Classical Poincaré conjecture via 4D topology

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ABSTRACT

The classical Poincaré conjecture that every homotopy 3-sphere is diffeomorphic to the 3-sphere is proved by G. Perelman by solving Thurston's program on geometrizations of 3-manifolds. A new confirmation of this conjecture is given by combining R. H. Bing's old result on this conjecture with Smooth Unknotting Conjecture for a 2-link and Smooth 4D Poincaré Conjecture.

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1. Introduction

A *homotopy 3-sphere* is a smooth 3-manifold M homotopy equivalent to the 3-sphere S^3. It is well-known that a simply connected closed connected 3-manifold is a smooth homotopy 3-sphere. The following theorem, called the classical Poincaré Conjecture coming from [22, 23] is positively shown by Perelman [20, 21] solving positively Thurston's program [24] on geometrizations of 3-manifolds (see [19] for detailed historical notes).

Theorem 1.1. Every homotopy 3-sphere M is diffeomorphic to the 3-sphere S^3.

The purpose of this paper is to give an alternative proof to Theorem 1.1 by combining R. H. Bing's result in [2, 3] on the classical Poincaré conjecture with Smooth Unknotting Conjecture and Smooth 4D Poincaré Conjecture to be explained from now on. Let F be a smooth surface-link with a component system F_i, ($i =$

$1, 2, \ldots, n$) in the 4-sphere S^4. The fundamental group $\pi_1(S^4 \setminus F, v)$ (with v a base point) is a *meridian-based free group* if the group $\pi_1(S^4 \setminus F, v)$ is a free group with a basis represented by a meridian system m_i $(i = 1, 2, \ldots, n)$ of F_i, $(i = 1, 2, \ldots, n)$ with a base point v. The smooth surface-link F is a *trivial surface-link* if the components F_i, $(i = 1, 2, \ldots, n)$ bound a disjoint handlebody system smoothly embedded in S^4. Smooth Unknotting Conjecture for a surface-link is the following conjecture.

Smooth Unknotting Conjecture. Every smooth surface-link F in S^4 with a meridian-based free fundamental group $\pi_1(S^4 \setminus F, v)$ is a trivial surface-link.

The positive proof of this conjecture is claimed by [13, 15] with supplement [14]. The result when F is a 2-link (i.e. an S^2-link, a surface-link with only S^2-components) is applied in this paper. A *homotopy 4-sphere* is a smooth 4-manifold X homotopy equivalent to the 4-sphere S^4. Smooth 4D Poincaré Conjecture is the following conjecture.

Smooth 4D Poincaré Conjecture. Every 4D smooth homotopy 4-sphere X is diffeomorphic to the 4-sphere S^4.

The positive proof of this conjecture is claimed by [16, 17]. For the proof of Theorem 1.1, the following result of R. H. Bing in [2, 3] is used:

Bing's Theorem. A homotopy 3-sphere M is diffeomorphic to S^3 if, for every knot k in M, there is a 3-ball in M containing the knot k.

Thus, the main result of this paper is to prove the following lemma.

Lemma 1.2. For every knot k in M, there is a 3-ball in M containing the knot k.

For the proof of Lemma 1.2, Artin's spinning construction of a knot in S^3 in [1] is generalized into a connected graph in a homotopy 3-sphere M to produce a spun S^2-link in S^4 with free fundamental group (not always meridian-based free group). This explanation is done in Section 2. In Section 3, it is shown that every S^2-link in S^4 with free fundamental group is a ribbon S^2-link by using Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture. In Section 4, the proof of Lemma 1.2 is done. To do this, it is shown that the spun torus-knot of every knot k in M is a ribbon-torus knot in S^4 by which it is shown that the knot k is a tangle sum of a proper arc system a_* in a boundary collar of a compact once-punctured manifold $M^{(o)}$ of M and a proper arc system e_* in $M^{(o)}$ with meridian-based free fundamental group $\pi_1(M^{(o)} \setminus e_*, v)$. To see this, an argument of a chord diagram of the spun S^2-link of a proper arc system a_* in a boundary collar of $M^{(o)}$ in [12] is used. The knot k with the last condition is shown to be in a 3-ball of M. In this way, the proof of Lemma 1.2 is completed. Thus, the proof of Theorem 1.1 is completed.

Conventions. The unit n-disk is denoted by D^n with the origin $\mathbf{0}$ as a standard notation, but the unit 2-disk D^2 is fixed in the complex plane $\mathbb{C}$. A smooth n-manifold diffeomorphic to the unit n-disk D^n is called an *n-ball* for $n \geq 3$ or *n-disk* for $n = 2$. A point $\mathbf{1}$ is fixed in the n-sphere $S^n = \partial D^{n+1}$.

2. Artin's spinning construction of a connected graph in a homotopy 3-sphere

For a homotopy 3-sphere M, let $M^{(o)}$ be the compact once-punctured manifold $\mathrm{cl}(M \setminus B)$ of M for a 3-ball B in M. Let

$$S = \partial B = \partial M^{(o)}$$

be the boundary 2-sphere of $M^{(o)}$. The closed smooth 4-manifold $X(M)$ defined by

$$X(M) = M^{(o)} \times S^1 \cup S \times D^2$$

is called the *spun manifold* of M with *axis* 4-submanifold $S \times D^2$. As a convention, the 3-submanifold $M^{(o)} \times 1$ of the product $M^{(o)} \times S^1$ is identified with $M^{(o)}$. In particular, a point $(q, 1) \in M^{(o)} \times 1$ is identified with the point $q \in M^{(o)}$. This 4-manifold $X(M)$ is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument and hence $X(M)$ is diffeomorphic to the 4-sphere S^4 by Smooth 4D Poincaré Conjecture. A *legged loop* with *base point* v is the union $k \cup \omega$ of a loop k and an arc ω joining the base point v with a point of k. The arc ω is called a *leg*. A *legged loop system* with base point v is the union

$$\gamma = \cup_{i=1}^{n} k_i \cup \omega_i$$

of n legged loops $k_i \cup \omega_i \, (i = 1, 2, \ldots, n)$ meeting only at the same base point v. Let $k(\gamma) = \cup_{i=1}^{n} k_i = k_*$ denote the loop system of the legged loop system γ. Let $\omega_* = \cup_{i=1}^{n} \omega_i$ and $v_* = k_* \cap \omega_*$. A regular neighborhood B of ω_* in M is taken as a 3-ball B used for the compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of M. Deform the subgraph $\gamma \cap B$ of γ so that

$$\omega_* \subset B, \quad \omega_* \cap S = \{v\} \cup v_* \quad \text{and} \quad k_* \cap B = k_* \cap S = a'_*$$

for a regular neighborhood arc system a'_* of v_* in k_*. Let

$$a(\gamma) = \cup_{i=1}^{n} a_i = a_*$$

for a proper arc $a_i = \mathrm{cl}(k_i \setminus a'_i)$ $(i = 1, 2, \ldots, n)$ in $M^{(o)}$. Let

$$\dot{a}(\gamma) = \partial a_* = \partial a'_*$$

be the set of $2n$ points in the boundary 2-sphere S of $M^{(o)}$. The *spun* S^2-link of the graph γ is the S^2-link $S(\gamma)$ in the 4-sphere $X(M)$ defined by

$$S(\gamma) = a(\gamma) \times S^1 \cup \dot{a}(\gamma) \times D^2.$$

Lemma 2.1. The inclusion $M^{(o)} \setminus a(\gamma) \subset X(M) \setminus S(\gamma)$ induces an isomorphism

$$\sigma : \pi_1(M \setminus \gamma, v) \to \pi_1(X(M) \setminus S(\gamma), v)$$

sending a meridian system of the proper arc system $a(\gamma)$ in $M^{(o)}$ to a meridian system of $S(\gamma)$.

Proof of Lemma 2.1. Note that there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus a(\gamma), v) \cong \pi_1(M \setminus \gamma, v).$$

Then the desired isomorphism σ is obtained by applying the van Kampen theorem between $(M^{(o)} \setminus a(\gamma)) \times S^1$ and $(S \setminus \dot{a}(\gamma)) \times D^2$. This completes the proof of Lemma 2.1. □

Here is a note on Lemma 2.1.

Note 2.2. A general connected graph γ with Euler characteristic $\chi(\gamma) = 1 - n$ in M is deformed into a legged loop system γ in M by choosing a maximal tree to shrink to a base point v. Note that there are only finitely many maximal trees of γ such that the loop systems $k(\gamma)$ of the resulting legged loop systems γ are distinct as links. By Lemma 2.1, we can obtain finitely many distinct spun S^2-links in S^4 with isomorphic fundamental groups obtained by taking different maximal trees of the connected graph γ. This is a detailed explanation on the spun S^2-link of a connected graph associated with a maximal tree in [7, p.204] when $M = S^3$.

An argument on Lemma 2.1 is further developed when the homotopy 3-sphere M is given by a Heegaard spitting $V \cup V'$ pasting along a Heegaard surface $F = \partial V = \partial V'$ of genus n. A *spine* of a handlebody V of genus n is a legged loop system γ in $F = \partial V$ with base point v such that the inclusion map $\gamma \to V$ induces an isomorphism $\pi_1(\gamma, v) \to \pi_1(V, v)$. A regular neighborhood $\dot{V}$ of γ in F is a planar surface in F.

By [5, Theorem 10.2], there is a diffeomorphism $(\dot{V} \times [0,1], \dot{V} \times 0) \to (V, \dot{V})$ sending every point $(x, 0) \in \dot{V} \times 0$ to $x \in \dot{V}$. The surface $\dot{V}$ is called a *spine surface* of V. Let γ and γ' be spines of the handlebodies V and V' in F with the same base point v, respectively. A *legged Heegaard loop system* in M is a legged loop system $\gamma\gamma'$ in M with base point v obtained by pushing $\gamma \setminus v$ and $\gamma' \setminus v$ into the interiors $\mathrm{Int}V$ and $\mathrm{Int}V'$, respectively. The fundamental groups of the spun S^2-links $S(\gamma)$, $S(\gamma')$ and $S(\gamma\gamma') = S(\gamma) \cup S(\gamma')$, in the 4-sphere $X(M)$ given by Lemma 2.1 are free groups, as shown in the following lemma:

Lemma 2.3. The fundamental groups $\pi_1(X(M) \setminus S(\gamma), v)$ and $\pi_1(X(M) \setminus S(\gamma'), v)$ are free groups of rank n and the fundamental group $\pi_1(X(M) \setminus S(\gamma\gamma'), v)$ is a free group of rank $2n$.

Proof of Lemma 2.3. The closed complements $\mathrm{cl}(M \setminus N(\gamma))$, $\mathrm{cl}(M \setminus N(\gamma'))$ and $\mathrm{cl}(M \setminus N(\gamma\gamma'))$ are diffeomorphic to the handlebodies V', V and $F^{(o)} \times [0,1]$ for the once-punctured surface $F^{(o)}$ of F, respectively. Since the fundamental groups $\pi_1(V', v)$, $\pi_1(V, v)$ and $\pi_1(F^{(o)} \times [0,1], v)$ are free groups of ranks n, n and $2n$, respectively, the desired result is obtained from Lemma 2.1. □

It should be noted that these free groups in Lemma 2.3 are not necessarily meridian-based free groups. Here is an example.

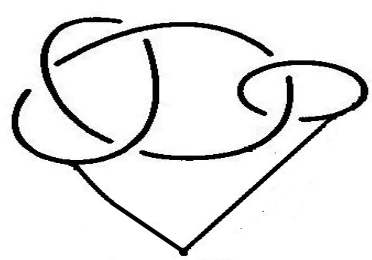

Figure 1: A legged loop system γ in S^3 with free fundamental group of rank 2

Example 2.4. Let γ be a legged loop system with base point v in S^3 illustrated in Fig. 1 with free fundamental group $\pi_1(S^3 \setminus \gamma, v)$ of rank 2. In fact, a trivial legged loop system is obtained by sliding an edge along another edge, so that the fundamental group $\pi_1(S^3 \setminus k(\gamma), v)$ is a free group of rank 2. A regular neighborhood V of γ in S^3 and the closed complement $V' = \mathrm{cl}(S^3 \setminus V)$ constitute a genus 2 Heegaard splitting $V \cup V'$ of S^3 by noting that the 3-manifold V' is a handlebody of genus 2 by the loop system theorem and the Alexander theorem (cf. e.g., [7]). Thus, the union $V \cup V'$ is a genus 2 Heegaard splitting of S^3. The legged loop system γ with vertex v is a spine of V by sliding the base point v into ∂V. By Lemma 2.3, the spun S^2-link $S(\gamma)$ in the 4-sphere $X(S^3) = S^4$ has the free fundamental group $\pi_1(X(S^3) \setminus S(\gamma), v)$ of rank 2, which does not admit any meridian basis because the S^2-link $S(\gamma)$ contains a component of the spun trefoil S^2-knot in S^4 whose fundamental group is known to be not infinite cyclic.

Given a proper arc system a_* in $M^{(o)}$, there is a legged loop system γ in M with the proper arc system $a(\gamma) = a_*$ in $M^{(o)}$. The spun S^2-link $S(\gamma)$ in $X(M)$ is uniquely determined by the arc system a_* and thus denoted by $S(a_*)$. The following lemma is directly used for the proof of Lemma 1.2.

Lemma 2.5. Let a_* be a proper arc system in a compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of a homotopy 3-sphere M. If the spun S^2-link $S(a_*)$ in the 4-sphere $X(M)$ is a trivial S^2-link, then the proper arc system a_* is in a boundary-collar $S \times [0, 1]$ of $M^{(o)}$.

Proof of Lemma 2.5. By Lemma 2.1, the fundamental group $\pi_1(M^{(o)} \setminus a(\gamma), v)$ is a meridian-based free group. Consider the 2-sphere S is the boundary of the product $d \times [0, 1]$ for a disk d so that $d \times 0$ contains one end of the proper arc system a_* and $d \times 1$ contains the other end of the proper arc system a_*. Let $(E; E_0, E_1)$ be the triplet obtained from $(M^{(o)}, d \times 0, d \times 1)$ by removing a tubular neighborhood of a_* in $M^{(o)}$. Then the inclusion $E_0 \subset E$ induces an isomorphism

$$\pi_1(E_0, v) \to \pi_1(E, v).$$

By [5, Theorem 10.2], E is diffeomorphic to the connected sum of the product $E_0 \times [0, 1]$ and a homotopy 3-sphere. This means that the proper arc system a_* is in a boundary-collar $S \times [0, 1]$. This completes the proof of Lemma 2.5. □

3. Ribbonness of an S^2-link with free fundamental group

The *4D handlebody* of *genus* n is the boundary 3-disk sum

$$Y^D = D^4 \natural_{i=1}^n S^1 \times D^3_i$$

obtained from n copies $S^1 \times D^3_i$ $(i = 1, 2, \ldots, n)$ of the 4D solid torus $S^1 \times D^3$ and the 4-disk D^4 by pasting a 3-disk system consisting of a boundary 3-disk in $(S^1 \setminus \{1\}) \times D^3_i$ for every i to a system of disjoint n boundary 3-disks of D^4. A legged loop system γ^D in the 4D handlebody Y^D of genus n is *standard* if the legged loop system γ^D has the following two conditions:

- The loop system $k(\gamma^D)$ is consistent with the system $S^1 \times \mathbf{1}_i\,(i = 1, 2, \ldots, n)$, and
- The base point v is in the 4-disk D^4 and the legs $\omega_i\,(i = 1, 2, \ldots, n)$ of γ^D do not meet the 3-disks $1 \times D^3_i\,(i = 1, 2, \ldots, n)$.

Note that the legs $(i = 1, 2, \ldots, n)$ of γ^D are ∂-relatively unique up to isotopies in Y^D. The *4D closed handlebody* of *genus* n is the double of the 4D handlebody Y^D of genus n, that is the 4-manifold

$$\partial(Y^D \times [0, 1]) = Y^D \times 0 \cup (\partial Y^D) \times [0, 1] \cup Y^D \times 1$$

which is canonically identified with the following 4-manifold

$$Y^S = S^4 \#_{i=1}^n S^1 \times S^3_i,$$

where the connected summands S^4 and $S^1 \times S^3_i$ correspond to the doubles of the 3-disk summands D^4 and $S^1 \times D^3_i$, respectively. The 4D handlebody $Y^D \times 0$ in Y^S is identified with Y^D. A legged loop system γ with vertex v of the 4D closed handlebody Y^S of genus n is *standard* if it is v-relatively isotopic to a standard legged loop system γ^D of $Y^D \subset Y^S$. A standard legged loop system of Y^S is denoted by γ^S. A homology 4-sphere is a smooth 4-manifold X with an isomorphism $H_*(X; \mathbf{Z}) \cong H_*(S^4; \mathbf{Z})$. A *4D closed homology handlebody* of *genus* n is a smooth 4-manifold Y with an isomorphism $H_*(Y; \mathbf{Z}) \cong H_*(Y^S; \mathbf{Z})$ for the 4D closed handlebody Y^S of genus n. For an S^2-link L in a homology 4-sphere X, take a normal disk bundle $L \times D^2$ in X and a 3-disk system D^3_L with $\partial D^3_L = L$. The transformation from X into the 4-manifold

$$Y = \mathrm{cl}(X \setminus L \times D^2) \cup D^3_L \times S^1$$

is called the *surgery* of X along the S^2-link L. Conversely, the transformation from Y into X is called the *surgery* of Y along the loop system $\mathbf{0}_* \times S^1$ by observing that $D^3_L \times S^1$ is a regular neighborhood of $\mathbf{0}_* \times S^1$ in Y. The following lemma is a more or less known fact.

Lemma 3.1. Let Y be the 4-manifold obtained from a homology 4-sphere X by surgery along any n-component S^2-link L. Then the 4-manifold Y is a 4D closed homology handlebody of genus n such that the inclusion $X \setminus L \times D^2 \subset Y$ induces an isomorphism

$$\pi_1(X \setminus L \times D^2, v) \to \pi_1(Y, v).$$

Proof of Lemma 3.1. To see that $H_2(Y; \mathbf{Z}) = 0$, use the Euler characteristic $\chi(Y) = 2n$. Since $H_1(Y; \mathbf{Z}) \cong \mathbf{Z}^n$, we have $H_2(Y; \mathbf{Z}) = 0$ by Poincaé duality, which shows that Y is a 4D closed homology handlebody of genus n. The isomorphism $i_* : \pi_1(X \setminus L \times D^2, v) \to \pi_1(Y, v)$ is obtained by a general position argument. □

A *meridian system* of an S^2-link L in X is a legged loop system γ_L in the closed complement $\mathrm{cl}(X \setminus L \times D^2)$ for a normal disk bundle $L \times D^2$ in X such that the loop system $k(\gamma_L)$ is the loop system $p_* \times S^1$ for a point system p_* in L with one point for every component of L. By Lemma 3.1, note that the meridian system γ_L induces a legged loop system γ in Y such that the loop system $k(\gamma)$ represents a homological basis of the homology group $H_1(Y; \mathbf{Z})$. Conversely, given any legged loop system γ in Y such that the loop system $k(\gamma)$ represents a homological basis of $H_1(Y; \mathbf{Z})$, then the 4-manifold X obtained from Y by surgery along the loop system $k(\gamma)$ is a homology 4-sphere and the legged loop system γ induces a meridian system γ_L of the S^2-link L in X obtained by surgery. A *4D closed homotopy handlebody* of *genus* n is a 4D closed homology handlebody Y of genus n such that the fundamental group $\pi_1(Y, p)$ is a free group of rank n. A legged loop system γ with base point v in a 4D closed homotopy handlebody Y of genus n is a *basis* if the inclusion $\gamma \subset Y$ induces an isomorphism

$$\pi_1(\gamma, v) \to \pi_1(Y, v).$$

For example, a standard legged loop system γ^S of the 4D closed handlebody Y^S is a basis. The following classification lemma is a result of Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture.

Lemma 3.2. Let Y^S be the 4D closed handlebody of genus n, and γ^S a standard legged loop system with base point v^S of Y^S. For every 4D closed homotopy handlebody Y of genus n and every basis γ in Y, there is an orientation-preserving diffeomorphism

$$f : Y \to Y^S$$

such that $f(\gamma) = \gamma^S$. Given any spin structures on Y and Y^S, the diffeomorphism f can be taken spin-structure-preserving.

Proof of Lemma 3.2. Let X be the 4-manifold obtained from Y by surgery along the loop system $k_* = k(\gamma)$. This 4-manifold X is diffeomorphic to the 4-sphere S^4 by Smooth 4D Poincaré Conjecture since it is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument. Since X is obtained from Y by replacing a normal 3-disk bundle $k_* \times D^3$ of k_* in Y with $D^2_* \times S^2$ for the disk system D^2_* bounded by k_*. Then there is an S^2-link $L = 0_* \times S^2$ in X. Since the basis γ of Y induces a meridian system of L in X, Lemma 3.1 implies that the fundamental group $\pi_1(X \setminus L, v)$ is a meridian based free group. By Smooth Unknotting Conjecture for an S^2-link, the S^2-link L is a trivial S^2-link in the 4-sphere X. By the back surgery replacing $D^2_* \times S^2$ in X with $k(\gamma) \times D^3$ in Y, there is an orientation-preserving diffeomorphism $f : Y \to Y^S$ with $f(k_*) = k(\gamma^S_*)$. Since a regular neighborhood $N(f(\gamma))$ of $f(\gamma)$ in Y^S is isotopic to Y^D in Y^S, the diffeomorphism $f : Y \to Y^S$ is modified to have $f(\gamma) = \gamma^S$. Given any spin structures on Y and Y^S, note that there is an orientation-preserving spin-structure-changing diffeomorphism : $S^1 \times S^3 \to S^1 \times S^3$ (see [4] for a similar diffeomorphism on $S^1 \times S^2$). Thus, by composing f with the orientation-preserving spin-structure-changing diffeomorphisms on some connected summands of Y^S which are copies of $S^1 \times S^3$, the diffeomorphism $f : Y \to Y'$ is modified into an orientation-preserving spin-structure-preserving diffeomorphism. This completes the proof of Lemma 3.2. □

The following corollary is directly obtained from Lemmas 2.3, 3.1 and 3.2.

Corollary 3.3. Let $\gamma\gamma'$ be a legged Heegaard loop system of a homotopy 3-sphere M associated with a Heegaard splitting $V \cup V'$ of genus n, and $Y(M;\gamma\gamma')$ the 4D closed homology handlebody obtained from the 4-sphere $X(M)$ by surgery along the spun S^2-link $S(\gamma\gamma')$ of $\gamma\gamma'$. Then the 4D closed homology handlebody $Y(M;\gamma\gamma')$ is diffeomorphic to the 4D closed handlebody Y^S of genus $2n$.

A surface-link L in S^4 is a *ribbon* surface-link if L is equivalent to a surface-link obtained from a trivial S^2-link L_0 in S^4 by surgery along embedded 1-handles on L_0 (see [18]). The following lemma is obtained by using lemma 3.2.

Lemma 3.4. Any S^2-link L in S^4 with free fundamental group $\pi_1(S^4 \setminus L, v)$ is a ribbon S^2-link.

Proof of Lemma 3.4. Let $K_i\,(i = 1, 2, \ldots, n)$ be the components of L. The following observation is used to determine a ribbon S^2-link.

(3.4.1) Let $(S^3_i)^{(1+m_i)}\,(i = 1, 2, \ldots, n)$ be a system of mutually disjoint compact $(1+m_i)$-punctured 3-spheres in S^4 such that the boundary $\partial(S^3_i)^{(1+m_i)}$ is the union of the component K_i and an S^2-link O_i of m_i components. If the union $O = \cup_{i=1}^n O_i$ is a trivial S^2-link in S^4, then the S^2-link $L = \cup_{i=1}^n K_i$ is a ribbon S^2-link in S^4.

Proof of (3.4.1). Let K'_i be a 2-sphere obtained from O_i by surgery along mutually disjoint 1-handles $h_i\,(i = 1, 2, \ldots, m_i - 1)$ in $(S^3_i)^{(1+m_i)}$, whose closed complement is diffeomorphic to the spherical shell $S^2 \times [0, 1]$. This means that the component K_i with reversed orientation is isotopic to the 2-sphere K'_i in $(S^3_i)^{(1+m_i)}$. This shows that $L = \cup_{i=1}^n K_i$ is a ribbon S^2-link in S^4, completing the proof of (3.4.1). □

Let Y be the 4-manifold obtained from S^4 by surgery along L. Let γ be a legged loop system in Y induced from a meridian system γ_L of L in S^4. Let $k(\gamma) = k_*$ be the loop system of γ in Y. By Lemma 3.2, Y is identified with the closed 4D handlebody Y^S of genus n. Let $S^1 \times S^3_i = k^S_i \times S^3 \, (i = 1, 2, \ldots, n)$ be the 4D closed handle summands in $Y = Y^S$. For every i, let $S^3_i = p_i \times S^3_i$ be the 3-sphere factor of the 4D closed handle $k^S_i \times S^3$ for a point $p_i \in k^S_i$, and $k^S_i = k^S_i \times p'_i$ the loop factor for a point $p'_i \in S^3$. Let γ^S be a standard legged loop system of $Y = Y^S$ with $k(\gamma^S) = k^S_*$ and with the same vertex v. The legged loop systems γ and γ^S are made disjoint except for the vertex v. Let $x_i = [k^S_i] \, (i = 1, 2, \ldots, n)$ be a basis of the free group $\pi_1(Y, v)$ of rank n represented by γ^S for every i. Let $y_i = [k_i] \, (i = 1, 2, \ldots, n)$ be an element system in $\pi_1(Y, v)$ represented by γ for every i. By a basis change of the basis $x_i \, (i = 1, 2, \ldots, n)$, assume that the product $x_i^{-1} y_i$ is in the commutator subgroup $[\pi_1(Y, v), \pi_1(Y, v)]$ of $\pi_1(Y, v)$ for every i.

Consider the meeting situation of the loop system k_* with the 3-sphere system S^3_*. There is a transverse intersecting point q_i between k_i and S^3_i for every i such that except for this point system q_* the loop k_i meets S^3_j transversely with a finite number of point pairs $(a_{ijs}, b_{ijs}) \, (s = 1, 2, \ldots, t_{ij})$ with opposite signs. Let α_{ijs} be the arc in the open arc $k_i \setminus \{q_i\}$ cut by the pair (a_{ijs}, b_{ijs}). For every i and j, the pair system $(a_{ijs}, b_{ijs}) \, (s = 1, 2, \ldots, t_{ij})$ is a *coupling system* if

$$\alpha_{ijs} \subset \alpha_{ijs'} \quad \text{or} \quad \alpha_{ijs} \cap \alpha_{ijs'} = \emptyset$$

for every $s < s'$. The pair system $(a_{ijs}, b_{ijs}) \, (s = 1, 2, \ldots, t_{ij})$ is always indexed to be a coupling system. For every $j \, (j = 1, 2, \ldots, n)$, let $B(q_j), B(a_{*j*}), B(b_{*j*})$ be the 3-ball neighborhoods $(k_i \times D^3) \cap S^3_j$ of the points q_j, a_{ijs}, b_{ijs} for all $i \, (1 \leq i \leq n)$ and $s \, (s = 1, 2, \ldots, t_{ij})$ in S^3_j, which are sections of the normal 3-disk bundle $k_i \times D^3$ over k_i. Let $U\{ijs\}$ be the 3-ball obtained from $B(a_{ijs})$ and $B(b_{ijs})$ by adding a 1-handle h_{ijs} in S^3_j thickening an arc β_{ijs} joining $B(a_{ijs})$ and $B(b_{ijs})$ whose interior does not meet $(k_* \times D^3) \cap S^3_j$, where note that the 1-handle h_{ijs} is unique by an isotopy keeping the attaching part. Let $(S^3)^{(0)}_j = \mathrm{cl}(S^3_j \setminus B(q_j))$. Let $(S^3)^{(1+m_j)}_j$ be the compact m_j-punctured 3-sphere of S^3_j with $m_j = \Sigma^n_{i=1} t_{ij}$ obtained by removing the interiors of the 3-balls $B(q_j), U_{ijs} \, (i = 1, 2, \ldots, n, s = 1, 2, \ldots, t_{ij})$. Let $S(q_j)$ and S_{ijs} be the boundary spheres of $B(q_j)$ and U_{ijs}, respectively. Do the surgery of Y along a normal 3-disk bundle system $k_* \times D^3$ of k_* in Y to obtain the 4-sphere $X = S^4$ where $k_i \times D^3$ is changed into the normal 2-disk bundle system $D^2 \times K_i$ of the component K_i of the S^2-link L in X. The compact m_j-punctured 3-sphere $(S^3)^{(1+m_j)}_j$ is canonically embedded in X where the 2-sphere $S(q_j)$ is identified with the component K_j of L and the 2-sphere S_{ijs} is a section of the S^2-bundle $(\partial D^2) \times K_i$ over the circle ∂D^2. The following claim is shown.

(3.4.2) The 2-spheres $S_{ijs}\,(i,j=1,2,\ldots,n, s=1,2,\ldots,t_{ij})$ form a trivial S^2-link in X.

By (3.4.1) and (3.4.2), the S^2-link $L=\cup_{j=1}^n K_j$ is shown to be a ribbon S^2-link in S^4.

Proof of (3.4.2). Let X^S be the 4-sphere S^4 obtained from Y by surgery along γ^S, and $L^S=\cup_{i=1}^n K_i^S$ be the trivial S^2-link in X^S with K_i^S the boundary of a once-punctured compact manifold $(S_i^3)^{(0)}$ of S_i^3.

First consider the meeting between the loop system k_* and the compact once-punctured 3-sphere $(S_1^3)^{(0)}$. There are disjoint disks $\Delta_i\,(i,j=1,2,\ldots,n)$ in the 4-sphere X^S with $\partial\Delta_i=k_i\,(i=1,2,\ldots,n)$ such that

(i) The intersection $\Delta_1\cap(S_1^3)^{(0)}$ is a disjoint union of the point q_1 and disjoint simple arcs $\alpha'_s\,(s=1,2,\ldots,n_{11})$ with $\partial\alpha'_{11}=\partial\alpha_{11}=\{a_{11},b_{11}\}\,(s=1,2,\ldots,t_{11})$,

(ii) For every $i\geq 2$, the intersection $\Delta_i\cap(S_1^3)^{(0)}$ is a transverse intersection arc system consisting of simple arcs $\alpha'_{i1s}\,(s=1,2,\ldots,n_{i1})$ with $\partial\alpha'_{i1s}=\partial\alpha_{i1s}=\{a_{i1},b_{i1}\}\,(s=1,2,\ldots,t_{i1})$, and

(iii) For every $i=1,2,\ldots,n$ and every $s=1,2,\ldots,t_{i1}$, the union $\ell_{i1s}=\alpha'_{i1s}\cup\alpha_{i1s}$ is a simple loop which bounds a disk δ_{i1s} in Δ_i.

Do the surgery of the 4-sphere X^S into the 4-sphere X along a normal 3-disk bundle system $k_*\times D^3$ of k_* in X^S and a normal 2-disk bundle system $K_*^S\times D^2$ of K_*^S in X^S where $k_*\times D^3$ is changed into a normal 2-disk bundle system $D^2\times L$ of the S^2-link L in X and $K_*^S\times D^2$ is changed into a normal 3-disk bundle system $D^3\times k_*^S$ in X. Note that the boundary sphere system $\partial(S_1^3)^{(1+m_1)}$ in X consists of the spheres $K_1=S(q_1)$ and $S_{i1s}\,(i=1,2,\ldots,n, s=1,2,\ldots,t_{i1})$. Let $\hat{\alpha}'_{i1s}=\alpha'_{i1s}\cap(S_1^3)^{(1+m_1)}$ for every j and s. Let $\hat{\Delta}_i$ be the 2-disk obtained from Δ_i by removing the annulus $\Delta_i\cap(k_i\times D^3)$ and then taking the closure for every i. Let $(\hat{\alpha}_{i1s};\hat{a}_{i1s},\hat{b}_{i1s})$ be a translation of the arc $(\alpha_{i1s};a_{js'},b_{i1s})$ in the loop $k_i=\partial\Delta_i$ into the loop $\partial\hat{\Delta}_i$ so that $\partial\hat{\alpha}_{i1s}=\partial\hat{\alpha}'_{i1s}=\{\hat{a}_{i1s},\hat{b}_{i1s}\}$ where $\hat{a}_{i1s}\in S(a_{i1s})$ and $\hat{b}_{i1s}\in S(b_{i1s})$. Note that the arcs $\hat{\alpha}_{i1s}$ and $\hat{\alpha}'_{i1s}$ are isotopic through the disk $\delta_{i1s}\cap\hat{\Delta}_i$ relative to the 2-spheres $S(a_{i1s})$ and $S(b_{i1s})$. The following observation on the spherical shell $S^2\times[0,1]$ is used for the proof of (3.4.2) (whose proof is omitted since it is an easy exercise).

Observation 3.4.3 The 2-sphere S' obtained from the 2-spheres $S^2\times\{0,1\}$ by surgery along a 1-handle h' thickening the arc $p\times[0,1]\,(p\in S^2)$ bounds the unique 3-ball $B'=\mathrm{cl}(S^2\times[0,1]\setminus h')$. Further, let S'' obtained from the 2-spheres $S^2\times\{\frac{1}{4},\frac{3}{4}\}$ by surgery along a 1-handle h'' thickening the arc $p \times [\frac{1}{4}, \frac{3}{4}]$, and $B'' = \mathrm{cl}(S^2 \times [\frac{1}{4}, \frac{3}{4}] \setminus h'')$ the 3-ball bounded by S''. If the 1-handle h' is thinner than the 1-handle h'', then the 3-ball B'' is in the interior of the 3-ball B'.

Consider the 2-sphere S'_{i1s} obtained from the 2-spheres $S(a_{i1s})$ and $S(b_{i1s})$ by surgery along a 1-handle thickening the arc $\hat{\alpha}_{i1s}$ in the spherical shell $S^2 \times \hat{\alpha}_{i1s}$ bounded by $S(a_{i1s})$ and $S(b_{i1s})$ in the S^2-bundle $(\partial D^2) \times K_i$. By Observation 3.4.3, the 2-spheres S_{i1s} for all $i, s\,(i = 1, 2, \ldots, n, s = 1, 2, \ldots, t_{i1})$ form a trivial S^2-link in X. Also, consider the 2-sphere S''_{i1s} obtained from the 2-spheres $S(a_{i1s})$ and $S(b_{i1s})$ by surgery along a 1-handle thickening the arc $\hat{\alpha}'_{i1s}$. By the isotopy between the arcs $\hat{\alpha}_{i1s}$ and $\hat{\alpha}'_{i1s}$ through the disk $\delta_{i1s} \cap \hat{\Delta}_i$ relative to the 2-spheres $S(a_{i1s})$ and $S(b_{i1s})$, the 2-spheres $\tilde{S}'_{i1s}$ for all $i, s\,(i = 1, 2, \ldots, n, s = 1, 2, \ldots, t_{i1})$ form a trivial S^2-link in X. Since $\tilde{S}'_{i1s}$ is identified with S_{i1s}, it is shown that the 2-spheres S_{i1s} for all $i, s\,(i = 1, 2, \ldots, n, s = 1, 2, \ldots, t_{i1})$ form a trivial S^2-link in X.

Next consider the meeting between the loop system k_* and the compact once-punctured 3-sphere $(S^3_2)^{(0)}$. For this purpose, take a 3-disk fiber D^3 of the normal 3-disk bundle system $k_* \times D^3$ of k_* in X^S larger than the 3-disk fiber D^3 of the normal 3-disk bundle system $k_* \times D^3$ of k_* in X^S used for the meeting between k_* and $(S^3_1)^{(0)}$. Then by a similar argument on the meeting between k_* and $(S^3_1)^{(0)}$, it is shown that the 2-spheres S_{ijs} for all $i, j, s\,(i = 1, 2, \ldots, n; j = 1, 2, s = 1, 2, \ldots, t_{ij})$ form a trivial S^2-link in X, where note that there is an isotopic deformation from $\hat{\alpha}'_{i2s}$ to $\hat{\alpha}_{i2s}$ avoiding intersection with the arc system $\hat{\alpha}_{*1*}$ by general position.

By continuing this process, the 2-spheres $S_{ijs}\,(i, j = 1, 2, \ldots, n, s = 1, 2, \ldots, t_{ij})$ form a trivial S^2-link in X. This completes the proof of (3.4.2). □

This completes the proof of Lemma 3.4. □

A group presentation $(y_1, y_2, \ldots, y_{n+s} | \, r_1, r_2, \ldots, r_s)$ of deficiency n is a *Wirtinger presentation* if every relator r_i is written as a form $y_{j_i}^{-1} w_j y_{j'_i} w_i^{-1}$ for two generators $y_j j_i, y_{j'_i}$ with distinct indexes j_i, j'_i and a word w_i in the letters $y_j\,(j = 1, 2, \ldots, n + s)$. It is known that the fundamental group of an n-component ribbon S^2-link has a Wirtinger presentation of deficiency n for some s (cf. [7, p. 193], [18, pp. 56-60]). An algebraic version of Lemma 3.4 means the following result in combinatorial group theory.

Corollary 3.5. Let $\mathbf{F}_n$ be the free group of rank n with a basis $x_i\,(i = 1, 2, \ldots, n)$. Let $x'_i\,(i = 1, 2, \ldots, n)$ be a set of elements normally generating the free group $\mathbf{F}_n$ written as words in the letters $x_i\,(i = 1, 2, \ldots, n)$ such that the products $x'_i x_i^{-1}\,(i = 1, 2, \ldots, n)$ belong to the commutator subgroup $[\mathbf{F}_n, \mathbf{F}_n]$ of $\mathbf{F}_n$. Then the free group $\mathbf{F}_n$ admits a Wirtinger presentation

$$(y_1, y_2, \dots, y_{n+s} | \, r_1, r_2, \dots, r_s)$$

of deficiency n for some s such that the elements $y_i \, (i = 1, 2, \dots, n + s)$ are written as words in the letters $x_i \, (i = 1, 2, \dots, n)$ containing the elements $x'_i \, (i = 1, 2, \dots, n)$ as the given words.

4. Main result: Proof of Lemma 1.2

The following observation relates a knot to a Heegaard splitting of a closed connected orientable 3-manifold.

Lemma 4.1. For any knot k in any closed connected orientable 3-manifold M, there is a Heegaard splitting $V \cup V'$ of M such that the knot k is equivalent to a component of the loop system $k(\gamma)$ of a spine γ of V in M.

Proof of Lemma 4.1. By considering k as a polygonal loop in M, there is a triangulation $\mathcal{T}$ of M whose 1-skeleton $\mathcal{T}^{(1)}$ contains the knot k. The graph $\mathcal{T}^{(1)}$ is deformed into a legged loop system γ in M so that k is a component of the loop system $k(\gamma)$. Let V be a regular neighborhood of γ in M which is a handlebody. The closed complement $V' = \mathrm{cl}(M \setminus V)$ is also a handlebody, so that we have a Heegaard splitting $V \cup V'$ of M. The legged loop system γ is deformed into a spine of the handlebody V. $\square$

By combining Lemmas 2.3, 3.4 with Lemma 4.1, the following corollary is obtained, because any component of a ribbon S^2-link in S^4 is a ribbon S^2-knot in S^4.

Corollary 4.2. For any knot k in any homotopy 3-sphere M, the spun S^2-knot $S(k)$ of k in $X(M) = S^4$ is a ribbon S^2-knot in S^4.

A chord diagram is a diagram C in S^2 consisting of a based loop system o (i.e., a trivial oriented link diagram) and a chord system α joining the based loops where intersections among the chords are permitted (see [8, 9, 10, 11, 12] for the detailed arguments). From a chord diagram $C = C(o, \alpha)$ in S^2, a ribbon surface-link $R(C)$ in the 4-sphere S^4 is constructed in a unique way. In fact, the ribbon surface-link $R(C)$ is obtained from a trivial oriented S^2-link L^0 in S^4 constructed from the based loop system o by surgery along an embedded 1-handle system $h(\alpha)$ on L^0 thickening the chord system α. The ribbon surface-link $R(C)$ in S^4 is uniquely constructed from the chord diagram C by using the Horibe-Yanagawa's lemma in [18] for uniqueness of the trivial S^2-link L^0 constructed from the based loop system o and an argument in

[6] for uniqueness of the embedded 1-handle system $h(\alpha)$ constructed from the chord system α. For a disk δ in S^2, a *chord diagram* in δ is the intersection $C \cap \delta$ for a chord diagram $C = C(o, \alpha)$ in S^2 such that the circle $\partial\delta$ does not meet the based loop system o and meets the chord system α transversely.

Lemma 4.3. Let a_* be an oriented arc system in a compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \backslash B)$ of a homotopy 3-sphere M which is obtained from an oriented proper arc diagram D_δ in a disk δ contained in the boundary 2-sphere S of $M^{(o)}$ by pushing the interior of an upper-arc around every crossing point of D_δ into the interior of $M^{(o)}$. Then the spun S^2-link $S(a_*)$ in $X(M)$ is a ribbon S^2-link in $X(M)$ with a chord diagram C_δ in δ obtained from the arc diagram D_δ by changing every crossing point as in Fig. 2.

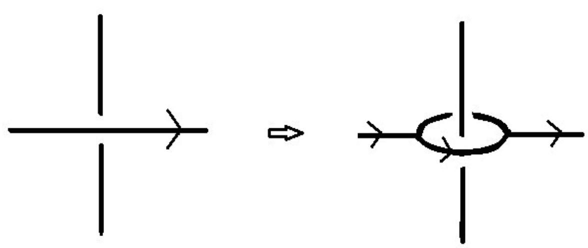

Figure 2: Changing a crossing point into a based loop with chords

Proof of Lemma 4.3. This fact is observed in [12, Theorem 2.3 (3)] for an inbound arc diagram whose closure is a knot chord diagram. The present claim is similarly shown for any oriented arc diagram. □

In Lemma 4.3, note that the arc diagram D_δ is recovered from the chord diagram C_δ by taking the upper-arc of every based loop. The proof of Lemma 1.2 is given as follows.

4.4: Proof of Lemma 1.2. Let k be a non-trivial knot in a homotopy 3-sphere M. By Corollary 4.2, the spun S^2-knot $S(k)$ in the 4-sphere $X(M) = S^4$ is a ribbon S^2-knot. The *spun torus-knot* of k in the 4-sphere $X(M)$ is given by the inclusions

$$T(k) = k \times S^1 \subset M^{(o)} \times S^1 \subset M^{(o)} \times S^1 \cup S \times D^2 = X(M).$$

The spun S^2-knot $S(k)$ in $X(M)$ is obtained from $T(k)$ by a 2-handle surgery and conversely the spun torus-knot $T(k)$ is obtained from the spun S^2-knot $S(k)$ by 1-

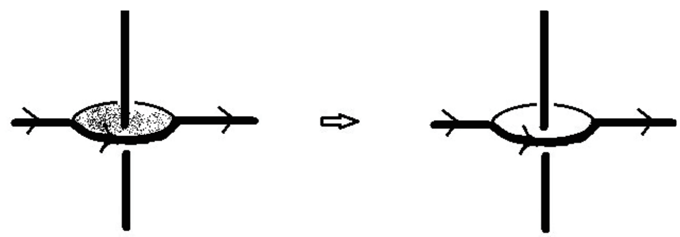

Figure 3: A diagram of the two arcs of k on the disk d_i

handle surgery. By definition, the spun torus-knot $T(k)$ is a ribbon torus-knot and hence bounds a ribbon solid torus V_R in $X(M)$. Let

$$V_R = \cup_{i=1}^n B_i \cup h_i$$

for a disjoint 3-ball system $B_i\,(i = 1, 2, \ldots, n)$ in $X(M)$ and an embedded disjoint 1-handle system $h_i\,(i = 1, 2, \ldots, n)$, denoted by h_*, on the 2-sphere system $\partial B_i\,(i = 1, 2, \ldots, n)$ in $X(M)$ so that the 1-handle h_i spans ∂B_i and ∂B_{i+1} for every i by taking $B_{n+1} = B_1$, and every 3-ball B_i meets transversely h_* with just a meridian disk d'_i of a 1-handle of h_* embedded in the interior of B_i. Since the knot k is non-trivial in $M^{(o)}$ and there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus k, v) \to \pi_1(X(M) \setminus T(k), v)$$

by the van Kampen theorem, the longitude of k in $M^{(o)}$ represents an infinite order element in the fundamental group $\pi_1(X(M) \setminus T(k), v)$, which implies that a meridian loop of V_R (i.e., a simple loop of $T(k)$ bounding a meridian disk of V_R) is a uniquely determined loop in $T(k)$ up to isotopies of $T(k)$. Fix an orientation of knot k. Then the construction of $T(k)$ determines uniquely the meridian disk orientation of the ribbon solid torus V_R and the ribbon solid torus V_R determines uniquely a disjoint oriented meridian disk system $d_i\,(i = 1, 2, \ldots, n)$ in V_R deformed from the 3-ball system $B_i\,(i = 1, 2, \ldots, n)$ so that the knot k meets the disk d_i with just one boundary arc orientation-coherently and just one interior point transversely and the union $k(d_*) = k \cup_{i=1}^n d_i$ (called a *disk-chord system*) recovers V_R uniquely by thickening k and $d_i\,(i = 1, 2, \ldots, n)$ (see the left figure of Fig. 3). The following observation is used.

Observation 4.4.1. The disk system $d_i\,(i = 1, 2, \ldots, n)$ is deformed into $M^{(o)}$ by an isotopy of $X(M)$ keeping the knot k fixed.

Proof of Observation 4.4.1. Llet α_i be a simple arc in d_i joining the point $k \cap \mathrm{Int}(d_i)$ with a point in the arc $k \cap \partial d_i$ for all i. The arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ is deformed into a bi-collar neighborhood $M^{(o)} \times [-1,1]$ of $M^{(o)}$ with $M^{(o)} \times 0 = M^{(o)}$ in $X(M)$ by an isotopy keeping $M^{(o)}$ fixed. Then the arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ is projected into $M^{(o)}$ by a general position argument. A deformed disk system $d_i\,(i = 1, 2, \ldots, n)$ in $M^{(o)}$ is obtained from the arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ in $M^{(o)}$ by extending them as a small disk system, completing the proof of Observation 4.4.1. □

By Observation 4.4.1, consider that the disk-chord system $k(d_*)$ is in $M^{(o)}$. Let $k(d_*)^{\times}$ be the graph in $M^{(o)}$ obtained from the disk-chord system $k(d_*)$ by shrinking every disk d_i into a 4-degree vertex for every i. By taking a maximal tree $\tau(k(d_*)^{\times})$ of $k(d_*)^{\times}$, one finds a disk δ in $M^{(o)}$ containing the maximal tree $\tau(k(d_*)^{\times})$. Let $e_i\,(i = 1, 2, \ldots, n+1)$ be the arc system $\mathrm{cl}(k(d_*)^{\times} \setminus \tau(k(d_*)^{\times}))$ where the number $n+1$ is uniquely determined by the Euler characteristic $\chi(k(d_*)^{\times}) = -n$. Then the disk-chord system

$$k(d_*)_\delta = \mathrm{cl}(k(d_*) \setminus (\cup_{i=1}^{n+1} e_i))$$

gives a chord diagram C_δ in the disk δ with the based loop system $o_i = \partial d_i\,(i = 1, 2, \ldots, n)$ and with two arcs of k on the disk d_i for every i drawn as bold lines in the right figure of Fig. 3. Let $a_j\,(j = 1, 2, \ldots, n+1)$ be the arc system $\mathrm{cl}(k \setminus \cup_{i=1}^{n+1} e_i)$. The chord diagram C_δ in the disk δ consists of the basel loop system $o_i\,(i = 1, 2, \ldots, n)$ and the chord system obtained from $a_j\,(j = 1, 2, \ldots, n+1)$ by deleting the interiors of the intersection arcs between $o_i\,(i = 1, 2, \ldots, n)$ and $a_j\,(j = 1, 2, \ldots, n+1)$. In Fig. 2, leave the two arcs of k by forgetting the disk or the based loop. Then the chord diagram C_δ changes into a proper arc diagram D_δ of the arc system $a_j\,(j = 1, 2, \ldots, n+1)$ in the disk δ with the crossing points corresponding to the based loop system $o_i\,(i = 1, 2, \ldots, n)$. Deform the disk δ into the 2-sphere $S = \partial M^{(o)}$ so that a collar $\delta \times [0,1]$ of δ in $M^{(o)}$ with $\delta \times 0 = \delta$ belongs to a boundary collar $S \times [0,1]$ of S in $M^{(o)}$ with $S \times 0 = S$. The arc system $a_i\,(i = 1, 2, \ldots, n+1)$ is realized as a proper arc system in the collar $\delta \times [0,1]$ from the arc diagram D_δ by pushing the interior of an upper-arc around every crossing point of D_δ into the interior of $\delta \times [0,1]$ and then by further pushing the interior of the arc system $a_i\,(i = 1, 2, \ldots, n+1)$ into the interior of $\delta \times [0,1]$. Then the disk-chord system $k(d_*)_\delta$ is also realized in $\delta \times [0,1]$. By Lemma 4.3, the spun S^2-link $S(a_*)$ of the arc system $a_i\,(i = 1, 2, \ldots, n+1)$ in $M^{(o)}$ in $X(M)$ is given by the chord system C_δ in δ constructed from the arc diagram D_δ and hence given by the disk-chord system $k(d_*)_\delta$ in $\delta \times [0,1]$. Let $M_\delta^{(o)} = \mathrm{cl}(M^{(o)} \setminus \delta \times [0,1])$ be a once-punctured manifold of M with $S' = \partial M_\delta^{(o)}$ the boundary 2-sphere, and $X'(M)$ the spun 4-sphere of M on $M_\delta^{(o)}$. The arc system $e_i\,(i = 1, 2, \ldots, n+1)$ is a prper arc system in $M_\delta^{(o)}$. The ribbon solid torus V_R bounded by the spun torus-knot $T(k)$ can be considered as a thickenning of the disk-chord system $k(d_*)$ in $M^{(0)}$. Since the disk-chord system $k(d_*)$ contains the disk-chord system $k(d_*)_\delta$, the spun S^2-link $S(e_*)$ in $X'(M)$ of the arc system $e_i\,(i = 1, 2, \ldots, n+1)$ in $M_\delta^{(o)}$ bounds disjoint 3-balls in V_R and hence a trivial S^2-link in $X'(M)$. By Lemma 2.5, the proper arc system $e_i\,(i=1,2,\dots,n+1)$ is in a boundary-collar $S'\times[0,1]$ of $M^{(o)}_\delta$. For a small disk d' in $S'\setminus\delta\times 1$ such that $b\times[0,1]$ does not meet the arc system $e_i\,(i=1,2,\dots,n+1)$, the union $\mathrm{cl}(S'\setminus d')\times[0,1]\cup\delta\times[0,1]$ is a 3-ball in $M^{(o)}$ containing the knot k. This completes the proof of Lemma 1.2. □

This completes the proof of Theorem 1.1.

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