Classical Poincaré conjecture via 4D topology

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ABSTRACT

The classical Poincaré conjecture that every homotopy 3-sphere is diffeomorphic to the 3-sphere is proved by G. Perelman by solving Thurston's program on geometrizations of 3-manifolds. A new confirmation of this conjecture is given by combining R. H. Bing's result on this conjecture with Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture.

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1. Introduction

A *homotopy 3-sphere* is a smooth 3-manifold M homotopy equivalent to the 3-sphere S^3. It is well-known that a simply connected closed connected 3-manifold is a smooth homotopy 3-sphere. The following theorem, called the classical Poincaré Conjecture coming from [22, 23] is positively shown by Perelman [20, 21] solving positively Thurston's program [24] on geometrizations of 3-manifolds (see [19] for detailed historical notes).

Theorem 1.1. Every homotopy 3-sphere M is diffeomorphic to the 3-sphere S^3.

The purpose of this paper is to give an alternative proof to Theorem 1.1 by combining R. H. Bing's result in [2, 3] on the classical Poincaré conjecture with Smooth Unknotting Conjecture and Smooth 4D Poincaré Conjecture to be explained from now on. Let F be a smooth surface-link with a component system F_i, ($i =$

$1, 2, \ldots, n$) in the 4-sphere S^4. The fundamental group $\pi_1(S^4 \setminus F, v)$ (with v a base point) is a *meridian-based free group* if the group $\pi_1(S^4 \setminus F, v)$ is a free group with a basis represented by a meridian system m_i $(i = 1, 2, \ldots, n)$ of F_i, $(i = 1, 2, \ldots, n)$ with a base point v. The smooth surface-link F is a *trivial surface-link* if the components F_i, $(i = 1, 2, \ldots, n)$ bound a disjoint handlebody system smoothly embedded in S^4. Smooth Unknotting Conjecture for a surface-link is the following conjecture.

Smooth Unknotting Conjecture. Every smooth surface-link F in S^4 with a meridian-based free fundamental group $\pi_1(S^4 \setminus F, v)$ is a trivial surface-link.

The positive proof of this conjecture is claimed by [13, 15] with supplement [14]. The result when F is an S^2-link (i.e., a surface-link with only S^2-components) is applied in this paper. A *homotopy 4-sphere* is a smooth 4-manifold X homotopy equivalent to the 4-sphere S^4. Smooth 4D Poincaré Conjecture is the following conjecture.

Smooth 4D Poincaré Conjecture. Every 4D smooth homotopy 4-sphere X is diffeomorphic to the 4-sphere S^4.

The positive proof of this conjecture is claimed by [16, 17]. For the proof of Theorem 1.1, the following result of R. H. Bing in [2, 3] is used:

Bing's Theorem. A homotopy 3-sphere M is diffeomorphic to S^3 if, for every knot k in M, there is a 3-ball in M containing the knot k.

Thus, the main result of this paper is to prove the following lemma.

Lemma 1.2. For every knot k in M, there is a 3-ball in M containing the knot k.

For the proof of Lemma 1.2, Artin's spinning construction of a knot in S^3 in [1] is generalized into a connected graph in a homotopy 3-sphere M to produce a spun S^2-link in S^4 with free fundamental group (not always meridian-based free group). This explanation is done in Section 2. In Section 3, it is shown that every S^2-link in S^4 with free fundamental group is a ribbon S^2-link by using Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture. In Section 4, the proof of Lemma 1.2 is done. To do this, it is shown that the spun torus-knot of a knot in M is a ribbon-torus knot in S^4 which is a sum of the spun S^2-link of a proper arc system a_* in a boundary collar of a compact once-punctured manifold $M^{(o)}$ of M and the spun S^2-link of a proper arc system e_* in $M^{(o)}$ with meridian-based free fundamental group $\pi_1(M^{(o)} \setminus e_*, v)$. To see this, an argument of a chord diagram of the spun S^2-link of a proper arc system a_* in a boundary collar of $M^{(o)}$ in [12] is used. In this way, it is shown that the knot k is in a 3-ball of M completing the proof of Lemma 1.2 and the proof of Theorem 1.1 is completed.

Conventions. The unit n-disk is denoted by D^n with the origin $\mathbf{0}$ as a standard notation, but the unit 2-disk D^2 is fixed in the complex plane $\mathbb{C}$. A smooth n-manifold diffeomorphic to the unit n-disk D^n is called an *n-ball* for $n \geq 3$ or *n-disk* for $n = 2$. A point $\mathbf{1}$ is fixed in the n-sphere $S^n = \partial D^{n+1}$.

2. Artin's spinning construction of a connected graph in a homotopy 3-sphere

For a homotopy 3-sphere M, let $M^{(o)}$ be the compact once-punctured manifold $\mathrm{cl}(M \setminus B)$ of M for a 3-ball B in M. Let

$$S = \partial B = \partial M^{(o)}$$

be the boundary 2-sphere of $M^{(o)}$. The closed smooth 4-manifold $X(M)$ defined by

$$X(M) = M^{(o)} \times S^1 \cup S \times D^2$$

is called the *spun manifold* of M with *axis* 4-submanifold $S \times D^2$. As a convention, the 3-submanifold $M^{(o)} \times 1$ of the product $M^{(o)} \times S^1$ is identified with $M^{(o)}$. In particular, a point $(q, 1) \in M^{(o)} \times 1$ is identified with the point $q \in M^{(o)}$. This 4-manifold $X(M)$ is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument and hence $X(M)$ is diffeomorphic to the 4-sphere S^4 by Smooth 4D Poincaré Conjecture. A *legged loop* with *base point* v is the union $k \cup \omega$ of a loop k and an arc ω joining the base point v with a point of k. The arc ω is called the *leg*. A *legged loop system* with base point v is the union

$$\gamma = \cup_{i=1}^{n} k_i \cup \omega_i$$

of n legged loops $k_i \cup \omega_i \, (i = 1, 2, \ldots, n)$ meeting only at the same base point v. Let $k(\gamma) = \cup_{i=1}^{n} k_i = k_*$ denote the loop system of the legged loop system of γ. Let $\omega_* = \cup_{i=1}^{n} \omega_i$ and $v_* = k_* \cap \omega_*$. For a maximal tree τ of γ containing the base point v, a regular neighborhood B of τ in M with $\gamma \cap B$ a regular neighborhood of τ in γ is taken as 3-ball B used for the compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of M. Deform the subgraph $\gamma \cap B$ of γ so that

$$\omega_* \subset B, \quad \omega_* \cap S = \partial \omega_* \quad \text{and} \quad k_* \cap B = k_* \cap S = a'_*$$

for an arc system a'_* in k_*, where note that the base point v is moved into S. Let

$$a(\gamma) = \cup_{i=1}^{n} a_i = a_*$$

for a proper arc $a_i = \mathrm{cl}(k_i \setminus a'_i)$ $(i = 1, 2, \ldots, n)$ in $M^{(o)}$. Let

$$\dot{a}(\gamma) = \partial a_* = \partial a'_*$$

be the set of $2n$ points in the boundary 2-sphere S of $M^{(o)}$. The *spun* S^2-link of the graph γ is the S^2-link $S(\gamma)$ in the 4-sphere $X(M)$ defined by

$$S(\gamma) = a(\gamma) \times S^1 \cup \dot{a}(\gamma) \times D^2.$$

Lemma 2.1. The inclusion $M^{(o)} \setminus a(\gamma) \subset X(M) \setminus S(\gamma)$ induces an isomorphism

$$\sigma : \pi_1(M \setminus \gamma, v) \to \pi_1(X(M) \setminus S(\gamma), v)$$

sending a meridian system of the proper arc system $a(\gamma)$ in $M^{(o)}$ to a meridian system of $S(\gamma)$.

Proof of Lemma 2.1. Note that there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus a(\gamma), v) \cong \pi_1(M \setminus \gamma, v).$$

Then the desired isomorphism σ is obtained by applying the van Kampen theorem between $(M^{(o)} \setminus a(\gamma)) \times S^1$ and $(S \setminus \dot{a}(\gamma)) \times D^2$. This completes the proof of Lemma 2.1. □

Here is a note on Lemma 2.1.

Note 2.2. A general connected graph γ with Euler characteristic $\chi(\gamma) = 1 - n$ in M is deformed into a legged loop system γ in M by choosing a maximal tree to shrink to a base point v. Note that there are only finitely many maximal trees of γ such that the loop systems $k(\gamma)$ of the resulting legged loop systems γ are distinct as links. By Lemma 2.1, we can obtain finitely many distinct spun S^2-links in S^4 with isomorphic fundamental groups obtained by taking different maximal trees of the connected graph γ. This is a detailed explanation on the spun S^2-link of a connected graph associated with a maximal tree in [7, p.204] when $M = S^3$.

An argument on Lemma 2.1 is further developed when the homotopy 3-sphere M is given by a Heegaard spitting $V \cup V'$ pasting along a Heegaard surface $F = \partial V = \partial V'$ of genus n. A *spine* of a handlebody V of genus n is a legged loop system γ with base point v in $F = \partial V$ such that the inclusion map $\gamma \to V$ induces an isomorphism $\pi_1(\gamma, v) \to \pi_1(V, v)$. A regular neighborhood $\dot{V}$ of γ in F is a planar surface in F.

By [5, Theorem 10.2], there is a diffeomorphism $(\dot{V} \times [0,1], \dot{V} \times 0) \to (V, \dot{V})$ sending every point $(x,0) \in \dot{V} \times 0$ to $x \in \dot{V}$. The surface $\dot{V}$ is called a *spine surface* of V. Let γ and γ' be spines of the handlebodies V and V' with the same base point $v \in F$, respectively. A *legged Heegaard loop system* in M is the legged loop system $\gamma\gamma'$ in M with base point v obtained by pushing $\gamma \backslash v$ and $\gamma' \backslash v$ into the interiors Int$V$ and IntV', respectively. The fundamental groups of the spun S^2-links $S(\gamma\gamma') = S(\gamma) \cup S(\gamma)$, $S(\gamma)$ and $S(\gamma)$ in the 4-sphere $X(M)$ given by Lemma 2.1 are free groups, as shown in the following lemma:

Lemma 2.3. The fundamental groups $\pi_1(X(M) \setminus S(\gamma), v)$ and $\pi_1(X(M) \setminus S(\gamma'), v)$ are free groups of rank n and the fundamental group $\pi_1(X(M) \setminus S(\gamma\gamma'), v)$ is a free group of rank $2n$.

Proof of Lemma 2.3. The closed complements $\mathrm{cl}(M \setminus N(\gamma))$, $\mathrm{cl}(M \setminus N(\gamma'))$ and $\mathrm{cl}(M \setminus N(\gamma))$ are diffeomorphic to the handlebodies V', V and $F^{(o)} \times [0,1]$ for the once-punctured surface $F^{(o)}$ of F , respectively. Since the fundamental groups $\pi_1(V', v)$, $\pi_1(V, v)$ and $\pi_1(F^{(o)} \times [0,1], v)$ are free groups of ranks n, n and $2n$, respectively, the desired result is obtained from Lemma 2.1. □

It should be noted that these free groups in Lemma 2.3 are not necessarily meridian-based free groups. Here is an example.

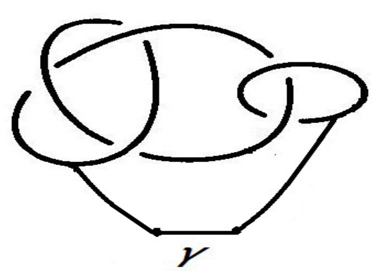

Figure 1: A legged loop system γ in S^3 with free fundamental group of rank 2

Example 2.4. Let γ be a legged loop system with base point v in S^3 illustrated in Fig. 1 with free fundamental group $\pi_1(S^3 \backslash \gamma, v)$ of rank 2. In fact, a trivial legged loop system is obtained by sliding an edge along another edge, so that the fundamental group $\pi_1(S^3 \setminus k(\gamma), v)$ is a free group of rank 2. A regular neighborhood V of γ in S^3 and the closed complement $V' = \mathrm{cl}(S^3 \setminus V)$ constitute a genus 2 Heegaard splitting

$V \cup V'$ of S^3 by noting that the 3-manifold V' is a handlebody of genus 2 by the loop system theorem and the Alexander theorem (cf. e.g., [7]). Thus, the union $V \cup V'$ is a genus 2 Heegaard splitting of S^3. The legged loop system γ with vertex v is a spine of V by sliding the base point v into ∂V. By Lemma 2.3, the spun S^2-link $S(\gamma)$ in the 4-sphere $X(S^3) = S^4$ has the free fundamental group $\pi_1(X(S^3) \setminus S(\gamma), v)$ of rank 2, which does not admit any meridian basis because the S^2-link $S(\gamma)$ contains a component of the spun trefoil S^2-knot in S^4 whose fundamental group is known to be not infinite cyclic.

Given a proper arc system a_* in $M^{(o)}$, there is a legged loop system γ in M with the proper arc system $a(\gamma) = a_*$ in $M^{(o)}$. The S^2-link $S(\gamma)$ in $X(M)$ is uniquely determined by the arc system a_* and thus denoted by $S(a_*)$. The following lemma is directly used for the proof of Lemma 1.2.

Lemma 2.5. Let a_* be a proper arc system in a compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of a homotopy 3-sphere M. If the S^2-link $S(a_*)$ in the 4-sphere $X(M)$ is a trivial S^2-link, then the proper arc system a_* is in a boundary-collar $S \times [0, 1]$ of $M^{(o)}$.

Proof of Lemma 2.5. By Lemma 2.1, the fundamental group $\pi_1(M^{(o)} \setminus a(\gamma), v)$ is a meridian-based free group. Consider the 2-sphere S is the boundary of the product $d \times [0, 1]$ for a disk d so that $d \times 0$ contains one end of the proper arc system a_* and $d \times 1$ contains the other end of the proper arc system a_*. Let $(E; E_0, E_1)$ be the triplet obtained from $(M^{(o)}, d \times 0, d \times 1)$ by removing a tubular neighborhood of a_* in $M^{(o)}$. Then the inclusion $E_0 \subset E$ induces an isomorphism

$$\pi_1(E_0, v) \to \pi_1(E, v).$$

By [5, Theorem 10.2], E is diffeomorphic to the connected sum of the product $E_0 \times [0, 1]$ and a homotopy 3-sphere. This means that the proper arc system a_* is in a boundary-collar $S \times [0, 1]$. This completes the proof of Lemma 2.5. □

3. Ribbonness of an S^2-link with free fundamental group The *4D handlebody* of *genus* n is the boundary 3-disk sum

$$Y^D = D^4 \natural_{i=1}^n S^1 \times D^3_i$$

obtained from n copies $S^1 \times D^3_i$ $(i = 1, 2, \ldots, n)$ of the 4D solid torus $S^1 \times D^3$ and the 4-disk D^4 by pasting a 3-disk system consisting of a boundary 3-disk in $(S^1 \setminus \{1\}) \times D^3_i$ for every i to a system of disjoint n boundary 3-disks of D^4. A legged loop system

γ^D in the 4D handlebody Y^D of genus n is *standard* if the legged loop system γ^D has the following two conditions:

- The loop system $k(\gamma^D)$ is consistent with the system $S^1 \times \mathbf{1}_i \, (i = 1, 2, \ldots, n)$, and
- The base point v is in the 4-disk D^4 and the legs $\omega_i \, (i = 1, 2, \ldots, n)$ of γ^D do not meet the 3-disks $1 \times D^3_i \, (i = 1, 2, \ldots, n)$.

Note that the legs $(i = 1, 2, \ldots, n)$ of γ^D are ∂-relatively unique up to isotopies in Y^D. The *4D closed handlebody* of *genus* n is the double of the 4D handlebody Y^D of genus n, that is the 4-manifold

$$\partial(Y^D \times [0,1]) = Y^D \times 0 \cup (\partial Y^D) \times [0,1] \cup Y^D \times 1$$

which is canonically identified with the following 4-manifold

$$Y^S = S^4 \#_{i=1}^n S^1 \times S^3_i,$$

where the connected summands S^3 and $S^1 \times S^3_i$ correspond to the doubles of the 3-disk summands D^4 and $S^1 \times D^3_i$, respectively. The 4D handlebody $Y^D \times 0$ in Y^S is identified with Y^D. A legged loop system γ with vertex v of the 4D closed handlebody Y^S of genus n is *standard* if it is v-relatively isotopic to a standard legged loop system γ^D of $Y^D \subset Y^S$. A standard legged loop system of Y^S is denoted by γ^S. A homology 4-sphere is a smooth 4-manifold X with an isomorphism $H_*(X; \mathbf{Z}) \cong H_*(S^4; \mathbf{Z})$. A *4D closed homology handlebody* of *genus* n is a smooth 4-manifold Y with an isomorphism $H_*(Y; \mathbf{Z}) \cong H_*(Y^S; \mathbf{Z})$ for the 4D closed handlebody Y^S of genus n. For an S^2-link L in X, take a normal disk bundle $L \times D^2$ in X and a 3-disk system D^3_L with $\partial D^3_L = L$. This transformation from X into the 4-manifold

$$Y = \mathrm{cl}(X \setminus L \times D^2) \cup D^3_L \times S^1$$

is called the *surgery* of X along the S^2-link L. Conversely, the transformation from Y into X is called the *surgery* of Y along the loop system $\mathbf{0}_* \times S^1$ by observing that $D^3_L \times S^1$ is a regular neighborhood of $\mathbf{0}_* \times S^1$ in Y. The following lemma is a more or less known fact.

Lemma 3.1. Let Y be the 4-manifold obtained from a homology 4-sphere X by surgery along any n-component S^2-link L. Then the 4-manifold Y is a 4D closed homology handlebody of genus n such that the inclusion $X \setminus L \times D^2 \subset Y$ induces an isomorphism

$$\pi_1(X \setminus L \times D^2, v) \to \pi_1(Y, v).$$

Proof of Lemma 3.1. To see that $H_2(Y;\mathbf{Z})=0$, use the Euler characteristic $\chi(Y)=2n$. Since $H_1(Y;\mathbf{Z})\cong\mathbf{Z}^n$, we have $H_2(Y;\mathbf{Z})=0$ by Poincaé duality, which shows that Y is a 4D closed homology handlebody of genus n. The isomorphism $i_*:\pi_1(X\setminus L\times D^2,v)\to\pi_1(Y,v)$ is obtained by a general position argument. □

A *meridian system* of an S^2-link L in X is a legged loop system γ_L in the closed complement $\mathrm{cl}(X\setminus L\times D^2)$ for a normal disk bundle $L\times D^2$ in X such that the loop system $k(\gamma_L)$ is the loop system $p_*\times S^1$ for a point system p_* in L with one point for every component of L. By Lemma 3.1, note that the meridian system γ_L induces a legged loop system γ in Y such that the loop system $k(\gamma)$ represents a homological basis of the homology group $H_1(Y;\mathbf{Z})$. Conversely, given any legged loop system γ in Y such that the loop system $k(\gamma)$ represents a homological basis of $H_1(Y;\mathbf{Z})$, then the 4-manifold X obtained from Y along the loop system $k(\gamma)$ is a homology 4-sphere and the legged loop system γ induces a meridian system γ_L of an S^2-link L in X. A *4D closed homotopy handlebody* of *genus* n is a 4D closed homology handlebody Y of genus n such that the fundamental group $\pi_1(Y,p)$ is a free group of rank n. A legged loop system γ with base point v in a 4D closed homotopy handlebody Y of genus n is a *basis system* if the inclusion $\gamma\subset Y$ induces an isomorphism

$$\pi_1(\gamma,v)\to\pi_1(Y,v).$$

For example, a standard legged loop system γ^S of the 4D closed handlebody Y^S is a basis system. The following classification lemma is a result of Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture.

Lemma 3.2. Let Y^S be the 4D closed handlebody of genus n, and γ^S a standard legged loop system with base point v^S of Y^S. For every 4D closed homotopy handlebody Y of genus n and every basis system γ in Y, there is an orientation-preserving diffeomorphism

$$f:Y\to Y^S$$

such that $f(\gamma)=\gamma^S$. Given any spin structures on Y and Y^S, the diffeomorphism f can be taken spin-structure-preserving.

Proof of Lemma 3.2. Let X be the 4-manifold obtained from Y by surgery along the loop system $k_*=k(\gamma)$. This 4-manifold X is diffeomorphic to the 4-sphere S^4 by Smooth 4D Poincaré Conjecture since it is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument. Since X is obtained from Y by replacing a normal disk bundle $k_*\times D^3$ of k_* in Y with $D^2_*\times S^2$ for the disk system D^2_* bounded by k_*. Then there is an S^2-link $L=0_*\times S^2$ in X. Since the basis system γ

of Y induces a meridian system of L in X, Lemma 3.1 implies that the fundamental group $\pi_1(X \setminus L, v)$ is a meridian based free group. By Smooth Unknotting Conjecture for an S^2-link, the S^2-link L is a trivial S^2-link in the 4-sphere X. By the back surgery replacing $D^2_* \times S^2$ in X with $k(\gamma) \times D^3$ in Y, there is an orientation-preserving diffeomorphism $f : Y \to Y^S$ with $f(k_*) = k(\gamma^S_*)$. Since a regular neighborhood $N(f(\gamma))$ of $f(\gamma)$ in Y^S is isotopic to Y^D in Y^S, the diffeomorphism $f : Y \to Y^S$ is modified to have $f(\gamma) = \gamma^S$. Given any spin structures on Y and Y^S, note that there is an orientation-preserving spin-structure-changing diffeomorphism : $S^1 \times S^3 \to S^1 \times S^3$ (see [4] for a similar diffeomorphism on $S^1 \times S^2$). Thus, by composing f with the orientation-preserving spin-structure-changing diffeomorphisms on some connected summands of Y^S which are copies of $S^1 \times S^3$, the diffeomorphism $f : Y \to Y'$ is modified into an orientation-preserving spin-structure-preserving diffeomorphism. This completes the proof of Lemma 3.2. □

The following corollary is directly obtained from Lemmas 2.3, 3.1 and 3.2.

Corollary 3.3. Let $\gamma\gamma'$ be a legged Heegaard loop system of a homotopy 3-sphere M associated with a Heegaard.splitting $V \cup V'$ of genus n, and $Y(M; \gamma\gamma')$ the 4D closed homology handlebody obtained from the 4-sphere $X(M)$ by surgery along the spun S^2-link $L(\gamma\gamma')$ of $\gamma\gamma'$. Then the 4D closed homology handlebody $Y(M; \gamma\gamma')$ is diffeomorphic to the 4D closed handlebody Y^S of genus $2n$.

A surface-link L in S^4 is a *ribbon* surface-link if L is equivalent to a surface-link obtained from a trivial S^2-link L^S in S^4 by surgery along embedded 1-handles on L^S (see [18]). The following lemma is obtained.

Lemma 3.4. Any S^2-link L in S^4 with free fundamental group $\pi_1(S^4 \setminus L, v)$ is a ribbon S^2-link.

Proof of Lemma 3.4. Let $K_i \,(i = 1, 2, \dots, n)$ be the components of L. Let Y be the 4-manifold obtained from S^4 by surgery along L. Let γ be a legged loop system in Y induced from a meridian system γ_L of L in S^4. Let $k(\gamma) = k_*$ be the loop system of γ in Y. The surgery manifold X of Y along k_* is identified with the 4-sphere S^4. In precise, let $X = \mathrm{cl}(Y \setminus N(k_*)) \cup D_* \times S^2$ for a regular neighborhood $N(k_*) = k_* \times D^3$ of k_* in Y and the disk system D_* with $\partial D_* = k_*$, where the 2-sphere system $0_* \times S^2$ is identified with L. By Lemma 3.2, Y is identified with the closed 4D handlebody Y^S of genus n. Let γ^S be a standard legged loop system of $Y = Y^S$ with the same vertex v as γ. Let $k(\gamma^S) = k^S_*$ be the loop system of γ^S in Y, which is disjoint from k_*. Let $x_i \,(i = 1, 2, \dots, n)$ be a basis of the free group $\pi_1(Y, v)$ of rank n represented by γ^S. Let $y_i\,(i=1,2,\dots,n)$ be an element system in $\pi_1(Y,v)$ represented by γ. By a basis change of the basis $x_i\,(i=1,2,\dots,n)$, assume that the product $x_i^{-1}y_i$ is in the commutator subgroup $[\pi_1(Y,v),\pi_1(Y,v)]$ of $\pi_1(Y,v)$ for every i. Let

$$Y^0 = \mathrm{cl}(Y \setminus N(k_*^S))$$

for a regular neighborhood $N(k_*^S) = k_*^S \times D^3$ of k_*^S in Y. Also, let

$$X^0 = \mathrm{cl}(X \setminus N(k_*^S))$$

by considering $N(k_*^S)$ in X. Since the loop system k_*^S is a trivial loop system in the 4-sphere X, there is a disjoint disk system Ω_* with $\partial\Omega_* = k_*^S$ smoothly embedded in X. Note that the intersection $N(k_*^S)\cap\Omega_*$ is a boundary collar of Ω_*. Let

$$\Omega'_* = \mathrm{cl}(\Omega_* \setminus (N(k_*^S)\cap\Omega_*)$$

which is a proper disk system in X^0. Let $S^1\times S^3_i = k_i^S \times S^3\,(i=1,2,\dots,n)$ be the connected summands of the closed 4D handlebody $Y=Y^S$. For every i, let $S^3_i = p_i \times S^3_i$ for a point $p_i \in k_i^S$. Let $V_i = S^3_i \cap Y^0$ be a 3-ball obtained from S^3_i by removing the interior of a 3-ball neighborhood of the point $p_i = p_i \times \mathbf{1}$ with $\partial V_i \subset \partial Y^0$. Let

$$Y^+ = Y^0 \cup_{i=1}^n \widetilde{\Omega}_i \times d$$

be the 4-manifold obtained from Y^0 by attaching 2-handles $\widetilde{\Omega}_i \times d\,(i=1,2,\dots,n)$ to the boundary $\partial Y^0 = \cup_{i=1}^n k_i^S \times S^2$ of Y^0 where $\widetilde{\Omega}_i$ is a disk with $\partial\widetilde{\Omega}_i = \partial\Omega'_i$ and a disk d in the 2-sphere S^2. Similarly, let

$$X^+ = X^0 \cup_{i=1}^n \widetilde{\Omega}_i \times d$$

be the 4-manifold obtained from X^0 by attaching 2-handles $\widetilde{\Omega}_i \times d\,(i=1,2,\dots,n)$ to the boundary ∂X^0 identical to ∂Y^0. Let (k_*^{S+}, p_*^+) be a moving of the pair (k_*^S, p_*) into the boundary pair $(\partial Y^0, \partial V_*)$. Let $k_i^{S+}\times[0,1]$ be an annulus in $k_i^{S+}\times S^2 \subset \partial Y^0$ for an arc $[0,1]$ in S^2. Consider that the element x_i^{-1} is represented by the loop $k_i^{S+}\times 0$ in Y^0. Since y_i is a word of the letters $x_j\,(j=1,2,\dots,n)$ in the fundamental group $\pi_1(Y,v)$, the element y_i is represented in Y^0 by a band sum k_i of the loop $k_i^{S+}\times 1$ and the boundary loop system ∂P_i of a disk system P_i consisting of suitably oriented parallel disks of $\widetilde{\Omega}_j$ in $\widetilde{\Omega}_j \times d\,(j=1,2,\dots,n)$ along a band system μ_i. Let b_i be a band in the anulus $k_i^{S+}\times[0,1]$ spanning the loop k_i^{S+} and the loop k_i with the centerline $\dot{b}_i = p_i^+ \times [0,1]$. Let k'_i be the loop in Y^0 obtained by a band sum of $k_i^{S+}\times 0$ and k_i along the band b_i. The union

$$\Delta_i = \mathrm{cl}(k_i^{S+}\times[0,1]\setminus b_i)\cup_{i=1}^n P_i \cup \mu_i$$

is considered as a disk smoothly embedded in Y^+ whose boundary loop $\partial\Delta_i$ represents the element $x_i^{-1}y_i$ in Y^0. Further, the disk system $\Delta_i\,(i=1,2,\dots,n)$ is made disjoint. By construction, the disk Δ_i meets the 3-ball system V_* only with the isolated finite point set $P_i\cap\partial V_*$ and with simple proper arcs $\beta_{i,j}\,(j=1,2,\dots,n_i)$ in Δ_i coming from the transverse intersection of the band system μ_i and the interior $\mathrm{Int}V_*$ of the 3-ball system V_*. Let $B_{i,j}\,(j=1,2,\dots,n_i)$ be disjoint 3-ball neighborhoods of the arcs $\beta_{i,j}\,(j=1,2,\dots,n_i)$ in $\mathrm{Int}V_i$, and $S_{i,j}\,(j=1,2,\dots,n_i)$ the boundary 2-spheres of $B_{i,j}\,(j=1,2,\dots,n_i)$. Then the following claim (#) is obtained.

(#) The S^2-link $\cup_{i=1}^n\cup_{j=1}^{n_i}S_{i,j}$ in Y becomes a trivial S^2-link in the 4-sphere X after the surgery of Y along the loop system k_*.

By assuming the proof of the claim (#), the proof of Lemma 3.4 is completed as follows. Let $(S^3)_i^{(*)}$ be a multi-punctured 3-ball obtained from S_i^3 by removing the interiors of the 3-balls $B_{i,j}\,(j=1,2,\dots,n_i)$ and a 3-ball neighborhood $N(q_i)=q_i\times D^3$ of the point $q_i=p_i^+\times 1\in k_i$ in V_i. Note that the S^2-link $\cup_{i=1}^n\partial N(q_i)$ in Y changes into the S^2-link $L=\cup_{i=1}^nK_i$ in X after the surgery of Y along k_*. Since K_i is equivalent to a 2-sphere in $(S^3)_i^{(*)}$ obtained from the trivial S^2-link $\partial V_i\cup_{i=1}^n\cup_{j=1}^{n_i}S_{i,j}$ in X by surgery along disjoint embedded 1-handles in $(S^3)_i^{(*)}$, it is shown that the S^2-link L is a ribbon S^2-link in the 4-sphere X. This completes the proof of Lemma 3.4 assuming the claim (#).

Proof of (#). Let V_*' be the 3-ball system obtained from the 3-ball system V_* by removing an open boundary collar which remains containing all the arcs $\beta_{i,j}$, so that $V_*'\cap\widetilde{\Omega}_j=\emptyset$. Since every arc $\beta_{i,j}$ splits the disk Δ_h containing the arc $\beta_{i,j}$ into two regions, there is an arc $\beta_{i',j'}$ such that a region Δ_h' of the disk Δ_h splitted by the $\beta_{i',j'}$ does not contain any other arc $\beta_{i'',j''}$ and does not meet the arc system $b_*\cap k_*$. The boundary of a regular neighborhood relative to V_*' of the region Δ_h' in Y^+ is a 3-sphere containing the 3-ball $B_{i',j'}$ whose complementary 3-ball is denoted by $\widetilde{B}_{i',j'}$. Let V_*'' be the 3-ball system obtained from V_*' by replacing the 3-ball $B_{i',j'}$ with the 3-ball $\widetilde{B}_{i',j'}$. Then $V_*''\cap\Delta_h'=\emptyset$. Continue this process on V_*'' instead of V_*'. Finally, a system of disjoint 3-balls $\widetilde{B}_{i,j}\,(i=1,2,\dots,n;j=1,2,\dots,n_i)$ bounded by the 2-spheres $S_{i,j}\,(i=1,2,\dots,n;j=1,2,\dots,n_i)$ and a 3-ball system V_*''' disjoint from the union $\Delta_*\cup b_*$ are obtained in Y^+. Consider that X^+ is obtained from Y^+ by a surgery along a loop system k_*^+ disjointedly parallel to the loop system k_* in Y^+ so that k_*^+ is in the interior $\mathrm{Int}(Y^0)$ of Y^0 and disjoint from the disk system Δ_*. The disk system Δ_* is now embedded into X^+ and the 3-ball $\widetilde{B}_{i,j}$ for any i,j is embedded into a regular neighborhood of Δ_* in the 4-manifold $\mathrm{cl}(Y^+\setminus N(k_*^+))=\mathrm{cl}(X^+\setminus N(L))$. Since the band system μ_i except for the attaching part is made disjoint from the disk system Ω'_*, the loop system k^+_* is made disjoint from the disk system Ω'_*. For a normal disk bundle $\Omega'_* \times d$ of Ω'_* in $\mathrm{cl}(Y^0 \setminus N(k^+_*)) = \mathrm{cl}(X^0 \setminus N(L))$, the union $U = \Omega'_* \times d \cup \widetilde{\Omega}_* \times d = (\Omega'_* \cup \widetilde{\Omega}_*) \times d$ in $\mathrm{cl}(Y^+ \setminus N(k^+_*)) = \mathrm{cl}(X^+ \setminus N(L))$ is diffeomorphic to the product $S^2 \times d$ and the intersection $U \cap \Delta_*$ coincides with the disk system P_*. By an isotopy of X^+ keeping U setwise fixed and keeping the outside of a neighborhood of U in X^+ fixed, the disk system P_* is deformed into a disk system P^X_* in $\Omega'_* \times d \subset X^0$, so that the disk system Δ_* is deformed into a disk system Δ^X_* in $\Omega'_* \times d \subset X^0$. Since the 3-ball $\widetilde{B}_{i,j}$ for any i, j is embedded in a regular neighborhood of Δ_* in the 4-manifold X^+, the 3-ball system $\widetilde{B}_{i,j}$ is isotopically deformed into a 3-ball system$\widetilde{B}^X_{i,j}$ in X^0 while the 2-spheres $S_{i,j}\,(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n_i)$ are fixed. This means that the 2-spheres $S_{i,j}\,(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n_i)$ are a trivial S^2-link in the surgery manifold X. This completes the proof of (#). □

This completes the proof of Lemma 3.4. □

A group presentation $(y_1, y_2, \ldots, y_{n+s} | r_1, r_2, \ldots, r_s)$ of deficiency n is a *Wirtinger presentation* if every relator r_i is written as a form $y_{j_i}^{-1} w_j y_{j'_i} w_i^{-1}$ for two generators $y_j j_i, y_{j'_i}$ with distinct indexes j_i, j'_i and a word w_i in the letters $y_j\,(j = 1, 2, \ldots, n+s)$. It is known that the fundamental group of an n-component ribbon S^2-link has a Wirtinger presentation of deficiency n for some s (cf. [7, p. 193], [18, pp. 56-60]). An algebraic version of Lemma 3.4 means the following result in combinatorial group theory.

Corollary 3.5. Let $\mathbf{F}_n$ be the free group of rank n with a basis $x_i\,(i = 1, 2, \ldots, n)$. Let $x'_i\,(i = 1, 2, \ldots, n)$ be a set of elements normally generating the free group $\mathbf{F}_n$ written as words in the letters $x_i\,(i = 1, 2, \ldots, n)$ such that the products $x'_i x_i^{-1}\,(i = 1, 2, \ldots, n)$ belong to the commutator subgroup $[\mathbf{F}_n, \mathbf{F}_n]$ of $\mathbf{F}_n$. Then the free group $\mathbf{F}_n$ admits a Wirtinger presentation

$$(y_1, y_2, \ldots, y_{n+s} | r_1, r_2, \ldots, r_s)$$

of deficiency n for some s such that the elements $y_i\,(i = 1, 2, \ldots, n+s)$ are written as words in the letters $x_i\,(i = 1, 2, \ldots, n)$ containing the elements $x'_i\,(i = 1, 2, \ldots, n)$ as the given words.

4. Main result: Proof of Lemma 1.2

The following observation relates a knot to a Heegaard splitting of a closed connected orientable 3-manifold.

Lemma 4.1. For any knot k in any closed connected orientable 3-manifold M, there is a Heegaard splitting $V \cup V'$ of M such that the knot k is equivalent to a component of the loop system $k(\gamma)$ of a spine γ of V in M.

Proof of Lemma 4.1. By considering k as a polygonal loop in M, there is a triangulation $\mathcal{T}$ of M whose 1-skeleton $\mathcal{T}^{(1)}$ contains the knot k. The graph $\mathcal{T}^{(1)}$ is deformed into a legged loop system γ in M so that k is a component of the loop system $k(\gamma)$. Let V be a regular neighborhood of γ in M which is a handlebody. The closed complement $V' = \mathrm{cl}(M \setminus V)$ is also a handlebody, so that we have a Heegaard splitting $V \cup V'$ of M. The legged loop system γ is deformed into a spine of the handlebody V. □

By combining Lemmas 2.3, 3.4 with Lemma 4.1, the following corollary is obtained, because any component of a ribbon S^2-link in S^4 is a ribbon S^2-knot in S^4.

Corollary 4.2. For any knot k in any homotopy 3-sphere M, the spun-S^2-knot $S(k)$ of k in $X(M) = S^4$ is a ribbon S^2-knot in S^4.

A chord diagram is a diagram C in S^2 consisting of a based loop system o (i.e., a trivial oriented link diagram) and a chord system α joining the based loops where intersections among the chords are permitted (see [8, 9, 10, 11, 12] for the detailed arguments). For a disk δ in S^2, a *chord diagram* in the delta δ is the intersection $C \cap \delta$ for a chord diagram $C = C(o, \alpha)$ in S^2 such that the circle $\partial\delta$ does not meet the based loop system o and meets the chord system α transversely. From a chord diagram $C = C(o, \alpha)$ in S^2, a ribbon surface-link $R(C)$ in the 4-sphere S^4 is constructed in a unique way. In fact, the ribbon surface-link $R(C)$ is obtained from a trivial oriented S^2-link L^0 in S^4 constructed from the based loop system o by surgery along an embedded 1-handle system $h(\alpha)$ on L^0 thickening the chord system α. The ribbon surface-link $R(C)$ in S^4 is uniquely constructed from the chord diagram C by using the Horibe-Yanagawa's lemma in [18] for uniqueness of the trivial S^2-link L^0 constructed from the based loop system o and an argument in [6] for uniqueness of the embedded 1-handle system $h(\alpha)$ constructed from the chord system α.

Lemma 4.3. Let a_* be a proper oriented arc system in a compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of a homotopy 3-sphere M which is obtained from an oriented proper arc diagram D in a disk δ contained in the boundary 2-sphere S of $M^{(o)}$ by pushing the interior of an upper-arc around every crossing point of D into the interior of $M^{(o)}$. Then the S^2-link $S(a_*)$ in $X(M)$ is a ribbon S^2-link in $X(M)$ with a chord diagram C in δ obtained from the arc diagram D by changing every crossing point as in Fig. 2.

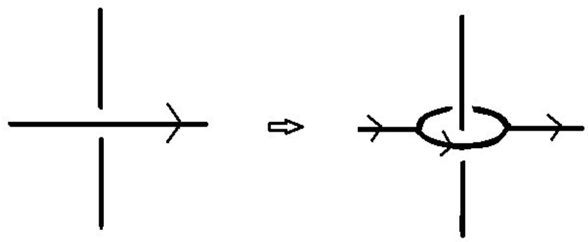

Figure 2: Changing a crossing point into a based loop with chords

Proof of Lemma 4.3. This fact is observed in [12, Theorem 2.3 (3)] for an inbound arc diagram whose closure is a knot chord diagram. The present claim is similarly shown for any oriented arc diagram. □

In Lemma 4.3, note that the arc diagram D is recovered from the chord diagram C by taking the upper-arc of every based loop. The proof of Lemma 1.2 is given as follows.

4.4: Proof of Lemma 1.2. Let k be a non-trivial knot in a homotopy 3-sphere M. By Corollary 4.2, the spun S^2-knot $S(k)$ in the 4-sphere $X(M) = S^4$ is a ribbon S^2-knot. The *spun torus-knot* of k in the 4-sphere $X(M)$ is given by the inclusion

$$T(k) = k \times S^1 \subset M^{(o)} \times S^1 \subset M^{(o)} \times S^1 \cup S \times D^2 = X(M).$$

The spun S^2-knot $S(k)$ in $X(M)$ is obtained from $T(k)$ by a 2-handle surgery and conversely the spun torus-knot $T(k)$ is obtained from the spun S^2-knot $S(k)$ by 1-handle surgery. By definition, the spun torus-knot $T(k)$ is a ribbon torus-knot and hence bounds a ribbon solid torus V_R in $X(M)$. Let

$$V_R = \cup_{i=1}^n B_i \cup h_i$$

for a disjoint 3-ball system $B_i\,(i = 1, 2, \ldots, n)$ in $X(M)$ and an embedded disjoint 1-handle system $h_i\,(i = 1, 2, \ldots, n)$ on the 2-sphere system $\partial B_i\,(i = 1, 2, \ldots, n)$ in $X(M)$ so that the 1-handle h_i spans ∂B_i and ∂B_{i+1} for every i with $B_{n+1} = B_1$ and every 3-ball B_i meets just one 1-handle h_{j_i} for some $j_i\,(1 \leq j_i \leq n)$ with a transverse disk d_{j_i} in the interior of B_i. Since the knot k is non-trivial in $M^{(o)}$ and there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus k, v) \to \pi_1(X(M) \setminus T(k), v)$$

by the van Kampen theorem, the longitude of k in $M^{(o)}$ represents an infinite order element in the fundamental group $\pi_1(X(M)\setminus T(k), v)$, which implies that the meridian loop of V_R (i.e., the simple loop of $T(k)$ bounding a meridian disk of V_R) is a uniquely specified loop in $T(k)$ up to isotopies of $T(k)$. Fix an orientation of knot k. Then by the construction of $T(k)$, the meridian disk orientation of the ribbon solid torus V_R is uniquely specified and the ribbon solid torus V_R specifies uniquely a disjoint oriented deformed meridian disk system $d_i\,(i = 1, 2, \ldots, n)$ in V_R so that the knot k meets the disk d_i with just one boundary arc orientation-coherently and just one interior point transversely and the union $k \cup_{i=1}^{n} d_i$ (called a *chord-disk system*) recovers V_R uniquely by thickening k and $d_i\,(i = 1, 2, \ldots, n)$ (see the left figure of Fig. 3). The disk system $d_i\,(i = 1, 2, \ldots, n)$ is isotopically deformed into $M^{(o)}$ by an isotopy of $X(M)$ keeping k fixed, so that the chord-disk system $k \cup_{i=1}^{n} d_i$ is in $M^{(o)}$. To show this claim, let α_i be a simple arc in d_i joining the point $k \cap \mathrm{Int} d_i$ with a point in the arc $k \cap \partial d_i$ for all i. The arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ is deformed into a bi-collar neighborhood $M^{o)} \times [-1, 1]$ of $M^{(o)}$ with $M^{(o)} \times 0 = M^{(o)}$ in $X(M)$ by an isotopy keeping $M^{(o)}$ fixed. Then the arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ is projected into $M^{(o)}$ by a general position argument. A deformed disk system $d_i\,(i = 1, 2, \ldots, n)$ in $M^{(o)}$ is obtained from the arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ in $M^{(o)}$ by extending them as a small disk system, completing the proof of the claim. Let $k^{\times}$ be the graph in $M^{(o)}$ obtained from the chord-disk system $k \cup_{i=1}^{n} d_i$ by shrinking every disk d_i into a 4-degree vertex for every i. By taking a maximal tree $\tau(k^{\times})$ of $k^{\times}$, one finds a disk δ in $M^{(o)}$ containing the maximal tree $\tau(k^{\times})$. Let $e_i\,(i = 1, 2, \ldots, n+1)$ be the arc system $\mathrm{cl}(k^{\times} \setminus \tau(k^{\times}))$ where the number $n + 1$ is uniquely determined by the Euler characteristic $\chi(K^{\times}) = -n$. Then the chord-disk system

$$k^{\times\times} = \mathrm{cl}((k \cup_{i=1}^{n} d_i) \setminus (\cup_{i=1}^{n+1} e_i))$$

can be drawn as a chord diagram C in the disk δ with the based loop system $o_i = \partial d_i\,(i = 1, 2, \ldots, n)$ so that the chord diagram of the two arcs of k on the disk d_i for every i are drawn with the two arcs as bold lines transversely meeting as in the right figure of Fig. 3. Let $a_i\,(i = 1, 2, \ldots, n{+}1)$ be the arc system $\mathrm{cl}(k \setminus \cup_{i=1}^{n+1} e_i)$. By replacing the chord diagram of the two arcs of k on the disk d_i for every i with an arc diagram, that is, by replacing the right diagram of Fig. 2 with the left diagram of Fig. 2, the diagram C changes into an arc diagram D of the arc system $a_i\,(i = 1, 2, \ldots, n)$ in the disk δ. Deform the disk δ into the 2-sphere $S = \partial M^{(o)}$ so that a collar $\delta \times [0, 1]$ of δ in $M^{(o)}$ with $\delta \times 0 = \delta$ belongs to a boundary collar $S \times [0, 1]$ of S in $M^{(o)}$ with $S \times 0 = S$. The arc system $a_i\,(i = 1, 2, \ldots, n)$ is realized in the collar $\delta \times [0, 1]$

from the arc diagram D by pushing the interiors of the upper-arcs of D into the interior of $\delta \times [0,1]$. By Lemma 4.3, the spun S^2-link $\cup_{i=1}^n S(a_i)$ in $X(M)$ with the chord system C in δ is obtained as in Fig. 2. This means that the spun S^2-link $\cup_{i=1}^n S(a_i)$ bounds a part V'_R of the ribbon solid torus V_R belonging to the 4-ball $A = (\delta \times [0,1]) \times S^1 \cup \delta \times D^2$ in $X(M)$. Since the spun torus-knot $T(k)$ is the union of the spun S^2-link $\cup_{i=1}^n S(e_i)$ and the spun S^2-link $\cup_{i=1}^n S(a_i)$ by deleting the common disk interiors, the spun S^2-link $\cup_{i=1}^n S(e_i)$ in $X(M)$ bounds disjoint 3-balls $\mathrm{cl}(V_R \setminus V'_R)$ in the 4-ball $A' = \mathrm{cl}(X(M) \setminus A)$. Let $X'(M)$ be the spun 4-sphere of M on the once-punctured manifold $M^{(o)}_\delta = \mathrm{cl}(M^{(o)} \setminus \delta \times [0,1])$ of M, and $S' = \partial M^{(o)}_\delta$ the boundary 2-sphere. The spun S^2-link $\cup_{i=1}^n S(e_i)$ is a trivial S^2-link in the 4-sphere $X'(M)$. By Lemma 2.5, the proper arc system e_i $(i = 1, 2, \dots, n)$ is in a boundary-collar $S' \times [0,1]$ of the once-punctured manifold $M^{(o)}_\delta$. This means that there is a 3-ball in $M^{(o)}$ containing the knot k. This completes the proof of Lemma 1.2. □

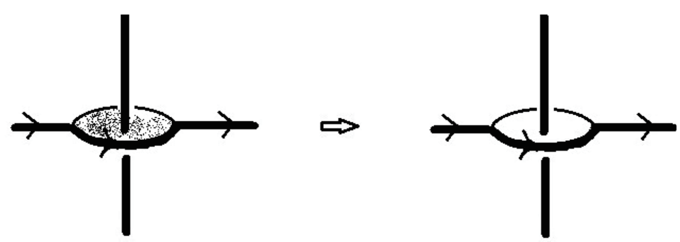

Figure 3: A diagram of the two arcs of k on the disk d_i

This completes the proof of Theorem 1.1.

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