Characterizations of Lie groups via finiteness conditions on their zero-dimensional subgroups

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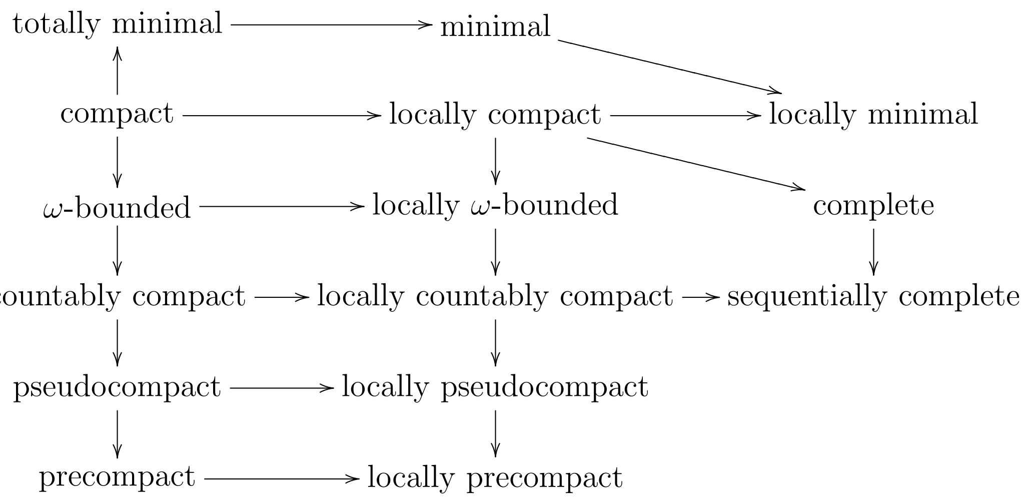
1. Compactness-like properties in topological groups

In this section we recall definitions of well-known compactness-like properties connections between which are summarized in the diagram below. (None of the arrows in this diagram are reversible.)



Recall that a topological space X is:

ω-bounded if the closure of every countable subset of X is compact,
countably compact if every countable open cover of X has a finite subcover,
pseudocompact if every real-valued continuous function defined on X is bounded.

A topological group G is *locally ω-bounded* (*locally countably compact*, *locally pseudocompact*) if it has an open neighbourhood U of its identity element whose closure $\overline{U}$ is ω-bounded (countably compact, pseudocompact, respectively). We say that a topological group is *(locally) precompact* if its two-sided uniformity completion is (locally) compact. Recall that a topological group G is called *sequentially complete* if every Cauchy sequence in G with respect to the two-sided uniformity of G converges to some element of G [14, 15].

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Recall that a topological group G is called *minimal* if every continuous isomorphism $f : G \to H$, where H is Hausdorff topological group, is a topological isomorphism [28]. A topological group G is said to be *totally minimal* if all Hausdorff quotients of G are minimal. Clearly, all compact groups are (totally) minimal. Easy examples show that the converse does not hold in general. Nevertheless, a somewhat weaker implication holds in the case of abelian groups:

Fact 1.1 (Prodanov and Stoyanov; see [9]) *A minimal abelian group G is precompact.*

A common generalization of locally compact groups and minimal groups was proposed by Morris and Pestov. A topological group (G,τ) is *locally minimal* if there exists a neighborhood V of e_G such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is a σ-neighborhood of e_G, then $\sigma = \tau$ [24]. We refer the reader to [2, 3] for some recent progress in this area.

2. Principal results

Definition 2.1 For a topological group G, consider the following conditions:

($\mathscr{Z}$) every closed zero-dimensional subgroup of G is discrete,
($\mathscr{Z}_m$) every closed zero-dimensional metric subgroup of G is discrete,
($\mathscr{Z}_{cm}$) every compact metrizable zero-dimensional subgroup of G is finite.

Remark 2.2 One may wonder why this definition omits the following natural condition:

($\mathscr{Z}_c$) every compact zero-dimensional subgroup of G is finite.

It turns out that this property is equivalent to $\mathscr{Z}_{cm}$. Indeed, the implication $\mathscr{Z}_c \to \mathscr{Z}_{cm}$ is trivial, and the converse implication easily follows from a result in [22].

The three properties from Definition 2.1 are ultimately related to Lie groups, as the following proposition shows:

Proposition 2.3 *For every topological group G, the following implications hold:*

$$\textit{Lie} \to \mathscr{Z} \to \mathscr{Z}_m \to \mathscr{Z}_{cm}. \tag{1}$$

We study in detail the question of when these implications can be reversed, for various classes of groups that are close to being compact.

Our first result shows that the three properties from Definition 2.1 are equivalent for locally ω-bounded groups.

Theorem 2.4 *For a locally ω-bounded group G the following conditions are equivalent:*

(i) G is a Lie group;
(ii) G satisfies $\mathscr{Z}$;
(iii) G satisfies $\mathscr{Z}_m$;
(iv) G satisfies $\mathscr{Z}_{cm}$.

Corollary 2.5 *A locally ω-bounded group without infinite compact metric zero-dimensional subgroups is a Lie group.*

Since Lie groups are locally compact and locally compact groups are locally ω-bounded, we get the following characterization of Lie groups in terms of their closed zero-dimensional compact metric subgroups.

Corollary 2.6 *A topological group is a Lie group if and only if it is locally ω-bounded and has no infinite compact metric zero-dimensional subgroups.*

Even a locally compact version of this corollary is new.

Corollary 2.7 *A topological group is a Lie group if and only if it is locally compact and has no infinite compact metric zero-dimensional subgroups.*

One cannot replace "locally ω-bounded" by "countably compact" in Theorem 2.4 and Corollaries 2.5 and 2.6; see Example 3.1.

Theorem 2.8 *For a locally minimal, locally precompact abelian group G the following conditions are equivalent:*

(i) G is a Lie group;
(ii) G satisfies $\mathscr{Z}$;
(iii) G satisfies $\mathscr{Z}_m$.

Moreover, if G is additionally assumed to be sequentially complete, then the following condition can be added to this list:

(iv) G satisfies $\mathscr{Z}_{cm}$.

Since minimal abelian groups are precompact by Fact 1.1, the conclusion of our next corollary follows from Theorem 2.8 and the fact that precompact Lie groups are compact.

Corollary 2.9 *For a minimal abelian group G the following conditions are equivalent:*

(i) G is a compact Lie group;
(ii) G satisfies $\mathscr{Z}$;
(iii) G satisfies $\mathscr{Z}_m$.

As we shall see in Example 3.6(ii), one cannot add $\mathscr{Z}_{cm}$ to the list of equivalent conditions in Corollary 2.9 and cannot drop the additional assumption of sequential completeness in the final part of Theorem 2.8.

Since Lie groups are locally compact (so, locally minimal) and satisfy property $\mathscr{Z}_m$, Theorem 2.8 gives the following characterization of abelian Lie groups:

Corollary 2.10 *An abelian topological group group G is a Lie group if and only if G is locally minimal, locally precompact and all closed metric zero-dimensional subgroups of G are discrete.*

Since compact groups are minimal and precompact discrete groups are finite, Corollary 2.9 yields a characterization of compact abelian Lie groups:

Corollary 2.11 *An abelian topological group is a compact Lie group if and only if it is minimal and has no infinite closed metric zero-dimensional subgroups.*

Our third theorem extends Theorem 2.8 beyond the abelian case.

Theorem 2.12 *For a connected, locally minimal, precompact sequentially complete group G, the following conditions are equivalent:*

(i) G is a compact Lie group;
(ii) G satisfies $\mathscr{Z}$;
(iii) G satisfies $\mathscr{Z}_m$;
(iv) G satisfies $\mathscr{Z}_{cm}$.

Example 3.1 below shows that local minimality cannot be omitted in Corollary 2.10 and Theorem 2.12.

Corollary 2.13 *A connected topological group is a compact Lie group if and only if it is sequentially complete, precompact, locally minimal and all its compact metric zero-dimensional subgroups are finite.*

The particular version of our results deserves special attention.

Corollary 2.14 *Let G be a countably compact minimal group satisfying $\mathscr{L}_{cm}$. If G is either abelian or connected, then G is a compact Lie group.*

Indeed, G is precompact and sequentially complete. Now the conclusion of this corollary follows from Theorem 2.8 (in the abelian case) and Theorem 2.12 (in the connected case).

The property $\mathscr{L}$ has been well studied in functional analysis.

Remark 2.15 The additive group of a Banach space B has property $\mathscr{L}$ (equivalently, property $\mathscr{L}_m$) if and only if B contains no subspace isomorphic to c_0; see [1, Theorem 4.1]. In particular, the additive group of the Hilbert space has property $\mathscr{L}$ (equivalently, property $\mathscr{L}_m$) [16].

Since the additive group of the Hilbert space is locally minimal and (sequentlially) complete, it follows that local precompactness of G cannot be omitted from the assumptions of both Theorem 2.8 and Corollary 2.10, minimality cannot be replaced with local minimality in Corollaries 2.9 and 2.11, and precompactness of G is necessary in Theorem 2.12 and cannot be dropped from its Corollary 2.13.

The following curious "automatic closedness" result is of independent interest.

Theorem 2.16 *Let G be a subgroup of an abelian Lie group K. If G satisfies $\mathscr{L}_m$, then G is closed in K; in particular G is a Lie group itself.*

One can consider the weaker versions of the three conditions $\mathscr{L}$, $\mathscr{L}_m$ and $\mathscr{L}_{cm}$ from Definition 2.1 obtained by replacing the word "subgroup" with the word "normal subgroup". The following example shows that (with the trivial exception of purely "abelian" results) most of our results spectacularly fail for these weaker versions of the three properties.

Example 2.17 Let $L = SO_3(\mathbb{R})$ be a compact connected simple Lie group. Then $G = L^{\mathbb{N}}$ *is a compact connected metric group without non-trivial closed zero-dimensional normal subgroups, yet G is not a Lie group*. Indeed, by a well-known theorem of Hofmann [23], a closed zero-dimensional normal subgroup of a connected compact group must be central, and the conclusion follows from the fact that G has the trivial center.

It is worth mentioning here the TAP property from [26] defined by requiring that no sequence in a topological group is multiplier convergent; see [12]. This property is weaker than NSS [26], and therefore, is possessed by every Lie group. Since TAP groups satisfy $\mathscr{L}_{cm}$, the results in this section can be applied to obtain characterizations of (compact) Lie groups in terms of multiplier convergence of sequences; see our forthcoming paper [13].

3. Examples distinguishing $\mathscr{Z}$, $\mathscr{Z}_m$ and $\mathscr{Z}_{cm}$

In this section we exhibit a series of examples showing that the arrows in (1) are not reversible in general.

Example 3.1 *Under Continuum Hypothesis, there exists a countably compact connected abelian group which satisfies $\mathscr{Z}$ but is not Lie.* In order to get such an example, we shall need the notion of an HFD set. Recall that a subset G of $\mathbb{T}^{\omega_1}$ is called an *HFD set* (an abbreviation for hereditarily finally dense) provided that for every countably infinite subset X of G one can find an ordinal $\alpha < \omega_1$ such that $q_\alpha(X)$ is dense in $\mathbb{T}^{\omega\setminus\alpha}$, where $q_\alpha : \mathbb{T}^{\omega_1} \to \mathbb{T}^{\omega_1\setminus\alpha}$ be the natural projection defined by $q_\alpha(h) = h\restriction_{\omega_1\setminus\alpha}$ for $h \in \mathbb{T}^{\omega_1}$.

(i) It is known that every HFD subset of $\mathbb{T}^{\omega_1}$ is hereditarily separable, countably compact, connected and does not contain any non-trivial convergent sequences.

(ii) We claim that *every HFD subgroup G of $\mathbb{T}^{\omega_1}$ satisfies $\mathscr{Z}$ but is not Lie*. Indeed, let N be an infinite closed zero-dimensional subgroup of G. Fix a countably infinite subset X of N. Let K be the closure of N in $\mathbb{T}^{\omega_1}$. Since N is a closed subgroup of the countably compact group G, it is countably compact as well. Therefore, $\dim K = \dim N = 0$ by Tkachenko's theorem ([30]). Since G is an HFD subset of $\mathbb{T}^{\omega_1\setminus\alpha}$, there exists an ordinal $\alpha < \omega_1$ such that $q_\alpha(X)$ is dense in $\mathbb{T}^{\omega_1\setminus\alpha}$. Since K is a compact group containing X, it follows that $q_\alpha(K) = \mathbb{T}^{\omega_1\setminus\alpha}$. Since continuous homomorphic images of compact zero-dimensional groups are zero-dimensional, we conclude that $\mathbb{T}^{\omega_1\setminus\alpha}$ must be zero-dimensional, in contradiction with its connectedness. This finishes the proof of the fact that G satisfies $\mathscr{Z}$. Since G contains no non-trivial convergent sequences, G is non-metrizable, and so cannot be a Lie group.

(iii) Tkachenko [29] gave an example, under the Continuum Hypothesis, of an HFD subgroup of $\mathbb{T}^{\omega_1}$.

It follows from our next proposition that local minimality of G cannot be omitted in Theorem 2.8.

Proposition 3.2 *For every infinite abelian group G there exists a zero-dimensional sequentially complete precompact group topology τ on G such that (G,τ) satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$.*

Proof: Indeed the group $G^\#$ (this is G equipped with its Bohr topology) is an infinite zero-dimensional [25] non-metrizable group, every infinite subgroup H of which is topologically isomorphic to $H^\#$, so H is not metrizable. Therefore, $G^\#$ satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$. Finally note that $G^\#$ is always sequentially complete [14, 15].

Under additional set-theoretic axioms, one can even strengthen precompactness to countable compactness in a counter-example to the implication $\mathscr{Z}_m \to \mathscr{Z}$.

Example 3.3 (i) *If G is an infinite zero-dimensional group without non-trivial convergent sequences, then G satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$.* Indeed, since G has no non-trivial convergent sequences, it satisfies $\mathscr{Z}_m$. Since G is infinite and zero-dimensional, it does not satisfy $\mathscr{Z}$.

(ii) Let G be dense pseudocompact subgroup of $\mathbb{Z}(2)^{\mathfrak{c}}$ without non-trivial convergent sequences constructed in [27]. It follows from (i) that *G is a pseudocompact abelian group which satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$.*

(iii) *Under Martin's Axiom, there exists a countably compact abelian group which satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$.* Indeed, let G be an infinite Boolean countably compact group without non-trivial convergent sequences built by van Douwen under the assumption of MA [18]. Since G is a countably compact group of finite exponent, G is zero-dimensional; see [7]. The rest follows from item (i).

This example shows that "ω-bounded" cannot be weakened to "countably compact" in Corollary 2.5, even in the "global" version.

Example 3.4 *There exists a pseudocompact abelian group satisfying* $\mathscr{Z}_{cm}$ *which does not satisfy* $\mathscr{Z}_m$. Indeed, let G be a pseudocompact abelian group of cardinality $\mathfrak{c}$ such that G contains an infinite cyclic metrizable subgroup N and all countable subgroups of G are closed; such a group is constructed in [31, Theorem 2.8]. Since N is countable, it is a closed zero-dimensional subgroup of G. Since N is metrizable, G does not satisfy $\mathscr{Z}_m$. It follows from [31, Corollary 2.7] that all (countable) compact subgroups of G are finite. Thus, G trivially satisfies $\mathscr{Z}_{cm}$.

The reader may want to compare the next proposition with Proposition 3.2 and Example 3.3.

Proposition 3.5 *Conditions* $\mathscr{Z}_m$ *and* $\mathscr{Z}_{cm}$ *are equivalent for locally precompact, sequentially complete groups. In particular, these two conditions coincide for countably compact groups.*

Proof: Let G be a locally precompact, sequentially complete group. The implication $\mathscr{Z}_m \to \mathscr{Z}_{cm}$ is established in Proposition 2.3. To prove the reverse implication suppose that G does not satisfy $\mathscr{Z}_m$. Then G has a non-discrete closed zero-dimensional metric subgroup N. Since N is a closed subgroup of G, it is locally precompact and sequentially complete. Since N is also metrizable, N is complete. Being also locally precompact, N is locally compact. Being a non-discrete locally compact zero-dimensional group, N contains an infinite open compact subgroup C, by van Dantzig's theorem. This shows that G does not satisfy $\mathscr{Z}_{cm}$.

Item (ii) of our next example shows that sequential completeness in Proposition 3.5 is essential, while Example 3.8(ii) shows that local precompactness is essential as well.

Example 3.6 (i) *Every non-discrete countable metrizable group G satisfies* $\mathscr{Z}_{cm}$ *but does not satisfy* $\mathscr{Z}_m$. Indeed, since infinite compact groups have size $\geq \mathfrak{c}$, all compact metric subgroups of G are finite, so G satisfies $\mathscr{Z}_{cm}$. Furthermore, G itself is zero-dimensional, so fails to satisfy $\mathscr{Z}_m$.

(ii) Let G be any countably infinite minimal metric abelian group (one can take, for example, $\mathbb{Q}/\mathbb{Z}$ as G; see [17, 28]). Then G *is a (precompact) minimal abelian group satisfying* $\mathscr{Z}_{cm}$ *but failing* $\mathscr{Z}_m$. Indeed, G is precompact by Fact 1.1. Since G is infinite, it is non-discrete. The rest follows from item (i).

Remark 3.7 (i) If continuous homomorphisms from a topological group G to $\mathbb{R}$ separate points of G, then G contains no non-trivial compact subgroups; in particular, G satisfies $\mathscr{Z}_{cm}$. Indeed, assume that K is a non-trivial compact subgroup of G. By our assumption, there exists a continuous homomorphism $f : G \to \mathbb{R}$ such that $f(K)$ is non-trivial. Since $f(K)$ is compact, this contradicts the fact that $\mathbb{R}$ has no non-trivial compact subgroups.

(ii) It follows from (i) that *the additive group G of every topological vector space over $\mathbb{R}$ has no non-trivial compact subgroups; in particular, G satisfies* $\mathscr{Z}_{cm}$.

Example 3.8 (i) *The complete metric group* $\mathbb{R}^{\mathbb{N}}$ *satisfies* $\mathscr{Z}_{cm}$ *but does not satisfy* $\mathscr{Z}_m$. Indeed, since $\mathbb{Z}^{\mathbb{N}}$ is an infinite non-discrete closed zero-dimensional subgroup of $\mathbb{R}^{\mathbb{N}}$, the group $\mathbb{R}^{\mathbb{N}}$ fails property $\mathscr{Z}_m$. The rest follows from Remark 3.7(ii).

(ii) *The additive group* G *of the Banach space* c_0 *is a locally minimal (sequentially) complete metric abelian group satisfying* $\mathscr{Z}_{cm}$ *but failing* $\mathscr{Z}_m$. Indeed, G satisfies $\mathscr{Z}_{cm}$ by Remark 3.7(ii). On the other hand, G does not satisfy $\mathscr{Z}_m$ by Remark 2.15.

It is worth noticing that the group $\mathbb{R}^{\mathbb{N}}$ from item (i) of this example is not locally minimal; see [8, Example 7.44].

All examples constructed so far are abelian. We shall now produce a series of examples which are minimal groups G close to being abelian (actually, they are nilpotent of class two, i.e., $G/Z(G)$ is abelian). In order to do so, we shall need the following general theorem inspired by and extending [8, Lemma 5.16].

Theorem 3.9 *Let* $m > 1$ *be an integer and let* X *be an infinite precompact abelian group with* $exp(X) = m$. *Let* $K = \mathbb{T}[m]$. *Then the discrete Pontryagin dual* $D = \widetilde{X}^\wedge$ *of* X *acts on* $K \times X$ *via automorphisms* $(t, x) \mapsto (t + \chi(x), x)$, $(t, x) \in K \oplus X$, $\chi \in D$. *The resulting semi-direct product* $L_X = (K \times X) \leftthreetimes D$ *is a minimal group group with the following properties:*

(i) $K \times X$ *is an open subgroup of* L_X, *so* L_X *is locally precompact;*

(ii) if X *is connected then* $m = 0$ *(i.e.,* $K = \mathbb{T}$*) and* $c(L_X) = \mathbb{T} \times X$ *(in particular,* L_X *is locally connected if and only if* X *is locally connected);*

(iii) L_X *is locally (countably) compact if and only if* X *is (countably) compact;*

(iv) L_X *is locally* ω*-bounded, whenever* X *is* ω*-bounded;*

(v) L_X *is locally pseudocompact if and only if* X *is pseudocompact;*

(vi) L_X *satisfies satisfies* $\mathscr{Z}$ *(resp.,* $\mathscr{Z}_m$. $\mathscr{Z}_{cm}$*) if and only if* X *satisfies satisfies* $\mathscr{Z}$ *(resp.,* $\mathscr{Z}_m$. $\mathscr{Z}_{cm}$*);*

(vii) L_X *is a Lie group precisely when* X *is a Lie group.*

(viii) L_X *is (sequentially) complete precisely when* X *is (sequentially) complete. (ix)* L_X *has no convergent sequences if and only if* X *has no convergent sequences and* $m > 0$*;*

(x) $Z(L_X) = K$ *and* $L_X/Z(L_X) \cong X \times D$, *so* L_X *is nilpotent (of class two).*

Our first group of examples shows that one cannot omit "abelian" in Theorem 2.8 and its corollaries, or replace it by the slightly weaker property "nilpotent". More precisely, we give an example of a minimal locally precompact (consistently, also locally countably compact) nilpotent sequentially complete group satisfying $\mathscr{Z}_m$, that does not satisfy $\mathscr{Z}$.

Example 3.10 (i) Take X to be an infinite precompact Boolean group without non-trivial convergent sequences (this can be easily obtained by taking X to have the Bohr topology $X^{\#}$ as in Proposition 3.2). Then L_X *is a locally precompact minimal sequentially complete nilpotent group that satisfies* $\mathscr{Z}_m$ *but does not satisfy* $\mathscr{Z}$. Since X has no non-trivial convergent sequences, X is satisfying $\mathscr{Z}_m$ and satisfies $\mathscr{Z}_m$. According to Theorem 3.9, L_X has the desired properties.

(ii) *Under the assumption of MA, there exists a locally countably compact minimal nilpotent group that satisfies* $\mathscr{Z}_m$ *but does not satisfy* $\mathscr{Z}$. Indeed, one can take as X a Boolean countably compact group without non-trivial convergent sequences mentioned in Example 3.3(iii). Then L_X becomes locally countably compact, in addition to the rest of properties from item (i).

Item (i) of the next example should be compared with Theorem 2.12, and item (ii) should be compared with Corollary 2.13.

Example 3.11 (i) *Under the assumption of the Continuum Hypothesis, there exists a locally countably compact locally connected minimal nilpotent group which satisfies $\mathscr{Z}$ but is not Lie*. Indeed, take as X the connected countably compact HFD subgroup of $\mathbb{T}^{\omega_1}$ constructed in Example 3.1. According to (ii), (iii) and (vi) of Theorem 3.9, the group L_X has the desired properties as X satisfies $\mathscr{Z}$.

(ii) *There exists a locally pseudocompact locally connected minimal sequentially complete nilpotent group which satisfies $\mathscr{Z}_m$ but is not Lie*. According to [20, Corollary 5.6], every abelian group of size $\leq 2^{2^{\mathfrak{c}}}$ admitting a pseudocompact group topology, admits also a pseudocompact group topology without non-trivial convergent sequences. On the other hand, every divisible pseudocompact group is connected by a theorem of Wilcox [33], while every connected compact group is divisible [23]. Hence, every connected compact abelian group of size $\leq 2^{2^{\mathfrak{c}}}$ admits a connected pseudocompact group topology without non-trivial convergent sequences. Take any abelian connected pseudocompact group X without convergent sequences (to this end one can use as a starting connected compact abelian group the circle $\mathbb{T}$). Clearly, X is sequentially complete. By items (ii), (v), (vi) and (viii) of Theorem 3.9, L_X is a locally pseudocompact locally connected minimal sequentially complete nilpotent group which satisfies $\mathscr{Z}_m$ but is not metrizable, hence not Lie.

4. Open questions

We do not know whether one can extend Theorem 2.16 to the non-abelian case.

Question 4.1 If a subgroup G of a Lie group K satisfies $\mathscr{Z}_m$, must then G be closed in K? Is this true at least when K is compact?

Question 4.2 Is there a pseudocompact (totally) minimal (abelian) group which satisfies $\mathscr{Z}_{cm}$ but fails $\mathscr{Z}_m$?

Question 4.3 (i) Does there exist a ZFC example of a countably compact (abelian) group which satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$?

(ii) Does there exist a pseudocompact (totally) minimal (abelian) group which satisfies $\mathscr{Z}_m$ but does not satisfy $\mathscr{Z}$?

(iii) Must a countably compact (totally) minimal group satisfying $\mathscr{Z}_m$ also satisfy $\mathscr{Z}$?

It is not known if every countably compact minimal group contains a non-trivial convergent sequence; see [11, Question ??]. Note that a counter-example to this question would be a countably compact minimal group that satisfies $\mathscr{Z}_m$ but is not a Lie group.

Question 4.4 (i) Does there exist a pseudocompact abelian group G which satisfies $\mathscr{Z}$ but is not metrizable (and thus, is not a Lie group)? What is the answer if one additionally assumes that G is sequentially complete?

(ii) Does there exist a ZFC example of a countably compact abelian group which satisfies $\mathscr{Z}$ but is not metrizable (and thus, is not a Lie group)?

(iii) Must a pseudocompact (totally) minimal (abelian) group satisfying $\mathscr{Z}$ be a Lie group?

(iv) Must a countably compact (totally) minimal group satisfying $\mathscr{Z}$ be a Lie group?

The answer to Questions 4.3(iii) and 4.4(iv) is positive when the group in question is either abelian or connected; see Corollary 2.14. We conjecture that the totally minimal versions of Questions 4.3(iii) and 4.4(iv) both have a positive answer.

Conjecture 4.5 *A countably compact totally minimal group satisfying $\mathscr{Z}_{cm}$ is a Lie group.*

Our next remark collects some comments that may be useful in the proof of this conjecture.

Remark 4.6 (i) *A countably compact totally minimal group G contains every normal closed metrizable subgroup N of its completion K.* Indeed, as G is totally dense in K, the intersection $N \cap G$ must be both countably compact and dense in N. Since N is metrizable, it must coincide with N. So, G must contain the metrizable closed normal subgroup N of K.

(ii) Let G be a countably compact totally minimal group satisfying $\mathscr{Z}_{cm}$. Then the center $Z(G)$ is both countably compact and totally minimal, so by the theorem from [10], $Z(G)$ is compact. By Theorem 2.4, $Z(G)$ is a Lie group. Therefore, *in order to prove Conjecture 4.5, it suffices to show that the quotient $G/Z(G)$ is a Lie group.*

(iii) *A totally disconnected countably compact totally minimal group satisfying $\mathscr{Z}_{cm}$ is torsion.* Being a totally disconnected countably compact group, G is zero-dimensional [7]. Under these circumstances $\mathscr{Z}_{cm}$ implies that all (closed) metrizable subgroups of G are finite. To see that G is torsion, it is enough to see that all cyclic subgroups of G are metrizable. By Tkachenko's theorem [29], the completion K of G is zero-dimensional as well, so the neutral element e_K has a base of neighborhoods formed by open subgroups of K. Hence, G has the same property. In particular, every cyclic subgroup of G, having the topology generated by open subgroups, must be metrizable.

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