Abe's lectures on Koszul rings and the Koszul duality
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Abstract

This is a record of Abe's lectures on Koszul rings and the Koszul duality during the week of 2017/10/16-20, with a few more details and references added. Applications to modular representations of Lie algebras or to the BGG category are not included. We have divided each section numbered by the days of the week the lectures were delivered into subsections.

月曜日 Preliminaries

For a ring A we let AMod denote the category of left A-modules. Unless otherwise specified by an A-module we will mean a left A-module. The first subsections 月.1-6 review some basics of homological algebras, before we introduce the derived category of A, precisely, of the category of A-modules.

月.1. Let G be a finite group and $\Bbbk$ an algebraically closed field of characteristic 0. Recall

Mascheke's theorem [服部, Th. 20.1, p. 119]: *Any $\Bbbk[G]$-module is semisimple, i.e., is a direct sum of simples.*

If V and V' are two simple $\Bbbk[G]$-modules, by Schur's lemma

$$\Bbbk[G]\mathrm{Mod}(V,V') \simeq \begin{cases} \Bbbk & \text{if } V \simeq V', \\ 0 & \text{else.} \end{cases}$$

Thus, the category $\Bbbk[G]$Mod is determined by the number of simples, which is equal to the number of conjugacy classes of G [CR, 3.37, p. 52]. Note, however, that if G' is another finite group with the same number of conjugacy classes, that may not infer an isomorphism between $\Bbbk[G]$ and $\Bbbk[G']$ as $\Bbbk$-algebras; e.g., G may be abelian while G' not.

Thus, 2 non-isomorphic rings may have equivalent module categories. There are even more fascinating phenomena in derived categories, which we presently introduce.

月.2. In what follows throughout 月, A will denote a unital ring.

Definition: *A complex* $(M^\bullet, d^\bullet)$ *of* A*-modules consists of a data* $M^i \in A\mathrm{Mod}$ *and* $d^i \in A\mathrm{Mod}(M^i, M^{i+1})$, $i \in \mathbb{Z}$, *such that* $d^{i+1} \circ d^i = 0$. *The* i*-th cohomology of* $(M^\bullet, d^\bullet)$, $i \in \mathbb{Z}$, *is* $\mathrm{H}^i(M^\bullet) = (\ker d^i)/(\mathrm{im} d^{i-1})$.

Roughly speaking, a derived category is where 2 complexes be isomorphic if their cohomology agree. If 2 rings are isomorphic, their module categories are equivalent, and hence also their derived categories, but not conversely in general. Koszul rings and Koszul duality provide a general framework for derived equivalences.

月.3. *Extensions*

Definition: *We say a sequence* $L \xrightarrow{f} M \xrightarrow{g} N$ *in* $A\mathrm{Mod}$ *is exact iff* $\mathrm{im} f = \ker g$. *Thus, a sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is exact iff* f *is injective,* $\mathrm{im} f = \ker g$, *and* g *is surjective, in which case we call the sequence short exact.*

We let $\mathrm{Ext}^1_A(N, L)$ *denote the set of short exact sequences* $0 \to L \to M \to N \to 0$ *modulo an equivalence relation such that that* $0 \to L \to M \to N \to 0$ *and* $0 \to L \to M' \to N \to 0$ *are equivalent iff there is a commutative diagram, CD for short in the following,*

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, dotted] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & M' \arrow[r] & N \arrow[r] & 0,
\end{tikzcd}$$

in which case $M \simeq M'$ *by the 5-lemma.*

E.g. Let $A = \Bbbk[x]$ be the polynomial ring in x over a field $\Bbbk$. Let $L = N = \Bbbk$ with x acting by 0, and let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence. Thus, $\dim M = 2$. If $v_1, v_2 \in M$ with $v_1 = f(1)$ and $g(v_2) = 1$, then (v_1, v_2) forms a $\Bbbk$-linear basis of M. One has

$$xv_1 = xf(1) = f(x \bullet 1) = f(0) = 0, \quad g(xv_2) = xg(v_2) = x \bullet 1 = 0.$$

Then $xv_2 \in \ker g = \mathrm{im} f$, and hence $xv_2 = \lambda v_1\ \exists \lambda \in \Bbbk$. Thus, the matrix of x on M with respect to the basis (v_1, v_2) is given by $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$. There follows a bijection

$$\mathrm{Ext}^1_A(N, L) \simeq \Bbbk \quad \text{via} \quad \text{“}0 \to L \to M \to N \to 0\text{”} \mapsto \lambda.$$

月.4. One can compute Ext^1_A more easily with projective resolutions.

Definition: *We say* $P \in A\mathrm{Mod}$ *is projective iff* $\forall f \in A\mathrm{Mod}(M, N)$ *surjective,* $\forall g \in A\mathrm{Mod}(P, N)$,

$$\begin{tikzcd}
 & P \arrow[d, "g"] \arrow[dl, dotted] \\
M \arrow[r, two heads, "f"'] & N.
\end{tikzcd}$$

Proposition: *(i) A free A-module is projective.*

(ii) $\forall M \in A\mathrm{Mod}$, $\exists$ *free* $P \in A\mathrm{Mod}$ *such that* $P \twoheadrightarrow M$.

Proof: (ii) Take a generating set $(m_\lambda|\lambda \in \Lambda)$ of M over A. Then $P = A^{\oplus \Lambda}$ with $e_\lambda \mapsto m_\lambda$, $\forall \lambda \in \Lambda$, will do, where e_λ is a basis element $(0, \dots, 0, 1, 0 \dots, 0)$ of P with 1 in the λ-th place.

月.5. **Definition:** *A projective resolution of* $M \in A\mathrm{Mod}$ *is an exact sequence* $\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$ *in* $A\mathrm{Mod}$ *with all* $P_i, i \in \mathbb{N}$, *projective.*

E.g. (i) Let $A = \Bbbk[x]$ and $M = \Bbbk[x]/(x) \simeq \Bbbk$. Consider an exact sequence $\Bbbk[x] \xrightarrow{f} \Bbbk[x]/(x) \to 0$ with $f : a \mapsto a + (x)$. As $\ker f = (x)$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Bbbk[x] & \longrightarrow & \Bbbk[x] & \xrightarrow{f} & \Bbbk[x]/(x) & \longrightarrow & 0 \\ & & a & \longmapsto & ax & & & & \end{array}$$

gives a projective resolution of $\Bbbk[x]/(x) \simeq \Bbbk$.

(ii) Let $A = \Bbbk[x, y]$ the polynomial ring over $\Bbbk$ in x and y and $M = \Bbbk[x, y]/(x, y) \simeq \Bbbk$. Consider an exact sequence $\Bbbk[x, y] \xrightarrow{f_1} \Bbbk[x, y]/(x, y) \to 0$ with $f_1 : a \mapsto a + (x, y)$. Then $\ker f_1 = (x, y)$. Define $f_2 : \Bbbk[x, y]^{\oplus 2} \to \Bbbk[x, y]$ via $(a, b) \mapsto ax + by$. One has

$$\ker f_2 = \{(a, b)|ax + by = 0\} = \{(ay, -ax)|a \in \Bbbk[x, y]\} \simeq \Bbbk[x, y],$$

and hence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Bbbk[x, y] & \longrightarrow & \Bbbk[x, y]^{\oplus 2} & \xrightarrow{f_2} & \Bbbk[x, y] & \xrightarrow{f_1} & \Bbbk[x, y]/(x, y) & \longrightarrow & 0 \\ & & a & \longmapsto & (ay, -ax) & & & & & & \end{array}$$

forms a projective resolution of $\Bbbk[x, y]/(x, y) \simeq \Bbbk$.

Ex. Let $n \in \mathbb{Z}$. Construct a projective resolution in the following cases:

(i) $A = \mathbb{Z}$, $M = \mathbb{Z}/n\mathbb{Z}$.

(ii) $A = \mathbb{Z}[x]$, $M = \mathbb{Z}[x]/(n, x)$.

月.6. **Definition:** *Let* $L, N \in A\mathrm{Mod}$. *Take a projective resolution* $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} N \to 0$ *of* N *and set,* $\forall i \in \mathbb{N}$,

$$\mathrm{Ext}^i_A(N, L) = \mathrm{H}^i(0 \to A\mathrm{Mod}(P_0, L) \xrightarrow{A\mathrm{Mod}(d_0, L)} A\mathrm{Mod}(P_1, L) \xrightarrow{A\mathrm{Mod}(d_1, L)} \dots).$$

For $i = 1$ the present definition agrees with the previous one in 月.3 [Rot, Th. 7.30, p. 425]. For given an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ one obtains a CD

$$\begin{tikzcd} P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dotted, "\phi_1"'] & P_0 \arrow[r, "d_0"] \arrow[d, dotted, "\phi_0"'] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r] & 0. \end{tikzcd}$$

Then $f\circ\phi_1\circ d_2=\phi_0\circ d_1\circ d_2=0$. As f is monic, $\phi_1\circ d_2=0$, and hence $\phi_1\in\ker(A\mathrm{Mod}(d_2,L))$. Define now a map

$$\text{“}0\to L\to M\to N\to 0\text{”}\mapsto[\phi_1]\in \mathrm{H}^1(A\mathrm{Mod}(P^\bullet,L)). \tag{1}$$

Conversely, given $[\phi]\in \mathrm{H}^1(A\mathrm{Mod}(P^\bullet,L))$ with $\phi\in A\mathrm{Mod}(P_1,L)$ such that $\phi\circ d_2=0$, let M' be the pushout of d_1 and ϕ: $M'=(L\oplus P_0)/\{(\phi(x),-d_1(x))|x\in P_1\}$. Then an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0\\
 & & l & \longmapsto & \overline{(l,0)} & & & & \\
 & & & & \overline{(l,y)} & \longmapsto & d_0y & &
\end{array}$$

gives an inverse to (1); define $M'\to M$ via $\overline{(l,y)}\mapsto f(l)+\phi_0(y)$.

月.7. Rather than taking cohomology, however, efforts of endowing complexes themselves with a structure lead to an introduction of derived categories.

Definition: *A morphism $f^\bullet:(M^\bullet,d^\bullet_M)\to(N^\bullet,d^\bullet_N)$ of complexes in* AMod *is a family $(f^i\in A\mathrm{Mod}(M^i,N^i)|i\in\mathbb{Z})$ such that, $\forall i\in\mathbb{Z}$,*

$$\begin{tikzcd}
M^i \arrow[r, "d^i_M"] \arrow[d, "f^i"'] \arrow[dr, phantom, "\circlearrowright"] & M^{i+1} \arrow[d, "f^{i+1}"] \\
N^i \arrow[r, "d^i_N"'] & N^{i+1}.
\end{tikzcd}$$

Together, the complexes of A-modules form a category, denoted $\mathrm{C}(A)$.

Given $f^\bullet\in\mathrm{C}(A)(M^\bullet,N^\bullet)$ one has, $\forall i\in\mathbb{Z}$,

$$\begin{tikzcd}
\mathrm{H}^i(M^\bullet) \arrow[r, dotted, "\mathrm{H}^i(f^\bullet)"] \arrow[d, equal] & \mathrm{H}^i(N^\bullet) \arrow[d, equal] \\
(\ker d^i_M)/(\mathrm{im} d^{i-1}_M) \arrow[r] & (\ker d^i_N)/(\mathrm{im} d^{i-1}_N).
\end{tikzcd}$$

We say $f^\bullet$ is a quasi-isomorphism, qis for short, iff $\mathrm{H}^i(f^\bullet)$ is invertible $\forall i\in\mathbb{Z}$.

月.8. The derived category of A is a localization of $\mathrm{C}(A)$ at qis'. Precisely, however, we need an auxiliary category, the homotopy category of $\mathrm{C}(A)$.

Definition: *We say $f^\bullet\in\mathrm{C}(A)(M^\bullet,N^\bullet)$ is homotopic to 0 iff $\exists\sigma^i\in A\mathrm{Mod}(M^i,N^{i-1})$, $i\in\mathbb{Z}$, such that, $\forall i$, $f^i=\sigma^{i+1}\circ d^i_{M^\bullet}+d^{i-1}_{N^\bullet}\circ\sigma^i$.*

$$\begin{tikzcd}
 & M^i \arrow[r, "d^i_M"] \arrow[dl, "\sigma^i"'] \arrow[d, "f^i"] & M^{i+1} \arrow[dl, "\sigma^{i+1}"] \\
N^{i-1} \arrow[r, "d^{i-1}_N"'] & N^i &
\end{tikzcd}$$

Lemma: *If $f^\bullet$ is homotopic to 0,* $\mathrm{H}^i(f^\bullet) = 0 \ \forall i \in \mathbb{Z}$.

Proof: $\forall m \in \ker d_M^i$,

$$f^i(m) = (\sigma^{i+1} \circ d_M^i + d_N^{i-1} \circ \sigma^i)(m) = (d_N^{i-1} \circ \sigma^i)(m) \in \operatorname{im} d_N^{i-1}.$$

月.9. Let $\mathrm{Ht}_0(M^\bullet, N^\bullet) = \{f^\bullet \in \mathrm{C}(A)(M^\bullet, N^\bullet) | f^\bullet \text{ is homotopic to } 0\}$.

Lemma: $\mathrm{Ht}_0(M^\bullet, N^\bullet)$ *is an abelian subgroup of* $\mathrm{C}(A)(M^\bullet, N^\bullet)$ *such that* $\forall f^\bullet \in \mathrm{Ht}_0(M^\bullet, N^\bullet)$, $\forall g^\bullet \in \mathrm{C}(A)(N^\bullet, L^\bullet)$, $\forall h^\bullet \in \mathrm{C}(A)(L^\bullet, M^\bullet)$, $g^\bullet \circ f^\bullet \in \mathrm{Ht}_0(M^\bullet, L^\bullet)$ *and* $f^\bullet \circ h^\bullet \in \mathrm{Ht}_0(L^\bullet, N^\bullet)$.

月.10. **Definition:** *The homotopy category* $\mathrm{K}(A)$ *of* A *has the same objects as* $\mathrm{C}(A)$ *with morphisms*

$$\mathrm{K}(A)(M^\bullet, N^\bullet) = \mathrm{C}(A)(M^\bullet, N^\bullet)/\mathrm{Ht}_0(M^\bullet, N^\bullet) \quad \forall M^\bullet, N^\bullet \in \mathrm{K}(A).$$

Ex. Check that the compositions of morphisms in $\mathrm{K}(A)$ are well-defined.

Remark: *(i)* $\forall f^\bullet \in \mathrm{K}(A)(M^\bullet, N^\bullet)$, $\mathrm{H}^i(f^\bullet)$, $i \in \mathbb{Z}$, *is well-defined as* $\mathrm{H}^i(g^\bullet) = 0 \ \forall g^\bullet \in \mathrm{Ht}_0(M^\bullet, N^\bullet)$. *One may thus say that* $f^\bullet \in \mathrm{K}(A)$ *is a qis iff* $\mathrm{H}^i(f^\bullet) = 0 \ \forall i \in \mathbb{Z}$.

(ii) There is a fully faithful functor $\iota : A\mathrm{Mod} \to \mathrm{K}(A)$ *such that* $M \mapsto (\cdots \to 0 \to M \to 0 \to \dots)$ *with* M *placed in degree 0, with* $\mathrm{H}^0 \circ \iota \simeq \mathrm{id}$.

月.11. Recall the localization of commutative rings. Let R be a commutative ring and S a multiplicative set of R: $\forall s, t \in S$, $st \in S$. Localization of R with respect to S is $S^{-1}R = (S \times R)/\sim$ with $\sim$ an equivalence relation such that $(s, a) \sim (t, b)$ iff $\exists u \in S$ with $u(sb - at) = 0$. To see the transitivity of $\sim$, we use the commutativity of R; if $(s_1, a_1) \sim (s_2, a_2)$ and $(s_2, a_2) \sim (s_3, a_3)$, $\exists t_1, t_2 \in S$ such that $t_1(s_1a_2 - a_1s_2) = 0 = t_2(s_2a_3 - a_2s_3)$. Then $s_1s_2t_1t_2(s_1a_3 - a_1s_3) = s_1^2t_1t_2s_3a_2 - s_1t_2t_1s_1a_2s_3 = 0$.

Let now $\mathcal{S}$ be the set of qis' of $\mathrm{K}(A)$. We'd like to define the derived category $\mathrm{D}(A)$ of A to have the same objects as $\mathrm{K}(A)$ with morphisms

$$\mathrm{D}(A)(M^\bullet, N^\bullet) = \{(s, f) | \ M^\bullet \xleftarrow[\mathrm{qis}]{s} X^\bullet \xrightarrow{f} N^\bullet \ \}/\sim,$$

where $\sim$ is an equivalence relation, which requires a more elaborate construction due to the lack of commutativity.

We first define a shift functor $[n]$, $n \in \mathbb{Z}$, on complexes as follows: $(X^\bullet[n])^i = X^{i+n}$, $(d_{X^\bullet[n]})^i = (-1)^n d_X^{i+n}$, and for $f^\bullet \in \mathrm{C}(A)(M^\bullet, N^\bullet)$ we set $(f[n])^i = f^{i+n} \ \forall i \in \mathbb{Z}$ [中岡, Def. 3.4.13, p. 189]. Thus, $[0] = \mathrm{id}$, $[n][m] = [n+m]$, and $\mathrm{H}^i(M^\bullet[n]) = \mathrm{H}^{i+n}(M^\bullet)$. The shift functors are induced on $\mathrm{K}(A)$, denoted by the same letters. We show

Lemma: *In* $\mathrm{K}(A)$

$$\forall \quad \begin{tikzcd} & X_1^\bullet \arrow[d, "f", "\mathrm{qis}"'] \\ X_2^\bullet \arrow[r, "g"'] & M^\bullet, \end{tikzcd} \qquad \exists \text{ a } CD \quad \begin{tikzcd} Y^\bullet \arrow[r, dotted] \arrow[d, dotted, "s"', "\mathrm{qis}"] & X_1^\bullet \arrow[d, "f"] \\ X_2^\bullet \arrow[r, "g"'] & M^\bullet. \end{tikzcd}$$

Proof: We make use of mapping cones. The mapping cone $\mathrm{cone}(f)$ of f is a complex [中岡, Prop. 3.4.15, p. 190] such that $\mathrm{cone}(f)^i = (X_1[1])^{i+1} \oplus M^i = X_1^{i+1} \oplus M^i$ and

$$d^i = \begin{pmatrix} d^i_{X_1[1]} & 0 \\ (f[1])^i & d^i_M \end{pmatrix} = \begin{pmatrix} -d^{i+1}_{X_1} & 0 \\ f^{i+1} & d^i_M \end{pmatrix} : \begin{matrix} X_1^{i+1} \\ \oplus \\ M^i \end{matrix} \to \begin{matrix} X_1^{i+2} \\ \oplus \\ M^{i+1}. \end{matrix}$$

One thus obtains a semi-split sequence $M^\bullet \xrightarrow{\binom{0}{1}} \mathrm{cone}(f) \xrightarrow{(1\ 0)} X_1^\bullet[1]$, i.e., the sequence reads at each $i \in \mathbb{Z}$ as a split exact sequence $0 \to M^i \to \mathrm{cone}(f)^i \to (X_1[1])^i \to 0$, which induces by the snake lemma [中岡, Lem. 4.2.21, p. 244]/[Iv, 1.6, p. 4] a long exact sequence, LES for short in what follows, [Iv, I.2.8, p. 9]

(1) $$\ldots \to \mathrm{H}^{i-1}(\mathrm{cone}(f)) \to \underset{\substack{\| \\ \mathrm{H}^i(X_1^\bullet)}}{\mathrm{H}^{i-1}(X_1^\bullet[1])} \xrightarrow[\mathrm{H}^i(f)]{} \mathrm{H}^i(M^\bullet) \to \mathrm{H}^i(\mathrm{cone}(f)) \to \underset{\substack{\| \\ \mathrm{H}^{i+1}(X_1^\bullet)}}{\mathrm{H}^i(X_1^\bullet[1])} \xrightarrow[\mathrm{H}^{i+1}(f)]{} \mathrm{H}^{i+1}(M^\bullet) \to \ldots.$$

As $f \in \mathcal{S}$, $\mathrm{H}^i(f)$ is invertible $\forall i \in \mathbb{Z}$, and hence

(2) $$\mathrm{H}^i(\mathrm{cone}(f)) = 0.$$

Consider next the mapping cone of $\binom{0}{1} \circ g \in \mathrm{K}(A)(X_2^\bullet, \mathrm{cone}(f))$ to obtain a semi-split sequence

$$\mathrm{cone}(f) \xrightarrow{\binom{0}{1}} \mathrm{cone}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ g) \xrightarrow{(1\ 0)} X_2^\bullet[1]$$

and a LES

$$\cdots \to \mathrm{H}^{i-1}(X_2^\bullet[1]) \to \mathrm{H}^i(\mathrm{cone}(f)) \to \mathrm{H}^i(\mathrm{cone}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ g)) \to \mathrm{H}^i(X_2^\bullet[1]) \to \mathrm{H}^{i+1}(\mathrm{cone}(f)) \to \ldots.$$

As $\mathrm{H}^i(\mathrm{cone}(f)) = 0$ $\forall i \in \mathbb{Z}$, $\mathrm{H}^i(\mathrm{cone}(\binom{0}{1} \circ g)) \xrightarrow{\mathrm{H}^i((0\ 1))} \mathrm{H}^i(X_2^\bullet[1])$ invertible. Thus, if we let $Y^\bullet = \mathrm{cone}(\binom{0}{1} \circ g)[-1]$, $s = (1\ 0) : Y^\bullet \to X_2^\bullet$ is a qis. Explicitly,

$$Y^i = \{(X_2^\bullet[1] \oplus \mathrm{cone}(f))[-1]\}^i = X_2^i \oplus \{(X_1^\bullet[1] \oplus M^\bullet))[-1]\}^i = X_2^i \oplus X_1^i \oplus M^{i-1},$$

$$d^i_Y = (d_{\mathrm{cone}(\binom{0}{1}\circ g)}[-1])^i = -d^{i-1}_{\mathrm{cone}(\binom{0}{1}\circ g)} = \begin{pmatrix} d^i_{X_2} & 0 \\ -(\binom{0}{1} \circ g)^i & -d^{i-1}_{\mathrm{cone}(f)} \end{pmatrix}$$

$$= \begin{pmatrix} d^i_{X_2} & 0 & 0 \\ 0 & d^i_{X_1} & 0 \\ -g^i & -f^i & -d^{i-1}_M \end{pmatrix} : \begin{matrix} X_2^i \\ \oplus \\ X_1^i \\ \oplus \\ M^{i-1} \end{matrix} \to \begin{matrix} X_2^{i+1} \\ \oplus \\ X_1^{i+1} \\ \oplus \\ M^i. \end{matrix}$$

Let $\pi_1 = (0\ 1\ 0) : Y^\bullet \to X_1^\bullet$ and $\pi_2 = (0\ 0\ 1) : Y^\bullet \to M^\bullet[-1]$ be the projections. Then

$$\begin{aligned}(f \circ (-\pi_1))^i - (g \circ s)^i &= -f^i \circ (0\ 1\ 0) - g^i \circ (1\ 0\ 0) = (-g^i\ \ -f^i\ \ 0)\\ &= (-g^i\ \ -f^i\ \ -d_M^i) + (0\ 0\ d_M^i) = \pi_2^{i+1} \circ d_Y^i + d_M^i \circ \pi_2^{i-1},\end{aligned}$$

and hence one has a CD in $\mathrm{K}(A)$

$$\begin{tikzcd} Y^\bullet \arrow[r, "-\pi_1"] \arrow[d, "s"'] & X_1^\bullet \arrow[d, "f"] \\ X_2^\bullet \arrow[r, "g"'] & M^\bullet. \end{tikzcd}$$

月.12. Given $M \underset{\text{qis}}{\overset{s_1}{\longleftarrow}} X_1 \xrightarrow{f_1} N$ and $M \underset{\text{qis}}{\overset{s_2}{\longleftarrow}} X_2 \xrightarrow{f_2} N$ in $\mathrm{K}(A)$, one has from 月.11 a CD

$$\begin{tikzcd} Y \arrow[r, dotted, "g_1"] \arrow[d, dotted, "g_2"', "\text{qis}"] & X_1 \arrow[d, "s_1"] \\ X_2 \arrow[r, "s_2"'] & M. \end{tikzcd}$$

Then $s_1 \circ g_1 = s_2 \circ g_2 \in \mathcal{S}$, and hence $g_1 \in \mathcal{S}$ also. We now define an equivalence relation by setting $(s_1, f_1) \sim (s_2, f_2)$ iff there is a CD in $\mathrm{K}(A)$ [中岡, Def. 2.4.30, p. 117]

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ M & Y \arrow[u, "g_1"'] \arrow[d, "g_2"] & N. \\ & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

月.13. Given $M \underset{\text{qis}}{\overset{s}{\longleftarrow}} X \xrightarrow{f} N$ and $N \underset{\text{qis}}{\overset{t}{\longleftarrow}} Y \xrightarrow{g} L$ in $\mathrm{K}(A)$ there is by 月.11 a CD

$$\begin{tikzcd} & & Z \arrow[dl, dotted, "u"', "\text{qis}"] \arrow[dr, dotted, "h"] & & \\ & X \arrow[dl, "s"'] \arrow[dr, "f"] & & Y \arrow[dl, "t"'] \arrow[dr, "g"] & \\ M & & N & & L. \end{tikzcd}$$

Define the composite in $\mathrm{D}(A)$ by

$$[N \underset{\text{qis}}{\overset{t}{\longleftarrow}} Y \xrightarrow{g} L] \circ [M \underset{\text{qis}}{\overset{s}{\longleftarrow}} X \xrightarrow{f} N] = [M \underset{\text{qis}}{\overset{s\circ u}{\longleftarrow}} Z \xrightarrow{g\circ h} L].$$

月.14. We will denote $M \underset{\text{qis}}{\overset{s}{\longleftarrow}} Z \xrightarrow{f} N$ in $\mathrm{D}(A)$ by $[f/s]$. In particular, $M \overset{\text{id}}{\longleftarrow} M \xrightarrow{f} N.$ by $[f/1]$.

Theorem [HRD, Prop. 1.3.1, p. 29]/[Gri, Th. 6.5, p. 53]: *Define a functor* $Q : \mathrm{K}(A) \to \mathrm{D}(A)$ *via* $[f : M \to N] \mapsto [f/1] = [M \xleftarrow{\mathrm{id}} M \xrightarrow{f} N]$.

(i) $\forall s \in \mathcal{S}$, $Q(s) = [s/1]$ *is invertible with inverse* $[\mathrm{id}_M/s]$*:*

$$\begin{tikzcd}[column sep=small]
 & & M \arrow[dl, equal] \arrow[dr, equal] & & \\
 & M \arrow[dl, equal] \arrow[dr, "s"] & & M \arrow[dl, "s"'] \arrow[dr, equal] & \\
M & & N & & M
\end{tikzcd} = [\mathrm{id}_M/1],$$

$$\begin{tikzcd}[column sep=small]
 & & M \arrow[dl, equal] \arrow[dr, equal] & & \\
 & M \arrow[dl, "s"'] \arrow[dr, equal] & & M \arrow[dl, equal] \arrow[dr, "s"] & \\
N & & M & & N
\end{tikzcd} = [\mathrm{id}_N/1].$$

(ii) $\forall$ *functor* $F : \mathrm{K}(A) \to \mathcal{C}$ *with* $F(s)$ *invertible* $\forall s \in \mathcal{S}$,

$$\exists \begin{tikzcd}
\mathrm{K}(A) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C} \\
\mathrm{D}(A) \arrow[ur, dotted, "!G"'] &
\end{tikzcd} \quad \textit{such that, } \forall [f/s],\ G([f/s]) = F(f)F(s)^{-1} : F(M) \to F(N).$$

月.15. **Remark:** On $\mathcal{S}$ defined as in 月.11 the following holds:

(a) $\mathrm{id} \in \mathcal{S}$,

(b) $\forall s, t \in \mathcal{S}$, $st \in \mathcal{S}$,

(c) $\forall f, g \in \mathrm{K}(A)(M^\bullet, N^\bullet)$, $s \circ f = s \circ g$ for some $s \in \mathcal{S}$ iff there is $t \in \mathcal{S}$ such that $f \circ t = g \circ t$ [Gri, FR3, p. 53]/[中岡, MS3, p. 401].

More generally, we call a family $\mathcal{S}$ of morphisms in a category $\mathcal{C}$ left multiplicative iff (a)-(c) and Lem. 月.11 hold, in which case one can define likewise localization $\mathcal{C}_{\mathcal{S}}$ such that Th. 月.14 hold with $\mathrm{K}(A) \to \mathrm{D}(A)$ replaced by $\mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ [Gri, Th. 1.2, p. 8]/[中岡, Prop. 2.4.32, p. 118].

火曜日 Derived functors

We define functors between derived categories of modules, derived from ones between homotopy categories. Starting with 火.11 we will introduce a variant, dg-algebras and dg-modules. This may be better suited to 森田-theory, which we introduce in 火.19. Fix a ring A.

火.1. We start with some remarks on $\mathrm{D}(A)$.

(i) Cohomology functor on $\mathrm{K}(A)$ carries over to $\mathrm{D}(A)$: $\forall i \in \mathbb{Z}$,

$$\begin{tikzcd}
\mathrm{K}(A) \arrow[r, "\mathrm{H}^i"] \arrow[d, "Q"'] & A\mathrm{Mod}. \\
\mathrm{D}(A) \arrow[ur, dotted, "\mathrm{H}^i"'] &
\end{tikzcd}$$

(ii) The fully faithful imbedding of AMod into $\mathrm{K}(A)$ from Rmk. 月.10.(ii) remiains fully faithful into $\mathrm{D}(A)$ with quasi-inverse $\mathrm{H}^0 : \mathrm{D}(A) \to A\mathrm{Mod}$.

(iii) The shift functor $[n]$, $n \in \mathbb{Z}$, on $\mathrm{K}(A)$ carries over to $\mathrm{D}(A)$ [中岡, Th. 6.2.49, p. 374] by setting

$$(M \underset{\mathrm{qis}}{\overset{s}{\leftarrow}} X \xrightarrow{f} N)[n] \ = \ (M \underset{\mathrm{qis}}{\overset{s[n]}{\leftarrow}} X[n] \xrightarrow{f[n]} N[n]) :$$

$$\begin{tikzcd}
\mathrm{K}(A) \arrow[r, "{[n]}"] \arrow[d, "Q"'] & \mathrm{K}(A) \arrow[d, "Q"] \\
\mathrm{D}(A) \arrow[r, dotted, "{[n]}"'] & \mathrm{D}(A).
\end{tikzcd}$$

火.2. We now introduce triangles in $\mathrm{K}(A)$ [中岡, Prop. 7.1.14, p. 406/Eg. 6.1.10, p.342].

Definition: *A distinguished triangle, d.t. for short, in* $\mathrm{K}(A)$ *is a sequence* $L^\bullet \xrightarrow{f} M^\bullet \xrightarrow{g} N^\bullet \xrightarrow{h} L^\bullet[1]$ *isomorphic in* $\mathrm{K}(A)$ *to a sequence* $L^\bullet \xrightarrow{f} M^\bullet \xrightarrow{\binom{0}{1}} \mathrm{cone}(f) \xrightarrow{(1\ 0)} L^\bullet[1]$.

D.t.'s are invariant under shifts [中岡, Prop. 6.1.2, p. 336]; $M^\bullet \xrightarrow{g} N^\bullet \xrightarrow{h} L[1]^\bullet \xrightarrow{-f[1]} M^\bullet[1]$ remains a d.t. Also, $\forall n \in \mathbb{Z}$, $L[n]^\bullet \xrightarrow{(-1)^n f[n]} M[n]^\bullet \xrightarrow{(-1)^n g[n]} N[n]^\bullet \xrightarrow{(-1)^n h[n]} L^\bullet[n+1]$ is a d.t. [Iv, I.4.18, p. 29]/[中岡, Prop. 6.2.14, p. 352]. If $X^\bullet \xrightarrow{f'} Y^\bullet \xrightarrow{g'} Z^\bullet \xrightarrow{h'} X^\bullet[1]$ is another d.t. with $\phi \in \mathrm{K}(A)(L^\bullet, X^\bullet)$, $\psi \in \mathrm{K}(A)(M^\bullet, Y^\bullet)$ such that $\psi \circ f = f' \circ \phi$, one has a CD in $\mathrm{K}(A)$ [中岡, Prop. 6.1.3, p. 336]

$$\begin{tikzcd}
L^\bullet \arrow[r, "f"] \arrow[d, "\phi"'] & M^\bullet \arrow[r, "g"] \arrow[d, "\psi"'] & N^\bullet \arrow[r, "h"] \arrow[d, dotted] & L^\bullet[1] \arrow[d, "{\phi[1]}"] \\
X^\bullet \arrow[r, "f'"'] & Y^\bullet \arrow[r, "g'"'] & Z^\bullet \arrow[r, "h'"'] & X^\bullet[1].
\end{tikzcd}$$

Also, the octahedron axiom [中岡, p. 341] holds.

E.g., Given an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in AMod imbed f and g into $\mathrm{K}(A)$ as in

Rmk. 月.10.(ii):

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots.
\end{tikzcd}$$

On the other hand, one has a CD

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{cone}(f)^{-2} \arrow[r] \arrow[dd, equal] & \mathrm{cone}(f)^{-1} \arrow[r] \arrow[d, equal] & \mathrm{cone}(f)^{0} \arrow[r] \arrow[d, equal] & \mathrm{cone}(f)^{1} \arrow[r] \arrow[dd, equal] & \dots \\
& & L \oplus 0 \arrow[d, "\wr"] & 0 \oplus M \arrow[d, "\wr"] & & \\
\dots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & L \arrow[r, "f"] \arrow[d] & M \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & N \arrow[r] & 0 \arrow[r] & \dots,
\end{tikzcd}$$

which gives a qis $\mathrm{cone}(f) \simeq N$.

More generally [Gri, Prop. 4.10, p. 42], $\forall$ exact sequence $0 \to L^\bullet \xrightarrow{f} M^\bullet \xrightarrow{g} N^\bullet \to 0$ in $\mathrm{C}(A)$,

(1) $$(0\ g) : \mathrm{cone}(f) \to N^\bullet \text{ is a qis.}$$

For one has a CD

$$\begin{tikzcd}
0 \arrow[r] & L^\bullet \arrow[r, "f"] & M^\bullet \arrow[r, "g"] & N^\bullet \arrow[r] & 0 \\
& L^\bullet \arrow[u, equal] \arrow[r, "-f"'] & M^\bullet \arrow[u, "-\mathrm{id}_{M^\bullet}"] \arrow[r, "-\binom{1}{0}"'] & \mathrm{cone}(f) \arrow[u, "(0\ g)"] \arrow[r, "(1\ 0)"'] & L^\bullet[1],
\end{tikzcd}$$

which induces a CD of LES's

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^i(L^\bullet) \arrow[r, "\mathrm{H}^i(f)"] & \mathrm{H}^i(M^\bullet) \arrow[r, "\mathrm{H}^i(g)"] & \mathrm{H}^i(N^\bullet) \arrow[r] & \mathrm{H}^{i+1}(L^\bullet) \arrow[r] & \mathrm{H}^{i+1}(M^\bullet) \arrow[r] & \dots \\
\dots \arrow[r] & \mathrm{H}^i(L^\bullet) \arrow[u, equal] \arrow[r, "-\mathrm{H}^i(f)"'] & \mathrm{H}^i(M^\bullet) \arrow[u, "-\mathrm{id}_{\mathrm{H}^i(M^\bullet)}"] \arrow[r, "-\mathrm{H}^i(\binom{1}{0})"'] & \mathrm{H}^i(\mathrm{cone}(f)) \arrow[u, "\mathrm{H}^i((0\ g))"] \arrow[r, "\mathrm{H}^i((1\ 0))"'] & \mathrm{H}^i(L^\bullet[1]) \arrow[u, equal] \arrow[r] & \mathrm{H}^i(M^\bullet[1]) \arrow[u, "-\mathrm{id}_{\mathrm{H}^i(M^\bullet[1])}"'] \arrow[r] & \dots,
\end{tikzcd}$$

and hence $(0\ g)$ is a qis by the 5-lemma.

Moreover [Gri, loc. cit.], if the sequence is semi-split, $(0\ g) \in \mathrm{C}(A)(\mathrm{cone}(f), N^\bullet)$ is invertible in $K(A)$:

(2) $$N^\bullet \simeq \mathrm{cone}(f) \text{ in } \mathrm{K}(A).$$

For let $s : N^\bullet \to M^\bullet$ be a splitting of g: $g \circ s = \mathrm{id}_{N^\bullet}$. Define $h \in \mathrm{C}(A)(N^\bullet, \mathrm{cone}(f))$ via $N^k \ni n \mapsto \binom{-l}{s(n)}$ with $l \in L^{k+1}$ such that $f(l) = (d_M \circ s - s \circ d_N)(n)$; $g((d_Y \circ s)(n) - (s \circ d_N)(n)) =$

$(d_N \circ g \circ s)(n) - (g \circ s \circ d_N)(n) = d_N(n) - d_N(n) = 0,$

$$\begin{aligned}
(d_{\mathrm{cone}(f)} \circ h)(n) &= \begin{pmatrix} -d_L & 0 \\ f & d_M \end{pmatrix} \begin{pmatrix} -l \\ s(n) \end{pmatrix} = \begin{pmatrix} d_L l \\ -f(l) + (d_M \circ s)(n) \end{pmatrix} \\
&= \begin{pmatrix} d_L l \\ -(d_M \circ s)(n) + (s \circ d_N)(n) + (d_M \circ s)(n) \end{pmatrix} = \begin{pmatrix} d_L l \\ (s \circ d_N)(n) \end{pmatrix} \\
&= (h \circ d_N)(n)
\end{aligned}$$

as $f(-d_L l) = -(f \circ d_L)(l) = -(d_M \circ f)(l) = -d_M((d_M \circ s)(n) - (s \circ d_N)(n)) = (d_M \circ s \circ d_N)(n) = (d_M \circ s - s \circ d_N)(d_N n)$. Then $((0\ g) \circ h)(n) = (g \circ s)(n) = n$, and hence $(0\ g) \circ h = \mathrm{id}_{N^\bullet}$. We show finally that $h \circ (0\ g) = \mathrm{id}_{\mathrm{cone}(f)}$ in $\mathrm{K}(A)$. $\forall \binom{x}{y} \in \mathrm{cone}(f)^k$, $(h \circ (0\ g))\binom{x}{y} = h(g(y)) = \binom{-l'}{s(g(y))}$ with $l' \in L^{k+1}$ such that $f(l') = (d_M \circ s - s \circ d_N)(g(y))$. Define $\sigma : \mathrm{cone}(f) \to \mathrm{cone}(f)[-1]$ via $\mathrm{cone}(f)^k \ni \binom{x}{y} \mapsto \binom{l''}{0}$ with $l'' \in L^k$ such that $f(l'') = y - s(g(y))$. Then

$$\begin{aligned}
d_{\mathrm{cone}(f)} \circ \sigma \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -d_L & 0 \\ f & d_M \end{pmatrix} \begin{pmatrix} l'' \\ 0 \end{pmatrix} = \begin{pmatrix} -d_L l'' \\ f(l'') \end{pmatrix} = \begin{pmatrix} -d_L l'' \\ y - s(g(y)) \end{pmatrix}, \\
\sigma \circ d_{\mathrm{cone}(f)} \begin{pmatrix} x \\ y \end{pmatrix} &= \sigma \circ \begin{pmatrix} -d_L & 0 \\ f & d_M \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \sigma \begin{pmatrix} -d_L x \\ f(x) + d_M(y) \end{pmatrix} = \begin{pmatrix} l' + x + d_L l'' \\ 0 \end{pmatrix}
\end{aligned}$$

as

$$\begin{aligned}
f(l' + x + d_L l'') &= (d_M \circ s - s \circ d_N)(g(y)) + f(x) + f(d_L l'') \\
&= d_M(s \circ g(y)) - (s \circ d_N)(g(y)) + f(x) + (d_M \circ f)(l'') \\
&= d_M(s \circ g(y)) - (s \circ d_N)(g(y)) + f(x) + d_M(y - s(g(y))) \\
&= f(x) + d_M(y) - (s \circ d_N)(g(y)) = f(x) + d_M(y) - s(g(f(x) + d_M(y))).
\end{aligned}$$

Thus,

$$\begin{aligned}
\{\mathrm{id}_{\mathrm{cone}(f)} - h \circ (0\ g)\} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -l' \\ s(g(y)) \end{pmatrix} = \begin{pmatrix} x + l' \\ y - s(g(y)) \end{pmatrix} \\
&= \{d_{\mathrm{cone}(f)} \circ \sigma + \sigma \circ d_{\mathrm{cone}(f)}\} \begin{pmatrix} x \\ y \end{pmatrix},
\end{aligned}$$

and hence $h \circ (0\ g) = \mathrm{id}_{\mathrm{cone}(f)}$ in $\mathrm{K}(A)$, as desired.

火.3. **Definition:** *A d.t. of* $\mathrm{D}(A)$ *is a sequence isomorphic to the image of a d.t. in* $\mathrm{K}(A)$ *under the localization* [中岡, Prop. 6.2.49, p. 374].

Thus, an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in $A\mathrm{Mod}$ yields a d.t. in $\mathrm{D}(A)$

$$\begin{tikzcd}
L \arrow[r] & M \arrow[r] & N \arrow[r, dotted] & L[1]. \\
 & & \mathrm{cone}(f) \arrow[u, "\wr"'] \arrow[ur, "(1\ 0)"'] &
\end{tikzcd}$$

Given a d.t. $L^\bullet \to M^\bullet \to N^\bullet \to L^\bullet[1]$ in $\mathrm{D}(A)$ one obtains from 月.11(1) an exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^i(L^\bullet) \arrow[r] & \mathrm{H}^i(M^\bullet) \arrow[r] & \mathrm{H}^i(N^\bullet) \arrow[r] & \mathrm{H}^i(L^\bullet[1]) \arrow[r] \arrow[d, equal] & \dots \\
 & & & & \mathrm{H}^{i+1}(L^\bullet) &
\end{tikzcd}$$

More generally, we say a functor $F : \mathrm{D}(A) \to A\mathrm{Mod}$ is cohomological iff F sends a d.t. to an exact sequence. If $L^\bullet \to M^\bullet \to N^\bullet \to L^\bullet[1]$ is a d.t. in $\mathrm{D}(A)$ and if F is cohomological, as $M^\bullet \to N^\bullet \to L^\bullet \to M^\bullet[1]$ remains a d.t., the sequence $F(M^\bullet) \to F(N^\bullet) \to F(L^\bullet[1])$ is also exact, and hence results an exact sequence

$$\cdots \to F(N^\bullet[-1]) \to F(L^\bullet) \to F(M^\bullet) \to F(N^\bullet) \to F(L^\bullet[1]) \to \dots.$$

Both $\mathrm{D}(A)(X^\bullet, ?)$ and $\mathrm{D}(A)(?, X^\bullet)$ are cohomological [中岡, Prop. 6.2.3, p. 347]. Together with d.t.'s $\mathrm{D}(A)$ forms a triangulated category [中岡, Def. 6.1.7, p. 339].

火.4. *Bounded derived categories*

Definition: *We let* $\mathrm{C}^+(A) = \{M^\bullet \in \mathrm{C}(A) | M^i = 0 \ \forall i \ll 0\}$ *a full subcategory of* $\mathrm{C}(A)$. $\mathrm{D}^+(A) = \{M^\bullet \in \mathrm{D}(A) | \mathrm{H}^i(M^\bullet) = 0 \ \forall i \ll 0\}$ *forms a full subcategory of* $\mathrm{D}(A)$*, called the subcategory bounded above. If* $\mathrm{K}^+(A) = \{M^\bullet \in \mathrm{K}(A) | M^i = 0 \ \forall i \ll 0\}$, $\mathrm{D}^+(A)$ *is equivalent to the localization of* $\mathrm{K}^+(A)$ *with respect to* $\mathcal{S}^+ = \mathrm{K}^+(A) \cap \mathcal{S}$ *[中岡, Prop. 7.1.20, p. 408].*

Likewise, we let $\mathrm{C}^-(A) = \{M^\bullet \in \mathrm{C}(A) | M^i = 0 \ \forall i \gg 0\}$, $\mathrm{K}^-(A) = \{M^\bullet \in \mathrm{K}(A) | M^i = 0 \ \forall i \gg 0\}$. *We call* $\mathrm{D}^-(A) = \{M^\bullet \in \mathrm{D}(A) | \mathrm{H}^i(M^\bullet) = 0 \ \forall i \gg 0\}$ *(resp.* $\mathrm{D}^b(A) = \{M^\bullet \in \mathrm{D}(A) | \mathrm{H}^i(M^\bullet) = 0$ *except for finitely many* $i\}$*) the subcategory bounded above (resp. bounded).*

Ex. If $\mathrm{H}^i(M^\bullet) = 0 \ \forall i > n$, there is a qis

$$\begin{tikzcd}
\dots \arrow[r] & M^{n-1} \arrow[r] \arrow[d, equal] & \ker(d_M^n) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r, "d_M^n"'] & M^{n+1} \arrow[r] & M^{n+2} \arrow[r] & \dots,
\end{tikzcd}$$

the top row of which is denote $\tau^{\leq n}(M^\bullet)$.

If $\mathrm{H}^i(M^\bullet) = 0 \ \forall i < n$, there is a qis

$$\begin{tikzcd}
\dots \arrow[r] & M^{n-1} \arrow[r] \arrow[d] & M^n \arrow[r, "d_M^n"] \arrow[d, two heads] & M^{n+1} \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathrm{coker}(d_M^{n-1}) \arrow[r] & M^{n+1} \arrow[r] & \dots,
\end{tikzcd}$$

the bottom row of which is denote $\tau^{\geq n}(M^\bullet)$.

Lemma: $\forall M^\bullet \in \mathrm{D}^-(A)$, $\exists P^\bullet \in \mathrm{D}^-(A)$ *with all* P^i *projective:* $P^\bullet \xrightarrow{\text{qis}} M^\bullet$ *in* $\mathrm{C}(A)$.

Proof: In case $M^\bullet$ is the image of $M \in A\mathrm{Mod}$, i.e., $M^\bullet = \cdots \to 0 \to M \to 0 \to \dots$ with M

in degree 0, take a projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$. Then

$$\begin{tikzcd}
\dots \arrow[r] & P^{-1} \arrow[r] \arrow[d, equal] & P^0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[dd] & \dots \\
\dots & P_1 \arrow[d] & P_0 \arrow[d] & & \\
\dots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

is a qis in $\mathrm{C}(A)$.

In general, we construct a qis $f^\bullet : P^\bullet \to M^\bullet$ by descending induction as follows. To start the induction, we may by Ex. above assume that $M^i = 0\ \forall i \gg 0$. Assume that we have constructed a CD

$$\begin{tikzcd}
P^n \arrow[r] \arrow[d, "f^n"'] & P^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \dots \\
M^n \arrow[r] & M^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

such that $\mathrm{H}^i(f^\bullet)$ is invertible $\forall i \geq n+1$ with a CD

$$\begin{tikzcd}
 & \ker(d_P^n) \arrow[r, hook] \arrow[dl, dotted, two heads] \arrow[d, dotted] & P^n \arrow[d, "f^n"] \\
\mathrm{H}^n(M^\bullet) & \ker(d_M^n) \arrow[l, two heads] \arrow[r, hook] & M^n.
\end{tikzcd}$$

We will construct a CD

$$\begin{tikzcd}
P^{n-1} \arrow[r, dotted, "d_P^{n-1}"] \arrow[d, dotted, "f^{n-1}"'] & P^n \arrow[d, "f^n"] \\
M^{n-1} \arrow[r, "d_M^{n-1}"'] & M^n
\end{tikzcd}$$

to make $\mathrm{H}^n(f^\bullet)$ invertible and to induce a CD

$$\begin{tikzcd}
 & \ker(d_P^{n-1}) \arrow[dl, two heads] \arrow[d, "f^{n-1}|_{\ker(d_P^{n-1})}"] \\
\mathrm{H}^{n-1}(M^\bullet) & \ker(d_M^{n-1}). \arrow[l]
\end{tikzcd}$$

The induction hypothesis allows one to construct a CD

$$\begin{tikzcd}
P^{n-1} \arrow[r, two heads] \arrow[d, dotted, "f^{n-1}"'] \arrow[rrr, dotted, bend left, "d_P^{n-1}"] & Y^{n-1} \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\square"] & \ker(d_P^n) \arrow[r, hook] \arrow[d] & P^n \arrow[d, "f^n"] \\
M^{n-1} \arrow[r, two heads] \arrow[rrr, bend right, "d_M^{n-1}"'] & M^{n-1}/\mathrm{im}(d_M^{n-2}) \arrow[r] & \ker(d_M^n) \arrow[r, hook] & M^n,
\end{tikzcd}$$

where $Y^{n-1} = \{(\bar{m}, x) \in (M^{n-1}/\mathrm{im}(d_M^{n-2})) \oplus \ker(d_P^n) | d_M^{n-1}(m) = f^n(x)\}$ with $\bar{m} = m + \mathrm{im}(d_M^{n-2})$, $m \in M^{n-1}$.

Let now $x \in \ker(d_P^n)$ with $f^n(x) = 0$ in $\mathrm{H}^n(M^\bullet)$. Thus, $f^n(x) \in \mathrm{im}(d_M^{n-1})$, say $f^n(x) = d_M^{n-1}(m), m \in M^{n-1}$. Then $(\bar{m}, x) \in Y^{n-1}$, and hence $x \in \mathrm{im}(d_P^{n-1})$. Then $x = 0$ in $\mathrm{H}^n(P^\bullet)$, and hence $\mathrm{H}^n(f^\bullet) : \mathrm{H}^n(P^\bullet) \to \mathrm{H}^n(M^\bullet)$ is invertible.

Let next $z \in \ker(d_P^{n-1})$. Then the image of z in $\ker(d_P^n)$ vanishes, and hence the image of z in Y^{n-1} is of the form $(\bar{m}, 0)$ with $\bar{m} = 0$ in $\ker(d_M^n)$. Thus, $f^{n-1}(z) \in \ker(d_M^{n-1})$. Finally, let $\bar{m} \in \mathrm{H}^{n-1}(M^\bullet)$ with $m \in \ker(d_M^{n-1})$. Then $m = 0$ in $\ker(d_M^n)$, and hence $(\bar{m}, 0) \in Y^{n-1}$. Take $y \in P^{n-1}$ such that $y \mapsto (\bar{m}, 0)$. Then $f^{n-1}(y) = \bar{m}$ in $M^{n-1}/\mathrm{im}(d_M^{n-2})$, and hence one has obtained a CD

$$\begin{tikzcd}
 & \ker(d_P^{n-1}) \arrow[dl, two heads] \arrow[d] \\
\mathrm{H}^{n-1}(M^\bullet) & \ker(d_M^{n-1}). \arrow[l]
\end{tikzcd}$$

火.5. We define a bifunctor $A\mathrm{Mod}^\bullet(?,?) : \mathrm{C}(A) \times \mathrm{C}(A) \to \mathrm{C}(\mathbb{Z})$, precisely, $\mathrm{C}(A)^{\mathrm{op}} \times \mathrm{C}(A) \to \mathrm{C}(\mathbb{Z})$, as follows [Gri, 10.1, p. 91]: $\forall i \in \mathbb{Z}$, set $A\mathrm{Mod}^i(M^\bullet, N^\bullet) = \prod_{j \in \mathbb{Z}} A\mathrm{Mod}(M^j, N^\bullet[i]^j)$ and $d^i : A\mathrm{Mod}^i(M^\bullet, N^\bullet) \to A\mathrm{Mod}^{i+1}(M^\bullet, N^\bullet)$ such that for each $\phi \in A\mathrm{Mod}^i(M^\bullet, N^\bullet)$ by $d^i\phi = d_N \circ \phi - (-1)^i \phi \circ d_M \in A\mathrm{Mod}^{i+1}(M^\bullet, N^\bullet)$

$$\begin{tikzcd}
 & N^{i+j} \arrow[dr, "d_N"] & \\
M^j \arrow[ur, "\phi_j"] \arrow[dr, "d_M"'] & & N^{i+j+1}. \\
 & M^{j+1} \arrow[ur, "\phi_j"'] &
\end{tikzcd}$$

Note that the complex is a complex of abelian groups, not of A-modules. If $f \in \mathrm{C}(A)(X^\bullet, M^\bullet)$ and $g \in \mathrm{C}(A)(N^\bullet, Y^\bullet)$, we define $A\mathrm{Mod}^\bullet(f^\bullet, g^\bullet) : A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet) \to A\mathrm{Mod}^\bullet(X^\bullet, Y^\bullet)$ by setting $\phi \mapsto g[i] \circ \phi \circ f$ $\forall \phi \in A\mathrm{Mod}^i(M^\bullet, N^\bullet)$.

One has

$$\begin{tikzcd}
\mathrm{C}(A) \times \mathrm{C}(A) \arrow[r, "A\mathrm{Mod}^\bullet(?,?)"] \arrow[d] & \mathrm{C}(Z) \arrow[d] \\
\mathrm{K}(A) \times \mathrm{K}(A) \arrow[r, dotted] & \mathrm{K}(Z).
\end{tikzcd}$$

Lemma [Gri, 10.2, p. 92]: $\forall i \in \mathbb{Z}$, $\forall M^\bullet, N^\bullet \in \mathrm{K}(A)$,

$$\mathrm{H}^i(A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet)) \simeq \mathrm{K}(A)(M^\bullet, N^\bullet[i]).$$

Proof: One has

$$\begin{aligned}
\ker(d^i_{A\mathrm{Mod}^\bullet(M^\bullet,N^\bullet)}) &= \{\phi \in A\mathrm{Mod}^i(M^\bullet, N^\bullet) | d_N \circ \phi = (-1)^i \phi \circ d_M\} \\
&= \{\phi \in A\mathrm{Mod}^i(M^\bullet, N^\bullet) | d_{N[i]} \circ \phi = \phi \circ d_M\} = \mathrm{C}(A)(M^\bullet, N^\bullet[i]).
\end{aligned}$$

火.6. $\forall N^\bullet \in \mathcal{K}(A)$, the functor $A\mathrm{Mod}^\bullet(?, N^\bullet) : \mathrm{K}^-(A) \to \mathrm{K}(\mathbb{Z})$ is triangulated, i.e., sends a d.t. to a d.t. [Gri, 10.4, p. 93]. For it to induce a functor $\mathrm{R}^-_I A\mathrm{Mod}^\bullet(?, N^\bullet) : \mathrm{D}^-(A) \to \mathrm{D}(\mathbb{Z})$ compatible with the localization functors $\mathrm{K}^-(A) \to \mathrm{D}^-(A)$ and $\mathrm{K}(\mathbb{Z}) \to \mathrm{D}(\mathbb{Z})$, one must take care that it be well-defined on morphisms. Thus, let $\mathrm{P}^-(A) = \{P^\bullet \in \mathrm{C}^-(A) | P^i \text{ projective } \forall i\}$.

Lemma [Iv, I.6.2, p. 41]: *Let $P^\bullet \in \mathrm{P}^-(A)$. $\forall f \in \mathrm{C}(A)(M^\bullet, N^\bullet)$ qis,*

$$\mathrm{K}(A)(P^\bullet, M^\bullet) \simeq \mathrm{K}(A)(P^\bullet, N^\bullet) \quad \textit{via} \quad \phi \mapsto f \circ \phi.$$

Proof: Consider a d.t. $M^\bullet \xrightarrow{f} N^\bullet \to \mathrm{cone}(f) \to M^\bullet[1]$. As $\mathrm{K}(A)(P^\bullet, ?)$ is cohomological [中岡, Prop. 6.2.3, p. 347], one has a LES

$$\begin{aligned}
\cdots \to \mathrm{K}(A)(P^\bullet, \mathrm{cone}(f)[-1]) \to \mathrm{K}(A)(P^\bullet, M^\bullet) \xrightarrow{\mathrm{K}(A)(P^\bullet, f)} \mathrm{K}(A)(P^\bullet, N^\bullet) \to \\
\mathrm{K}(A)(P^\bullet, \mathrm{cone}(f)) \to \dots
\end{aligned}$$

As $\mathrm{H}^n(\mathrm{cone}(f)) = 0$ $\forall n \in \mathbb{Z}$ from 月.11(2), we have only to show that

(1) $$\mathrm{K}(A)(P^\bullet, X^\bullet) = 0 \quad \forall X^\bullet \in \mathrm{C}^-(A) \text{ with all } \mathrm{H}^n(X^\bullet) = 0, n \in \mathbb{Z}.$$

Given $g \in \mathrm{C}(A)(P^\bullet, X^\bullet)$, we construct a homotopy $\sigma : P^\bullet \to X^\bullet$ such that $g^n = d_X^{n-1} \circ \sigma^n + \sigma^{n+1} \circ d_P^n$ by decsending induction on n. Assume done up to $n+1$: $g^i = d_X^{i-1} \circ \sigma^i + \sigma^{i+1} \circ d_P^i$ $\forall i \geq n+1$. We now construct $\sigma^n : P^n \to X^{n-1}$

$$\begin{tikzcd}
 & & P^n \arrow[r, "d_P^n"] \arrow[d, "f^n"] \arrow[dll, dotted, "\sigma^n"'] \arrow[dl, dotted, "s"] & P^{n+1} \arrow[r, "d_P^{n+1}"] \arrow[d, "f^{n+1}"] \arrow[dl, "\sigma^{n+1}"] & P^{n+2} \arrow[d, "f^{n+2}"] \arrow[dl, "\sigma^{n+2}"] \\
X^{n-1} \arrow[r] \arrow[rr, bend right, "d_X^{n-1}"'] & \ker(d_X^n) \arrow[r, hook] & X^n \arrow[r, "d_X^n"'] & X^{n+1} \arrow[r, "d_X^{n+1}"'] & X^{n+2}.
\end{tikzcd}$$

One has

$$\begin{aligned} d_X^n \circ (f^n - \sigma^{n+1} \circ d_P^n) &= f^{n+1} \circ d_P^n - d_X^n \circ \sigma^{n+1} \circ d_P^n \\ &= (d_X^n \circ \sigma^{n+1} + \sigma^{n+2} \circ d_P^{n+1}) \circ d_P^n - d_X^n \circ \sigma^{n+1} \circ d_P^n = 0, \end{aligned}$$

and hence there is $s : P^n \to \ker(d_X^n)$ such that $f^n - \sigma^{n+1} \circ d_P^n = s$. As $H^n(X) = 0$, $\ker(d_X^n) = \mathrm{im}(d_X^{n-1})$. As P^n is projective, s factors through $X^{n-1} \twoheadrightarrow \ker(d_X^n)$ to yield σ^n with $f^n = \sigma^{n+1} \circ d_P^n + d_X^{n-1} \circ \sigma^n$.

火.7. **Corollary:** *(i)* $\forall P^\bullet \xrightarrow[\text{qis}]{f} M^\bullet \xleftarrow[\text{qis}]{g} Q^\bullet$ *in* $\mathrm{K}^-(A)$,

$$\begin{tikzcd} P^\bullet \arrow[rr, dotted, "\sim"] \arrow[dr, "f"'] & & Q^\bullet \arrow[dl, "g"] \\ & M^\bullet. & \end{tikzcd}$$

(ii) $\mathrm{P}^-(A) \simeq \mathrm{D}^-(A)$.

Proof: (i) Let $\phi \in \mathrm{K}(A)(P^\bullet, Q^\bullet)$ with $g \circ \phi = f$ and $\psi \in \mathrm{K}(A)(Q^\bullet, P^\bullet)$ with $f \circ \psi = g$ after 火.6. Then $f \circ \psi \circ \phi = g \circ \phi = f = f \circ \mathrm{id}_{P^\bullet}$. As $\mathrm{K}(A)(P^\bullet, f) : \mathrm{K}(A)(P^\bullet, P^\bullet) \xrightarrow{\sim} \mathrm{K}(A)(P^\bullet, M^\bullet)$, we must have $\psi \circ \phi = \mathrm{id}_{P^\bullet}$. Likewise, $\phi \circ \psi = \mathrm{id}_{Q^\bullet}$.

(ii) See [Gri, Th. 8.10, p. 73].

火.8. $\forall Y^\bullet \in \mathrm{K}(A)$ with all $\mathrm{H}^i(Y^\bullet) = 0$, $i \in \mathbb{Z}$, all $\mathrm{H}^i(A\mathrm{Mod}^\bullet(P^\bullet, Y^\bullet)) = 0$ [Gri, 10.5, p. 93]. One then obtains from [Gri, 10.7, p. 95/9.8, p. 82], $\forall N^\bullet \in \mathrm{K}(A)$,

$$\begin{tikzcd} \mathrm{P}^-(A) \arrow[r, "A\mathrm{Mod}^\bullet(?,N^\bullet)"] \arrow[d, "\sim"'] & \mathrm{K}(\mathbb{Z}) \arrow[d, "Q"] \\ \mathrm{D}^-(A) \arrow[r, dotted, "\mathrm{R}_I^- A\mathrm{Mod}^\bullet(?,N^\bullet)"'] & \mathrm{D}(\mathbb{Z}). \end{tikzcd}$$

Thus, $\forall M^\bullet \in \mathrm{D}^-(A)$, with $P^\bullet \in \mathrm{P}^-(A)$ such that $P^\bullet \xrightarrow{\text{qis}} M^\bullet$ in $\mathrm{K}^-(A)$, $\forall N^\bullet \in \mathrm{K}(A)$,

$$\mathrm{R}_I^- A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet) \simeq A\mathrm{Mod}^\bullet(P^\bullet, N^\bullet). \tag{1}$$

In particular, $\forall M, N \in A\mathrm{Mod}$, regarding M as $\cdots \to 0 \to M \to 0 \to \cdots \in \mathrm{K}^-(A)$ with M located in degree 0 and N in $\mathrm{K}(A)$ likewise, one has

$$\begin{aligned} \mathrm{H}^i(\mathrm{R}_I^- A\mathrm{Mod}^\bullet(M, N)) &\simeq \mathrm{H}^i(A\mathrm{Mod}^\bullet(P^\bullet, N)) \quad \text{for some } P^\bullet \in \mathrm{P}^-(A) \text{ with } P^\bullet \xrightarrow{\text{qis}} M \\ &\simeq \mathrm{H}^i(\cdots \to A\mathrm{Mod}(P^{-n}, N) \to A\mathrm{Mod}(P^{-n-1}, N) \to \dots) \\ &\qquad \text{as } A\mathrm{Mod}^n(P^\bullet, N) = \prod_j A\mathrm{Mod}(P^j, N[n]^j) = A\mathrm{Mod}(P^{-n}, N) \\ &\simeq \mathrm{Ext}_A^i(M, N) \end{aligned}$$

$$\begin{tikzcd} A\mathrm{Mod} \arrow[r, hook] \arrow[ddr, dotted, "\mathrm{Ext}^i_A(?,N)"'] & \mathrm{D}^-(A) \arrow[d, "\mathrm{R}_I^- A\mathrm{Mod}(?,N)"] \\ & \mathrm{D}(\mathbb{Z}) \arrow[d, "\mathrm{H}^i"] \\ & \mathbb{Z}\mathrm{Mod}. \end{tikzcd}$$

From (1) one obtains a bifunctor $\mathrm{R}_I^- A\mathrm{Mod}^\bullet(?,?) : \mathrm{D}^-(A) \times \mathrm{K}^+(A) \to \mathrm{D}(\mathbb{Z})$. If $M^\bullet \in \mathrm{D}^-(A)$, the functor $\mathrm{R}_I^- A\mathrm{Mod}^\bullet(M^\bullet, ?) : \mathrm{K}^+(A) \to \mathrm{D}(\mathbb{Z})$ induces a functor $\mathrm{R}_{II}^+\mathrm{R}_I^- A\mathrm{Mod}^\bullet(M^\bullet, ?) : \mathrm{D}^+(A) \to \mathrm{D}(\mathbb{Z})$ [Gri, 10.7, p. 95]: let $\mathrm{I}^+(A) = \{I^\bullet \in \mathrm{K}^+(A) | \text{all } I^n,\ n \in \mathbb{Z}, \text{ are injective}\}$. $\forall N^\bullet \in \mathrm{D}^+(A)$,

$$\mathrm{R}_{II}^+\mathrm{R}_I^- A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet) = \mathrm{R}_I^- A\mathrm{Mod}^\bullet(M^\bullet, I^\bullet) \quad \text{with } I^\bullet \in \mathrm{I}^+(A) \text{ such that } N^\bullet \xrightarrow{\text{qis}} I^\bullet.$$

One has likewise, $\forall M^\bullet \in \mathrm{K}^-(A)$, a functor $A\mathrm{Mod}^\bullet(M^\bullet, ?) : \mathrm{K}^+(A) \to \mathrm{K}(\mathbb{Z})$, which induces $\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(M^\bullet, ?) : \mathrm{D}^+(A) \to \mathrm{D}(\mathbb{Z})$, and a bifunctor $\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(?,?) : \mathrm{K}^-(A) \times \mathrm{D}^+(A) \to \mathrm{D}(A)$. $\forall N^\bullet \in \mathrm{D}^+(A)$, $\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(?, N^\bullet) : \mathrm{K}^-(A) \to \mathrm{D}(A)$ induces $\mathrm{R}_I^-\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(?,?) : \mathrm{D}^-(A) \times \mathrm{D}^+(A) \to \mathrm{D}(\mathbb{Z})$ [Gri, 10.6, p. 94]: $\forall M^\bullet \in \mathrm{D}^-(A)$,

$$\mathrm{R}_I^-\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet) = \mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(P^\bullet, N^\bullet) \quad \text{with } P^\bullet \in \mathrm{P}^-(A) \text{ such that } P^\bullet \xrightarrow{\text{qis}} M^\bullet.$$

Theorem [Gri, 10.8, p. 95]: *On* $\mathrm{D}^-(A) \times \mathrm{D}^+(A)$

$$\mathrm{R}_{II}^+\mathrm{R}_I^- A\mathrm{Mod}^\bullet(?,?) \simeq \mathrm{R}_I^-\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(?,?).$$

火.9. **Proposition [Gri, 10.9, p. 96]:** $\forall M^\bullet \in \mathrm{D}^-(A)$, $\forall N^\bullet \in \mathrm{D}^+(A)$, $\forall i \in \mathbb{Z}$,

$$\mathrm{H}^i(\mathrm{R}_{II}^+\mathrm{R}_I^- A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet)) \simeq \mathrm{D}(A)(M^\bullet, N^\bullet[i]).$$

Proof: Let $P^\bullet \in \mathrm{P}^-(A)$ with qis $P^\bullet \to M^\bullet$ and $J^\bullet \in \mathrm{I}^+(A)$ with qis $N^\bullet \to J^\bullet$. Then

$$\begin{aligned} \text{(1)} \qquad \text{LHS} &= \mathrm{H}^i(\mathrm{R}_{II}^+ A\mathrm{Mod}^\bullet(P^\bullet, N^\bullet)) = \mathrm{H}^i(A\mathrm{Mod}^\bullet(P^\bullet, J^\bullet)) \quad \text{by construction} \\ &= \mathrm{K}(A)(P^\bullet, J^\bullet[i]) \quad \text{by 火.5} \\ &= \mathrm{K}(A)(P^\bullet, N^\bullet[i]) \quad \text{by 火.6.} \end{aligned}$$

We now claim

$$\text{(2)} \qquad \mathrm{D}(A)(P^\bullet, N^\bullet) \simeq \mathrm{K}(A)(P^\bullet, N^\bullet).$$

Let $[P^\bullet \underset{\text{qis}}{\overset{s}{\leftarrow}} X^\bullet \xrightarrow{f} N^\bullet] \in \text{LHS}$. As $P^\bullet \in \mathrm{P}^-(A)$, there is $Y^\bullet \in \mathrm{K}^-(A)$ with qis $Y^\bullet \to X^\bullet$. Take a qis $Q^\bullet \to Y^\bullet$ with $Q^\bullet \in \mathrm{P}^-(A)$, so

$$\begin{tikzcd} Q^\bullet \arrow[r] \arrow[dr, dotted, "t"', "\text{qis}"] & Y^\bullet \arrow[d] \\ & X^\bullet. \end{tikzcd}$$

Then $[P^\bullet \xleftarrow{s} X^\bullet \xrightarrow{f} N^\bullet] = [P^\bullet \xleftarrow{s\circ t} Q^\bullet \xrightarrow{f\circ t} N^\bullet]$ with $s \circ t$ invertible in $\mathrm{K}(A)$ by 火.7. Thus, one obtains a map $\mathrm{D}(A)(P^\bullet, N^\bullet) \to \mathrm{K}(A)(P^\bullet, N^\bullet)$ via $[f/s] \mapsto (f \circ t) \circ (s \circ t)^{-1}$ with inverse $[h/1] \leftarrow\!\shortmid h$.

火.10. $\forall M^\bullet, N^\bullet \in \mathrm{D}(A)$, $\forall i \in \mathbb{Z}$, set

$$\mathrm{Ext}^i_A(M^\bullet, N^\bullet) = \mathrm{D}(A)(M^\bullet, N^\bullet[i]).$$

In case $M^\bullet \in \mathrm{D}^-(A)$ and $N^\bullet \in \mathrm{D}^+(A)$ one has from 火.9

$$\mathrm{Ext}^i_A(M^\bullet, N^\bullet) = \mathrm{H}^i(\mathrm{R}^+_{II}\mathrm{R}^-_I A\mathrm{Mod}^\bullet(M^\bullet, N^\bullet)).$$

$\forall M, N \in A\mathrm{Mod}$,

$$\begin{aligned} \mathrm{H}^i(\mathrm{R}^+_{II}\mathrm{R}^-_I A\mathrm{Mod}^\bullet(M, N)) &\simeq \mathrm{K}(A)(P^\bullet, N[i]) \quad \text{by 火.9.1} \\ &\simeq \mathrm{H}^i(A\mathrm{Mod}^\bullet(P^\bullet, N)) \quad \text{by 火.5} \\ &\simeq \mathrm{H}^i(A\mathrm{Mod}(P^\bullet, N)) = \mathrm{Ext}^i_A(M, N), \end{aligned}$$

consistant with the notation in 月.6.

火.11. *A variant: dg-algebras and dg-modules*

Let $\Bbbk$ be a field, and let $\mathbf{Alg}_\Bbbk$ denote the category of $\Bbbk$-algebras

Definition: *A $\mathbb{Z}$-graded $\Bbbk$-algebra is a $\Bbbk$-algebra A such that $A = \coprod_{i\in\mathbb{Z}} A^i$ as $\Bbbk$-linear spaces with $A^iA^j \subseteq A^{i+j}$ $\forall i, j$ and $1 \in A^0$; A^i should not be confused with $\underbrace{A \dots A}_{i-times}$. We will often suppress $\mathbb{Z}$ and refer to a graded $\Bbbk$-algebra or even to a graded algebra.*

E.g. The polynomial $\Bbbk$-algebra $\Bbbk[x]$ in x is a graded algebra $\Bbbk[x] = \coprod_{i\in\mathbb{N}} \Bbbk x^i$ with

$$\Bbbk[x]^i = \begin{cases} \Bbbk x^i & i \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

火.12. Let A be a graded $\Bbbk$-algebra.

Definition: *A graded A-module is an A-module M such that $M = \coprod_{i\in\mathbb{Z}} M^i$ as $\Bbbk$-linear spaces with $A^iM^j \subseteq M^{i+j}$ $\forall i, j$. We say $m \in M$ is of degree i iff $m \in M^i$, in which case we write $\deg(m) = i$. We say $m \in M$ is homogeneous iff $m \in M^i$ for some $i \in \mathbb{Z}$.*

If M, N are graded A-modules, we say $f \in \mathrm{AMod}(M, N)$ is of degree k iff $f(M^i) \subseteq N^{i+k}$ $\forall i$. We let AModgr denote the category of graded A-modules with morphisms of degree 0.

火.13. **Definition:** *A dg-algebra is a pair $(\mathcal{A}, \mathrm{d})$ of a graded $\Bbbk$-algebra $\mathcal{A} = \coprod_{i\in\mathbb{Z}} \mathcal{A}^i$ and a $\Bbbk$-linear map $\mathrm{d}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ of degree 1 such that* *(i)* $\mathrm{d}_{\mathcal{A}}^2 = 0$,

(ii) $\forall a \in \mathcal{A}$ *homogeneous,* $\forall b \in \mathcal{A}$, $\mathrm{d}_{\mathcal{A}}(ab) = (\mathrm{d}_{\mathcal{A}}a)b + (-1)^{\deg a}a(\mathrm{d}_{\mathcal{A}}b)$.

In particular, $\mathrm{d}_{\mathcal{A}}1 = 0$.

A dg $\mathcal{A}$*-module is a pair* $(\mathcal{M}, \mathrm{d}_{\mathcal{M}})$ *of a graded* $\mathcal{A}$*-module* $\mathcal{M}$ *and a* $\Bbbk$*-linear map* $\mathrm{d}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ *of degreee 1 such that*

(i) $\mathrm{d}_{\mathcal{M}}^2 = 0$,

(ii) $\forall a \in \mathcal{A}$ *homogeneous,* $\forall m \in \mathcal{M}$, $\mathrm{d}_{\mathcal{M}}(am) = (\mathrm{d}_{\mathcal{A}}a)m + (-1)^{\deg a}a(\mathrm{d}_{\mathcal{M}}m)$,

in which case we set $\mathrm{H}^i(\mathcal{M}) = \ker(\mathrm{d}_{\mathcal{M}}|_{\mathcal{M}^i})/\mathrm{im}(\mathrm{d}_{\mathcal{M}}|_{\mathcal{M}^{i-1}})$ $\forall i \in \mathbb{Z}$.

A morphism of dg $\mathcal{A}$*-modules is a homomorphism* $f : \mathcal{M} \to \mathcal{N}$ *of* $\mathcal{A}$*-modules of degree 0 such that* $\mathrm{d}_{\mathcal{N}} \circ f = f \circ \mathrm{d}_{\mathcal{M}}$, *in which case one obtains* $\mathrm{H}^i(f) : \mathrm{H}^i(\mathcal{M}) \to \mathrm{H}^i(\mathcal{N})$ $\forall i \in \mathbb{Z}$. *We say* f *is a qis iff* $\mathrm{H}^i(f)$ *is invertible* $\forall i$. *We denote the category of dg* $\mathcal{A}$*-modules by* $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$. *For* $\mathcal{M} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$ *and* $i \in \mathbb{Z}$ *let* $\mathcal{M}[i] \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$ *such that* $\mathcal{M}[i]^j = \mathcal{M}^{j+i}$ $\forall j \in \mathbb{Z}$ *and* $\mathrm{d}_{\mathcal{M}[i]} = (-1)^i\mathrm{d}_{\mathcal{M}}$

E.g. (i) The dg-algebra $\mathcal{A}$ itself is a dg $\mathcal{A}$-module with the same differential.

(ii) Let A be a $\Bbbk$-algebra. Set $\mathcal{A} = \coprod_{i\in\mathbb{Z}} \mathcal{A}^i$ with $\mathcal{A}^i = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$ Then $\mathcal{A}$ forms a dg-algebra with $\mathrm{d}_{\mathcal{A}} = 0$. A dg $\mathcal{A}$-module $(\mathcal{M}, \mathrm{d}_{\mathcal{M}})$ is $\mathcal{M} = \coprod_{i\in\mathbb{Z}} \mathcal{M}^i$ such that each $\mathcal{M}^i$, $i \in \mathbb{Z}$, is an A-module with $\mathrm{d}_{\mathcal{M}} \in A\mathrm{Mod}(\mathcal{M}^i, \mathcal{M}^{i+1})$ such that $\mathrm{d}_{\mathcal{M}}^2 = 0$ and $\mathrm{d}_{\mathcal{M}}(am) = a\mathrm{d}_{\mathcal{M}}m$ $\forall a \in A = \mathcal{A}^0$. Thus, $\mathrm{d}_{\mathcal{M}}$ is A-linear and $(\mathcal{M}, \mathrm{d}_{\mathcal{M}})$ is just a complex of A-modules.

火.14. For a dg-algebra $\mathcal{A}$ one defines $\mathrm{K}_{\mathrm{dg}}(\mathcal{A}), \mathrm{D}_{\mathrm{dg}}(\mathcal{A}), \mathrm{D}^b_{\mathrm{dg}}(\mathcal{A})$, etc. from $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$ in the standard way; we say $f, g \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N})$ are homotopic iff there is $s \in \mathcal{A}\mathrm{Modgr}(\mathcal{M}, \mathcal{N}[-1])$, which need not belong to $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$, such that $f - g = s\mathrm{d}_{\mathcal{M}} + \mathrm{d}_{\mathcal{N}}s$. We define the homotopy category $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$ as the ideal quotient of $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$ by the null homotopic morphisms: $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N}) = \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N})/\mathrm{Ht}_{\mathrm{dg}}(\mathcal{M}, \mathcal{N})$ with $\mathrm{Ht}_{\mathrm{dg}}(\mathcal{M}, \mathcal{N}) = \{f \in \mathrm{K}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N}) | f \text{ is homotopic to } 0\}$ [BL, 10.3.1], [中岡, Def. 3.2.43, p. 147]. For $f \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N})$ the cone of f is $\mathrm{cone}(f) = \mathcal{M}[1] \oplus \mathcal{N}$ with differential $\begin{pmatrix} -\mathrm{d}_{\mathcal{M}} & 0 \\ f & \mathrm{d}_{\mathcal{N}} \end{pmatrix}$. We call the sequence $\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{(0,1)} \mathrm{cone}(f) \xrightarrow{(1,0)} \mathcal{M}[1]$ a standard triangle. A distinguished triangle in $\mathcal{K}_{\mathrm{dg}}(\mathcal{A})$ is a sequence isomorphic to a standard one in $\mathcal{K}_{\mathrm{dg}}(\mathcal{A})$. We say an exact sequence $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ in $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$ is semi-split iff it splits as graded $\mathcal{A}$-modules. As in 火.2.(2) any semi-split $f \in \mathrm{K}_{\mathrm{dg}}(\mathcal{A})(\mathcal{L}, \mathcal{M})$ can be completed to form a d.t. $\mathcal{L} \to \mathcal{M} \to \mathcal{N} \to \mathcal{L}[1]$ in $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$. Together with the d.t.'s $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$ form a triangulated category. The localization of $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$ by the qis's form $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ with triangulation induced from one on $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$.

Definition: $\forall \mathcal{M}, \mathcal{N}$ *dg* $\mathcal{A}$*-modules, let* $\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{M}, \mathcal{N})$ *denote a complex of* $\Bbbk$*-linear spaces such that*

(i) $\forall i \in \mathbb{Z}$, $\mathcal{A}\mathrm{Mod}^i(\mathcal{M}, \mathcal{N}) = \mathcal{A}\mathrm{Modgr}(\mathcal{M}, \mathcal{N}[i]) = \{f \in \mathcal{A}\mathrm{Mod}(\mathcal{M}, \mathcal{N}[i]) | f(\mathcal{M}^j) \subseteq \mathcal{N}[i]^j$

$\forall j \in \mathbb{Z}\} = \mathcal{A}\mathrm{Modgr}(\mathcal{M}[-i], \mathcal{N})$,

(ii) $\forall f \in \mathcal{A}\mathrm{Mod}^i(\mathcal{M}, \mathcal{N})$, $df = \mathrm{d}_{\mathcal{N}} \circ f - (-1)^i f \circ \mathrm{d}_{\mathcal{M}}$.

In particular, as $\mathrm{d}_{\mathcal{A}} 1 = 0$, one has a bijection $\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{A}, \mathcal{M}) \to \mathcal{M}$ via $f \mapsto f(1)$ such that $(df)(1) = d_{\mathcal{M}}(f(1))$. If we let $af = f(?a)$ $\forall a \in \mathcal{A}$ $f \in \mathcal{A}\mathrm{Mod}^\bullet(\mathcal{A}, \mathcal{M})$, $\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{A}, \mathcal{M})$ comes equipped with a structure of dg $\mathcal{A}$-module by d to make the bijection into an isomorphism of dg $\mathcal{A}$-modules

$$\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{A}, \mathcal{M}) \simeq \mathcal{M}. \tag{1}$$

One has only to check that, $\forall f \in \mathcal{A}\mathrm{Mod}^i(\mathcal{A}, \mathcal{M})$, $\forall a \in \mathcal{A}^j$, $d(af) = (\mathrm{d}_{\mathcal{A}} a) f + (-1)^j a(df)$. In $\mathcal{M}^{i+j}$, however, one has

$$\begin{aligned}
\{d(af)\}(1) &= \{d(f(?a))\}(1) = \{\mathrm{d}_{\mathcal{M}} \circ f(?a) - (-1)^{i+j} f(?a) \circ \mathrm{d}_{\mathcal{A}}\}(1) = \mathrm{d}_{\mathcal{M}}(f(a)) = \mathrm{d}_{\mathcal{M}}(af(1)) \\
&= (\mathrm{d}_{\mathcal{A}} a) f(1) + (-1)^j a \mathrm{d}_{\mathcal{M}}(f(1)) = (\mathrm{d}_{\mathcal{A}} a) f(1) + (-1)^j a(df)(1) \\
&= \{(\mathrm{d}_{\mathcal{A}} a) f + (-1)^j a(df)\}(1),
\end{aligned}$$

as desired.

Ex. (i) $\ker(d : \mathcal{A}\mathrm{Mod}^0(\mathcal{M}, \mathcal{N}) \to \mathcal{A}\mathrm{Mod}^1(\mathcal{M}, \mathcal{N})) = \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N})$.

(ii) $\mathrm{H}^0(\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{M}, \mathcal{N})) = \mathrm{K}_{\mathrm{dg}}(\mathcal{A})(\mathcal{M}, \mathcal{N})$.

火.15. For a dg-algebra $\mathcal{A}$ a right dg $\mathcal{A}$-module $(\mathcal{M}, \mathrm{d}_{\mathcal{M}})$ is a right graded $\mathcal{A}$-module $\mathcal{M} = \coprod_{i \in \mathbb{Z}} \mathcal{M}^i$ with a $\Bbbk$-linear map $\mathrm{d}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ of degree 1 such that $\mathrm{d}_{\mathcal{M}}^2 = 0$ and that

$$\mathrm{d}_{\mathcal{M}}(ma) = (\mathrm{d}_{\mathcal{M}} m)a + (-1)^{\deg m} m(\mathrm{d}_{\mathcal{A}} a) \quad \forall a \in \mathcal{A}, \forall m \in \mathcal{M} \text{ homogeneous,} \tag{1}$$

We denote the category of right dg $\mathcal{A}$-module by $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}}$, and define $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}}$, $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}}$ as for the dg $\mathcal{A}$-modules [BL, 10.6.1].

We define the opposite $\mathcal{A}^{\mathrm{op}}$ to be a dg-algebra whose ambient $\Bbbk$-linear space and the differential are the same as those of $\mathcal{A}$, but with new multiplication [BL, 10.6.2]

$$a_{\mathcal{A}^{\mathrm{op}}} b = (-1)^{\deg(a) \deg(b)} ba \quad \forall a, b \text{ homogeneous.} \tag{2}$$

Then $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}} \simeq \mathrm{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}})$ [BL, 10.6.3] by assignning $\mathcal{M} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}}$ a structure of left dg $\mathcal{A}^{\mathrm{op}}$-module such that

$$am = (-1)^{\deg(a) \deg(m)} ma \quad \forall a, m \text{ homogeneous.} \tag{3}$$

$\forall \mathcal{M} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}}$, $\forall \mathcal{N} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$, define a complex $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ of $\Bbbk$-linear spaces with the differential such that

$$\mathrm{d}(m \otimes n) = (\mathrm{d}_{\mathcal{M}} m) \otimes n + (-1)^{\deg(m)} m \otimes \mathrm{d}_{\mathcal{N}} n \quad \forall m \in \mathcal{M}, \forall n \in \mathcal{N} \text{ homogeneous.} \tag{4}$$

In particular,

$$\mathcal{A} \otimes_{\mathcal{A}} \mathcal{N} \simeq \mathcal{N} \quad \text{via} \quad a \otimes n \mapsto an, \tag{5}$$
$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{M} \quad \text{via} \quad m \otimes a \mapsto ma. \tag{6}$$

火.16. *Bar construction*

Let $\mathcal{A}$ be a dg-algebra. One has that bifunctors $\mathcal{A}\mathrm{Mod}^\bullet(?,?) : \mathrm{K}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{K}_{\mathrm{dg}}(\Bbbk) = \mathrm{C}(\Bbbk)$ and $?\otimes_{\mathcal{A}}? : \mathrm{K}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}} \times \mathrm{K}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{K}_{\mathrm{dg}}(\Bbbk)$ are both triangulated, i.e., sends a d.t. to a d.t. [BL, 10.8.1, 10.9.1], [中岡, 6.2.2, p. 364]. In order to define derived functors of $\mathcal{A}\mathrm{Mod}^\bullet(?,?)$ and $?\otimes_{\mathcal{A}}?$ we introduce bar construction [BL, 10.12.2.4].

Let $\mathcal{M}$ be a dg $\mathcal{A}$-module. Let $\mathcal{P}_0 = \mathcal{A}\otimes_\Bbbk\mathcal{M} = \coprod_{i\in\mathbb{Z}}\mathcal{P}_0^i$ with $\mathcal{P}_0^i = (\mathcal{P}_0)^i = \coprod_{s+t=i}\mathcal{A}^s\otimes_\Bbbk\mathcal{M}^t$. $\forall a \in \mathcal{A}$ homogeneous, $\forall m \in \mathcal{M}$, define $\mathrm{d}_{\mathcal{P}_0}(a\otimes m) = (\mathrm{d}_{\mathcal{A}}a)\otimes m + (-1)^{\deg a}a\otimes \mathrm{d}_{\mathcal{M}}m$. Then $(\mathcal{P}_0, \mathrm{d}_{\mathcal{P}_0})$ forms a dg $\mathcal{A}$-module. If $\delta_0 : \mathcal{P}_0 \to \mathcal{M}$ via $a\otimes m \mapsto am$, $\delta_0 \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(\mathcal{P}_0,\mathcal{M})$, and hence $(\ker(\delta_0), \mathrm{d}_{\mathcal{P}_0}|_{\ker(\delta_0)}) \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$. Let next $\mathcal{P}_{-1} = \mathcal{A}\otimes_\Bbbk \ker(\delta_0)$ with $\mathrm{d}_{\mathcal{P}_{-1}}$ defined just like $\mathrm{d}_{\mathcal{P}_0}$ replacing $\mathrm{d}_{\mathcal{M}}$ by $\mathrm{d}_{\mathcal{P}_0}|_{\ker(\delta_0)}$. Then $(\mathcal{P}_{-1}, \mathrm{d}_{\mathcal{P}_{-1}}) \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$. If $\delta_{-1} : \mathcal{P}_{-1} \to \mathcal{P}_0$ via $a\otimes p \mapsto ap$, $\delta_{-1} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(\mathcal{P}_{-1},\mathcal{P}_0)$, and hence $(\ker(\delta_{-1}), \mathrm{d}_{\mathcal{P}_{-1}}|_{\ker(\delta_{-1})}) \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$. Repeat to get an exact sequence in $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$

$$\cdots \to \mathcal{P}_{-2} \xrightarrow{\delta_{-2}} \mathcal{P}_{-1} \xrightarrow{\delta_{-1}} \mathcal{P}_0 \xrightarrow{\delta_0} \mathcal{M} \to 0.$$

Definition: *Set* $B(\mathcal{M}) = \coprod_{i\in\mathbb{Z}}\mathcal{P}_{-i}[i] = \coprod_{j\in\mathbb{Z}} B(\mathcal{M})^j$ *with* $B(\mathcal{M})^j = \coprod_{i\in\mathbb{Z}}(\mathcal{P}_{-i}[i])^j = \coprod_{i\in\mathbb{Z}}\mathcal{P}_{-i}^{i+j}$. *As* $\mathcal{A}^k(\mathcal{P}_{-i}[i])^j = \mathcal{A}^k\mathcal{P}_{-i}^{i+j} \subseteq \mathcal{P}_{-i}^{i+j+k} = \mathcal{P}_{-i}[i]^{j+k}$, $B(\mathcal{M})$ *is a graded* $\mathcal{A}$*-module. For* $p \in \mathcal{P}_{-i}[i]$ *homogeneous let* $\mathrm{d}_{B(\mathcal{M})}(p) = \mathrm{d}_{\mathcal{P}_{-i}}(p) + (-1)^{\deg p}\delta_{-i}(p)$. *If* $p \in (\mathcal{P}_{-i}[i])^j \subseteq B(\mathcal{M})^j$, $\mathrm{d}_{\mathcal{P}_{-i}}(p) \in \mathcal{P}_{-i}^{i+j+1} = (\mathcal{P}_{-i}[i])^{j+1} \subseteq B(\mathcal{M})^{j+1}$ *and* $\delta_{-i}(p) \in \mathcal{P}_{-i+1}^{i+j} = (\mathcal{P}_{-(i-1)}[i-1])^{j+1}$, *and hence* $\mathrm{d}_{B(\mathcal{M})}(p) \in B(\mathcal{M})^{j+1}$. *If* $a \in \mathcal{A}^k$,

$$\begin{aligned}\mathrm{d}_{B(\mathcal{M})}(ap) &= \mathrm{d}_{\mathcal{P}_{-i}}(ap) + (-1)^{k+j}\delta_{-i}(ap) = \mathrm{d}_{\mathcal{A}}(a)p + (-1)^k a\mathrm{d}_{\mathcal{P}_{-i}}(p) + (-1)^{k+j}a\delta_{-i}(p)\\ &= \mathrm{d}_{\mathcal{A}}(a)p + (-1)^k a\{\mathrm{d}_{\mathcal{P}_{-i}}(p) + (-1)^j\delta_{-i}(p)\} = \mathrm{d}_{\mathcal{A}}(a)p + (-1)^k a\mathrm{d}_{B(\mathcal{M})}(p).\end{aligned}$$

Thus, $(B(\mathcal{M}), \mathrm{d}_{B(\mathcal{M})})$ *forms a dg* $\mathcal{A}$*-module.*

E.g. Assume the set up of E.g. 火.13.(ii), and let M be an A-module, regarded as a dg $\mathcal{A}$-module $\mathcal{M} = \coprod_{i\in\mathbb{Z}}\mathcal{M}^i$ with $\mathrm{d}_{\mathcal{M}} = 0$ and $\mathcal{M}^i = \begin{cases} M & \text{if } i = 0,\\ 0 & \text{else.}\end{cases}$ Then $B(M) = \coprod_{i\in\mathbb{N}}\mathcal{P}_{-i}[i]$ reads

$$\begin{aligned}\mathcal{P}_0 &= \coprod_{i\in\mathbb{Z}}\mathcal{P}_0^i \quad \text{with} \quad \mathcal{P}_0^i = (\mathcal{P}_0)^i = \begin{cases} A\otimes_\Bbbk M & \text{if } i = 0,\\ 0 & \text{else,}\end{cases}\\ &= \mathcal{P}_0^0, \quad \mathrm{d}_{\mathcal{P}_0} = 0,\end{aligned}$$

$$\begin{tikzcd} \mathcal{P}_0 \arrow[r, dotted, "\delta_0"] \arrow[d, equal] & \mathcal{M} \arrow[d, equal] \\ \mathcal{P}_0^0 \arrow[d, equal] & \mathcal{M}^0 \arrow[d, equal] \\ A\otimes_\Bbbk M \arrow[r, "\delta_0^0"] & M \\ a\otimes m \arrow[r, mapsto] & am, \end{tikzcd}$$

$$
\begin{aligned}
\mathcal{P}_{-1} &= \mathcal{A} \otimes_{\Bbbk} \ker(\delta_0) \\
&= \coprod_{i\in\mathbb{Z}} \mathcal{P}^i_{-1} \quad \text{with} \quad \mathcal{P}^i_{-1} = (\mathcal{P}_{-1})^i = \begin{cases} A \otimes_{\Bbbk} \ker(\delta_0) = A \otimes_{\Bbbk} \ker(\delta^0_0) & \text{if } i = 0, \\ 0 & \text{else,} \end{cases} \\
&= \mathcal{P}^0_{-1}, \quad \mathrm{d}_{\mathcal{P}_{-1}} = 0,
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{P}_{-1} \arrow[r, dotted, "\delta_{-1}"] \arrow[d, equal] & \mathcal{P}_0 \arrow[d, equal] \\
\mathcal{P}^0_{-1} \arrow[d, equal] & \mathcal{P}^0_0 \arrow[d, equal] \\
A \otimes_{\Bbbk} \ker(\delta^0_0) \arrow[r, "\delta^0_{-1}"] & A \otimes_{\Bbbk} M \\
a \otimes x \arrow[r, mapsto] & ax,
\end{tikzcd}
$$

$$
\begin{aligned}
\mathcal{P}_{-2} &= \mathcal{A} \otimes_{\Bbbk} \ker(\delta_{-1}) \\
&= \coprod_{i\in\mathbb{Z}} \mathcal{P}^i_{-2} \quad \text{with} \quad \mathcal{P}^i_{-2} = (\mathcal{P}_{-2})^i = \begin{cases} A \otimes_{\Bbbk} \ker(\delta_{-1}) = A \otimes_{\Bbbk} \ker(\delta^0_{-1}) & \text{if } i = 0, \\ 0 & \text{else,} \end{cases} \\
&= \mathcal{P}^0_{-2}, \quad \mathrm{d}_{\mathcal{P}_{-2}} = 0,
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{P}_{-2} \arrow[r, dotted, "\delta_{-2}"] \arrow[d, equal] & \mathcal{P}_{-1} \arrow[d, equal] \\
\mathcal{P}^0_{-2} \arrow[d, equal] & \mathcal{P}^0_{-1} \arrow[d, equal] \\
A \otimes_{\Bbbk} \ker(\delta^0_{-1}) \arrow[r, "\delta^0_{-2}"] & A \otimes_{\Bbbk} \ker(\delta^0_0) \\
a \otimes y \arrow[r, mapsto] & ay,
\end{tikzcd}
$$

$\dots$, $\mathcal{P}_{-i} = \mathcal{P}^0_{-i} = (\mathcal{P}_{-i})^0 = \mathcal{A} \otimes_{\Bbbk} \ker(\delta^0_{-i+1})$, $\mathrm{d}_{\mathcal{P}_{-i}} = 0$,

$$
\begin{tikzcd}
\mathcal{P}_{-i} \arrow[r, dotted, "\delta_{-i}"] \arrow[d, equal] & \mathcal{P}_{-i+1} \arrow[d, equal] \\
\mathcal{P}^0_{-i} \arrow[d, equal] & \mathcal{P}^0_{-i+1} \\
A \otimes_{\Bbbk} \ker(\delta^0_{-i+1}) \arrow[ur, "\delta^0_{-i}"] & az. \\
a \otimes z \arrow[ur, mapsto] &
\end{tikzcd}
$$

Thus, $\mathcal{P}_{-i}[i] = \mathcal{P}^0_{-i}[i] = (\mathcal{P}_{-i}[i])^{-i}$. Then, $\forall j \in \mathbb{Z}$,

$$
B(M)^j = (\coprod_i \mathcal{P}_{-i}[i])^j = \coprod_i \mathcal{P}^{i+j}_{-i} = \begin{cases} \mathcal{P}^0_{-i} & \text{if } j = -i, \\ 0 & \text{else.} \end{cases}
$$

If $p \in (\mathcal{P}_{-i}[i])^{-i} = \mathcal{P}^0_{-i}$, $\mathrm{d}_{B(M)}(p) = \mathrm{d}_{\mathcal{P}_{-i}}(p) + (-1)^{-i}\delta_{-i}(p) = (-1)^i\delta_{-i}(p)$. $\forall j \in \mathbb{N}$,

$$
\begin{tikzcd}
B(M)^{-j} \arrow[r, dotted, "\mathrm{d}^{-j}_{B(M)}"] \arrow[d, phantom, "\shortparallel"] & B(M)^{-j+1} \arrow[d, phantom, "\shortparallel"] \\
\mathcal{P}^0_{-j} \arrow[d, equal] & \mathcal{P}^0_{-j+1} \\
A \otimes_{\Bbbk} \ker(\delta^0_{-j+1}) \arrow[ur, "\delta^0_{-i}"] & (-1)^j ax. \\
a \otimes x \arrow[ur, mapsto] &
\end{tikzcd}
$$

Thus, $B(M) = \coprod_{i\in\mathbb{Z}} B(M)^i = \coprod_{i\in\mathbb{N}} B(M)^{-i}$ with $B(M)^{-i} = A \otimes_{\Bbbk} \ker(\delta^0_{-i+1})$, and $\mathrm{d}^{-i}_{B(M)} : B(M)^{-i} \to B(M)^{-i+1}$ A-linear. As $B(M)^0 = \mathcal{P}^0_0 = A \otimes_{\Bbbk} M \xrightarrow{\delta^0_0} M$ is surjective, we may regard $(B(M), \mathrm{d}_{B(M)})$ as a free A-linear resolution of M.

Lemma: *One has [BL, 10.12.2.5]*

$$
\begin{tikzcd}
B(\mathcal{M}) \arrow[r, "\mathrm{proj}"] \arrow[dr, dotted, "\mathrm{qis}"'] & \mathcal{P}_0 \arrow[d] \\
& \mathcal{M}.
\end{tikzcd}
$$

火.17. Analogously to 火. 6.1, one has

Lemma: *Let $\mathcal{M}, \mathcal{N} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A})$.*

(i) If $\mathrm{H}^i(\mathcal{N}) = 0$ $\forall i \in \mathbb{Z}$, $\mathrm{H}^i(\mathcal{A}\mathrm{Mod}^\bullet(B(\mathcal{M}), \mathcal{N})) = 0$ $\forall i \in \mathbb{Z}$, which suggests the "projectivity" of $B(\mathcal{M})$ [BL, 10.12.2.6].

(ii) $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{M}), \mathcal{N}) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{M}), \mathcal{N})$ [BL, 10.12.2.2].

火.18. As in 火.8

Definition: *We define* $\mathrm{R}\mathcal{A}\mathrm{Mod}^\bullet(?, ?) : \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \times \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{D}(\mathbb{Z})$ *by setting*

$$\mathrm{R}\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{M}, \mathcal{N}) = \mathcal{A}\mathrm{Mod}^\bullet(B(\mathcal{M}), \mathcal{N}) \quad \forall \mathcal{M}, \mathcal{N} \in \mathrm{C}_{\mathrm{dg}}(\mathcal{A}) \quad \text{[BL, 10.12.3.1]},$$

and $?^{\mathbb{L}}\otimes_{\mathcal{A}}? : \mathrm{D}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}} \times \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{D}(\mathbb{Z})$ *by setting*

$$\mathcal{L}^{\mathbb{L}}\otimes_{\mathcal{A}}\mathcal{M} = \mathcal{L}^{\mathbb{L}}\otimes_{\mathcal{A}}B(\mathcal{M}) \quad \forall \mathcal{M} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A}), \forall \mathcal{L} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})^{\mathrm{r}} \quad \text{[BL, 10.12.4.5]}.$$

Theorem: *Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of dg-algebras. If f is a qis,*

$$\mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) \quad \textit{via} \quad \mathcal{M} \mapsto \mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M})$$

with quasi-inverse $\mathcal{N} \mapsto f^*\mathcal{N}$ which is $\mathcal{N}$ regarded as a dg $\mathcal{A}$-module through f.

Proof: As f is a qis and as $B(\mathcal{M})$ is flat over $\mathcal{A}$, one has by 火.16

$$\mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M}) \xleftarrow[\text{qis}]{f \otimes_{\mathcal{A}} B(\mathcal{M})} \mathcal{A} \otimes_{\mathcal{A}} B(\mathcal{M}) \simeq B(\mathcal{M}) \xrightarrow{\text{qis}} \mathcal{M},$$

and hence $f^*(\mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M})) \simeq \mathcal{M}$ in $D_{\text{dg}}(\mathcal{A})$.

Likewise, one has a CD

$$\begin{tikzcd} b \otimes y \arrow[r, mapsto] & by \\ \mathcal{B} \otimes_{\mathcal{A}} B(f^*\mathcal{N}) \arrow[r] & \mathcal{N} \\ \mathcal{A} \otimes_{\mathcal{A}} B(f^*\mathcal{N}) \arrow[u, "f \otimes_{\mathcal{A}} B(f^*\mathcal{N})", "\text{qis}"'] \arrow[r, no head, "\sim"'] & B(\mathcal{N}), \arrow[u, "\text{qis}"'] \end{tikzcd}$$

and hence $\mathcal{B} \otimes_{\mathcal{A}} B(f^*\mathcal{N}) \simeq \mathcal{N}$ in $\mathrm{D}_{\text{dg}}(\mathcal{B})$.

火.19. 森田-*theory*

Let $\Bbbk$ be a field and A a $\Bbbk$-algebra.

Definition: *A projective A-module P is called a progenerator iff $\forall M \in A\mathrm{Mod}$, $P^{\oplus \Lambda} \twoheadrightarrow M$ for some Λ.*

E.g. (i) A is a progenerator.

(ii) If $A = \Bbbk[G]$ for a finite group G with $\mathrm{ch}\, \Bbbk = 0$, $P = \coprod V$ with V running over a complete set Irr of representatives of the non-isomorphic irreducible $\Bbbk[G]$-modules is a progenerator, as

$$\begin{aligned} \Bbbk[G] &\simeq \coprod_{V \in \mathrm{Irr}} V^{\oplus \Bbbk[G]\mathrm{Mod}(\Bbbk[G], V)} \quad \text{by Maschke} \\ &= \coprod_{V \in \mathrm{Irr}} V^{\oplus \dim V}. \end{aligned}$$

Theorem: *Let $P \in A\mathrm{Mod}$ be a progenerator of finite type, $B = A\mathrm{Mod}(P, P)$, and $\mathrm{Mod}B$ the category of right B-modules. There is an equivalence*

$$A\mathrm{Mod} \simeq \mathrm{Mod}B \quad \textit{via} \quad M \mapsto A\mathrm{Mod}(P, M) \quad \textit{with quasi-inverse} \quad N \otimes_B P \hookleftarrow N,$$

where $A\mathrm{Mod}(P, M)$ is a right B-module via $(fb)(p) = f(b(p))$ while $N \otimes_B P$ is a left A-module via $a(n \otimes p) = n \otimes ap$; if $\phi \in B$, $a(n\phi \otimes p) = n\phi \otimes ap = n \otimes \phi \bullet (ap) = n \otimes \phi(ap) = n \otimes a\phi(p) = a(n \otimes \phi(p)) = a(n \otimes \phi \bullet p)$.

Proof: We check first that

$$A\mathrm{Mod}(P, M) \otimes_B P \simeq M \quad \text{via} \quad \phi \otimes p \mapsto \phi(p). \tag{1}$$

Assume first that $M = P^{\oplus\Lambda}$ for some Λ. Then

$$\begin{aligned} A\mathrm{Mod}(P,M) &= A\mathrm{Mod}(P,P^{\oplus\Lambda}) \\ &\simeq A\mathrm{Mod}(P,P)^{\oplus\Lambda} \quad \text{as } P \text{ is of finite type over } A \\ &= B^{\oplus\Lambda}, \end{aligned}$$

and hence

$$A\mathrm{Mod}(P,M)\otimes_B P \simeq B^{\oplus\Lambda}\otimes_B P \simeq P^{\oplus\Lambda} = M. \tag{2}$$

In general, take a resolution of M, an exact sequence of A-modules $P^{\oplus\Lambda_1} \to P^{\oplus\Lambda_0} \to M \to 0$, to obtain a CD

$$\begin{tikzcd} A\mathrm{Mod}(P,P^{\oplus\Lambda_1})\otimes_B P \arrow[r] \arrow[d, "\sim"', "(2)"] & A\mathrm{Mod}(P,P^{\oplus\Lambda_0})\otimes_B P \arrow[r] \arrow[d, "\sim"', "(2)"] & A\mathrm{Mod}(P,M)\otimes_B P \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ P^{\oplus\Lambda_1} \arrow[r] & P^{\oplus\Lambda_0} \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

As P is projective, the top row is exact, and hence $A\mathrm{Mod}(P,M)\otimes_B P \simeq M$ by the 5-lemma.

We show next that

$$N \simeq A\mathrm{Mod}(P, N\otimes_B P) \quad \text{via} \quad n \mapsto n\otimes \mathrm{id}_P(?). \tag{3}$$

Take a resolution $B^{\oplus\Lambda_1} \to B^{\oplus\Lambda_0} \to N \to 0$ of N. Then $B^{\oplus\Lambda_1}\otimes_B P \to B^{\oplus\Lambda_0}\otimes_B P \to N\otimes_B P \to 0$ remains exact. As P is projective, one has a CD of exact sequences

$$\begin{tikzcd} A\mathrm{Mod}(P,B^{\oplus\Lambda_1}\otimes_B P) \arrow[r] & A\mathrm{Mod}(P,B^{\oplus\Lambda_0}\otimes_B P) \arrow[r] & A\mathrm{Mod}(P,N\otimes_B P) \arrow[r] & 0 \\ B^{\oplus\Lambda_1} \arrow[r] \arrow[u] & B^{\oplus\Lambda_0} \arrow[r] \arrow[u] & N \arrow[r] \arrow[u] & 0. \arrow[u, equal] \end{tikzcd}$$

As $B \simeq A\mathrm{Mod}(P, B\otimes_B P)$ via $b \mapsto b\otimes \mathrm{id}_P(?)$, the left 2 vertical arrows are invertible, so therefore is the 3rd.

火.20. **Remarks:** (i) Any categorical equivalence $F : \mathrm{Mod}B \simeq A\mathrm{Mod}$ is realized as above with $P = F(B)$:

$$A\mathrm{Mod}(P,P) = A\mathrm{Mod}(F(B),F(B)) \simeq B\mathrm{Mod}(B,B) \simeq B.$$

(ii) One can set up the theorem in terms of right modules entirely as follows [中岡, Cor. 4.4.10, p. 281]: let P be a progenerator of finite type in $\mathrm{Mod}A$ and let $B = \mathrm{Mod}A(P,P)$. Then

$$\mathrm{Mod}A \simeq \mathrm{Mod}B \quad \text{via} \quad M \mapsto \mathrm{Mod}A(P,M) \quad \text{with quasi-inverse} \quad N\otimes_B P \leftarrow\!\shortmid N,$$

where the right B-module structure on $\mathrm{Mod}A(P,M)$ is given by $(fb)(p) = f(b(p))$.

火.21. **E.g.** Let G be a finite group, $\Bbbk$ an algebraically closed field of characteristic 0. Put $A = \Bbbk[G]$, $P = \coprod V$ with V running over a complete set Irr of representatives of the non-siomorphic irreducible $\Bbbk[G]$-modules, and $B = A\mathrm{Mod}(P,P)$. By Schur's lemma $B \simeq \prod_{V\in\mathrm{Irr}} \Bbbk$. In particular, B is commutative, and hence one obtains an equivalence

$$\Bbbk[G]\mathrm{Mod} \simeq (\prod_{V\in\mathrm{Irr}} \Bbbk)\mathrm{Mod}.$$

水曜日 森田-theory for dg-algebras

In 水.1-5 we describe 森田-theory of dg-algebras. We then define Koszul rings in 水.6.

水.1. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a morphsm of dg-algebras. Put $\phi^* = \mathcal{B}^{\mathbb{L}}\otimes_{\mathcal{A}}? : \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$, and let $\phi_* : \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) \to \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ denote the restriction of scalars. As $B(\mathcal{M})$, $\mathcal{M} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$, is $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$-flat [BL, 10.12.4.4], $\phi^*(\mathcal{M}) \simeq \mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M})$ [BL, 10.12.4.5]. We say ϕ is a qis iff $\mathrm{H}^i(\phi) : \mathrm{H}^i(\mathcal{A}) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{B})\ \forall i \in \mathbb{Z}$.

Theorem [BL, 10.12.5.1]: *If ϕ is a qis, $\phi^* : \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$ is an equivalence with quasi-inverse ϕ_*.*

Proof: Let $\mathcal{M} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ with bar resolution $\delta : B(\mathcal{M}) \xrightarrow{\text{qis}} \mathcal{M}$. Then $(\phi_* \circ \phi^*)(\mathcal{M}) = \mathcal{B}^{\mathbb{L}}\otimes_{\mathcal{A}}\mathcal{M} = \mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M})$. Define a natural transformation $\alpha : \mathrm{id}_{\mathrm{D}_{\mathrm{dg}}(\mathcal{A})} \to \phi_*\phi^*$ via

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\delta^{-1}"] \arrow[dr, dotted, "\alpha"'] & B(\mathcal{M}) \arrow[d, "f"] & x \arrow[d, mapsto] \\
& \mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M}) & 1 \otimes x.
\end{tikzcd}$$

As ϕ is a qis and as $B(\mathcal{M})$ is $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$-flat [BL, 10.12.4.3], $\phi \otimes_{\mathcal{A}} B(\mathcal{M})$ remains a qis; consider a d.t. $\mathcal{A} \xrightarrow{\phi} \mathcal{B} \to \mathrm{cone}(\phi) \to \mathcal{A}[1]$ with $\mathrm{cone}(\phi)$ acyclic as ϕ is a qis. Then $\mathcal{A} \otimes_{\mathcal{A}} B(\mathcal{M}) \xrightarrow{\phi\otimes_{\mathcal{A}}B(\mathcal{M})} \mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{M}) \to \mathrm{cone}(\phi) \otimes_{\mathcal{A}} B(\mathcal{M}) \to \mathcal{A} \otimes_{\mathcal{A}} B(\mathcal{M})[1]$ remains a d.t. with $\phi \otimes_{\mathcal{A}} B(\mathcal{M})$ qis as $\mathrm{cone}(\phi) \otimes_{\mathcal{A}} B(\mathcal{M})$ remains acyclic. Thus, f is a qis, so therefore is α.

Consider next a natural transformation $\beta : \phi^*\phi_* \to \mathrm{id}_{\mathrm{D}_{\mathrm{dg}}(\mathcal{B})}$ such that

$$\begin{tikzcd}
\phi^*\phi_*\mathcal{N} \arrow[rr, dotted, "\beta"] \arrow[d, equal] & & \mathcal{N} & b\delta'(y). \\
\mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{N}) \arrow[urr] & b \otimes y \arrow[urr, mapsto] & &
\end{tikzcd}$$

with $\delta' : B(\mathcal{N}) \to \mathcal{N}$ denoting the bar resolution of $\mathcal{N}$ regarded as dg $\mathcal{A}$-module $\phi_*\mathcal{N}$. In particular, $B(\mathcal{N}) = B(\phi_*\mathcal{N})$ is $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$-flat. One has a CD

$$\begin{tikzcd}
B(\mathcal{N}) \arrow[d, phantom, "\wr|"] \arrow[ddr, bend left, "\delta'"] & \\
\mathcal{A} \otimes_{\mathcal{A}} B(\mathcal{N}) \arrow[d, "\phi\otimes_{\mathcal{A}}B(\mathcal{N})"'] & \\
\mathcal{B} \otimes_{\mathcal{A}} B(\mathcal{N}) \arrow[r, "\beta"'] & \mathcal{N}.
\end{tikzcd}$$

As $\phi \otimes_{\mathcal{A}} B(\mathcal{N})$ and δ' are both qis's, so is β. Thus, ϕ^* and ϕ_* are quasi-inverse to each other.

水.2. Let $\mathcal{A}$ be a dg-algebra.

Definition: *A dg $\mathcal{A}$-module $\mathcal{M}$ is called a generator iff $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ coincides with the smallest thick full triangulated subcategory containing $\mathcal{M}$ and closed under infinite direct sums, i.e., if $\langle\!\langle \mathcal{M} \rangle\!\rangle$ denotes the smallest full subcategory of $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ such that* *(i)* $\mathcal{M} \in \langle\!\langle \mathcal{M} \rangle\!\rangle$*,*

(ii) $\langle\!\langle \mathcal{M} \rangle\!\rangle$ *is closed taking infinite direct sums, direct summands, and shifts,*

(iii) $\forall$ *d.t.* $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_1[1]$ *in* $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$*, if* $\mathcal{M}_1, \mathcal{M}_2 \in \langle\!\langle \mathcal{M} \rangle\!\rangle$*,* $\mathcal{M}_3 \in \langle\!\langle \mathcal{M} \rangle\!\rangle$*,*

then $\mathrm{D}_{\mathrm{dg}}(\mathcal{A}) = \langle\!\langle \mathcal{M} \rangle\!\rangle$*.*

Note that $\langle\!\langle \mathcal{M} \rangle\!\rangle$ is closed under isomorphism in $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ as it is closed taking direct summands.

Lemma: *$\mathcal{A}$ is a generator of* $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$*.*

Proof: Let $\mathcal{M} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$. As $\mathcal{M} \simeq B(\mathcal{M})$ in $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$, we have only to show $B(\mathcal{M}) \in \langle\!\langle \mathcal{A} \rangle\!\rangle$. As $B(\mathcal{M}) = \coprod_{i \geq 0} \mathcal{P}_{-i}[i]$, it suffices to show that each $\mathcal{P}_{-i}$ belongs to $\langle\!\langle \mathcal{A} \rangle\!\rangle$.

One has an exact sequence $0 \to \mathcal{A} \otimes_{\Bbbk} \ker \mathrm{d}_{\mathcal{M}} \to \mathcal{A} \otimes_{\Bbbk} \mathcal{M} \to \mathcal{A} \otimes_{\Bbbk} (\mathcal{M}/\ker \mathrm{d}_{\mathcal{M}}) \to 0$ in $\mathrm{C}_{\mathrm{dg}}(\mathcal{A})$, which induces a d.t. $\mathcal{A} \otimes_{\Bbbk} \ker \mathrm{d}_{\mathcal{M}} \to \mathcal{A} \otimes_{\Bbbk} \mathcal{M} \to \mathcal{A} \otimes_{\Bbbk} (\mathcal{M}/\ker \mathrm{d}_{\mathcal{M}}) \to (\mathcal{A} \otimes_{\Bbbk} \mathcal{M})[1]$ in $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$ in $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$; as the short exact sequence is semi-split, the d.t. actually is realized in $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$ already 火.2(2). As $\mathrm{d} = 0$ on $\ker \mathrm{d}_{\mathcal{M}}$, $\mathcal{A} \otimes_{\Bbbk} \ker \mathrm{d}_{\mathcal{M}} \simeq \mathcal{A}^{\oplus \ker \mathrm{d}_{\mathcal{M}}}$, and hence $\mathcal{A} \otimes_{\Bbbk} \ker \mathrm{d}_{\mathcal{M}} \in \langle\!\langle \mathcal{A} \rangle\!\rangle$. Likewise $\mathcal{A} \otimes_{\Bbbk} (\mathcal{M}/\ker \mathrm{d}_{\mathcal{M}}) \in \langle\!\langle \mathcal{A} \rangle\!\rangle$. Then $\mathcal{P}_0 = \mathcal{A} \otimes_{\Bbbk} \mathcal{M} \in \langle\!\langle \mathcal{A} \rangle\!\rangle$. Likewise, $\mathcal{P}_{-1} = \mathcal{A} \otimes_{\Bbbk} \ker(\delta_0) \in \langle\!\langle \mathcal{A} \rangle\!\rangle$, and all $\mathcal{P}_{-i} \in \langle\!\langle \mathcal{A} \rangle\!\rangle$.

水.3. Let $\mathcal{X} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$. We equip the complex $\mathcal{C} = \mathcal{A}\mathrm{Mod}^{\bullet}(B(\mathcal{X}), B(\mathcal{X}))$ with a structure of dg algebra with $\mathcal{C}^i = \mathcal{A}\mathrm{Mod}^i(B(\mathcal{X}), B(\mathcal{X}))$ and differential $\mathrm{d}_{\mathcal{C}}$ given by the differential d on $\mathcal{A}\mathrm{Mod}^{\bullet}(B(\mathcal{X}), B(\mathcal{X}))$; we have to check that $\mathrm{d}_{\mathcal{C}}(fg) = (\mathrm{d}_{\mathcal{C}} f)g + (-1)^i f \mathrm{d}_{\mathcal{C}} g \ \forall f \in \mathcal{C}^i \ \forall g \in \mathcal{C}^j$. Indeed,

$$\begin{aligned}
\mathrm{RHS} &= (\mathrm{d}_{\mathcal{C}} f) \circ g + (-1)^i f \circ \mathrm{d}_{\mathcal{C}} g = (d \circ f - (-1)^i f \circ d) \circ g + (-1)^i f \circ (d \circ g - (-1)^j g \circ d) \\
&= d \circ (f \circ g) - (-1)^{i+j} f \circ g \circ d = \mathrm{LHS}.
\end{aligned}$$

Let now $\mathcal{B} = \mathcal{A}\mathrm{Mod}^{\bullet}(B(\mathcal{X}), B(\mathcal{X}))^{\mathrm{op}}$ be the dg-algebra opposite to $\mathcal{C}$. The functor $\mathcal{A}\mathrm{Mod}^{\bullet}(\mathcal{X}, ?) : \mathrm{K}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{K}_{\mathrm{dg}}(\mathcal{B})$ induces $\mathrm{R}\mathcal{A}\mathrm{Mod}^{\bullet}(\mathcal{X}, ?) : \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$, which reads $\mathcal{A}\mathrm{Mod}^{\bullet}(B(\mathcal{X}), ?)$ as $B(\mathcal{X})$ is $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$-projective [BL, 10.12.3.1].

Definition: *We say $\mathcal{X} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ is small iff $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})(\mathcal{X}, ?)$ commutes with arbitrary direct sums, i.e., $\mathcal{X}$ is of "finite type".*

Theorem: *If $\mathcal{X}$ is a small generator of* $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$*,*$B(\mathcal{X})^{\mathbb{L}} \otimes_{\mathcal{B}} ? : \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) \to \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ *is an equivalence with quasi-inverse* $\mathrm{R}\mathcal{A}\mathrm{Mod}^{\bullet}(\mathcal{X}, ?)$*.*

Proof: As $\mathcal{B}$ is $\mathrm{K}_{\mathrm{dg}}(\mathcal{B})$-flat [BL, 10.12.4.1], $B(\mathcal{X})^{\mathbb{L}} \otimes_{\mathcal{B}} \mathcal{B} \simeq B(\mathcal{X}) \otimes_{\mathcal{B}} \mathcal{B} \simeq B(\mathcal{X}) \simeq \mathcal{X}$ in $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$. Then, as $\mathrm{D}_{\mathrm{dg}}(\mathcal{A}) = \langle\!\langle \mathcal{X} \rangle\!\rangle$ by the hypothesis, $B(\mathcal{X})^{\mathbb{L}} \otimes_{\mathcal{B}} ?$ is essentially surjective/dense [中岡, Def. 2.2.19, p. 71], i.e., $\forall \mathcal{M} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \ \exists \mathcal{N} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) : \mathcal{M} \simeq B(\mathcal{X})^{\mathbb{L}} \otimes_{\mathcal{B}} \mathcal{N}$. Thus, we are left to show that $B(\mathcal{X})^{\mathbb{L}} \otimes_{\mathcal{B}}$ is fully faithfully.

Put $\mathfrak{Y} = \{\mathcal{N} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) | \mathrm{D}_{\mathrm{dg}}(\mathcal{B})(\mathcal{B}, \mathcal{N}[i]) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{B}, B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{N}[i]) \ \forall i \in \mathbb{Z}\}$. One has

$$\begin{aligned}
\mathrm{D}_{\mathrm{dg}}(\mathcal{B})(\mathcal{B}, \mathcal{B}[i]) &\simeq \mathrm{K}_{\mathrm{dg}}(\mathcal{B})(\mathcal{B}, \mathcal{B}[i]) \quad \text{as } \mathcal{B} \text{ is } \mathrm{K}_{\mathrm{dg}}(\mathcal{B})\text{-projective [BL, 10.12.2.2] by 火.14.1} \\
&\simeq \mathrm{H}^i(\mathcal{B}\mathrm{Mod}^\bullet(\mathcal{B}, \mathcal{B})) \quad \text{by Ex. 火.14.(ii)} \\
&\simeq \mathrm{H}^i(\mathcal{B}) \quad \text{by 火.14.1 again} \\
&= \mathrm{H}^i(\mathcal{A}\mathrm{Mod}^\bullet(B(\mathcal{X}), B(\mathcal{X}))^{\mathrm{op}}) \\
&\simeq \mathrm{K}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X}), B(\mathcal{X})[i]) \quad \text{by Ex. 火.14.(ii) again} \\
&\simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X}), B(\mathcal{X})[i]) \quad \text{by 火.17.(ii)} \\
&\simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{B}, B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{B}[i]) \quad \text{as } \mathcal{B} \text{ is } \mathrm{K}_{\mathrm{dg}}(\mathcal{B})\text{-flat},
\end{aligned}$$

and hence $\mathcal{B} \in \mathfrak{Y}$. One has also $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{B}, ?) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X}), ?) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(\mathcal{X}, ?)$. As $\mathcal{X}$ is small, $\mathfrak{Y}$ is a thick triangulated subcategory of $\mathrm{D}_{\mathrm{dg}}(\mathcal{B})$ closed under taking arbitrary direct sums. Then $\mathfrak{Y} \supseteq \langle\!\langle \mathcal{B} \rangle\!\rangle = \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$ by 水.2. Thus, $\forall \mathcal{N} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$,

$$\mathrm{D}_{\mathrm{dg}}(\mathcal{B})(\mathcal{B}, \mathcal{N}[i]) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{B}, B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{N}[i]).$$

Then, as $\langle\!\langle \mathcal{B} \rangle\!\rangle = \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$ again, $\forall \mathcal{M} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{B})$,

$$\mathrm{D}_{\mathrm{dg}}(\mathcal{B})(\mathcal{M}, \mathcal{N}[i]) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{M}, B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}\mathcal{N}[i]),$$

as desired.

水.4. **Remarks:** (i) If $\mathcal{X}$ is just small, the functor $B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}?$ is already fully faithful.

(ii) Let A be a $\Bbbk$-algebra and let $\mathcal{A} = \coprod_{i \in \mathbb{Z}} \mathcal{A}^i$ be a dg-algebra such that $\mathcal{A}^i = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{else} \end{cases}$ with $\mathrm{d}_{\mathcal{A}} = 0$. Recall from Eg. 火.13 that

$$\mathrm{C}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathrm{C}(A). \tag{1}$$

Let now $X^\bullet \in \mathrm{D}(A)$ be a small generator with $\mathrm{D}(A)(X^\bullet, X^\bullet[i]) = 0 \ \forall i \neq 0$. Put $B = \mathrm{D}(A)(X^\bullet, X^\bullet)^{\mathrm{op}}$, and define a dg-algebra $\mathcal{B}$ from B as for $\mathcal{A}$ from A. Let $\mathcal{B}'$ be the dg-algebra $\mathcal{A}\mathrm{Mod}^\bullet(B(X^\bullet), B(X^\bullet))^{\mathrm{op}}$ using (1). Then $\mathrm{D}_{\mathrm{dg}}(\mathcal{B}') \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ by 水.3 via $B(X^\bullet)^{\mathbb{L}}\otimes_{\mathcal{B}'}?$, and hence

$$\mathrm{D}(A) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{B}'). \tag{2}$$

On the other hand, $\forall i \in \mathbb{Z}$,

$$\begin{aligned}
\mathrm{H}^i(\mathcal{B}') &\simeq \mathrm{K}(A)(B(X^\bullet), B(X^\bullet)[i]) \quad \text{by 火.5} \\
&\simeq \mathrm{D}(A)(X^\bullet, B(X^\bullet)[i]) \quad \text{by 火.9(2) as } B(X^\bullet) \to X^\bullet \text{ is a projective resolution} \\
&\simeq \mathrm{D}(A)(X^\bullet, X^\bullet[i]) = \begin{cases} \mathrm{D}(A)(X^\bullet, X^\bullet) & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

As $\mathcal{A}\mathrm{Mod}^0(B(X^\bullet), B(X^\bullet)) = \mathrm{C}_{\mathrm{dg}}(\mathcal{A})(B(X^\bullet), B(X^\bullet)) \simeq \mathrm{C}(A)(B(X^\bullet), B(X^\bullet))$ and as $\mathrm{D}(A)(B(X^\bullet), B(X^\bullet)) \simeq \mathrm{D}(A)(X^\bullet, X^\bullet)$, the quotient $(\mathcal{B}')^{\mathrm{op}} \to \mathcal{B}$ induces a qis $\mathcal{B}' \to \mathcal{B}$ of dg-algebras. Then

$$\begin{aligned} \mathrm{D}_{\mathrm{dg}}(\mathcal{B}') &\simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) \quad \text{by 水.3} \\ &\simeq \mathrm{D}(B) \quad \text{by (1)}, \end{aligned}$$

and hence together with (2) one obtains Rickard's theorem

$$\mathrm{D}(A) \simeq \mathrm{D}(\mathrm{D}(A)(X^\bullet, X^\bullet)^{\mathrm{op}}).$$

水.5. More generally, let $\mathcal{X} \in \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ and put $\mathcal{B} = \mathcal{A}\mathrm{Mod}^\bullet(B(\mathcal{X}), B(\mathcal{X}))^{\mathrm{op}}$. Let $\langle \mathcal{X} \rangle$ denote the smallest triangulated subcategory of $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ containing $\mathcal{X}$ closed under taking direct summands; it may not be closed under infinite diret sums.

Theorem: *One has a CD*

$$\begin{tikzcd} \mathrm{D}_{\mathrm{dg}}(\mathcal{B}) \arrow[r, "B(\mathcal{X})^{\mathbb{L}}\otimes_{\mathcal{B}}?"] & \mathrm{D}_{\mathrm{dg}}(\mathcal{A}) \\ \langle \mathcal{B} \rangle \arrow[u, hook] \arrow[r, dotted, "\sim"'] & \langle \mathcal{X} \rangle \arrow[u, hook] \end{tikzcd}$$

with quasi-inverse $\mathrm{R}\mathcal{A}\mathrm{Mod}^\bullet(\mathcal{X}, ?) : \langle \mathcal{X} \rangle \xrightarrow{\sim} \langle \mathcal{B} \rangle$.

水.6. *Koszul rings*

Let $A = \coprod_{i\in\mathbb{N}} A_i$ be a positively graded $\Bbbk$-algebra with A_0 semisimple as a $\Bbbk$-algebra and $\dim A_i < \infty\ \forall i$.

Ex. (i) $\coprod_{i>0} A_i \lhd A$.

(ii) If $M = \coprod_{i\in\mathbb{Z}} M_i$ is a graded A-module, pure of weight n: $M_i = 0\ \forall i \neq n$, M is semisimple.

Let $M = \coprod_{i\in\mathbb{Z}} M_i$ be a graded A-module, and $n \in \mathbb{Z}$. We let $M\langle n\rangle$ denote another graded A-module such that $M\langle n\rangle_i = M_{i-n}\ \forall i \in \mathbb{Z}$; we alter the notation from 火.12. Earlier, for a graded A-module $N = \coprod_{i\in\mathbb{Z}} N^i$ we let $N[n]$ denote another graded A-module such that$N[n]^i = N^{i+n}$ $\forall i \in \mathbb{Z}$.

Definition: *We say A is Koszul iff A_0, regarded as $A/\coprod_{i>0} A_i$, admits a resolution by graded projective A-modules*

$$\cdots \to P^{-i} \to P^{-i+1} \to \cdots \to P^0 \to A_0 \to 0$$

such that each P^{-i}, $i \in \mathbb{N}$, is generated by its i-th degree piece: $P^{-i} = A(P^{-i})_i$.

E.g. Let $A = \Bbbk[x, y]$ be a polynomial $\Bbbk$-algebra in indeterminates x and y, graded in such a way that $A_i = \coprod_{s+t=i} \Bbbk x^s y^t$. Then $A_0 = \Bbbk$ admits a resolution by graded projective A-modules

$$\begin{tikzcd} & & & A \arrow[d, equal] & A/\coprod_{i>0} A_i \arrow[d, "\wr"] & \\ 0 \arrow[r] & \Bbbk[x,y]\langle 2\rangle \arrow[r] & \Bbbk[x,y]^{\oplus_2}\langle 1\rangle \arrow[r] & \Bbbk[x,y] \arrow[r] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$
$$h \longmapsto (-yh, xh)$$
$$(f,g) \longmapsto xf + yg.$$

Thus, $\Bbbk[x,y]$ is Koszul.

水.7. Any polynomial $\Bbbk$-algebra turns out Koszul, which we presently demonstrate.

For a finite dimensional $\Bbbk$-linear space V let $A = \mathrm{S}(V)$ denote the symmetric algebra of V over $\Bbbk$, graded such that $\deg V = 1$. We will write its degree i-piece as $A_i = \mathrm{S}^i(V)$, $i \in \mathbb{N}$. If $x_1, \dots, x_n$ is a $\Bbbk$-linear basis of V, $\mathrm{S}(V) \simeq \Bbbk[x_1, \dots, x_n]$ the polynomial algebra in $x_1, \dots, x_n$.

Let also $\mathrm{T}(V) = \coprod_{i \in \mathbb{N}} \mathrm{T}^i(V)$ with $\mathrm{T}^i(V) = V^{\otimes_i}$ $\forall i$ denote the tensor algebra of V over $\Bbbk$: the multiplication on $\mathrm{T}(V)$ is given by

$\mathrm{T}^i(V) \times \mathrm{T}^j(V) \to \mathrm{T}^{i+j}(V)$ via $(v_1 \otimes \cdots \otimes v_i, w_1 \otimes \cdots \otimes w_j) \mapsto v_1 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_j$.

Thus, $\mathrm{S}(V) \simeq \mathrm{T}(V)/(v_1 \otimes v_2 - v_2 \otimes v_1 | v_1, v_2 \in V)$.

Set $\Lambda(V) = \mathrm{T}(V)/(v \otimes v | v \in V)$, the exterior algebra of V over $\Bbbk$. We will denote its degree i-piece by $\Lambda^i(V)$. We will write the image of $v \otimes w \in \mathrm{T}^2(V)$ in $\mathrm{S}^2(V)$ (resp. $\Lambda^2(V)$) as vw (resp. $v \wedge w$). Thus, $\Lambda^i(V) = \sum_{v_1, \dots, v_i \in V} \Bbbk(v_1 \wedge \cdots \wedge v_i)$. In terms of basis $x_1, \dots, x_n$,

$$\Lambda^i(V) = \coprod_{j_1 < \cdots < j_i} \Bbbk(x_{j_1} \wedge \cdots \wedge x_{j_i}).$$

Definition: *$\forall i > 0$, define* $\mathrm{d} : \mathrm{S}(V) \otimes_\Bbbk \Lambda^i(V) \to \mathrm{S}(V) \otimes_\Bbbk \Lambda^{i-1}(V)$ *via* $x \otimes (v_1 \wedge \cdots \wedge v_i) \mapsto \sum_{j=1}^{i} (-1)^{j+1} x v_j \otimes (v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_i)$. *We call a sequence, with $n = \dim V$,*

$$0 \longrightarrow \mathrm{S}(V) \otimes_\Bbbk \Lambda^n V \xrightarrow{d} \mathrm{S}(V) \otimes_\Bbbk \Lambda^{n-1} V \xrightarrow{\quad d \quad} \dots$$

$$\begin{tikzcd} \dots \arrow[r, "d"] & \mathrm{S}(V) \otimes_\Bbbk \Lambda^1 V \arrow[r, "d"] \arrow[d, "\sim" {anchor=east}, dash] & \mathrm{S}(V) \otimes_\Bbbk \Lambda^0 V \arrow[r, "d"] \arrow[d, "\sim" {anchor=east}, dash] & \Bbbk \arrow[r] \arrow[d, "\sim" {anchor=east}, dash] & 0 \\ & \mathrm{S}(V) \otimes_\Bbbk V & \mathrm{S}(V) & \mathrm{S}(V)/\coprod_{i>0} \mathrm{S}^i(V) & \end{tikzcd}$$

Koszul complex of V.

Ex. d is well-defined and $d^2 = 0$.

E.g. Assume that V is 2-dimensional with a basis x and y. Thus, $\mathrm{S}(V) \simeq \Bbbk[x,y]$, $\wedge^2 V =$

$\Bbbk(x \wedge y)$, $\wedge^1 V = V$, $\wedge^0 V = \Bbbk$, and the Koszul complex of V reads as a CD

$$f \otimes (x \wedge y) \longmapsto fx \otimes y - fy \otimes x$$

$$f \otimes v \longmapsto fv$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{S}(V) \otimes_{\Bbbk} \Bbbk(x \wedge y) \arrow[r] \arrow[d, "\sim"'] & \mathrm{S}(V) \otimes_{\Bbbk} (\Bbbk x \oplus \Bbbk y) \arrow[r] \arrow[d, "\sim"'] & \mathrm{S}(V) \otimes_{\Bbbk} \Bbbk \arrow[r] \arrow[d, "\sim"'] & \Bbbk \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Bbbk[x,y]\langle 2 \rangle \arrow[r] & \Bbbk[x,y]^{\oplus_2}\langle 1 \rangle \arrow[r] & \Bbbk[x,y] \arrow[r] & \Bbbk \arrow[r] & 0,
\end{tikzcd}$$

the bottom row of which coincides with the one in E.g. 水.6, and hence exact.

We will show

Theorem: *The Koszul complex of V is exact, and hence $\mathrm{S}(V)$ is a Koszul ring.*

木曜日

We first establish that $\mathrm{S}(V)$ forms a Koszul ring. We then give a criterion for a $\Bbbk$-algebra to be Koszul in 木.4., and move on to Koszul duality.

木.1. We are to show that $\mathrm{S}(V)$ is Koszul, i.e., that the Koszul complex of V is exact. Let $V_0 \leq V$, and let $K(V, V_0)$ denote the sequence

$$0 \longrightarrow \mathrm{S}(V) \otimes_{\Bbbk} \wedge^{\dim V} V_0 \xrightarrow{d^{-\dim V_0}} \mathrm{S}(V) \otimes_{\Bbbk} \wedge^{\dim V - 1} V_0 \xrightarrow{d^{-\dim V_0 + 1}} \dots$$

$$f \otimes (v_1 \wedge \dots \wedge v_{\dim V_0}) \mapsto \sum_{j=1}^{\dim V_0} (-1)^{j+1} f v_j \otimes (v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_{\dim V_0})$$

$$\dots \xrightarrow{d^{-2}} \mathrm{S}(V) \otimes_{\Bbbk} V_0 \xrightarrow{d^{-1}} \mathrm{S}(V) \longrightarrow 0,$$

where $\hat{v}_j$ is meant to delete the j-th term v_j. it suffices to show

claim: $\forall i \in \mathbb{N}$,

$$\mathrm{H}^{-i}(K(V, V_0)) \simeq \begin{cases} \mathrm{S}(V/V_0) & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

木.2. We argue by induction on $\dim V_0$. If $V_0 = 0$, $K(V, V_0)$ reads $0 \to \mathrm{S}(V) \to 0$, and hence the assertion holds.

Assume $V_0 > 0$ and write $V_0 = V_1 \oplus \Bbbk v_0$. Consider the d.t.

$$K(V, V_1) \xrightarrow{v_0} K(V, V_1) \to \mathrm{cone}(v_0) \to K(V, V_1)[1] \tag{1}$$

with v_0 denoting the multiplication by v_0 on $\mathrm{S}(V)$.

Lemma: $\mathrm{cone}(v_0) \simeq K(V, V_0)$.

Proof: Recall that $\wedge^i V_0 = \wedge^i(V_1 \oplus \Bbbk v_0) \simeq (\wedge^i V_1) \oplus \{\wedge^{i-1} V_1) \otimes_\Bbbk \Bbbk v_0\}$ via

$$v_1 \wedge \cdots \wedge v_i + v_0 \wedge w_1 \wedge \cdots \wedge w_{i-1} \hookleftarrow (v_1 \wedge \cdots \wedge v_i, w_1 \wedge \cdots \wedge w_{i-1} \otimes v_0).$$

Define $\phi^{-i} : \mathrm{cone}(v_0)^{-i} = K(V, V_1)^{-i+1} \oplus K(V, V_1)^{-i} = \{\mathrm{S}(V) \otimes_\Bbbk \wedge^{i-1} V_1\} \oplus \{\mathrm{S}(V) \otimes_\Bbbk \wedge^i V_1\} \xrightarrow{\sim} \mathrm{S}(V) \otimes_\Bbbk \wedge^i V_0$ via

$$(f \otimes (v_1 \wedge \cdots \wedge v_{i-1}), g \otimes (w_1 \wedge \cdots \wedge w_i)) \mapsto f \otimes (v_0 \wedge v_1 \wedge \cdots \wedge v_{i-1}) + g \otimes (w_1 \wedge \cdots \wedge w_i).$$

Then

$$\begin{pmatrix} f \otimes (v_1 \wedge \cdots \wedge v_{i-1}) \\ g \otimes (w_1 \wedge \cdots \wedge w_i) \end{pmatrix} \xmapsto{d^{-i}_{\mathrm{cone}(v_0)} = \begin{pmatrix} -d & 0 \\ v_0 & d \end{pmatrix}}$$
$$\begin{pmatrix} -\sum_{j=1}^{i-1} (-1)^{j+1} f v_j \otimes (v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-1}) \\ f v_0 \otimes (v_1 \wedge \cdots \wedge v_{i-1}) + \sum_{k=1}^{i} (-1)^{k+1} g w_k \otimes (w_1 \wedge \cdots \wedge \hat{w}_k \wedge \cdots \wedge w_i) \end{pmatrix}$$
$$\xmapsto{\phi^{-i+1}} -\sum_{j=1}^{i-1} (-1)^{j+1} f v_j \otimes (v_0 \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-1}) + f v_0 \otimes (v_0 \wedge v_1 \wedge \cdots \wedge v_{i-1})$$
$$+ \sum_{k=1}^{i} (-1)^{k+1} g w_k \otimes (v_0 \wedge w_1 \wedge \cdots \wedge \hat{w}_k \wedge \cdots \wedge w_i)$$

which coincides with

$$d^{-i}_{K(V,V_0)}(f \otimes (v_0 \wedge v_1 \wedge \cdots \wedge v_{i-1}) + g \otimes (w_1 \wedge \cdots \wedge w_i))$$
$$= (d^{-i}_{K(V,V_0)} \circ \phi^{-i})(\begin{pmatrix} f \otimes (v_1 \wedge \cdots \wedge v_{i-1}) \\ g \otimes (w_1 \wedge \cdots \wedge w_i) \end{pmatrix}).$$

Thus $\phi : \mathrm{cone}(v_0) \to K(V, V_0)$ gives an isomorphism in $\mathrm{C}(\mathrm{S}(V))$.

木.3. To finish the proof of claim 木.1, the d.t. 木.2.(1) now induces, as it is $\mathrm{S}(V)$-linear, a LES

(1) $$\cdots \to \mathrm{H}^{-i}(K(V, V_1)) \xrightarrow{v_0} \mathrm{H}^{-i}(K(V, V_1)) \to \mathrm{H}^{-i}(K(V, V_0)) \xrightarrow{v_0} \mathrm{H}^{-i+1}(K(V, V_1)) \to \ldots$$

By the induction hypothesis

$$\mathrm{H}^{-i}(K(V, V_1)) \simeq \begin{cases} \mathrm{S}(V/V_1) & \text{if } i = 0, \\ 0 & \text{else,} \end{cases}$$

and hence (1) yields that $\mathrm{H}^{-i}(K(V, V_0)) = 0 \ \forall i \geq 2$, and an exact sequence

$$0 \to \mathrm{H}^{-1}(K(V, V_0)) \to \mathrm{S}(V/V_1) \xrightarrow{v_0} \mathrm{S}(V/V_1) \to \mathrm{H}^0(K(V, V_0)) \to 0.$$

Then $\mathrm{H}^{-1}(K(V, V_0)) = 0$, $\mathrm{H}^0(K(V, V_0)) \simeq \mathrm{S}(V/V_1)/v_0\mathrm{S}(V/V_1) \simeq \mathrm{S}(V/V_0)$, and claim 木.1 holds, as desired.

木.4. Let $A = \coprod_{i\in\mathbb{N}} A_i$ be a positively graded $\Bbbk$-algebra with A_0 semisimple. $\forall M, N \in A\mathrm{Modgr}$, recall that $A\mathrm{Modgr}(M,N) = \{f \in A\mathrm{Mod}(M,N) | f(M_i) \subseteq N_i \ \forall i \in \mathbb{N}\}$.

For $j \in \mathbb{Z}$ put $M_{\geq j} = \coprod_{i\geq j} M_j$ and $M_{>j} = \coprod_{i>j} M_j$. We regard A_0 as a graded A-module $A/A_{>0}$.

Proposition: *The following are equivalent:*

(i) A is Koszul.

(ii) $\forall i, j \in \mathbb{Z}$ with $i \neq j$, $\mathrm{Ext}^i_{A\mathrm{Modgr}}(A_0, A_0\langle j\rangle) = 0$.

Proof: (i) $\Rightarrow$ (ii) Let $\cdots \to P^{-1} \to P^0 \to A_0 \to 0$ be a Koszul resolution of A_0. Thus, each P^{-i}, $i \in \mathbb{N}$, is graded projective over A with $P^{-i} = A(P^{-i})_i$. Then, $\forall i \in \mathbb{N}$, $\forall j \in \mathbb{Z}$, $\mathrm{Ext}^i_{A\mathrm{Modgr}}(A_0, A_0\langle j\rangle) = \mathrm{H}^i(A\mathrm{Modgr}(P^\bullet, A_0\langle j\rangle)$.

For $j \neq i$ let $f \in A\mathrm{Modgr}(P^{-i}, A_0\langle j\rangle)$. As $(A_0\langle j\rangle)_i = (A_0)_{i-j} = 0$, $f|_{(P^{-i})_i} = 0$. Then $f = 0$ as $P^{-i} = A(P^{-i})_i$.

(ii) $\Rightarrow$ (i) We will construct a Koszul resolution $P^\bullet$ of A_0 by induction on $i \in \mathbb{N}$ in such a way that $\{\ker(d^{-i} : P^{-i} \to P^{-i+1})\}_i = 0$.

First define

$$\begin{tikzcd}
P^0 = A \arrow[r, dotted, "d^0"] \arrow[dr, "\mathrm{qt}"'] & A_0 \arrow[r] \arrow[d, "\wr"] & 0. \\
& A/A_{>0} &
\end{tikzcd}$$

Then $\ker(d^0) = A_{>0}$, and hence $\{\ker(d^0)\}_0 = 0$.

Assume done up to i: one has an exact sequence

$$P^{-i} \xrightarrow{d^{-i}} P^{-i+1} \xrightarrow{d^{-i+1}} \cdots \to P^0 \to A_0 \to 0$$

with all $P^{-j} = A(P^{-j})_j$, $j \in [0,i]$, graded projective over A and $\{\ker(d^{-i})\}_i = 0$. We will construct a graded projective $P^{-(i+1)}$ and $d^{-(i+1)} : P^{-(i+1)} \to P^{-i}$ such that $\{\ker(d^{-(i+1)})\}_{i+1} = 0$. Put $K = \ker(d^{-i})$. We claim

(1) $$K = AK_{i+1}.$$

Just suppose not. Put $K' = AK_{i+1}$, and let $s = \min\{j > i+1 | K_j > K'_j\}$. As $K = K_{>j}$ by the induction hypothesis, the A-module structure on K/K' factors through $A/A_{>0} \simeq A_0$, and hence K/K' is a semisimple A_0-module. Then, K_s/K'_s is an A_0-direct summand of K/K'. Let L be a simple A_0-module such that $L\langle s\rangle$ is an A_0-direct summand of K_s/K'_s. One then obtains in $A\mathrm{Modgr}$

$$\begin{tikzcd}
K \arrow[r, "\mathrm{qt}"] \arrow[rrrr, dotted, bend right, "\neq 0"'] & K/K' \arrow[r, "\mathrm{proj}"] & K_s/K'_s \arrow[r, "\mathrm{proj}"] & L\langle s\rangle \arrow[r, hook] & A_0\langle s\rangle.
\end{tikzcd}$$

On the other hand,

$$\begin{aligned} A\mathrm{Modgr}(K, A_0\langle s\rangle) &\simeq \mathrm{Ext}^{i+1}_{A\mathrm{Modgr}}(A_0, A_0\langle s\rangle) \quad \text{par décalage} \\ &= 0 \quad \text{by hypothesis as } s > i+1 \text{, absurd.} \end{aligned}$$

Namely, one has exact sequences

$$0 \to K \to P^{-i} \to \mathrm{im}(d^{-i}) \to 0, \tag{2}$$

$$0 \to \mathrm{im}(d^{-i}) \to P^{-i+1} \to \mathrm{im}(d^{-i+1}) \to 0, \tag{3}$$

$$\dots,$$

$$0 \to \mathrm{im}(d^{-1}) \to P^{0} \to A_0 \to 0.$$

From (2) one obtains a LES

$$\begin{aligned} A\mathrm{Modgr}(P^{-i}, A_0\langle s\rangle) \to A\mathrm{Modgr}(K, A_0\langle s\rangle) \to \mathrm{Ext}^{1}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i}), A_0\langle s\rangle) \\ \to \mathrm{Ext}^{1}_{A\mathrm{Modgr}}(P^{-i}, A_0\langle s\rangle) \end{aligned}$$

with

$$\begin{aligned} A\mathrm{Modgr}(P^{-i}, A_0\langle s\rangle) &= A\mathrm{Modgr}(A(P^{-i})_i, A_0\langle s\rangle) \\ &= 0 \quad \text{as } (A_0\langle s\rangle)_i = (A_0)_{i-s} = 0 \\ &= \mathrm{Ext}^{1}_{A\mathrm{Modgr}}(P^{-i}, A_0\langle s\rangle) \quad \text{as } P^{-i} \text{ is projective,} \end{aligned}$$

and hence $A\mathrm{Modgr}(K, A_0\langle s\rangle) \simeq \mathrm{Ext}^{1}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i}), A_0\langle s\rangle)$. In turn, from (3) one obtains a LES

$$\begin{aligned} \mathrm{Ext}^{1}_{A\mathrm{Modgr}}(P^{-i+1}, A_0\langle s\rangle) \to \mathrm{Ext}^{1}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i}), A_0\langle s\rangle) \to \mathrm{Ext}^{2}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i+1}), A_0\langle s\rangle) \\ \to \mathrm{Ext}^{2}_{A\mathrm{Modgr}}(P^{-i+1}, A_0\langle s\rangle), \end{aligned}$$

and hence $\mathrm{Ext}^{1}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i}), A_0\langle s\rangle) \simeq \mathrm{Ext}^{2}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i+1}), A_0\langle s\rangle)$. Repeat to get

$$\mathrm{Ext}^{2}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-i+1}), A_0\langle s\rangle) \simeq \dots \simeq \mathrm{Ext}^{i}_{A\mathrm{Modgr}}(\mathrm{im}(d^{-1}), A_0\langle s\rangle) \simeq \mathrm{Ext}^{i+1}_{A\mathrm{Modgr}}(A_0, A_0\langle s\rangle).$$

Define now

$$\begin{tikzcd} P^{-(i+1)} \arrow[r, dotted, "d^{-(i+1)}"] \arrow[d, equal] & P^{-i} \\ A \otimes_{A_0} K_{i+1} \arrow[r, two heads] & AK_{i+1} = K. \arrow[u, hook] \end{tikzcd}$$

It remains to check that $\{\ker(d^{-(i+1)})\}_{i+1} = 0$, which follows from a CD

$$\begin{tikzcd} P^{-(i+1)} \arrow[r, "d^{-(i+1)}"] \arrow[rd] & P^{-i} \\ (P^{-(i+1)})_{i+1} \arrow[u, hook] & K \arrow[u, hook] \\ (A \otimes_{A_0} K_{i+1})_{i+1} \arrow[u, equal] & \\ (A_0 \otimes_{A_0} K_{i+1})_{i+1} \arrow[u, equal] \arrow[r, dotted, "\sim"'] & K_{i+1}. \arrow[uu, hook] \end{tikzcd}$$

木.5. Let A be a Koszul ring, $\mathcal{A} = \coprod_{i\in\mathbb{Z}} \mathcal{A}^i$ a dg-algebra with $\mathrm{d}_{\mathcal{A}} = 0$ and $\mathcal{A}^i = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$
Let $\mathcal{E} = \mathcal{A}\mathrm{Mod}^\bullet(B(A_0), B(A_0)) = \coprod_{i\in\mathbb{Z}} \mathcal{E}^i$ with

$$\begin{aligned} \mathcal{E}^i &= \mathcal{A}\mathrm{Mod}^i(B(A_0), B(A_0)) \\ &= \mathcal{A}\mathrm{Modgr}(B(A_0), B(A_0)[i]) = \{f \in \mathcal{A}\mathrm{Mod}(B(A_0), B(A_0)| f(B(A_0)^j) \subseteq B(A_0)^{j+i}\ \forall j \in \mathbb{Z}\} \\ &= A\mathrm{Mod}^i(B(A_0), B(A_0)) \quad \text{by E.g. 火.16} \\ &= \prod_{j\in\mathbb{Z}} A\mathrm{Mod}(B(A_0)^j, B(A_0)^{j+i}) \end{aligned}$$

and $\mathrm{d}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ such that $\forall f \in \mathcal{E}^i$, $\mathrm{d}_{\mathcal{E}} f = d_{B(A_0)} \circ f - (-1)^i f \circ d_{B(A_0)}$. Under the composition product $(\mathcal{E}, \mathrm{d}_{\mathcal{E}})$ forms a dg-algebra as in 水.3 [Iv, I.8.3, p. 60]:

$$\begin{array}{ccc} \mathcal{E}^i \times \mathcal{E}^j & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\rightarrow & \mathcal{E}^{i+j} \\ \| & & \| \\ \mathcal{A}\mathrm{Modgr}(B(A_0), B(A_0)[i]) \times \mathcal{A}\mathrm{Modgr}(B(A_0), B(A_0)[j]) & \longrightarrow & \mathcal{A}\mathrm{Modgr}(B(A_0), B(A_0)[i+j]) \\ (f, g) & \longmapsto & f \smile g = f[j] \circ g, \end{array}$$

$(\mathcal{E}, \mathrm{d}_{\mathcal{E}})$ forms a dg-algebra: $\forall f \in \mathcal{E}^i$, $g \in \mathcal{E}^j$,

$$\begin{aligned} \mathrm{d}_{\mathcal{E}}(f \smile g) &= d_{B(A_0)} \circ (f \smile g) - (-1)^{i+j}(f \smile g) \circ d_{B(A_0)} \\ &= d_{B(A_0)} \circ f[j] \circ g - (-1)^{i+j} f[j] \circ g \circ d_{B(A_0)} \end{aligned}$$

while

$$\begin{aligned} (\mathrm{d}_{\mathcal{E}} f) \smile g + (-1)^i f \smile (\mathrm{d}_{\mathcal{E}} g) &= (\mathrm{d}_{\mathcal{E}} f)[j] \circ g + (-1)^i f[j+1] \circ \mathrm{d}_{\mathcal{E}} g \\ &= (d_{B(A_0)} \circ f - (-1)^i f \circ d_{B(A_0)})[j] \circ g + (-1)^i f[j+1] \circ (\mathrm{d}_{B(A_0)} \circ g - (-1)^j g \circ d_{B(A_0)}) \\ &= (d_{B(A_0)} \circ f)[j] \circ g - (-1)^i (f \circ d_{B(A_0)})[j] \circ g + (-1)^i f[j+1] \circ \mathrm{d}_{B(A_0)} \circ g \\ &\quad - (-1)^{i+j} f[j+1] \circ g \circ d_{B(A_0)}), \end{aligned}$$

and hence $\mathrm{d}_{\mathcal{E}}(f \smile g) = (\mathrm{d}_{\mathcal{E}} f) \smile g + (-1)^i f \smile (\mathrm{d}_{\mathcal{E}} g)$.

Recall also from 火.16 that $B(A_0)^j = 0$ unless $j \leq 0$: $B(A_0)^j = (\coprod_{i\in\mathbb{Z}} \mathcal{P}_{-i}[i])^j = \coprod_{i\in\mathbb{Z}} \mathcal{P}_{-i}^{i+j} = \mathcal{P}_j^0$, and hence $\mathcal{E}^i = \prod_{l\leq 0} A\mathrm{Mod}(B(A_0)^l, B(A_0)^{l+i})$. Also,

$$\begin{aligned} \mathrm{H}^i(\mathcal{E}) &= \mathrm{H}^i(\mathcal{E}^\bullet) = \mathrm{H}^i(\mathcal{A}\mathrm{Mod}^\bullet(B(A_0), B(A_0))) \\ &\simeq \mathrm{K}_{\mathrm{dg}}(\mathcal{A})(B(A_0), B(A_0)[i]) \quad \text{by Ex. 火.14.(ii)} \\ &\simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(A_0), B(A_0)[i]) \quad \text{by Ex. 火.17.(ii)} \\ &\simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})(B(A_0), A_0[i]) \\ &\simeq \mathrm{K}_{\mathrm{dg}}(\mathcal{A})(B(A_0), A_0[i]) \simeq \mathrm{H}^i(\mathcal{A}\mathrm{Mod}^\bullet(B(A_0), A_0)) \\ &\qquad \text{by Ex. 火.17.(ii) and Ex. 火.14.(ii) again} \\ &\simeq \mathrm{H}^i(A\mathrm{Mod}^\bullet(B(A_0), A_0)) \end{aligned}$$

regarding $B(A_0)$ as a projective resolution of A_0:

$$\begin{array}{ccccccccccc}
\dots \longrightarrow & B(A_0)^{-i} & \xrightarrow{d^{-i}_{B(A_0)}} & B(A_0)^{-i+1} & \xrightarrow{d^{-i+1}_{B(A_0)}} & \dots & \xrightarrow{d^{-1}_{B(A_0)}} & B(A_0)^0 & \longrightarrow 0 \longrightarrow \dots \\
& \| & & \| & & & & \| & \\
& \mathcal{P}^0_{-i} & & \mathcal{P}^0_{-i+1} & & & & \mathcal{P}^0_0 &
\end{array}$$

with $d_{A\mathrm{Mod}^\bullet(B(A_0),A_0)}f = d_{A_0} \circ f - (-1)^k f \circ d_{B(A_0)} = d_{A_0} \circ f + (-1)^{k+1} f \circ d_{B(A_0)}$ if $f \in A\mathrm{Mod}^k(B(A_0), A_0) = A\mathrm{Mod}(B(A_0)^{-k}, A_0)$. Thus,

$$\begin{aligned}
\mathrm{H}^i(\mathcal{E}) &\simeq \mathrm{Ext}^i_A(A_0, A_0) \\
&\simeq \mathrm{K}(A)(B(A_0), A_0[i]) \\
&\simeq \mathrm{K}(A)(B(A_0), B(A_0)[i]) \quad \text{by 火.6} \\
&\simeq \mathrm{H}^i(A\mathrm{Mod}^\bullet(B(A_0), B(A_0))).
\end{aligned} \tag{1}$$

Now, using the grading on A, $\forall k, j \in \mathbb{Z}$, put

$$\begin{aligned}
\mathcal{E}^k_j &= A\mathrm{Mod}^k_j(B(A_0), B(A_0)) = \{f \in A\mathrm{Mod}(B(A_0), B(A_0)) | f(B(A_0)^l_p) \subseteq B(A_0)^{l+k}_{p+j} \ \forall l, j \in \mathbb{Z}\} \\
&= \{f \in A\mathrm{Modgr}(B(A_0)\langle j\rangle, B(A_0)) | f(B(A_0)^l) \subseteq B(A_0)^{l+k} \ \forall l \in \mathbb{Z}\}.
\end{aligned}$$

Let $\mathcal{E}^\bullet_j = \coprod_{i\in\mathbb{Z}} \mathcal{E}^i_j$, and $'\mathcal{E} = \coprod_{j\in\mathbb{Z}} \mathcal{E}^\bullet_j$ a dg subalgebra of $\mathcal{E}$. Regarding each $\mathcal{E}^\bullet_j$ as a complex such that $d_{\mathcal{E}^\bullet_j} f = d_{B(A_0)} \circ f + (-1)^{k+1} f \circ d_{B(A_0)}$ $\forall f \in \mathcal{E}^k_j$, one has

$$\begin{aligned}
\mathrm{H}^i(\mathcal{E}^\bullet_j) &= \mathrm{K}_{\mathrm{gr}}(A)(B(A_0)\langle j\rangle, B(A_0)[i]) \\
&\simeq \mathrm{K}_{\mathrm{gr}}(A)(B(A_0)\langle j\rangle, A_0[i]) = \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0\langle j\rangle, A_0) \\
&= 0 \quad \text{unless } i \neq -j \text{ by the Koszulity of } A \text{ 木.4},
\end{aligned}$$

and hence, $\forall i, j \in \mathbb{Z}$,

$$\mathrm{H}^i(\mathcal{E}^\bullet_j) = \begin{cases} \mathrm{H}^i(\mathcal{E}^\bullet_{-i}) = \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle) & \text{if } j = -i \leq 0, \\ 0 & \text{else.} \end{cases}$$

Then, again,

$$\mathrm{H}^i('\mathcal{E}) = \begin{cases} \mathrm{H}^i(\coprod_j \mathcal{E}^\bullet_j) = \mathrm{H}^i(\mathcal{E}^\bullet_{-i}) = \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0\langle -i\rangle, A_0) = \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle) & \text{if } i \geq 0, \\ 0 & \text{else.} \end{cases}$$

Moreover, letting $\mathcal{E}^{\leq j}_{-j}$ denote the truncation of $\mathcal{E}^\bullet_{-j}$ at degree j, one has qis's

$$\begin{tikzcd}
\dots \arrow[r] & \mathcal{E}^{j-1}_{-j} \arrow[r] & \mathcal{E}^j_{-j} \arrow[r] & \mathcal{E}^{j-1}_{-j} \arrow[r] & \dots \\
\dots \arrow[r] & \mathcal{E}^{j-1}_{-j} \arrow[r] \arrow[u, equal] \arrow[d] & \ker(\mathrm{d}^j_{\mathcal{E}^\bullet_{-j}}) \arrow[r] \arrow[u, hook] \arrow[d] & 0 \arrow[r] \arrow[u, hook] \arrow[d, equal] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathrm{H}^j(\mathcal{E}^\bullet_{-j}) \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Thus,

$$
{}'\mathcal{E} = \textstyle\coprod_{j\in\mathbb{Z}} \mathcal{E}_j^{\bullet} \xleftarrow{\text{qis}} \coprod_{j\in\mathbb{Z}} \mathcal{E}_j^{\leq -j} \xrightarrow{\text{qis}} \coprod_{j\in\mathbb{Z}} \mathrm{H}^{-j}(\mathcal{E}_j^{\bullet})[j] \tag{2}
$$

with

$$
\mathrm{H}^{-j}(\mathcal{E}_j^{\bullet})[j] = \begin{cases} \mathrm{H}^{-j}({}'\mathcal{E})[j] = \mathrm{Ext}^{-j}_{A,\mathrm{gr}}(A_0, A_0\langle -j\rangle)[j] & \text{if } j \leq 0, \\ 0 & \text{else.} \end{cases}
$$

Letting $\mathcal{C} = \coprod_{j\in\mathbb{Z}} \mathcal{E}_j^{\leq -j}$ be a dg-subalgebra of ${}'\mathcal{E}$, (2) reads as qis's of dg-algebras

$$
{}'\mathcal{E} \xleftarrow{\text{qis}} \mathcal{C} \xrightarrow{\text{qis}} \textstyle\coprod_{i\in\mathbb{Z}} \mathrm{H}^i({}'\mathcal{E})[-i], \tag{3}
$$

in which case we say dg-algebra ${}'\mathcal{E}$ is formal.

Assume now that A is left noetherian. Then, taking a graded A-projective resolution of A_0 of finite type, one obtains

$$
\begin{aligned}
\mathrm{Ext}^i_A(A_0, A_0) &= \coprod_{j\in\mathbb{Z}} \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0\langle j\rangle, A_0) \quad \text{[NvO, 2.4.7, p. 29]} \\
&= \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0\langle -i\rangle, A_0),
\end{aligned} \tag{4}
$$

where $\mathrm{Ext}^i_{A,\mathrm{gr}}$ denotes $\mathrm{Ext}^i_{A\mathrm{Modgr}}$ for short. Then $\mathrm{H}^i(\mathcal{E}) \simeq \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0\langle -i\rangle, A_0)$ from (1), and hence $\coprod_{j\in\mathbb{Z}} \mathrm{H}^{-j}(\mathcal{E}_j^{\bullet})[j] = \coprod_{i\in\mathbb{Z}} \mathrm{H}^i(\mathcal{E})[-i]$ in (2). Equipping $\coprod_{i\in\mathbb{Z}} \mathrm{H}^i(\mathcal{E})[-i]$ with a structure of dg-algebra with $\mathrm{d} = 0$ such that

$$
\begin{tikzcd}
\mathrm{H}^i(\mathcal{E})[-i] \times \mathrm{H}^j(\mathcal{E})[-j] \arrow[r, dotted] \arrow[d, phantom, "\wr"] & \mathrm{H}^{i+j}(\mathcal{E})[-i-j] \arrow[d, phantom, "\wr"] \\
\mathrm{Ext}^i_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle)[-i] \times \mathrm{Ext}^j_{A,\mathrm{gr}}(A_0, A_0\langle j\rangle)[-j] \arrow[d, phantom, "\wr"] & \mathrm{Ext}^{i+j}_{A,\mathrm{gr}}(A_0, A_0\langle i+j\rangle)[-i-j] \\
\mathrm{K}_{\mathrm{dg}}(A)(B(A_0), B(A_0)\langle i\rangle[-i]) \times \mathrm{K}_{\mathrm{dg}}(A)(B(A_0), B(A_0)\langle j\rangle[-j]) \arrow[r] & \mathrm{K}_{\mathrm{dg}}(A)(B(A_0), B(A_0)\langle i+j\rangle[-i-j]) \arrow[d, phantom, "\wr"] \\
(f, g) \arrow[r, mapsto] & f\langle j\rangle[-j] \circ g.
\end{tikzcd} \tag{5}
$$

yields

Corollary: *If A is left noetherian, $\mathcal{E}$ is formal:*

$$
\mathcal{E} \xleftarrow{\text{qis}} {}'\mathcal{E} \xleftarrow{\text{qis}} \mathcal{C} \xrightarrow{\text{qis}} \textstyle\coprod_{i\in\mathbb{Z}} \mathrm{H}^i(\mathcal{E})[-i].
$$

木.6. *Koszul duality*

Keep the notation of 木.5 with A left noetherian. Thus, $\mathrm{H}(\mathcal{E}) = \coprod_{i\in\mathbb{Z}} \mathrm{H}(\mathcal{E})^i$ is a dg-algebra with $\mathrm{H}(\mathcal{E})^i = \mathrm{H}^i(\mathcal{E})[-i]$ $\forall i$ and $\mathrm{d}_{\mathrm{H}(\mathcal{B})} = 0$. One can further $\mathbb{Z}$-grade $\mathrm{H}(\mathcal{E})$, written with subscripts, such that $\mathrm{H}(\mathcal{E})^i_j = \mathrm{H}^i(\mathcal{E})[j]$. Then

$$
\mathrm{H}(\mathcal{E})^i_j = \begin{cases} \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle)[-i] & \text{if } i = -j \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}
$$

The multiplication on $\mathrm{H}(\mathcal{E})$ is given by the composition product among the $\mathrm{H}(\mathcal{E})^i = \mathrm{H}^i(\mathcal{E})[-i] = \mathrm{Ext}_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle)[-i]$ as described in 木.5(5). We let $\mathrm{C}_{\mathrm{dg,gr}}(\mathrm{H}(\mathcal{E}))$ denote the category of graded dg $\mathrm{H}(\mathcal{E})$-modules

Let now $E(A) = \coprod_{i\in\mathbb{Z}} E(A)_i$ be a graded algebra with $E(A)_i = \mathrm{Ext}^i_A(A_0, A_0)$ under the composition product. By the Koszulity of A one has $E(A)_i = \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle)$, and hence one may identify $\mathrm{H}(\mathcal{B})$ and $E(A)$ as rings. Let $\mathrm{C}_{\mathrm{gr}}(E(A))$ denote the category of complexes of graded $E(A)$-modules, and let $\mathrm{D}_{\mathrm{gr}}(E(A))$ denote the derived category of graded $E(A)$-modules. Define a functor $F : \mathrm{C}_{\mathrm{dg,gr}}(\mathrm{H}(\mathcal{E})) \to \mathrm{C}_{\mathrm{gr}}(E(A))$ by setting, $\forall \mathcal{M} \in \mathrm{C}_{\mathrm{dg,gr}}(\mathrm{H}(\mathcal{E}))$,

$$F(\mathcal{M})^i = \coprod_{j\in\mathbb{Z}} F(\mathcal{M})^i_j \quad \text{with} \quad F(\mathcal{M})^i_j = \mathcal{M}^{i+j}_{-j}, \tag{1}$$

where $\mathcal{M}^{i+j}_{-j}$ is just an abelian group. Let $m \in F(\mathcal{M})^i_j = \mathcal{M}^{i+j}_{-j}$ and $x \in E(A)_k$. Under the identification of $E(A)$ and $\mathrm{H}(\mathcal{E})$ as rings, x lies in $\mathrm{H}(\mathcal{E})^k_{-k}$. Then $xm \in \mathcal{M}^{i+j+k}_{-j-k} = F(\mathcal{M})^i_{j+k}$, and F is well-defined. In particular, $\forall i \in \mathbb{Z}$,

$$F(\mathrm{H}(\mathcal{E}))^i = \coprod_j \mathrm{H}(\mathcal{E})^{i+j}_{-j} = \begin{cases} \coprod_j \mathrm{H}(\mathcal{E})^j_{-j} = E(A) & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

Thus, $F(\mathrm{H}(\mathcal{E})) = E(A)$. As $\mathrm{D}_{\mathrm{dg}}(\mathrm{H}(\mathcal{E})) = \langle\!\langle \mathrm{H}(\mathcal{E})\rangle\!\rangle$ and $\mathrm{D}_{\mathrm{gr}}(E(A)) = \langle\!\langle E(A)\rangle\!\rangle$ by 水.2, one obtains an isomorphism

$$DF : \mathrm{D}_{\mathrm{dg,gr}}(\mathcal{E}) \xrightarrow{\sim} \mathrm{D}_{\mathrm{gr}}(E(A)). \tag{2}$$

Let $n \in \mathbb{Z}$. $\forall i, j \in \mathbb{Z}$,

$$(\mathrm{D}F)(\mathcal{M}[n])^i_j = (\mathcal{M}[n])^{i+j}_{-j} = \mathcal{M}^{i+j+n}_{-j} = (\mathrm{D}F)(\mathcal{M})^{i+n}_j = \{(\mathrm{D}F)(\mathcal{M})[n]\}^i_j,$$

and hence

$$(\mathrm{D}F)(\mathcal{M}[n]) = (\mathrm{D}F)(\mathcal{M})[n], \tag{3}$$

while

$$\begin{aligned}(\mathrm{D}F)(\mathcal{M}\langle n\rangle)^i_j &= (\mathcal{M}\langle n\rangle)^{i+j}_{-j} = \mathcal{M}^{i+j}_{-j-n} = (\mathrm{D}F)(\mathcal{M})^{i-n}_{j+n} = \{(\mathrm{D}F)(\mathcal{M})\langle -n\rangle\}^{i-n}_j \\ &= \{(\mathrm{D}F)(\mathcal{M})\langle -n\rangle[-n]\}^i_j \quad \text{by (3)},\end{aligned}$$

and hence

$$(\mathrm{D}F)(\mathcal{M}\langle n\rangle) = (\mathrm{D}F)(\mathcal{M})\langle -n\rangle[-n]. \tag{4}$$

Put $A^! = E(A)^{\mathrm{op}}$. Recall now equivalence $\mathrm{D}(A) \simeq \mathrm{D}_{\mathrm{dg}}(\mathcal{A})$ from 水.4.(ii), under which regarding A_0 as living in $\mathrm{D}_{\mathrm{dg}}(\mathcal{A})$, $\langle A_0\rangle \xrightarrow[\sim]{\mathrm{R}\mathcal{A}\mathrm{Mod}^\bullet(A_0,?)} \langle \mathcal{E}^{\mathrm{op}}\rangle$ from 水.3, and $\mathrm{D}_{\mathrm{dg,gr}}(\mathcal{E}^{\mathrm{op}}) \simeq \mathrm{D}_{\mathrm{dg,gr}}(\mathrm{H}(\mathcal{E})^{\mathrm{op}})$

from 木.5. Composing with (2) one obtains Koszul duality $\mathcal{K}$:

(5)
$$\begin{tikzcd}
\mathrm{D}_{\mathrm{gr}}(A) \arrow[r, dash, "\sim"] & \mathrm{D}_{\mathrm{dg,gr}}(\mathcal{A}) & \mathrm{D}_{\mathrm{dg,gr}}(\mathcal{E}^{\mathrm{op}}) \arrow[l, "B(A_0)^{\mathbb{L}}\otimes_{\mathcal{E}^{\mathrm{op}}}?"', "\sim"] \arrow[r, dash, "\sim"] & \mathrm{D}_{\mathrm{dg,gr}}(\mathrm{H}(\mathcal{E})^{\mathrm{op}}) \arrow[r, "(2)", "\sim"'] & \mathrm{D}_{\mathrm{gr}}(A^!) \\
\langle A_0\rangle \arrow[u, hook] \arrow[r, dash, "\sim"] \arrow[rrrr, dotted, bend right, "\mathcal{K}"'] & \langle A_0\rangle \arrow[u, hook] \arrow[r, "\sim", "\mathrm{R}\mathcal{A}\mathrm{Mod,gr}^{\bullet}(A_0,?)"'] & \langle \mathcal{E}^{\mathrm{op}}\rangle \arrow[u, hook] \arrow[rr, "\sim"] & & \langle A^!\rangle. \arrow[u, hook]
\end{tikzcd}$$

Theorem: $\mathcal{K}(A_0) = A^!$. $\forall M \in \langle A_0\rangle, \forall n \in \mathbb{Z}$,

$$\mathcal{K}(M[n]) = \mathcal{K}(M)[n], \quad \mathcal{K}(M\langle n\rangle) = \mathcal{K}(M)\langle -n\rangle[-n].$$

木.7. Keep the notation of 木.6. One has isomorphisms of rings

(1)
$$\begin{array}{rcl}
(A^!)_0 = A\mathrm{Modgr}(A_0, A_0)^{\mathrm{op}} \simeq A_0\mathrm{Mod}(A_0, A_0)^{\mathrm{op}} & \xleftarrow{\sim} & A_0 \\
?a & \longleftarrow\!\shortmid & a.
\end{array}$$

Equip $A_0\mathrm{Mod}(A, A_0)$ with a structure of (A, A_0)-bimodule such that $afb = f(?a)b$ $\forall f \in A_0\mathrm{Mod}(A, A_0)$, $\forall a \in A$, $b \in A_0$, and let $A^{\circledast} = \coprod_{i\in\mathbb{Z}}(A^{\circledast})_i$ with $(A^{\circledast})_i = A_0\mathrm{Mod}(A_{-i}, A_0)$ $\forall i$, which is a graded A-module: $A_j(A^{\circledast})_i \subseteq (A^{\circledast})_{i+j}$.

As A_0 is semisimple, A_0 is injective over A_0. $\forall M \in A\mathrm{Mod}$, one has

(2)
$$A\mathrm{Mod}(M, A_0\mathrm{Mod}(A, A_0)) \xrightarrow{\sim} A_0\mathrm{Mod}(M, A_0) \quad \text{via} \quad \phi \mapsto \text{“}m \mapsto \phi(m)(1)\text{”}$$

with inverse $f \mapsto$ “$m \mapsto f(?m)$”, and hence $A_0\mathrm{Mod}(A, A_0)$ is injective in $A\mathrm{Mod}$. Moreover, (2) induces, $\forall M \in A\mathrm{Modgr}$,

(3)
$$\begin{tikzcd}
A\mathrm{Mod}(M, A_0\mathrm{Mod}(A, A_0)) \arrow[r, "\sim"] & A_0\mathrm{Mod}(M, A_0) \\
A\mathrm{Modgr}(M, A^{\circledast}) \arrow[u, hook] \arrow[r, dotted, "\sim"'] & A_0\mathrm{Mod}(M_0, A_0). \arrow[u, hook]
\end{tikzcd}$$

Thus, $A^{\circledast}$ is injective in $A\mathrm{Modgr}$.

Lemma: $\mathcal{K}(A^{\circledast}) = (A^!)_0$.

Proof: One has

$$\mathcal{K}(A^{\circledast}) = \mathrm{R}\mathcal{A}\mathrm{Mod},\mathrm{gr}^{\bullet}(A_0, A^{\circledast}) = \mathrm{R}A\mathrm{Modgr}^{\bullet}(A_0, A^{\circledast}) = \coprod_{i\in\mathbb{Z}} \mathrm{D}_{\mathrm{gr}}(A)(A_0, A^{\circledast}[i])$$

$$= \coprod_{i\in\mathbb{Z}} \mathrm{Ext}^i_{A,\mathrm{gr}}(A_0, A^{\circledast}[i])$$

$= A\mathrm{Modgr}(A_0, A^{\circledast})$ as $A^{\circledast}$ is injective in $A\mathrm{Modgr}$

$\simeq A_0\mathrm{Mod}(A_0, A_0)$ by (3)

$\simeq A_0$ as left A_0-modules, where the structure on $A_0\mathrm{Mod}(A_0, A_0)$ is given by $af = f(?a)$, which is compatible with the structure of left $A_0\mathrm{Mod}(A_0, A_0)^{\mathrm{op}}$-module such that $\varphi f = f(\varphi(?))$, $\varphi \in A_0\mathrm{Mod}(A_0, A_0)^{\mathrm{op}}$

$\simeq (A^!)_0$ by (1).

木.8. Keep the notation of 木.7.

Corollary: *If A is left noetherian Koszul, $A^!$ remains Koszul.*

Proof: $\forall i, j \in \mathbb{Z}$,

$$\begin{aligned}
\mathrm{Ext}^i_{A^!,\mathrm{gr}}((A^!)_0, (A^!)_0\langle j\rangle) &= \mathrm{D}_{\mathrm{gr}}(A^!)((A^!)_0, (A^!)_0\langle j\rangle[i]) \quad \text{by 火.10} \\
&= \mathrm{D}_{\mathrm{gr}}(A)(A^{\circledast}, A^{\circledast}\langle -j\rangle[i-j]) \quad \text{by 木.6, 7} \\
&= \mathrm{Ext}^{i-j}_{A,\mathrm{gr}}(A^{\circledast}, A^{\circledast}\langle -j\rangle) \quad \text{by 火.10} \\
&= 0 \quad \text{unless } i-j=0 \text{ as } A^{\circledast} \text{ is injective 木.7.}
\end{aligned}$$

木.9. Let $B = \coprod_{i\in\mathbb{N}} B_i$ be a positively graded ring with B_0 a semisimple subring. We say B is quadratic iff

(i) B is generated by B_1 over B_0,

(ii) $\ker(\mathrm{T}_{B_0}(B_1) \twoheadrightarrow B) = (\ker(\mathrm{T}_{B_0}(B_1) \twoheadrightarrow B) \cap (B_1 \otimes_{B_0} B_1))$,

in which case let us denote $\ker(\mathrm{T}_{B_0}(B_1) \twoheadrightarrow B) \cap (B_1 \otimes_{B_0} B_1)$ by R_B.

We say B is left (resp. right) finite iff each B_i, $i \in \mathbb{N}$, is of finite type over B_0 as left (resp. right) module.

For a B_0-bimodule V let $V^\vee = B_0\mathrm{Mod}(V, B_0)$ (resp. ${}^\vee V = \mathrm{Mod}B_0(V, B_0)$) equipped with a structure of B_0-bimodule such that $afb = f(?a)b$ (resp. $afb = af(b?)$) $\forall a, b \in B_0$ $\forall f \in V^\vee$. As B_0 is semisimple, if V is of finite type over B_0 as left (resp. right) module, $V^\vee$ (resp. ${}^\vee V$) is of finite type over B_0 as right (resp. left) module.

As B_0 is semisimple, one has bijections

$$\begin{aligned}
(B_1)^\vee \otimes_{B_0} (B_1)^\vee \to (B_1 \otimes_{B_0} B_1)^\vee \quad &\text{via} \quad f\otimes g \mapsto \text{“}a\otimes b \mapsto g(af(b))\text{”}, \\
{}^\vee(B_1) \otimes_{B_0} {}^\vee(B_1) \to {}^\vee(B_1 \otimes_{B_0} B_1) \quad &\text{via} \quad f\otimes g \mapsto \text{“}a\otimes b \mapsto f(g(a)b)\text{”}.
\end{aligned}$$

Put $R_B^\perp = \{\phi \in (B_1 \otimes_{B_0} B_1)^\vee | \phi(R_B) = 0\}$, ${}^\perp R_B = \{\phi \in {}^\vee(B_1 \otimes_{B_0} B_1) | \phi(R_B) = 0\}$, and let $B^! = \mathrm{T}_{B_0}((B_1)^\vee)/(R_B^\perp)$, ${}^!B = \mathrm{T}_{B_0}({}^\vee(B_1))/({}^\perp R_B)$. Thus, if B is a left (resp. right) finite quadratic ring, $B^!$ (resp. ${}^!B$) is a right (resp. left) finite quadratic ring.

Lemma [BGS, Rmk. 2.8]: *If B is a left (resp. right) quadratic ring, ${}^!(B^!) \simeq B$ (resp. $({}^!B)^! \simeq B$).*

Proof: Assume that B is left finite quadratic. Thus, $B = \mathrm{T}_{B_0}(B_1)/(R_B)$, $B^! = \mathrm{T}_{B_0}((B_1)^\vee)/(R_B^\perp)$, and ${}^!(B^!) = \mathrm{T}_{B_0}({}^\vee((B_1)^\vee))/({}^\perp(R_{B^!}))$. As B_0 is semisimple and as B_1 is of finite type over B_0, one has an isomorphism of left B_0-modules

$$B_1 \to {}^\vee((B_1)^\vee) \quad \text{via} \quad b \mapsto \mathrm{ev}_b \quad \text{with} \quad \mathrm{ev}_b(f) = f(b)\ \forall f \in (B_1)^\vee.$$

Likewise, $B_1 \otimes_{B_0} B_1 \xrightarrow{\sim} {}^\vee((B_1 \otimes_{B_0} B_1)^\vee)$. Under those identifications one has

$$\begin{aligned} R_B^\perp &\leq B_1^\vee \otimes_{B_0} B_1^\vee = (B_1 \otimes_{B_0} B_1)^\vee, \\ {}^\perp(R_{B^!}) = {}^\perp(R_B^\perp) &\leq {}^\vee((B_1)^\vee) \otimes_{B_0} {}^\vee((B_1)^\vee) = B_1 \otimes_{B_0} B_1. \end{aligned}$$

Then ${}^\perp(R_B^\perp) = \{x \in B_1 \otimes_{B_0} B_1 | \mathrm{ev}_x(R_B^\perp) = 0\} = \{x \in B_1 \otimes_{B_0} B_1 | f(x) = 0\ \forall f \in R_B^\perp\} \geq R_B$. As R_B is a direct summand of $B_1 \otimes_{B_0} B_1$ and as $B_1 \otimes_{B_0} B_1 = {}^\perp((B_1 \otimes_{B_0} B_1)^\perp)$, we must have ${}^\perp(R_B^\perp) = R_B$. Thus, ${}^!(B^!) = B$.

Likewise, if B is right finite quadratic.

木.10. One has

Theorem [BGS, Cor. 2.3.3]: *A Koszul ring is quadratic.*

木.11. Moreover,

Theorem [BGS, Th. 2.10.1]: *For a left finite Koszul ring B one has $B^! \simeq E(B)^{\mathrm{op}} = \mathrm{Ext}_B^\bullet(B_0, B_0)^{\mathrm{op}}$.*

木.12. Back to our Koszul algebra A, our notation $A^!$ is compatible by 木.11 with the one given in 木.9 to yield

Corollary: *If A is left finite, one has an isomorphism ${}^!(A^!) \simeq A$ of graded $\Bbbk$-algebras.*

木.13. **Remarks:** (i) For our Koszul algebra A, $\mathrm{ev}_a : A^\circledast \to A_0$, $a \in A$, is not left A_0-linear. Neither is the map $A^\circledast \to A^\circledast$ via $f \mapsto af$ is left A-linear.

(ii) Assume for the moment that $\Bbbk$ is perfect. Then semisimple A_0 is separable over $\Bbbk$ [CR, Cor. 7.6, p. 145], and hence the reduced trace form $\mathrm{tr}_{A_0/\Bbbk} : A_0 \times A_0 \to \Bbbk$ is nndegenerate [CR, Prop. 7.41, p. 165]. Thus, one obtains an isomorphism of left A_0-modules

$$A_0 \to \mathrm{Mod}_\Bbbk(A_0, \Bbbk) \quad \text{via} \quad a \mapsto \mathrm{tr}_{A_0/\Bbbk}(?a), \tag{1}$$

where the left A_0-module structure on $\mathrm{Mod}_\Bbbk(A_0, \Bbbk)$ (resp. A_0) is given by $a\gamma = \gamma(?a)$ (resp. the left regular action $a?$); it is injective by the nondegeneracy of the reduced trace form, and hence bijective by dimension. Then in 木.7(3)

$$(2)\qquad \begin{array}{rcl} A_0\mathrm{Mod}(M_0, A_0) \simeq A_0\mathrm{Mod}(M_0, \mathrm{Mod}_{\Bbbk}(A_0, \Bbbk)) & \xrightarrow{\sim} & \mathrm{Mod}_{\Bbbk}(M_0, \Bbbk) \\ \phi & \longmapsto & \text{“}m \mapsto \phi(m)(1)\text{”} \\ \text{“}m \mapsto f(?m)\text{”} & \longleftarrow & f. \end{array}$$

In particular, $A^{\circledast} = \coprod_i (A^{\circledast})_i$ with

$$(3)\qquad (A^{\circledast})_i = A_o\mathrm{Mod}(A_{-i}, A_0) \simeq \mathrm{Mod}_{\Bbbk}(A_{-i}, \Bbbk) = (A_{-i})^*.$$

One show as in [BGS, Prop. 2.2.1] that $(A^!)^{\mathrm{op}}$ remians Koszul: $\forall i, j$,

$$(4)\qquad \begin{aligned} 0 &= \mathrm{Ext}^i_{(A^!)^{\mathrm{op}},\mathrm{gr}}((A^!)^{\mathrm{op}}_0, (A^!)^{\mathrm{op}}_0\langle j\rangle) \quad \text{unless } i = j \\ &\simeq \mathrm{Ext}^i_{(A^!)^{\mathrm{op}},\mathrm{gr},\mathrm{rgt}}((A^!)^{\mathrm{op}}_{0,\mathrm{rgt}}, (A^!)^{\mathrm{op}}_{0,\mathrm{rgt}}\langle j\rangle), \end{aligned}$$

where rgt stands for regarding those as right modules. Set

$$(5)\qquad \begin{aligned} {}^!(A^!) &= E((A^!)^{\mathrm{op}}) = \{\{(A^!)^{\mathrm{op}}\}^!\}^{\mathrm{op}} = E(E(A)) \\ &= \mathrm{Ext}^{\bullet}_{(A^!)^{\mathrm{op}}}((A^!)^{\mathrm{op}}_0, (A^!)^{\mathrm{op}}_0) = \coprod_i \mathrm{Ext}^i_{(A^!)^{\mathrm{op}}}((A^!)^{\mathrm{op}}_0, (A^!)^{\mathrm{op}}_0) \\ &= \coprod_i \coprod_j \mathrm{Ext}^i_{(A^!)^{\mathrm{op}},\mathrm{gr}}((A^!)^{\mathrm{op}}_0, (A^!)^{\mathrm{op}}_0)^{\mathrm{op}}\langle j\rangle) \\ &= \coprod_i \mathrm{Ext}^i_{(A^!)^{\mathrm{op}},\mathrm{gr}}((A^!)^{\mathrm{op}}_0, (A^!)^{\mathrm{op}}_0)^{\mathrm{op}}\langle i\rangle) \quad \text{by (4).} \end{aligned}$$

One shows as in 木.6 an equivalence

$$(6)\qquad \mathrm{D}_{\mathrm{gr}}((A^!)^{\mathrm{op}}) \simeq \mathrm{D}_{\mathrm{gr},\mathrm{rgt}}(A),$$

where $\mathrm{D}_{\mathrm{gr},\mathrm{rgt}}(A)$ denotes the derived category of graded right A-modules, under which

$$(7)\qquad (A^!)^{\mathrm{op}}_0 \mapsto A^{\circledast}_{\mathrm{rgt}} = \coprod_i (A_{-i})^*$$

with a graded right A-module structure on the RHS given by $fa = f(?a)$, $f \in A^{\circledast}$, $a \in A$. For a graded right A-module M with each M_i finite dimensional let $M^{\circledast} = \coprod_i (M^{\circledast})_i$ with $(M^{\circledast})_i = \mathrm{Mod}A_0(M_{-i}, A_0) \simeq \mathrm{Mod}_{\Bbbk}(M_{-i}, \Bbbk) = (M_{-i})^*$ as in (2), $\mathrm{Mod}A_0$ denoting the category of right A_0-modules. Letting $\mathrm{Modgr}A$ denote the category of graded right A-modules, one has

$$(8)\qquad \begin{aligned} \mathrm{Modgr}A(M, A^{\circledast}_{\mathrm{rgt}}) &\simeq \mathrm{Mod}A_0(M_0, A_0) \quad \text{as in 木.7(3)} \\ &\simeq (M_0)^* \quad \text{as in (2)} \\ &= (M^{\circledast})_0 \simeq \mathrm{Mod}A_0(A_0, (M^{\circledast})_0) \simeq \mathrm{Modgr}A(A, M^{\circledast}). \end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Ext}^i_{(A^!)^{\mathrm{op}},\mathrm{gr}}((A^!)_0^{\mathrm{op}},(A^!)_0^{\mathrm{op}})^{\mathrm{op}}\langle j\rangle) &\simeq \mathrm{D}_{\mathrm{gr}}((A^!)^{\mathrm{op}})((A^!)_0^{\mathrm{op}},(A^!)^{\mathrm{op}}\langle i\rangle[i]) \\
&\simeq \mathrm{D}_{\mathrm{gr,rgt}}(A)(A^{\circledast}_{\mathrm{rgt}},A^{\circledast}_{\mathrm{rgt}}\langle -i\rangle) \quad \text{by (6), (7)} \\
&\simeq \mathrm{D}_{\mathrm{gr,rgt}}(A)(A,(A^{\circledast}_{\mathrm{rgt}})^{\circledast}_{\mathrm{rgt}}\langle -i\rangle) \quad \text{by (8)} \\
&\simeq \mathrm{Modgr}A(A,(A^{\circledast}_{\mathrm{rgt}})^{\circledast}_{\mathrm{rgt}}\langle -i\rangle) \quad \text{as } A \text{ is projective in } \mathrm{Modgr}A \\
&= \{(A^{\circledast}_{\mathrm{rgt}})^{\circledast}_{\mathrm{rgt}}\langle -i\rangle\}_0 = \{(A^{\circledast}_{\mathrm{rgt}})^{\circledast}_{\mathrm{rgt}}\}_i = \{(A^{\circledast}_{\mathrm{rgt}})_{-i}\}^* \simeq (A_i)^{**} \simeq A_i.
\end{aligned}$$

Thus, one obtains an isomorphism of graded $\Bbbk$-algebras

$$^!(A^!) \simeq A, \tag{9}$$

which is consistent with 木.12.

木.14. **E.g:** Let V be a finite dimensional $\Bbbk$-linear space, and let $A = \mathrm{S}(V) = \mathrm{S}_\Bbbk(V)$ the symmetric $\Bbbk$-algebra over V. Recall from 水.7 the Koszul complex of V, a projective A-resolution $\mathrm{S}(V)\otimes_\Bbbk \wedge^\bullet V \to \Bbbk$ of $A_0 = \Bbbk$ with $d^{-(i+1)} : \mathrm{S}(V)\otimes_\Bbbk \wedge^{i+1}V \to \mathrm{S}(V)\otimes_\Bbbk \wedge^i V$ via

$$x\otimes(v_1\wedge\cdots\wedge v_{i+1}) \mapsto \sum_{j=0}^{i+1}(-1)^{j+1}xv_j\otimes(v_1\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_{i+1}),$$

where $\widehat{v}_j$ is to denote deleting the term v_j. Thus, $\forall i\in\mathbb{N}$,

$$\mathrm{Ext}^i_A(A_0,A_0) = \mathrm{H}^i(\mathrm{S}(V)\mathrm{Mod}(\mathrm{S}(V)\otimes_\Bbbk\wedge^\bullet V,\Bbbk)) \simeq \mathrm{H}^i(\mathrm{Mod}(\wedge^\bullet V,\Bbbk)) \simeq \mathrm{H}^i((\wedge^\bullet V)^*).$$

$\forall \phi \in \mathrm{S}(V)\mathrm{Mod}(\mathrm{S}(V)\otimes_\Bbbk\wedge^\bullet V,\Bbbk)$,

$$\begin{aligned}
\mathrm{S}(V)\mathrm{Mod}(d^{-(i+1)},\Bbbk)(\phi) = \phi\circ d^{-(i+1)} &= \sum_{j=0}^{i+1}(-1)^{j+1}\phi(xv_j\otimes(v_1\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_{i+1}) \\
&= \sum_{j=0}^{i+1}(-1)^{j+1}v_j\phi(x\otimes(v_1\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_{i+1}) \quad \text{as } xv_j = v_jx \\
&\qquad \text{and as } \phi \text{ is } \mathrm{S}(V)\text{-linear} \\
&= 0 \quad \text{as } v_j\in V = A_1 \text{ annihilates } A_0 = \Bbbk,
\end{aligned}$$

and hence $d_{\mathrm{S}(V)\mathrm{Mod}(\mathrm{S}(V)\otimes_\Bbbk\wedge V,\Bbbk)} = 0$. Then [BA, III.11.5.(30)]

$$\begin{array}{ccccc}
\mathrm{H}^i((\wedge^\bullet V)^*) \simeq \{(\wedge^\bullet V)^*\}^i & \xrightarrow{\sim} & (\wedge^i V)^* & \xleftarrow{\sim} & \wedge^i(V^*) \\
& & v_1\wedge\cdots\wedge v_i & & \\
& & \downarrow & \longleftarrow & f_1\wedge\cdots\wedge f_i. \\
& & \det[\![(f_k(v_l))]\!] & &
\end{array}$$

Now, the multiplication on $\mathrm{Ext}^\bullet_A(A_0, A_0)$ is given by the composition product such that

$$\begin{tikzcd}[column sep=small]
\mathrm{S}(V)\mathrm{Mod}(\mathrm{S}(V)\otimes_{\Bbbk}\wedge^\bullet V, \mathrm{S}(V)\otimes_{\Bbbk}\wedge^\bullet V[i]) \times \mathrm{S}(V)\mathrm{Mod}(\mathrm{S}(V)\otimes_{\Bbbk}\wedge^\bullet V, \mathrm{S}(V)\otimes_{\Bbbk}\wedge^\bullet V[j]) \arrow[r] \arrow[d] & \mathrm{S}(V)\mathrm{Mod}(\mathrm{S}(V)\otimes_{\Bbbk}\wedge^\bullet V, \mathrm{S}(V)\otimes_{\Bbbk}\wedge^\bullet V[i+j]) \arrow[d] \\
\mathrm{Mod}_{\Bbbk}(\wedge^\bullet V, \Bbbk[i]) \times \mathrm{Mod}_{\Bbbk}(\wedge^\bullet V, \Bbbk[j]) \arrow[r] \arrow[d, no head, "\wr"'] & \mathrm{Mod}_{\Bbbk}(\wedge V, \Bbbk[i+j]) \arrow[d, no head, "\wr"] \\
(\wedge^i V)^* \times (\wedge^j V)^* \arrow[r] \arrow[d, no head, "\wr"'] & (\wedge^{i+j} V)^* \arrow[d, no head, "\wr"] \\
\wedge^i(V^*) \times \wedge^j(V^*) \arrow[r] & \wedge^{i+j}(V^*).
\end{tikzcd}$$

Thus, $E(A) \simeq (\wedge V)^*$, and hence

(1) $$A^! \simeq \{(\wedge V)^{*\mathrm{gr}}\}^{\mathrm{op}} \simeq \wedge(V^*)$$

under $\wedge(V^*) \xrightarrow{\sim} \{(\wedge V)^{*\mathrm{gr}}\}^{\mathrm{op}}$ via $f_1 \wedge \cdots \wedge f_r \longmapsto \left(v_1 \wedge \cdots \wedge v_r \mapsto (-1)^{\binom{r}{2}} \det[\![f_i(v_j)]\!] \right)$ [BA, Prop. III.5.7].

Then by 木.12

$$^!(\wedge(V^*)) \simeq {}^!(\mathrm{S}(V)^!) \simeq \mathrm{S}(V).$$

Alternatively, $\mathrm{S}(V) = \mathrm{T}_{\Bbbk}(V)/(R_A)$ with $R_A = \{v \otimes w - w \otimes v | v, w \in V\}$. Then $\mathrm{S}(V)^! = \mathrm{T}_{\Bbbk}(V^*)/(R_A^\perp)$ by 木.9. Let $(v_1, \dots, v_n)$ is a $\Bbbk$-linear basis of V and $(f_1, \dots, f_n)$ its dual basis. If $\sum_{i,j} \xi_{ij} f_i \otimes f_j \in R_A^\perp$, $\xi_{ij} \in \Bbbk$, $0 = (\sum_{i,j} \xi_{ij} f_i \otimes f_j)(v_k \otimes v_l - v_l \otimes v_k) = \xi_{kl} - \xi_{lk}$. Thus,

$$\begin{aligned} R_A^\perp &= \{f \otimes f | f \in V^*\} \\ &= {}^\perp R_A \quad \text{likewise,} \end{aligned}$$

and hence 木.9 yields

(2) $$\begin{aligned} &\mathrm{S}(V)^! \simeq \wedge(V^*) \simeq \wedge({}^*V) \simeq {}^!\mathrm{S}(V), \quad {}^!(\wedge(V^*)) \simeq \mathrm{S}(V) \simeq (\wedge({}^*V))^!, \\ &{}^!(\wedge V) \simeq \mathrm{S}(V^*) \simeq \mathrm{S}({}^*V) \simeq (\wedge V)^!. \end{aligned}$$

Or, writing $\wedge V = \mathrm{T}_{\Bbbk}(V)/(R_B)$ with $R_B = \{v \otimes v | v \in V\}$, if $\sum_{i,j} \xi_{ij} f_i \otimes f_j \in R_B^\perp$, $0 = (\sum_{i,j} \xi_{ij} f_i \otimes f_j)(v_k \otimes v_k) = \xi_{kk}$, and $0 = (\sum_{i,j} \xi_{ij} f_i \otimes f_j)\{(v_k + v_l) \otimes (v_k + v_l)\} = \xi_{kl} + \xi_{lk}$ for $k \neq l$, and hence the 3rd isomorphisms.

Let now $\mathrm{D}^b_{\mathrm{gr,fl}}(A)$ (resp. $\mathrm{D}^b_{\mathrm{gr,f}}(A)$) denote the bounded category of graded A-modules consisting of those of finite length (resp. of finite type). The simples of $A\mathrm{Modgr}$ and those of $\mathrm{D}^b_{\mathrm{gr}}(A)$ coincide [Z, Prop. 6.3.3]. Any simple graded A-module is annihilated by $A_{>0}$, and hence an A_0-module. Let M be an A-module and $\langle M \rangle$ the smallest triangulated subcategory of $\mathrm{D}(A)$ containing M and closed under taking direct summands. If $f \in A\mathrm{Mod}(X, Y)$ with $X, Y \in \langle M \rangle$, $Y \oplus X[1] = \mathrm{cone}(f) \in \langle M \rangle$. One has qis' of rows

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & \dots & \mathrm{cone}(f) \\
\dots \arrow[r] & 0 \arrow[r] & X \arrow[r] & \mathrm{coker} f \arrow[r] & 0 \arrow[r] & \dots & \\
\dots \arrow[r] & 0 \arrow[r] \arrow[u, equal] & \ker f \arrow[r] \arrow[u, hook] & \mathrm{coker} f \arrow[r] \arrow[u, equal] & 0 \arrow[r] \arrow[u, equal] & \dots &
\end{tikzcd}$$

As the bottom complex is $\mathrm{coker} f \oplus (\ker f)[1]$, both $\mathrm{coker}(f)$ and $\ker f \in \langle M \rangle$. Thus, $\mathrm{D}^b_{\mathrm{gr,fl}}(\mathrm{S}(V)) = \langle \Bbbk \rangle$ and, as $\wedge(V^*)$ is finite dimensional, $\mathrm{D}^b_{\mathrm{gr,fl}}(\wedge(V^*)) = \langle \wedge(V^*) \rangle = \mathrm{D}^b_{\mathrm{gr,f}}(\wedge(V^*))$:

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{gr}}(\mathrm{S}(V)) & \mathrm{D}_{\mathrm{gr}}(\wedge(V^*)) & \mathrm{D}_{\mathrm{gr}}(\wedge V) & \mathrm{D}_{\mathrm{gr}}(\mathrm{S}(V^*)) \\
\langle \Bbbk \rangle \arrow[u, hook] \arrow[r, "\sim", "\mathcal{K}"'] & \langle \wedge(V^*) \rangle \arrow[u, hook] & \langle \Bbbk \rangle \arrow[u, hook] \arrow[r, "\sim", "\mathcal{K}"'] & \langle \mathrm{S}(V^*) \rangle \arrow[u, hook] \\
\mathrm{D}^b_{\mathrm{gr,fl}}(\mathrm{S}(V)) \arrow[u, equal] & \mathrm{D}^b_{\mathrm{gr,fl}}(\wedge(V^*)), \arrow[u, equal] & \mathrm{D}^b_{\mathrm{gr,f}}(\wedge V) \arrow[u, equal] & \mathrm{D}^b_{\mathrm{gr,f}}(\mathrm{S}(V^*)). \arrow[u, equal]
\end{tikzcd}$$

木.15. More generally,

Theorem [BGS, Th. 2.12.6]: *Let A be a Koszul ring. Assume that A is of finite type over A_0 both as left and right modules. In particular, $A_i = 0\ \forall i \gg 0$. Assume in addition that $A^!$ is left noetherian. Then the Koszul duality induces an isomorphism*

$$\mathrm{D}^b_{\mathrm{gr,f}}(A) \simeq \mathrm{D}^b_{\mathrm{gr,f}}(A^!).$$

木.16. Our next objective is to find conditions for a $\Bbbk$-algebra A to admit a Koszul algebra B 森田-equivalent to A.

Fix a finite dimensional $\Bbbk$-algebra A. Set $\mathrm{rad}A = \bigcap_{\substack{\text{maximal left}\\ \text{ideals of } A}} \mathfrak{m}$, called the radical of A. Thus, $\mathrm{rad}A = \bigcap_{\substack{\text{maximal right}\\ \text{ideals of } A}} \mathfrak{m}$ [AF, 15.3, p. 166]. As A is finite dimensional,

$$A/\mathrm{rad}A \text{ is semisimple [AF, 15.16, p. 170],} \tag{1}$$
$$\mathrm{rad}A \text{ is nilpotent [AF, 15.19, p. 172].} \tag{2}$$

Recall also that

(3) a ring B is called local iff $\mathrm{rad}B$ is a left maximal ideal iff $B \setminus B^\times \lhd B$ iff $B \setminus B^\times = \mathrm{rad}B$ iff $\forall b \in B$, either b or $1 - b \in B^\times$ [AF, 15.15, p. 170],

(4) for an A-module M of finite type M is indecomposable iff $A\mathrm{Mod}(M, M)$ is local [AF, a remark, p. 144 and Lem. 12.8, p. 146]; M is finite dimensional as A is,

(5) if M is a finite dimensional A-module, $\mathrm{rad}M = (\mathrm{rad}A)M$ [AF, 15.18, p. 171], and hence $M/\mathrm{rad}M$ is semisimple as an A-module,

(6) a projective A-module P of finite type P is indecomposable iff $P/\mathrm{rad}P$ is simple [AF, 17.19, p. 201].

Thus, one has a bijection between the set of indecomposable A-projectives of finite type and the set of A-simples.

木.17. Throughout the rest of 木 assume that our finite dimensional $\Bbbk$-algebra A is graded: $A = \coprod_{i \in \mathbb{Z}} A_i$.

Definition: *We say an A-module M is gradable iff there exists a graded A-module $\tilde{M}$ such that $\tilde{M} \simeq M$ as A-modules, in which case we call $\tilde{M}$ a lift of M.*

Proposition: *Let M be an A-module of finite type. If M is indecomposable, a lift of M if any is unique up to isomorphism and a degree shift.*

Proof: Let $\tilde{X}$, $\tilde{Y}$ be two lifts of M. As M is of finite type over A, $A\mathrm{Mod}(M,M) \simeq \coprod_{i\in\mathbb{Z}} A\mathrm{Modgr}(\tilde{X}, \tilde{Y}\langle i\rangle)$, under which write $\mathrm{id}_M = \sum_i x_i$. As $A\mathrm{Mod}(M,M)$ is local, some x_i must be invertible; $A\mathrm{Mod}(M,M) \setminus A\mathrm{Mod}(M,M)^\times$ forms an abelian group 木.16(3). Then the corresponding $\tilde{x}_i \in A\mathrm{Modgr}(\tilde{X}, \tilde{Y}\langle i\rangle)$ is bijective, and hence $\tilde{X} \simeq \tilde{Y}\langle i\rangle$.

木.18. **Proposition:** *If M is an indecomposable graded A-module of finite type, M remains indecomposable as an A-module by degradation.*

Proof: Put $E = A\mathrm{Mod}(M,M)$. As M is of finite type over A, $E \simeq \coprod_{i\in\mathbb{Z}} E_i$ with $E_i = A\mathrm{Modgr}(M\langle i\rangle, M)$. Thus, E_0 is local. $\forall i \in \mathbb{Z}$, one has

$$E_iE_{-i} = A\mathrm{Modgr}(M\langle i\rangle, M)A\mathrm{Modgr}(M\langle -i\rangle, M) = A\mathrm{Modgr}(M\langle i\rangle, M)A\mathrm{Modgr}(M, M\langle i\rangle) \subseteq E_0.$$

We claim, $\forall i \neq 0$,

$$(E_iE_{-i}) \cap (E_0)^\times = \emptyset. \tag{1}$$

For let $a \in E_i$ and $b \in E_{-i} \setminus 0$. As E is finite dimensional, $E_{-j} = 0\ \forall j \gg 0$, and hence there is $N \in \mathbb{N}$ with $b^N = 0$ but $b^{N-1} \neq 0$. Then $ab^N = 0$, and hence $ab \notin (E_0)^\times$.

Now, put $\mathfrak{m} = \mathrm{rad}(E_0) = E_0 \setminus (E_0)^\times \lhd E_0$ 木.16(3). One has by (1)

$$\mathfrak{m} + \sum_{i\neq 0} E_i \lhd E. \tag{2}$$

Put $I = \mathfrak{m} + \sum_{i\neq 0} E_i$. Thus, E/I is a quotient of E_0, and hence local; as $\mathrm{rad}(E_0)$ is a maximal left ideal of E_0, the radical of any quotient of E_0 remains a maximal left ideal of the quotient, and hence the quotient is local 木.16(3).

It now suffices to show that I is nilpotent; for let $x \in E$. If $\bar{x} \in (E/I)^\times$, $\exists y \in E$: $\bar{x}\bar{y} = 1$. Then $xy = 1 - z$ for some $z \in I$. If $z^n = 0$, $xy(1 + z + \cdots + z^{n-1}) = 1 - z^n = 1$, and hence $x \in E^\times$. If $1 - \bar{x} \in (E/I)^\times$, $\exists y \in E$: $(1 - \bar{x})\bar{y} = 1$. Then $(1-x)y = 1 - z$ for some $z \in I$, and hence $1 - x \in E^\times$. To see that I is nilpotent, we follow an argument from [GG, Th. 3.1]. As E is finite dimensional, we may assume $E = \coprod_{i\in[-N,N]} E_i$. As E_0 is finite dimensional, $\mathfrak{m}$ is nilpotent 木.16(2), say $\mathfrak{m}^r = 0$. We have only to show that, $\forall$ homogeneous $x_1, \ldots, x_{(2N+1)(r+1)} \in I$, $x_1 \ldots x_{(2N+1)(r+1)} = 0$. Just suppose not. Put $d_i = \deg(x_1 \ldots x_i)$, $i \in [1, (2N+1)(r+1)]$. As $d_i \in [-N, N]$, $\forall i \in [1, (2N+1)(r+1)]$, $\exists i_1 < \cdots < i_{r+1}$: $d_{i_1} = \cdots = d_{i_{r+1}}$, equal to, say j. Put $y_1 = x_1 \ldots x_{i_1}$, $y_2 = x_{i_1+1} \ldots x_{i_2}, \ldots$, $y_{r+1} = x_{i_r+1} \ldots x_{i_{r+1}}$. Then

$$\begin{aligned} j &= \deg(x_1 \ldots x_{i_{r+1}}) = \deg(y_1 \ldots y_{r+1}) = \deg(x_1 \ldots x_{i_r}) = \deg(y_1 \ldots y_r) = \ldots \\ &= \deg(x_1 \ldots x_{i_1}) = \deg(y_1), \end{aligned}$$

and hence $0 = \deg(y_{r+1}) = \cdots = \deg(y_2)$. Then $y_2 \dots y_{r+1} \in \mathfrak{m}^r = 0$, and hence $x_1 \dots x_{(2N+1)(r+1)} = y_1 \dots y_{r+1} x_{i_{r+1}+1} \dots x_{(2N+1)(r+1)} = 0$, absurd.

木.19. **Proposition:** *For a graded A-module M of finite type any direct summand of M remains gradable.*

Proof: Write $M = \coprod_i M_i$ with each M_i graded indecomposable. By 木.18 each M_i remains indecomposable upon degradation. On the other hand, any direct summand of M as an A-module is by Krull-Schmidt-東屋 [AF, 12.6, p. 144] isomorphic to a direct sum of some M_i's.

木.20. **Corollary:** *Any projective A-module of finite type is gradable.*

木.21. **Proposition:** $\mathrm{rad}A$ *is homogeneous:* $\mathrm{rad}A = \coprod_{i \in \mathbb{Z}} (A_i \cap \mathrm{rad}A)$.

Proof: As A is finite dimensional, we may assume $A = \sum_{i=-N}^{N} A_i$ for some N with $N \neq 0$ in $\Bbbk$.

Assume for the moment that $\Bbbk$ admits a primitive N-th root ζ of 1. Define a $\Bbbk$-algebra automorphism σ of A via $\sum_i a_i \mapsto \sum_i \zeta^i a_i$, $a_i \in A_i$. Then $\sigma(\mathrm{rad}A) = \mathrm{rad}A$. Let $a = \sum_i a_i \in \mathrm{rad}A$. Fix $j \in [-N, N]$; in particular, $B(\mathcal{N}) = B(\phi_* \mathcal{N})$ is $\mathrm{K}_{\mathrm{dg}}(\mathcal{A})$-flat. One has

$$\mathrm{rad}A \ni \sum_{k=0}^{N-1} \zeta^{-jk} \sigma^k(a) = \sum_{k=0}^{N-1} \zeta^{-jk} \sum_i \zeta^{ki} a_i = \sum_{k=0}^{N-1} \sum_i \zeta^{(i-j)k} a_i = \sum_i \sum_k \zeta^{(i-j)k} a_i.$$

As $(\sum_{k=0}^{N-1} \zeta^{(i-j)k})(1 - \zeta^{i-j}) = 1 - (\zeta^{i-j})^N = 0$, $\sum_{k=0}^{N-1} \zeta^{(i-j)k} = \delta_{i,j} N$, and hence $a_j \in \mathrm{rad}A$, as desired.

In the general case, see [GG, Prop. 3.5].

金曜日

We give a sufficient condition for a $\Bbbk$-algebra to admit a Koszul algebra that is 森田-equivalent to the algebra. Finally, we show that a Koszul grading if any on a ring is unique.

金.1. Let $A = \coprod_{i \in \mathbb{Z}} A_i$ be a graded $\Bbbk$-algebra. Assume that there is a finite dimensional $\Bbbk$-linear space V such that A is a graded $\mathrm{S}(V)$-algebra of finity type with $\mathrm{S}(V)$ central in A and positively graded such that $\deg V = 1$. In particular,

(1) $$A_i = 0 \quad \forall i \ll 0.$$

Put $\bar{A} = A/VA$. As $\Bbbk \simeq \mathrm{S}(V)/\mathrm{S}(V)_{>0}$, $\bar{A}$ is a finite dimensional $\Bbbk$-algebra. By 木.21

(2) $\mathrm{rad}(\bar{A})$ is homogeneous.

Each simple $\bar{A}$-module is a direct summand of $\bar{A}/\mathrm{rad}\bar{A}$, and hence gradable by 木.19, and hence

(3) each graded simple $\bar{A}$-module remains simple over $\bar{A}$ upon degradation.

By (1)

(4) any simple graded A-module is annihilated by V,

and hence

(5) the simple graded A-modules are in bijective correspondence with

the simple graded $\bar{A}$-modules.

Let $L_1 \dots, L_r$ be a complete set of the representatives of simple $\bar{A}$-modules. As $\mathrm{rad}(\bar{A})$ is homogeneous, $\bar{A}/\mathrm{rad}(\bar{A})$ is graded. Each L_i, $i \in [1, r]$, is a direct summand of $\bar{A}/\mathrm{rad}(\bar{A})$, and hence gradable by 木.19, and let $\tilde{L}_i$ be a lift of L_i to a graded simple A-module by (5). In particular,

$$A\mathrm{Modgr}(\tilde{L}_i, \tilde{L}_j\langle k\rangle) = 0 \quad \text{unless } i = j \text{ and } k = 0. \tag{6}$$

We will show

Theorem [Ri, Th. 9.2.1]: *Assume that* $\mathrm{Ext}^k_{A,\mathrm{gr}}(\tilde{L}_i, \tilde{L}_j\langle l\rangle) = 0$, $\forall i, j \in [1, r]$ $\forall k, l \in \mathbb{Z}$ *with* $k \neq l$. *Then* A *admits a Koszul* $\Bbbk$*-algebra* B *graded* 森田*-equivalent to* A.

金.2. Keep the notation of 金.1. $\forall M \in A\mathrm{Modgr}$, let $\mathrm{radgr}M = \bigcap_{\substack{\phi \in A\mathrm{Modgr}(M,L) \\ L \text{ simple}}} \ker\phi$, and define inductively $\mathrm{radgr}^{n+1}M = \mathrm{radgr}(\mathrm{radgr}^n M)$ $\forall n \in \mathbb{N}$.

E.g. If $\Bbbk[x]$ is the polynomial $\Bbbk$-algebra in x with x having degree 1,

$$\mathrm{rad}\,\Bbbk[x] \subseteq \bigcap_{a \in \Bbbk} (x - a) = 0$$

while

$$\mathrm{radgr}\,\Bbbk[x] = (x) \quad \text{as } (x) \text{ is a unique maximal graded ideal.}$$

$\forall f \in A\mathrm{Modgr}(M, N)$,

$$f(\mathrm{radgr}M) \subseteq \mathrm{radgr}N \quad \text{[AF, 8.16, p. 110]}. \tag{1}$$

If N is graded simple, $f(VM) \subseteq VN = 0$ by 金.1.4, and hence

$$VM \subseteq \mathrm{radgr}M. \tag{2}$$

Also,

$$\mathrm{radgr}(M/\mathrm{radgr}M) = 0 \quad [\mathrm{AF}, 8.17, p.110]. \tag{3}$$

Lemma: *Let M be a graded A-module of finite type and put $\bar{M} = M/VM$.*

(i) $\mathrm{rad}\bar{M} = \mathrm{radgr}\bar{M}$ *as $\bar{A}$-modules.*

(ii) $\bigcap_{n \in \mathbb{N}} \mathrm{radgr}^n M = 0$.

(iii) The quotient $M \to \bar{M}$ induces an isomorphism of graded modules

$$M/\mathrm{radgr}M \xrightarrow{\sim} \bar{M}/\mathrm{radgr}\bar{M}.$$

Proof: (i) As $\bar{M}$ is finite dimensional, $\mathrm{radgr}\bar{M}$ is a finite intersection of $\ker\phi$'s, $\phi \in \bar{A}\mathrm{Modgr}(\bar{M}, \bar{L})$, $\bar{L}$ graded $\bar{A}$-simple. In particular, $\bar{M}/\mathrm{radgr}\bar{M}$ is graded semisimple over $\bar{A}$. Then $\bar{M}/\mathrm{radgr}\bar{M}$ remains semisimple over $\bar{A}$ by 金.1(3), and hence $\mathrm{rad}\bar{M} \subseteq \mathrm{radgr}\bar{M}$.

On the other hand, $\mathrm{rad}\bar{M} = (\mathrm{rad}\bar{A})\bar{M}$ by 木.16(5), and hence homogeneous as $\mathrm{rad}\bar{A}$ is by 木.21. Then $\bar{M}/\mathrm{rad}\bar{M}$ is graded $\bar{A}$-semisimple, and hence $\mathrm{radgr}\bar{M} \subseteq \mathrm{rad}\bar{M}$.

(ii) One has

$$\begin{aligned} \mathrm{radgr}^i\bar{M} &= \mathrm{rad}^i\bar{M} \quad \text{by (i)} \\ &= 0\ \forall i \gg 0 \quad \text{as } \mathrm{rad}\bar{M} = (\mathrm{rad}\bar{A})\bar{M} \text{ and as } \mathrm{rad}\bar{A} \text{ is nilpotent 木.16(2)}, \end{aligned}$$

and hence $\mathrm{radgr}^i M \subseteq VM$ $\forall i \gg 0$ by (1). Then

$$\mathrm{radgr}^{2i}M = \mathrm{radgr}^i(\mathrm{radgr}^i M) \subseteq \mathrm{radgr}^i(VM) \subseteq V^2M \quad \forall i \gg 0.$$

On the other hand, $M_i = 0$ $\forall i \ll 0$ as M is of finite type over A. Then $\forall i \in \mathbb{Z}$, $\exists k \in \mathbb{N}$: $(V^kM)_i = 0$, and hence $\cap_{k\in\mathbb{N}}(V^kM) = 0$. Thus, $\cap_n(\mathrm{radgr}^n M) \subseteq \cap_k(V^kM) = 0$.

(iii) By (1) one has a surjection $M/\mathrm{radgr}M \xrightarrow{\sim} \bar{M}/\mathrm{radgr}\bar{M}$. On the other hand, from (2) one has

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M/\mathrm{radgr}M, \\ M/VM = \bar{M} \arrow[ur, dotted] & \end{tikzcd}$$

and hence one obtains by (1) again and by (3) a surjection

$$\bar{M}/\mathrm{radgr}\bar{M} \to (M/\mathrm{radgr}M)/\mathrm{radgr}(M/\mathrm{radgr}M) = M/\mathrm{radgr}M.$$

As $\bar{M}$ is finite dimensional, $M/\mathrm{radgr}M \simeq \bar{M}/\mathrm{radgr}\bar{M}$ by dimension.

金.3. Keep the notation of 金.1. By 金.2(iii) one has that $A/\mathrm{radgr}A$ is a direct sum of $\tilde{L}_i\langle n\rangle$'s, $i \in [1, r]$, and for some $n \in \mathbb{Z}$. By renumbering if necessary we may assume each $\tilde{L}_i$, $i \in [1, r]$, appears in $A/\mathrm{radgr}A$. Recall now from [AJS, E.6] that

(1) if M is an indecomposable graded A-module of finite type, $A\mathrm{Modgr}(M, M)$ is local.

Thus, $\forall i \in [1, r]$, there is a graded indecomposable direct summand $\tilde{P}_i$ of A which is a projective cover of $\tilde{L}_i$ [AF, 17.19, p. 201]. For let P be a graded indecomposable direct summand of A, which exists by 金.2.(iii). Put $E(P) = A\mathrm{Modgr}(P, P)$, and let M be a maximal graded submodule of P. We show that the quotient $\pi : P \to P/M$ is a projective cover, i.e., $M \ll P$; $\forall L \leq P$ with $P = M + L$, $L = P$. Write

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, dotted, "s"'] & P/M \arrow[d, "\sim"] \\ L \arrow[r, two heads] & L/L \cap M. \end{tikzcd}$$

Then $\mathrm{im}s \not\leq M$. As $\mathrm{im}s + M = P$, $\mathrm{im}s \not\ll P$. Thus, $s \notin \mathrm{rad}(E(P))$; write

$$\begin{tikzcd} P \arrow[r, "s"] \arrow[dr, dotted, two heads] & P \arrow[d, two heads, "\pi"] \\ P \arrow[u, dotted, "t"] \arrow[r, two heads, "\pi"'] & P/M. \end{tikzcd}$$

Then $\pi = \pi \circ s \circ t$, and hence $\pi \circ (1 - st) = 0$. If $s \in \mathrm{rad}(E(P))$, $1 - st \in E(P)^\times$ by 木.16(3), and hence $\pi = 0$, absurd. Then $s \in E(P)^\times$ by 木.16.(3) again, and hence $L = P$, as desired.

One has

$$\tilde{P}_i/\mathrm{radgr}\tilde{P}_i \simeq \tilde{L}_i, \tag{2}$$

$$A\mathrm{Modgr}(\tilde{P}_i, \tilde{L}_j\langle k\rangle) = 0 \ \forall j \in [1, r] \ \forall k \in \mathbb{Z} \quad \text{unless } j = i \text{ and } k = 0. \tag{3}$$

Put $\tilde{L} = \coprod_{i=1}^r \tilde{L}_i$, $\tilde{P} = \coprod_{i=1}^r \tilde{P}_i$, and $B = A\mathrm{Mod}(\tilde{P}, \tilde{P})^{\mathrm{op}}$. As $\tilde{P}$ is of finite type over A,

$$B = \coprod_{j\in\mathbb{Z}} B_j \quad \text{with} \quad B_j = A\mathrm{Modgr}(\tilde{P}\langle j\rangle, \tilde{P}). \tag{4}$$

By a graded version [AJS, E.4] of 火.19 one has B graded 森田-equivalent to A:

$$A\mathrm{Mod}(\tilde{P}, ?) : A\mathrm{Modgr} \xrightarrow{\sim} B\mathrm{Modgr}. \tag{5}$$

In particular, the graded B-simples are the $\tilde{S}_i\langle n\rangle$, $i \in [1, r], n \in \mathbb{Z}$, with $\tilde{S}_i = A\mathrm{Modgr}(\tilde{P}, \tilde{L}_i)$.

We show that B is Koszul.

Lemma: *Assume the hypothesis of Th. 金.1. $\forall n \in \mathbb{N}$,*

$$\mathrm{radgr}^n(\tilde{P})/\mathrm{radgr}^{n+1}(\tilde{P}) \simeq \coprod_{i=1}^r \tilde{L}_i\langle n\rangle^{\oplus m(i,n)} \quad \textit{for some } m(i, n) \in \mathbb{N}.$$

Proof: As LHS is graded semisimple by 金.2.(iii), we have only to show that

$$A\mathrm{Modgr}(\mathrm{radgr}^n(\tilde{P})/\mathrm{radgr}^{n+1}(\tilde{P}), \tilde{L}_i\langle m\rangle) = 0 \quad \forall i \in [1, r], \forall m \in \mathbb{Z} \setminus \{n\}. \tag{6}$$

$\forall \phi \in A\mathrm{Modgr}(\mathrm{radgr}^n(\tilde{P}), \tilde{L}_i\langle m\rangle)$, $\phi(\mathrm{radgr}^{n+1}(\tilde{P})) \subseteq \mathrm{radgr}(\tilde{L}_i\langle m\rangle) = 0$ by 金.2(1), and hence

$$A\mathrm{Modgr}(\mathrm{radgr}^n(\tilde{P})/\mathrm{radgr}^{n+1}(\tilde{P}), \tilde{L}_i\langle m\rangle) \simeq A\mathrm{Modgr}(\mathrm{radgr}^n(\tilde{P}), \tilde{L}_i\langle m\rangle). \tag{7}$$

We argue by induction on n. If $n = 0$, the assertion holds by (2). Let $n > 0$ and suppose $A\mathrm{Modgr}(\mathrm{radgr}^n(\tilde{P})/\mathrm{radgr}^{n+1}(\tilde{P}), \tilde{L}_i\langle m\rangle)) \neq 0$. The exact sequence $0 \to \mathrm{radgr}^n\tilde{P} \to \mathrm{radgr}^{n-1}\tilde{P} \to \mathrm{radgr}^{n-1}\tilde{P}/\mathrm{radgr}^n\tilde{P} \to 0$ yields by (7) a LES

$$\begin{aligned} 0 \to A\mathrm{Modgr}(\mathrm{radgr}^{n-1}\tilde{P}/\mathrm{radgr}^n\tilde{P}, \tilde{L}_i\langle m\rangle) &\xrightarrow{\sim} A\mathrm{Modgr}(\mathrm{radgr}^{n-1}\tilde{P}, \tilde{L}_i\langle m\rangle) \\ &\to A\mathrm{Modgr}(\mathrm{radgr}^n\tilde{P}, \tilde{L}_i\langle m\rangle) \to \mathrm{Ext}^1_{A,\mathrm{gr}}(\mathrm{radgr}^{n-1}\tilde{P}/\mathrm{radgr}^n\tilde{P}, \tilde{L}_i\langle m\rangle). \end{aligned}$$

Then

$$
\begin{aligned}
A\mathrm{Modgr}(\mathrm{radgr}^n\tilde{P}, \tilde{L}_i\langle m\rangle) &\leq \mathrm{Ext}^1_{A,\mathrm{gr}}(\mathrm{radgr}^{n-1}\tilde{P}/\mathrm{radgr}^n\tilde{P}, \tilde{L}_i\langle m\rangle) \\
&\simeq \coprod_{j=1}^{r} \mathrm{Ext}^1_{A,\mathrm{gr}}(\tilde{L}_j\langle n-1\rangle^{\oplus m(j,n-1)}, \tilde{L}_i\langle m\rangle) \quad \text{by the induction hypothesis} \\
&= 0 \quad \text{unless } m-(n-1)=1 \text{ by the standing hypothesis,}
\end{aligned}
$$

and hence

$$
\begin{aligned}
0 &= A\mathrm{Modgr}(\mathrm{radgr}^n\tilde{P}, \tilde{L}_i\langle m\rangle) \quad \text{unless } m = n-1+1 = n \\
&\simeq A\mathrm{Modgr}(\mathrm{radgr}^n(\tilde{P})/\mathrm{radgr}^{n+1}(\tilde{P}), \tilde{L}_i\langle m\rangle) \quad \text{by (7).}
\end{aligned}
$$

金.4. Assume the hypothesis of Th. 金.1. Koszulity of B now follows from

Proposition: *(i)* $B_0 = \coprod_{i=1}^r \tilde{S}_i$.

(ii) $\forall n, m \in \mathbb{Z}$, $\mathrm{Ext}^n_{B,\mathrm{gr}}(B_0, B_0\langle m\rangle) = 0$ *unless* $n = m$.

(iii) $B_n = 0\ \forall j < 0$.

Proof: (i) As $\tilde{P}/\mathrm{radgr}\tilde{P} = \coprod_{i=1}^r \tilde{L}_i$ and as $\tilde{P}$ is projective, we have only to show that $A\mathrm{Modgr}(\tilde{P}, \mathrm{radgr}\tilde{P}) = 0$. Just suppose not, and let $f \in A\mathrm{Modgr}(\tilde{P}, \mathrm{radgr}\tilde{P})\backslash 0$. As $\cap_{n\in\mathbb{N}}\mathrm{radgr}^n(\tilde{P}) = 0$ by 金.2.(ii), $\mathrm{im}f \not\subseteq \mathrm{radgr}^n(\tilde{P})$ for some $n \geq 2$. Take minimal such n. Then $\mathrm{im}f \subseteq \mathrm{radgr}^{n-1}(\tilde{P})$, and hence f induces by 金.3

$$
\begin{tikzcd}
\tilde{P} \arrow[r] \arrow[dr, dotted, "\neq 0"'] & \mathrm{radgr}^{n-1}\tilde{P} \arrow[d] \\
& \mathrm{radgr}^{n-1}\tilde{P}/\mathrm{radgr}^n\tilde{P} = \coprod_i \tilde{L}_i\langle n-1\rangle^{\oplus m(i,n-1)}.
\end{tikzcd}
$$

Then $n-1=0$ by 金.3(3), absurd.

(ii) One has

$$
\begin{aligned}
\mathrm{Ext}^n_{B\mathrm{gr}}(B_0, B_0\langle m\rangle) &= \mathrm{Ext}^n_{B,\mathrm{gr}}(\coprod_i \tilde{S}_i, \coprod_i \tilde{S}_i\langle m\rangle) \quad \text{by (i)} \\
&\simeq \mathrm{Ext}^n_{A,\mathrm{gr}}(\tilde{L}, \tilde{L}\langle m\rangle) \quad \text{by the equivalence} \\
&= 0 \quad \text{unless } n = m \text{ by the standing hypothesis.}
\end{aligned}
$$

(iii) Let $f \in B_k \setminus 0$. We argue as in (i). As $\cap_{n\in\mathbb{N}}\mathrm{radgr}^n(\tilde{P}) = 0$, $\mathrm{im}f \not\subseteq \mathrm{radgr}^n\tilde{P}$ for some $n \geq 1$. Take minimal such n. Then $\mathrm{im}f \subseteq \mathrm{radgr}^{n-1}(\tilde{P})$, and hence f induces by 金.3

$$
\begin{tikzcd}
\tilde{P}\langle k\rangle \arrow[r] \arrow[dr, dotted, "\neq 0"'] & \mathrm{radgr}^{n-1}\tilde{P} \arrow[d] \\
& \mathrm{radgr}^{n-1}\tilde{P}/\mathrm{radgr}^n\tilde{P} = \coprod_i \tilde{L}_i\langle n-1\rangle^{\oplus m(i,n-1)}.
\end{tikzcd}
$$

Then $n-1-k=0$ by 金.3.(3), and hence $k=n-1\geq 0$.

金.5. *Unicity of Koszul gradings*

We show finally

Theorem: *Let A be a finite dimensional $\Bbbk$-algebra. A Koszul grading on A, if any, is unique; if $A=\coprod_{i\in\mathbb{N}}A_i=\coprod_{i\in\mathbb{N}}A'_i$ are 2 Koszul gradings on A, there is $\sigma\in\mathbf{Alg}_{\Bbbk}(A,A)^{\times}$ such that $\sigma(A_i)=A'_i$ $\forall i$.*

金.6. Throughout the rest let $A=\coprod_{i\in\mathbb{N}}A_i$ denote a positively graded $\Bbbk$-algebra.

Proposition: *If A is generated by A_1 as A_0-ring, $\forall j\in\mathbb{N}$, $(A_{>0})^j=\coprod_{n\geq j}A_n$; we do not assume A_0 is central in A. If, in addition, A is is finite dimensional over $\Bbbk$ with A_0 semisimple, $A_{>0}=\mathrm{rad}A$, and hence $A\simeq\coprod_{i\in\mathbb{N}}(\mathrm{rad}^iA/\mathrm{rad}^{i+1}A)$ as graded $\Bbbk$-algebras.*

Proof: Put $I=A_{>0}$. By the hypothesis we must have $I^n=\coprod_{i\geq n}A_i$ $\forall n\in\mathbb{N}$. Then $I^n/I^{n+1}\simeq A_n$.

Assume now that A is finite dimensional over $\Bbbk$. Then $I^n=0$ $\forall n\gg 0$, and hence $I\subseteq\mathrm{rad}A$ [AF, 15.19, p. 172]. On the other hand, $A/I\simeq A_0$ is semisimple, and hence $\mathrm{rad}A\subseteq I$.

金.7. We finish the proof of Th. 金.5. As a Koszul grading on A guarantees the semisimplicity of A_0, we have by 金.6 only to show that A is generated by A_1 over A_0. For that it is enough to show that

(1) $$A_{>0}=AA_1\quad\text{left ideal of } A \text{ generated by } A_1.$$

Indeed, (1) will yield

$$\begin{aligned}A&=A_0+A_{>0}=A_0+(A_0+A_{>0})A_1=A_0+A_0A_1+A_{>0}A_1=A_0+A_0A_1+AA_1^2\\&=A_0+A_0A_1+A_0A_1^2+A_{>0}A_1^2=A_0+A_0A_1+A_0A_1^2+A_0A_1^3+\dots.\end{aligned}$$

Put $I=A_{>0}$. We claim

(2) $$I=AA_1\quad\text{iff}\quad\forall i\in\mathbb{Z}\setminus\{1\}, A\mathrm{Modgr}(I,A_0\langle i\rangle)=0.$$

"only if" Let $f\in A\mathrm{Modgr}(I,A_0\langle i\rangle)\setminus 0$. Then $0\neq f(AA_1)=Af(A_1)\subseteq(A_0\langle i\rangle)_1$, and hence $i=1$. "if" Just suppose not. There is $n>1$ with $I_n>(AA_1)_n$, and let $s>1$ be minimal such. Then one has graded A-linear maps

$$\begin{tikzcd} I \arrow[r] \arrow[dr, dotted, "0\neq"'] & I/AA_1 \arrow[d] \\ & I/(AA_1+I_{>s}) \end{tikzcd}$$

with $A_{>0}$ annihilating $I/(AA_1+I_{>s})$. As A_0 is semisimple, $A\mathrm{Modgr}(I/(AA_1+I_{>s}),A_0\langle s\rangle)\neq 0$, yielding a nonzero graded$A$-linear map $I\to A_0\langle s\rangle$, absurd.

Now let $i \neq 1$. Consider an exact sequence $0 \to A_{>0} \to A \to A_0 \to 0$ of graded A-linear modules. As $A_{>0}$ annihilates $A_0\langle i\rangle$, it induces $A\mathrm{Modgr}(A_0, A_0\langle i\rangle) \simeq A\mathrm{Modgr}(A, A_0\langle i\rangle)$, and an LES

$$0 \to A\mathrm{Modgr}(A_0, A_0\langle i\rangle) \xrightarrow{\sim} A\mathrm{Modgr}(A, A_0\langle i\rangle) \to A\mathrm{Modgr}(A_{>0}, A_0\langle i\rangle) \to \mathrm{Ext}^1_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle)$$

with $\mathrm{Ext}^1_{A,\mathrm{gr}}(A_0, A_0\langle i\rangle) = 0$ as A is Koszul. Thus, $A\mathrm{Modgr}(A_{>0}, A_0\langle i\rangle) = 0$, as desired.

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