and Bounded pointwise variation

In what follows, we fix $1 < p < N$ and put $p_* = \frac{(N-1)p}{N-p}$.
Concerning the monotonicity of $q \mapsto S_q$, first, we prove the following lemma:

**Lemma 2.1.** The function $q \mapsto |\partial \Omega|^{p/q} S_q$ is monotone decreasing on $[1, p_*]$. In particular, the function $q \in [1, p_*] \mapsto S_q$ is monotone decreasing if $|\partial \Omega| \leq 1$ and strictly monotone decreasing if $|\partial \Omega| < 1$.

**Proof.** Let $1 \leq q_1 < q_2 \leq p_*$. By Hölder’s inequality, we have

$$|\partial \Omega|^{p/q_2} \left( \int_{\partial \Omega} |u|^{q_2} \, dH^{N-1} \right)^{-p/q_2} \leq |\partial \Omega|^{p/q_1} \left( \int_{\partial \Omega} |u|^{q_1} \, dH^{N-1} \right)^{-p/q_1}.$$

Multiplying $\int_{\Omega} (|\nabla u|^p + |u|^p) \, dx$ to both sides and taking infimum, we see that $q \in [1, p_*] \mapsto |\partial \Omega|^{p/q} S_q$ is a monotone decreasing function. Thus

$$S_{q_1} \geq |\partial \Omega|^{(1/q_2 - 1/q_1)p} S_{q_2} > S_{q_2}$$

if $|\partial \Omega| < 1$. \qed

In Lemma 2.1, we see that the function $q \mapsto |\partial \Omega|^{p/q} S_q$ is strictly monotone decreasing on $[1, p_*]$ if $|\partial \Omega| < 1$. However, we can say more. In the next lemma, the Rayleigh quotient associated with the trace Sobolev embedding $W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega) \to \mathbb{R}$ is denoted by

$$R_q(u) = \frac{\int_\Omega (|\nabla u|^p + |u|^p) \, dx}{\left( \int_{\partial \Omega} |u|^q \, dH^{N-1} \right)^{\frac{p}{q}}} = \frac{\|u\|_{W^{1,p}(\Omega)}^p}{\|u\|_{L^q(\partial \Omega)}^q}.$$

**Lemma 2.2.** Let $u \in (W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)) \cap L^\infty(\partial \Omega)$, $u \neq$ constant. Then for each $1 \leq q_1 < q_2 \leq p_*$

$$|\partial \Omega|^{\frac{p}{q_1}} R_{q_1}(u) = |\partial \Omega|^{\frac{p}{q_2}} R_{q_2}(u) \exp \left( p \int_{q_1}^{q_2} \frac{K(t, u)}{t^2} \, dt \right) \quad (2.1)$$

where

$$K(t, u) = \frac{\int_{\partial \Omega} |u|^t \log |u|^t \, dH^{N-1}}{\|u\|_{L^t(\partial \Omega)}^t} + \log \left( \frac{|\partial \Omega|}{\|u\|_{L^t(\partial \Omega)}^t} \right) > 0 \quad (2.2)$$

Before the proof, we remark that the assumption of $u \in L^\infty(\partial \Omega)$ is used to assure the finiteness of the integral $\int_{\partial \Omega} |u|^p \log |u| \, dH^{N-1}$.

**Proof.** The proof will be done by differentiating $\log \left( \frac{|\partial \Omega|^{\frac{p}{q_1}}}{\|u\|_{L^q(\partial \Omega)}^q} \right)$ with respect to $t$. 


Fix $q_0 < p_*$ and consider $t \in [1, q_0]$. For $u \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)$, we have an estimate
\[
|u|^t \log |u| = \chi_{|u|\leq 1}|u|^t \log |u| + \chi_{|u|>1}|u|^t \log |u| \\
\leq \chi_{|u|\leq 1}(te)^{-1} + \chi_{|u|>1} \frac{1}{p_* - t} |u|^{p_*} \\
\leq e^{-1} + \frac{1}{p_* - q_0} |u|^{p_*} \in L^1(\partial \Omega),
\]
here we have used $x^t \log x \leq (te)^{-1}$ for $0 < x \leq 1$ and $|\log x| \leq \beta^{-1} x^\beta$ for any $x \geq 1$ and $\beta > 0$. Thus we see $|u|^t \log |u| \in L^1(\partial \Omega)$. Since $q_0$ can be chosen arbitrarily near to $p_*$, we may differentiate under the integral symbol to get
\[
\frac{d}{dt} \int_{\partial \Omega} |u|^t \, d\mathcal{H}^{N-1} = \int_{\partial \Omega} |u|^t \log |u| \, d\mathcal{H}^{N-1}
\]
for any $1 \leq t < p_*$ by Lebesgue’s dominated convergence theorem. Thus
\[
\frac{d}{dt} \left( \log \frac{|\partial \Omega|^t}{|u|^t L^1(\partial \Omega)} \right) = \frac{d}{dt} \left( \frac{1}{t} \log |\partial \Omega| \right) - \frac{d}{dt} \left( \frac{1}{t} \log \int_{\partial \Omega} |u|^t \, d\mathcal{H}^{N-1} \right) \\
= -\frac{1}{t^2} \log |\partial \Omega| + \frac{1}{t^2} \log \int_{\partial \Omega} |u|^t \, d\mathcal{H}^{N-1} - \frac{1}{t} \int_{\partial \Omega} |u|^t \log |u| \, d\mathcal{H}^{N-1} \\
= -\frac{K(t, u)}{t^2}.
\]
Integrate the above on $[q_1, q_2]$ with respect to $t$, we obtain
\[
\frac{|\partial \Omega|^\frac{t}{q_1}}{|u|^{q_1} L^1(\partial \Omega)} = \frac{|\partial \Omega|^\frac{t}{q_2}}{|u|^{q_2} L^1(\partial \Omega)} \exp \int_{q_1}^{q_2} \frac{K(t, u)}{t^2} \, dt
\]
Multiplying $|u|_{W^{1,p}(\Omega)}$, and taking $p$-th power, we get (2.1).

Next, we claim $K(t, u) > 0$. Define $h : [0, \infty) \to \mathbb{R}$ as
\[
h(\xi) = \begin{cases} 
\xi \log \xi & (\xi > 0) \\
0 & (\xi = 0).
\end{cases}
\]
Then \( h \) is convex, and Jensen’s inequality implies
\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} |u|^t \, d\mathcal{H}^{N-1} \leq \frac{1}{|\partial \Omega|} \int_{\partial \Omega} h(|u|^t) \, d\mathcal{H}^{N-1}
\]
\[
\Leftrightarrow |\partial \Omega|^{-1} \left( \int_{\partial \Omega} |u|^t \, d\mathcal{H}^{N-1} \right) \log \left( \int_{\partial \Omega} |u|^t \, d\mathcal{H}^{N-1} \right)
\]
\[
\leq |\partial \Omega|^{-1} \int_{\partial \Omega} |u|^t \log |u|^t \, d\mathcal{H}^{N-1}
\]
\[
\Leftrightarrow \int_{\partial \Omega} |u|^t \log |u|^t \, d\mathcal{H}^{N-1} \leq \log \left( \frac{|\partial \Omega|}{\|u\|_{L^t(\partial \Omega)}^t} \right) \geq 0
\]

By the equality cases for Jensen’s inequality (see [12]), if the equality holds for the above inequality, then \( |u|^t \) must be a constant, which is excluded. Thus the equalities do not hold and \( K(t, u) > 0 \).

\[\square\]

From Lemma 2.2, we easily see the next corollary:

**Corollary 2.3.** The function \( q \in [1, p_\ast] \mapsto |\partial \Omega|^{p/q} S_q \) is strictly monotone decreasing. In particular, The function \( q \in [1, p_\ast] \mapsto S_q \) is strictly monotone decreasing if \( |\partial \Omega| \leq 1 \).

**Proof.** Let \( 1 \leq q_1 < q_2 \leq p_\ast \) and let \( u_{q_1} \in W^{1, p}(\Omega) \setminus W^{1, p}_0(\Omega) \) denote an extremal function for \( S_{q_1} \). Then the regularity theorem assures that \( u_{q_1} \in C^\alpha(\overline{\Omega}) \) and \( u_{q_1} \) must not be a constant. It follows from Lemma 2.2 that
\[
|\partial \Omega|^{p/q_1} S_{q_1} = |\partial \Omega|^{p/q_2} R_{q_2}(u_{q_1}) \exp \left( p \int_{q_1}^{q_2} \frac{K(t, u_{q_1})}{t^2} \, dt \right)
\]
\[
> |\partial \Omega|^{p/q_2} R_{q_2}(u_{q_1})
\]
\[
\geq |\partial \Omega|^{p/q_2} S_{q_2}.
\]

The latter claim is trivial. \[\square\]

Let \( I \subset \mathbb{R} \) be an interval. In what follows, a finite set \( P = \{x_0, \cdots, x_n\} \subset I, x_0 < x_1 < \cdots < x_n \), is called a partition of \( I \). Following [10] Chapter 2, we say that a function \( f : I \to \mathbb{R} \) has bounded pointwise variation if
\[
\sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\} < \infty
\]
where the supremum is taken over all partitions \( P = \{x_0, \cdots, x_n\} \) of \( I, n \in \mathbb{N} \). The space of all functions \( f : I \to \mathbb{R} \) with bounded pointwise variation is denoted by \( BPV(I) \).

**Corollary 2.4.** The function \( q \mapsto S_q \) is in \( BPV(I) \) where \( I = [1, p_\ast] \).
Proof. Since a bounded monotone function on $I$ is in $BPV(I)$ ([10] Proposition 2.10), and the product of a bounded function and a function in $BPV(I)$ is again in $BPV(I)$, we have $S_q = (|\partial \Omega|^{|p/q}S_q)|\partial \Omega|^{-p/q}$ is in $BPV(I)$.

3. SOME ESTIMATES FOR THE EXTREMALS

First by utilizing level set techniques, we derive some pointwise estimates for any positive solution to (1.3).

Lemma 3.1. Let $u$ be a positive weak solution to (1.3) with $1 \leq q < p_*$. Then for any $\sigma \geq 1$, it holds
\begin{equation}
\left(\frac{1}{2}\right)^{\sigma+N-1} C_q \|u\|_{L^\infty(\partial\Omega)}^{-(N-1)(p-q)+(p-1)\sigma} \leq \|u\|^{\sigma}_{L^p(\partial\Omega)}
\end{equation}
where
\[ C_q = \left(\frac{S_{p_*}}{S_q}\right)^{\frac{N-p}{p-1}} N^{-\frac{N-1}{p-1}}. \]

Proof. As $u > 0$ solves (1.3) weakly, it holds
\begin{equation}
- \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx + S_q \int_{\partial\Omega} u^{p-1} \phi \, dH^{N-1} = \int_{\Omega} u^{p-1} \phi dx
\end{equation}
for all $\phi \in W^{1,p}(\Omega)$.

By a regularity theory (see [15], [17]), we may assume $u \in C^\alpha(\bar{\Omega})$ for some $0 < \alpha < 1$. Fix $t \in \mathbb{R}$ such that $0 < t < \|u\|_{L^\infty(\partial\Omega)}$. Put
\[ A_t = \{ x \in \Omega \mid u(x) > t \}, \quad a_t = \{ x \in \partial\Omega \mid u(x) > t \}. \]
We take the function
\[ \phi = (u - t)^+ \in W^{1,p}(\Omega), \quad \phi = \begin{cases} u - t & \text{in } A_t \cup a_t, \\ 0 & \text{otherwise} \end{cases} \]
in (3.2), then we have
\begin{equation}
- \int_{A_t} |\nabla u|^p dx + S_q \int_{a_t} u^{p-1}(u - t) \, dH^{N-1} = \int_{A_t} u^{p-1}(u - t) dx.
\end{equation}
Rewriting this, we have
\begin{equation}
\int_{A_t} (|\nabla u|^p + u^{p-1}(u - t)) \, dx = S_q \int_{a_t} u^{p-1}(u - t) \, dH^{N-1}
\leq S_q \|u\|_{L^\infty(\partial\Omega)}^{\sigma-1}(\|u\|_{L^\infty(\partial\Omega)} - t)|a_t|.
\end{equation}
Now, put
\[ g(t) = \int_{\partial\Omega} (u - t)^+ \, dH^{N-1} = \int_{a_t} (u - t) \, dH^{N-1}. \]
and recall the layer cake representation: Let \( v \geq 0 \) be a \( \mathcal{H}^{N-1} \)-measurable function on \( \partial \Omega \). Then for any \( \sigma \geq 1 \), it holds

\[
\int_{\partial \Omega} v^\sigma \, d\mathcal{H}^{N-1} = \sigma \int_0^\infty s^{\sigma-1} \mathcal{H}^{N-1}(\{ x \in \partial \Omega \mid v(x) > s \}) \, ds.
\]

Thus, we see

\[
g(t) = \int_0^\infty \mathcal{H}^{N-1}(\{ x \in \partial \Omega \mid (u + t)^+ > s \}) \, ds = \int_t^\infty |a_s| \, ds,
\]

here the last equality follows from a change of variables \( t + s \mapsto s \). This implies \( g'(t) = -|a_t| \). By Hölder’s inequality, (1.1) and (3.3), we have

\[
g(t)^p = \left( \int_{\partial \Omega} (u + t)^+ \, d\mathcal{H}^{N-1} \right)^p \\
\leq \left( \int_{\partial \Omega} \{(u + t)^+\}^{p*} \, d\mathcal{H}^{N-1} \right)^{\frac{p}{p*}} |a_t|^{p(1 - \frac{1}{p})} \\
\leq \frac{1}{S_{p*}} |a_t|^{p(1 - \frac{1}{p})} \int_{A_u} (|\nabla (u + t)^+|^p + \{(u + t)^+\}^p) \, dx \\
= \frac{1}{S_{p*}} |a_t|^{p(1 - \frac{1}{p})} \int_{A_u} (|\nabla u|^p + (u + t)^{p-1}(u - t)) \, dx \\
\leq \frac{1}{S_{p*}} |a_t|^{p(1 - \frac{1}{p})} \int_{A_u} (|\nabla u|^p + u^{p-1}(u - t)) \, dx \\
\leq \frac{S_q}{S_{p*}} \|u\|_{L^\infty(\partial \Omega)}^{-1} \|u\|_{L^\infty(\partial \Omega)} (\|u\|_{L^\infty(\partial \Omega)} - t) |a_t|^{p(1 - \frac{1}{p})+1} \\
= \frac{S_q}{S_{p*}} \|u\|_{L^\infty(\partial \Omega)}^{-1} \|u\|_{L^\infty(\partial \Omega)} (\|u\|_{L^\infty(\partial \Omega)} - t) (-g'(t)) \frac{N-1}{p-1},
\]

which results in

\[
\left[ \frac{S_q}{S_{p*}} \|u\|_{L^\infty(\partial \Omega)}^{-1} (\|u\|_{L^\infty(\partial \Omega)} - t) \right]^{-\frac{N-1}{p-1}} \leq -g(t) \frac{N-1}{p-1} g'(t). \tag{3.4}
\]

Changing a variable from \( t \) to \( s \), and integrating the both sides of (3.4) on \( [t, \|u\|_{L^\infty(\partial \Omega)}] \), we get

\[
C_q \|u\|_{L^\infty(\partial \Omega)}^{-\frac{(N-1)(q-1)}{p-1}} (\|u\|_{L^\infty(\partial \Omega)} - t)^N \leq g(t). \tag{3.5}
\]

Since \( g(t) \leq (\|u\|_{L^\infty(\partial \Omega)} - t) |a_t| \), we have from (3.5) that

\[
C_q \|u\|_{L^\infty(\partial \Omega)}^{-\frac{(N-1)(q-1)}{p-1}} (\|u\|_{L^\infty(\partial \Omega)} - t)^{N-1} \leq |a_t|. \tag{3.6}
\]
We multiply $\sigma t^{\sigma-1}$ to the both sides of (3.6) and integrate them on $[0, \|u\|_{L^\infty(\partial\Omega)}]$. Then the right hand side becomes $\|u\|_{L^p(\partial\Omega)}$ by layer cake representation. By changing variables $t \mapsto \|u\|_{L^\infty(\partial\Omega)},$ we observe

$$(LHS) = C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(q-1)}{p-1}} \sigma \int_0^{\|u\|_{L^\infty(\partial\Omega)}} t^{\sigma-1}(\|u\|_{L^\infty(\partial\Omega)} - t)^{N-1}dt$$

$$= C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(q-1)}{p-1}} \sigma \int_0^1 s^{\sigma-1}(1-s)^N ds$$

$$\geq C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(q-1)}{p-1}} \sigma \int_0^\frac{1}{2} s^{\sigma-1} 2^{-(N-1)} ds$$

$$= \left(\frac{1}{2}\right)^{\sigma+N-1} C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(q-1)}{p-1}}.$$

Thus we get the conclusion. \hfill \Box

By the Lemma 3.1, we have the uniform boundedness of the extremizers for the subcritical range.

**Lemma 3.2.** Let $\varepsilon > 0$ sufficiently small and let $u_\varepsilon$ be a positive $L^q(\partial\Omega)$-normalized extremal for $S_q$ where $1 \leq q \leq p_* - \varepsilon$. Then we have

$$|\partial\Omega|^{-1/q} \leq \|u_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C_\varepsilon$$

where $C_\varepsilon > 0$ is a constant which depends only on $\varepsilon > 0$.

**Proof.** Hölder’s inequality and the fact $\|u_\varepsilon\|_{L^q(\partial\Omega)} = 1$ yield the first inequality.

Next, suppose $1 \leq q \leq p$. Taking $\sigma = 1$ in (3.1), we have

$$\left(\frac{1}{2}\right)^N C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(q-1)+(p-1)}{p-1}} \leq \|u\|_{L^1(\partial\Omega)} \leq |\partial\Omega|^{1-1/q} \|u_\varepsilon\|_{L^q(\partial\Omega)} = |\partial\Omega|^{1-1/q}.$$

Thus

$$\|u_\varepsilon\|_{L^\infty(\partial\Omega)} \leq \max_{1 \leq q \leq p} \left(\frac{2^N |\partial\Omega|^{1-1/q}}{C_q} \right)^{\frac{p-1}{(N-1)(q-1)+(p-1)}} =: A.$$

If $p \leq q \leq p_* - \varepsilon$, then take $\sigma = q$ in (3.1) to obtain

$$\left(\frac{1}{2}\right)^{q+N-1} C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(q-1)+(p-1)q}{p-1}} \leq \|u\|_{L^q(\partial\Omega)}^q = 1.$$

Thus

$$\|u_\varepsilon\|_{L^\infty(\partial\Omega)} \leq \max_{p \leq q \leq p_* - \varepsilon} \left(\frac{2^{q+N}+1}{C_q} \right)^{\frac{(N-1)(q-1)+(p-1)q}{(p-1)}} =: B_\varepsilon.$$
since \((N-1)(p-q) + (p-1)q = (N-p)(p_* - q)\). Put \(C_\varepsilon = \max\{A, B_\varepsilon\}\).

By combining Lemma 3.2 and Proposition 2.7 in [7], we have the following fact:

**Proposition 3.3.** (Bonder-Rossi [7] Proposition 2.8.) The function \(q \in [1, p_*] \mapsto S_q\) is continuous.

For the proof, we refer the readers to [7].

4. Local Lipschitz and absolute continuity

In this section, by combining the arguments in [3] and [2], we prove the local Lipschitz continuity of \(S_q\) on \((1, p_*)\) and the absolute continuity of \(S_q\) on the whole closed interval \([1, p_*]\).

**Theorem 4.1.** The function \(q \mapsto S_q\) is locally Lipschitz continuous on the interval \((1, p_*)\).

**Proof.** Fix \(u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)\). Since \(x^t(\log |x|)^2 \leq (te)^{-2}\) for \(0 < x \leq 1\) and \(t > 0\), we see for \(1 \leq t \leq q_0 < p_*\),

\[
|u|^t (\log |u|)^2 = (\chi_{|u| \leq 1} + \chi_{|u| > 1}) |u|^t \log |u|^2 \\
= \chi_{|u| \leq 1} |u|^t \log |u|^2 + \chi_{|u| > 1} |u|^t \log |u|^2 \\
\leq \chi_{|u| \leq 1} (te)^{-2} + \chi_{|u| > 1} \frac{1}{p_* - t} |u|^{p_*} \\
\leq e^{-2} + \frac{1}{p_* - q_0} |u|^{p_*} \in L^1(\partial \Omega).
\]

Since \(q_0\) can be chosen arbitrarily close to \(p_*\), we have \(\|u\|_{L^q(\partial \Omega)}^q\) is at least twice differentiable and

\[
\frac{d^2}{dq^2}\|u\|_{L^q(\partial \Omega)}^q = \int_\Omega |u|^q (\log |u|)^2 \, dx \geq 0
\]

for any \(q \in (1, p_*)\) by dominated convergence theorem. Thus \(q \in (1, p_*) \mapsto \|u\|_{L^q(\partial \Omega)}^q\) is a convex function. Now, set

\[S = \{ u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \mid \|u\|_{W^{1,p}(\Omega)} = 1 \}\]

and define

\[h(q) = \sup_{u \in S} \|u\|_{L^q(\partial \Omega)}^q.\]

Since \(h\) is a supremum of convex functions \(\|u\|_{L^q(\partial \Omega)}^q\), it is also convex and locally Lipschitz continuous on \((1, p_*)\) (see [5] pp.236), which yields
that \(|h(q)| < \infty\) and \(|h'(q)| < \infty\) a.e. in \(q \in (1, p_*)\). Note that \(S_q = h(q)^{-\frac{1}{q}} = e^{-\frac{1}{q} \log h(q)}\), so

\[
S'_q = S_q \left(-\frac{1}{q} \log h(q)\right). 
\]

It is easy to see that \(h(q)\) is bounded from above and below by a positive constant on \(q \in (1, p_*)\). Thus

\[
|S'_q| = S_q \left|\left(\frac{1}{q} \log h(q)\right)\right| 
\leq S_q \left(\frac{1}{q^2} |\log h(q)| + \frac{1}{q} \left|\frac{h'(q)}{h(q)}\right|\right) < \infty \quad \text{a.e. in } (1, p_*)
\]

From this, we have the conclusion.

Theorem 4.2. The function \(q \mapsto S_q\) is absolutely continuous on the whole interval \([1, p_*]\).

Proof. Since we know that \(S_q\) is of bounded pointwise variation on \([1, p_*]\) by Corollary 2.4, we have

\[
S_q = S_1 = \int_1^q S'_t \, dt + S_C(q) + S_J(q)
\]

where \(S_C\) is the Cantor part of \(S_q\) and \(S_J\) is the jump part of \(S_q\), see [10] Theorem 3.73. Then the claim that \(S_q\) is absolutely continuous on \([1, p_*]\) is equivalent to \(S_C \equiv S_J \equiv 0\). Since \(S_q\) is continuous on \([1, p_*]\) by Proposition 3.3, we see that the discontinuous part \(S_J \equiv 0\). The Cantor part of \(S_q\), that is \(S_C\), is continuous, differentiable a.e., and \(S'_C(q) = 0\) a.e. \(q \in [1, p_*]\). Since \(S_q\) is Lipschitz continuous on any interval of the form \([1, p_* - \varepsilon]\), \(\varepsilon > 0\), it is absolutely continuous on the same interval, thus the support of \(S_C\) must be concentrated on \(\{p_*\}\). Therefore \(S_C \equiv 0\) since \(S_C\) is continuous at \(p_*\). 

5. A characterization of differentiability

Let us define the functional \(I_q : (W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)) \to \mathbb{R}\) as

\[
I_q(u) = \int_{\partial \Omega} |u|^q \log |u| \, dH^{N-1}
\]

and the set of \(L^q(\partial \Omega)\)-normalized extremal functions

\[
E_q = \{u \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega) \mid \|u\|_{L^q(\partial \Omega)} = 1, \|u\|^p_{W^{1,p}(\Omega)} = S_q\}
\]

for \(q \in [1, p_*]\).
Theorem 5.1. For each \( q \in [1, p_*) \) let \( u_q \) be arbitrarily chosen in \( E_q \). Then we have
\[
\limsup_{t \to q^+0} \frac{S_q - S_t}{q - t} \leq -\frac{p}{q} I_q(u_q) S_q \leq \liminf_{t \to q^-0} \frac{S_q - S_t}{q - t}.
\]
Therefore for \( q \in (1, p_*) \) on which \( S'_q \) exists, it holds
\[
S'_q + \frac{p}{q} I_q(u_q) S_q = 0.
\]
(5.1)

Proof. Take \( q \in (1, p_*) \) and let \( u_q \) an extremal for \( S_q \) in \( E_q \). Put
\[
J(t) = \int_{\partial \Omega} |u_q|^t \, d\mathcal{H}^{N-1}.
\]
Then we see \( J(q) = 1 \) and \( J'(t) \bigg|_{t=q} = \int_{\partial \Omega} |u_q|^q \log |u_q| \, d\mathcal{H}^{N-1} = I_q(u_q) \).
Since
\[
(J(t)^{p/t})' = J(t)^{p/t} \left( -\frac{p}{t^2} \log J(t) + \frac{p}{t} \frac{J'(t)}{J(t)} \right),
\]
we see
\[
\frac{d}{dt} \bigg|_{t=q} (J(t)^{p/t}) = \frac{p}{q} J'(t) \bigg|_{t=q} = \frac{p}{q} I_q(u_q).
\]
Also testing \( S_t \) by \( u_q \), we see
\[
S_q = \|u_q\|_{W^{1,p} (\Omega)}^{p} \geq S_t \left( \int_{\partial \Omega} |u_q|^t \, d\mathcal{H}^{N-1} \right)^{p/t} = S_t J(t)^{p/t}.
\]
Thus L’Hopital’s rule and the continuity of \( S_q \) imply that
\[
\limsup_{t \to q^+0} \frac{S_q - S_t}{q - t} \leq \limsup_{t \to q^+0} S_t \frac{J(t)^{p/t} - 1}{q - t}
= -S_q \lim_{t \to q^-0} \frac{d}{dt} \bigg|_{t=q} (J(t)^{p/t})
= -\frac{p}{q} I_q(u_q) S_q.
\]
The similar argument yields
\[
\liminf_{t \to q^-0} \frac{S_q - S_t}{q - t} \geq -\frac{p}{q} I_q(u_q) S_q.
\]

If \( S'_q \) exists for \( q \), the value \( S'_q \) is independent of the choice of \( u_q \in E_q \). Therefore, the above theorem implies that the value \( I_q(u_q) \) is also independent of the choice of \( u_q \in E_q \), which proves the next corollary. Indeed, \( I_q(u_q) = -\frac{q}{p} S'_q \) for any choice of \( u_q \) in \( E_q \).
Corollary 5.2. Let \( q \in (1, p_*) \) be such that \( S'_q \) exists. Then the functional \( I_q \) takes a constant value on \( E_q \); \( I_q(u_1) = I_q(u_2) \) for any \( u_1, u_2 \in E_q \).

Now, let us define \( f \) as

\[
    f(q) := \begin{cases} 
      \frac{p}{q} I_q(u_q) & \text{when } S'_q \text{ exists,} \\
      0 & \text{when } S'_q \text{ does not exist.}
    \end{cases} \tag{5.2}
\]

\( f \) is well-defined on \([1, p_*]\) by Corollary 5.2 and \( f(q) = \frac{S'_q}{S_q} \) when \( S'_q \) exists by (5.1).

We have a representation formula for \( S_q \) by using \( f \) in (5.2).

Theorem 5.3. It holds

\[
    S_q = S_1 \exp \left( - \int_1^q f(t) \, dt \right) \tag{5.3}
\]

for \( 1 \leq q \leq p_* \).

Proof. Since the function \( q \mapsto S_q \) is absolutely continuous on \([1, p_*]\) by Theorem 4.2, we have also the function \([1, p_*] \ni q \mapsto \log S_q \) is absolutely continuous. Thus by (5.1),

\[
    \log S_q - \log S_1 = \int_1^q \left( \frac{d}{dt} \log S_t \right) dt = \int_1^q \frac{S'_t}{S_t} dt = - \int_1^q f(t) dt
\]

for all \( q \in [1, p_*] \), which yields the result. \( \square \)

Theorem 5.3 implies also

\[
    S_q = S_1 \exp \left( - \int_1^{p_*} f(t) dt + \int_q^{p_*} f(t) dt \right) \\
    = S_1 \exp \left( - \int_1^{p_*} f(t) dt \right) \exp \left( \int_q^{p_*} f(t) dt \right) = S_{p_*} \exp \left( \int_q^{p_*} f(t) dt \right).
\]

As an immediate corollary of Theorem 5.3, we have the following:

Corollary 5.4. Let \( q \in [1, p_*) \) be a point of continuity of \( f \). Then \( \frac{d}{dq} S_q \) exists and

\[
    S'_q = -S_q f(q)
\]

holds.

Proposition 5.5. Suppose \( I_q \) is constant on \( E_q \) for some \( q \in [1, p_*) \). Then \( f \) is continuous on such \( q \). Especially \( f \) is continuous on \( q \) where \( S'_q \) exists.
Proof. Take $q \in [1, p_*)$ and a sequence $q_n \to q$ as $n \to \infty$. Since $q \mapsto S_q$ is continuous, we see $S_{q_n} \to S_q$. Also by elliptic regularity and the fact that $\|u_{q_n}\|_{L^\infty(\Omega)}$ is uniformly bounded in $n$, we have a subsequence (again denoted by $q_n$) and $u \in E_q$ such that $u_{q_n} \to u$ in $C^1(\overline{\Omega})$ and $\|u\|_{L^q(\partial\Omega)} = 1$. Therefore, we have

$$f(q_n) = \frac{p}{q_n} \int_{\partial\Omega} |u_{q_n}|^{q_n} \log |u_{q_n}| \, d\mathcal{H}^{N-1} \to \frac{p}{q} \int_{\partial\Omega} |u|^q \log |u| \, d\mathcal{H}^{N-1}$$

$$= \frac{p}{q} I_q(u) = \frac{p}{q} I_q(u_{q_n}) = f(q),$$

since $I_q(u) = I_q(u_{q_n})$ for $u, u_{q_n} \in E_q$. \hfill \Box

Now, we obtain a characterization of the differentiability of the function $q \mapsto S_q$.

**Theorem 5.6.** The following 3 assertions on a point $q \in [1, p_*)$ are equivalent:

(i) $S_q'$ exists.

(ii) $I_q$ is constant on $E_q$.

(iii) The function $t \in [1, p_*] \mapsto I_t(u_t)$ is continuous at $t = q$.

Proof. (i) $\implies$ (ii): Corollary 5.2.

(ii) $\implies$ (iii): Since the continuity of $f(t)$ at $t = q$ is equivalent to the continuity of $t \mapsto I_t(u_t)$ is continuous at $t = q$, the proof follows from Proposition 5.5.

(iii) $\implies$ (i): Corollary 5.4. \hfill \Box

It is known that $S_q$ is simple when $q = p$ and $E_p = \{ \pm u_p \}$ for some $u_p \in E_p$ ([13]). Thus we see $S_q' = \frac{d}{dq} S_q|_{q=p}$ exists and $t \mapsto I_t(u_t)$ is continuous at $t = p$. Also if $\Omega$ is a ball with sufficiently small radius and $p = 2$, then $S_q$ is simple for any $1 \leq q < 2_* = \frac{2(N-1)}{N-2}$ and the unique normalized extremizer for $S_q$ is radial (see [6] Theorem 2.1). Thus $q \mapsto S_q$ is differentiable on $1 \leq q < 2_*$ on small balls. Moreover the abstract approach using a variational principle in [9] could be applied to obtain the uniqueness of the positive solution of

$$\begin{cases} 
\Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^q u & \text{on } \partial\Omega,
\end{cases}$$

where $\lambda > 0$, $1 < p < N$ and $1 \leq q < p$. If this is the case, then we see that the function $q \mapsto S_q$ is differentiable for $1 \leq q < p$ on any bounded domain. However, the simplicity of $S_q$ for $p < q < p_*$ on a general bounded smooth domain is unknown.
Acknowledgments.

This work was supported by the Research Institute of Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University, and partly by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

The second author (F.T.) was supported by JSPS Grant-in-Aid for Scientific Research (B), No.19H01800.

References

