Hurwitz integrality of the power series expansion of the sigma function for telescopic curves

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Abstract

A telescopic curve is a certain algebraic curve defined by $m-1$ equations in the affine space of dimension m, which can be a hyperelliptic curve and an (n,s) curve as a special case. The sigma function $\sigma(u)$ associated with a telescopic curve of genus g is a holomorphic function on $\mathbb{C}^g$. For a subring R of $\mathbb{C}$ and variables $u={}^t(u_1,\dots,u_g)$, let

$$R\langle\langle u\rangle\rangle=\left\{\sum_{k_1,\dots,k_g\geq 0}\zeta_{k_1,\dots,k_g}\frac{u_1^{k_1}\cdots u_g^{k_g}}{k_1!\cdots k_g!}\ \middle|\ \zeta_{k_1,\dots,k_g}\in R\right\}.$$

If the power series expansion of a holomorphic function $f(u)$ on $\mathbb{C}^g$ around the origin belongs to $R\langle\langle u\rangle\rangle$, then $f(u)$ is said to be Hurwitz integral over R. In this paper, we show that the sigma function $\sigma(u)$ associated with a telescopic curve is Hurwitz integral over the ring generated by the coefficients of the defining equations of the curve and $\frac{1}{2}$ over $\mathbb{Z}$, and its square $\sigma(u)^2$ is Hurwitz integral over the ring generated by the coefficients of the defining equations of the curve over $\mathbb{Z}$. Our results are a generalization of the results of Y. Ônishi for the (n,s) curves to the telescopic curves.

1 Introduction

The Weierstrass's elliptic sigma function plays important roles in the theory of the Weierstrass's elliptic function. F. Klein [23, 24] generalized the Weierstrass's elliptic sigma function to the multivariate sigma function associated with the hyperelliptic curves. V. M. Buchstaber, V. Z. Enolski, and D. V. Leykin developed the theory of the Klein's hyperelliptic sigma function and generalized it to the more general plane

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algebraic curves called (n, s) curves (e.g. [10, 11, 12, 13, 14, 15, 16, 17, 20]). The sigma function is obtained by modifying the Riemann's theta function so as to be modular invariant, i.e., it does not depend on the choice of a canonical homology basis. Further the sigma function has some remarkable algebraic properties that it is directly related with the defining equations of an algebraic curve. Namely, the coefficients of the power series expansion of the sigma function around the origin become polynomials of the coefficients of the defining equations of the algebraic curve. This property is important in the study of differential structure of Abelian functions (cf. [19, 28]). Further, from this property of the sigma function, the sigma function has a limit when the coefficients of the defining equations of a curve are specialized in any way, which is important in the study of integrable systems (cf. [7, 29]). It is the central problem to determine the coefficients of the power series expansion of the sigma function. This problem is studied in many papers (e.g. [5, 8, 9, 16, 21, 22, 26, 27, 32]).

Throughout the present paper, we denote the sets of positive integers, non-negative integers, integers, rational numbers, and complex numbers by $\mathbb{N}, \mathbb{Z}_{\geq 0}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{C}$, respectively. For a subring R of $\mathbb{C}$ and a set of some complex numbers $\mathcal{A}$, we denote by $R[\mathcal{A}]$ the ring generated by elements in $\mathcal{A}$ over R. For positive integers $k_1, \dots, k_n$, let $\langle k_1, \dots, k_n \rangle = \{\ell_1 k_1 + \cdots + \ell_n k_n \mid \ell_1, \dots, \ell_n \in \mathbb{Z}_{\geq 0}\}$ and we denote by $\gcd(k_1, \dots, k_n)$ the greatest common divisor of $k_1, \dots, k_n$. For a subring R of $\mathbb{C}$ and variables $z = {}^t(z_1, \dots, z_n)$, let

$$R\langle\langle z \rangle\rangle = R\langle\langle z_1, \dots, z_n \rangle\rangle = \left\{ \sum_{k_1, \dots, k_n \geq 0} \zeta_{k_1, \dots, k_n} \frac{z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!} \;\middle|\; \zeta_{k_1, \dots, k_n} \in R \right\}.$$

If the power series expansion of a holomorphic function $f(z) = f(z_1, \dots, z_n)$ on $\mathbb{C}^n$ around the origin belongs to $R\langle\langle z \rangle\rangle$, then we write $f(z) \in R\langle\langle z \rangle\rangle$ and $f(z)$ is said to be *Hurwitz integral* over R.

For relatively prime positive integers n and s such that $n, s \geq 2$, the (n, s) curve is the algebraic curve defined by the following equation in $\mathbb{C}^2 = (x, y)$

$$y^n = x^s + \sum_{ni+sj<ns} \lambda_{i,j} x^i y^j, \quad \lambda_{i,j} \in \mathbb{C}$$

(cf. [12]). The $(2, s)$ curves are equal to the hyperelliptic curves. The sigma function $\sigma(u)$ associated with an (n, s) curve of genus g is a holomorphic function on $\mathbb{C}^g$. We denote by $\{\lambda_{i,j}\}$ the set of all $\lambda_{i,j}$. In [26], the expression of the sigma function associated with the (n, s) curves in terms of the prime function and algebraic functions is derived. In [27], the expression of the sigma function associated with the (n, s) curves in terms of the tau function of the KP-hierarchy is derived. In [26, 27], by using these expressions of the sigma function of the (n, s) curves, it is proved that $\sigma(u) \in \mathbb{Q}[\{\lambda_{i,j}\}]\langle\langle u \rangle\rangle$ for the (n, s) curves. We set $\lambda'_{i,j} = \lambda_{i,j}/2$ if both i and j are odd, and $\lambda'_{i,j} = \lambda_{i,j}$ otherwise. Moreover, we denote by $\{\lambda'_{i,j}\}$ the set of all $\lambda'_{i,j}$. In [32], a special local parameter of the (n, s) curves around ∞ is introduced, which is called *arithmetic local parameter*, and by using the arithmetic local parameter and the expression of the sigma function associated with the (n,s) curves in terms of the tau function of the KP-hierarchy derived in [27], it is proved that $\sigma(u) \in \mathbb{Z}[\{\lambda'_{i,j}\}]\langle\langle u\rangle\rangle$ and $\sigma(u)^2 \in \mathbb{Z}[\{\lambda_{i,j}\}]\langle\langle u\rangle\rangle$ for the (n,s) curves. In [31], in the case of $(n,s)=(2,3)$, the Hurwitz integrality of the elliptic sigma function is proved by an approach different from [32]. In [31, 32], the relationships of the Hurwitz integrality of the sigma functions with number theory are discussed.

On the other hand, in [25], Miura introduced a certain canonical form, Miura canonical form, for defining equations of any non-singular algebraic curve. A telescopic curve [25] is a special curve for which Miura canonical form is easy to determine. For an integer $m \geq 2$, let $A_m = (a_1, \ldots, a_m)$ be a sequence of positive integers such that $\gcd(a_1, \ldots, a_m) = 1$, $a_i \geq 2$ for any i, and

$$\frac{a_i}{d_i} \in \left\langle \frac{a_1}{d_{i-1}}, \ldots, \frac{a_{i-1}}{d_{i-1}} \right\rangle, \quad 2 \leq i \leq m,$$

where $d_i = \gcd(a_1, \ldots, a_i)$. Let

$$B(A_m) = \left\{ (\ell_1, \ldots, \ell_m) \in \mathbb{Z}_{\geq 0}^m \;\middle|\; 0 \leq \ell_i \leq \frac{d_{i-1}}{d_i} - 1 \;\text{ for }\; 2 \leq i \leq m \right\}.$$

For any $2 \leq i \leq m$, there exists a unique sequence $(\ell_{i,1}, \ldots, \ell_{i,m}) \in B(A_m)$ satisfying

$$\sum_{j=1}^{m} a_j \ell_{i,j} = a_i \frac{d_{i-1}}{d_i}.$$

For any $2 \leq i \leq m$, we have $\ell_{i,j} = 0$ for $j \geq i$. Then the telescopic curve associated with A_m is the algebraic curve defined by the following $m-1$ equations in $\mathbb{C}^m = (x_1, \ldots, x_m)$

$$x_i^{d_{i-1}/d_i} = \prod_{j=1}^{i-1} x_j^{\ell_{i,j}} + \sum \lambda^{(i)}_{j_1,\ldots,j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad 2 \leq i \leq m,$$

where $\lambda^{(i)}_{j_1,\ldots,j_m} \in \mathbb{C}$ and the sum of the right hand side is over all $(j_1, \ldots, j_m) \in B(A_m)$ such that

$$\sum_{k=1}^{m} a_k j_k < a_i \frac{d_{i-1}}{d_i}.$$

For $m=2$, the telescopic curves are equal to the (n,s) curves. We denote by $\boldsymbol{\lambda}$ the set of all $\lambda^{(i)}_{j_1,\ldots,j_m}$. In [1], the sigma function of the (n,s) curves is generalized to case of the telescopic curves. The sigma function $\sigma(u)$ associated with a telescopic curve of genus g is a holomorphic function on $\mathbb{C}^g$. In [4], the expression of the sigma function associated with the telescopic curves in terms of the prime function and algebraic functions is derived. Further, in [4], the expression of the sigma function associated with the telescopic curves in terms of the tau function of the KP-hierarchy is also derived. In [4], by using these expressions of the sigma function of the telescopic curves, it is proved that $\sigma(u) \in \mathbb{Q}[\boldsymbol{\lambda}]\langle\langle u\rangle\rangle$ for the telescopic curves. We assign degrees as

$$\deg \lambda^{(i)}_{j_1,\ldots,j_m} = a_i d_{i-1}/d_i - \sum_{k=1}^{m} a_k j_k.$$

We set $\widetilde{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}/2$ if $\deg \lambda^{(i)}_{j_1,\dots,j_m}$ is odd and $\widetilde{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}$ if $\deg \lambda^{(i)}_{j_1,\dots,j_m}$ is even. We denote by $\widetilde{\boldsymbol{\lambda}}$ the set of all $\widetilde{\lambda}^{(i)}_{j_1,\dots,j_m}$. In this paper, we generalize the arithmetic local parameter of the (n,s) curves to the case of the telescopic curves (Section 3). By using the arithmetic local parameter of the telescopic curves and the expression of the sigma function associated with the telescopic curves in terms of the tau function of the KP-hierarchy, we show that $\sigma(u) \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}]\langle\langle u\rangle\rangle$ and $\sigma(u)^2 \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u\rangle\rangle$ for the telescopic curves (Theorem 4.6). For a non-negative integer n, if n is even, then we set $\chi(n) = 0$, and if n is odd, then we set $\chi(n) = 1$. We set $\overline{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}/2$ if $\sum_{k=1}^m \chi(j_k) \geq 2$ and $\overline{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}$ otherwise. We denote by $\overline{\boldsymbol{\lambda}}$ the set of all $\overline{\lambda}^{(i)}_{j_1,\dots,j_m}$. If $\sum_{j=1}^{i-1}\chi(\ell_{i,j}) \leq 1$ for any $2 \leq i \leq m$, then we show that $\sigma(u) \in \mathbb{Z}[\overline{\boldsymbol{\lambda}}]\langle\langle u\rangle\rangle$ for the telescopic curves (Theorem 4.8). We can apply Theorem 4.8 to the (n,s) curves. The result obtained by applying Theorem 4.8 to the (n,s) curves is equal to [32, Theorem 2.3].

In the case of the hyperelliptic curves, more precise properties on the power series expansion of the sigma function are known. Let V_g be the hyperelliptic curve of genus g defined by

$$y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \lambda_6 x^{2g-2} + \cdots + \lambda_{4g}x + \lambda_{4g+2}, \quad \lambda_i \in \mathbb{C}.$$

The sigma function $\sigma(u)$ associated with V_g is a holomorphic function on $\mathbb{C}^g$. By applying [32, Theorem 2.3] to the curve V_g, we obtain $\sigma(u) \in \mathbb{Z}[\{\lambda_{2i}\}_{i=2}^{2g+1}]\langle\langle u\rangle\rangle$. In [18, 31], it is proved that $\sigma(u) \in \mathbb{Z}[2\lambda_4, 8\lambda_6]\langle\langle u\rangle\rangle$ for $g=1$. In [18], it is conjectured that $\sigma(u) \in \mathbb{Z}[2\lambda_4, 24\lambda_6]\langle\langle u\rangle\rangle$ for $g=1$. In [3, Corollary 2], it is proved that $\sigma(u) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]\langle\langle u\rangle\rangle$ for $g=2$.

2 Preliminaries

2.1 Telescopic curves

In this section we briefly review the definition of telescopic curves following [25, 1, 4].

For an integer $m \geq 2$, let $A_m = (a_1, \dots, a_m)$ be a sequence of positive integers such that $\gcd(a_1,\dots,a_m) = 1$, $a_i \geq 2$ for any i, and

$$\frac{a_i}{d_i} \in \left\langle \frac{a_1}{d_{i-1}}, \dots, \frac{a_{i-1}}{d_{i-1}} \right\rangle, \quad 2 \leq i \leq m,$$

where $d_i = \gcd(a_1,\dots,a_i)$. Let

$$B(A_m) = \left\{ (\ell_1,\dots,\ell_m) \in \mathbb{Z}^m_{\geq 0} \,\middle|\, 0 \leq \ell_i \leq \frac{d_{i-1}}{d_i} - 1 \ \text{ for } \ 2 \leq i \leq m \right\}.$$

Lemma 2.1 ([25, 1]). *For any $a \in \langle a_1,\dots,a_m\rangle$, there exists a unique element $(k_1,\dots,k_m)$ of $B(A_m)$ such that*

$$\sum_{i=1}^m a_i k_i = a.$$

By this lemma, for any $2 \le i \le m$, there exists a unique sequence $(\ell_{i,1}, \ldots, \ell_{i,m}) \in B(A_m)$ satisfying

$$\sum_{j=1}^{m} a_j \ell_{i,j} = a_i \frac{d_{i-1}}{d_i}. \tag{2.1}$$

Lemma 2.2 ([4]). *For any $2 \le i \le m$, we have $\ell_{i,j} = 0$ for $j \ge i$.*

Consider $m-1$ polynomials in m variables $x = (x_1, \ldots, x_m)$ given by

$$F_i(x) = x_i^{d_{i-1}/d_i} - \prod_{j=1}^{i-1} x_j^{\ell_{i,j}} - \sum \lambda^{(i)}_{j_1,\ldots,j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad 2 \le i \le m, \tag{2.2}$$

where $\lambda^{(i)}_{j_1,\ldots,j_m} \in \mathbb{C}$ and the sum of the right hand side is over all $(j_1, \ldots, j_m) \in B(A_m)$ such that

$$\sum_{k=1}^{m} a_k j_k < a_i \frac{d_{i-1}}{d_i}.$$

We assign degrees as

$$\deg x_k = a_k, \quad \deg \lambda^{(i)}_{j_1,\ldots,j_m} = a_i d_{i-1}/d_i - \sum_{k=1}^{m} a_k j_k.$$

We denote by $\boldsymbol{\lambda}$ the set of all $\lambda^{(i)}_{j_1,\ldots,j_m}$. For $2 \le i \le m$, the polynomial $F_i(x)$ is homogeneous of degree $a_i d_{i-1}/d_i$ with respect to the coefficients $\boldsymbol{\lambda}$ and the variables $x_1, \ldots, x_m$. Let X^{aff} be the common zeros of $F_2, \ldots, F_m$:

$$X^{\mathrm{aff}} = \{(x_1, \ldots, x_m) \in \mathbb{C}^m \mid F_i(x_1, \ldots, x_m) = 0,\ 2 \le i \le m\}.$$

In [25, 1], X^{aff} is proved to be an affine algebraic curve. We assume that X^{aff} is non-singular. Let X be the compact Riemann surface corresponding to X^{aff}. Then X is obtained from X^{aff} by adding one point, say ∞ [25, 1]. The genus of X is given by

$$g = \frac{1}{2} \left\{ 1 - a_1 + \sum_{i=2}^{m} \left(\frac{d_{i-1}}{d_i} - 1 \right) a_i \right\}$$

(cf. [25, 1]). We call X the telescopic curve associated with A_m. The numbers $a_1, \ldots, a_m$ are a generator of the semigroup of non-gaps at ∞.

Example 2.3. (i) Let n and s be integers such that $n, s \ge 2$ and $\gcd(n, s) = 1$. The telescopic curve associated with $A_2 = (n, s)$ is the (n, s) curve introduced in [12].

(ii) For $m = 3$ and $A_3 = (4, 6, 5)$, the polynomials F_2 and F_3 are given by

$$\begin{aligned} F_2(x) &= x_2^2 - x_1^3 - \lambda^{(2)}_{0,1,1} x_2 x_3 - \lambda^{(2)}_{1,1,0} x_1 x_2 - \lambda^{(2)}_{1,0,1} x_1 x_3 - \lambda^{(2)}_{2,0,0} x_1^2 - \lambda^{(2)}_{0,1,0} x_2 - \lambda^{(2)}_{0,0,1} x_3 \\ &\quad - \lambda^{(2)}_{1,0,0} x_1 - \lambda^{(2)}_{0,0,0}, \\ F_3(x) &= x_3^2 - x_1 x_2 - \lambda^{(3)}_{1,0,1} x_1 x_3 - \lambda^{(3)}_{2,0,0} x_1^2 - \lambda^{(3)}_{0,1,0} x_2 - \lambda^{(3)}_{0,0,1} x_3 - \lambda^{(3)}_{1,0,0} x_1 - \lambda^{(3)}_{0,0,0}. \end{aligned}$$

(iii)[1] Let a and b be integers such that $a > b \geq 2$ and $\gcd(a, b) = 1$. For $A_m = (a_1, \ldots, a_m)$, where $a_i = a^{m-i}b^{i-1}$, the polynomials F_i, $2 \leq i \leq m$, are given by

$$F_i(x) = x_i^a - x_{i-1}^b - \sum_{a_1 j_1 + \cdots + a_m j_m < a a_i} \lambda^{(i)}_{j_1, \ldots, j_m} x_1^{j_1} \cdots x_m^{j_m}.$$

For a meromorphic function f on X, we denote by $\mathrm{ord}_\infty(f)$ the order of a pole at ∞. Then we have $\mathrm{ord}_\infty(x_i) = a_i$. We enumerate the monomials $x_1^{k_1} \cdots x_m^{k_m}$, $(k_1, \ldots, k_m) \in B(A_m)$, according as the order of a pole at ∞ and denote them by φ_i, $i \geq 1$. In particular we have $\varphi_1 = 1$. Let $(w_1, \ldots, w_g)$ be the gap sequence at ∞:

$$\{w_i \mid 1 \leq i \leq g\} = \mathbb{Z}_{\geq 0} \backslash \langle a_1, \ldots, a_m \rangle, \quad w_1 < \cdots < w_g.$$

In particular, we have $w_1 = 1$. The set $\{\varphi_i\}_{i=1}^{\infty}$ is a basis of the vector space consisting of meromorphic functions on X which are holomorphic at any point except ∞. Let G be the $(m-1) \times m$ matrix defined by

$$G = \left(\frac{\partial F_i}{\partial x_j} \right)_{2 \leq i \leq m,\, 1 \leq j \leq m}$$

and G_k be the $(m-1) \times (m-1)$ matrix obtained by deleting the k-th column from G. Then a basis of the vector space consisting of holomorphic one forms on X is given by

$$\omega_i = -\frac{\varphi_{g+1-i}}{\det G_1} dx_1, \quad 1 \leq i \leq g,$$

where $\det G_1$ is the determinant of G_1 (cf. [1]). The following lemma is proved in [1].

Lemma 2.4. *We have $w_g = 2g - 1$. In particular, the holomorphic one form ω_g has a zero of order $2g - 2$ at ∞.*

From Lemma 2.4, we find that the vector of Riemann constants for a telescopic curve with the base point ∞ is a half-period.

2.2 Fundamental differential of second kind

A fundamental differential of second kind plays important roles in the theory of the sigma functions. We recall its definition.

Let X be a telescopic curve of genus g and K_X be the canonical bundle of X. For $i = 1, 2$, let $\pi_i : X \times X \to X$ be the projection to the i-th component. A section of $\pi_1^* K_X \otimes \pi_2^* K_X$ is called a bilinear form on $X \times X$ and a bilinear form $\omega(P, Q)$ is called symmetric if $\omega(Q, P) = \omega(P, Q)$.

[1] This example is given in [30, 4].

Definition 2.5. A meromorphic symmetric bilinear form $\omega(P,Q)$ on $X \times X$ is called a fundamental differential of second kind if the following conditions are satisfied.

(i) $\omega(P,Q)$ is holomorphic at any point except $\{(R,R) \mid R \in X\}$.

(ii) For $R \in X$, take a local parameter t around R. Then $\omega(P,Q)$ has the following form around (R,R) :

$$\omega(P,Q) = \left(\frac{1}{(t_P - t_Q)^2} + f\left(t_P, t_Q\right) \right) dt_P dt_Q,$$

where $t_P = t(P)$, $t_Q = t(Q)$, and $f\left(t_P, t_Q\right)$ is a holomorphic function of t_P and t_Q.

For a fundamental differential of second kind $\omega(P,Q)$ and complex numbers $\{c_{i,j}\}_{i,j=1}^{g}$ such that $c_{i,j} = c_{j,i}$,

$$\omega(P,Q) + \sum_{i,j=1}^{g} c_{i,j} \omega_i(P) \omega_j(Q)$$

is also a fundamental differential of second kind.

For the telescopic curve X, a fundamental differential of second kind is algebraically constructed in [1]. We recall its construction. Note that the construction inherits all steps of classical construction in [6] that was recently recapitulated and generalized in [20, 26] for the (n,s) curves. We define the meromorphic bilinear form $\widehat{\omega}(P,Q)$ on $X \times X$ by

$$\widehat{\omega}(P,Q) = d_Q \Omega(P,Q) + \sum_{i=1}^{g} \omega_i(P) \eta_i(Q),$$

where $P = (x_1, \ldots, x_m)$ and $Q = (y_1, \ldots, y_m)$ are points on X,

$$\Omega(P,Q) = \frac{\det H(P,Q)}{(x_1 - y_1) \det G_1(P)} dx_1,$$

$H = (h_{i,j})_{2 \le i,j \le m}$ with

$$h_{i,j} = \frac{F_i(y_1, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_m) - F_i(y_1, \ldots, y_{j-1}, y_j, x_{j+1}, \ldots, x_m)}{x_j - y_j},$$

and η_i is a meromorphic one form on X which is holomorphic at any point except ∞. Here $d_Q \Omega(P,Q)$ means the derivative of $\Omega(P,Q)$ with respect to Q.

Lemma 2.6 ([1, Lemma 4.7], [26, Lemma 6])**.** *The set*

$$\left\{ \frac{\varphi_i(P)}{\det G_1(P)} dx_1 \right\}_{i=1}^{\infty}$$

is a basis of the vector space consisting of meromorphic one forms on X which are holomorphic at any point except ∞.

Let

$$\sum_{i=1}^{g}\omega_i(P)\eta_i(Q)=\frac{\sum c_{i_1,\ldots,i_m;j_1,\ldots,j_m}x_1^{i_1}\cdots x_m^{i_m}y_1^{j_1}\cdots y_m^{j_m}}{\det G_1(P)\det G_1(Q)}dx_1dy_1,$$

where $(i_1,\ldots,i_m),(j_1,\ldots,j_m)\in B(A_m)$ and $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}\in\mathbb{C}$.

Lemma 2.7 ([1, Theorem 4.1 (i)], [26, Proposition 2 (ii)]). *It is possible to take* $\{\eta_i\}_{i=1}^g$ *such that* $\widehat{\omega}(Q,P)=\widehat{\omega}(P,Q)$, $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}\in\mathbb{Q}[\boldsymbol{\lambda}]$, *and* $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ *is homogeneous of degree* $2(2g-1)-\sum_{k=1}^m a_k(i_k+j_k)$ *with respect to* $\boldsymbol{\lambda}$ *if* $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}\neq 0$.

Lemma 2.8 ([1, Theorem 4.1 (ii)], [26, Proposition 2 (i)]). *If we take* $\{\eta_i\}_{i=1}^g$ *as in Lemma 2.7, then* $\widehat{\omega}(P,Q)$ *becomes a fundamental differential of second kind.*

2.3 Sigma function of telescopic curves

Let X be a telescopic curve of genus g associated with $A_m=(a_1,\ldots,a_m)$. We take $\{\eta_i\}_{i=1}^g$ as in Lemma 2.7. We take a canonical basis $\{\mathfrak{a}_i,\mathfrak{b}_i\}_{i=1}^g$ in the one-dimensional homology group of the curve X and define the matrices of periods by

$$2\omega'=\left(\int_{\mathfrak{a}_j}\omega_i\right),\quad 2\omega''=\left(\int_{\mathfrak{b}_j}\omega_i\right),\quad -2\eta'=\left(\int_{\mathfrak{a}_j}\eta_i\right),\quad -2\eta''=\left(\int_{\mathfrak{b}_j}\eta_i\right).$$

The matrix of normalized periods is given by $\tau=(\omega')^{-1}\omega''$. Let $\delta=\tau\delta'+\delta''$, $\delta',\delta''\in\mathbb{R}^g$, be the vectors of Riemann's constants with respect to $(\{\mathfrak{a}_i,\mathfrak{b}_i\}_{i=1}^g,\infty)$. We set $\delta={}^t({}^t\delta',{}^t\delta'')$. We denote the imaginary unit by $\mathbf{i}$. The sigma function $\sigma(u)$ associated with the curve X, $u={}^t(u_1,\ldots,u_g)$, is defined by

$$\sigma(u)=C\exp\left(\frac{1}{2}{}^tu\eta'(\omega')^{-1}u\right)\theta[\delta]\left((2\omega')^{-1}u,\tau\right),$$

where $\theta[\delta](u)$ is the Riemann's theta function with the characteristics δ defined by

$$\theta[\delta](u)=\sum_{n\in\mathbb{Z}^g}\exp\{\pi\mathbf{i}\,{}^t(n+\delta')\tau(n+\delta')+2\pi\mathbf{i}\,{}^t(n+\delta')(u+\delta'')\},$$

and C is a constant. Since δ is a half-period from Lemma 2.4, $\sigma(u)$ vanishes on the Abel-Jacobi image of the $(g-1)$-th symmetric products of the telescopic curve. We have the following proposition.

Proposition 2.9 ([1, 26]). *For* $m_1,m_2\in\mathbb{Z}^g$ *and* $u\in\mathbb{C}^g$, *we have*

$$\frac{\sigma(u+2\omega'm_1+2\omega''m_2)}{\sigma(u)}=(-1)^{2({}^t\delta'm_1-{}^t\delta''m_2)+{}^tm_1m_2}\exp\{{}^t(2\eta'm_1+2\eta''m_2)(u+\omega'm_1+\omega''m_2)\}.$$

A sequence of non-negative integers $\mu=(\mu_1,\mu_2,\ldots,\mu_\ell)$ such that $\mu_1\geq\mu_2\geq\cdots\geq\mu_\ell$ is called a *partition*. For a partition $\mu=(\mu_1,\mu_2,\ldots,\mu_\ell)$, let $|\mu|=\mu_1+\mu_2+\cdots+\mu_\ell$. For $n\geq 0$, let $p_n(T)$ be the polynomial of $T_1,T_2,\ldots$ defined by

$$\sum_{i=0}^{\infty}\frac{1}{i!}\left(\sum_{j=1}^{\infty}T_jk^j\right)^i=\sum_{n=0}^{\infty}p_n(T)k^n, \tag{2.3}$$

where k is a variable, i.e., $p_n(T)$ is the coefficient of k^n in the left hand side of (2.3). For example, we have

$$p_0(T) = 1, \quad p_1(T) = T_1, \quad p_2(T) = T_2 + \frac{T_1^2}{2}, \quad p_3(T) = T_3 + T_1T_2 + \frac{T_1^3}{6}.$$

For $n < 0$, let $p_n(T) = 0$.

Lemma 2.10. *For $n \geq 1$, we have*

$$p_n(T) = \sum \frac{T_1^{k_1} \cdots T_n^{k_n}}{k_1! \cdots k_n!},$$

where the sum is over all $(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n$ satisfying

$$\sum_{j=1}^{n} jk_j = n.$$

Proof. By comparing the coefficients of k^n in (2.3), we obtain the statement of the lemma. □

For an arbitrary partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, the Schur function $S_\mu(T)$ is defined by

$$S_\mu(T) = \det\left(p_{\mu_i - i + j}(T)\right)_{1 \leq i,j \leq \ell}.$$

For the telescopic curve X associated with $A_m = (a_1, \ldots, a_m)$, we define the partition by

$$\mu(A_m) = (w_g, \ldots, w_1) - (g - 1, \ldots, 0).$$

Lemma 2.11. *The Schur function $S_{\mu(A_m)}(T)$ is a polynomial of the variables $T_{w_1}, \ldots, T_{w_g}$.*

Proof. We can prove this lemma as in the case of (n, s) curves (cf. [12, Section 4]). □

Theorem 2.12 ([4, Theorem 7])**.** *The sigma function $\sigma(u)$ is a holomorphic function on $\mathbb{C}^g$ and we have the unique constant C such that the series expansion of $\sigma(u)$ around the origin has the following form :*

$$\sigma(u) = S_{\mu(A_m)}(T)|_{T_{w_i} = u_i} + \sum_{w_1n_1 + \cdots + w_gn_g > |\mu(A_m)|} \varepsilon_{n_1,\ldots,n_g} \frac{u_1^{n_1} \cdots u_g^{n_g}}{n_1! \cdots n_g!}, \tag{2.4}$$

where $\varepsilon_{n_1,\ldots,n_g} \in \mathbb{Q}[\boldsymbol{\lambda}]$ and $\varepsilon_{n_1,\ldots,n_g}$ is homogeneous of degree $w_1n_1 + \cdots + w_gn_g - |\mu(A_m)|$ with respect to $\boldsymbol{\lambda}$ if $\varepsilon_{n_1,\ldots,n_g} \neq 0$.

We take the constant C such that the expansion (2.4) holds, see the expression for the sigma function above, which involves the constant C. Then the sigma function $\sigma(u)$ does not depend on the choice of a canonical basis $\{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^g$ in the one-dimensional homology group of the curve X and is determined by the coefficients $\boldsymbol{\lambda}$ of the defining equations of the curve X.

3 Arithmetic local parameter for telescopic curves

Since $\gcd(a_1, \ldots, a_m) = 1$, we can take $(b_1, \ldots, b_m) \in \mathbb{Z}^m$ such that

$$a_1 b_1 + \cdots + a_m b_m = -1.$$

We consider the defining equations (2.2) of the telescopic curve X. Let $M_m(\mathbb{Z})$ be the set of $m \times m$ matrices such that all the components are integers. We consider the matrix

$$D = \begin{pmatrix} -\ell_{2,1} & d_1/d_2 & 0 & \cdots & 0 \\ -\ell_{3,1} & -\ell_{3,2} & d_2/d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\ell_{m,1} & -\ell_{m,2} & \cdots & -\ell_{m,m-1} & d_{m-1}/d_m \\ b_1 & b_2 & \cdots & b_{m-1} & b_m \end{pmatrix} \in M_m(\mathbb{Z}).$$

Lemma 3.1. *We have* $\det(D) = (-1)^m$.

Proof. From (2.1) and Lemma 2.2, we have

$$D \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}. \tag{3.1}$$

By multiplying some elementary matrices whose determinants are 1 on the left, the equation (3.1) becomes

$$\widetilde{D} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix},$$

where

$$\widetilde{D} = \begin{pmatrix} e_2 & d_1/d_2 & 0 & \cdots & 0 \\ e_3 & 0 & d_2/d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_m & 0 & 0 & \cdots & d_{m-1}/d_m \\ e & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for certain $e_2, \ldots, e_m, e \in \mathbb{Q}$. From the above equation, we obtain $e = -1/a_1$. We have

$$\det(D) = \det(\widetilde{D}) = (-1)^{m-1} \frac{d_1}{d_2} \cdot \frac{d_2}{d_3} \cdots \frac{d_{m-1}}{d_m} \cdot e = (-1)^m.$$

□

Let

$$t = x_1^{b_1} \cdots x_m^{b_m}. \tag{3.2}$$

Since t has a zero of order 1 at ∞, we can regard t as a local parameter of X around ∞. We call t the *arithmetic local parameter* as in the case of [32]. For $1 \le i \le m$, we consider the expansion of x_i around ∞ with respect to t

$$x_i = \frac{1}{t^{a_i}} \sum_{k=0}^{\infty} p_{i,k} t^k, \qquad p_{i,k} \in \mathbb{C}. \tag{3.3}$$

By substituting (3.3) into the defining equations of X, for $2 \le i \le m$, we obtain

$$\begin{aligned} \left(\sum_{k=0}^{\infty} p_{i,k} t^k\right)^{d_{i-1}/d_i} &= \prod_{j=1}^{i-1} \left(\sum_{k=0}^{\infty} p_{j,k} t^k\right)^{\ell_{i,j}} \\ &+ \sum \lambda_{j_1,\ldots,j_m}^{(i)} t^{a_i d_{i-1}/d_i - \sum_{k=1}^m a_k j_k} \left(\sum_{k=0}^{\infty} p_{1,k} t^k\right)^{j_1} \cdots \left(\sum_{k=0}^{\infty} p_{m,k} t^k\right)^{j_m}, \end{aligned} \tag{3.4}$$

where the sum of the right hand side is over all $(j_1, \ldots, j_m) \in B(A_m)$ such that

$$\sum_{k=1}^{m} a_k j_k < a_i \frac{d_{i-1}}{d_i}.$$

Proposition 3.2. *We have* $p_{1,0} = p_{2,0} = \cdots = p_{m,0} = 1$.

Proof. By comparing the coefficients of t^0 in (3.4), we obtain

$$p_{i,0}^{d_{i-1}/d_i} = \prod_{j=1}^{i-1} p_{j,0}^{\ell_{i,j}} \tag{3.5}$$

for $2 \le i \le m$. By substituting (3.3) into (3.2), we obtain

$$1 = \left(\sum_{k=0}^{\infty} p_{1,k} t^k\right)^{b_1} \cdots \left(\sum_{k=0}^{\infty} p_{m,k} t^k\right)^{b_m}.$$

We divide the set $\{1, 2, \ldots, m\}$ into the two sets $\{\alpha_1, \ldots, \alpha_s\}$ and $\{\alpha_{s+1}, \ldots, \alpha_m\}$, where $b_{\alpha_1}, \ldots, b_{\alpha_s}$ are negative integers and $b_{\alpha_{s+1}}, \ldots, b_{\alpha_m}$ are non-negative integers. Then we have

$$\prod_{j=1}^{s} \left(\sum_{k=0}^{\infty} p_{\alpha_j,k} t^k\right)^{-b_{\alpha_j}} = \prod_{j=s+1}^{m} \left(\sum_{k=0}^{\infty} p_{\alpha_j,k} t^k\right)^{b_{\alpha_j}}. \tag{3.6}$$

By comparing the coefficients of t^0 in (3.6), we obtain

$$p_{1,0}^{b_1} \cdots p_{m,0}^{b_m} = 1. \tag{3.7}$$

For a complex number z satisfying $z \neq 0$, the principal value of the complex logarithm of z is defined by $\mathrm{Log}(z) = \log|z| + \mathbf{i}\mathrm{Arg}(z)$, where $\mathrm{Arg}(z)$ is the argument of z satisfying $-\pi < \mathrm{Arg}(z) \leq \pi$. For complex numbers z_1 and z_2 satisfying $z_1 z_2 \neq 0$, the following relations hold:

$$\mathrm{Log}(z_1 z_2) - \mathrm{Log}(z_1) - \mathrm{Log}(z_2) \in 2\pi\mathbf{i}\mathbb{Z},$$
$$\mathrm{Log}(z_1/z_2) - \mathrm{Log}(z_1) + \mathrm{Log}(z_2) \in 2\pi\mathbf{i}\mathbb{Z},$$

where $2\pi\mathbf{i}\mathbb{Z} = \{2\pi\mathbf{i}n \mid n \in \mathbb{Z}\}$. From (3.5), we have

$$\frac{d_{i-1}}{d_i}\mathrm{Log}(p_{i,0}) - \sum_{j=1}^{i-1} \ell_{i,j}\mathrm{Log}(p_{j,0}) \in 2\pi\mathbf{i}\mathbb{Z}, \qquad 2 \leq i \leq m.$$

From (3.7), we have

$$b_1\mathrm{Log}(p_{1,0}) + \cdots + b_m\mathrm{Log}(p_{m,0}) \in 2\pi\mathbf{i}\mathbb{Z}.$$

Therefore we have

$$D\begin{pmatrix} \mathrm{Log}(p_{1,0}) \\ \vdots \\ \mathrm{Log}(p_{m,0}) \end{pmatrix} \in 2\pi\mathbf{i}\mathbb{Z}^m,$$

where $2\pi\mathbf{i}\mathbb{Z}^m = \{2\pi\mathbf{i}n \mid n \in \mathbb{Z}^m\}$. From Lemma 3.1, we have $D^{-1} \in M_m(\mathbb{Z})$, where D^{-1} is the inverse matrix of D. Therefore we have $\mathrm{Log}(p_{j,0}) \in 2\pi\mathbf{i}\mathbb{Z}$ for $1 \leq j \leq m$. Thus we obtain $p_{j,0} = 1$ for $1 \leq j \leq m$. □

We set $\deg t = -1$.

Proposition 3.3. *We have $p_{i,k} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and the expansion of x_i around ∞ with respect to t is homogeneous of degree a_i with respect to $\boldsymbol{\lambda}$ and t.*

Proof. From Proposition 3.2, for any $1 \leq i \leq m$, we have $p_{i,0} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $p_{i,0}$ is homogeneous of degree 0 with respect to $\boldsymbol{\lambda}$. We take an integer $\ell \geq 1$. For any $1 \leq i \leq m$ and $0 \leq k \leq \ell - 1$, we assume that $p_{i,k} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $p_{i,k}$ is homogeneous of degree k with respect to $\boldsymbol{\lambda}$ if $p_{i,k} \neq 0$. By comparing the coefficients of t^ℓ in (3.4), there exist $f_i(\boldsymbol{\lambda}) \in \mathbb{Z}[\boldsymbol{\lambda}]$ for $2 \leq i \leq m$ such that $f_i(\boldsymbol{\lambda})$ are homogeneous of degree ℓ with respect to $\boldsymbol{\lambda}$ if $f_i(\boldsymbol{\lambda}) \neq 0$ and

$$\frac{d_{i-1}}{d_i}p_{i,\ell} - \sum_{j=1}^{i-1} \ell_{i,j}p_{j,\ell} = f_i(\boldsymbol{\lambda}), \qquad 2 \leq i \leq m.$$

By comparing the coefficients of t^ℓ in (3.6), there exists $f(\boldsymbol{\lambda}) \in \mathbb{Z}[\boldsymbol{\lambda}]$ such that $f(\boldsymbol{\lambda})$ is homogeneous of degree ℓ with respect to $\boldsymbol{\lambda}$ if $f(\boldsymbol{\lambda}) \neq 0$ and

$$b_1 p_{1,\ell} + \cdots + b_m p_{m,\ell} = f(\boldsymbol{\lambda}).$$

Therefore, we obtain

$$D\begin{pmatrix} p_{1,\ell} \\ \vdots \\ p_{m,\ell} \end{pmatrix} = \begin{pmatrix} f_2(\boldsymbol{\lambda}) \\ \vdots \\ f_m(\boldsymbol{\lambda}) \\ f(\boldsymbol{\lambda}) \end{pmatrix}.$$

Since $D^{-1} \in M_m(\mathbb{Z})$, for any $1 \le i \le m$, we have $p_{i,\ell} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $p_{i,\ell}$ is homogeneous of degree ℓ with respect to $\boldsymbol{\lambda}$ if $p_{i,\ell} \neq 0$. By mathematical induction, for any $1 \le i \le m$ and $k \ge 0$, we have $p_{i,k} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $p_{i,k}$ is homogeneous of degree k with respect to $\boldsymbol{\lambda}$ if $p_{i,k} \neq 0$. □

Lemma 3.4. *For any $(k_1, \dots, k_m) \in \mathbb{Z}_{\ge 0}^m$, the meromorphic function $x_1^{k_1} \cdots x_m^{k_m}$ on X can be expressed by the linear combination of φ_i uniquely as follows :*

$$x_1^{k_1} \cdots x_m^{k_m} = \sum_{i=1}^{n} \rho_i \varphi_i, \tag{3.8}$$

where $ord_\infty(\varphi_n) = \sum_{i=1}^m a_i k_i$, $\rho_n = 1$, $\rho_i \in \mathbb{Z}[\boldsymbol{\lambda}]$ for $1 \le i \le n-1$, and the right hand side of (3.8) is homogeneous of degree $\sum_{i=1}^m a_i k_i$ with respect to $\boldsymbol{\lambda}$ and $x_1, \dots, x_m$.

Proof. Since the set $\{\varphi_i\}_{i=1}^\infty$ is a basis of the vector space consisting of meromorphic functions on X which are holomorphic at any point except ∞, the meromorphic function $x_1^{k_1} \cdots x_m^{k_m}$ on X can be expressed by the linear combination of φ_i uniquely as follows :

$$x_1^{k_1} \cdots x_m^{k_m} = \sum_{i=1}^{n} \rho_i \varphi_i, \quad \rho_i \in \mathbb{C}. \tag{3.9}$$

For $1 \le i \le n$, let $j_i = \mathrm{ord}_\infty(\varphi_i)$. We have $j_n = \sum_{i=1}^m a_i k_i$. By expanding the both sides of (3.9) around ∞ with respect to t and comparing the coefficients of t^{-j_n}, from Proposition 3.2, we obtain $\rho_n = 1$. We take an integer $1 \le \ell \le n-1$. For any $\ell + 1 \le i \le n$, we assume that $\rho_i \in \mathbb{Z}[\boldsymbol{\lambda}]$ and ρ_i is homogeneous of degree $j_n - j_i$ with respect to $\boldsymbol{\lambda}$ if $\rho_i \neq 0$. From Proposition 3.3, the expansion of $x_1^{k_1} \cdots x_m^{k_m} - \sum_{i=\ell+1}^n \rho_i \varphi_i$ around ∞ with respect to t is homogeneous of degree j_n with respect to $\boldsymbol{\lambda}$ and t. By expanding the both sides of (3.9) around ∞ with respect to t and comparing the coefficients of t^{-j_ℓ}, from Proposition 3.2, we have $\rho_\ell \in \mathbb{Z}[\boldsymbol{\lambda}]$ and ρ_ℓ is homogeneous of degree $j_n - j_\ell$ with respect to $\boldsymbol{\lambda}$ if $\rho_\ell \neq 0$. By mathematical induction, for any $1 \le i \le n-1$, we have $\rho_i \in \mathbb{Z}[\boldsymbol{\lambda}]$ and ρ_i is homogeneous of degree $j_n - j_i$ with respect to $\boldsymbol{\lambda}$ if $\rho_i \neq 0$. □

Lemma 3.5 ([2, Lemma 3.4])**.** *For $1 \le k \le m$, we have*

$$\det G_k(P) = (-1)^{k+1} a_k x_1^{\gamma_1} \cdots x_m^{\gamma_m} + \sum \beta_{i_1,\dots,i_m} x_1^{i_1} \cdots x_m^{i_m}, \tag{3.10}$$

where $(\gamma_1, \dots, \gamma_m)$ is the unique element of $B(A_m)$ such that $\sum_{j=1}^m a_j \gamma_j = \sum_{j=2}^m a_j d_{j-1}/d_j - \sum_{j=1}^m a_j + a_k$ and the sum of the right hand side of (3.10) is over all $(i_1, \dots, i_m) \in B(A_m)$ such that $\sum_{j=1}^m a_j i_j < \sum_{j=2}^m a_j d_{j-1}/d_j - \sum_{j=1}^m a_j + a_k$. We have $\beta_{i_1,\dots,i_m} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and the right hand side of (3.10) is homogeneous of degree $\sum_{j=2}^m a_j d_{j-1}/d_j - \sum_{j=1}^m a_j + a_k$ with respect to $\boldsymbol{\lambda}$ and $x_1, \dots, x_m$.

In the same way as [4, Proposition 1], for $1 \leq i \leq g$, we can prove that the expansion of ω_i around ∞ with respect to t has the following form :

$$\omega_i = t^{w_i - 1}\left(1 + \sum_{j=1}^{\infty} b_{i,j} t^j\right) dt, \quad b_{i,j} \in \mathbb{C}.$$

Proposition 3.6. *We have $b_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and the expansion of ω_i around ∞ with respect to t is homogeneous of degree $1 - w_i$ with respect to $\boldsymbol{\lambda}$ and t.*

Proof. From Propositions 3.2, 3.3, and Lemma 3.5, for $1 \leq k \leq m$, we have the following expansion :

$$\frac{dx_k}{\det G_k(P)} = (-1)^k t^{2g-2}\left(1 + \sum_{j=1}^{\infty} b_j^{(k)} t^j\right) dt,$$

where $b_j^{(k)} \in \mathbb{Z}[1/a_k, \boldsymbol{\lambda}]$ and $b_j^{(k)}$ is homogeneous of degree j with respect to $\boldsymbol{\lambda}$ if $b_j^{(k)} \neq 0$. For any $1 \leq k \leq m$, we have

$$\frac{dx_1}{\det G_1(P)} = (-1)^{k-1} \frac{dx_k}{\det G_k(P)}. \tag{3.11}$$

(cf. the proof of [1, Lemma 3.2]). Therefore we have $b_j^{(1)} = b_j^{(k)}$ for any $j \geq 1$ and $2 \leq k \leq m$. Since $\gcd(a_1, \ldots, a_m) = 1$, for any $j \geq 1$, we have

$$b_j^{(1)} \in \bigcap_{k=1}^{m} \mathbb{Z}[1/a_k, \boldsymbol{\lambda}] = \mathbb{Z}[\boldsymbol{\lambda}].$$

From Propositions 3.2 and 3.3, we obtain the statement of the proposition. □

We take $\{\eta_i\}_{i=1}^g$ as in Lemma 2.7.

Proposition 3.7. *It is possible to take $\{\eta_i\}_{i=1}^g$ such that $c_{i_1,\ldots,i_m;j_1,\ldots,j_m} = 0$ if $\sum_{k=1}^m a_k i_k \geq \sum_{k=1}^m a_k j_k$.*

Proof. If $\sum_{k=1}^m a_k i_k > \sum_{k=1}^m a_k j_k$ and $c_{i_1,\ldots,i_m;j_1,\ldots,j_m} \neq 0$, we add

$$-\frac{c_{i_1,\ldots,i_m;j_1,\ldots,j_m} x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m}}{\det G_1(P) \det G_1(Q)} dx_1 dy_1 - \frac{c_{i_1,\ldots,i_m;j_1,\ldots,j_m} x_1^{j_1} \cdots x_m^{j_m} y_1^{i_1} \cdots y_m^{i_m}}{\det G_1(P) \det G_1(Q)} dx_1 dy_1$$

to $\widehat{\omega}(P, Q)$. If $\sum_{k=1}^m a_k i_k = \sum_{k=1}^m a_k j_k$, which is equivalent to $(i_1, \ldots, i_m) = (j_1, \ldots, j_m)$, and $c_{i_1,\ldots,i_m;j_1,\ldots,j_m} \neq 0$, we add

$$-\frac{c_{i_1,\ldots,i_m;i_1,\ldots,i_m} x_1^{i_1} \cdots x_m^{i_m} y_1^{i_1} \cdots y_m^{i_m}}{\det G_1(P) \det G_1(Q)} dx_1 dy_1$$

to $\widehat{\omega}(P, Q)$. Then we can take $\{\eta_i\}_{i=1}^g$ in the form of this proposition. □

Hereafter, we take $\{\eta_i\}_{i=1}^g$ as in Proposition 3.7.

Lemma 3.8 ([1, p. 470], [2, p. 6]). *We have*

$$d_Q\Omega(P,Q) = \frac{\{\sum_{i=1}^m (-1)^{i+1}(x_1-y_1)\frac{\partial \det H}{\partial y_i}(P,Q)\det G_i(Q)\} + \det G_1(Q)\det H(P,Q)}{(x_1-y_1)^2 \det G_1(P)\det G_1(Q)} dx_1 dy_1,$$

where the numerator is homogeneous of degree $2\sum_{i=2}^m (d_{i-1}/d_i - 1)a_i = 2(2g-1+a_1)$ *with respect to* $\boldsymbol{\lambda}$ *and* $x_1,\ldots,x_m,y_1,\ldots,y_m$.

We define $\tilde{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ by

$$d_Q\Omega(P,Q) = \frac{\sum \tilde{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m} x_1^{i_1}\cdots x_m^{i_m} y_1^{j_1}\cdots y_m^{j_m}}{(x_1-y_1)^2\det G_1(P)\det G_1(Q)} dx_1 dy_1,$$

where $(i_1,\ldots,i_m),(j_1,\ldots,j_m) \in B(A_m)$.

Lemma 3.9. *We have* $\tilde{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m} \in \mathbb{Z}[\boldsymbol{\lambda}]$ *and* $\tilde{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ *is homogeneous of degree* $2(2g-1+a_1) - \sum_{k=1}^m a_k(i_k+j_k)$ *with respect to* $\boldsymbol{\lambda}$ *if* $\tilde{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m} \neq 0$.

Proof. From Lemmas 3.4 and 3.8, we obtain the statement of the lemma. □

We define $F(P,Q)$ by

$$\widehat{\omega}(P,Q) = \frac{F(P,Q)}{(x_1-y_1)^2\det G_1(P)\det G_1(Q)} dx_1 dy_1.$$

We can determine the coefficients $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ by the following recurrence relations.

Proposition 3.10. *We take* $(i_1,\ldots,i_m),(j_1,\ldots,j_m) \in B(A_m)$ *such that* $\sum_{k=1}^m a_k i_k < \sum_{k=1}^m a_k j_k$.

(i) If $i_1 = 0$, *we have*

$$c_{0,i_2,\ldots,i_m;j_1,\ldots,j_m} = \tilde{c}_{j_1+2,j_2,\ldots,j_m;0,i_2,\ldots,i_m} - \tilde{c}_{0,i_2,\ldots,i_m;j_1+2,j_2,\ldots,j_m}.$$

(ii) If $i_1 = 1$, *we have*

$$\begin{aligned} c_{1,i_2,\ldots,i_m;j_1,\ldots,j_m} &= 2\tilde{c}_{j_1+3,j_2,\ldots,j_m;0,i_2,\ldots,i_m} - 2\tilde{c}_{0,i_2,\ldots,i_m;j_1+3,j_2,\ldots,j_m} + \tilde{c}_{j_1+2,j_2,\ldots,j_m;1,i_2,\ldots,i_m} \\ &\quad - \tilde{c}_{1,i_2,\ldots,i_m;j_1+2,j_2,\ldots,j_m}. \end{aligned}$$

(iii) If $i_1 \geq 2$, *we have*

$$\begin{aligned} c_{i_1,\ldots,i_m;j_1,\ldots,j_m} &= 2c_{i_1-1,i_2,\ldots,i_m;j_1+1,j_2,\ldots,j_m} - c_{i_1-2,i_2,\ldots,i_m;j_1+2,j_2,\ldots,j_m} \\ &\quad + \tilde{c}_{j_1+2,j_2,\ldots,j_m;i_1,\ldots,i_m} - \tilde{c}_{i_1,\ldots,i_m;j_1+2,j_2,\ldots,j_m}. \end{aligned}$$

Proof. (i) The coefficient of $x_2^{i_2}\cdots x_m^{i_m}y_1^{j_1+2}y_2^{j_2}\cdots y_m^{j_m}$ in $F(P,Q)$ is

$$c_{0,i_2,\ldots,i_m;j_1,\ldots,j_m}+\tilde{c}_{0,i_2,\ldots,i_m;j_1+2,j_2,\ldots,j_m}.$$

From $\sum_{k=2}^m a_k i_k<\sum_{k=1}^m a_k j_k$ and Proposition 3.7, the coefficient of $x_1^{j_1+2}x_2^{j_2}\cdots x_m^{j_m}y_2^{i_2}\cdots y_m^{i_m}$ in $F(P,Q)$ is $\tilde{c}_{j_1+2,j_2,\ldots,j_m;0,i_2,\ldots,i_m}$. From $\widehat{\omega}(Q,P)=\widehat{\omega}(P,Q)$, we obtain the statement of (i).

(ii) The coefficient of $x_1x_2^{i_2}\cdots x_m^{i_m}y_1^{j_1+2}y_2^{j_2}\cdots y_m^{j_m}$ in $F(P,Q)$ is

$$c_{1,i_2,\ldots,i_m;j_1,\ldots,j_m}-2c_{0,i_2,\ldots,i_m;j_1+1,j_2,\ldots,j_m}+\tilde{c}_{1,i_2,\ldots,i_m;j_1+2,j_2,\ldots,j_m}.$$

From $a_1+\sum_{k=2}^m a_k i_k<\sum_{k=1}^m a_k j_k$ and Proposition 3.7, the coefficient of $x_1^{j_1+2}x_2^{j_2}\cdots x_m^{j_m}y_1y_2^{i_2}\cdots y_m^{i_m}$ in $F(P,Q)$ is $\tilde{c}_{j_1+2,j_2,\ldots,j_m;1,i_2,\ldots,i_m}$. From $\widehat{\omega}(Q,P)=\widehat{\omega}(P,Q)$ and (i), we obtain the statement of (ii).

(iii) The coefficient of $x_1^{i_1}\cdots x_m^{i_m}y_1^{j_1+2}y_2^{j_2}\cdots y_m^{j_m}$ in $F(P,Q)$ is

$$c_{i_1,\ldots,i_m;j_1,\ldots,j_m}-2c_{i_1-1,i_2,\ldots,i_m;j_1+1,j_2,\ldots,j_m}+c_{i_1-2,i_2,\ldots,i_m;j_1+2,j_2,\ldots,j_m}+\tilde{c}_{i_1,\ldots,i_m;j_1+2,j_2,\ldots,j_m}.$$

From $\sum_{k=1}^m a_k i_k<\sum_{k=1}^m a_k j_k$ and Proposition 3.7, the coefficient of $x_1^{j_1+2}x_2^{j_2}\cdots x_m^{j_m}y_1^{i_1}\cdots y_m^{i_m}$ in $F(P,Q)$ is $\tilde{c}_{j_1+2,j_2,\ldots,j_m;i_1,\ldots,i_m}$. From $\widehat{\omega}(Q,P)=\widehat{\omega}(P,Q)$, we obtain the statement of (iii). □

Lemma 3.11. *We have $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}\in\mathbb{Z}[\boldsymbol{\lambda}]$ and $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ is homogeneous of degree $2(2g-1)-\sum_{k=1}^m a_k(i_k+j_k)$ with respect to $\boldsymbol{\lambda}$ if $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}\neq 0$.*

Proof. From Proposition 3.10, we obtain the statement of the lemma. □

We define $\overline{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ by

$$F(P,Q)=\sum \overline{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}x_1^{i_1}\cdots x_m^{i_m}y_1^{j_1}\cdots y_m^{j_m},$$

where $(i_1,\ldots,i_m),(j_1,\ldots,j_m)\in B(A_m)$.

Lemma 3.12. *We have $\overline{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}\in\mathbb{Z}[\boldsymbol{\lambda}]$ and $\overline{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ is homogeneous of degree $2(2g-1+a_1)-\sum_{k=1}^m a_k(i_k+j_k)$ with respect to $\boldsymbol{\lambda}$ if $\overline{c}_{i_1,\ldots,i_m;j_1,\ldots,j_m}\neq 0$.*

Proof. From Lemmas 3.9 and 3.11, we obtain the statement of the lemma. □

The fundamental differential of second kind $\widehat{\omega}(P,Q)$ is expanded around $\infty\times\infty$ with respect to the arithmetic local parameter t as follows :

$$\widehat{\omega}(P,Q)=\left(\frac{1}{(t_P-t_Q)^2}+\sum_{i,j\geq 1}q_{i,j}t_P^{i-1}t_Q^{j-1}\right)dt_Pdt_Q,\quad q_{i,j}\in\mathbb{C},\tag{3.12}$$

where $t_P=t(P)$ and $t_Q=t(Q)$. From $\widehat{\omega}(Q,P)=\widehat{\omega}(P,Q)$, we have $q_{j,i}=q_{i,j}$ for any i,j.

Proposition 3.13. *We have $q_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $q_{i,j}$ is homogeneous of degree $i+j$ with respect to $\boldsymbol{\lambda}$ if $q_{i,j} \neq 0$.*

Proof. From (3.3) and Proposition 3.2, we have

$$t_P^{2a_1} t_Q^{2a_1} (x_1 - y_1)^2 = \left(t_Q^{a_1} + \sum_{k=1}^{\infty} p_{1,k} t_P^k t_Q^{a_1} - t_P^{a_1} - \sum_{k=1}^{\infty} p_{1,k} t_P^{a_1} t_Q^k \right)^2. \tag{3.13}$$

From Proposition 3.3, we have

$$t_P^{2a_1} t_Q^{2a_1} (x_1 - y_1)^2 = (t_P - t_Q)^2 \sum_{i,j \geq 0} \nu_{i,j} t_P^i t_Q^j, \tag{3.14}$$

where $\nu_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $\nu_{i,j}$ is homogeneous of degree $2 - 2a_1 + i + j$ with respect to $\boldsymbol{\lambda}$ if $\nu_{i,j} \neq 0$. From (3.12), (3.13), and (3.14), around $\infty \times \infty$, we have

$$t_P^{2a_1} t_Q^{2a_1} (x_1 - y_1)^2 \widehat{\omega}(P, Q) =$$
$$\left\{ \sum_{i,j \geq 0} \nu_{i,j} t_P^i t_Q^j + \left(t_Q^{a_1} + \sum_{k=1}^{\infty} p_{1,k} t_P^k t_Q^{a_1} - t_P^{a_1} - \sum_{k=1}^{\infty} p_{1,k} t_P^{a_1} t_Q^k \right)^2 \left(\sum_{i,j \geq 1} q_{i,j} t_P^{i-1} t_Q^{j-1} \right) \right\} dt_P dt_Q. \tag{3.15}$$

Therefore, around $\infty \times \infty$, we have

$$t_P^{2a_1} t_Q^{2a_1} (x_1 - y_1)^2 \widehat{\omega}(P, Q) = \left(\sum_{i,j \geq 0} \widetilde{q}_{i,j} t_P^i t_Q^j \right) dt_P dt_Q, \quad \widetilde{q}_{i,j} \in \mathbb{C}. \tag{3.16}$$

From Propositions 3.3, 3.6, and Lemma 3.12, we have $\widetilde{q}_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $\widetilde{q}_{i,j}$ is homogeneous of degree $2 - 2a_1 + i + j$ with respect to $\boldsymbol{\lambda}$ if $\widetilde{q}_{i,j} \neq 0$. From (3.15) and (3.16), we have

$$\sum_{i,j \geq 0} (\widetilde{q}_{i,j} - \nu_{i,j}) t_P^i t_Q^j$$
$$= \left(t_Q^{a_1} + \sum_{k=1}^{\infty} p_{1,k} t_P^k t_Q^{a_1} - t_P^{a_1} - \sum_{k=1}^{\infty} p_{1,k} t_P^{a_1} t_Q^k \right)^2 \left(\sum_{i,j \geq 1} q_{i,j} t_P^{i-1} t_Q^{j-1} \right). \tag{3.17}$$

By comparing the coefficients of $t_Q^{2a_1}$ in the both sides of (3.17), we have $q_{1,1} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $q_{1,1}$ is homogeneous of degree 2 with respect to $\boldsymbol{\lambda}$ if $q_{1,1} \neq 0$. We take a pair of positive integers (i_0, j_0). For any $(i,j) \in \mathbb{N}^2$ such that

- $i + j < i_0 + j_0$ or
- $i + j = i_0 + j_0$ and $i < i_0$,

we assume that $q_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $q_{i,j}$ is homogeneous of degree $i+j$ with respect to $\boldsymbol{\lambda}$ if $q_{i,j} \neq 0$. By comparing the coefficients of $t_P^{i_0-1} t_Q^{j_0+2a_1-1}$ in the both sides of (3.17), we have $q_{i_0,j_0} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and q_{i_0,j_0} is homogeneous of degree $i_0 + j_0$ with respect to $\boldsymbol{\lambda}$ if $q_{i_0,j_0} \neq 0$. By mathematical induction, we obtain the statement of the proposition. □

For $k \geq 1$, we define c_k by

$$\sum_{k=1}^{\infty} c_k t^{k-1} = \frac{1}{2} \frac{\frac{d}{dt}\left(1 + \sum_{j=1}^{\infty} b_{g,j} t^j\right)}{1 + \sum_{j=1}^{\infty} b_{g,j} t^j}. \tag{3.18}$$

We set $\widetilde{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}/2$ if $\deg \lambda^{(i)}_{j_1,\dots,j_m}$ is odd and $\widetilde{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}$ if $\deg \lambda^{(i)}_{j_1,\dots,j_m}$ is even. We denote by $\widetilde{\boldsymbol{\lambda}}$ the set of all $\widetilde{\lambda}^{(i)}_{j_1,\dots,j_m}$. We set $\deg \widetilde{\lambda}^{(i)}_{j_1,\dots,j_m} = \deg \lambda^{(i)}_{j_1,\dots,j_m}$. For a domain R, we denote by $R[[t]]$ the set consisting of formal power series over R.

Lemma 3.14. *We have $c_k \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}]$ and c_k is homogeneous of degree k with respect to $\widetilde{\boldsymbol{\lambda}}$ if $c_k \neq 0$.*

Proof. From Proposition 3.6, we have $b_{g,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ and $b_{g,j}$ is homogeneous of degree j with respect to $\boldsymbol{\lambda}$ if $b_{g,j} \neq 0$. Therefore, if j is odd and $b_{g,j} \neq 0$, any term of $b_{g,j}$ contains a coefficient $\lambda^{(i)}_{j_1,\dots,j_m}$ such that $\deg \lambda^{(i)}_{j_1,\dots,j_m}$ is odd. Thus, we have

$$\frac{1}{2}\frac{d}{dt}\left(1 + \sum_{j=1}^{\infty} b_{g,j} t^j\right) \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}][[t]]$$

and it is homogeneous of degree 1 with respect to $\widetilde{\boldsymbol{\lambda}}$ and t. On the other hand, we have

$$\frac{1}{1 + \sum_{j=1}^{\infty} b_{g,j} t^j} = 1 + \sum_{\ell=1}^{\infty} \left(-\sum_{j=1}^{\infty} b_{g,j} t^j\right)^{\ell} \in \mathbb{Z}[\boldsymbol{\lambda}][[t]]$$

and it is homogeneous of degree 0 with respect to $\boldsymbol{\lambda}$ and t. Therefore we have

$$\sum_{k=1}^{\infty} c_k t^{k-1} \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}][[t]]$$

and it is homogeneous of degree 1 with respect to $\widetilde{\boldsymbol{\lambda}}$ and t. Thus we obtain the statement of the lemma. □

4 Hurwitz integrality of the power series expansion of the sigma function for telescopic curves

Definition 4.1. For a subring R of $\mathbb{C}$ and variables $z = {}^t(z_1, \dots, z_n)$, let

$$R\langle\langle z \rangle\rangle = R\langle\langle z_1, \dots, z_n \rangle\rangle = \left\{ \sum_{k_1,\dots,k_n \geq 0} \zeta_{k_1,\dots,k_n} \frac{z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!} \;\middle|\; \zeta_{k_1,\dots,k_n} \in R \right\}.$$

If the power series expansion of a holomorphic function $f(z) = f(z_1, \dots, z_n)$ on $\mathbb{C}^n$ around the origin belongs to $R\langle\langle z \rangle\rangle$, then we write $f(z) \in R\langle\langle z \rangle\rangle$ and $f(z)$ is said to be *Hurwitz integral* over R.

Let $W = \{w_1, \ldots, w_g\}$ and $u = {}^t(u_1, \ldots, u_g)$. For any partition μ and the Schur function $S_\mu(T)$, we substitute $T_{w_i} = u_i$ for $1 \le i \le g$ and $T_j = 0$ for any j satisfying $j \notin W$, and denote it by $S_\mu(u)$.

Lemma 4.2. *For any partition μ, we have $S_\mu(u) \in \mathbb{Z}\langle\langle u \rangle\rangle$.*

Proof. For an integer $n \ge 1$ and the polynomial $p_n(T)$ (cf. Section 2.3), we substitute $T_{w_i} = u_i$ for $1 \le i \le g$ and $T_j = 0$ for any j satisfying $j \notin W$, and denote it by $p_n(u)$. Let $p_0(u) = 1$ and $p_n(u) = 0$ for $n < 0$. From Lemma 2.10, for $n \ge 0$, we have

$$p_n(u) = \sum \frac{u_1^{n_1} \cdots u_g^{n_g}}{n_1! \cdots n_g!},$$

where the sum is over all $(n_1, \ldots, n_g) \in \mathbb{Z}_{\ge 0}^g$ satisfying $w_1 n_1 + \cdots + w_g n_g = n$. We have

$$S_\mu(u) = \det\left(p_{\mu_i - i + j}(u)\right)_{1 \le i,j \le \ell},$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. For integers $m_1, \ldots, m_g, n_1, \ldots, n_g \ge 0$, we have

$$\frac{u_1^{m_1} \cdots u_g^{m_g}}{m_1! \cdots m_g!} \frac{u_1^{n_1} \cdots u_g^{n_g}}{n_1! \cdots n_g!} = \binom{m_1 + n_1}{m_1} \cdots \binom{m_g + n_g}{m_g} \frac{u_1^{m_1+n_1} \cdots u_g^{m_g+n_g}}{(m_1+n_1)! \cdots (m_g+n_g)!}.$$

Since the binomial coefficients $\binom{m_1 + n_1}{m_1}, \ldots, \binom{m_g + n_g}{m_g}$ are integers, we obtain the statement of the lemma. □

Lemma 4.3. *Let R be a subring of $\mathbb{C}$, $f(u) = f(u_1, \ldots, u_g)$ be a holomorphic function on $\mathbb{C}^g$, and M be a $g \times g$ matrix such that all the components are included in R. If $f(u) \in R\langle\langle u \rangle\rangle$, then we have $f(Mu) \in R\langle\langle u \rangle\rangle$.*

Proof. Let $M = (m_{i,j})_{1 \le i,j \le g}$, where $m_{i,j} \in R$. For any integer $n \ge 0$, we have

$$\frac{(m_{i,1}u_1 + \cdots + m_{i,g}u_g)^n}{n!} = \sum m_{i,1}^{n_1} \cdots m_{i,g}^{n_g} \frac{u_1^{n_1} \cdots u_g^{n_g}}{n_1! \cdots n_g!},$$

where the sum is over all $(n_1, \ldots, n_g) \in \mathbb{Z}_{\ge 0}^g$ satisfying $n_1 + \cdots + n_g = n$. Thus we obtain the statement of the lemma. □

We expand $t^{g-1}\varphi_j$ around ∞ with respect to the arithmetic local parameter t

$$t^{g-1}\varphi_j = \sum_i \xi_{i,j} t^i.$$

From Proposition 3.3, we have $\xi_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$. For $j > g$, we have

$$\xi_{i,j} = \begin{cases} 0 & \text{if } i < -j \\ 1 & \text{if } i = -j. \end{cases}$$

For a partition $\mu = (\mu_1, \mu_2, \dots)$, we define

$$\xi_\mu = \det(\xi_{m_i,j})_{i,j\in\mathbb{N}} = \begin{vmatrix} \xi_{m_1,1} & \xi_{m_1,2} & \xi_{m_1,3} & \cdots \\ \xi_{m_2,1} & \xi_{m_2,2} & \xi_{m_2,3} & \cdots \\ \xi_{m_3,1} & \xi_{m_3,2} & \xi_{m_3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix},$$

where $m_i = \mu_i - i$ and the infinite determinant is well defined. Then we have $\xi_\mu \in \mathbb{Z}[\boldsymbol{\lambda}]$. The tau function $\tau(u)$ is defined by

$$\tau(u) = \sum_\mu \xi_\mu S_\mu(u),$$

where the sum is over all partitions.

Proposition 4.4. *We have* $\tau(u) \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u\rangle\rangle$.

Proof. From Lemma 4.2, we obtain the statement of the proposition. □

For $1 \le i \le g$, we expand ω_i around ∞ with respect to t as follows :

$$\omega_i = \sum_{j=1}^\infty \widetilde{b}_{i,j} t^{j-1} dt, \qquad \widetilde{b}_{i,j} \in \mathbb{C}.$$

From Proposition 3.6, we have

$$\widetilde{b}_{i,j} = \begin{cases} 0 & \text{if } j < w_i \\ 1 & \text{if } j = w_i \end{cases}$$

and $\widetilde{b}_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ if $j > w_i$. We define the $g \times g$ matrix

$$B = (\widetilde{b}_{i,w_j})_{1\le i,j\le g} = \begin{pmatrix} 1 & \widetilde{b}_{1,w_2} & \widetilde{b}_{1,w_3} & \cdots & \widetilde{b}_{1,w_g} \\ 0 & 1 & \widetilde{b}_{2,w_3} & \cdots & \widetilde{b}_{2,w_g} \\ 0 & 0 & 1 & \cdots & \widetilde{b}_{3,w_g} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$c = (c_{w_1}, \cdots, c_{w_g})$, and the $g \times g$ matrix

$$N = (q_{w_i,w_j})_{1\le i,j\le g} = \begin{pmatrix} q_{w_1,w_1} & q_{w_1,w_2} & \cdots & q_{w_1,w_g} \\ q_{w_2,w_1} & q_{w_2,w_2} & \cdots & q_{w_2,w_g} \\ \vdots & \vdots & \ddots & \vdots \\ q_{w_g,w_1} & q_{w_g,w_2} & \cdots & q_{w_g,w_g} \end{pmatrix}.$$

Theorem 4.5 ([4, Theorem 1], [27, Theorem 8])**.** *For* $v = {}^t(v_1, \dots, v_g) \in \mathbb{C}^g$, *the following relation holds :*

$$\tau(v) = \exp\left(-cv + \frac{1}{2}{}^t vNv\right)\sigma(Bv).$$

Theorem 4.6. *We have* $\sigma(u) \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}]\langle\langle u \rangle\rangle$ *and* $\sigma(u)^2 \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u \rangle\rangle$.

Proof. Since the determinant of B is 1, we have

$$B^{-1} = \widetilde{B},$$

where $\widetilde{B}$ is the adjugate matrix of B. Therefore all the components of B^{-1} are included in $\mathbb{Z}[\boldsymbol{\lambda}]$. We set $u = Bv$ in Theorem 4.5. Then we have

$$\sigma(u) = \exp\left(cB^{-1}u - \frac{1}{2}{}^t u \, {}^t(B^{-1})NB^{-1}u\right)\tau(B^{-1}u). \tag{4.1}$$

From Lemma 4.3 and Proposition 4.4, we have $\tau(B^{-1}u) \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u \rangle\rangle$. Let $\overline{c} = cB^{-1}$, $\overline{c} = (\overline{c}_1, \dots, \overline{c}_g)$, $\overline{N} = {}^t(B^{-1})NB^{-1}$, and $\overline{N} = (\overline{q}_{i,j})_{1 \le i,j \le g}$. From Lemma 3.14, we have $\overline{c}_i \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}]$ for any i. Since N is the symmetric matrix, $\overline{N}$ is also the symmetric matrix. From Proposition 3.13, we have $\overline{q}_{i,j} \in \mathbb{Z}[\boldsymbol{\lambda}]$ for any i, j. Thus we have $\exp(\overline{c}_i u_i) \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}]\langle\langle u_i \rangle\rangle$ and $\exp(\overline{q}_{i,j} u_i u_j) \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u_i, u_j \rangle\rangle$ for any i, j. For any non-negative integer n, we have

$$\frac{(2n)!}{2^n n!} \in \mathbb{Z}$$

(cf. [3, Lemma 11]). Therefore, we have $\exp(\overline{q}_{i,i} u_i^2/2) \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u_i \rangle\rangle$ for any i. Thus, from (4.1), we have $\sigma(u) \in \mathbb{Z}[\widetilde{\boldsymbol{\lambda}}]\langle\langle u \rangle\rangle$. From (4.1), we have

$$\sigma(u)^2 = \exp\left(2cB^{-1}u - {}^t u \, {}^t(B^{-1})NB^{-1}u\right)\tau(B^{-1}u)^2.$$

From (3.18), we have $2c_i \in \mathbb{Z}[\boldsymbol{\lambda}]$ for any i. Therefore we have $\sigma(u)^2 \in \mathbb{Z}[\boldsymbol{\lambda}]\langle\langle u \rangle\rangle$. □

For a non-negative integer n, if n is even, then we set $\chi(n) = 0$, and if n is odd, then we set $\chi(n) = 1$. We set $\overline{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}/2$ if $\sum_{k=1}^m \chi(j_k) \ge 2$ and $\overline{\lambda}^{(i)}_{j_1,\dots,j_m} = \lambda^{(i)}_{j_1,\dots,j_m}$ otherwise. We denote by $\overline{\boldsymbol{\lambda}}$ the set of all $\overline{\lambda}^{(i)}_{j_1,\dots,j_m}$. For a subring R of $\mathbb{C}$, let $2R = \{2r \mid r \in R\}$.

Lemma 4.7. *For even non-negative integers* $k_1, \dots, k_m$, *we define* $\widetilde{p}_n$ *by*

$$\sum_{n=0}^{\infty} \widetilde{p}_n t^n = \left(\sum_{k=0}^{\infty} p_{1,k} t^k\right)^{k_1} \cdots \left(\sum_{k=0}^{\infty} p_{m,k} t^k\right)^{k_m}, \tag{4.2}$$

where $p_{i,k} \in \mathbb{Z}[\boldsymbol{\lambda}]$ *is defined in (3.3). If* n *is odd, then we have* $\widetilde{p}_n \in 2\mathbb{Z}[\boldsymbol{\lambda}]$.

Proof. We differentiate the both sides of (4.2) with respect to t. Since $k_1, \dots, k_m$ are even non-negative integers, we have $n\widetilde{p}_n \in 2\mathbb{Z}[\boldsymbol{\lambda}]$ for any $n \ge 1$. Therefore, if n is odd, then we have $\widetilde{p}_n \in 2\mathbb{Z}[\boldsymbol{\lambda}]$. □

Theorem 4.8. *If* $\sum_{j=1}^{i-1} \chi(\ell_{i,j}) \le 1$ *for any* $2 \le i \le m$, *where* $\ell_{i,j}$ *is defined in (2.1), then we have* $\sigma(u) \in \mathbb{Z}[\overline{\boldsymbol{\lambda}}]\langle\langle u \rangle\rangle$.

Proof. We take an integer k such that a_k is odd. From $\sum_{j=1}^{i-1}\chi(\ell_{i,j}) \leq 1$ and the definition of $\overline{\lambda}^{(i)}_{j_1,\ldots,j_m}$, for any $2 \leq i \leq m$ and $1 \leq j \leq m$, we have the expression

$$\frac{\partial F_i}{\partial x_j} = \sum \kappa^{(i,j)}_{k_1,\ldots,k_m} x_1^{k_1}\cdots x_m^{k_m},$$

where $\kappa^{(i,j)}_{k_1,\ldots,k_m} \in 2\mathbb{Z}[\overline{\boldsymbol{\lambda}}]$ or all of $k_1,\ldots,k_m$ are even non-negative integers. Therefore, the determinant $\det G_k(P)$ has the following form

$$\det G_k(P) = \sum \kappa_{k_1,\ldots,k_m} x_1^{k_1}\cdots x_m^{k_m}, \tag{4.3}$$

where $\kappa_{k_1,\ldots,k_m} \in 2\mathbb{Z}[\overline{\boldsymbol{\lambda}}]$ or all of $k_1,\ldots,k_m$ are even non-negative integers. From Propositions 3.2, 3.3, and Lemma 3.5, $\det G_k(P)$ can be expanded around ∞ with respect to t as follows

$$\det G_k(P) = \frac{(-1)^{k+1}a_k}{t^{2g-1+a_k}}\left(1+\sum_{n=1}^{\infty}\overline{p}_n t^n\right),$$

where $\overline{p}_n \in \mathbb{Z}[1/a_k, \overline{\boldsymbol{\lambda}}]$ for any $n \geq 1$. From Lemma 4.7 and (4.3), if n is odd, then we have $\overline{p}_n \in 2\mathbb{Z}[1/a_k, \overline{\boldsymbol{\lambda}}]$. Let

$$\widetilde{f}(t) = -\sum_{n=1}^{\infty}\overline{p}_n t^n.$$

From

$$\frac{1}{\det G_k(P)} = \frac{t^{2g-1+a_k}}{(-1)^{k+1}a_k}\left(1+\sum_{n=1}^{\infty}\widetilde{f}(t)^n\right),$$

we have the expansion

$$\frac{1}{\det G_k(P)} = \frac{t^{2g-1+a_k}}{(-1)^{k+1}a_k}\left(1+\sum_{n=1}^{\infty}\overline{q}_n t^n\right), \tag{4.4}$$

where $\overline{q}_n \in \mathbb{Z}[1/a_k, \overline{\boldsymbol{\lambda}}]$ for any $n \geq 1$. Further, if n is odd, then we have $\overline{q}_n \in 2\mathbb{Z}[1/a_k, \overline{\boldsymbol{\lambda}}]$. On the other hand, we have the expansion around ∞ with respect to t

$$dx_k = \frac{-a_k}{t^{a_k+1}}\left(1+\sum_{n=1}^{\infty}\widetilde{q}_n t^n\right), \tag{4.5}$$

where $\widetilde{q}_n \in \mathbb{Z}[1/a_k, \overline{\boldsymbol{\lambda}}]$ for any $n \geq 1$. Further, if n is odd, then we have $\widetilde{q}_n \in 2\mathbb{Z}[1/a_k, \overline{\boldsymbol{\lambda}}]$. From Proposition 3.6, we have $b_{g,j} \in \mathbb{Z}[\overline{\boldsymbol{\lambda}}]$ for any $j \geq 1$. From (3.11), (4.4), and (4.5), if j is odd, then we have $b_{g,j} \in 2\mathbb{Z}[\overline{\boldsymbol{\lambda}}]$. Thus, we have

$$\frac{1}{2}\frac{d}{dt}\left(1+\sum_{j=1}^{\infty}b_{g,j}t^j\right) \in \mathbb{Z}[\overline{\boldsymbol{\lambda}}][[t]].$$

As in the case of Lemma 3.14, we have $c_k \in \mathbb{Z}[\overline{\boldsymbol{\lambda}}]$ for any $k \geq 1$. Therefore, as in the case of Theorem 4.6, we obtain the statement of the theorem. □

Remark 4.9. We can apply Theorem 4.8 to the (n, s) curves. The result obtained by applying Theorem 4.8 to the (n, s) curves is equal to [32, Theorem 2.3]. We can apply Theorem 4.8 to the telescopic curves considered in Example 2.3 (iii).

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References

[1] T. Ayano, Sigma functions for telescopic curves, Osaka J. Math., **51** (2014), 459–480.

[2] T. Ayano, On Jacobi Inversion Formulae for Telescopic Curves, SIGMA, **12** (2016), 086, 21 pages.

[3] T. Ayano and V. M. Buchstaber, Analytical and number-theoretical properties of the two-dimensional sigma function, Chebyshevskii Sb., **21** (2020), 9–50.

[4] T. Ayano and A. Nakayashiki, On Addition Formulae for Sigma Functions of Telescopic Curves, SIGMA, **9** (2013), 046, 14 pages.

[5] H. F. Baker, An introduction to the theory of multiply periodic functions, Cambridge University Press, 1907.

[6] H. F. Baker, Abelian Functions. Abel's theorem and the allied theory of theta functions, Cambridge University Press, 1995.

[7] J. Bernatska, V. Enolski and A. Nakayashiki, Sato Grassmannian and Degenerate Sigma Function, Commun. Math. Phys., **374** (2020), 627–660.

[8] O. Bolza, Proof of Brioschi's Recursion Formula for the Expansion of the Even σ-Functions of Two Variables, Amer. J. Math., **21** (1899), 175–190.

[9] O. Bolza, Remarks Concerning the Expansions of the Hyperelliptic Sigma-Functions, Amer. J. Math., **22** (1900), 101–112.

[10] V. M. Buchstaber, V. Z. Enolskii and D. V. Leykin, Hyperelliptic Kleinian Functions and Applications, in Solitons, Geometry, and Topology: On the Crossroad, Amer. Math. Soc. Transl. Ser. 2, **179**, Amer. Math. Soc., Providence, RI, 1997, 1–33.

[11] V. M. Buchstaber, V. Z. Enolskii and D. V. Leykin, Hyperelliptic Abelian Functions, 1997, available at

`https://www.researchgate.net/publication/266955336_Kleinian_functions_hyperelliptic_Jacobians_and_applications`

[12] V. M. Buchstaber, V. Z. Enolskii and D. V. Leykin, Rational Analogs of Abelian Functions, Funct. Anal. Appl., **33** (1999), 83–94.

[13] V. M. Buchstaber, V. Z. Enolskii and D. V. Leykin, Uniformization of Jacobi Varieties of Trigonal Curves and Nonlinear Differential Equations, Funct. Anal. Appl., **34** (2000), 159–171.

[14] V. M. Buchstaber, V. Z. Enolski and D. V. Leykin, Multi-Dimensional Sigma-Functions, arXiv:1208.0990, (2012).

[15] V. M. Buchstaber, V. Z. Enolski and D. V. Leykin, σ-Functions: Old and New Results, In R. Donagi and T. Shaska (Eds.), Integrable Systems and Algebraic Geometry (London Mathematical Society Lecture Note Series 459), Cambridge University Press, 2020, 175–214.

[16] V. M. Buchstaber and D. V. Leykin, Addition Laws on Jacobian Varieties of Plane Algebraic Curves, Proc. Steklov Inst. Math., **251** (2005), 49–120.

[17] V. M. Buchstaber, D. V. Leykin and V. Z. Enolskii, σ-functions of (n, s)-curves, Russ. Math. Surv., **54** (1999), 628–629.

[18] E. Yu. Bunkova, Weierstrass Sigma Function Coefficients Divisibility Hypothesis, arXiv:1701.00848, (2017).

[19] K. Cho and A. Nakayashiki, Differential Structure of Abelian Functions, Int. J. Math., **19** (2008), 145–171.

[20] J. C. Eilbeck, V. Z. Enolskii and D. V. Leykin, On the Kleinian Construction of Abelian Functions of Canonical Algebraic Curves, In proceedings of the Conference SIDE III: Symmetries and Integrability of Difference Equations (Sabaudia, 1998), CRM Proc. Lecture Notes, **25**, Amer. Math. Soc., Providence, RI, 2000, 121–138.

[21] J. C. Eilbeck, J. Gibbons, Y. Ônishi and S. Yasuda, Theory of Heat Equations for Sigma Functions, arXiv:1711.08395, (2017).

[22] J. C. Eilbeck and Y. Ônishi, Recursion Relations on the Power Series Expansion of the Universal Weierstrass Sigma Function, RIMS Kôkyûroku Bessatsu, **B78** (2020), 077–098.

[23] F. Klein, Ueber hyperelliptische Sigmafunctionen, Math. Ann., **27** (1886), 431–464.

[24] F. Klein, Ueber hyperelliptische Sigmafunctionen, Math. Ann., **32** (1888), 351–380.

[25] S. Miura, Linear Codes on Affine Algebraic Curves, IEICE Trans., **J81-A** (1998), 1398–1421 (in Japanese).

[26] A. Nakayashiki, On Algebraic Expressions of Sigma Functions for (n, s) Curves, Asian J. Math., **14** (2010), 175–212.

[27] A. Nakayashiki, Sigma Function as A Tau Function, Int. Math. Res. Not., **2010** (2010), 373–394.

[28] A. Nakayashiki, On hyperelliptic abelian functions of genus 3, J. Geom. Phys., **61** (2011), 961–985.

[29] A. Nakayashiki, Degeneration of trigonal curves and solutions of the KP-hierarchy, Nonlinearity, **31** (2018), 3567–3590.

[30] D. Nishijima, Order counting algorithm for Fermat curves and Klein curves using p-adic cohomology, Master's Thesis, Osaka University, 2008.

[31] Y. Ônishi, Universal Elliptic Functions, (in Japanese), 2016, available at

http://www2.meijo-u.ac.jp/~yonishi/#publications

[32] Y. Ônishi, Arithmetical Power Series Expansion of the Sigma Function for a Plane Curve, Proc. Edinburgh Math. Soc., **61** (2018), 995–1022.