- Notation

$$\mu, \nu \cdots = 0, 1, 2, 3$$

$$i, j \cdots = 1, 2, 3$$

$$g_{\mu\nu} = (+, -, -, -)$$

- Unit

$$c = \hbar = k_B = 1$$

Inflationary Universe

1. Problems in Standard Model

Standard Model

- Cosmological Principle
  - The universe is spatially homogeneous.
  - The universe is isotropic.

→ Robertson-Walker Metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

$a(t)$ scale factor

$$\begin{cases} K > 0 & \text{closed universe} \\ K = 0 & \text{flat universe} \\ K < 0 & \text{open universe} \end{cases}$$

Cosmological Principle

Energy-Momentum Tensor = Perfect Fluid Form

$$\boxed{T^{\mu\nu} = -pg^{\mu\nu} + (\rho + p)u^\mu u^\nu}$$

$u^0 = 1, \ u^i = 0$

Einstein Equation

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}}$$

Λ Cosmological Constant

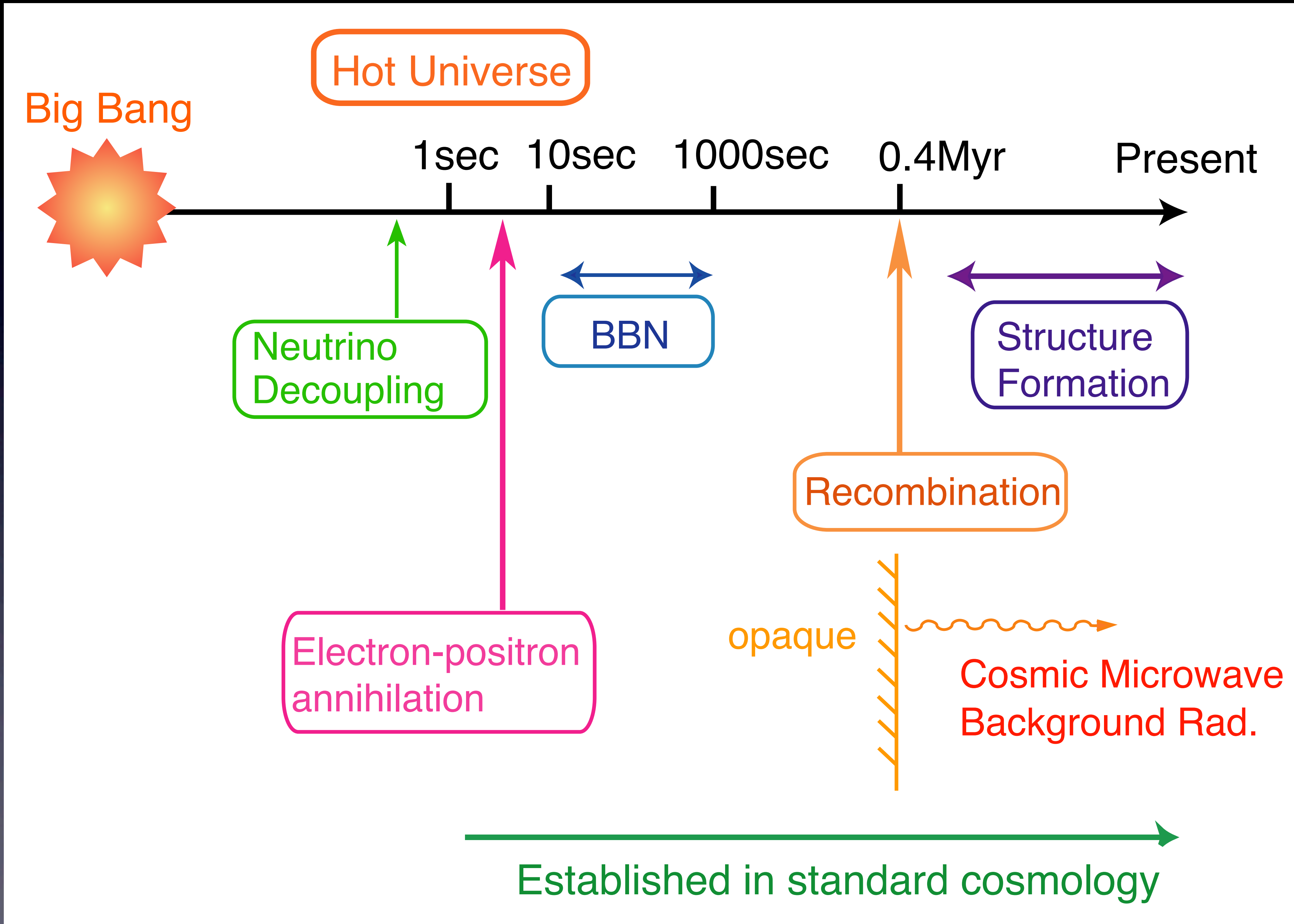
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a + \frac{\Lambda}{3}a$$

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

two independent equations

History of the Universe



1.1 Flatness Problem

Friedmann Equation $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\begin{cases} H = \frac{\dot{a}}{a} \\ \Omega = \frac{8\pi G\rho}{3H^2} \\ \lambda = \frac{\Lambda}{3H^2} \end{cases} \Rightarrow H^2 = \Omega H^2 - \frac{K}{a^2} + \lambda H^2$$

$$\boxed{\Omega + \lambda - 1 = \frac{K}{a^2H^2}}$$

$T_{eq} \sim 1\mathrm{eV} = 10^{-9}\mathrm{GeV}$

$$\Omega + \lambda - 1 = \frac{K}{a^2H^2} \propto \begin{cases} T^{-2} \quad (RD) \quad a \propto 1/T, \quad H \propto \rho^{1/2} \propto T^2 \\ T^{-1} \quad (MD) \quad a \propto 1/T, \quad H \propto \rho^{1/2} \propto T^{3/2} \end{cases}$$

At present

$$\Omega_0 \sim O(1) \quad \lambda \sim O(1) \qquad T_0 \simeq 2.7\mathrm{K} \simeq 3 \times 10^{-4}\mathrm{eV}$$

At Planck time $T \sim 10^{19}\mathrm{GeV}$

$$\Omega + \lambda - 1 \sim \left(\frac{10^{19}}{10^{-9}}\right)^{-2} \left(\frac{10^{-9}}{10^{-13}}\right)^{-1} \sim 10^{-60}$$

$\Rightarrow$ $\Omega + \lambda = 1$ with accuracy 10^{-60}

unnatural fine tuning !

$\Rightarrow$ $\boxed{\text{Flatness Problem}}$

Entropy of the Universe

$$S = a^3 s \quad \Rightarrow \quad a = (S/s)^{1/3}$$

$$\boxed{S = \left[\frac{K}{H^2(\Omega + \lambda - 1)}\right]^{3/2} s}$$

$K = 0, \pm 1 \quad {}^{(3)}R = \pm 1/a^2$

Re-definition of r, a

Entropy of the Universe

Thermodynamics $$dS(V,T) = \frac{1}{T}d(\rho V) + \frac{p}{T}dV$$

$\rho(T),\ p(T)$ function of T

$$\Rightarrow \quad \boxed{S(V,T) = \frac{V}{T}(\rho(T) + p(T))}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

The universe expands adiabatically

Einstein eq. $$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

$$dS = \frac{1}{T}d(\rho a^3) + \frac{p}{T}d(a^3) \quad \Rightarrow \quad \frac{dS}{dt} = 0 \Rightarrow S \text{ const}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

Relativistic particle in thermal equilibrium

$$\rho = \frac{\pi^2}{30} g T^4 \qquad p = \frac{1}{3}\rho$$

entropy density

$$s = \frac{\pi^2}{30} g \left(1 + \frac{1}{3}\right) T^3 = \frac{2\pi^2}{45} g T^3$$

$$S = a^3 s = \mathrm{const} \;\Rightarrow\; T^3 a^3 = \mathrm{const}$$

$$\boxed{T \propto a^{-1}}$$

$$S = \left[\frac{K}{H^2(\Omega + \lambda - 1)}\right]^{3/2} s$$

At Present

$$\begin{aligned} s_0 &= \frac{2\pi}{45}\left(2T_{\gamma,0}^3 + 2\frac{7}{8}3T_{\nu,0}^3\right) = \frac{2\pi}{45}\left(2T_{\gamma,0}^3 + 2\frac{7}{8}3\frac{4}{11}T_{\gamma,0}^3\right) \\ &= 1.715T_{\gamma,0}^3 = 2.8 \times 10^3\ \mathrm{cm}^{-3} \end{aligned}$$

$$H_0^{-1} \simeq 3000\ \mathrm{Mpc} \sim 10^{28}\ \mathrm{cm}$$

$\Rightarrow$ $S \gtrsim 10^{87}$

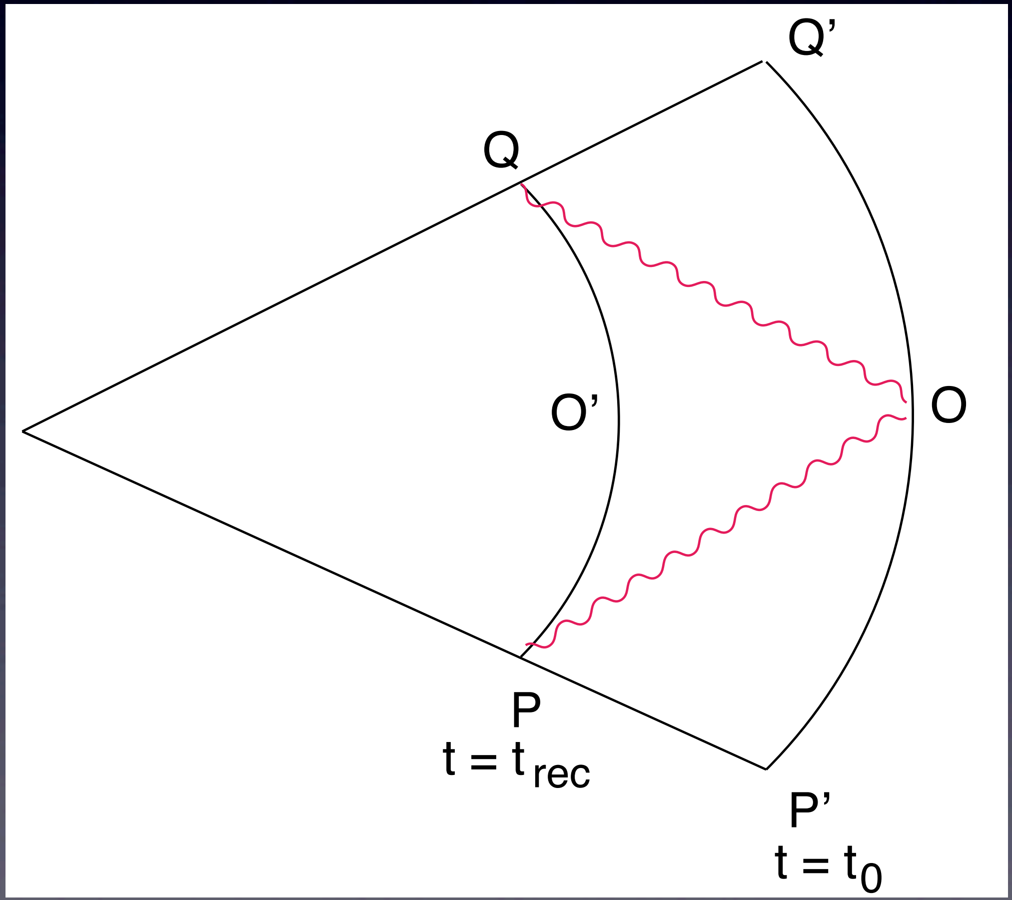
Large Entropy Problem

1.2 Horizon Problem

CMB is extremely isotropic

emitted at recombination (T ~ 3000K)

Observer at present sees CMB coming from two opposite directions



distance between P and O

$$\mathrm{PO}'(t_{rec}) = a(t_{rec}) \int_{t_{rec}}^{t_0} \frac{dt'}{a(t')}$$

$$= bt^{2/3} \int_{t_{rec}}^{t_0} \frac{dt'}{bt'^{2/3}}$$

$$= 3(t_{rec}^{2/3} t_0^{1/3} - t_{rec})$$

$$(a \propto t^{2/3} \quad MD)$$

Horizon

- Particle Horizon
  maximum travel distance of light emitted at t=0 until t

Geodesics of light $$ds^2 = 0 = dt^2 - a^2(t)\frac{dr^2}{1-Kr^2}$$

$$\boxed{\ell_H(t) \equiv a(t)\int_0^t \frac{dt'}{a(t')}}$$

$$a(t) \propto t^m \quad \begin{cases} m = 1/2 & (RD) \\ \quad = 2/3 & (MD) \end{cases}$$

$$\Rightarrow \quad \ell_H(t) = \frac{t}{1-m} = \begin{cases} 2t & (RD) \\ 3t & (MD) \end{cases}$$

Particle horizon at recombination

$$\ell_H(t_{rec}) = bt_{rec}^{2/3} \int_0^{t_{rec}} \frac{dt'}{bt'^{2/3}} = 3t_{rec}$$

$$N = \frac{PQ}{\ell_H} = 2\left[\left(\frac{t_0}{t_{rec}}\right)^{1/3} - 1\right]$$

$$= 2\left[\left(\frac{T_{rec}}{T_0}\right)^{1/2} - 1\right] = 2\left[\left(\frac{3000\mathrm{K}}{2.7\mathrm{K}}\right)^{1/2} - 1\right] \simeq 74$$

➡ P and Q are far away and have no causal relation

unnatural ➡ $\boxed{\text{Horizon Problem}}$

1.3 Monopole Problem

GUTs (Grand Unified Theories)

Interaction among elementary particles can be described by one gauge interaction with symmetry represented by group G

$$G \;\Rightarrow\; SU_c(3) \times SU_L(2) \times U_Y(1)$$

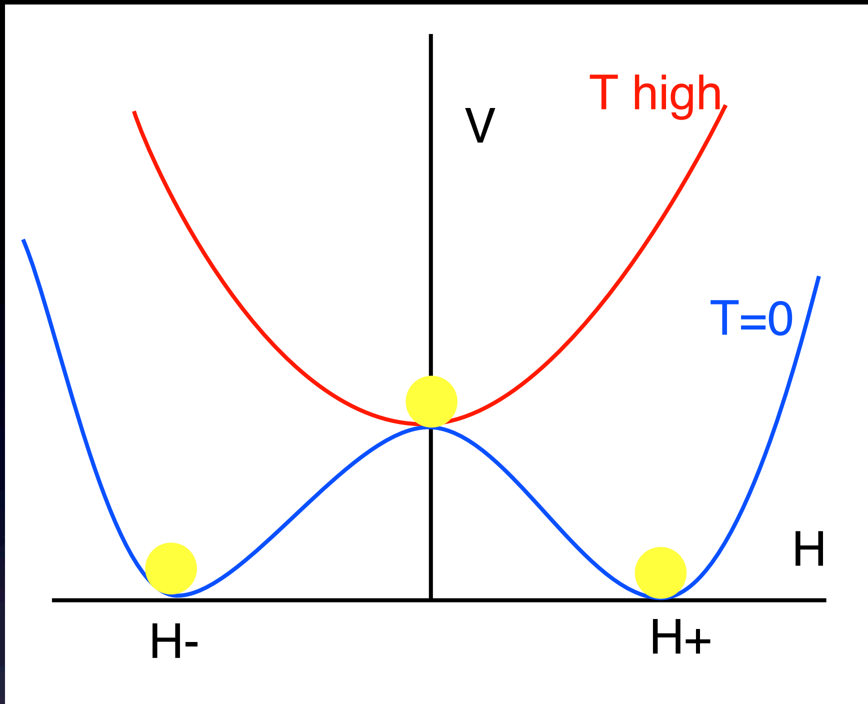
$\uparrow$

Higgs Mechanism   H: Higgs field $\langle H \rangle = 0 \;\rightarrow \langle H \rangle \neq 0$

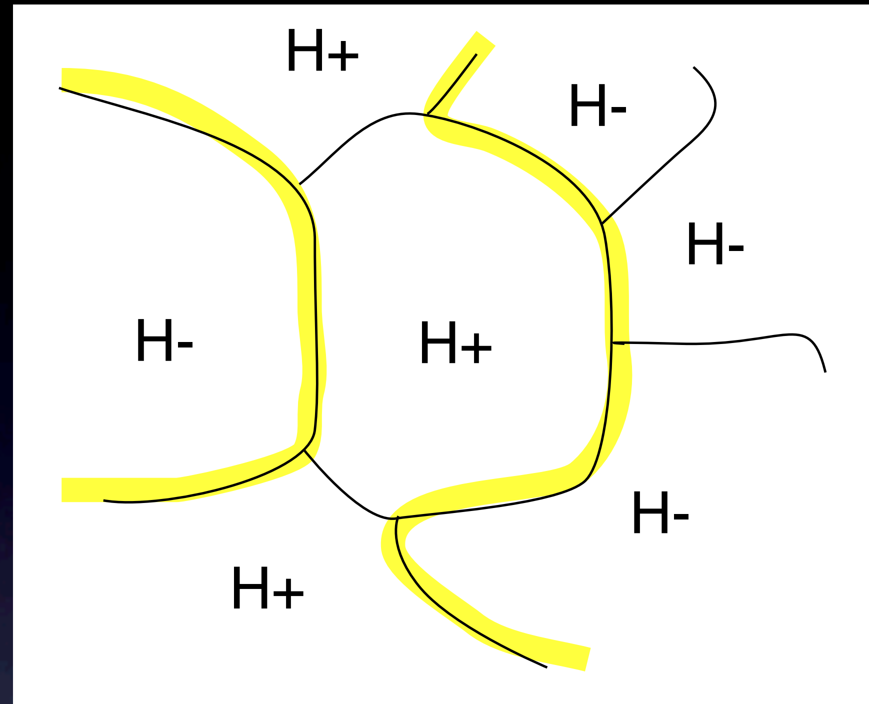
When symmetry G is spontaneously broken to U(1) , topological defects (monopoles) are produced (Kibble Mechanism)

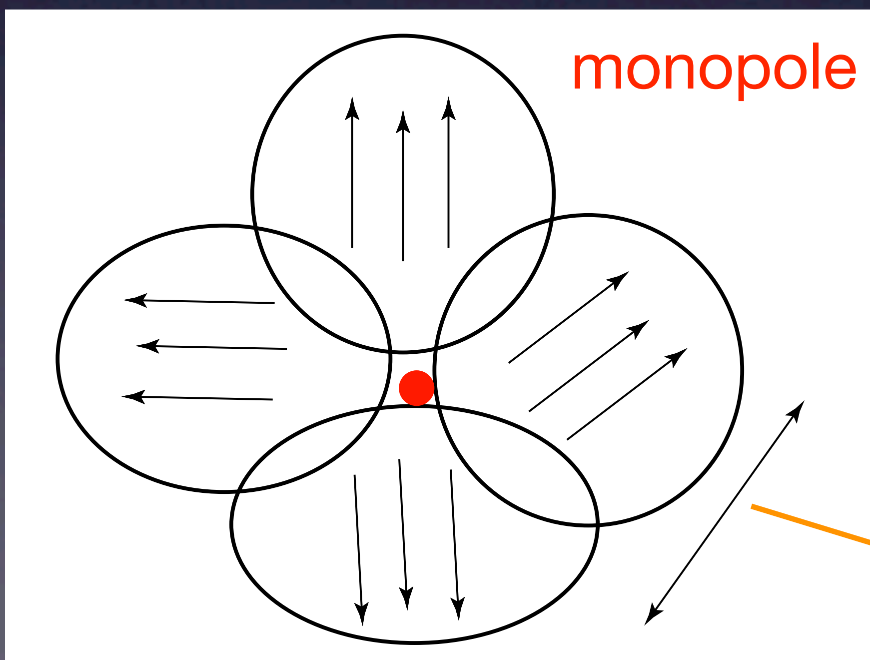
Topological Defect
- Domain Wall (2dim)
- Cosmic String (1dim)
- Monopole (0dim)

Example (Domain Wall)



Domain Wall





coherent length

Monopole density at formation epoch t_f

coherent length $\xi < \ell_H = 2t_f$

$$\Rightarrow \quad n_M = \xi^{-3} \geq \ell_H^{-3} = \frac{1}{8t_f^3} \qquad \leftarrow \quad \propto a^{-3}$$

Entropy density

$$s = \frac{2\pi^2}{45} g_* T_f^3 \qquad \leftarrow \quad \propto a^{-3}$$

$$\frac{n_M}{s} \geq \frac{1}{8t_f^3}\left(\frac{2\pi^2}{45} g_* T_f^3\right)^{-1} = \frac{1}{8}\left(\frac{2\pi^2 g_*}{45}\right)^{3/2} \frac{T_f^6}{M_G^3}\left(\frac{2\pi^2}{45} g_* T_f^3\right)^{-1}$$

$$= \frac{\pi}{4\sqrt{2}\sqrt{45}} g_*^{1/2} \frac{T_f^3}{M_G^3} \simeq 0.8 \left(\frac{T_f}{M_G}\right)^3$$

$$T_f = 10^{15}\ \mathrm{GeV} \quad \Rightarrow \quad \frac{n_M}{s} \geq 0.8 \left(\frac{10^{15}}{2.4 \times 10^{18}}\right)^3 \simeq 6 \times 10^{-11}$$

$$\left(\frac{n_M}{s}\right)_0 = \left(\frac{n_M}{s}\right)_f \qquad s_0 = 2.8 \times 10^3 \ \mathrm{cm}^{-3}$$

Monopole mass

$$m_M \simeq 10^{16} \ \mathrm{GeV}$$

Monopole mass density

$$\rho_M \geq 1.6 \times 10^9 \ \mathrm{GeVcm}^{-3}$$

$$\rho_c = 1.053 \times 10^{-5} h^2 \ \mathrm{GeVcm}^{-3}$$

$$\boxed{\Omega_M \geq 1.6 \times 10^{14} \gg 1}$$

$\Rightarrow$ Monopole Problem

1.4 Gravitino Problem

Supersymmetry (SUSY)

Fermion $\longleftrightarrow$ Boson

- Hierarchy Problem
  Keep electroweak scale against radiative correction
- Coupling Constant Unification in GUT

quark $\longleftrightarrow$ squarks
lepton $\longleftrightarrow$ slepton
photon $\longleftrightarrow$ photino

Gravitino $\psi_{3/2}$

Superpartner of graviton

mass of gravitino ~ O(100) GeV

Gravitinos are expected to be in thermal equilibrium at Planck time

$$\Rightarrow \quad n_{3/2} \sim n_\gamma$$

Lifetime of gravitino

$$\tau(\psi_{3/2} \to \tilde{\gamma} + \gamma) \simeq 4 \times 10^8 \,\mathrm{sec} \left(\frac{m_{3/2}}{100\mathrm{GeV}} \right)^{-3}$$

Entropy production

$$\frac{\rho_{3/2}}{\rho_\gamma} \sim \frac{m_{3/2} n_{3/2}}{T n_\gamma} \sim \frac{m_{3/2}}{T} \gg 1 \qquad \left(\leftarrow \begin{array}{l} t \sim 10^8 \,\mathrm{sec} \\ T \sim 100 \,\mathrm{eV} \end{array} \right)$$

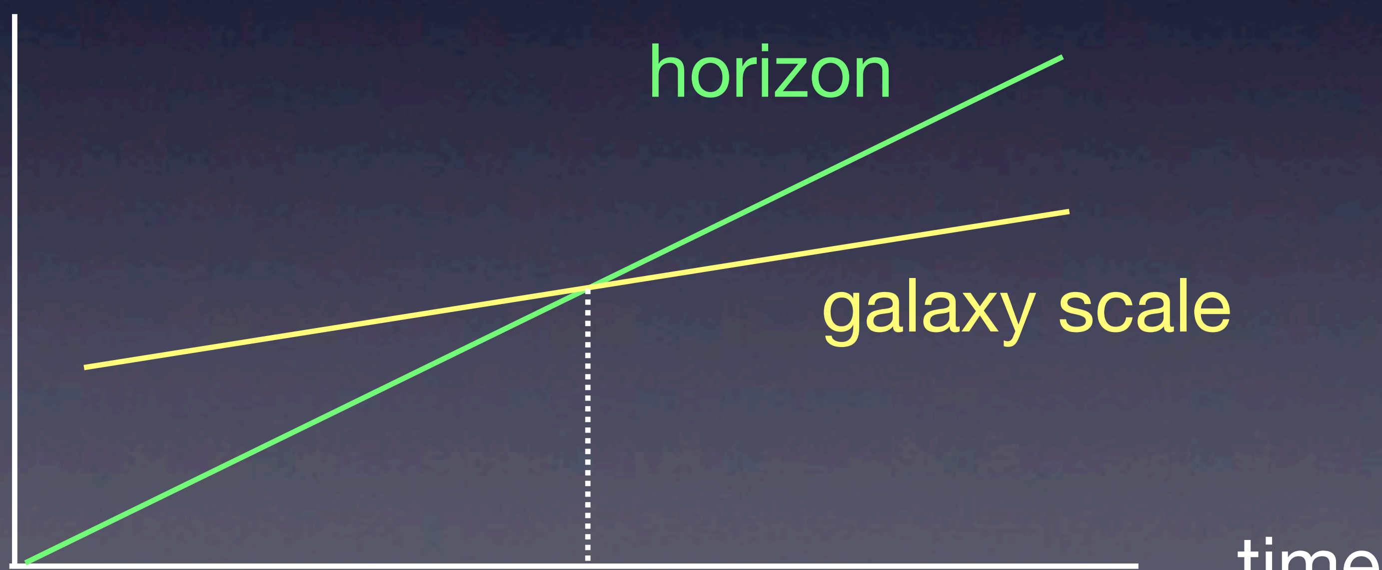
➡ Huge entropy production

➡ Too much dilute baryon density $\quad \mathrm{BBN} \Rightarrow \dfrac{n_B}{n_\gamma} \sim 10^{-10}$

$\boxed{\text{Gravitino Problem}}$

1.5 Density Fluctuation Problem

- Structures of the Universe (galaxies...) are formed from small density fluctuations through gravitational instability
- However, no mechanism to produce the fluctuations in the standard model is found



$t_{gal} \simeq 2 \times 10^9\,\mathrm{sec}$

2. Success of Inflation Model

2.1 Inflationary Universe

For some reason vacuum energy dominates the universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_v = H^2 \qquad \rho_v : \ \text{vacuum energy} \sim \text{const}$$

$$\Rightarrow \begin{cases} a(t) & \propto & \exp(Ht) \\ H & = & \left[\frac{8\pi G}{3}\rho_v\right]^{1/2} \end{cases}$$

Exponential expansion

Inflation Model

- Exponential (accelerated) expansion
- End of exponential expansion and release of vacuum energy into radiation

- Vacuum energy $p_v = -\rho_v$
  - Energy-momentum tensor

$$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu = \rho_v g_{\mu\nu}$$

  - Einstein equation

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3p) a = \frac{8\pi}{3} G \rho_v a$$

accelerated expansion

  - energy conservation?

$$T_{\mu\nu} = \rho_v g_{\mu\nu} \qquad T_{\mu\,;\nu}^{\nu} = \rho_v g_{\mu\,;\nu}^{\nu} = 0$$

(vacuum energy) + (gravitaional energy) = const.

Inflationary Universe

K. Sato (1981)

Mon. Not. R. astr. Soc. (1981) **195**, 467–479

First-order phase transition of a vacuum and the expansion of the Universe

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Received 1980 September 9; in original form 1980 February 21

In abstract

The following results are obtained: (1) If the nucleation rates are small and the vacuum stays at the metastable state for a long time, the Universe begins to expand exponentially. As a result, the progress of the phase transition is

Inflationary Universe

A. Guth (1981)

PHYSICAL REVIEW D VOLUME 23, NUMBER 2 RY 19

Inflationary universe: A possible solution to the horizon and flatness problems

Alan H. Guth*

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(Received 11 August 1980)

The standard model of hot big-bang cosmology requires initial conditions which are problematic in two ways: (1) The early universe is assumed to be highly homogeneous, in spite of the fact that separated regions were causally disconnected (horizon problem); and (2) the initial value of the Hubble constant must be fine tuned to extraordinary accuracy to produce a universe as flat (i.e., near critical mass density) as the one we see today (flatness problem). These problems would disappear if, in its early history, the universe supercooled to temperatures 28 or more orders of magnitude below the critical temperature for some phase transition. A huge expansion factor would then result from a period of exponential growth, and the entropy of the universe would be multiplied by a huge factor when the latent heat is released. Such a scenario is completely natural in the context of grand unified models of elementary-particle interactions. In such models, the supercooling is also relevant to the problem of monopole suppression. Unfortunately, the scenario seems to lead to some unacceptable consequences, so modifications must be sought.

Phase Transitions and Magnetic Monopole Production in the Very Early Universe

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(Received 26 December 1979)

Nuclear Physics B180 [FS2] (1981) 385–404

MONOPOLE PRODUCTION IN THE VERY EARLY UNIVERSE IN A FIRST-ORDER PHASE TRANSITION

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Received 30 July 1980
(Revised 10 November 1980)

2.2 Flatness Problem

During inflation the scale factor increases by

$$\boxed{\frac{a_f}{a_i} = \exp[H(t_f - t_i)] \equiv Z}$$

$$t = t_i \Rightarrow \quad \rho_{R,i} \sim \rho_v$$

$$t = t_f \Rightarrow \quad \rho_{R,f} \sim \rho_v$$

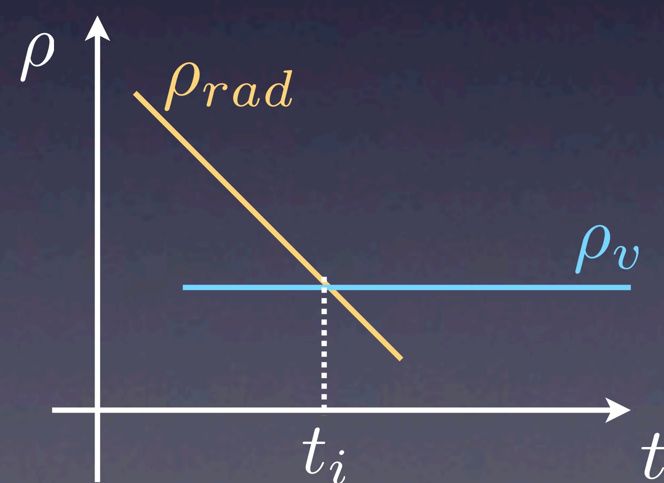
$\Rightarrow$ $\boxed{T_i = T_R}$

T_R reheating temp

$\Rightarrow$ entropy density $\quad s(t_i) = s(t_f)$

scale factor $\quad Za(t_i) = a(t_f)$

total entropy $\quad Z^3 S(t_i) = S(t_f)$



$$Z \gtrsim 10^{29} \quad \Rightarrow \quad S(t_f) \gtrsim 10^{87}$$

$$H\Delta t \gtrsim 67$$

even if S(ti)~1

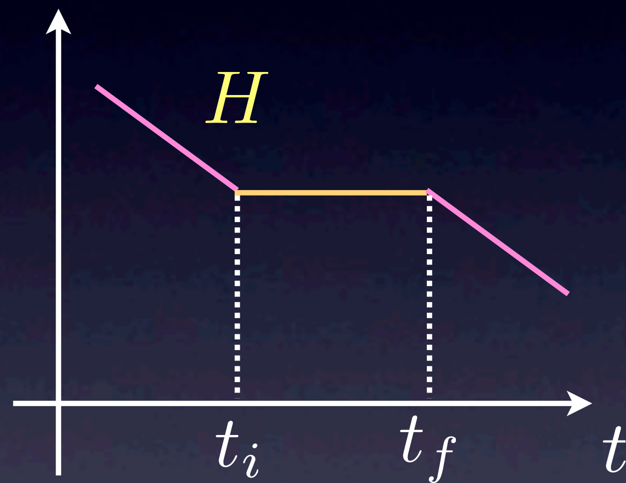
2.3 Horizon Problem

Horizon length just before inflation

$$\ell_H(t_i) \simeq 2t_i$$

Just after inflation

$$\ell_H(t_f) \simeq 2t_i Z$$



At present the scale corresponding the above horizon

$$L \simeq \frac{a(t_0)}{a(t_f)} 2t_i Z$$

$$t_i \sim H^{-1} \sim \frac{M_G}{\rho_v^{1/2}} \sim \frac{10^{18}\mathrm{GeV}}{(10^{16}\mathrm{GeV})^2} \simeq (10^{14}\mathrm{GeV})^{-1} \simeq 2\times 10^{-28}\mathrm{cm}$$

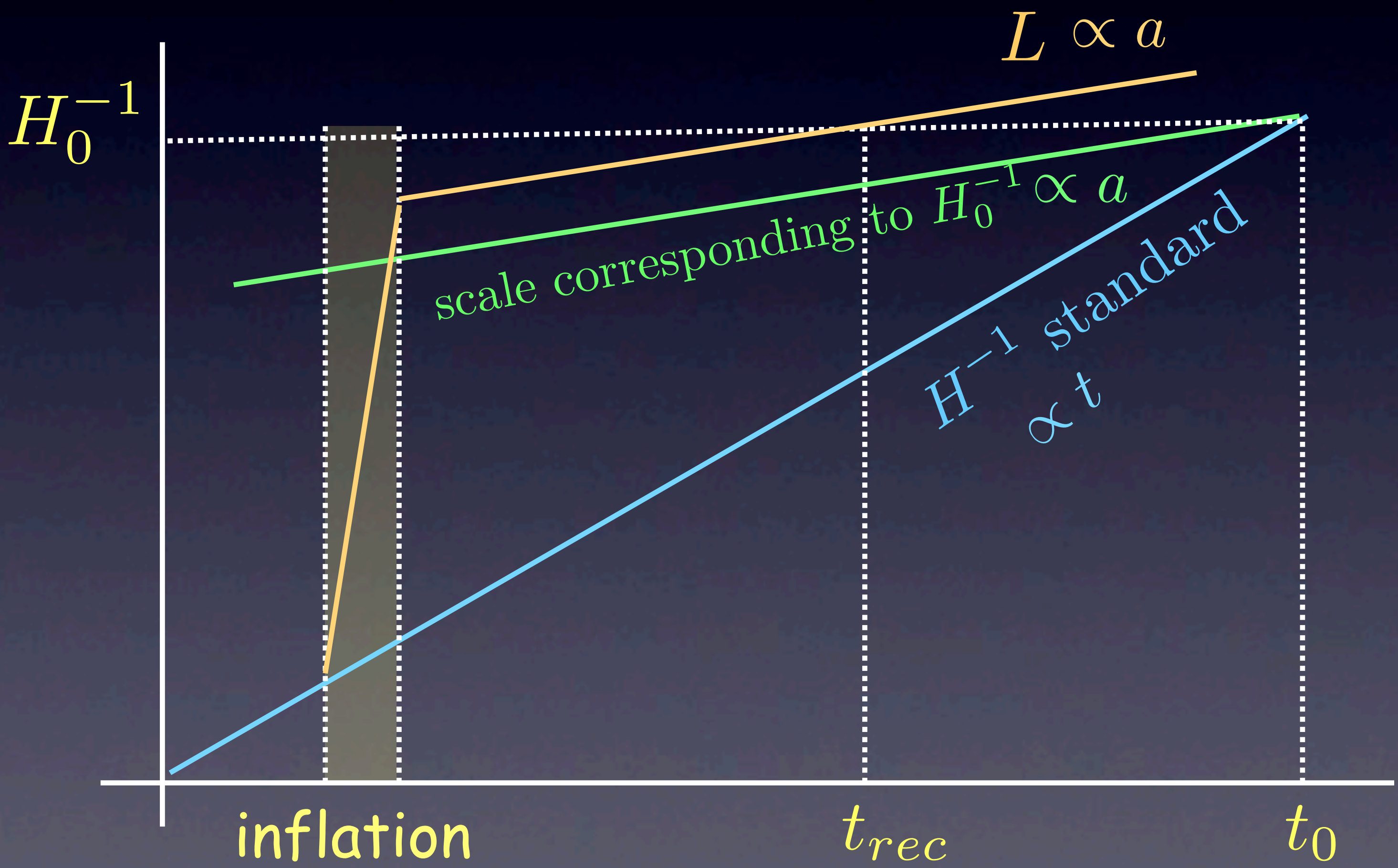
$$T_R \sim 10^{16}\mathrm{GeV} \quad \rightarrow \quad \frac{a(t_0)}{a(t_f)} \sim \left(\frac{10^{-4}\mathrm{eV}}{10^{16}\mathrm{GeV}}\right)^{-1} \sim 10^{29}$$

$$\rightarrow \quad L \sim 40Z \text{ cm} \gtrsim H_0^{-1} \sim 10^{28} \text{ cm} \quad \rightarrow \quad Z \gtrsim 10^{26}$$

size of the observable universe

$$= a(t_0) \int_{t_{rec}}^{t_0} \frac{dt}{a(t)} \sim H_0^{-1}$$

$$H\Delta t \gtrsim 60$$



- Particle Horizon at t=t$_f$

$$\ell_H(t_f) = a(t_f)\int_0^{t_f}\frac{dt'}{a(t')} = a(t_f)\left[\int_0^{t_i}\frac{dt'}{a(t')} + \int_{t_i}^{t_f}\frac{dt'}{a(t')}\right]$$

$$a(t) = \begin{cases} a(t_i)(t/t_i)^{1/2} & t < t_i \\ a(t_i)e^{H(t-t_i)} & t \geq t_i \end{cases}$$

$$\begin{aligned} \ell_H(t_f) &= a(t_f)\left[2\frac{t_i}{a}(t_i) + \frac{1}{a(t_i)H}\left(1 - e^{-H(t_f - t_i)}\right)\right] \\ &\simeq 4t_i\frac{a(t_f)}{a(t_i)} \sim e^{H\delta t}t_i \end{aligned}$$

$$H = 1/2t_i$$

- Particle Horizon at t$_0$

$$\begin{aligned} \ell_H(t_f) &= a(t_0)\left[\int_0^{t_i}\frac{dt'}{a(t')} + \int_{t_i}^{t_f}\frac{dt'}{a(t')} + \int_{t_f}^{t_0}\frac{dt'}{a(t')}\right] \\ &\simeq \frac{a(t_0)}{a(t_f)}e^{H\Delta t}t_i + 3t_0 \end{aligned}$$

2.5 Other Problems

Monopole

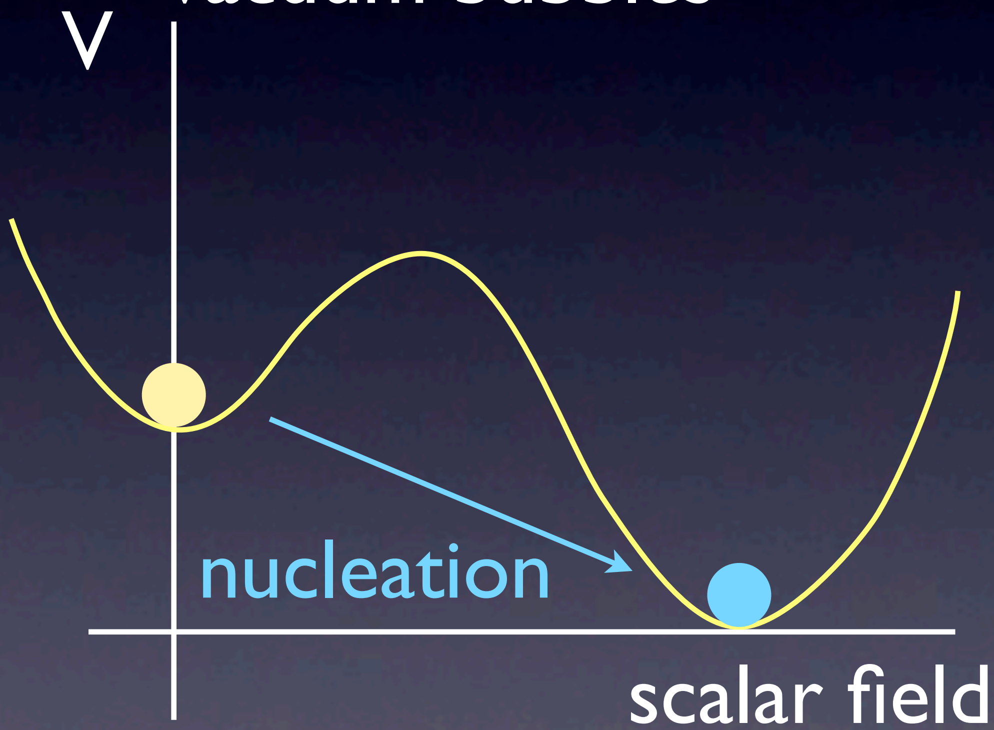
Gravitino

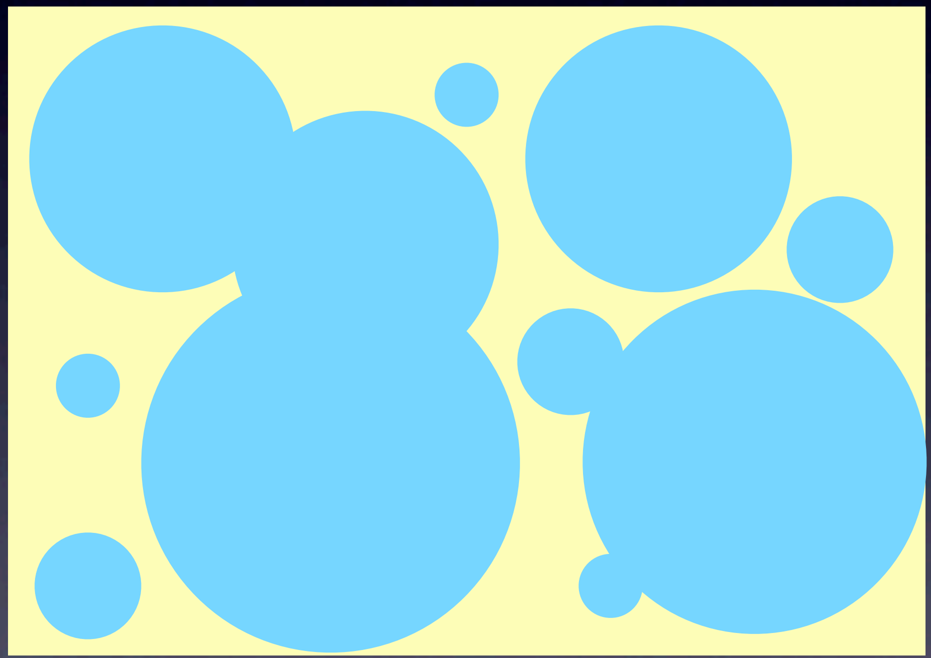
→ Diluted by $Z^{-3} \gtrsim 10^{-87}$

Density Fluctuations ← Inflation has mechanism for generating density fluctuations

Old (original) Inflation Model

- Inflation takes place in the false vacuum
- Inflation ends through production of true vacuum bubbles





Graceful Exit Problem

inflation never ends

Inflationary Universe

A. Guth (1981)

PHYSICAL REVIEW D VOLUME 23, NUMBER 2 RY 19

Inflationary universe: A possible solution to the horizon and flatness problems

Alan H. Guth*

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(Received 11 August 1980)

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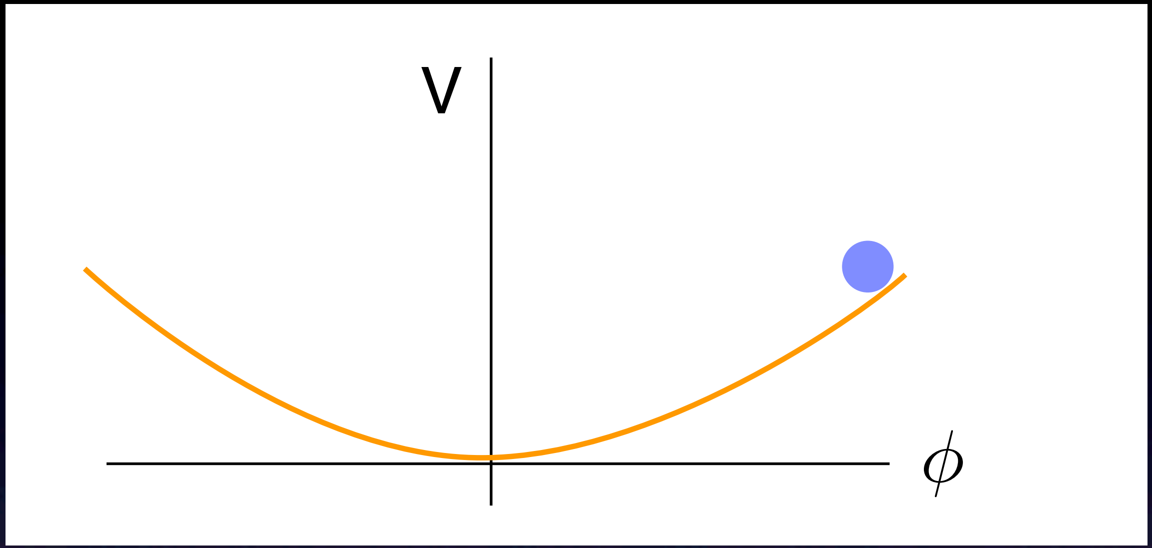
3. Chaotic Inflation Model



$$\boxed{\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \\ V(\phi) &= \frac{1}{2}m^2\phi^2 \end{aligned}}$$

In general $V(\phi) = \dfrac{\lambda\phi^n}{nM_G^{n-4}}$ $\quad 0 < \lambda \ll 1$

3.1 Chaotic Condition

Initial Condition? $t = t_{pl}$

Heisenberg Uncertainty Principle $\Delta E \; \Delta t \gtrsim 1$

Energy density

$$\Delta\rho \sim \Delta E/(\Delta x)^3 \gtrsim 1/(\Delta t)(\Delta x)^3 \gtrsim M_{pl}^4$$

$$\Delta x \lesssim \ell_H \sim M_{pl}^{-1}$$

$$\Delta t \lesssim t_{pl} \sim M_{pl}^{-1}$$

→ Energy density is determined only with accuracy O(M_{pl})

$$\Rightarrow \begin{cases} \partial_0\phi\partial_0\phi & \sim \quad M_{pl}^4 \\ \partial_i\phi\partial_i\phi & \sim \quad M_{pl}^4 \\ V(\phi) & \sim \quad M_{pl}^4 \end{cases}$$

← Chaotic Condition of the Early Universe

3.2 Dynamics of Inflaton

initial value of Φ is large

$$m^2\phi_i^2 \sim M_{pl}^4 \quad m \ll M_{pl} \quad \Rightarrow \quad \phi_i \sim M_{pl}^2/m \gg M_{pl}$$

Suppose, in some region

$$(\partial_0\phi)^2, (\partial_i\phi)^2 \ < \ V(\phi) \sim M_{pl}^4$$

Friedmann eq.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_G^2}\left(\frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2a^2} + V(\phi)\right)}$$

Evolution of inflaton $\quad S = \int \sqrt{-g}\mathcal{L}d^4x = \int a^3\mathcal{L}d^4x$

$$\boxed{\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\triangle\phi = -\frac{dV}{d\phi}}$$

$$\dot\phi^2, (\nabla\phi)^2 \ll V,\ \ddot\phi \ll \frac{dV}{d\phi}$$

slow roll approximation

$$\Rightarrow \begin{cases} \left(\frac{\dot a}{a}\right)^2 = \frac{1}{3M_G^2} V(\phi) \\ 3\frac{\dot a}{a}\dot\phi = -\frac{dV}{d\phi} \end{cases}$$

$$\frac{\dot a}{a} = \frac{V(\phi)^{1/2}}{\sqrt{3}M_G} \qquad 3\dot\phi\frac{\dot a}{a} = \frac{\sqrt{3}V(\phi)^{1/2}}{M_G}\dot\phi = -\frac{dV}{d\phi}$$

$$\boxed{V = m^2\phi^2/2} \Rightarrow \sqrt{3}\frac{1}{M_G}\frac{m}{\sqrt{2}}\phi\dot\phi = -m^2\phi$$

$$\dot\phi = -\frac{\sqrt{2}}{\sqrt{3}}M_G m$$

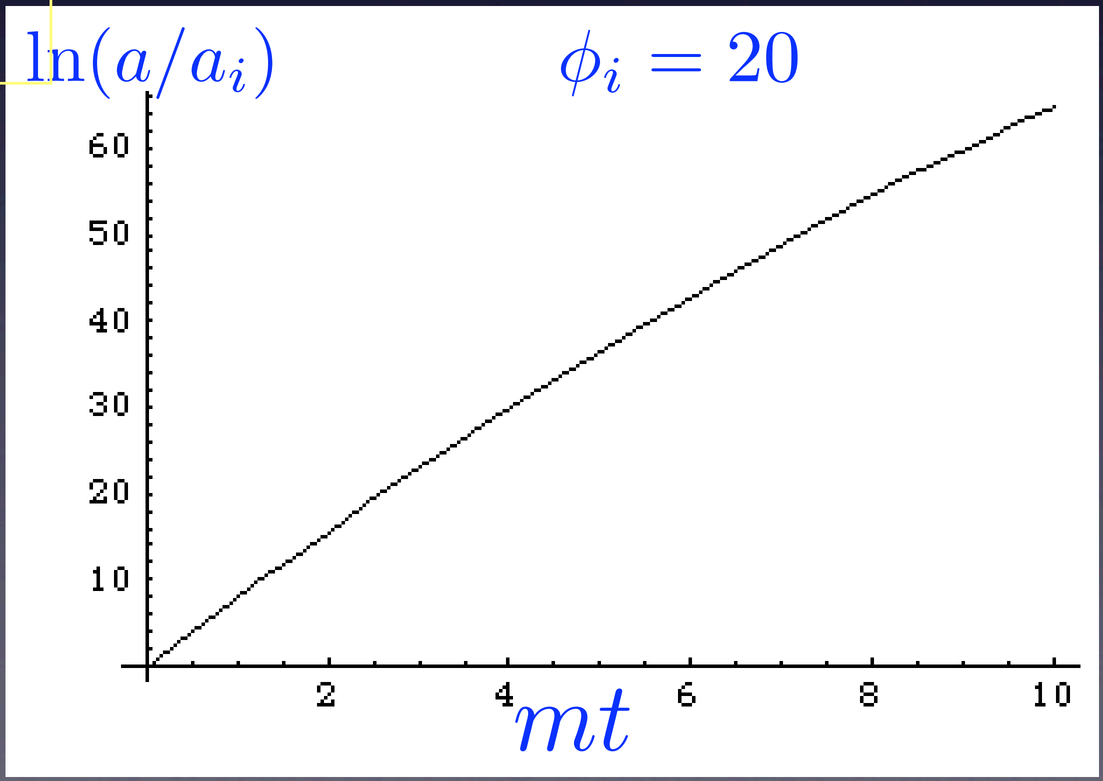
$$\boxed{\phi - \phi_i = -\frac{\sqrt{2}}{\sqrt{3}}M_G m(t - t_i)}$$

$$\frac{\dot{a}}{a} = \frac{1}{\sqrt{3}M_G}\frac{1}{\sqrt{2}}m\phi = \frac{m}{\sqrt{6}M_G}\left(\phi_i - \frac{\sqrt{2}}{\sqrt{3}}M_G m(t-t_i)\right)$$

$$\ln\frac{a}{a_i} = \frac{m}{\sqrt{6}M_G}\left(\phi_i(t-t_i) - \frac{1}{\sqrt{6}}M_G m(t-t_i)^2\right)$$

$$= \frac{1}{4M_G^2}(\phi_i^2 - \phi^2)$$

$$\boxed{a = a_i \exp\left[\frac{1}{4M_G^2}(\phi_i^2 - \phi^2)\right]}$$



3.3 Slow Roll Condition

$$\dot{\phi} \simeq -\frac{\frac{dV}{d\phi}}{3\frac{\dot{a}}{a}} \simeq -\frac{V'}{3H} \qquad \ddot{\phi} \simeq -\frac{V''}{3H}\dot{\phi} \simeq \frac{V''V'}{9H^2}$$

(1) $|\ddot{\phi}| \ll \left|\frac{dV}{d\phi}\right|$

$$\Rightarrow \quad \left|\frac{V''V'}{9H^2}\right| \ll |V'| \qquad |V''| \ll 9H^2 = 3\frac{V}{M_G^2}$$

$$\boxed{\eta \equiv \frac{V''}{V}M_G^2} \qquad \Rightarrow \quad |\eta| \ll 1$$

(2) $\frac{1}{2}\dot{\phi}^2 \ll V \quad \Rightarrow \quad \frac{(V')^2}{9H^2} \ll 2V \qquad (V')^2 \ll 18H^2V = 6\frac{V^2}{M_G^2}$

$$\boxed{\epsilon \equiv \frac{1}{2}\left(\frac{V'}{V}\right)^2 M_G^2} \qquad \Rightarrow \quad \epsilon \ll 1$$

For chaotic inflation $V = \frac{1}{2}m^2\phi^2$

$$\eta = \frac{m^2}{m^2\phi^2/2}M_G^2 \ll 1 \quad \Rightarrow \boxed{\phi \gg \sqrt{2}M_G}$$

$$\epsilon = \frac{1}{2}\left(\frac{m^2\phi}{m^2\phi^2/2}\right)^2 M_G^2 \ll 1 \quad \Rightarrow \boxed{\phi \gg \sqrt{2}M_G}$$

Inflation ends when $\eta \simeq 1 \text{ or } \epsilon \simeq 1$

$$\Rightarrow \quad \boxed{\phi_f \simeq \sqrt{2}M_G}$$

Accelerated Expansion

$$\ddot{a} = -\frac{1}{6M_G}(\rho + 3p)a > 0$$

$$\begin{cases} \rho = V + \frac{1}{2}\dot{\phi}^2 \\ p = -V + \frac{1}{2}\dot{\phi}^2 \end{cases}$$

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)}\partial_\nu \phi - g_{\mu\nu}\mathcal{L}$$

homogeneous $\partial_i \phi = 0$

$$\begin{aligned} T_{00} &= \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - V\right) \\ T_{ii} &= \frac{1}{2}\dot{\phi}^2 - V \end{aligned}$$

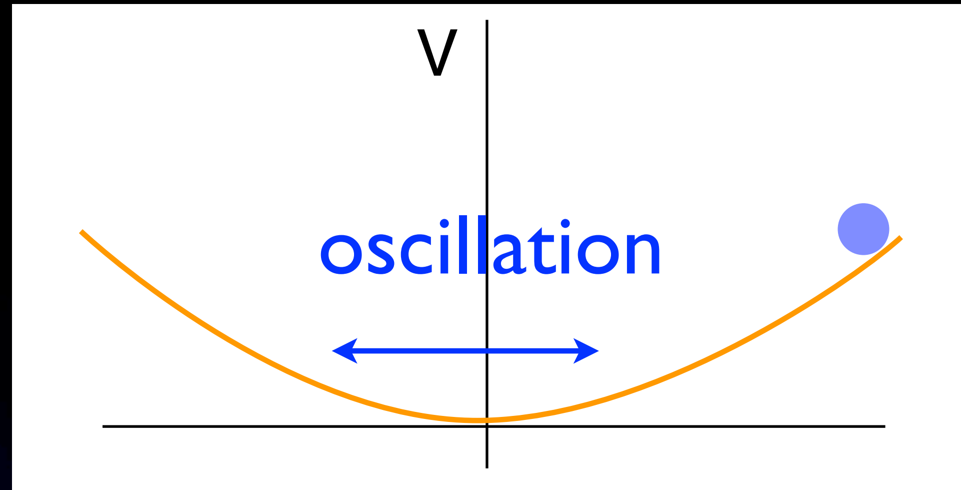
$$\ddot{a} = -\frac{1}{6M_G}(-2V + 2\dot{\phi}^2)a = \frac{1}{3M_G}(V - \dot{\phi}^2)a$$

$$\ddot{a} > 0 \quad \Leftrightarrow \quad V > \dot{\phi}^2$$

$$\rightarrow \boxed{\epsilon \lesssim 1}$$

3.4 After Inflation

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{dV}{d\phi}$$



$$\times\dot{\phi} \qquad \ddot{\phi}\dot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}^2 = -\frac{dV}{d\phi}\dot{\phi} \quad \Rightarrow \quad \frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{\phi}^2 + V\right)}_{\rho_\phi} = -3\frac{\dot{a}}{a}\dot{\phi}^2$$

$$V = \frac{1}{2}m^2\phi^2 \qquad \text{assume } H = \dot{a}/a \ll m$$

We can neglect the cosmic expansion

$$\ddot{\phi} = -m^2\phi \quad \Rightarrow \quad \phi = A\sin(mt + \delta)$$

$$\frac{1}{2}\dot{\phi}^2 = \frac{1}{2}m^2A^2\cos^2(mt + \delta)$$

To see the effect of cosmic expansion, take average over one oscillation

$$\left\langle \frac{1}{2}\dot{\phi}^2 \right\rangle = \frac{1}{4}m^2A^2 = \langle V(\phi)\rangle \ = \frac{1}{2}\rho_\phi$$

$$\Rightarrow \ \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = -3\frac{\dot{a}}{a}\dot{\phi}^2$$

$$\boxed{\frac{d}{dt}\rho_\phi = -3\frac{\dot{a}}{a}\rho_\phi \quad \Rightarrow \quad \rho_\phi \propto a^{-3}}$$

Inflaton oscillation behaves as matter

Oscillation $\Rightarrow$ Reheating

particle creation

Reheating Temperature

- Inflaton decays into other particles

$$\phi \quad \Rightarrow \quad \chi, \psi, \cdots$$

- Produced particles in the decay scatter and further decay

→ thermalized → form thermal plasma with T_R

Reheating

- Reheating Temperature

Γ_ϕ : inflaton decay rate

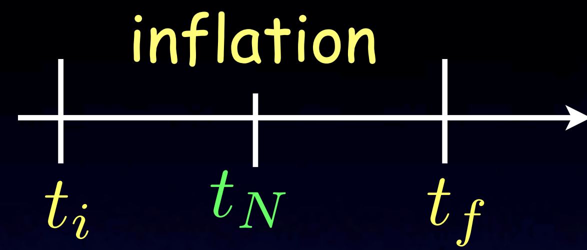
$$\Gamma_\phi \simeq 3H = 3\left(\frac{(\pi^2/30)g_* T_R^4}{3M_G^2}\right)^{1/2}$$

→ $$\boxed{T_R = \left(\frac{\pi^2 g_*}{10}\right)^{-1/4} \sqrt{\Gamma_\phi M_G}}$$

3.5 N-efold

- The scale factor increases by e^N from t_N to t_f

$$\frac{dN}{dt_N} = \frac{d}{dt_N}\ln\left(\frac{a(t_f)}{a(t_N)}\right) = -\frac{\dot{a}}{a} = -H$$



$$\Rightarrow \ N = \int_{t_N}^{t_f} dt(-H) \simeq \int_{\phi_N}^{\phi_f} \left(-\frac{3H}{V'}d\phi\right)(-H)$$

$$d\phi/dt = -V'/3H$$

$$\boxed{\Rightarrow \ N \simeq \int_{\phi_N}^{\phi_f} \frac{V}{V' M_G^2} d\phi}$$

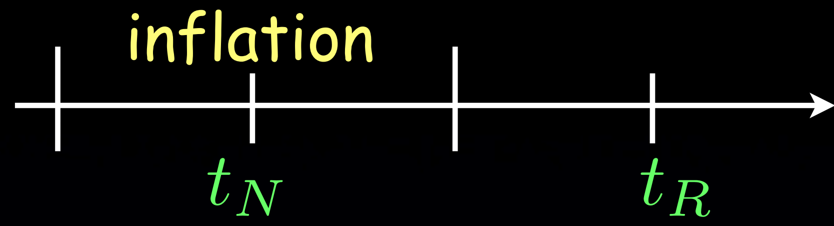
Chaotic inflation $V = \frac{1}{2}m^2\phi^2$

$$
\begin{aligned}
N &= -\int_{\phi_N}^{\phi_f} \frac{m^2\phi^2/2}{m^2\phi M_G^2} d\phi \\
&= \frac{-1}{2M_G^2} \int_{\phi_N}^{\phi_f} d\phi\phi \\
&= \frac{1}{4M_G^2}[\phi_N^2 - \phi_f^2] \\
&= \frac{1}{4M_G^2}[\phi_N^2 - 2M_G^2]
\end{aligned}
$$

$$
N_{tot} = \int_{\phi_i}^{\phi_f} \frac{V}{V' M_G^2} d\phi = \frac{1}{4M_G^2}[\phi_i^2 - 2M_G^2] \sim \left(\frac{M_G}{m}\right)^2
$$

$$
\sim 10^{10}
$$

3.5 Relation between N and Cosmological Scale L



Hubble radius at t_N corresponds to the present scale L

- At the end of inflation

$$H^{-1}(t_N) \;\to\; e^N H^{-1}(t_N)$$

- Between t_f and t_R (reheating epoch)
  Inflaton oscillation

$$\rho_\phi = \frac{1}{2}\Phi_f^2 m^2 \left(\frac{a(t)}{a(t_f)}\right)^{-3} \qquad \Phi \;\; : \text{amplitude of oscillation}$$

$$\rho_\phi(t_R) \simeq \frac{\pi^2}{30} g_* T_R^4$$

$$\Rightarrow \quad \left(\frac{a(t_R)}{a(t_f)}\right)^3 \simeq \frac{\frac{1}{2}\Phi_f^2 m^2}{\frac{\pi^2}{30} g_* T_R^4} = \frac{M_G^2 m^2}{\frac{\pi^2}{30} g_* T_R^4} \qquad \leftarrow \quad \Phi_f \simeq \sqrt{2} M_G$$

At t=t$_R$ the entropy of the region corresponding to $H^{-1}(t_N)$

$$S_N = H^{-3}(t_N) e^{3N} \left(\frac{a(t_R)}{a(t_f)}\right)^3 s(T_R) \qquad s(t_R) = \frac{2\pi^2}{45} g_* T_R^3$$

At present the entropy inside the region with scale L

$$S_L = 10^{87} \left(\frac{L}{3000 h^{-1}\mathrm{Mpc}}\right)^3$$

$$\boxed{\Rightarrow \quad N = 54 + \ln\left(\frac{L}{3000 h^{-1}\mathrm{Mpc}}\right) + \frac{1}{3}\ln\left(\frac{T_R}{10^{10}\mathrm{GeV}}\right) + \frac{1}{3}\ln\left(\frac{m}{10^{13}\mathrm{GeV}}\right)}$$

4. Generation of Density Fluctuations

4.1 Fluctuations of scalar field during inflation

massless ϕ

$$\boxed{\begin{aligned} &\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\triangle\phi = 0 \\ &ds^2 = dt^2 - e^{2Ht}d\vec{x}^2 \end{aligned}}$$

$$\boxed{\begin{aligned} &\Rightarrow \quad \phi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^3k[a_k^\dagger \psi_k^*(t)e^{-i\vec{k}\vec{x}} + a_k\psi_k(t)e^{i\vec{k}\vec{x}}] \\ &[a_k, a_q^\dagger] = \delta^{(3)}(\vec{k}-\vec{q}) \end{aligned}}$$

c.f. Minkowski space

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k_0}}[a_k^\dagger e^{+ik_0t - i\vec{k}\vec{x}} + a_k e^{-ik_0t + i\vec{k}\vec{x}}]$$

$$\psi_k \quad \rightarrow \quad \frac{e^{-ik_0t}}{\sqrt{2k_0}}$$

Remark:

k : comoving momentum

$p = e^{-Ht}k$: physical momentum

$$\boxed{\ddot{\psi}_k + 3H\dot{\psi}_k + k^2 e^{-2Ht}\psi_k = 0}$$

$$\eta = -H^{-1}e^{-Ht}, \quad \psi = \eta^{3/2}u$$

$$\Rightarrow \quad u'' + \frac{1}{\eta}u' + \left(k^2 - \frac{9}{4\eta^2}\right)u = 0$$ Bessel differential eq.

General sol.

$$\psi_k(t) = \frac{\sqrt{\pi}}{2}H\eta^{3/2}[C_1(k)H^{(1)}_{3/2}(k\eta) + C_2(k)H^{(2)}_{3/2}(k\eta)]$$

$H^{(1,2)}_{3/2}$: Hankel function

$$H^{(2)}_{3/2} = (H^{(1)}_{3/2})^* = \sqrt{\frac{2}{\pi x}}e^{-ix}\left(1 + \frac{1}{ix}\right)$$

$C_1(k), C_2(k)$?

Quantization in de Sitter space should be the same as that in Minkowski space in the limit of $k \to 0$

$$\psi_k \to \frac{-1}{\sqrt{2k}}\left[C_1(k)e^{-iH^{-1}k+ikt} + C_2(k)e^{iH^{-1}k-ikt}\right]$$

$$= \frac{e^{-ikt}}{\sqrt{2k}}$$

$$\Rightarrow \quad C_1(k) \to 0 \quad C_2(k) \to 1$$

$$\Rightarrow \boxed{\psi_k(t) = \frac{\sqrt{\pi}}{2} H\eta^{3/2} H^{(2)}_{3/2}}$$

For cosmologically interesting fluctuations

$k \gg H$ at the start of the inflation

$$\Rightarrow \boxed{C_2 = 1, \quad C_1 = 0}$$

$$\Rightarrow \psi_k(t) = \frac{-iH}{k\sqrt{2k}} \left(1 + \frac{k}{iH} e^{-Ht}\right) \exp\left(-\frac{ik}{H} e^{-Ht}\right)$$

Large t $\psi_k(t) \to \dfrac{-iH}{k\sqrt{2k}}$ no oscillation

$$\langle \phi^2 \rangle = \int |\psi_k|^2 d^3k = \frac{1}{(2\pi)^3} \int \left(\frac{e^{-2Ht}}{2k} + \frac{H^2}{2k^3} \right) d^3k$$

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int \left(\frac{1}{2} + \frac{H^2}{2p^2} \right) \frac{d^3p}{p} \qquad p = ke^{-Ht}$$

vacuum fluctuations

$$\langle \phi^2 \rangle = \frac{H^2}{(2\pi)^2} \int d \ln \left(\frac{p}{H} \right)$$

$$\Rightarrow \boxed{\delta\phi \simeq \frac{H}{2\pi}}$$

Fourier mode $\delta\phi_k = \dfrac{H}{\sqrt{2}k^{3/2}}$

Density Fluctuations

Inflaton fluctuations $\Rightarrow$ metric fluctuations

$\Rightarrow$ density fluctuations

$$\boxed{ds^2 = a^2[1+2\Phi]d\tau^2 - a^2[1+2\Psi](d\vec{x})^2}$$

newtonian gauge

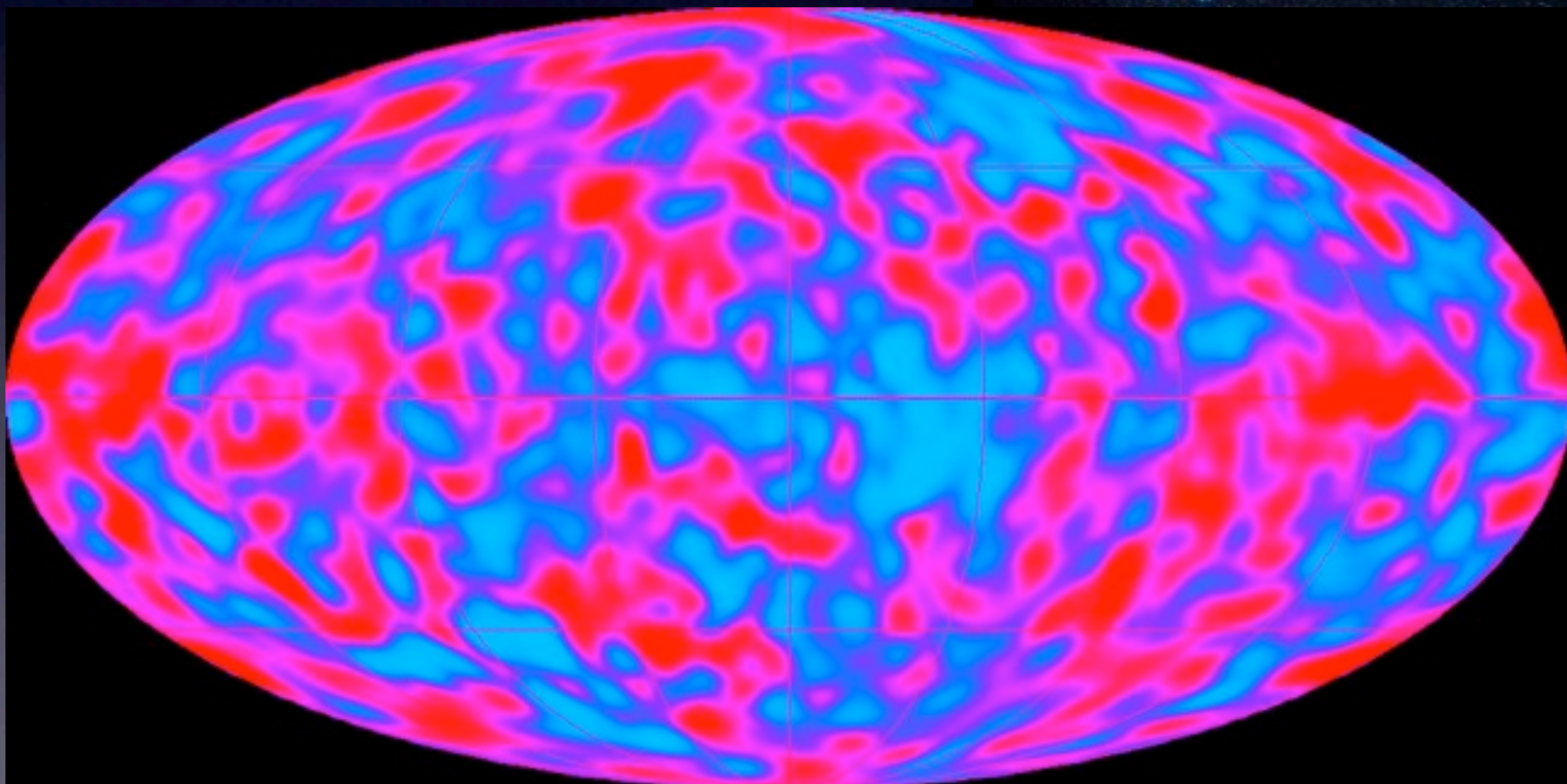
$$\Rightarrow \boxed{\begin{aligned} \Psi &= -\Phi \\ \Phi &= H_I \frac{\delta\phi}{\dot\phi}\left(1 - \frac{\dot a}{a^2}\int_o^t a dt\right) \end{aligned}} \qquad \Phi = \begin{cases} \frac{2}{3} H_\mathrm{I} \frac{\delta\phi}{\dot\phi} & \text{(RD)} \\ \frac{3}{5} H_\mathrm{I} \frac{\delta\phi}{\dot\phi} & \text{(RD)} \end{cases}$$

Poisson Equation

$$\Delta\Phi = 4\pi G a^2 \delta\rho \longrightarrow k^2 \Phi_k = -4\pi G a^2 \delta\rho_k$$

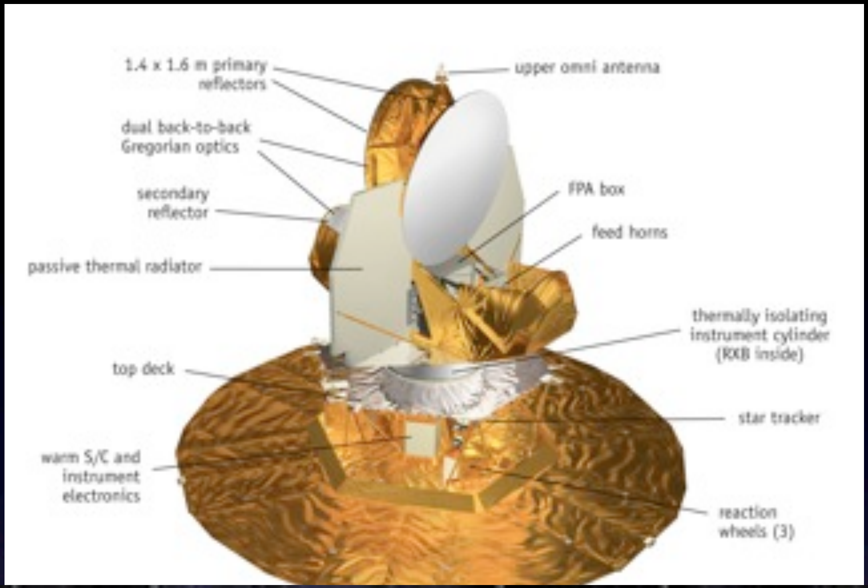
$$\Rightarrow \boxed{\frac{\delta\rho_k}{\rho} = \frac{2}{3}\frac{k^2}{a^2 H^2}\Phi_k = \frac{2}{5}\frac{k^2}{a^2 H^2} H_\mathrm{I} \frac{\delta\phi_k}{\dot\phi}}$$

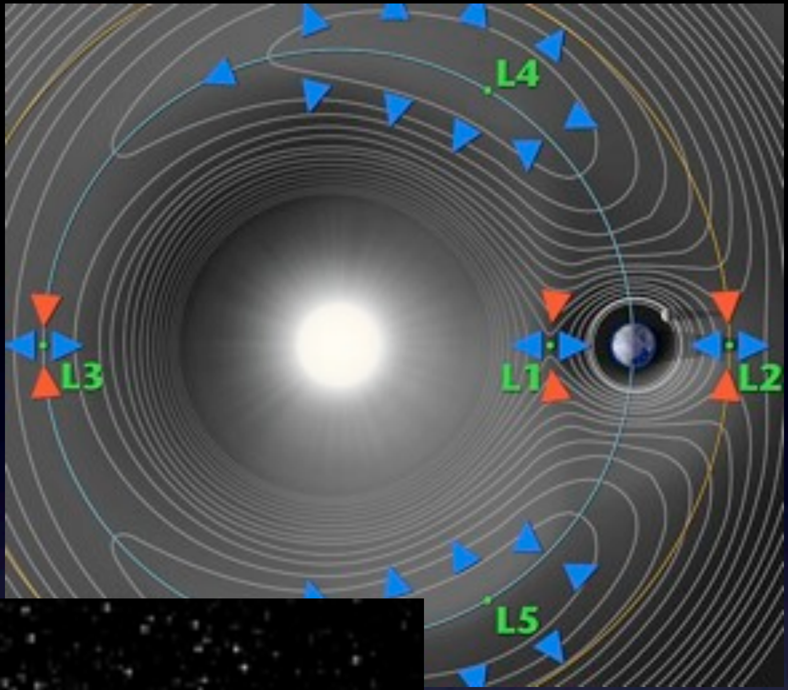
COBE

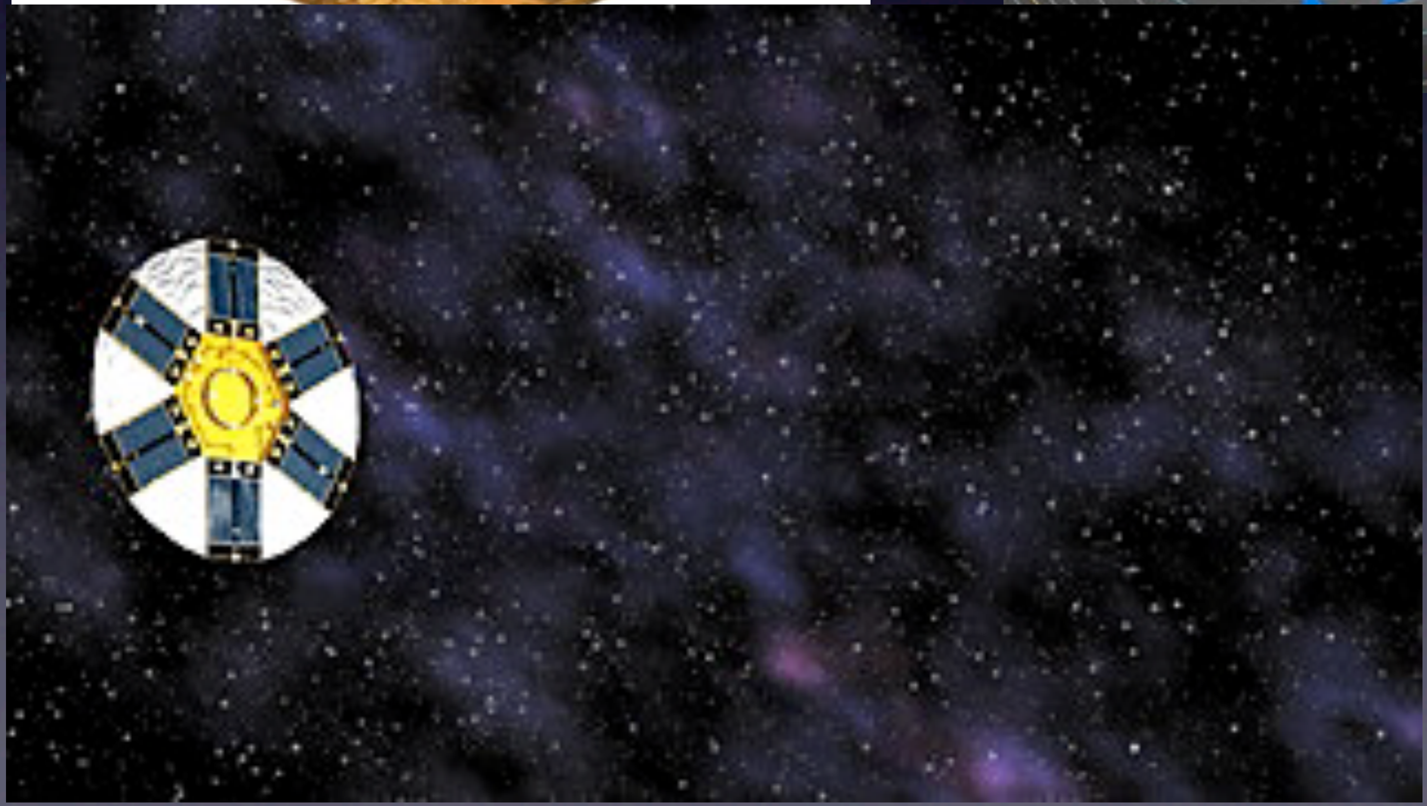


http://map.gsfc.nasa.gov/product/cobe

WMAP (Wilkinson Microwave Anisotropy Probe)

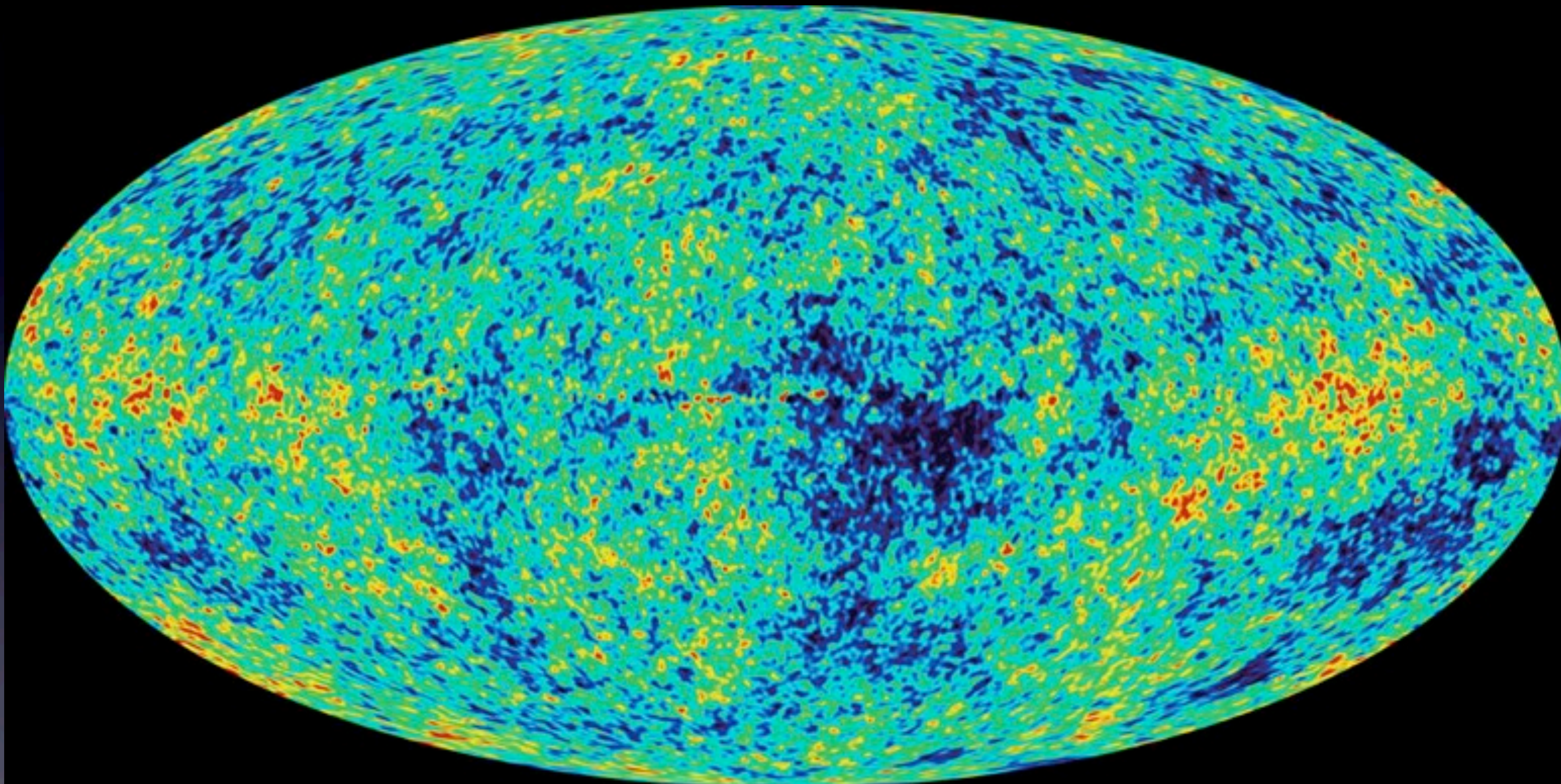






http://map.gsfc.nasa.gov/

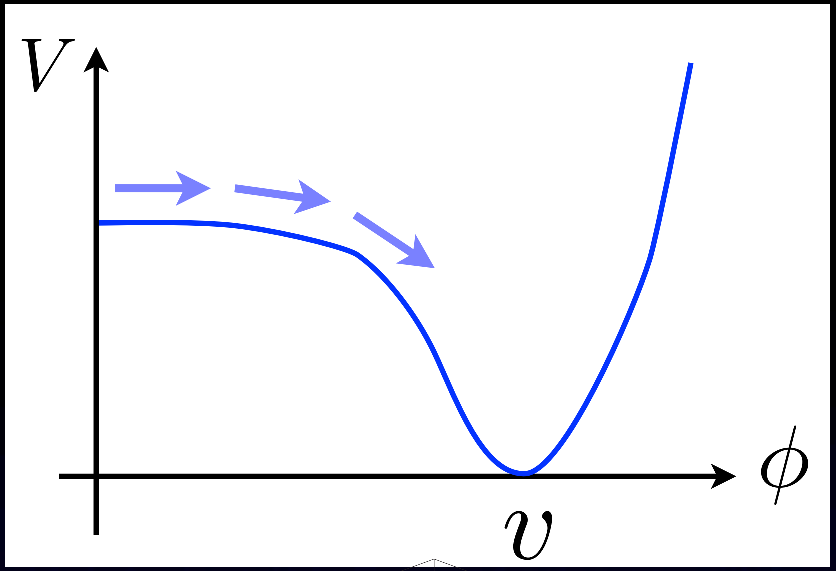
WMAP Full Sky Map



Other Inflation Models

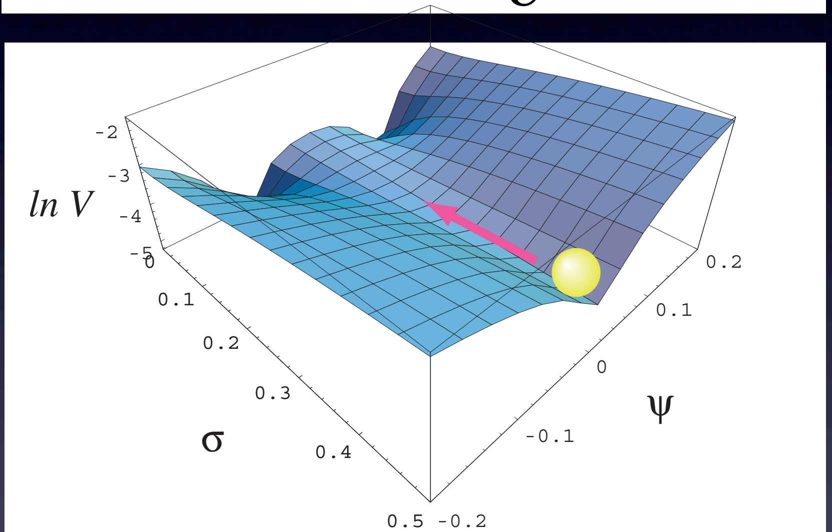
New Inflation Model

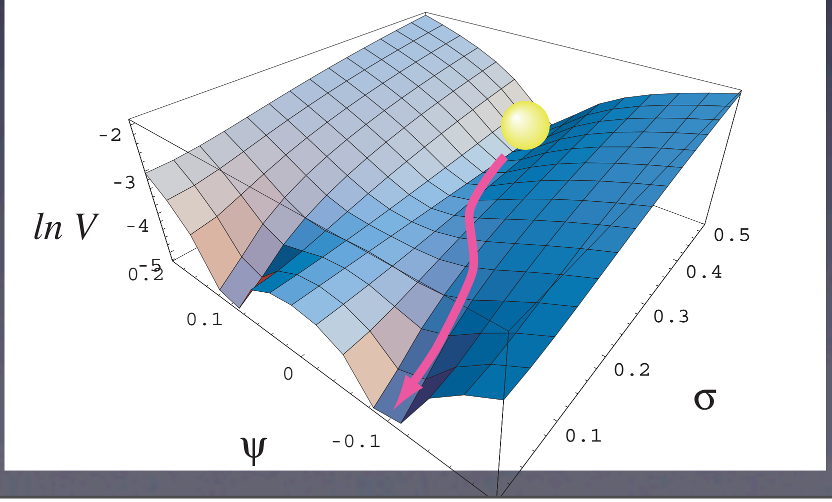
$$V = \mu^4 \left(1 - \left(\frac{\phi}{v}\right)^n\right)^2 \qquad n \geq 4$$



Hybrid Inflation Model

$$V = (\mu^2 - \lambda\psi^2)^2 + \kappa\psi^2\sigma^2 + \frac{1}{2}m_\sigma^2\sigma^2$$





Inflation models

Inflation

- Chaotic Inflation
  - natural (no initial value problem)
  - take place at planck time
  - inflaton $\phi > M_G$
- Hybrid Inflation
  - initial value problem
  - cosmic string
- New Inflation
  - severe initial value problem
  - flatness (longevity) problem

Hubble

High

Low

Flat Potential → SUSY η problem