Standard Cosmology

- Notation

$$\mu, \nu \cdots = 0, 1, 2, 3$$

$$i, j \cdots = 1, 2, 3$$

$$g_{\mu\nu} = (+, -, -, -)$$

- Unit

$$c = \hbar = k_B = 1$$

1. Standard Model

- Cosmological Principle

  - The universe is spatially homogeneous
  - The universe is isotropic.

→ Robertson-Walker Metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

$a(t)$ scale factor

$$\begin{cases} K > 0 & \text{closed universe} \\ K = 0 & \text{flat universe} \\ K < 0 & \text{open universe} \end{cases}$$

3dim curvature

$$= \frac{K}{a^2}$$

- Another Form

$$ds^2 = dt^2 - a^2(t)\left[d\vec{x}^2 + K\frac{(\vec{x}\cdot d\vec{x})^2}{1-K\vec{x}^2}\right]$$

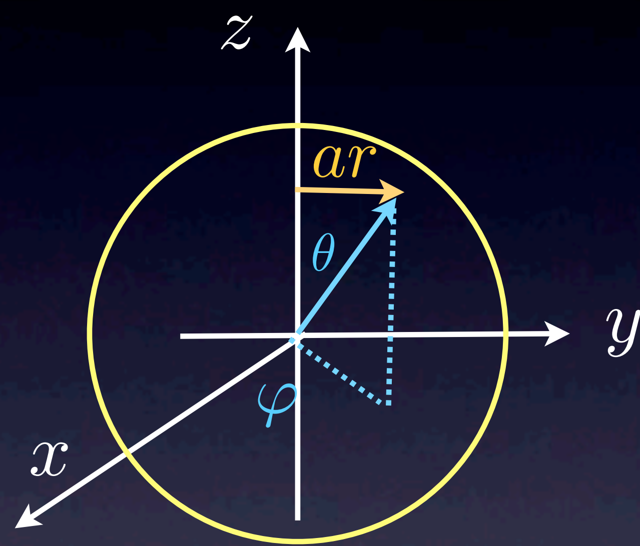
$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{ij} &= -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2K}{a^2}\right]g_{ij} \\
R &= -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right]
\end{aligned}$$

Element length in curved space

Sphere in Euclidian Space

$$x^2 + y^2 + z^2 = a^2$$

$$ds^2 = dx^2 + dy^2 + dz^2$$



$$\begin{cases} z = a\cos\theta \\ x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \end{cases}$$

$$\begin{cases} dz = -a\sin\theta d\theta \\ dx = a\cos\theta\cos\varphi d\theta - a\sin\theta\sin\varphi d\varphi \\ dy = a\cos\theta\sin\varphi d\theta + a\sin\theta\cos\varphi d\varphi \end{cases}$$

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$r = \sin\theta \quad \rightarrow \quad dr = \cos\theta d\theta \qquad d\theta^2 = \frac{dr^2}{\cos^2\theta} = \frac{dr^2}{1-r^2}$$

$$\boxed{ds^2 = a^2\left(\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right)}$$

Sphere in Euclidian Space (2)

$$\vec{x} = (x, y) \qquad \vec{x}^2 + z^2 = a^2$$

$$ds^2 = d\vec{x}^2 + dz^2$$

$$\vec{x} \cdot d\vec{x} + zdz = 0$$

$$ds^2 = d\vec{x}^2 + \frac{(\vec{x} \cdot d\vec{x})^2}{z^2} = d\vec{x}^2 + \frac{(\vec{x} \cdot d\vec{x})^2}{a^2 - x^2}$$

rescale $\vec{x} \rightarrow a\vec{x}$

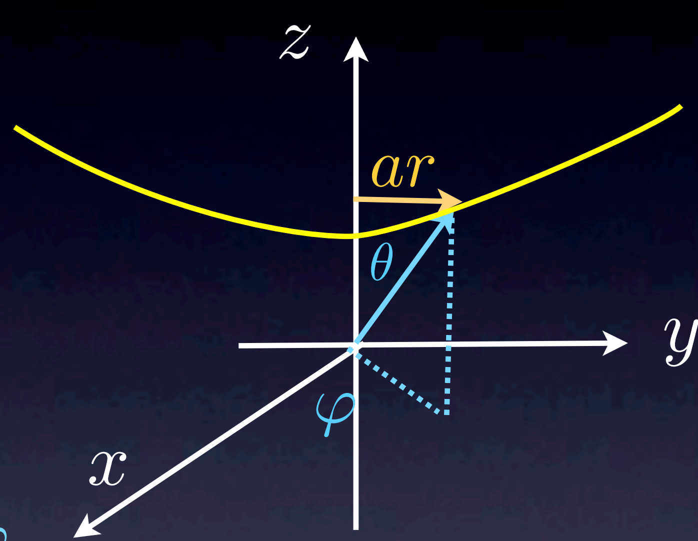
$$\boxed{ds^2 = a^2 \left(d\vec{x}^2 + \frac{(\vec{x} \cdot d\vec{x})^2}{1 - x^2} \right)}$$

Element length in curved space

Hypabolic surface in Minkowski space

a —> ia z —> iz θ —> iθ $x^2 + y^2 - z^2 = -a^2$

$$ds^2 = dx^2 + dy^2 - dz^2$$



$$\begin{cases} z = a\cosh\theta \\ x = a\sinh\theta\cos\varphi \\ y = a\sinh\theta\sin\varphi \end{cases}$$

$$\Rightarrow \begin{cases} dz = a\sinh\theta d\theta \\ dx = a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi \\ dy = a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi \end{cases}$$

$$ds^2 = a^2(d\theta^2 + \sinh^2\theta d\varphi^2)$$

$$r = \sinh\theta \ \rightarrow \ dr = \cosh\theta d\theta \qquad d\theta^2 = \frac{dr^2}{\cosh^2\theta} = \frac{dr^2}{1+r^2}$$

$$\Rightarrow \boxed{ds^2 = a^2\left(\frac{dr^2}{1+r^2} + r^2 d\varphi^2\right)}$$

Energy Momentum Tensor

Cosmological Principle $\Rightarrow$ Energy-Momentum Tensor = Perfect Fluid Form

$$T^{00} = \rho(t) \qquad T^{ij} = g^{ij} p(t)$$

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho & & & O \\ & -p & & \\ & & -p & \\ O & & & -p \end{pmatrix}$$

$$\boxed{T^{\mu\nu} = -p g^{\mu\nu} + (\rho + p) u^{\mu} u^{\nu}} \qquad u^0 = 1,\ u^i = 0$$

Einstein eq.

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= 8\pi G T_{\mu\nu} \end{aligned} \quad \Rightarrow \quad \begin{aligned} G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \\ G_{ij} &= \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) g_{ij} \end{aligned}$$

Einstein Equation

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}}$$

Λ Cosmological Constant

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)$$

$$G_{ij} = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) g_{ij}$$

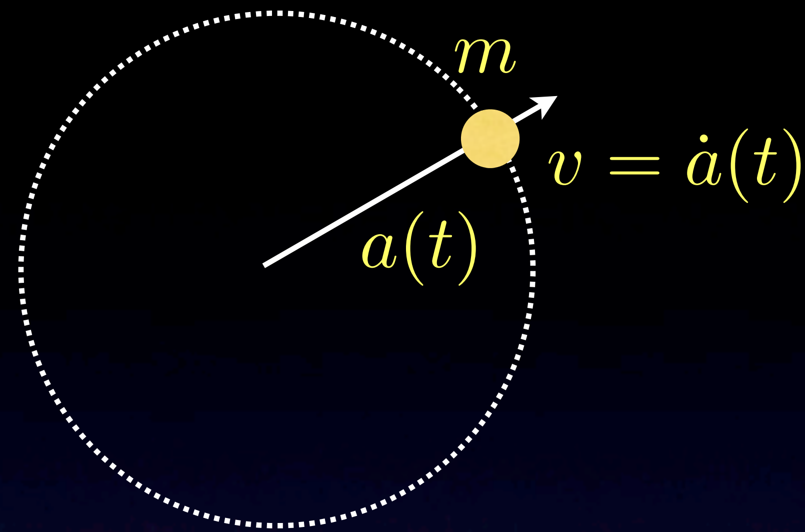
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a + \frac{\Lambda}{3}a$$

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

two independent equations

Newtonian Picture



Energy Conservation

$$\frac{1}{2}mv^2 - \frac{GMm}{a} = mE = \mathrm{const}$$

$$M = \frac{4}{3}\pi a^3 \rho \quad \text{total mass}$$

$$\frac{\dot{a}^2}{2} - \frac{4\pi G\rho}{3}a^2 = E \equiv -\frac{K}{2}$$

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi}{3}G\rho$$

1st Law in Thermodynamics

$$dE = -pdV$$

$$E = \rho a^3 \quad V = a^3$$

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$$

2. Cosmological Parameters

Friedmann Equation $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

- Hubble Parameter $$H \equiv \left(\frac{\dot{a}}{a}\right)$$

The present Hubble parameter = Hubble constant H_0

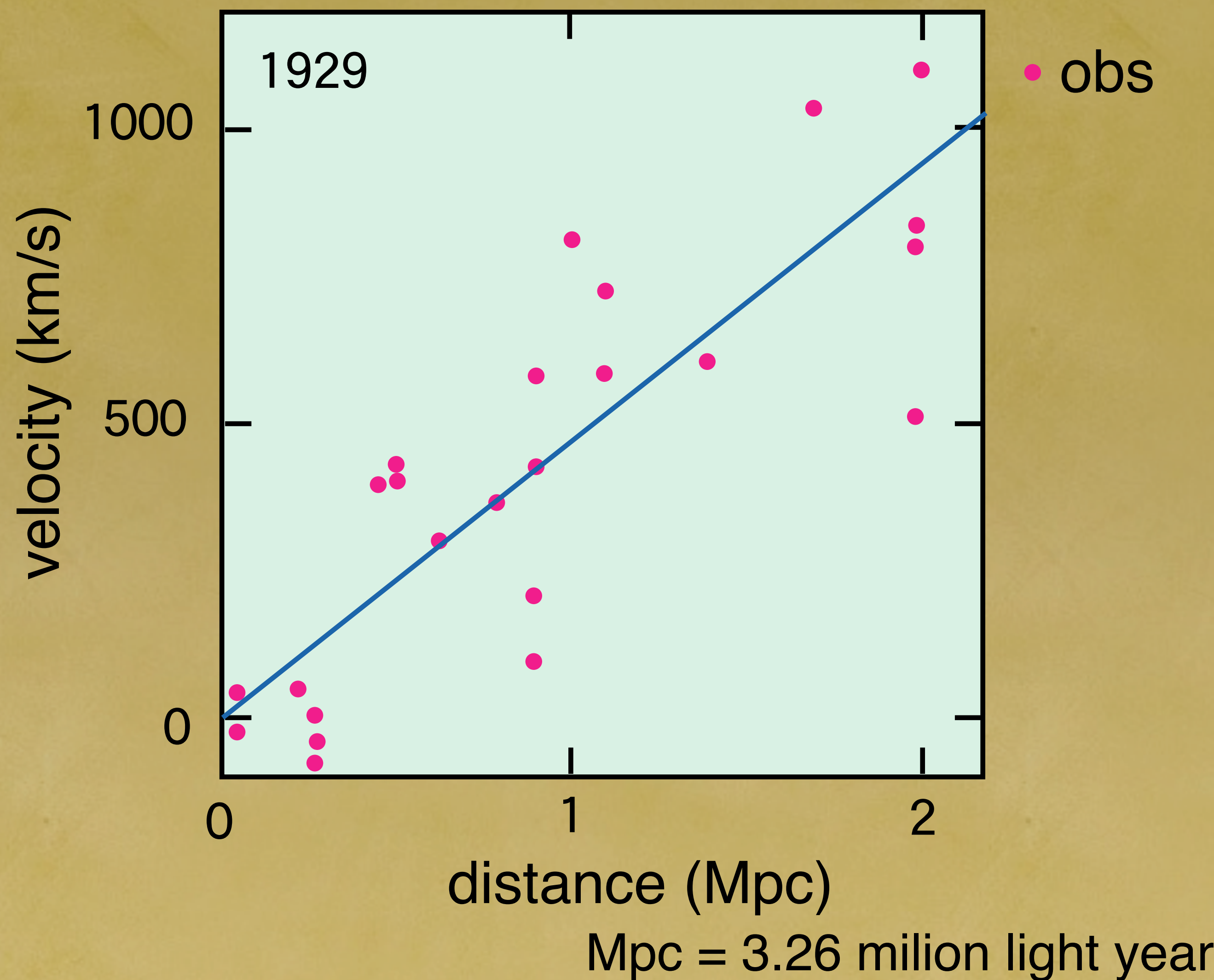
$\mathrm{Mpc} \simeq 3 \times 10^{24}\mathrm{cm}$

obs: $$H_0 = h_0 \times (100\mathrm{km/s/Mpc})$$

$$h_0 \simeq 0.7 \pm 0.1$$

distance to a galaxy: d

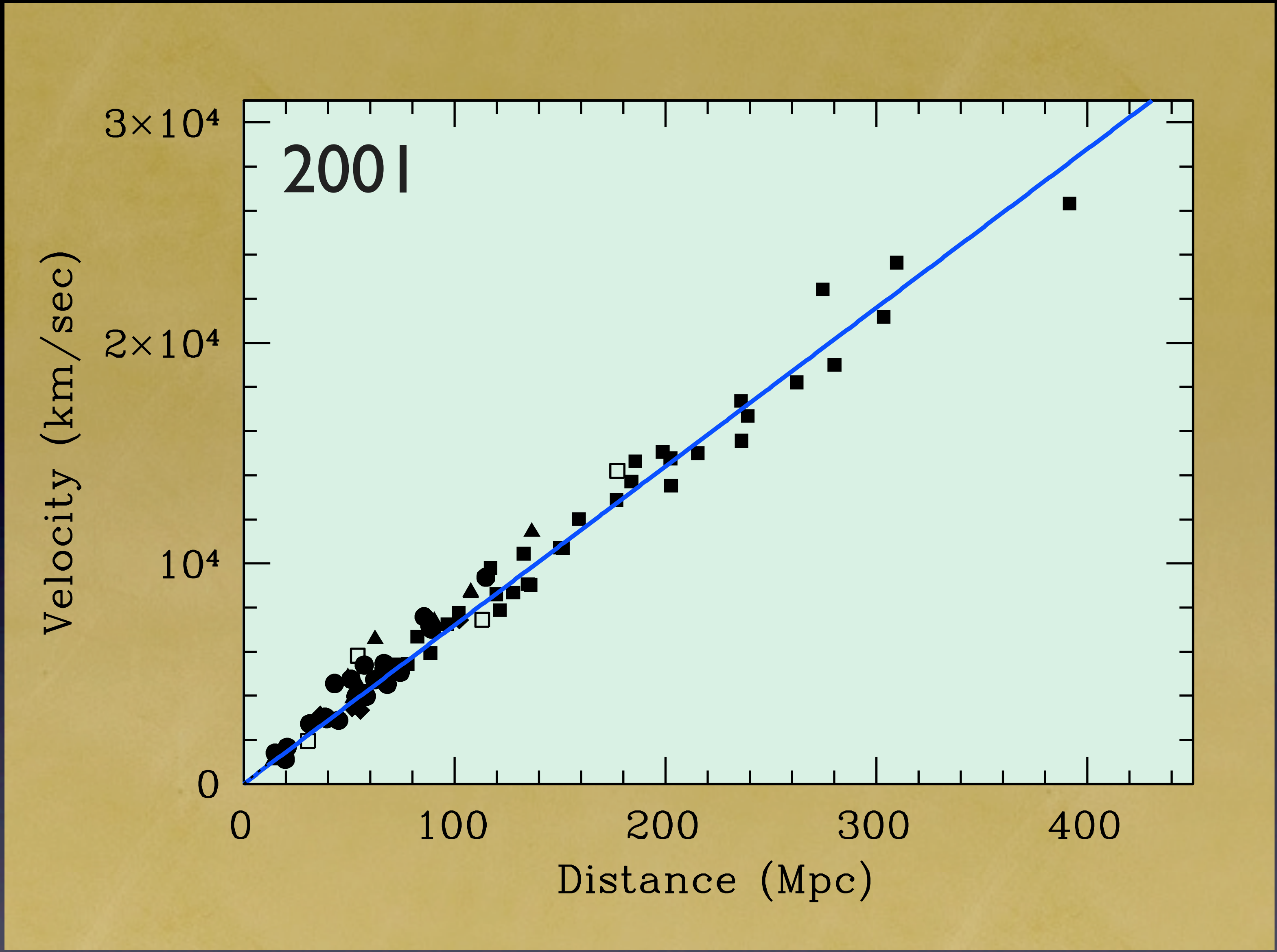
$$d = ar \ (r \ll 1) \ \Rightarrow \ v = \dot{d} = (\dot{a}/a)ar = Hd$$



$$(\text{velocity}) = H_0 \text{ X } (\text{distance})$$

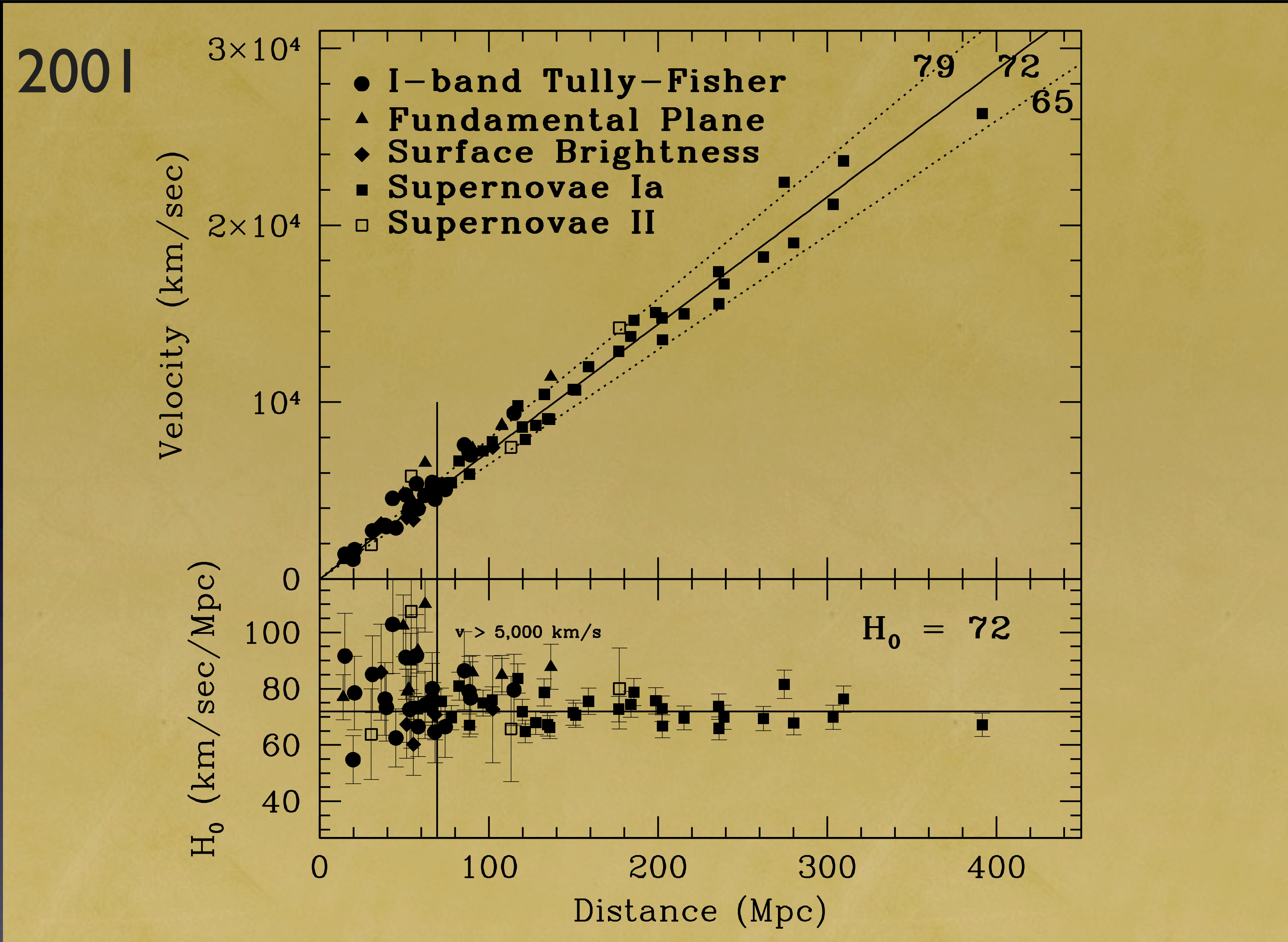
H_0 Hubble Constant = 465km/s/Mpc

Hubble Space Telescope



(velocity) = H_0 X (distance)

H_0 Hubble Constant = 68-75km/s/Mpc



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2. Cosmological Parameters

Friedmann Equation $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

- Density Parameter
  critical density

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}$$

$$\rho_c \equiv \frac{3H^2}{8\pi G}$$

$$\rho_{c,o} = 1.053 \times 10^{-5} h^2 \ \mathrm{GeV cm^{-3}}$$

- Cosmological constant

$$\lambda = \frac{\Lambda}{3H^2}$$

Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho \quad \Rightarrow \quad H^2 + \frac{K}{a^2} - \lambda H^2 = \Omega H^2$$

$$\Rightarrow \quad \boxed{(\Omega + \lambda - 1)H^2 = \frac{K}{a^2}}$$

$$\Omega + \lambda \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \iff \begin{cases} K > 0 & \text{closed universe} \\ K = 0 & \text{flat universe} \\ K < 0 & \text{open universe} \end{cases}$$

3. Density of the Universe

Einstein equation $\boxed{\dfrac{d}{dt}(a^3\rho) = -p\dfrac{d}{dt}(a^3)}$

equation of state $\boxed{p = w\rho}$

$$\rightarrow \quad \frac{d}{dt}(a^3\rho) = a^3\frac{d\rho}{dt} + \rho\frac{d}{dt}(a^3) = -w\rho\frac{d}{dt}(a^3)$$

$$\frac{d\rho}{\rho} = -(1+w)\frac{1}{a^3}d(a^3)$$

$$\rightarrow \quad \boxed{\rho \propto a^{-3(1+w)}}$$

In cosmology, three kinds of densities are important

- ρ_M : Matter (non-relativistic, non-thermal)

  $\rho_M \propto a^{-3}$ $\quad w = 0$

- ρ_R : Radiation (relativistic particles)

  $\rho_R \propto a^{-4}$ $\quad w = 1/3$

- ρ_{DE} : Dark Energy $\quad w < 0$

$w = -1 \Rightarrow$ Cosmological constant $\quad \rho_{DE} \propto a^0$

$p = -\rho$

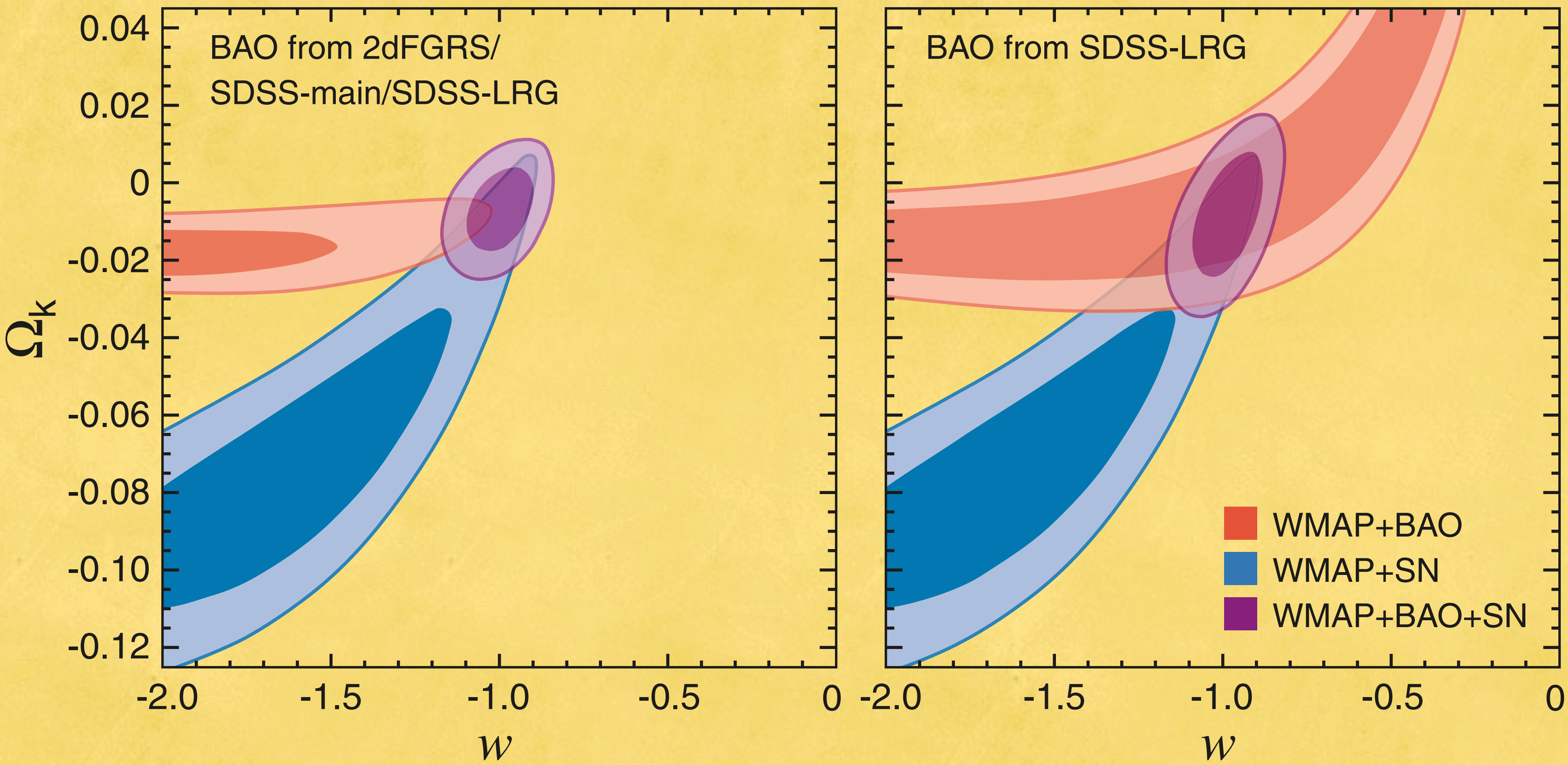
$$T^{\mu\nu} = \underline{\rho g^{\mu\nu}} \qquad [T^{\mu\nu} = -p g^{\mu\nu} + (\rho + p) u^\mu u^\nu]$$

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} + \underline{\Lambda g^{\mu\nu}}$$

cosmological const.
= vacuum energy

$$\Rightarrow \rho = \rho_M + \rho_R + \rho_{DE}$$

Observational Constraint on w



The present Universe

particle	temp (K)	density (1/cm3)	density (eV/cm3)	Ωh^2
photon	2.73	415	0.23	2.2×10^{-5}
neutrino	1.95	113X3	0.052X3	$4.9 \times 10^{-6} \times 3$
baryon	-	2.5×10^{-7}	250	0.02

neutrinos have masses $\Omega_\nu h^2 > 5 \times 10^{-4} \quad (m_{\nu_3} > 0.05\mathrm{eV})$

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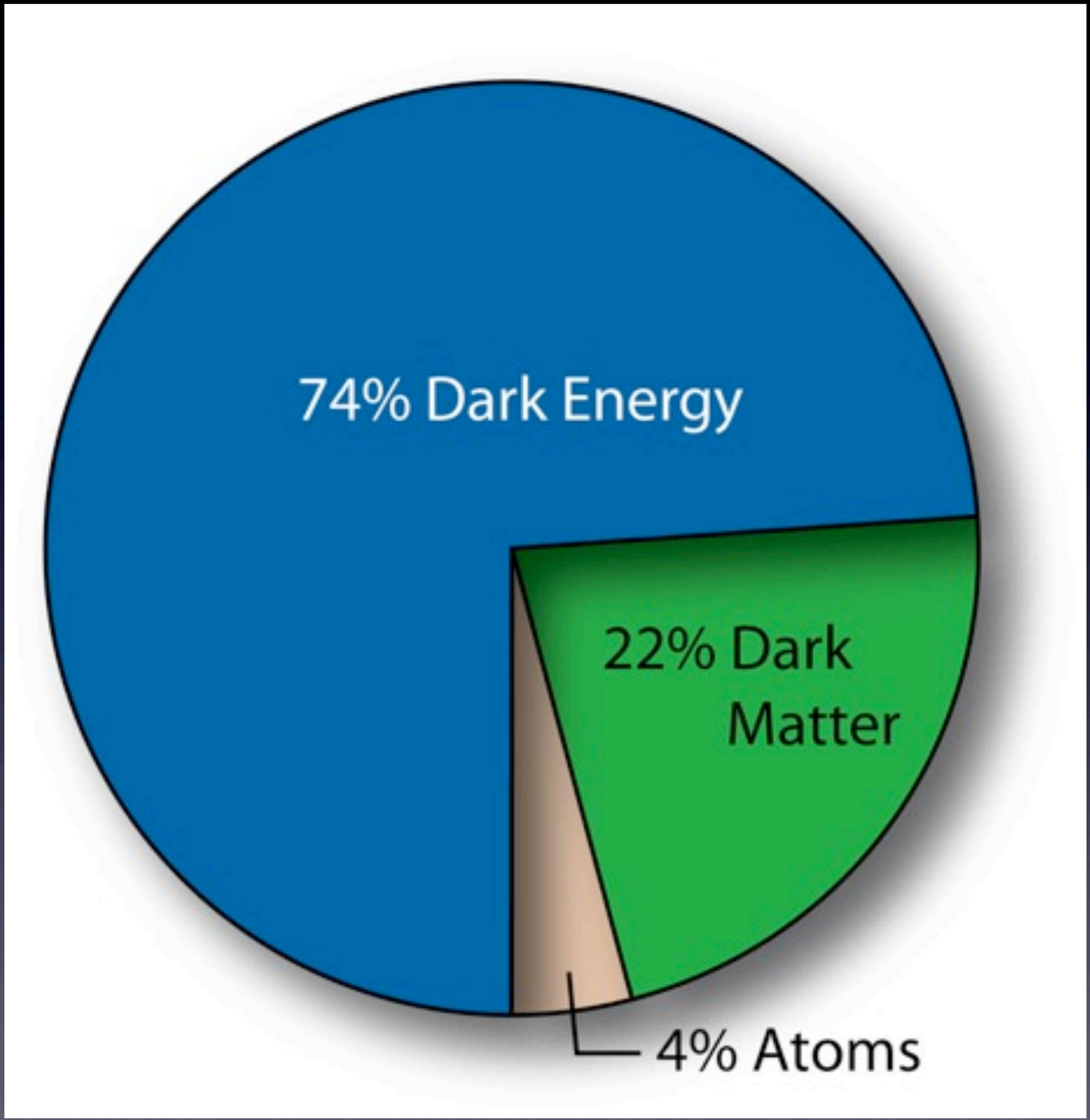
baryon	-	2.5×10^{-7}	250	0.02
dark matter	-	?	1300	0.105
dark energy	-	?	4800	0.38

neutrinos have masses $\Omega_\nu h^2 > 5 \times 10^{-4} \quad (m_{\nu_3} > 0.05\mathrm{eV})$

Resent WMAP result

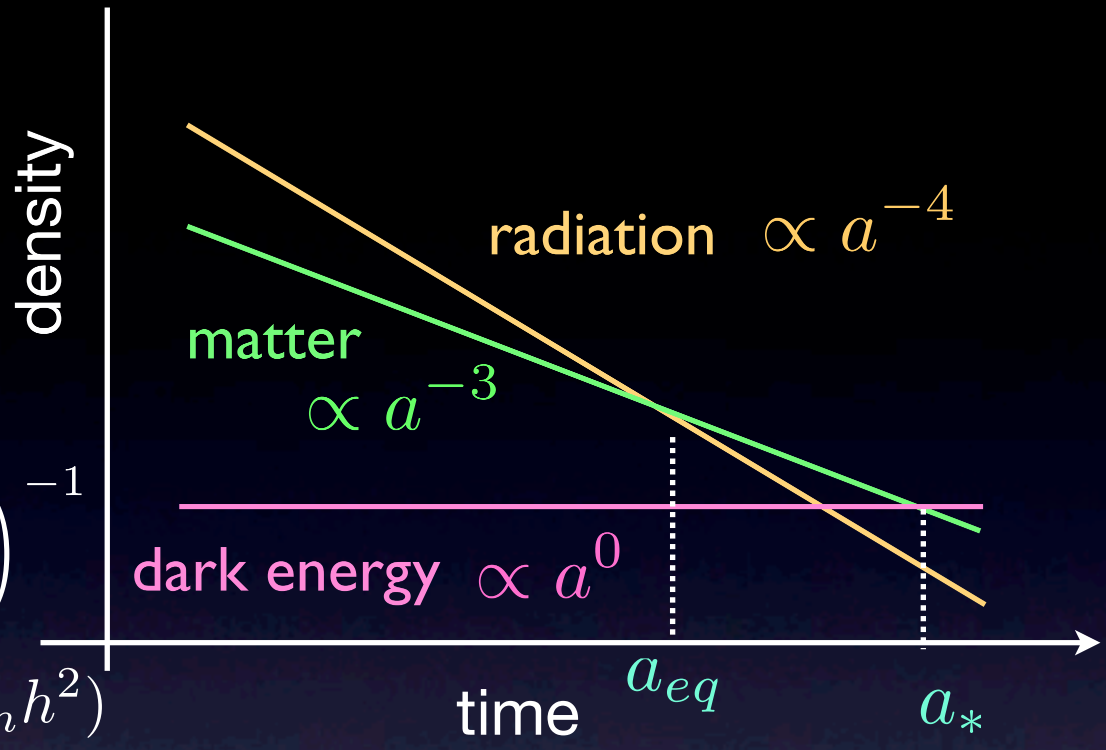


Matter vs Radiation

$$\frac{\rho_R}{\rho_M} = \frac{\Omega_{\gamma,0} + \Omega_{\nu,0}}{\Omega_{B,0} + \Omega_{DM,0}} \left(\frac{a}{a_0}\right)^{-1}$$

$$= 3.7 \times 10^{-5} (\Omega_m h^2)^{-1} \left(\frac{a}{a_0}\right)^{-1}$$

$$= 1 \quad \Rightarrow \quad \frac{a_0}{a} = 2.7 \times 10^4 (\Omega_m h^2)$$



$$(\Omega_m = \Omega_{B,0} + \Omega_{DM,0})$$

The early universe is radiation-dominated

$$\boxed{\frac{a_0}{a} > \frac{a_0}{a_{eq}} = 2.7 \times 10^4 (\Omega_m h^2)} \simeq 3000$$

4. Matter Dominated Universe

Friedmann Equation $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

$$\begin{cases} K = a_0^2(\Omega_m + \lambda - 1)\,H_0^2 \\ \Lambda = 3H_0^2\lambda \\ \frac{8\pi}{3}G\rho = \frac{8\pi}{3}G\rho_0\left(\frac{a}{a_0}\right)^{-3} = \Omega_m H_0^2\left(\frac{a}{a_0}\right)^{-3} \end{cases}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = -H_0^2(\Omega_m + \lambda - 1)\left(\frac{a_0}{a}\right)^2 + H_0^2\Omega_m\left(\frac{a_0}{a}\right)^3 + H_0^2\lambda}$$

- Dark Energy Dominated Universe $a > a_*$

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\lambda \quad \Rightarrow \quad a = a_0 \exp\left[H_0\lambda^{1/2}(t-t_0)\right]$$

- Matter Dominated Universe $a_{eq} < a < a_*$

$$\left(\frac{\dot a}{a}\right)^2 = -H_0^2(\Omega_m+\lambda-1)\left(\frac{a_0}{a}\right)^2 + H_0^2\Omega_m\left(\frac{a_0}{a}\right)^3 + H_0^2\lambda$$

$a_{eq} \ll a \ll a_*$
$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\Omega_m\left(\frac{a_0}{a}\right)^3$$

$$\left(\frac{a_0}{a}\right) d\left(\frac{a}{a_0}\right) = H_0\Omega_m^{1/2}\left(\frac{a_0}{a}\right)^{3/2} dt$$

$$\Rightarrow \left(\frac{a}{a_0}\right) = \left(\frac{3}{2}H_0\Omega_m^{1/2}t\right)^{2/3}$$

solved analytically for λ=0

- K < 0

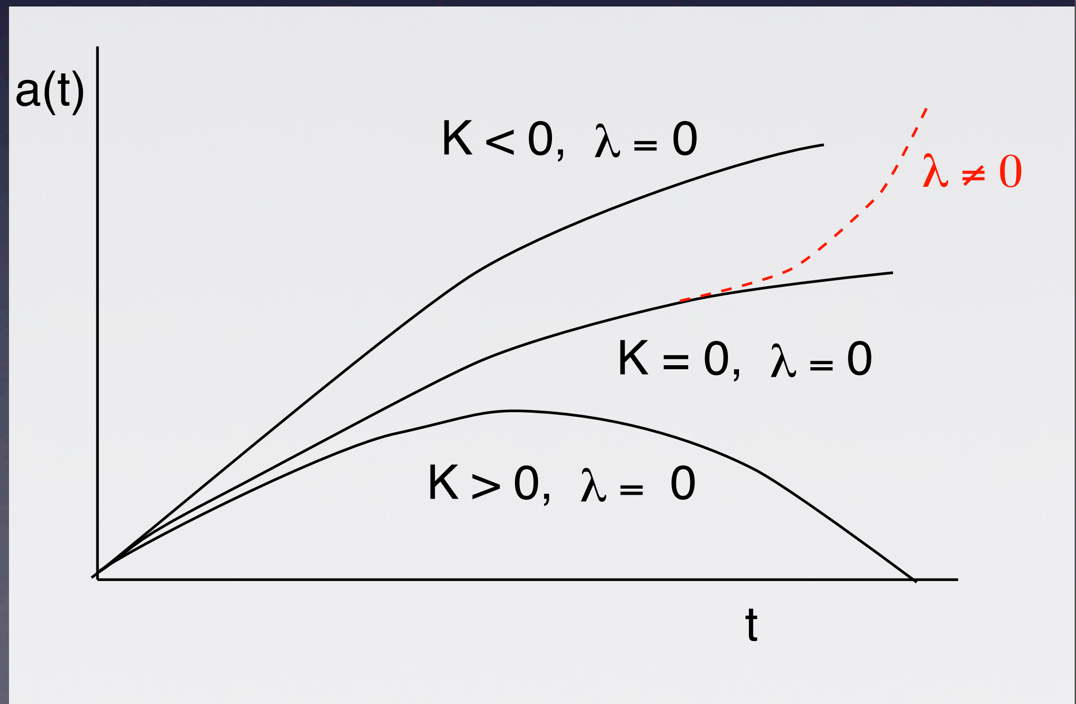
$$\begin{cases} \frac{a}{a_0} = \frac{1}{2}(1-\Omega_m)^{-1}\Omega_m(\cosh\eta - 1) \\ t = \frac{1}{2}H_0^{-1}\Omega_m(1-\Omega_m)^{-3/2}(\sinh\eta - \eta) \end{cases}$$

- K > 0

$$\begin{cases} \frac{a}{a_0} = \frac{1}{2}(\Omega_m - 1)^{-1}\Omega_m(1-\cos\eta) \\ t = \frac{1}{2}H_0^{-1}\Omega_m(\Omega_m - 1)^{-3/2}(\eta - \sin\eta) \end{cases}$$

- K = 0

$$\frac{a}{a_0} = \left(\frac{3H_0 t}{2}\right)^{2/3}$$



- Flat Universe $\quad \Omega_m + \lambda = 1$

$$\Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_m \left(\frac{a_0}{a}\right)^3 + H_0^2 (1 - \Omega_m)$$

$$\rightarrow \quad \boxed{\frac{a}{a_0} = \left(\frac{\Omega_m}{1-\Omega_m}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{1-\Omega_m} H_0 t\right)}$$

$$t \gg \frac{2}{3} H_0^{-1} \qquad \frac{a}{a_0} = \left(\frac{\Omega_m}{1-\Omega_m}\right)^{1/3} \frac{1}{2^{2/3}} \exp\left[\sqrt{1-\Omega_m} H_0 t\right]$$

$$t \ll \frac{2}{3} H_0^{-1} \qquad \frac{a}{a_0} = \left(\frac{\Omega_m}{1-\Omega_m}\right)^{1/3} \left[\frac{3}{2}\sqrt{1-\Omega_m} H_0 t\right]^{2/3} = \left[\frac{3}{2} \Omega_m^{1/2} H_0 t\right]^{2/3}$$

5. Entropy of the Universe

Thermodynamics $$dS(V,T) = \frac{1}{T} d(\rho V) + \frac{p}{T} dV$$

$\rho(T),\ p(T)$ function of T

Integrability condition

$$\frac{\partial S}{\partial V} = \frac{1}{T}(\rho + P), \quad \frac{\partial S}{\partial T} = \frac{V}{T}\frac{d\rho}{dT}$$

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial}{\partial T}\left(\frac{1}{T}(\rho + p)\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{d\rho}{dT}\right)$$

$$-\frac{1}{T^2}(\rho + p) + \frac{1}{T}\frac{d\rho}{dT} + \frac{1}{T}\frac{dp}{dT} = \frac{1}{T}\frac{d\rho}{dT}$$

$$\Rightarrow \quad \frac{dp}{dT} = \frac{1}{T}(\rho + p)$$

$$\begin{aligned} \Rightarrow \quad dS &= \tfrac{1}{T} d[(\rho+p)V] - \tfrac{V}{T} dp \\ &= \tfrac{1}{T} d[(\rho+p)V] - \tfrac{V}{T^2}(\rho+p)dT \\ &= d\left(\tfrac{V}{T}(\rho+p)\right) \end{aligned}$$

$$\Rightarrow \quad \boxed{S(V,T) = \frac{V}{T}(\rho(T)+p(T))}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T)+p(T))}$$

Einstein eq. $\quad \frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)$

$$dS = \frac{1}{T}d(\rho a^3) + \frac{p}{T}d(a^3) \quad \Rightarrow \quad \frac{dS}{dt} = 0 \;\Rightarrow\; S \ \mathrm{const}$$

$$\Rightarrow \quad \begin{aligned} dS &= \tfrac{1}{T} d[(\rho + p)V] - \tfrac{V}{T} dp \\ &= \tfrac{1}{T} d[(\rho + p)V] - \tfrac{V}{T^2}(\rho + p) dT \\ &= d\left(\tfrac{V}{T}(\rho + p)\right) \end{aligned}$$

$$\Rightarrow \quad \boxed{S(V,T) = \frac{V}{T}(\rho(T) + p(T))}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

The universe expands adiabatically

Einstein eq. $\quad \frac{d}{dt}(a^3 \rho) = -p \frac{d}{dt}(a^3)$

$$dS = \frac{1}{T} d(\rho a^3) + \frac{p}{T} d(a^3) \quad \Rightarrow \quad \frac{dS}{dt} = 0 \quad \Rightarrow \quad S \ \mathrm{const}$$

Entropy of the Universe

$$\boxed{S = \frac{a^3}{T}(\rho(T) + p(T))}$$

Relativistic particle in thermal equilibrium

$$\rho = \frac{\pi^2}{30} g T^4 \qquad p = \frac{1}{3}\rho$$

entropy density

$$s = \frac{\pi^2}{30} g \left(1 + \frac{1}{3}\right) T^3 = \frac{2\pi^2}{45} g T^3$$

$$S = a^3 s = \mathrm{const} \;\Rightarrow\; T^3 a^3 = \mathrm{const}$$

$$\boxed{T \propto a^{-1}}$$

6. Redshift

Wavelength of light becomes longer as the universe expands

Geodesics of light

$$ds^2 = 0 = dt^2 - a^2(t)\frac{dr^2}{1 - Kr^2}$$

- Light emitted at $t=t_1$ $r=r_1$ reaches $r=0$ at $t=t_0$

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - Kr^2}}$$

- Light emitted at $t=t_1+\delta t_1$ $r=r_1$ reaches $r=0$ at $t=t_0+\delta t_0$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - Kr^2}}$$

$$\Rightarrow \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{t_1}^{t_0} \frac{dt}{a(t)} \qquad \Rightarrow \frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

$$\Rightarrow \quad \frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

Redshift

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\lambda_0}{\lambda_1} - 1 = \frac{\delta t_0}{\delta t_1} - 1 = \frac{a(t_0)}{a(t_1)} - 1$$

$$\boxed{\frac{a(t)}{a(t_0)} = \frac{1}{1+z}}$$

Momentum of photon (relativistic particle) decreases as 1/a

$$p_0 = \frac{a(t)}{a_0} p$$

Redshift of Momentum

- Momentum of a particle with mass m

$$\boxed{p = m\sqrt{-g_{ij}\frac{dx^i}{d\eta}\frac{dx^j}{d\eta}}}$$

local inertial Cartesian coordinate

$$\begin{aligned} d\eta &= dt\sqrt{1-v^2} \qquad v^i = dx^i/dt \\ p^i &= mv^i/\sqrt{1-v^2} \end{aligned}$$

- Equation of motion

$$\boxed{\frac{d^2x^i}{d\eta^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\eta}\frac{dx^\nu}{d\eta} = -\frac{2}{a}\frac{da}{dt}\frac{dx^i}{d\eta}\frac{dt}{d\eta}}$$

spatial coordinate system in which the particle position is near the origin x=0

$$\begin{aligned} g_{ij} &= a^2\left(\delta_{ij} + K\frac{x^i x^j}{1-K\vec{x}^2}\right) \\ &= a^2(\delta_{ij} + O(\vec{x}^2)) \end{aligned} \qquad \Gamma^i_{j\ell} = 0 \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}$$

$$\times\frac{d\eta}{dt} \qquad \Rightarrow \qquad \frac{d}{dt}\frac{dx^i}{d\eta} = -\frac{2}{a}\frac{da}{dt}\frac{dx^i}{d\eta}$$

$$\frac{d}{dt}\frac{dx^i}{d\eta} = -\frac{2}{a}\frac{da}{dt}\frac{dx^i}{d\eta}$$

$$\Rightarrow \quad \frac{dx^i}{d\eta} \propto \frac{1}{a^2}$$

$$g_{ij} \propto a^2$$

$$p = m\sqrt{-g_{ij}\frac{dx^i}{d\eta}\frac{dx^j}{d\eta}} \quad \Rightarrow \quad \boxed{p(t) \propto 1/a(t)}$$

Photon temperature

Photon is in thermal equilibrium at $t = t_*$

$$f(p_*) = \frac{1}{\exp(p_*/T_*) - 1}$$

If the photon freely streams without interaction after t_*

$$\boxed{p = \frac{a(t_*)}{a(t)} p_*} \quad \Rightarrow \quad f(p) = \frac{1}{\exp(p/T_*(a(t)/a(t_*))) - 1}$$

Define $$T = \frac{a(t_*)}{a(t)} T_*$$

$$\boxed{f(p) = \frac{1}{\exp(p/T) - 1}}$$

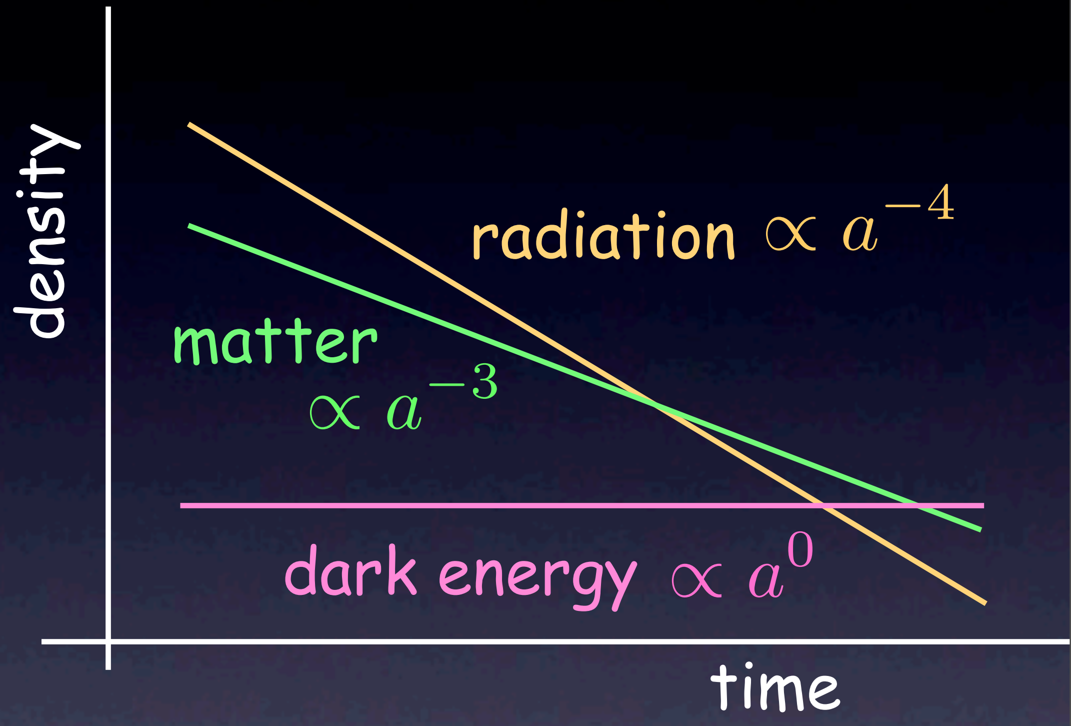
Same as equilibrium distribution with T

$$\boxed{T \propto a^{-1}}$$

7. Radiation Dominated Universe

The early universe is dominated by radiation

$$\boxed{\frac{a_0}{a} > \frac{a_0}{a_{eq}} = 2.7 \times 10^4 (\Omega_m h^2)}$$



Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \underbrace{\frac{K}{a^2}}_{\propto a^{-2}} - \underbrace{\frac{\Lambda}{3}}_{\propto a^0} = \underbrace{\frac{8\pi G}{3}\rho}_{\propto a^{-3 \text{ or } -4}}$$

We can neglect K- and Λ– terms

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \underset{T \propto 1/a}{\Rightarrow} \left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3}\rho$$

Total energy density

$$\boxed{\begin{aligned} \rho &= \frac{\pi^2}{30} g_* T^4 \\ g_* &= \sum_{\text{boson}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermion}} g_i \left(\frac{T_i}{T}\right)^4 \end{aligned}}$$

for example

T=1MeV (γ, e, 3ν)

$$g_* = \underset{\gamma}{2} + \frac{7}{8} \times \underset{e^\pm}{2} \times 2 + \frac{7}{8} \times \underset{3\nu\bar{\nu}}{3} \times 2 = \frac{43}{4}$$

$$\Rightarrow \left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3}\frac{\pi^2}{30}g_* T^4 = \frac{g_*}{3M_G^2}\frac{\pi^2}{30}T^4$$

$$M_G \equiv \frac{1}{\sqrt{8\pi G}} \simeq 2.4 \times 10^{18}\,\mathrm{GeV}$$

$$\boxed{\begin{aligned} t &= \left(\frac{45}{2\pi^2 g_*}\right)^{1/2} \frac{M_G}{T^2} \\ &\simeq 2.3\,\mathrm{sec}\, g_*^{-1/2} \left(\frac{T}{10^{10}\mathrm{K}}\right)^{-2} \\ &\simeq 1.7\,\mathrm{sec}\, g_*^{-1/2} \left(\frac{T}{\mathrm{MeV}}\right)^{-2} \end{aligned}}$$

$$\Rightarrow \; a(t) \propto t^{1/2}$$

8. Horizon

- Particle Horizon
  maximum travel distance of light emitted at t=0 until t

Geodesics of light $\quad ds^2 = 0 = dt^2 - a^2(t)\dfrac{dr^2}{1-Kr^2}$

$$\boxed{\ell_H(t) \equiv a(t)\int_0^t \frac{dt'}{a(t')}}$$

$$a(t) \propto t^m \quad \begin{cases} m = 1/2 & (RD) \\ \quad = 2/3 & (MD) \end{cases}$$

$$\Rightarrow \quad \ell_H(t) = \frac{t}{1-m} = \begin{cases} 2t & (RD) \\ 3t & (MD) \end{cases}$$

- Event Horozon

maximum travel distance of light emitted at t until t=t(max)

$$\boxed{\ell_{He}(t) \equiv a(t) \int_t^{t_{max}} \frac{dt'}{a(t')}}$$

For exponentially expanding universe $a \propto \exp(Ht)$

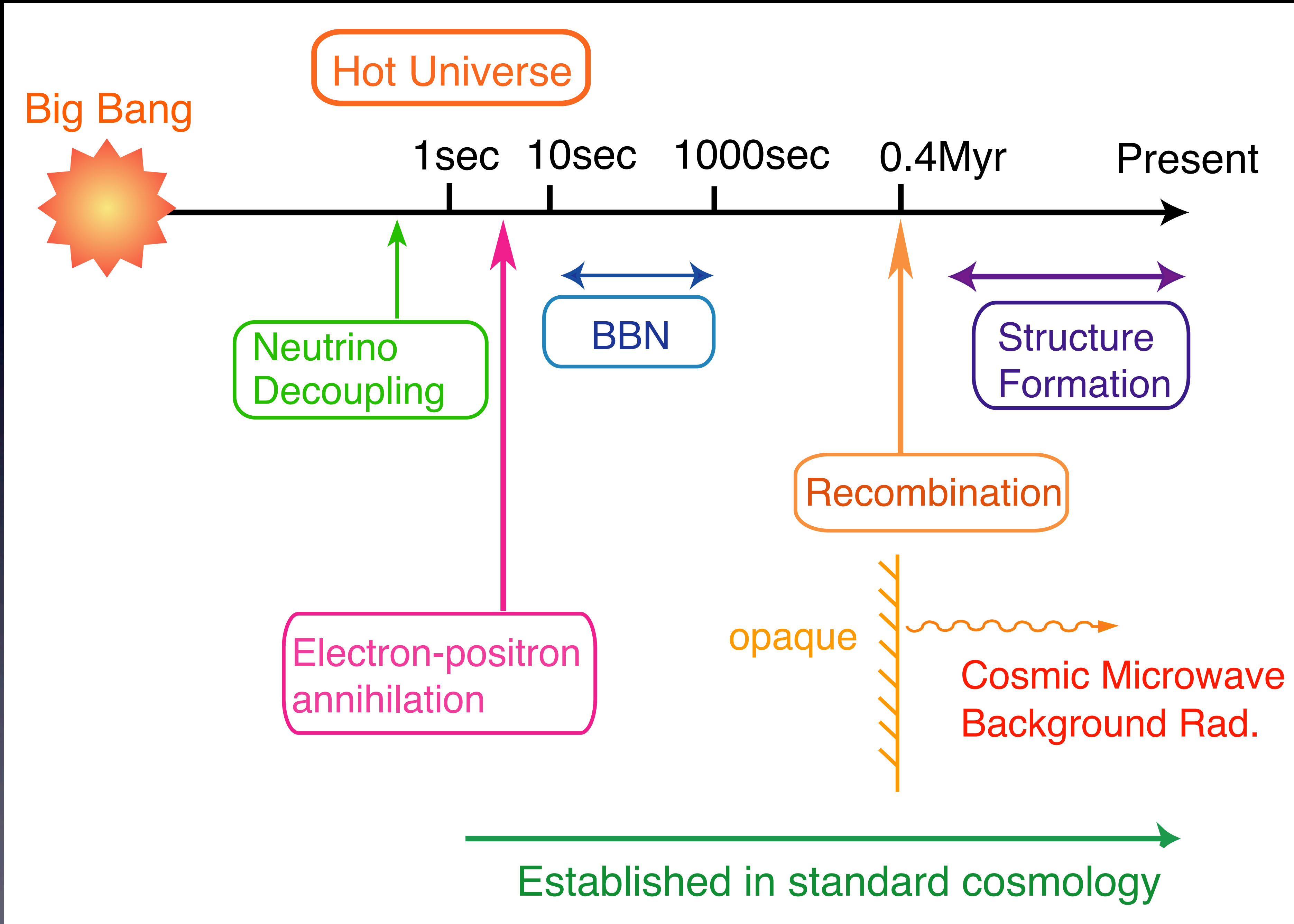
$$\ell_{He}(t) = e^{Ht} \int_t^{\infty} e^{-Ht'} dt' = 1/H$$

- Hubble Radius

$$\boxed{\equiv H^{-1}(t) = \frac{a}{\dot{a}}}$$

$$a \propto t^m \;\; \Rightarrow \;\; H(t)^{-1} = \frac{t}{m} = \begin{cases} 2t & (RD) \\ 3t/2 & (MD) \end{cases} \sim \ell_H$$

9. (Thermal) History of the Universe



10. Neutrino Decoupling

- T > 2 MeV

  Neutrinos are in thermal equilibrium via weak interaction

  $$\nu_i + \nu_i \leftrightarrow e^+ + e^-$$

  cross section:

  $$\langle \sigma v \rangle \simeq \frac{4G_F^2}{9\pi} \langle E^2 \rangle \simeq \frac{4G_F^2}{9\pi} T^2$$

  Boltzmann eq.

  $$\boxed{\frac{dn_\nu}{dt} + 3\frac{\dot{a}}{a} n_\nu = -\langle \sigma v \rangle (n_\nu^2 - n_{\nu,eq})}$$

  $$3(\dot{a}/a) \ll \langle \sigma v \rangle n_\nu \;\Rightarrow\; n_\nu = n_{\nu,eq}$$

  $$3(\dot{a}/a) \gg \langle \sigma v \rangle n_\nu \;\Rightarrow\; \nu \text{ decouple}$$

  $$3(\dot{a}/a) \simeq \langle \sigma v \rangle n_\nu \;\Rightarrow\; G_F^2 T^2 T^3 \simeq T^2 / M_G \;\Rightarrow\; T_d \simeq 2 \text{ MeV}$$

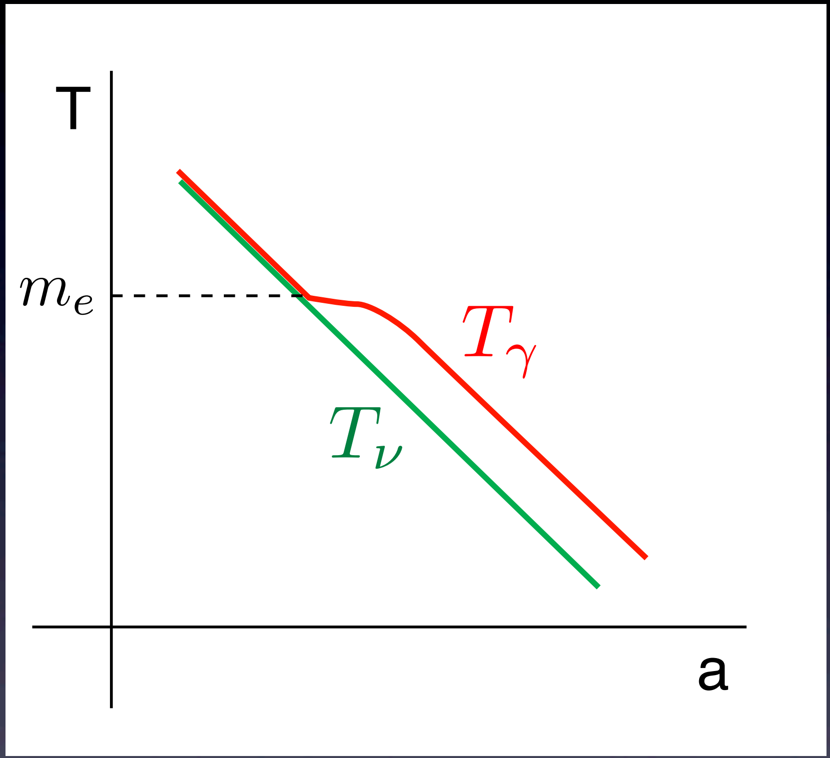
- T < 2 MeV

  $$\boxed{e, \gamma} \not\leftrightarrow \boxed{3\nu}$$

- $T_1 > m_e$ $\quad S_\gamma = a_1^3 T_1^3 \left(2 + \frac{7}{8} \times 2 \times 2\right) \frac{2\pi^2}{45}$

- $T \sim m_e$ $\quad \boxed{e^+ + e^- \;\rightarrow\; 2\gamma}$

- $T_2 < m_e$ $\quad S_\gamma = a_2^3 T_2^3 \,(2)\, \frac{2\pi^2}{45}$



Entropy conservation

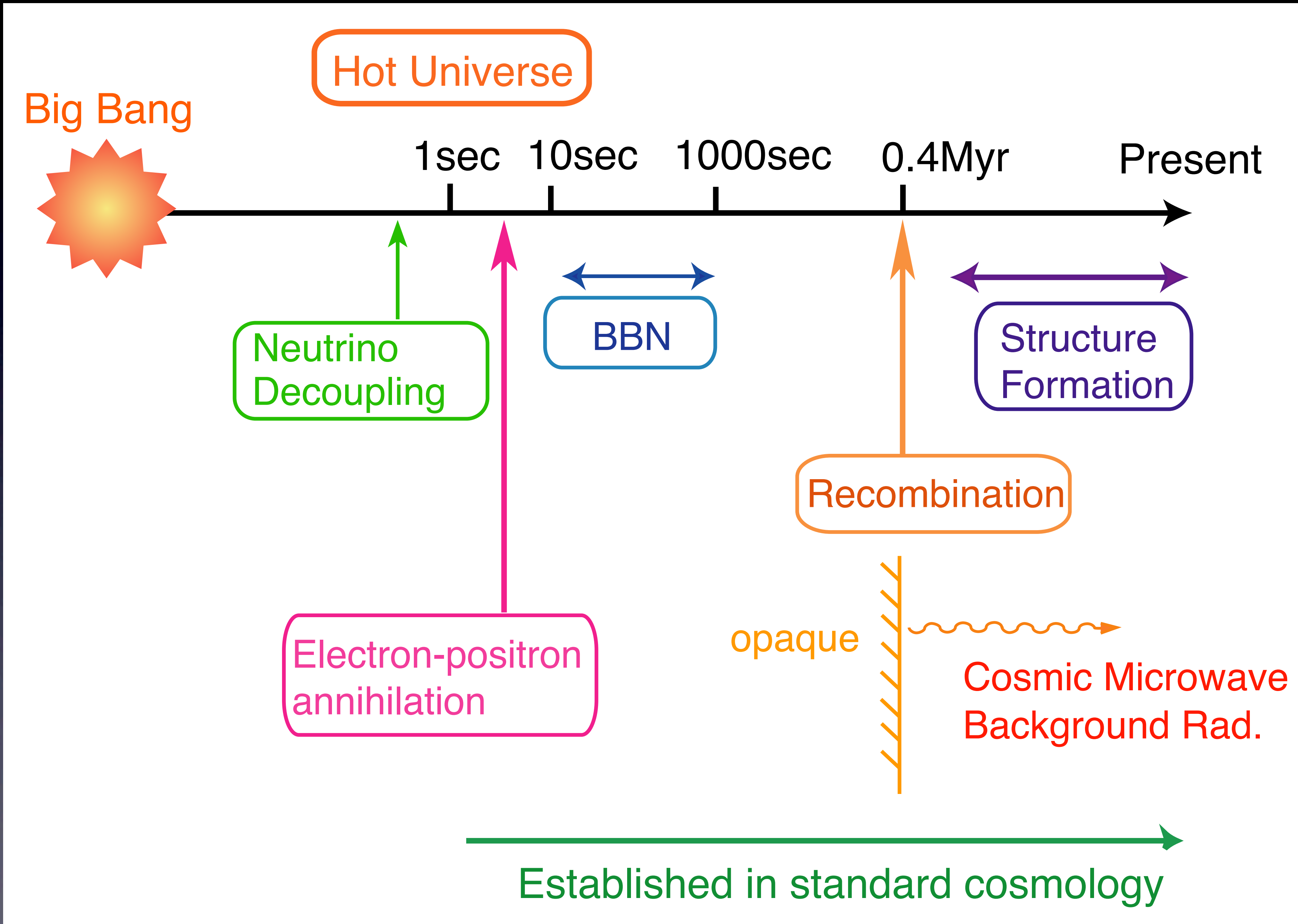
$$\Rightarrow \; T_2 = T_1 \left(\frac{a_1}{a_2}\right) \left(\frac{11}{4}\right)^{1/3}$$

On the other hand

$$T_{\nu,2} = T_{\nu,1} \left(\frac{a_1}{a_2}\right)$$

$$\rightarrow \quad \boxed{T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3}}$$

History of the Universe

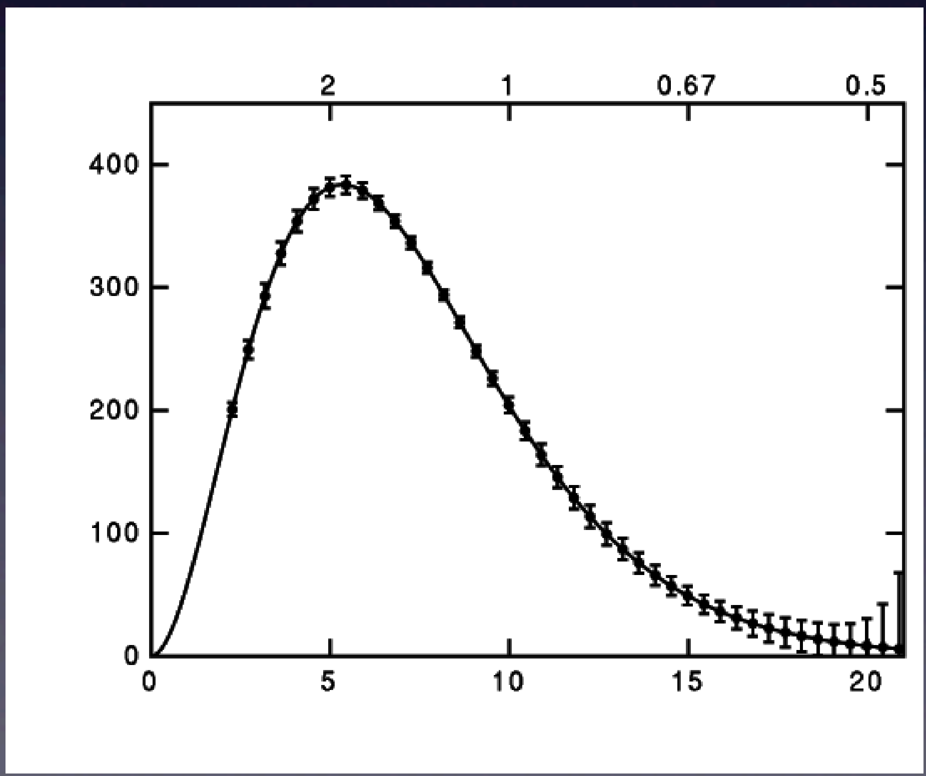


At present

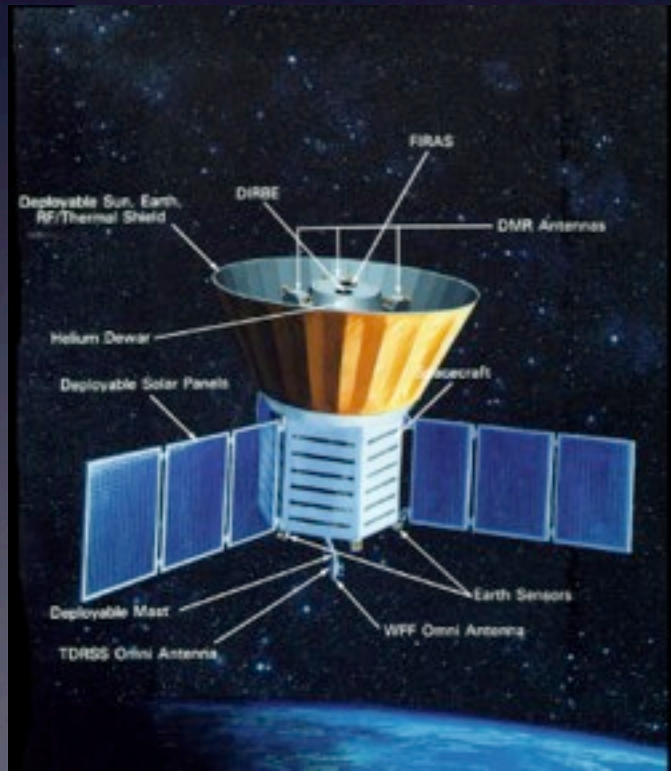
$$T_{\gamma,0} = 2.726\mathrm{K} \quad \Rightarrow \quad T_{\nu,0} = 1.95\mathrm{K}$$

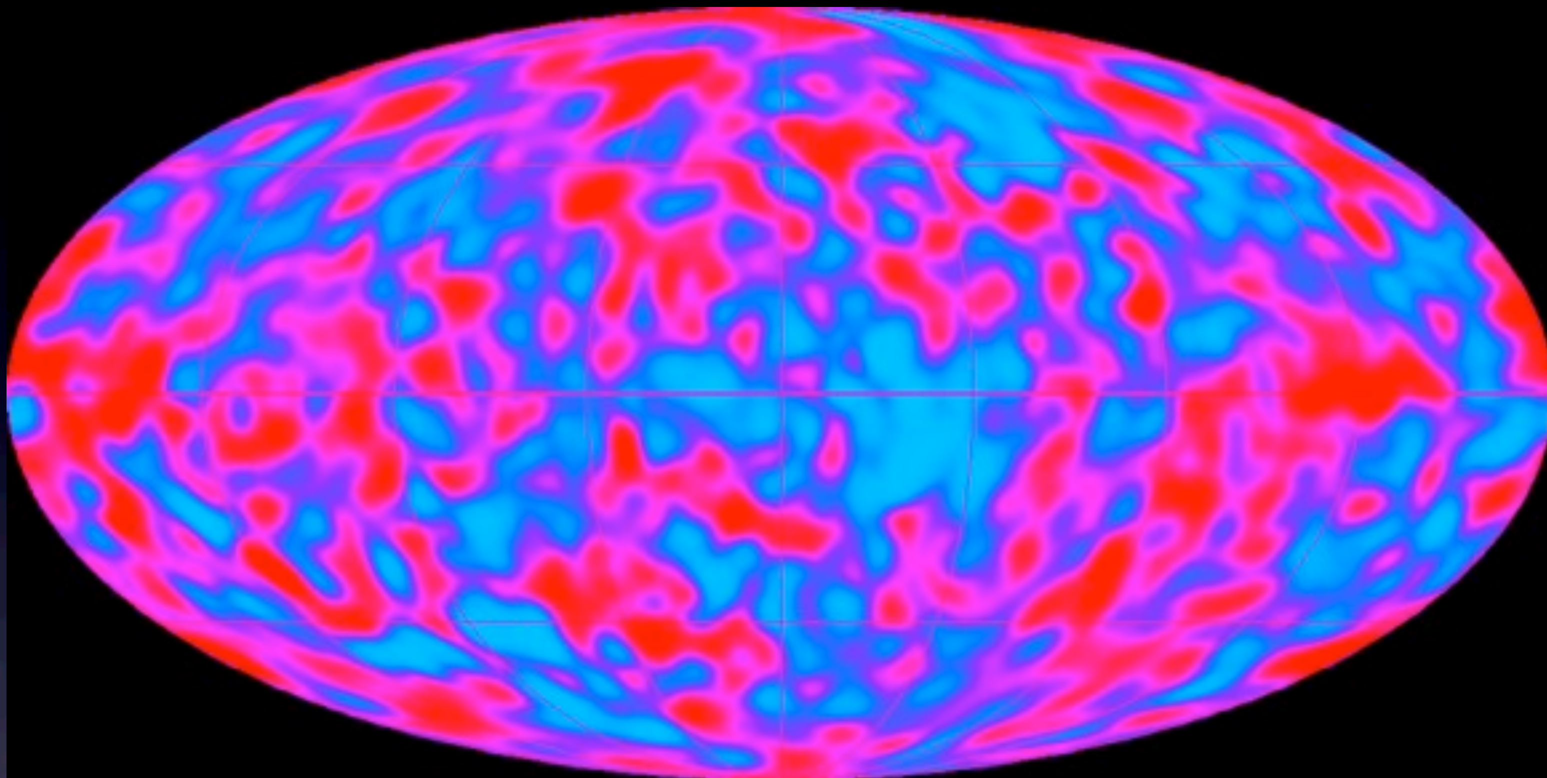
$$\frac{n_\nu}{n_\gamma} = \frac{3}{4}\left(\frac{T_\nu}{T_\gamma}\right)^3 = \frac{3}{4}\frac{4}{11} = \frac{3}{11}$$

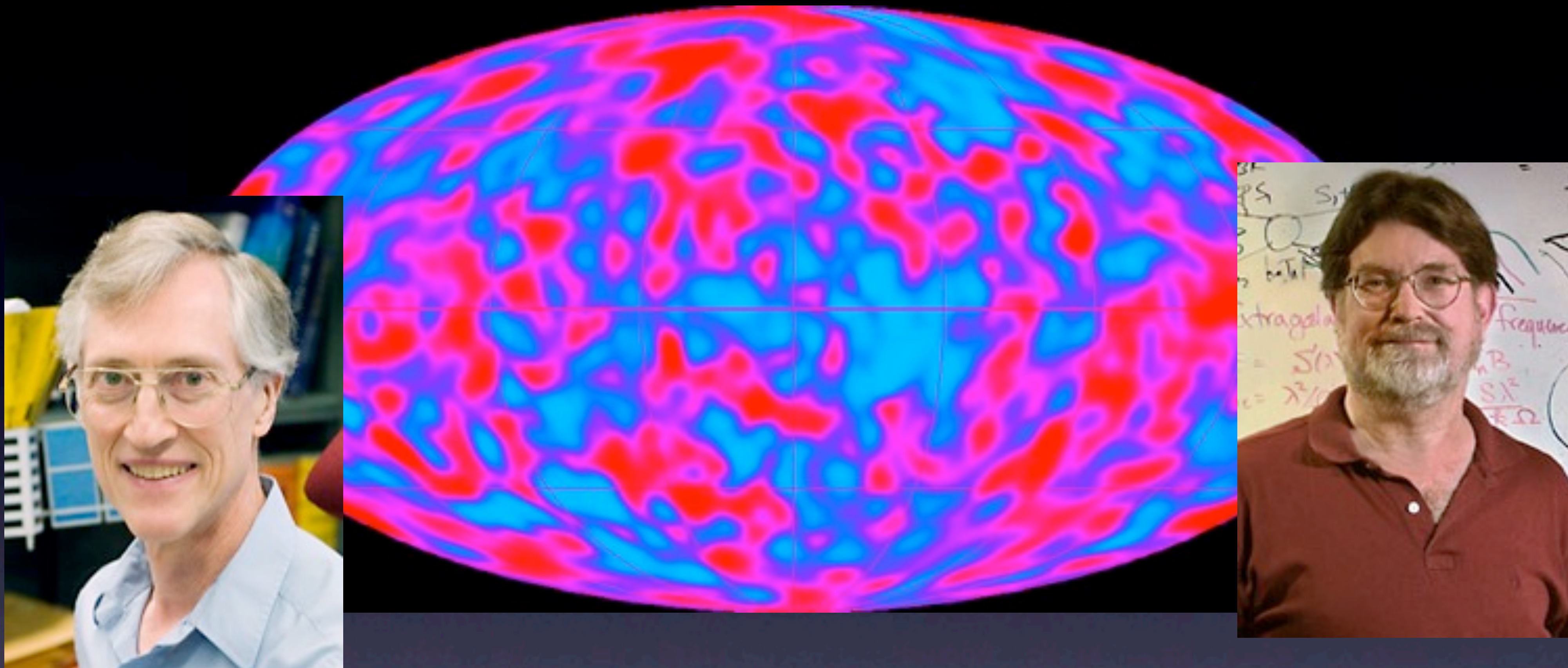
$$n_{\gamma,0} = 415\ \mathrm{cm}^{-3} \quad \Rightarrow \quad n_{\nu,0} = 113\ \mathrm{cm}^{-3}$$



error bars are multiplied by 400







2006 Nobel Prize in Physics

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

11. Big Bang Nucleosynthesis (BBN)

In the early universe (T=1 - 0.01MeV)

$$\boxed{2p + 2n \rightarrow {}^4\mathrm{He}}$$ + small D $\quad {}^3\mathrm{He}$ $\quad {}^7\mathrm{Li}$

A. Initial Condition

p and n interchange via weak interaction

$$\boxed{\begin{aligned} \nu_e + n &\leftrightarrow p + e^- \\ e^+ + n &\leftrightarrow p + \bar{\nu}_e \\ n &\leftrightarrow p + e^- + \bar{\nu}_e \end{aligned}}$$

Reaction Rate $\quad \Gamma \sim \sigma v n_e \sim G_F^2 T^2 T^3 \sim G_F^2 T^5$

$\Gamma \gg H$ $\Rightarrow$ Chemical Equilibrium

$$\boxed{\mu_{\nu_e} + \mu_n = \mu_p + \mu_{e^-}}$$

$$n = \frac{g}{2\pi^2}\int_0^\infty p^2 dp \frac{1}{\exp[(E-\mu)/T] \pm 1}$$

non-relativistic

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp[-(m-\mu)/T]$$

$\mu_e/T \sim 10^{-10} \ll 1$ $\qquad$ $n_{e^-} - n_{e^+} = \frac{1}{3}\mu_e T^2 = n_p$

$\mu_\nu/T \ll 1 \longleftarrow$ assumption

$$\boxed{\mu_n = \mu_p}$$

$$\frac{n_n}{n_p} = \exp[(m_p - m_n + \mu_n - \mu_p)/T]$$
$$= \exp[-Q/T + (\mu_n - \mu_p)/T]$$

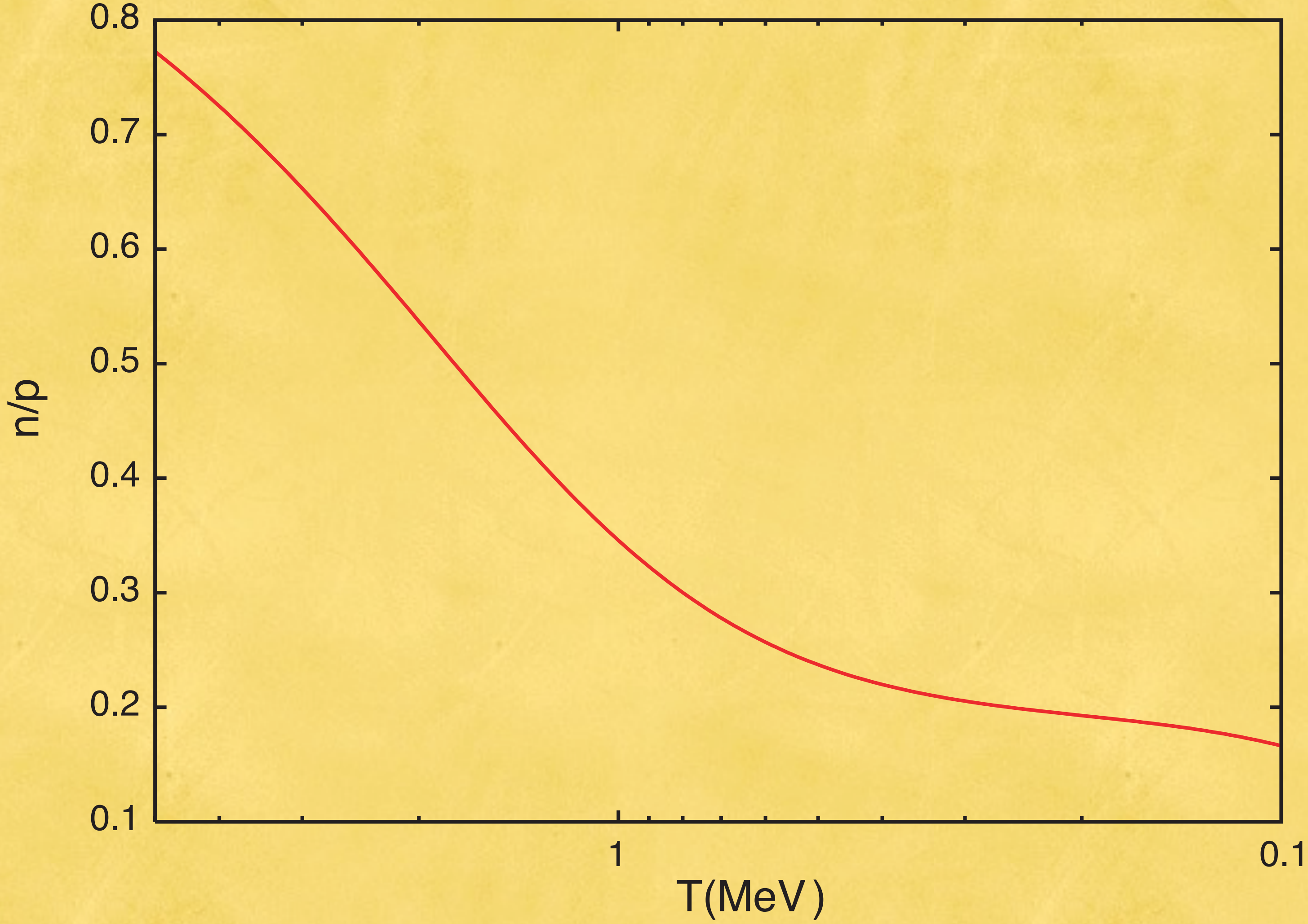
$$Q = m_n - m_p = 1.293 \text{ MeV}$$

$$\boxed{\left(\frac{n_n}{n_p}\right)_{eq} = \exp\left(-\frac{Q}{T}\right)}$$

$\Gamma \sim H \quad \Rightarrow \quad T_f$ **freeze-out temp**

$$G_F^2 T_f^5 \sim \frac{T_f^2}{\sqrt{3} M_G}\left(\frac{\pi^2}{30} g_*\right)^{1/2} \quad \Rightarrow \quad T_f \sim 1 \text{ MeV}$$

$$\Rightarrow \quad \boxed{\frac{n_n}{n_p} \simeq \exp\left(-\frac{Q}{T_f}\right) \simeq \frac{1}{7}}$$


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
n/p
1
0.1
T(MeV)

B. 0.1 MeV < T < 1 MeV

$$p + n \ \leftrightarrow \ D + \gamma \qquad Q_d = 2.22 \text{ MeV}$$

$n_\gamma \sim 10^{10} n_B \gg n_B$

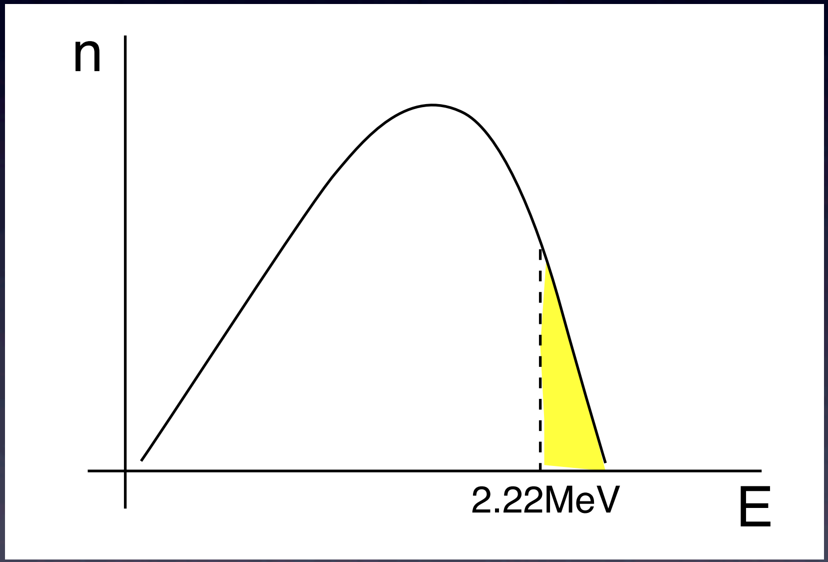
Produced D is destroyed

$$D + \gamma \to p + n$$

$T \simeq 0.1 \text{ MeV}$

$n_\gamma(E_\gamma > 2.22\text{MeV}) \searrow$

$$\Rightarrow \boxed{p + n \ \to \ D + \gamma}$$



C. T < 0.1 MeV

$$D + D \ \leftrightarrow \ {}^3\text{He} + p$$

$${}^3\text{He} + n \ \leftrightarrow \ {}^3\text{H} + p$$

$${}^3\text{H} + D \ \leftrightarrow \ {}^4\text{He} + n$$

$${}^4\text{He}$$

$+ \text{ small amount of } D, \ {}^3\text{He}, \ {}^3\text{H}$

$({}^3\text{H} \to {}^3\text{He} + e^- + \nu_e, \quad \tau_{1/2} \sim 12\text{yr})$

D. Heavier Light Elements?

No

- No stable nuclei with A=5 or 8
- Coulomb Barrier

But tiny amount of Li7

$$^4\mathrm{He} + {}^3\mathrm{H} \rightarrow {}^7\mathrm{Li} + \gamma$$

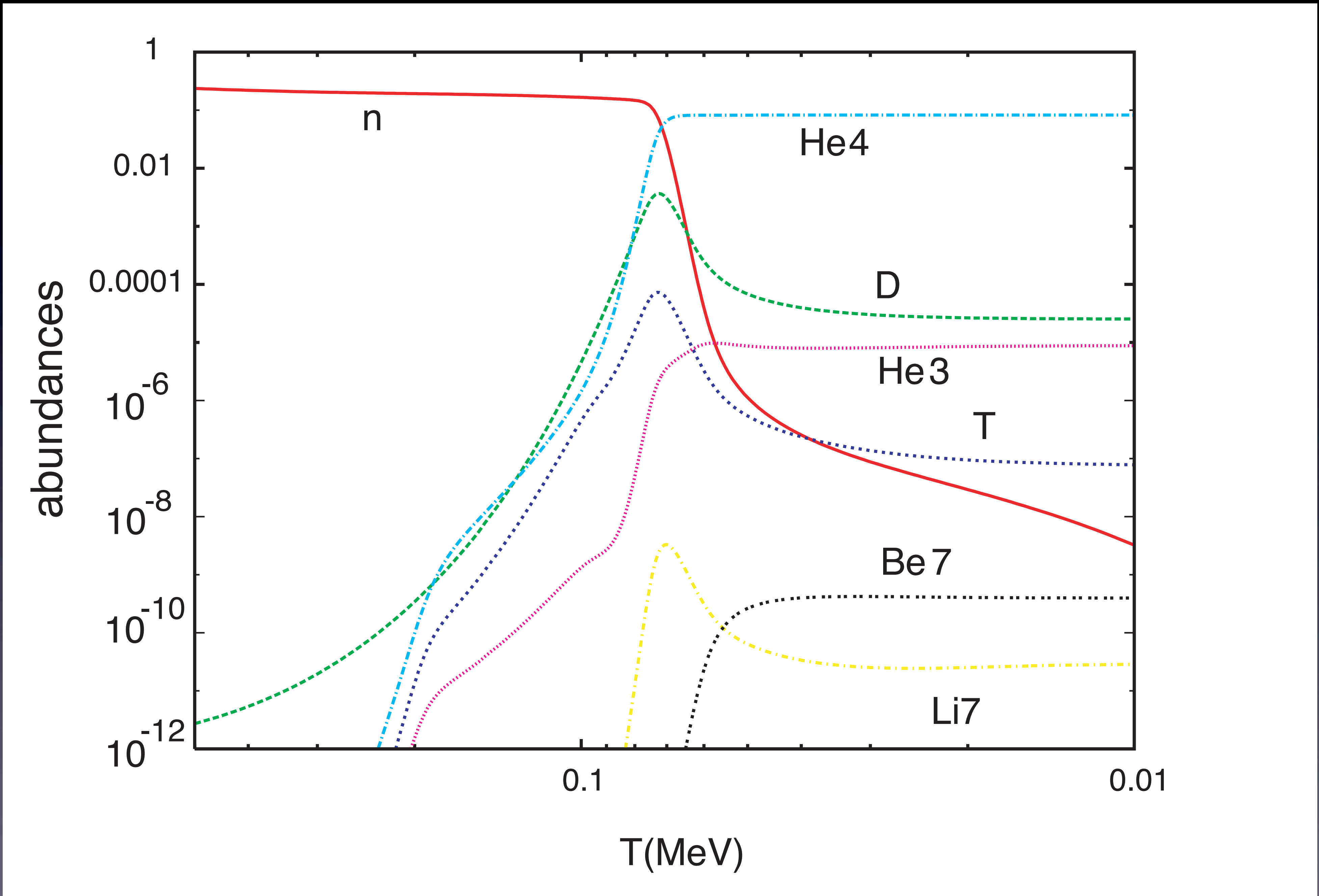
$$^4\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma$$

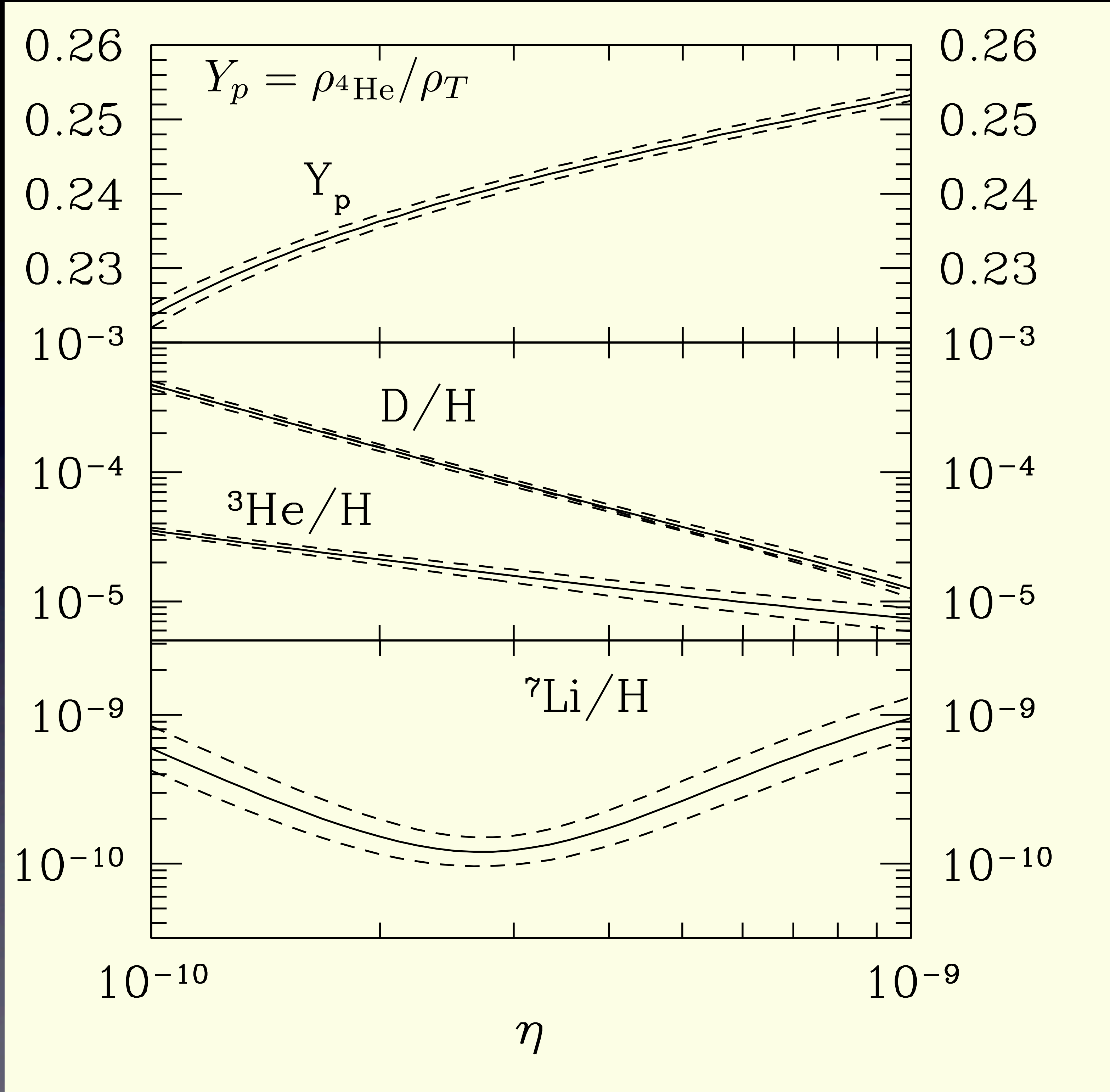
$$\hookrightarrow {}^7\mathrm{Be} + e^- \rightarrow {}^7\mathrm{Li} + \nu_e$$

Abundances of Light Elements only depend on baryon-to-photon ratio

$$\boxed{\eta_B \equiv \frac{n_B}{n_\gamma}}$$

Evolution of Light Elements




$Y_p = \rho_{^4\mathrm{He}}/\rho_T$
0.26
0.25
0.24
0.23
$\mathrm{Y_p}$
10^{-3}
D/H
10^{-4}
^{3}He/H
10^{-5}
^{7}Li/H
10^{-9}
10^{-10}
10^{-10}
10^{-9}
η

Observational Abundances of Light Elements (old)

- He4

$$Y_p = 0.238 \pm 0.002 \pm 0.005$$ Fields,Olive (1998)

$$Y_p = 0.242 \pm 0.002(\pm 0.005)$$ Izotov et al. (2003)

$$Y_p = 0.250 \pm 0.004$$ Fukugita,Kawasaki (2006)

- D/H

$$D/H = (2.8 \pm 0.4) \times 10^{-5}$$

Kirkman et al. (2003)

- Li7/H

$$\log_{10}(^7Li/H) = -9.66 \pm 0.056 \ (\pm 0.3)$$

Bonifacio et al. (2002)

Observational Abundances of Light Elements

- He4

$$Y_p = 0.2516 \pm 0.0040$$

Izotov et al. (2007)

- D/H

$$\mathrm{D/H} = (2.82 \pm 0.26) \times 10^{-5}$$

O'Meara et al. (2006)

- Li7/H

$$\log_{10}(^7\mathrm{Li/H}) = -9.90 \pm 0.09 + 0.3$$

Bonifacio et al. (2006)

- Li6/H

$$^6\mathrm{Li}/^7\mathrm{Li} < 0.046 \pm 0.022 + 0.084$$

Asplund et al. (2006)

- He3/D

$$^3\mathrm{He/D} < 0.83 \pm 0.27$$

Geiss and Gloeckler (2003)

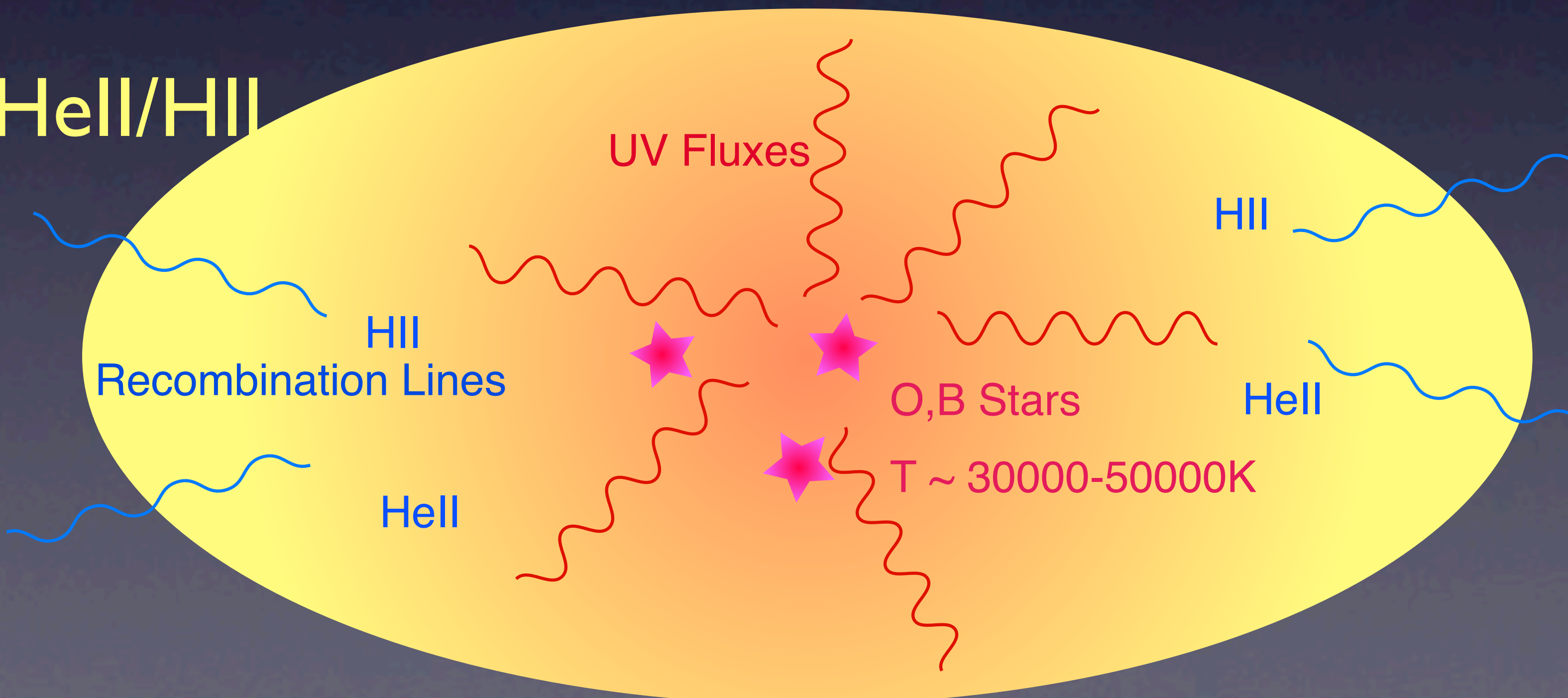
Measurement of He in HII region

- HII region
  - OB stars ionize H and He
  - E(HI)= 13.6eV, E(HeI)= 24.6eV,E(HeII)= 56.4eV
- Recombination lines

NGC 6611

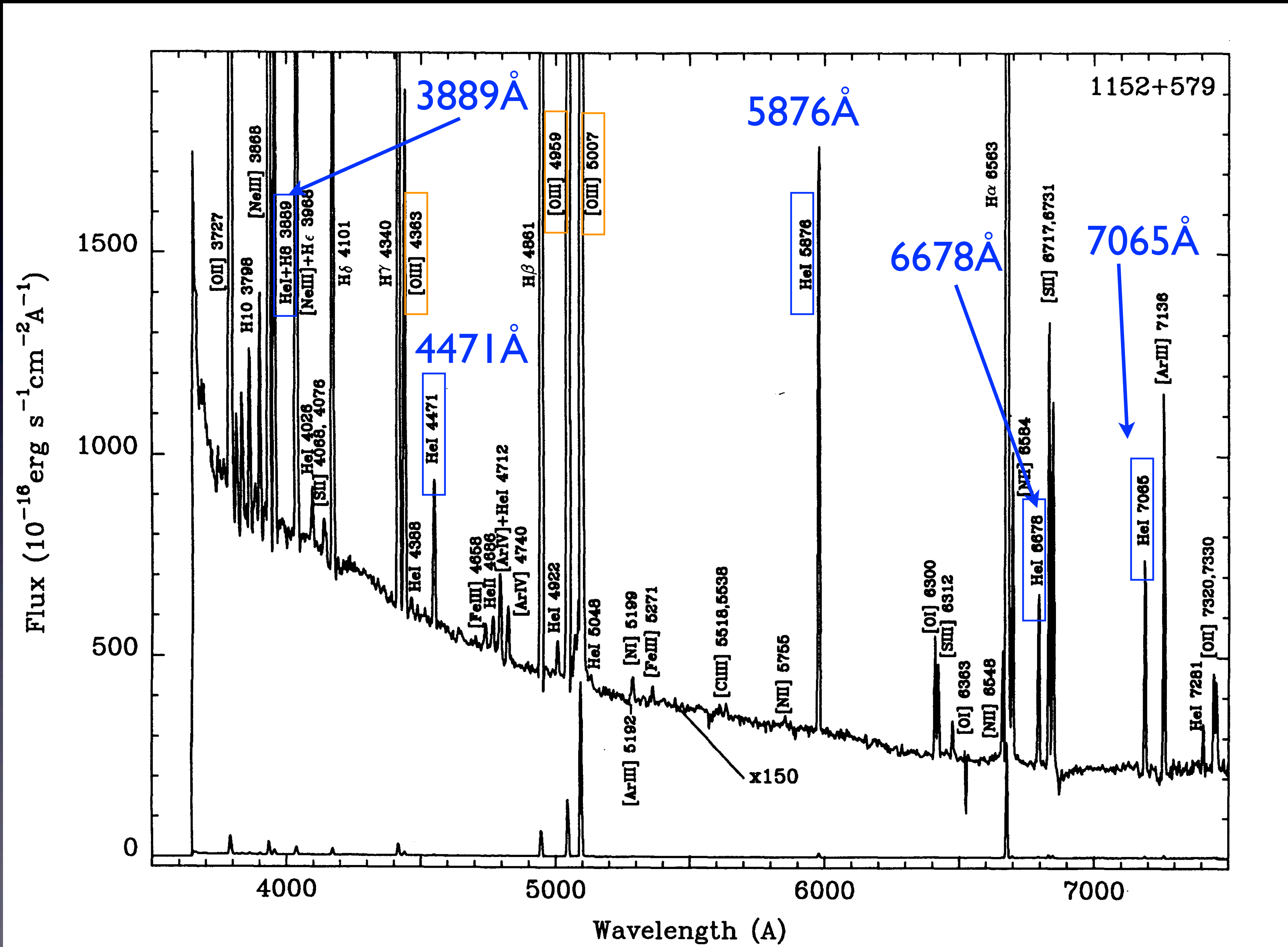
H II HeII

- measure HeII/HII



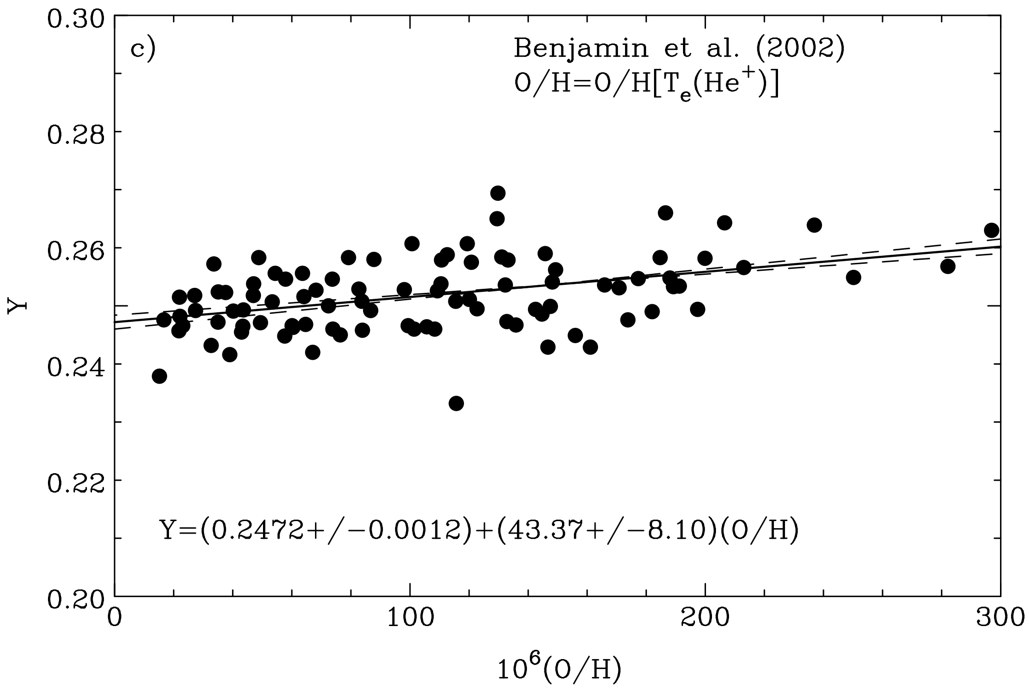
Spectrum

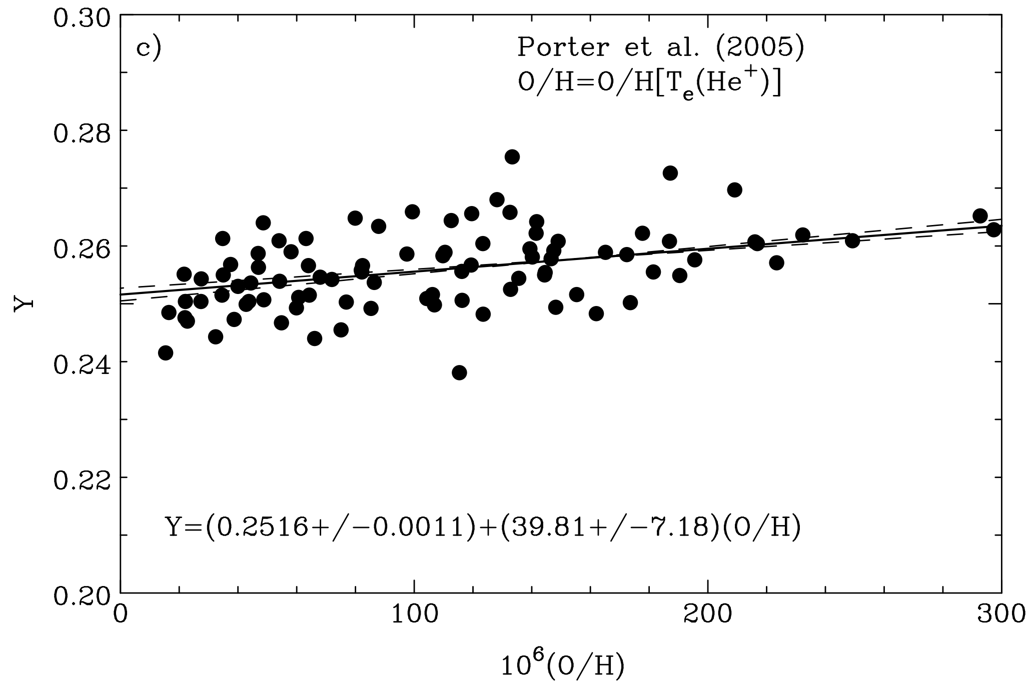
MRK 193 Izotov, Thuan, Lipovetsky (1994)



2009年11月11日水曜日

Izotov, Thuan, Stasinska 2007





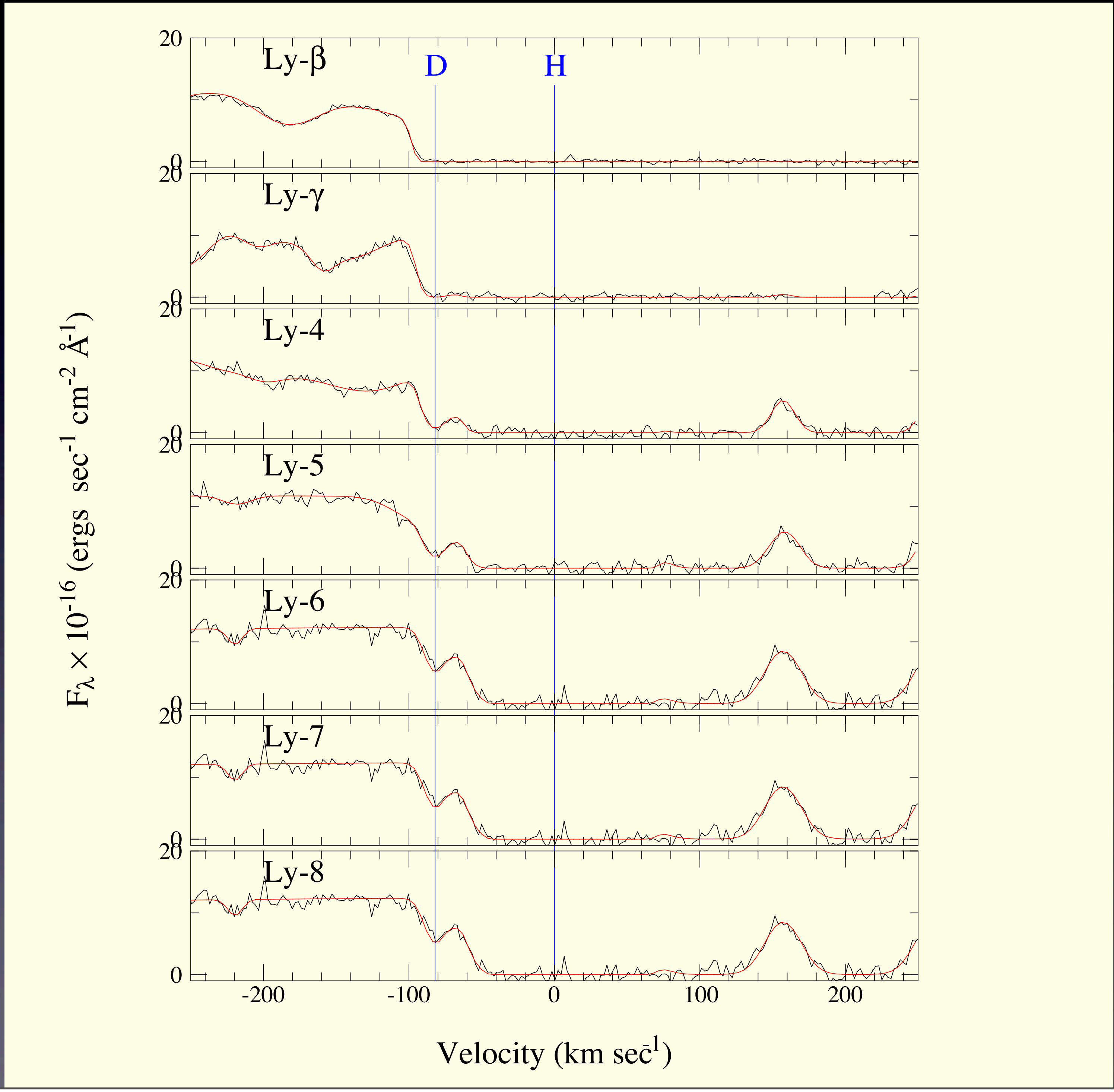
BBS

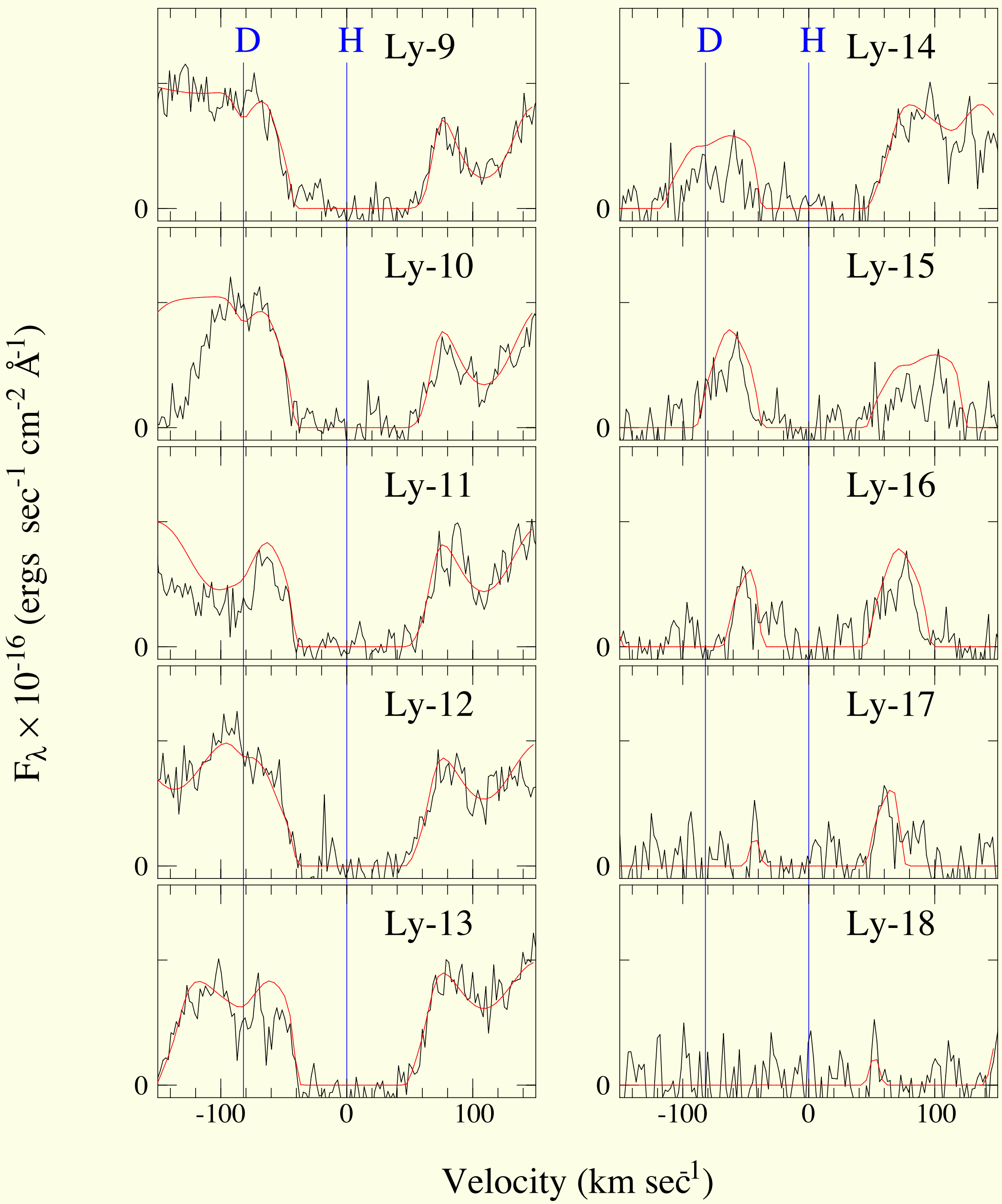
$$Y_p = 0.2472 \pm 0.0012$$

PBFM

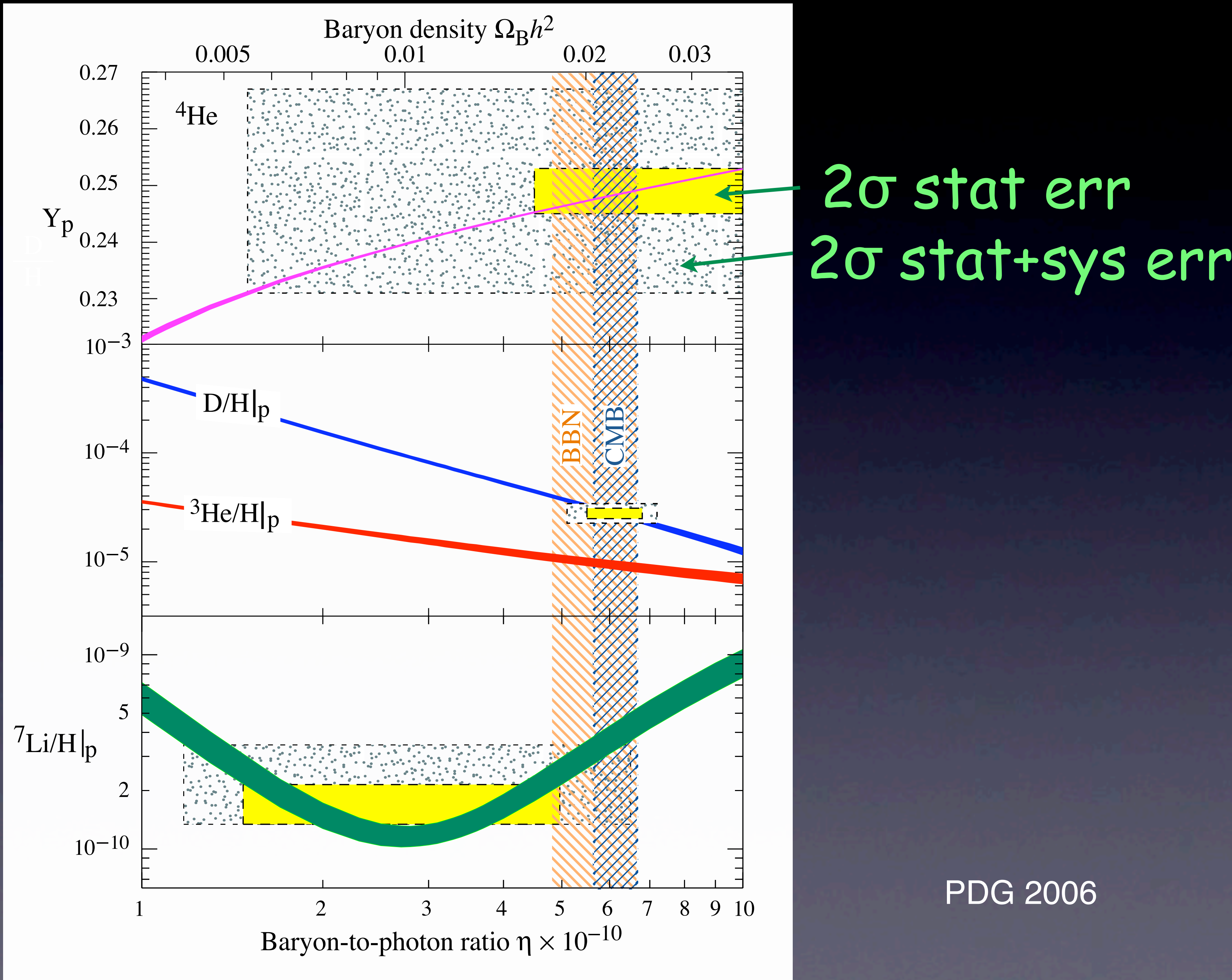
$$Y_p = 0.2516 \pm 0.0011$$

D absorption in QSO spectrum




D
H
Ly-9
Ly-10
Ly-11
Ly-12
Ly-13
Ly-14
Ly-15
Ly-16
Ly-17
Ly-18
0
-100
0
100
Fλ × 10^-16 (ergs sec^-1 cm^-2 Å^-1)
Velocity (km sec^-1)

Theory vs Observation



PDG 2006

Number of Neutrino Species N_ν

BBN can impose a stringent limit on N_ν

$$\boxed{N_\nu \nearrow \quad \Rightarrow \quad \rho(T) \nearrow \quad \Rightarrow \quad H \nearrow \quad \Rightarrow \quad n/p \nearrow}$$

$\Rightarrow$ ^{4}He $\nearrow$

$$\begin{aligned}
\rho_\nu &= \frac{7}{8} \times 2 \times N_\nu \frac{\pi^2}{30} T^4 = \frac{7}{4} N_\nu \frac{\pi^2}{30} T^4 \\
\rho_{\rm tot} &= \rho_\nu + \left(2 + \frac{7}{8} \times 2 \times 2\right) \frac{\pi^2}{30} T^4 \\
&= \left(\frac{7}{4} N_\nu + \frac{22}{4}\right) \frac{\pi^2}{30} T^4
\end{aligned}$$

He4 abundance

$$Y_p \equiv \rho_{^4\rm He}/\rho_{\rm tot} \simeq 0.245 + 0.014(N_\nu - 3)$$

$$(\eta_B = 6 \times 10^{-10})$$

He4 abundance

$$Y_p \equiv \rho_{^4\mathrm{He}}/\rho_{\mathrm{tot}} \simeq 0.245 + 0.014(N_\nu - 3)$$

$$(\eta_B = 6 \times 10^{-10})$$

Observation

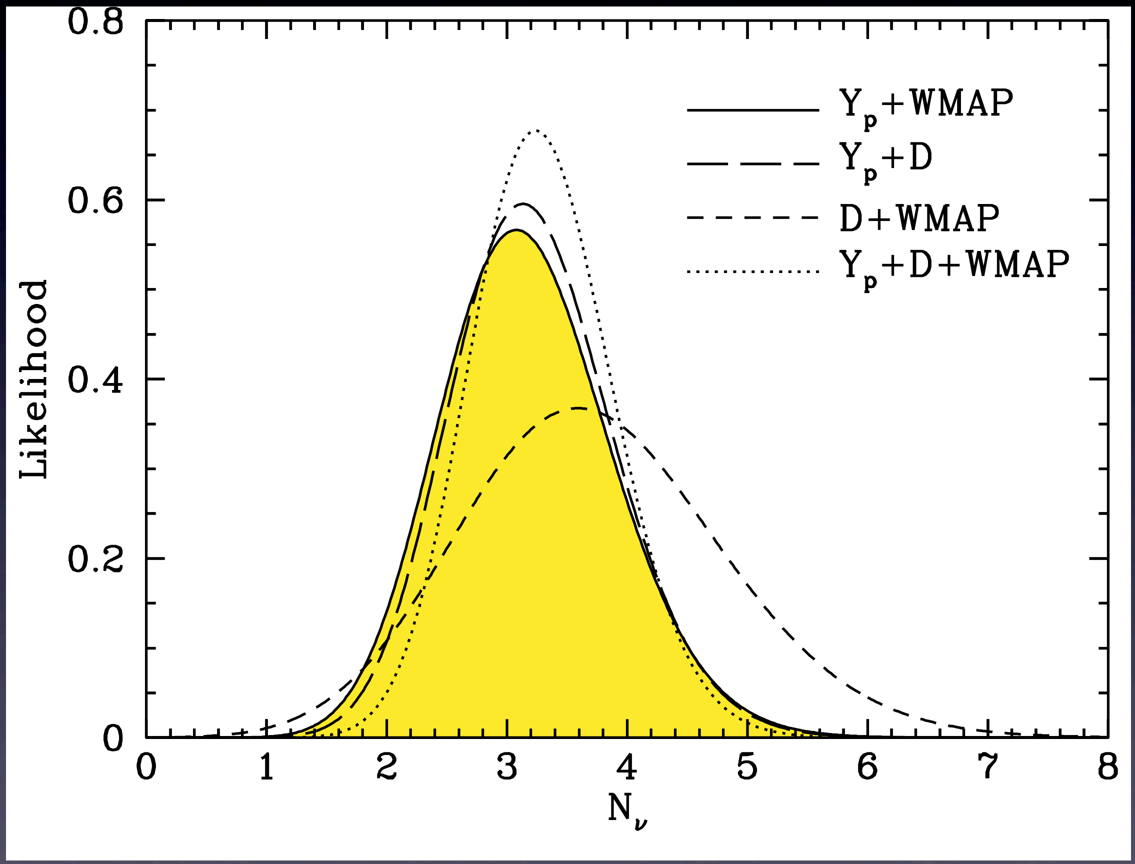
$$Y_{p,obs} < 0.26 \longleftarrow Y_p = 0.2516 \pm 0.0040$$

$$\Rightarrow \boxed{\Delta N_\nu \leq 1}$$

Number of Neutrino Species N_ν

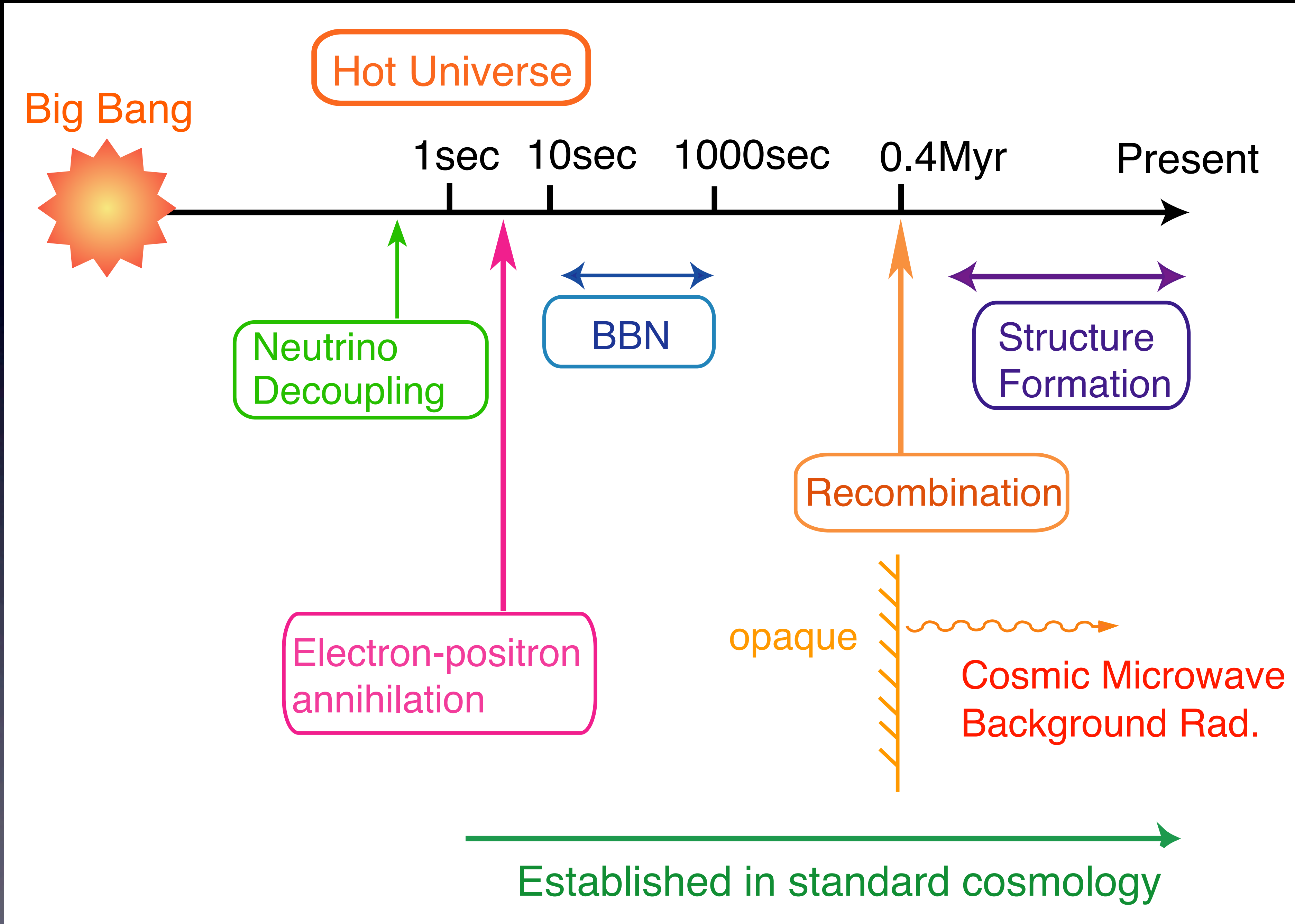
$$Y_p = 0.249 \pm 0.009$$ Olive, Skillman (2004)

Cyburt et al (2005)



$$N_\nu = 3.1 \pm 0.7$$

History of the Universe



12. Recombination

$$\boxed{p + e \ \rightarrow \ \mathrm{H} + \gamma}$$

at T = 4000K

ignoring He4

$$n_B = n_p + n_H \qquad n_p = n_e$$

Thermal density

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right) \quad (i = e, p, \mathrm{H})$$

Chemical equilibrium $\boxed{\mu_{\mathrm{H}} = \mu_e + \mu_p}$

$$\Rightarrow \quad \frac{n_{\mathrm{H}}}{n_p n_e} = \frac{g_{\mathrm{H}}}{g_p g_e} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left(\frac{B}{T}\right)$$

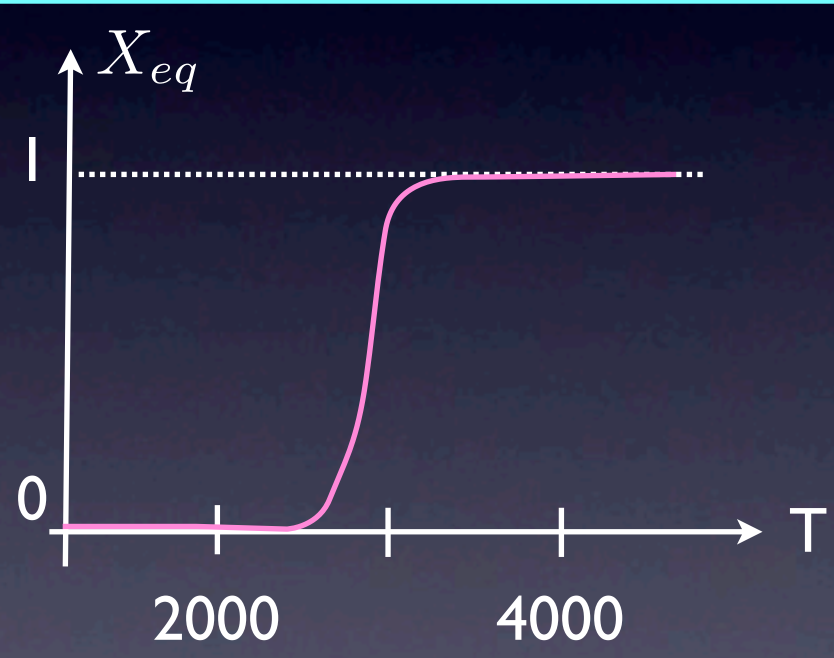
$$B = m_p + m_e - m_{\mathrm{H}} = 13.6 \text{ eV}$$

$$g_p = g_e = 2, \quad g_{\mathrm{H}} = 4$$

Define ionization fraction $X \equiv \frac{n_p}{n_B}$

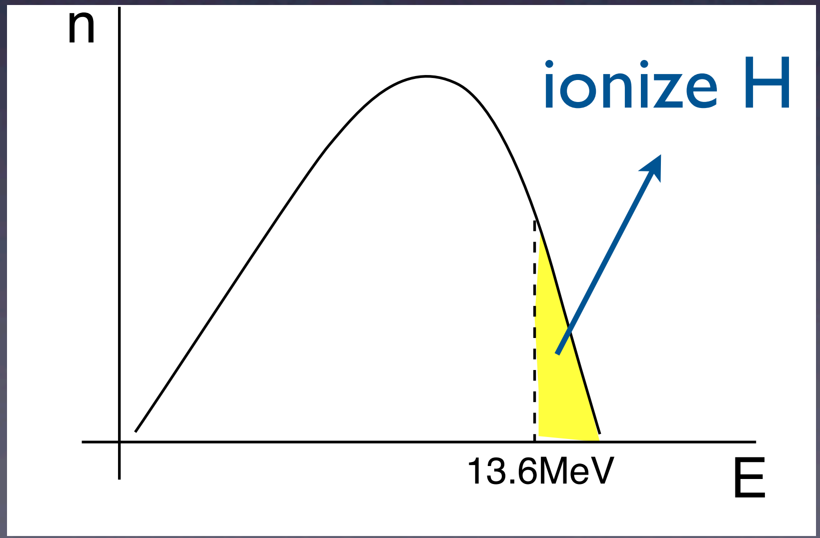
$$n_B = \eta_B n_\gamma = \eta_B \frac{2\zeta(3)}{\pi^2} T^3$$

$$\boxed{\frac{1-X_{eq}}{X_{eq}^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta_B \left(\frac{T}{m_e}\right)^{3/2} \exp\left(\frac{B}{T}\right) \quad \text{Saha formula}}$$

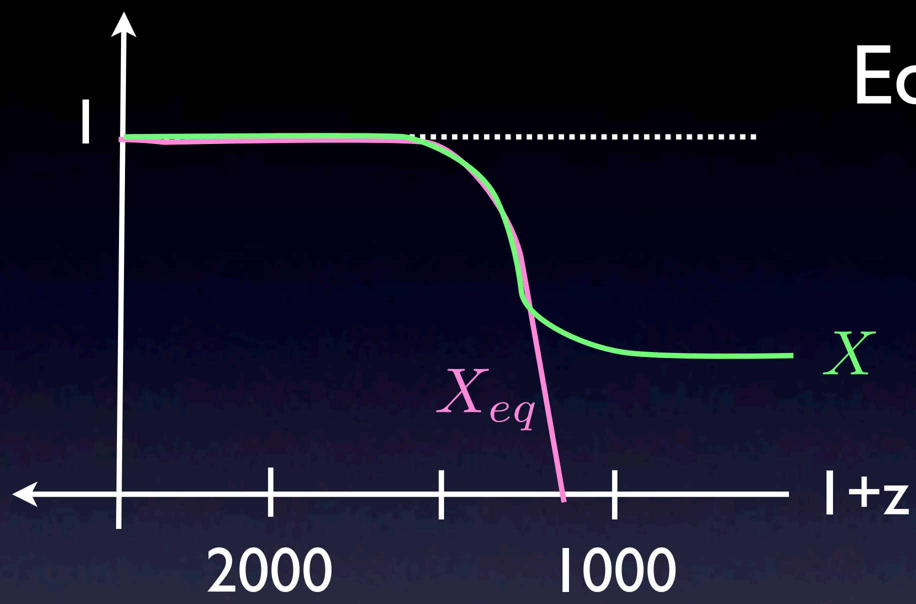


$T_{\rm rec} \simeq 3000\ \text{K} = 0.3\ \text{eV}$

$T_{\rm rec} \ll B = 13.6\text{eV}$



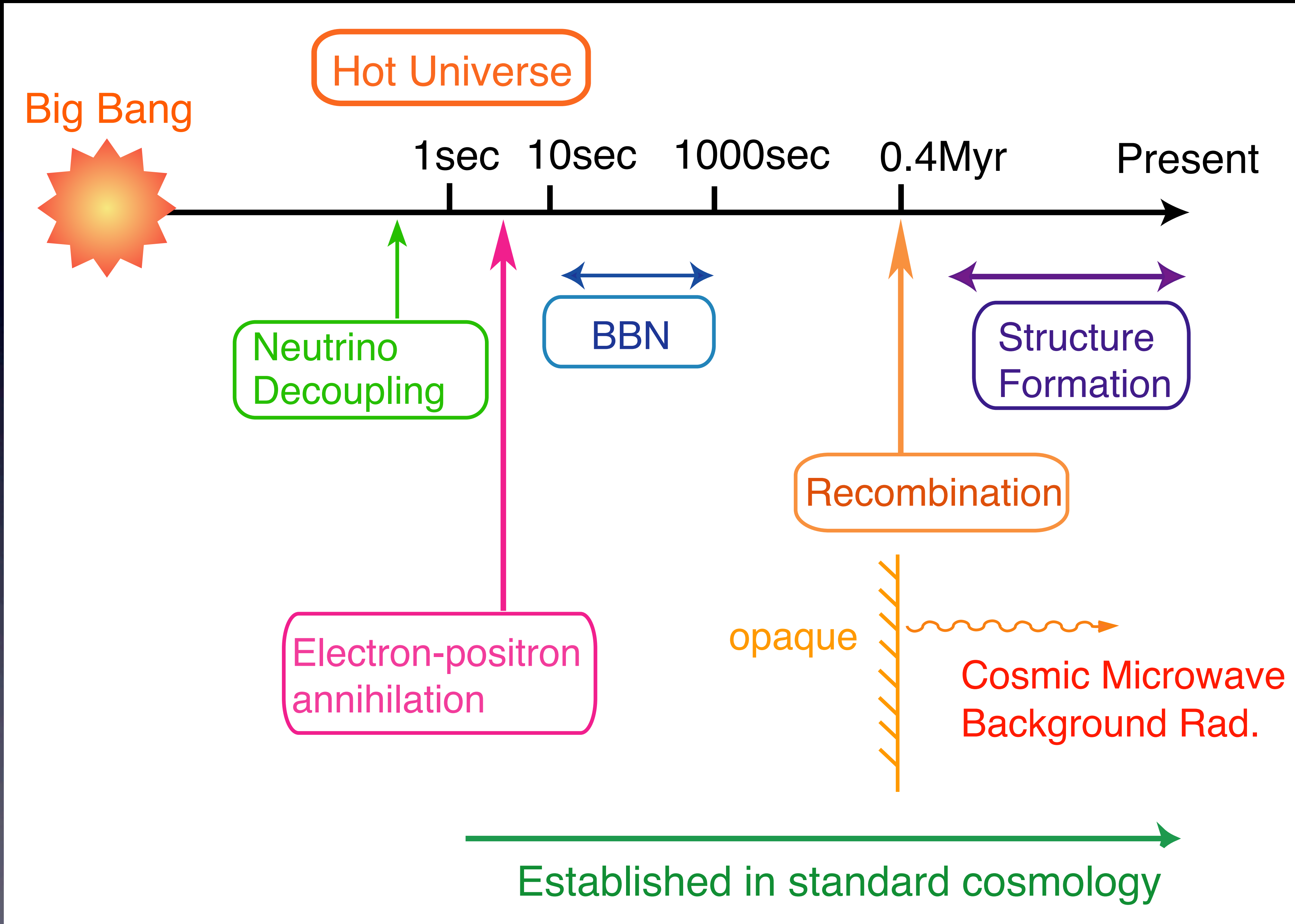
Saha formula cannot be used for X << 1



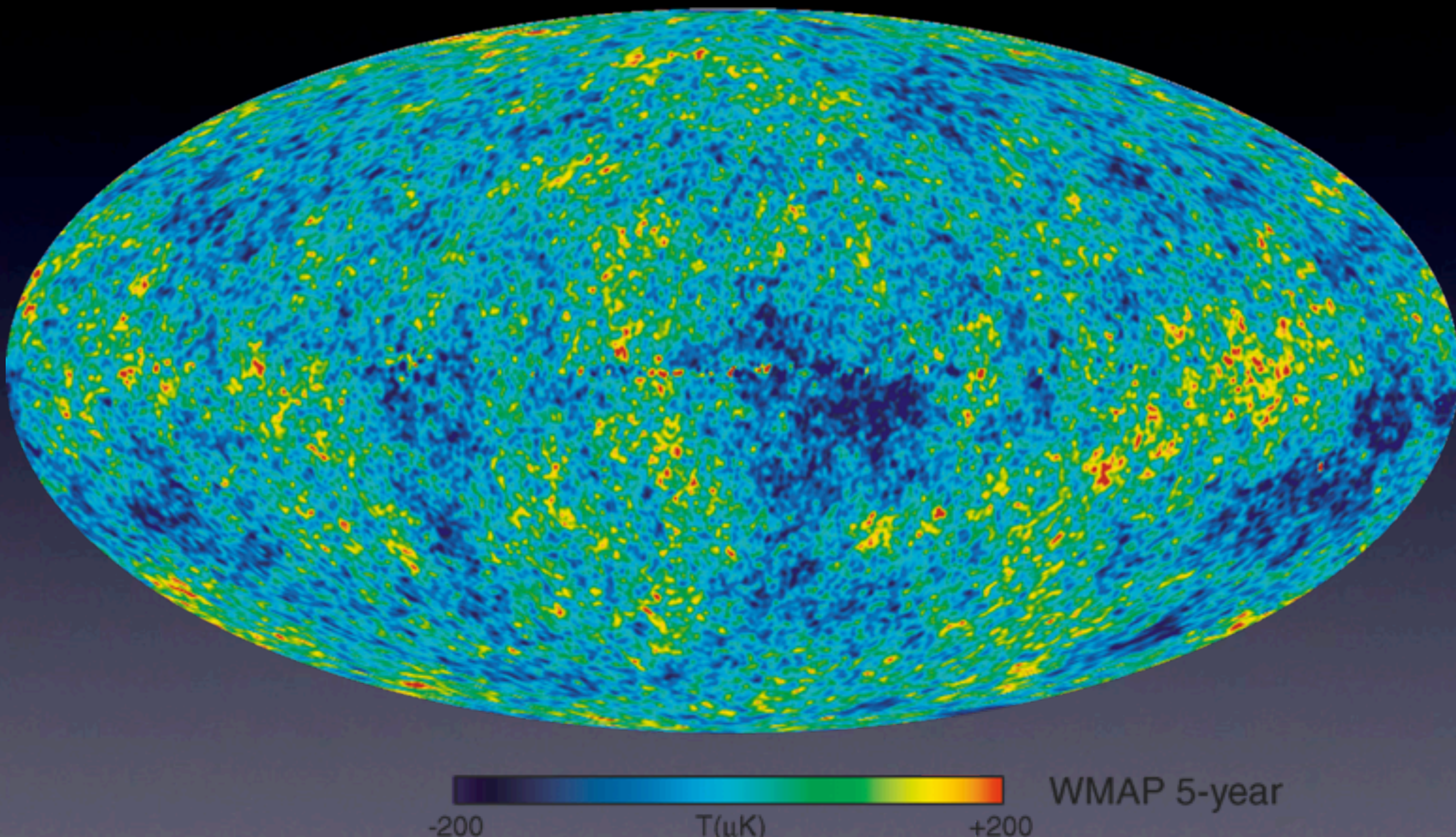
Equilibrium is not maintained

$$X_\infty \simeq 2 \times 10^{-5} \frac{\Omega_m}{\Omega_B}$$

History of the Universe

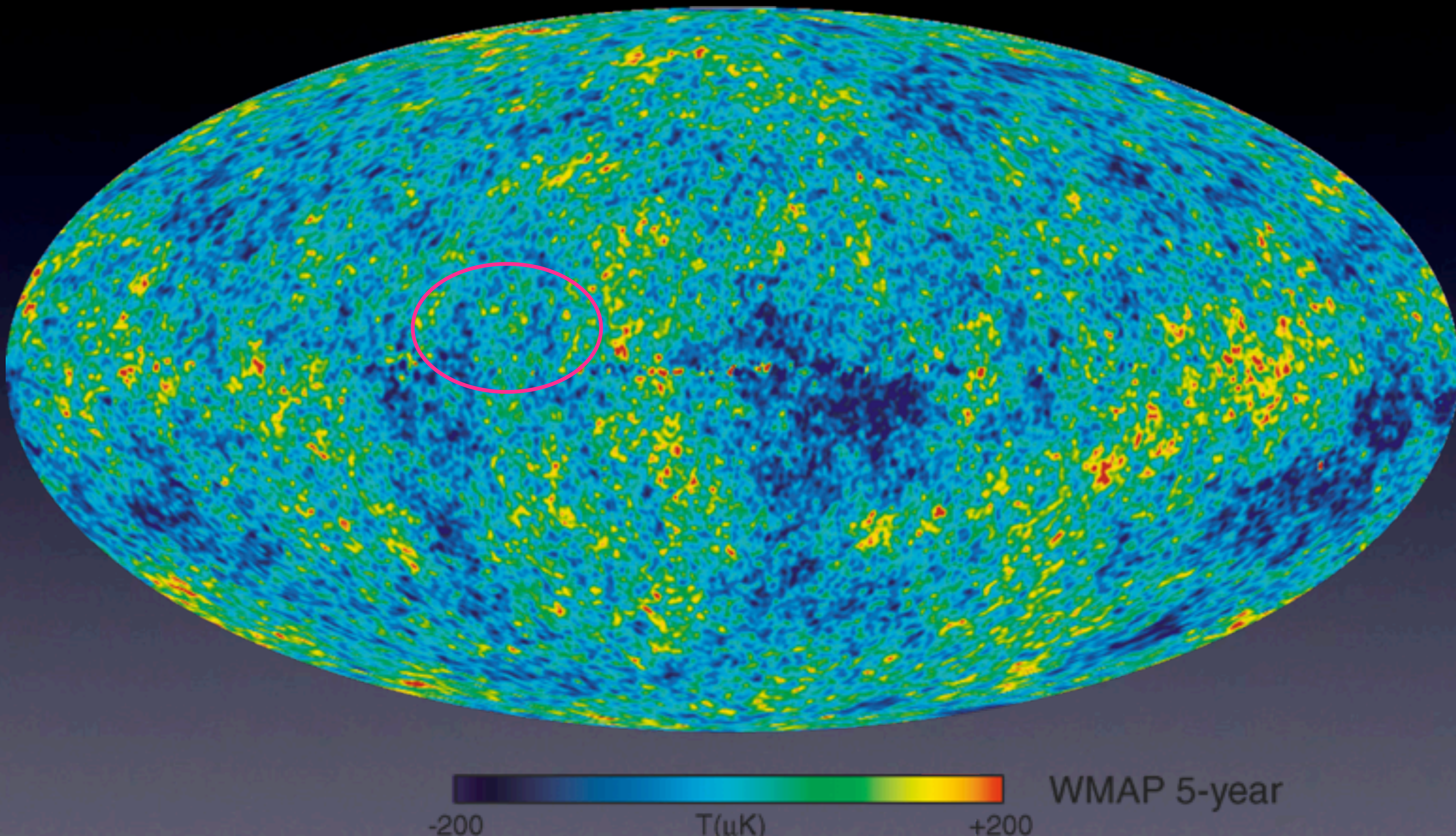


WMAPによる観測



http://map.gsfc.nasa.gov/

WMAPによる観測



http://map.gsfc.nasa.gov/