研 究 集 会

第18回可換環論シンポジウム

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序

この報告集は，第 18 回可換環論シンポジウムの講演記録として作成したものです。各講演者から提出された原稿を，そのまま縮小複写して印刷してあります。このシンポジウムは，1996 年 11 月 6 日から 11 月 9 日にかけて，インテック大山研修センター (富山県大山町) で開かれました。各地から参加した延べ 70 名近くの研究者・大学院生のもと，合計 18 もの興味深い講演が行なわれました。また，参加者の活発な討論もあり，例年どうり充実した 4 日間でした。

シンポジウム開催にあたり，旅費・会場費等の経費および本報告集の出版費について，京都大学の秋葉知温氏を代表者とする文部省科学研究費総合 A からの援助を受けました。また，期間中，インテック大山研修センターの方々には大変お世話になりました。ここにあらためて感謝いたします。

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福井大学　小野田信春

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Strange Curve の Hilbert 関数

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本稿の内容は, 筆者と E. Ballico 氏 (伊, Trento 大) との共同研究である. 但し, Rathmann ([10]) らによる既存の結果の紹介にも スペースを割いた.

序

K を代数閉体, $A = \bigoplus_{i \geq 0} A_i$ を K 上の斉次環とする. つまり, ネーター的次数付き可換環であり, $A_0 \simeq K$ 代数として A_1 で生成されているとする. この時, 整数 $h_0, \dots, h_s$, $h_0 = 1$, $h_s \neq 0$, が存在して,

$$\sum_{i \geq 0} \dim_K A_i \lambda^i = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d}$$

と出来る. ここで, d は A の Krull 次元である. $(h_0, \dots, h_s)$ を A の **h-列**と言う. A の Hilbert 関数の性質の記述に大変便利なものである (特に, A が Cohen–Macaulay の場合).

A が斉次環一般の時, 及び Cohen-Macaulay 斉次環の時, Hilbert 関数ないし h-列の振舞いは, Macaulay の古典的な定理によって完全に分かっている (厳密に言うと, 自然数から自然数への写像 ϕ が与えられた時, ϕ を Hilbert 関数に持つような, 斉次環ないし Cohen–Macaulay 斉次環が存在するか否かを判定する手順が有るという意味. [3] 参照). 例えば, A が Cohen-Macaulay の場合, その h-列の各成分は正の整数である. ところが, A が Cohen-Macaulay 斉次聖域の場合の h-列の振舞いについては, 多くは知られていない (一般の Cohen-Macaulay 環の場合に比べて かなり強い制約を受ける筈なのだが, どの程度に強い制約かが 良く分からない). 例えば, $\operatorname{char} K = 0$ の場合, Cohen-Macaulay 斉次聖域の h-列 $(h_0, h_1, \dots, h_s)$ は, 任意の $1 \leq i \leq s-1$ に対して $h_1 \leq h_i$ をみたす事が知られている. これは, 整域でない場合には全く見られない現象である.

以前 筆者は以下の結果を得た. これは, 単体的球面の数え上げ組合わせ論の基本的な定理である Barnette の "下限定理" のアナロジーを与えている ([7] 参照).

定理 1 ([11]) $\mathrm{char}\, K = 0$ とし, A を K 上の Cohen-Macaulay 斉次整域, $(h_0, h_1, \ldots, h_s)$ をその h-列とする. この時, ある $2 \leq i \leq s-2$ に対して, $h_i \leq h_1$ ならば, $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$ である. $h_s \geq 2$ ならば, $h_{s-1} \leq h_1$ でも同じ結論を導く.

これの証明に当たっては, Bertini の定理から, A を射影曲線 $C \subset \mathbf{P}^r$, $r = h_1 + 1$, の斉次座標環としてよい. ここまでは, 正標数でも成り立つ議論である. [11] では, ここから さらに "超平面切断"を取るという論法を用いている. $\mathbf{P}^r$ の一般の超平面 H と C との交わりは, 単に $\deg C$ 個の閉点であるから, それ自身は どうにもならない空虚な存在なのだが, これの $H \simeq \mathbf{P}^{r-1}$ の中での配置, つまり超平面や超曲面との包含関係に着目すると, 色々面白い現象が起こっている事に気づく. まず, 次の用語を準備する.

定義 2 $S \subset \mathbf{P}^n$ を閉点の有限集合とする. 部分集合 $T \subset S$, $\#T \leq n+1$, が常に $(\#T-1)$ 次元の線形空間を張る時 (つまり、目一杯 線形独立の時) , S は *linear general position* にあると言う.

定義 3 $S \subset \mathbf{P}^n$ を閉点の有限集合とする. $\#T = \#T'$ なる二つの部分集合 $T, T' \subset S$ の Hilbert 関数が常に一致する時, S は *uniform position* にあると言う.

S が非退化, つまり超平面に含まれなければ (本稿で考察するのは, 常にこのような場合であるが), uniform position は linear general position よりも強い条件である.

J. Harris による 次の結果が重要である.

定理 4 (Uniform Position Lemma). 基礎体の標数を 0 とする. 既約かつ被約で非退化な (i.e., 超平面に含まれない) 射影曲線 $C \subset \mathbf{P}^r$, $r \geq 3$, の一般の超平面切断は, その超平面の中で uniform position にある.

結局, 定理 1 は, 標数に関係無く, uniform position にある点集合の座標環に対して常に成り立つ (そして, これを経由して証明できた) 結果である. しかし, 定理 4 には, 正標数で反例が存在する (定理 1 自身の反例は見つかっていない).

例 5 $C \subset \mathbf{P}^n$, $n \geq 3$, を $X_0^q - X_1 X_n^{q-1}$, $X_1^q - X_2 X_n^{q-1}, \ldots$ の完全交叉として得られる曲線とする．ここで、$q > 1$ は $p = \mathrm{char}\, K > 0$ の巾とした．C は既約かつ被約である．C の一般の超平面切断 X は 位数 q の有限体上の $n-1$ 次元のアフィン空間と同じ配置になる．つまり, X の q 個の点で一本の直線の上に並ぶものが存在し, q^2 個の点で一つの平面に乗っているものが存在する．従って, $q = 2$, $n = 3$ の場合を唯一の例外として, X は linear general ですらない.

この為, [11] の論法は, 正標数では使えない．但し, 定理 4 が成り立たない射影曲線は, 極めて特殊な存在である事が知られている (次節以降参照).

$\mathrm{char}\, K > 0$ の時の h-列の振舞いが, 今回の研究対象である.

1 Uniform Position Lemma が成立しない射影曲線

前節の議論から Uniform Position Lemma をみたさない射影曲線 (正標数の場合にのみ存在する訳だが) のより詳しい分析の重要性が分かって頂けたと思う．この方面では, Rathmann [10] が最も重要な仕事である．Rathmann の理論については, 次節でやや詳しく紹介するが, この節では以下の結果のみを用いる.

命題 6 (Rathmann [10]) 既約かつ被約で非退化な射影曲線 $C \subset \mathbf{P}^r$, $r \geq 4$, の一般の超平面切断が, その超平面の中で uniform position になければ, C は strange curve である, つまり 任意の非特異点における接線は固定された一点 (C の**中心**と呼ぶ) を通る.

例 5 の曲線 C は, 斉次座標 $(1, 0, \ldots, 0)$ で表される点を中心とする strange curve である．$C \subset \mathbf{P}^r$, $r \geq 2$, を直線でない既約かつ被約な strange curve とする．C を その中心から $\mathbf{P}^{r-1}$ へ射影すると 関数体の非分離拡大が引き起こされる．従って, 直線でない既約かつ被約な strange curve は 正標数の場合にしか存在しない．また、被約かつ既約で非退化な射影曲線 $C \subset \mathbf{P}^r$, $r \geq 3$, の一般の超平面切断が linear general position にない時, C は strange curve である事が, 割合単純な幾何的議論で示される ([10] 参照). 特に, 標数 0 の時は, C の一般の超平面切断は linear general position にある事が分かる (これは, Uniform Position Lemma の弱形である). この事実は, Castelnuovo が既に着目し 利用している．実

は, 本稿の初めのほうで紹介した, Cohen-Macaulay 斉次整域の h-列が, $h_i \geq h_1$ を満たすという事実も, これを用いて示されるのである.

また, 非特異な strange curve は, 直線と標数 2 の時の二次曲線以外にない事も知られている (Samuel). つまり Uniform Position Lemma は, $r \geq 4$ の時, 正標数であっても非特異曲線に対しては成立する事が分かる.

この事から, $h_1 \geq 3$ の時, "A は正規" という条件を追加すれば, 定理 1 は正標数でも成立する事が分かる. 一方, $h_1 = 2$ の場合 (つまり, $r = 3$ の場合) は, Cohen–Macaulay 斉次整域の h-列は完全に特徴付けられている ([5, 8]. 道具としては, 余次元 2 の完全イデアルと言う事で, Hilbert-Burch の定理が用いられている). この結果は, 標数に依らない. 従って, $h_1 = 2$ の場合, 定理 1 は正標数でも正しい.

以上をまとめると, 次を得る.

命題 7 定理 1 に於いて, "A は正規" という条件を追加すると, この定理の主張は正標数でも成り立つ.

以後 $p := \mathrm{char} K > 0$ とする.

定理 8 $C \subset \mathbf{P}^r$, $r \geq 4$, を既約かつ被約で非退化な射影曲線で, その一般の超平面切断が uniform position にないものとする. C が射影的に Cohen–Macaulay であれば, その座標環の h-列は, 任意の $2 \leq i < p$ に対して $h_i > h_1$をみたす.

上で述べたように $r = 3$ に対応した場合は, 定理 1 は正標数でも文句無しに成り立つので, 次を得る.

系 9 A を K 上の Cohen-Macaulay 斉次整域, $(h_0, h_1, \ldots, h_s)$ をその h-列とする. この時, ある $2 \leq i \leq \min\{p-1, s-2\}$ に対して, $h_i \leq h_1$ ならば, $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$ である. $h_s \geq 2$ ならば, $i = \min\{p-1, s-1\}$ の場合でも同じ結論を導く.

定理 8 の証明 $T(C)$ を C の Tangent Varaiety, つまり, C の非特異点における接線全体の union の閉包であるとする. 今, C は strange であるから, $T(C)$ は, C の中心 v を頂点とする錘である. 次の補題が基本的.

補題 10 記号は上と同じとする. $F \subset \mathbf{P}^r$ を $\deg(F) < p$ なる超曲面とする. この時, $F \supset C$ ならば $F \supset T(C)$ である.

補題の証明 P を C の非特異点とし, D を C の P における接線とする. C は strange curve だから, D と C の P における交点数は 標数 p の巾である (C の v からの射影を考えよ). もし, $F \not\supset D$ ならば,

$$(C \text{ と } D \text{ の } P \text{ における交点数}) \leq (F \text{ と } D \text{ の } P \text{ における交点数}) \leq \deg(F) < p$$

となって矛盾. よって, $F \supset D$. 従って, $F \supset T(C)$ を得る. □

上の補題さえ認めれば, 以下の二つは, ほぼ明らかであろう (補題 12 の証明には, C の座標環の Cohen-Macaulay 性が効いている).

補題 11 全ての $i < p$ に対し,

$$H^0(\mathbf{P}^r, \mathcal{I}_C(i)) = H^0(\mathbf{P}^r, \mathcal{I}_{T(C)}(i)).$$

補題 12 $H \subset \mathbf{P}^r$ を一般の超平面とし, $X := C \cap H$, $C' := T(C) \cap H$ とおく. このとき $i < p$ に対して $H_X(i) = H_{C'}(i)$. 但し, H_X は $X \subset \mathbf{P}^{r-1}$ の Hilbert 関数とする. $H_{C'}$ も同様.

定理 8 の証明の続き $i < p$ に対して,

$$\begin{aligned} h_i &= H_X(i) - H_X(i-1) \\ &= H_{C'}(i) - H_{C'}(i-1) \qquad \text{(補題 12 より)} \\ &\geq \min\{r+i-2, \deg C'\} \end{aligned}$$

よって, $2 \leq i < p$ に対して $h_i \leq h_1 = r-1$ ならば, $\deg C' \leq r-1$ を得る. $C' \subset H \simeq \mathbf{P}^{r-1}$ は既約かつ非退化であるから, $\deg C' = r-1$ で, $C' \subset \mathbf{P}^{r-1}$ は rational normal curve とならざるを得ない. ところが, rational normal curve に含まれる点集合は必ず uniform position にあるので, $X \subset C' \subset \mathbf{P}^{r-1}$ は uniform position に在ることになって矛盾. □

2 Trisecant Lemma が成立しない射影曲線

唐突ではあるが, 有限群論からの準備をする.

d 次対称群 S_d の部分群 G が, 位数 d の集合 Ω に作用しているとする. Ω の t 個の順列 $\{a_1, a_2, \ldots, a_t\}$, $\{b_1, b_2, \ldots, b_t\}$ を任意に 2 つ取ったとき, 一方を他方に順序を保って移す G の元が存在するとき, すなわち $a_i^\sigma = b_i^\sigma$ $(i = 1, 2, \ldots, t)$ をみたす G の元 σ が常に存在するとき G の Ω への作用は **t-重可移** であるという. [1]

定義から明らかなように, S_d 自身は d 重可移, 交代群 A_d は $d-2$ 重可移である. この二つ以外は, **自明でない**多重可移群と呼ばれるが, これらの殆どは 高々2 重可移でしかなく, 3 重可移以上のものには 分類定理が存在する.

まず, 位数 2 の有限体上の n 次元ベクトル空間のアフィン変換群 (線形変換＋平行移動) $AGL(n, 2) \subset S_{2^n}$ は, 3 重可移である (位数 3 以上の有限体上のアフィン変換群は 2 重可移でしかない). また, 位数 q の有限体上の射影直線に作用している群 $G \subset S_{q+1}$ であって, $PSL(2, q) \leq G \leq P\Gamma L(2, q)$ をみたすものも 3 重可移である. 自明でない 3 重可移群は, これら有限幾何経由の群 (の無限系列) を除いては, 有限個しか存在しない.

4 重可移以上の自明でない多重可移群は, 同型を除いて有限個しか存在しない. M_{11}, M_{12}, M_{23}, M_{24} の四つで, Mathieu 群と総称される. $M_{11} \subset S_{11}$, $M_{23} \subset S_{23}$ は 4 重可移, $M_{12} \subset S_{12}$, $M_{24} \subset S_{24}$ は 5 重可移である (これらから派生した散在型の 3 重可移群も存在する). 自明でない 6 重可移群は存在しない.

曲線の話に戻る. $C \subset \mathbf{P}^r$ を被約で既約で非退化な射影曲線とする. $M = \{(x, H) \in C \times \mathbf{P}^{r*} \mid x \in H\}$ を考える. 射影 $M \to C$ のファイバーは $\mathbf{P}^{r-1}$ と同型なので, M は既約であると分かる. また, もう一方の射影 $M \to \mathbf{P}^{r*}$ は, 次数 $d := \deg C$ の有限射であるが, これによって引き起こされる有理関数体の拡大, $\pi^* : K(\mathbf{P}^{r*}) \to K(M)$ は次数 d の分離拡大である (一般の超平面切断は, $\deg C$ 個の相異なる閉点であるから. Bertini の定理).

よって, $K(M)$ は $K(\mathbf{P}^{r*})$ 上, 一個の元 $f \in K(M)$ で生成されている. $P(f) = 0$ なる $K(\mathbf{P}^{r*})$ 係数の d 次多項式 P を考える. P の最小分解体の $K(\mathbf{P}^{r*})$ 上の Galois 群

[1] 多重可移群については [9] に詳しい. 但し, 有限単純群の分類の完成以前の本なので 現在では解決済みの ([10] でも使用されている) 結果が, "予想" という形で取り上げられたりしている. では, 現在得られている最終的な解決をどの本ないし論文で調べれば良いかとなると, (Rathmann [10] によると) なかなか適切なものが無いらしい. 取りあえず, [10] の参考文献リストを参照して頂きたい.

Gal $(P, K(\mathbf{P}^{r*}))$ を, $C \subset \mathbf{P}^r$ の **monodromy 群** と呼び, G_C と記す. もちろん G_C は, f の取り方に依らない. G_C は, f の共役元全体の集合に置換群として作用している. つまり, $G_C \subset S_d$ と自然に見なせる.

G_C は, 基礎体が標数 0 の時, 特に複素数体の時にも定義出来るが, この場合 G_C は 普通の (?) monodromy 群であり, C の一般の超平面切断として得られる d 個の点の集合に 置換群として作用している事が, 直感的に捉え易いと思う.

Harris, Rathmann らにより, 次が示されている.

命題 13 記号を上の通りとする. この時, 以下が成り立つ.

(1) $\operatorname{char} K = 0$ の時は G_C は, 対称群 S_d に等しい ([6] 参照).

(2) G_C が交代群 A_d を含めば, C の一般の超平面切断は uniform position にある ([6, 10]).

(3) $1 \le t \le r$ なる任意の t に対して, $G_C \subset S_d$ が t-重可移である為の必要充分条件は, C の一般の超平面切断 X において, 任意の t 個の点が $t-1$ 次元のベクトル空間を張る事である. 特に G_C は常に 2 重可移以上ではある ([10]).

定理 14 $C \subset \mathbf{P}^r$, $r \ge 4$, を被約かつ既約で非退化な射影曲線とする. C の一般の超平面切断が uniform position にないならば, 次の 3 つの条件のどれかがみたされる.

(1) C の一般の 2 点を結ぶ直線は C の他の点を少なくとも もう一つ含む. つまり, いわゆる **Trisecant Lemma** が成立していない (G_C が高々2 重可移の時).

(2) C の一般の 3 点が張る平面は C の他の点を少なくとも もう一つ含む (G_C が高々3 重可移の時).

(3) $\deg C \in \{11, 12, 23, 24\}$ であり, G_C は Mathieu 群の何れかと同形である.

(1) や (2) の場合 ((3) の場合も r が充分大きければ), C の一般の超平面切断は, linear general position にすらない. 例 5 の曲線 C の $q > 2$ の場合が, 定理 14 の (1) の例を与え, $q = 2$ の場合が (1) ではない (2) の例を与えている ((1) でない (2) の例は, 基本的には この例だけのようである. [1] 参照). 実は, G_C は $AGL(n-1, q)$ と同形である ([10] 参照). (3)

の実例は知られていないようである. 特に, 一般の超平面切断が linear general position には在っても, uniform position にないような既約曲線の例は, 知られてなさそうである.

G_C をもう少し詳しく分析すると, 定理 14 の 3 つのケースの内, 殆どの場合には, (1) が成立している事が分かる.

定理 15 (Ballico [1]) $C \subset \mathbf{P}^r$, $r \geq 5$, を一般の超平面切断が uniform position にない既約かつ被約で非退化な射影曲線とする. さらに, $\mathrm{char}\, K \neq 2$ ($\mathrm{char}\, K = 2$ の時は 定理 8, 系 9 は何も主張していない) かつ $\deg C > 24$ ならば, C の一般の 2 点を結ぶ直線は, C の点を少なくとももう一つ含む. つまり, いわゆる Trisecant Lemma が成立していない.

上の結果を標語的に言うと, "Uniform Position Lemma を満たさない既約曲線は大抵の場合 Trisecant Lemma さえ満たしていない" という風になる. これは "自明で無い三重可移群は珍しい" という別の原理に起因している.

定理 16 $C \subset \mathbf{P}^r$, $r \geq 3$, を Trisecant Lemma をみたさない既約かつ被約で非退化な射影曲線とする. この時, C の斉次座標環の定義イデアルの 斉次元のみからなる極小生成系を考えると, 次数が p より小さい元は高々一つしかない.

証明の概略 C は既約かつ被約なので, その斉次座標環の定義イデアルに含まれる斉次式のうち次数の最も低いものは既約多項式である. よって, 斉次元のみからなる極小生成系のうち次数の低い方から 2 つ取ってくると, これらは正則列をなす.

C は, Trisecant Lemma をみたしてないので strange curve である. よって, $r = 3$ の場合, 定理は 補題 10 より直ちに従う. $r \geq 4$ の時は, 帰納法で示す. $C \subset \mathbf{P}^r$ を その中心から $\mathbf{P}^{r-1}$ へ射影した像 C' も Trisecant Lemma をみたさない事に注意. また C の定義イデアルの斉次極小生成系で次数が p より低いものが二つ以上あれば, C' もそうである (strange curve としての "中心" からの射影なので). よって, 帰納法の仮定に矛盾する. □

これにより, 定理 8 の不等式は, 緩やかな条件の下で著しく改良される事になる.

系 17 $C \subset \mathbf{P}^r$, $r \geq 5$, $\deg C > 24$, を一般の超平面切断が uniform position にない既約かつ被約で非退化な射影曲線とする (C は, 射影的に Cohen–Macaulay である必要はない). こ

の時, C の斉次座標環の定義イデアルの 斉次元のみからなる極小生成系を考えると, 次数が p より小さい元は高々一つしかない. 特に, $2 \leq i < p$ に対して次が成立する.

$$h_i = \binom{r-1+i}{i} - \binom{r-1+i-\sigma(C)}{i-\sigma(C)} \geq \binom{r+i-1}{i} - \binom{r+i-3}{i-2}$$

が成立する.

付記

本稿の方向性とはかなり異なっていますが, 射影空間の中の点集合の配置と これの斉次環の Hilbert 関数への応用に関する, Eisenbud 他による概説記事 [4] が出ていますので, 興味のある方は御覧になってください.

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A generalization of Matijevic-Roberts theorem

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Introduction

Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a commutative $\mathbb{Z}$-graded noetherian ring. In [9], J. Matijevic and P. Roberts proved the following.

Theorem 0.1 *If $A_{\mathfrak{p}}$ is Cohen-Macaulay for every graded prime ideal $\mathfrak{p}$ of A, then A is Cohen-Macaulay.*

This gives an affirmative answer for a problem given by Nagata [13]: If $A = \bigoplus_{n\geq 0} A_n$ is a commutative non-negatively graded noetherian ring, and if $A_{\mathfrak{m}}$ is Cohen-Macaulay for every graded maximal ideal $\mathfrak{m}$ of A, then A is Cohen-Macaulay, because any graded prime ideal is contained in a graded maximal ideal in non-negatively graded case. It was proved by Aoyama-Goto [1] and Matijevic [8] that the same is true for Gorenstein property.

After a while, S. Goto and K.-i. Watanabe [5] generalized these results as follows.

Theorem 0.2 *Let A be a $\mathbb{Z}^n$-graded ring and M a finitely generated graded A-module. Let $\mathfrak{p} \in \operatorname{Spec} A$, and we denote by $\mathfrak{p}^*$ the maximal graded (prime) ideal contained in $\mathfrak{p}$. Then, the following holds:*

1 *If $A_{\mathfrak{p}^*}$ is a regular local ring, then so is $A_{\mathfrak{p}}$.*

2 *If $M_{\mathfrak{p}^*}$ is Cohen-Macaulay (resp. Gorenstein), then so is $M_{\mathfrak{p}}$.*

In the theorem above, when we set $R := \mathbb{Z}$, the ring of integers and $G := \mathbb{G}^n_{m,R}$, the split torus over R of rank n, then the $\mathbb{Z}^n$-graded structure of A is nothing but an R-action of G to A, and the grading of M (resp. $\mathfrak{p}^*$) is nothing but a G-action on it which is compatible with the A-action.

The purpose of this note is to extend these results to more general G-action. In fact, we have the following:

Theorem 0.3 *Let R be a noetherian commutative ring, G a geometrically integral flat affine R-group scheme of finite type, and X a locally noetherian R-scheme. Assume that G acts on X (R-rationally from right). Let $x \in X$. Then, the closure of the image of the action $\overline{\{x\}} \times_{\operatorname{Spec} R} G \to X$ is integral, and we denote its generic point by x^*. Let M be a coherent $(G, \mathcal{O}_X)$-module. Moreover, we assume that the coordinate ring of G is a union of R-finite projective subcoalgebras. Then, the following hold:*

1 *If $\mathcal{O}_{X,x^*}$ is a regular local ring (resp. a complete intersection), then so is $\mathcal{O}_{X,x}$.*

2 *If M_{x^*} is Gorenstein (resp. Cohen-Macaulay, free), then so is M_x.*

For the definition of (quasi-coherent) $(G, \mathcal{O}_X)$-modules, which is a substitute for graded modules, see section 3. Although the proof is basically a translation of the proof in [5], there is considerable technical difficulty to overcome. For example, the technique of homogeneous localizations is not available in our case: If R is an algebraically closed field and $X = B\backslash G$, then X is an R-projective homogeneous space, and it is absurd to expect a purely local (or ring theoretical) treatment, where B is a Borel subgroup of G.

Geometric integrality assumption is necessary for our statement, see Example 1.3 and Example 1.4. However, Kamoi [7] discuss torsion-group graded rings, which is grasped as a diagonalizable group-scheme action. He used a remarkable notion of G-prime ideals.

Here the author confess that there was a *fatal error* in my talk at the symposium, and the statement of the main theorem has some *additional assumption* (Remark 3.5) here. I would like to apologize to all who listened to my talk.

In section 1, we discuss the generality of G-stability of subschemes, which seems to be well-known as a folklore. In section 2, we review the results on hyperalgebras. The results stated there should be compared with the results stated in [6]. In section 3, we review some generality of $(G, \mathcal{O}_X)$-modules. This notion appears in [12] using the 'linearlizable sheaf' condition. We also introduce the notion of $(\mathrm{Hyp}(G), \mathcal{O}_X)$-modules, which seems to be new here. Most of the proofs in section 2 and 3 are omitted, as the technical detail is irrelevant here and it does not fit into this short proceedings. In section 4, we prove Theorem 0.3.

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1 Stable subschemes

Throughout this note, R denotes a noetherian commutative ring. For an R-scheme Z, we say that Z is *geometrically integral* (resp. *reduced, irreducible*) when for any field K which is an R-algebra, $K \otimes_R Z$ is integral (resp. reduced, irreducible).

In this section, G denotes a geometrically integral flat R-group scheme of finite type, and X denotes a locally noetherian R-scheme which has a right R-rational G-action. Note that if Y is an irreducible (resp. reduced) R-scheme, then so is $Y \otimes_R G$, as G is flat and geometrically integral over R.

We denote the action (resp. the first projection) $X \times G \to X$ by $a = a_X$ (resp. $p = p_X$). The isomorphism $X \times G \to X \times G$ given by $(x, g) \mapsto (xg, g)$ is denoted by h_X. Note that it holds $p_X \circ h_X = a_X$.

For a subscheme Y of X, we say that Y is *G-stable* when the action $Y \times G \to X$ $((y, g) \mapsto yg)$ factors through $Y \hookrightarrow X$. In this case, Y has a unique G-action such that $Y \hookrightarrow X$ is a G-morphism (i.e., R-morphism which preserves G-action). We say that $x \in X$ is G-stable when $\overline{\{x\}}$ is G-stable.

Lemma 1.1 *The following holds.*

1 *G is R-smooth.*

2 *Let Y be a closed subscheme of X, and Y^* denotes the closure of the image of the action $Y \times G \to X$. Then, Y^* is the smallest G-stable closed subscheme containing Y. If Y is irreducible (resp. reduced), then so is Y^*.*

2' *For a reduced closed subscheme Y of X, the following are equivalent.*

a *Y is G-stable*

b *Any irreducible component (maximal integral closed subscheme) of Y is G-stable*

c *$X - Y$ is G-stable.*

3 *When $X = X_0 \times_R G$ is a principal G-bundle, then any G-stable open set of X is of the form $V \times_R G$, where V is an open set of X_0.*

4 *If $\varphi : X \to X'$ is a G-morphism of locally of finite type between locally noetherian R-schemes with G-actions, then the flat locus* $\mathrm{Flat}(\varphi)$ *is a G-stable open subset of X.*

5 *If the Cohen-Macaulay (resp. Gorenstein, l.c.i., regular) locus is an open set of X, then it is G-stable.*

Proof. **1** We may assume that R is an algebraically closed field. In this case, G is an algebraic group variety, and hence is R-smooth.

2 Straightforward.

2' a⇒b Let Y_i be an irreducible component of Y. Then, Y_i is integral, and we have $Y_i \subset Y_i^* \subset Y$, as Y is G-stable. Hence, we have $Y_i = Y_i^*$ by the maximality of Y_i. **b⇒c** If (U_i) is an affine open covering of X, then $(a_X(U_i \times G))$ is an open covering by G-stable open subsets of X, since a_X is an open map. As a finite intersection of G-stable subschemes is again G-stable and a union of G-stable open subschemes is again G-stable, we may replace X by $a_X(U_i \times G)$. Hence, we may assume that X is quasi-compact. As an intersection of finite G-stable open subsets is again G-stable, it suffices to prove **a⇒c**. We set $U := X - Y$. As G is universally open over R, the image W of the action $U \times G \to X$ is open, and we have $U \subset W$. It suffices to prove that $U = W$. Assume the contrary. Then, there is an algebraically closed field K which is an R-algebra such that $U(K) \to W(K)$ is not surjective. As G is of finite type over R, we have $(U \times G)(K) \to W(K)$ is surjective by Hilbert's theorem, and clearly we have $X(K) = U(K) \coprod Y(K)$. This shows that there exist $g \in G(K)$, $u \in U(K)$ such that $ug \in Y(K)$. As Y is G-stable, this shows that $u = (ug)g^{-1} \in Y(K)$, which is a contradiction. **c⇒a** Similarly, we have the image of $Y \times G \to X$ does not meet U, and hence Y^* is set-theoretically contained in Y. As both Y and Y^* are reduced, we have $Y = Y^*$, and Y is G-stable.

3 We denote the projection $X_0 \times G \to X_0$ by p. Let U be a G-stable open set of $X = X_0 \times G$. Then, $W = pU \times G = p^{-1}(pU)$ is an open set, since p is an open map. The same argument as in the proof of **2'** shows that $U = W$.

4 The flat locus $\mathrm{Flat}(\varphi)$ of φ is an open subset of X [10, Theorem 24.3]. Let $f' : Z' \to X'$ be a flat morphism, $Z := Z' \times_{X'} X$, $f : Z \to X$ the second projection, and $\psi : Z \to Z'$ the first projection. For $z \in Z$, ψ is flat at z if and only if φ is flat at fz. Or equivalently, $\mathrm{Flat}(\psi) = f^{-1}(\mathrm{Flat}(\varphi))$. We apply this observation to the flat morphisms $p_{X'}$ and $a_{X'}$. Consider the diagram

$$\begin{array}{ccccc} X \times G & \xrightarrow{a_X} & X & \xleftarrow{p_X} & X \times G \\ \downarrow{\scriptstyle \varphi \times 1_G} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi \times 1_G} \\ X' \times G & \xrightarrow{a_{X'}} & X' & \xleftarrow{p_{X'}} & X' \times G \end{array}$$

Obviously, the right square is a fiber square. As $(\varphi \times 1_G) \circ h_X = h_{X'} \circ (\varphi \times 1_G)$ and h_X and $h_{X'}$ are isomorphisms, the left square is also a fiber square. It follows that

$$\mathrm{Flat}(\varphi) \times G = p_X^{-1}(\mathrm{Flat}(\varphi)) = \mathrm{Flat}(\varphi \times 1_G) = a_X^{-1}(\mathrm{Flat}(\varphi)).$$

This shows that $\mathrm{Flat}(\varphi)$ is G-stable.

5 Let $f : X' \to X$ be a smooth morphism, $x' \in X'$ and $fx' = x$. Then, $\mathcal{O}_{X',x'}$ is Cohen-Macaulay (resp. Gorenstein, l.c.i., regular) if and only if so is $\mathcal{O}_{X,x}$. Or equivalently, when we denote the Cohen-Macaulay (resp. Gorenstein, l.c.i., regular) locus of X (resp. X') by $U(X)$ (resp. $U(X')$), then it holds $U(X') = f^{-1}(U(X))$. Applying this observation to $p_X : X \times G \to X$ and $a_X : X \times G \to X$, we have

$$U(X) \times G = p_X^{-1}(U(X)) = U(G \times X) = a_X^{-1}(U(X)).$$

This shows that $U(X)$ is G-stable. □

For $x \in X$, we denote the generic point of $\overline{\{x\}}^*$ by x^*. x is G-stable if and only if $x = x^*$.

Corollary 1.2 *Assume that the Cohen-Macaulay (resp. Gorenstein, l.c.i., regular) locus U of X is an open subset of X (e.g., X is excellent). If U contains all of the G-stable points of X, then $U = X$.*

Proof. Assume the contrary. Then, $Y := X - U$ with the reduced closed subscheme structure is a non-empty G-stable closed subscheme of X. It follows that the generic point η of any irreducible component of Y is G-stable, but $\eta \notin U$. This is a contradiction. □

Thus, Matijevic-Roberts type theorem is true for excellent X without assuming that G is affine. In fact, Theorem 0.3 is much easier to prove when we assume that X is excellent.

The following examples show that the integrality assumption is necessary for our theorem.

Example 1.3 Let k be a field, and X_0 a non Cohen-Macaulay variety over k. We set $X := X_0 \coprod X_0$, and the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on X via permutation. It is easy to see that there is no stable point on X. However, X is not Cohen-Macaulay.

Example 1.4 Let k be a field of characteristic $p > 0$. Then, $G := \operatorname{Spec} k[z]/(z^p)$ is a closed subgroup scheme of $\mathbb{G}_a = \operatorname{Spec} k[z]$ with z primitive (i.e., $\Delta(z) = z \otimes 1 + 1 \otimes z$ and $\varepsilon(z) = 0$). We set $\tilde{A} := k[x,y]$ and $\tilde{X} := \operatorname{Spec} \tilde{A} = \mathbb{A}_k^2$. G acts on $\tilde{X}$ by $\omega(x) := x \otimes 1 + 1 + z$ and $\omega(y) := 0$. Then, $I := (xy, y^2)\tilde{A}$ is a G-ideal of I. Then, G acts on $X := \operatorname{Spec} \tilde{A}/I$, and X is non-singular off the origin, but is not Cohen-Macaulay at the origin. However, the origin (defined by (x, y)) is not G-stable.

2 Hyperalgebra action

In this section, $G = \operatorname{Spec} H$ denotes an affine flat R-group scheme of finite type. Let A be a G-algebra (i.e., an R-algebra with a G-action).

Definition 2.1 We say that M is a (G, A)-module when M is a G-module (i.e., H-comodule) which is also an A-module, and the A-action $A \otimes M \to M$ is a G-homomorphism (i.e., H-comodule map). For (G, A)-modules M, N and a map $f : M \to N$, we say that f is (G, A)-linear when f is a G-homomorphism which is also A-linear.

This definition is in [16] and some important results are proved over arbitrary noetherian R. We obtain the category ${}_{G,A}\mathbb{M}$ of (G, A)-modules and (G, A)-linear maps. The category ${}_{G,A}\mathbb{M}$ is abelian with enough injectives and arbitrary inductive limit. However, this category is not closed under projective limits in general, and we need to extend the category in order to obtain various homological operations.

Definition 2.2 We define the *hyperalgebra* of G by

$$\varinjlim \operatorname{Hom}_R(H/I^n, R) \subset H^* = \operatorname{Hom}_R(H, R)$$

and denote it by Hyp(G), where I denotes the defining ideal of the unit element $\{e\} \hookrightarrow G$ (the kernel of the counit map $\varepsilon_H : H \to R$). We say that G is *infinitesimally flat* if H is normally flat along I.

The hyperalgebra Hyp(G) is a subalgebra of the dual algebra H^* of H. If R is a field of characteristic zero, then Hyp(G) is isomorphic to the universal enveloping algebra of the Lie algebra Lie(G). We set $U := \operatorname{Hyp}(G)$. As a G-module M is an H^*-module with the action given by

$$h^* m := \sum_{(m)} \langle h^*, m_1 \rangle m_0$$

(we use the Sweedler's notation, see [17]), we obtain a functor $\Phi : {}_G\mathbb{M} \to {}_U\mathbb{M}$. As $\Phi(M) = M$ as an R-module, Φ is exact.

If G is infinitesimally flat, then U is R-projective, and a Hopf algebra in a natural way. In this case, the identity map $\Phi(M \otimes N) = M \otimes N = \Phi M \otimes \Phi N$ is U-linear. If M is R-finite moreover, then the identity map $\Phi(\operatorname{Hom}_R(M, N)) = \operatorname{Hom}_R(M, N) = \operatorname{Hom}_R(\Phi M, \Phi N)$ is U-linear.

We say that U is *universally dense* if for any R-module M, the canonical map $\rho_M : M \otimes_R H \to \operatorname{Hom}_R(U, M)$ given by $\rho_M(m \otimes h)(u) = \langle u, h \rangle m$ is injective.

If U is universally dense and $M \in {}_U\mathbb{M}$, then we define the *rational part* M_{rat} of U as $\theta_M^{-1}(\operatorname{Im} \rho_M)$, where $\theta_M : M \to \operatorname{Hom}_R(U, M)$ is given by $\theta(m)(u) = um$. It is well-known that $M_{\mathrm{rat}} \hookrightarrow M \to \operatorname{Hom}_R(U, M)$ factors through $M_{\mathrm{rat}} \otimes_R H \hookrightarrow M \otimes_R H \to \operatorname{Hom}_R(U, M)$, and we have an H-comodule structure of $M_{\mathrm{rat}} \subset M$. For any H-comodule M, we have $(\Phi M)_{\mathrm{rat}} = M$. Letting this identification as the unit and letting the inclusion $\Phi(M_{\mathrm{rat}}) = M_{\mathrm{rat}} \subset M$ as the counit, $(?)_{\mathrm{rat}}$ is the right adjoint functor of Φ. This shows that Φ is fully-faithful. A U-module M is called *rational* when $M_{\mathrm{rat}} = M$. By the remark above, the full subcategory of rational U-modules in ${}_U\mathbb{M}$ is identified with the category of H-comodules ${}_G\mathbb{M}$.

The following is a useful criterion for infinitesimally flatness and universal density.

Lemma 2.3 *If G is R-smooth, then G is infinitesimally flat. If G is infinitesimally flat and geometrically irreducible, then U is universally dense. If G is geometrically integral, then G is infinitesimally flat, U is universally dense, and H is R-projective.*

The projectivity of H is due to Raynaud [14].

From now on, we assume that G is geometrically integral. As Φ preserves tensor products, U acts on A in a natural way. It is easy to verify that a G-module A-module M is a (G, A)-module if and only if $\Phi M = M$ is a U-module A-module which satisfy $u(am) = \sum_{(u)} (u_1 a)(u_2 m)$. Namely, ΦM is a module over the *smash product* $A \# U$. The smash product $A \# U$ is $A \otimes_R U$ as an R-module, and is an R-algebra whose product is given by

$$(a \otimes u)(a' \otimes u') = \sum_{(u)} a u_{(1)} a' \otimes u_{(2)} u'.$$

Thus, we obtain an exact functor $\Phi : {}_{G,A}\mathbb{M} \to {}_{A\#U}\mathbb{M}$ with the right adjoint $(?)_{\mathrm{rat}}$.

Lemma 2.4 *The canonical restriction functor ${}_{A\#U}\mathbb{M} \to {}_A\mathbb{M}$ preserves projectives (resp. injectives).*

This is because U is R-projective (resp. flat).

3 Construction of $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(M, N)$ and $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)$

Let $G = \operatorname{Spec} H$ be an affine flat geometrically integral R-group scheme of finite type. We define $\mathcal{G}_X$ to be the category of noetherian (G, X)-schemes of finite type. Namely, an object in $\mathcal{G}_X$ is a quasi-compact G-scheme Y together with a G-morphism $Y \to X$ locally of finite type. Note that $Y \in \mathcal{G}_X$ implies Y is noetherian and of finite type over X. A morphism $Y \to Y'$ is a G-morphism X-morphism. $\mathcal{G}_X^{\mathrm{fl}}$ denotes the full subcategory of $\mathcal{G}_X$ of flat affine X-schemes.

For a category $\mathcal{C}$, a $\mathcal{C}$-valued (G, X)-functor (resp. $(G, X)^{\mathrm{fl}}$-functor) is a contravariant functor from $\mathcal{G}_X$ (resp. $\mathcal{G}_X^{\mathrm{fl}}$) to $\mathcal{C}$ by definition. Note that the category $\mathcal{G}_X$ is skeletally small, so the category of (G, X)-functors with valued in any $\mathcal{C}$ is a category with small hom sets.

Definition 3.1 Assume that $\mathcal{C}$ has finite projective limits. A $\mathcal{C}$-valued (G, X)-functor F is called a (G, X)-faisceau if it satisfies the conditions:

1 If $Y = Y_1 \coprod \cdots \coprod Y_r \in \mathcal{G}_X$ is a finite direct product of (G, X)-schemes, then the canonical maps $F(Y) \to F(Y_i)$ yield an isomorphism $F(Y) \cong \prod_i F(Y_i)$.

2 If $Y \to Y'$ is a morphism in $\mathcal{G}_X$ which is faithfully flat, then the map $F(Y') \to F(Y)$ is a difference kernel of two maps $F(f_1)$ and $F(f_2)$, where $f_i : Y' \times_Y Y' \to Y'$ is the ith projection.

$(G, X)^{\mathrm{fl}}$-faisceau is defined similarly.

A (G, X)-faisceau is uniquely extended to a sheaf over the category $\tilde{\mathcal{G}_X}$ of (G, X)-schemes locally of finite type over X (with the fppf topology), if $\mathcal{C}$ has direct products.

The functor $Y \mapsto \Gamma(Y, \mathcal{O}_Y)$ is a faisceau of commutative rings, which we denote by $\mathcal{O}$. An $\mathcal{O}$-module M is said to be *quasi-coherent* if for any $A \to B$ such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is a $\mathcal{G}_X$-morphism, the canonical B-module map $B \otimes_A M(A) \to M(B)$ is an isomorphism. M is called *coherent* if it is quasi-coherent and $M(A)$ is A-finite for any A with $\operatorname{Spec} A \in \mathcal{G}_X$. A quasi-coherent $\mathcal{O}$-module is a faisceau. For a quasi-coherent sheaf (for usual Zariski topology) M on X, we define a (G, X)-functor of $\mathcal{O}$-modules $W(M)$ by $W(M)(A) = \Gamma(\operatorname{Spec} A, f^*M)$ for $A \in \mathcal{G}_X$, where $f : \operatorname{Spec} A \to X$ is the structure map. It is easy to check that $W(M)$ is quasi-coherent. These definition are done for $(G, X)^{\mathrm{fl}}$-functors similarly.

If M is a quasi-coherent (G, X)-faisceau, then, by restriction, we get a $(G, X)^{\mathrm{fl}}$-faisceau.

Finally, if M is a $(G, X)^{\mathrm{fl}}$-faisceau, then for any noetherian affine open set $U = \operatorname{Spec} A$ of X, $A \otimes H$ together with the action $U \times G \to X$ as the structure morphism lies in $\mathcal{G}_X^{\mathrm{fl}}$. When we set $Q(M)(U) := M(A \otimes H)/I(A \otimes H)M(A \otimes H)$, then by the quasi-coherent condition, we have $M(A \otimes H)$ is nothing but $Q(M)(U) \otimes H$ as an R-module. So the faisceau condition on M yields the usual Zariski sheaf condition on $Q(M)$, and $Q(M)$ is uniquely extended to a quasi-coherent sheaf, as the set of noetherian affine open subsets of X forms an open basis of X. It is easy to see that Q is a quasi-inverse of W. Thus, a quasi-coherent (G, X)-faisceau, a quasi-coherent $(G, X)^{\mathrm{fl}}$-faisceau and a quasi-coherent sheaf are one and the same thing.

Definition 3.2 A $(G, \mathcal{O})$-module (resp. $(U, \mathcal{O})$-module) (faisceau) M is a collection of data:

1 M is an $\mathcal{O}$-module (faisceau)

2 For $A \in \mathcal{G}_X$, $M(A)$ is a (G, A)-module (resp. $A\#U$-module).

3 For each $A \to B$, the canonical map $M(A) \to M(B)$ is a G-homomorphism (resp. U-linear), where $U = \mathrm{Hyp}(G)$ is the hyperalgebra of G. A $(G, \mathcal{O})^{\mathrm{fl}}$-module (faisceau) and a $(U, \mathcal{O})^{\mathrm{fl}}$-module (faisceau) are defined similarly.

Note that a quasi-coherent $(G, \mathcal{O})^{\mathrm{fl}}$-module is extended to a quasi-coherent $(G, \mathcal{O})$-module (in a natural way), uniquely up to isomorphisms.

We have the abelian categories of $(G, \mathcal{O})$-modules and $(U, \mathcal{O})$-modules in an obvious way, which we denote by ${}_{G,\mathcal{O}}\mathbb{M}^p$ and ${}_{U,\mathcal{O}}\mathbb{M}^p$, respectively. The category of $(G, \mathcal{O})$-module faisceaux and $(U, \mathcal{O})$-module faisceaux are denoted by ${}_{G,\mathcal{O}}\mathbb{M}$ and ${}_{U,\mathcal{O}}\mathbb{M}$, respectively. There is an obvious functor $\Phi : {}_{G,\mathcal{O}}\mathbb{M} \to {}_{U,\mathcal{O}}\mathbb{M}$. If $M \in {}_{U,\mathcal{O}}\mathbb{M}$, then M_{rat} defined by $M_{\mathrm{rat}}(A) = M(A)_{\mathrm{rat}}$ is a faisceau, and we obtain a functor $(?)_{\mathrm{rat}} : {}_{U,\mathcal{O}}\mathbb{M} \to {}_{G,\mathcal{O}}\mathbb{M}$. Note that $(?)_{\mathrm{rat}}$ is a right adjoint of Φ.

Lemma 3.3 *The category ${}_{U,\mathcal{O}}\mathbb{M}^p$ and ${}_{U,\mathcal{O}}\mathbb{M}$ have enough injectives.*

This is because each category has exact filtered inductive limits, direct products, and a family of generators (cf. [11, Lemma III.1.3]).

Hence, ${}_{G,\mathcal{O}}\mathbb{M}^p$ and ${}_{G,\mathcal{O}}\mathbb{M}$ also have enough injectives. For any $Y \in \mathcal{G}_X$ and $M \in {}_{U,\mathcal{O}}\mathbb{M}^p$, the Čech cohomology $\check{H}^i(Y, M)$ is defined, and it is a $(U, \check{H}^0(Y, \mathcal{O}))$-module. If $M \in {}_{G,\mathcal{O}}\mathbb{M}^p$, then it is rational.

The associated faisceau functor $a : {}_{U,\mathcal{O}}\mathbb{M}^p \to {}_{U,\mathcal{O}}\mathbb{M}$ is defined, as well as the direct image and its adjoint, and the 'extension by zero' functors. If $M \in {}_{G,\mathcal{O}}\mathbb{M}^p$, then $a(M) \in {}_{G,\mathcal{O}}\mathbb{M}$.

For $M, N \in {}_{U,\mathcal{O}}\mathbb{M}$, we define $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(M, N)$ by $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(M, N)(Y) := \mathrm{Hom}_{\mathcal{O}_Y}(M|_Y, N|_Y)$. The right derived functor of $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(M, ?)$ is denoted by $\underline{\mathrm{Ext}}_{\mathcal{O}_X}(M, ?)$.

As the 'extension by zero' functor is exact, we have:

Lemma 3.4 *Let $M, N \in {}_{U,\mathcal{O}}\mathbb{M}$.*

1 *For any $Y \in \mathcal{G}_X$, we have*

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_Y}(M|_Y, N_Y) \cong \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)|_Y.$$

2 *If M is coherent and N is quasi-coherent, then for any* $\mathrm{Spec}\, A \in \mathcal{G}_X$, *we have*

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)(A) \cong \mathrm{Ext}^i_A(M(A), N(A))$$

in a natural way.

3 *Assume that for any R-finite G-module V there exists some R-finite projective G-module W and a surjective G-homomorphism $W \to V$. Then, for coherent $M \in {}_{G,\mathcal{O}}\mathbb{M}$ and quasi-coherent $N \in {}_{G,\mathcal{O}}\mathbb{M}$, we have $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)$ is rational.*

Remark 3.5 The assumption in **3** of the lemma seems to be difficult to check unless the coordinate ring H of G is an inductive limit of finite R-projective subcoalgebras, where we say that an R-submodule H' is a subcoalgebra of H when $H' \hookrightarrow H$ is pure and $\Delta(H') \subset H' \otimes H'$. Note that this condition is satisfied if R is hereditary. This is proved easily in the line of [16, Proposition 4]. In particular, if G is a split reductive group (over any R), then this condition is satisfied, as any split reductive group is obtained by a base change of a split reductive group over $\mathbb{Z}$ [3, Sect 3.4].

In my talk at the symposium, I erroneously asserted that **3** is always true without this condition (Theorem A in the abstract distributed there). It seems that this is a fatal gap also for the main result, and I could not remove the assumption.

The importance of $(G, \mathcal{O}_X)$-module faisceau can be seen by the following lemma.

Lemma 3.6 *Let M be a coherent $(G, \mathcal{O})$-faisceau. Then, M (identified with $Q(M)$) is a coherent $\mathcal{O}_X$-module, and the set $\{x \in X \mid M_x$ is $\mathcal{O}_{X,x}$-free$\}$ is a G-stable open set of X.*

Such a good property can not be expected for $(U, \mathcal{O})$-faisceaux.

Finally, we mention the construction of the Tor functor. We do this only for *quasi-coherent faisceaux.* First, we consider the case $X = \operatorname{Spec} A$ is affine. Let M and N be $A\#U$-modules. In this case, $M \otimes_A N$ is a $A\#U$-module in a natural way, and we have

$$L_i(M \otimes_A ?)(N) \cong L_i(? \otimes_A N)(M) \cong \operatorname{Tor}_i^A(M, N),$$

as a $U\#A$-projective module is A-projective. This defines the $A\#U$-module $\operatorname{Tor}_i^A(M, N)$.

Lemma 3.7 *If H is a union of R-finite projective subcoalgebras and M and N are rational, then so is $\operatorname{Tor}_i^A(M, N)$ for any $i \geq 0$.*

Now consider general X and *quasi-coherent* $(G, \mathcal{O})$-modules M and N. Then by the lemma, we obtatin a quasi-coherent $(G, \mathcal{O})^{\mathrm{fl}}$-module $\underline{\operatorname{Tor}}_i^{\mathcal{O}_X}(M, N)$ for $i \geq 0$. By definition, for any $\operatorname{Spec} A \in \mathcal{G}_X^{\mathrm{fl}}$, we have

(3.8) $$\underline{\operatorname{Tor}}_i^{\mathcal{O}_X}(M, N)(A) := \operatorname{Tor}_i^A(M(A), N(A)).$$

As was mentioned above, the quasi-coherent $(G, \mathcal{O})^{\mathrm{fl}}$-module $\underline{\operatorname{Tor}}_i^{\mathcal{O}_X}(M, N)$ is extended to a quasi-coherent $(G, \mathcal{O})$-module in a natural way. However, note that (3.8) is *not* true for general $\operatorname{Spec} A \in \mathcal{G}_X$, as is clear.

The construction of $\underline{\operatorname{Ext}}$ and $\underline{\operatorname{Tor}}$ will be used only for X affine in the next section, as the general case is reduced to the affine case.

4 Main results

Let $G = \operatorname{Spec} H$ be an affine flat R-group scheme of finite type, and X a locally noetherian G-action.

Lemma 4.1 *Let Y be a closed integral subscheme of X with the generic point η. Then, the following hold:*

1 *$\mathcal{O}_{Y^*,\eta}$ is a regular local ring*

2 *Let M be a coherent $(G, \mathcal{O}_{Y^*})$-module and N a coherent $(G, \mathcal{O}_X)$-module. Assume that H is a union of R-finite projective subcoalgebras. Then, for any $i \geq 0$, the modules $\operatorname{Tor}_i^{\mathcal{O}_{X,\eta}}(M_\eta, N_\eta)$ and $\operatorname{Ext}^i_{\mathcal{O}_{X,\eta}}(M_\eta, N_\eta)$ are $\mathcal{O}_{Y^*,\eta}$-free. In particular, so is M_η.*

Proof. We take an affine open neighborhood $V = \operatorname{Spec} A$ of η in X. We set $W := V \cap Y^*$. Note that $W = \operatorname{Spec} A/P$ is a dense open neighborhood of η in Y^*, where P is some prime ideal in A.

Consider the morphism $\varphi : W \times G \to Y^*$ defined by $(w, g) \mapsto wg$. Then, $\operatorname{Flat}(\varphi)$ is a G-stable open subset of $W \times G$, and is of the form $F \times G$ for some open subset F of W. As φ is dominating and both $W \times G$ and Y^* are integral, we have that $\eta \in F$. Hence, the composite morphism

$$\psi : \operatorname{Spec} \kappa(\eta) \times G \to W \times G \xrightarrow{\varphi} Y^*$$

is flat, as the first arrow is a localization and is flat. As $\operatorname{Spec} \kappa(\eta) \times G$ is $\kappa(\eta)$-smooth and the unit element $\operatorname{Spec} \kappa(\eta) \times \{e\}$ is mapped to $\eta \in Y^*$ by ψ, the local ring $\mathcal{O}_{Y^*,\eta}$ is a regular local ring, and **1** is proved.

To prove **2**, we may replace X by $V \times G$ and Y by $W \times \{e\}$, by the previous paragraph, as the freeness of a finitely generated module can be checked after a faithfully flat extension. So we may assume that $X = \operatorname{Spec} B$ is affine, $Y = \operatorname{Spec} B/P$ and $Y^* = \operatorname{Spec} B/P^*$. Then, $\operatorname{Tor}_i^B(M, N)$ and $\operatorname{Ext}_B^i(M, N)$ are $(G, B/P^*)$-modules, as M has a (G, B)-resolution $\mathbb{F}$ with each term B-finite projective. As the free locus of a $(G, B/P^*)$-module is a G-stable open subset of Y^* and is clearly non-empty, it must contain η. □

Now we start the proof of the main theorem. We set $Y := \overline{\{x\}}$. We set $B := \mathcal{O}_{X,x}$, $B/P = \mathcal{O}_{Y^*,x}$, and $M := M_x$. We know the following:

1 B/P is a regular local ring (Lemma 4.1, **1**).

2 B is normally flat along P, as the defining ideal sheaf $\mathcal{P}$ of Y^* is a $(G, \mathcal{O})$-module and we have $(\mathcal{P}^n/\mathcal{P}^{n+1})_x = P^n/P^{n+1}$ (see Lemma 4.1 **2**). Similarly, M is normally flat along P (i.e., $\operatorname{Gr}_P M$ is B/P-flat).

3 By the same reason, we have that $\operatorname{Tor}_i^B(B/P, B/P)$ is B/P-free for $i \geq 0$.

4 B_P is a regular local ring (resp. l.c.i.) by the assumption of the theorem.

5 $\operatorname{Ext}_B^i(B/P, M)$ is B/P-free for $i \geq 0$.

6 M_P is Cohen-Macaulay (resp. Gorenstein, free) by assumption.

The first part of the main theorem is reduced to the following lemma.

Lemma 4.2 *Let $(B, \mathfrak{n})$ be a local ring, P a prime ideal of B. Assume that B is normally flat along P. Then, we have* $\dim B = \dim B_P + \dim B/P$. *If B/P is regular, then the following hold:*

1 *B is regular if and only if B_P is regular.*

2 *B is a l.c.i. if and only if B_P is a l.c.i. and $\operatorname{Tor}_2^B(B/P, B/P)$ is B/P-free. If so, then $\operatorname{Tor}_i^B(B/P, B/P)$ is B/P-free for any $i \geq 0$.*

Proof. We take $b_1, \ldots, b_d \in \mathfrak{n}$ so that their image in $\mathfrak{n}/P$ forms a system of parameters of B/P, where $d = \dim B/P$. As B is normally flat along P, we have

$$\dim B/(J + P) \otimes_{B/P} \operatorname{Gr}_P B = \dim \kappa(P) \otimes_{B/P} \operatorname{Gr}_P B = \dim \operatorname{Gr}_{PB_P} B_P = \dim B_P.$$

This shows that there exists some $c_1, \ldots, c_h \in \mathfrak{n}$ ($h := \dim B_P$) such that

$$b_1, \ldots, b_d, \operatorname{in}(c_1), \ldots, \operatorname{in}(c_h)$$

form a system of parameters of $\operatorname{Gr}_P B$. It is easy to see that $B/(b_1, \ldots, b_d, c_1, \ldots, c_h)$ is artinian. As it is obvious that $d + h \leq \dim B$, we have $d + h = \dim B$ and $b_1, \ldots, b_d, c_1, \ldots, c_h$ is a system of parameters of B. This proves the first part.

Now we assume that B/P is regular. In this case, we can take $b_1, \ldots, b_d$ so that they form a regular system of parameters of B/P. We set $J := (b_1, \ldots, b_d)$. Note that $b_1, \ldots, b_d$ is $\operatorname{Gr}_P B$-regular, and hence is B-regular. Note that $\dim B/J = h$.

Consider a finite free B-resolution $\mathbb{F}$ of B/P. We set $\mathbb{G} := \mathbb{F} \otimes_B \mathbb{F}$. Then, as $\underline{b}$ is B/P-regular, we have that $\bar{F} := B/J \otimes_B \mathbb{F}$ is a finite B/J-resolution of $B/\mathfrak{n}$. As $H_i(\mathbb{G}) = \operatorname{Tor}_i^B(B/P, B/P)$ and $H_i(B/J \otimes_B \mathbb{G}) = H_i(\bar{F} \otimes_{B/J} \bar{F}) = \operatorname{Tor}_i^{B/J}(B/\mathfrak{n}, B/\mathfrak{n})$, we have a spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^B(B/J, \operatorname{Tor}_q^B(B/P, B/P)) \Rightarrow \operatorname{Tor}^{B/J}(B/\mathfrak{n}, B/\mathfrak{n}).$$

As we have $\operatorname{Tor}_0^B(B/P, B/P) = B/P$ and $\operatorname{Tor}_1^B(B/P, B/P) = P/P^2$ are B/P-free and hence are Tor-independent of B/J, we have that

$$B/\mathfrak{n} \otimes_{B/P} \operatorname{Tor}_i^B(B/P, B/P) \cong E^2_{0,i} \cong E^\infty_{0,i} \cong E_i = \operatorname{Tor}_i^{B/J}(B/\mathfrak{n}, B/\mathfrak{n})$$

for $i \leq 2$. In particular, we have

$$\begin{aligned}\operatorname{emdim} B/J = \beta_1^{B/J}(B/\mathfrak{n}) &= \mu_B(\operatorname{Tor}_1^B(B/P, B/P)) \\ &= \operatorname{rank}_{B/P}(\operatorname{Tor}_1^B(B/P, B/P)) = \beta_1^{B_P}(\kappa(P)) = \operatorname{emdim} B_P.\end{aligned}$$

Hence, if B_P is regular, then so is B/J. In this case, B is also regular, because J is generated by a B-sequence. This proves **1**, because the converse is well-known.

We prove **2**. B is a l.c.i. if and only if so is B/J. On the other hand, B/J is l.c.i. if and only if

$$\beta_2^{B/J}(B/\mathfrak{n}) = \dim_{B/\mathfrak{n}} \operatorname{Tor}_2^{B/J}(B/\mathfrak{n}, B/\mathfrak{n}) = \binom{e}{2} + e - h.$$

On the other hand, we have

$$\begin{aligned}\beta_2^{B/J}(B/\mathfrak{n}) = \mu_{B/P} \operatorname{Tor}_2^B(B/P, B/P) &\geq \dim_{\kappa(P)} \operatorname{Tor}_2^{B_P}(\kappa(P), \kappa(P)) \\ &= \beta_2^{B_P}(\kappa(P)) \geq \binom{e}{2} + e - h.\end{aligned}$$

The first inequality is an equality if and only if $\operatorname{Tor}_2^B(B/P, B/P)$ is B/P-free, and the second inequality is an equality if and only if B_P is a l.c.i. This proves the equivalence of **2**.

Now we prove that $\operatorname{Tor}_i^B(B/P, B/P)$ is B/P free for any $i \geq 0$ by induction on i, assuming that B is l.c.i.

By induction assumption, we have that $E^2_{p,q} = 0$ for $p > 0$ and $q < i$ in the spectral sequence above. Hence, we have

$$B/\mathfrak{n} \otimes_{B/P} \operatorname{Tor}_i^B(B/P, B/P) \cong \operatorname{Tor}_i^{B/J}(B/\mathfrak{n}, B/\mathfrak{n}).$$

As the Betti numbers of the residue field of a l.c.i. is completely determined by the dimension and the embedding dimension [2], we have $\beta_i^{B_P}(\kappa(P)) = \beta_i^{B/J}(B/\mathfrak{n})$. Hence, $\operatorname{Tor}_i^B(B/P, B/P)$ is B/P-free, as desired. □

Remark 4.3 *A similar argument shows the following. Let $(B, \mathfrak{n})$ be a noetherian local ring, and $P \in \operatorname{Spec} B$. Assume the following:*

1 *B is normally flat along P.*

2 $\operatorname{emdim} B = \operatorname{emdim} B_P + \operatorname{emdim} B/P$.

3 $\operatorname{Tor}_2^B(B/P, B/P)$ *is B/P-free.*

4 *B/P is a complete intersection.*

Then, B is a complete intersection if and only if so is B_P. This statement is also sufficient to prove the main theorem, because **2** *and* **4** *are satisfied if B/P is regular.*

The second assertion of the main theorem is reduced to the following lemma.

Lemma 4.4 *Let $(B, \mathfrak{n})$ be a noetherian local ring, M a finite B-module. Let P be a prime ideal of B. Assume that M is normally flat along P. Then, the following hold:*

1 *We have* $\dim M = \dim M_P + \dim B/P$.

2 *Assume that B is normally flat along P. Then, M is B-free if and only if M_P is B_P-free.*

3 *If* $\mathrm{Tor}_i^B(B/P, M)$ *is B/P-free for $i \geq 0$, then we have*

$$\beta_i^B(M) = \beta_i^{B_P}(M_P)$$

for $i \geq 0$. In particular, we have $\mathrm{proj.dim}_B M = \mathrm{proj.dim}_{B_P} M_P$ *in this case.*

If $\mathrm{Ext}_B^i(B/P, M)$ *is B/P-free for $i \geq 0$ moreover, then, the following hold:*

4 *We have* $\mathrm{depth}\, M = \mathrm{depth}\, M_P + \mathrm{depth}\, B/P$. *In particular, M is Cohen-Macaulay if and only if so are M_P and B/P. If M is Cohen-Macaulay, then*

$$\mathrm{type}\, M = \mathrm{type}\, M_P \cdot \mathrm{type}\, B/P,$$

where type *denotes the Cohen-Macaulay type. In particular, M is Gorenstein if and only if so are M_P and B/P.*

5 *Assume that B/P is Gorenstein. Then, we have $\mu_B^{i+\dim B/P}(M) = \mu_{B_P}^i(M_P)$, where μ^i denotes the Bass number.*

Proof. **1** is proved as in Lemma 4.2. We prove **2**. If M is B-free, then M_P is B_P-free. We prove the converse. Assume that M_P is B_P-free. Note that $\mathrm{Gr}_P M$ is a graded $\mathrm{Gr}_P B$-module generated by degree zero component. If $m_1, \ldots, m_r$ are elements of M such that their image in M/PM is a B/P-basis, then $m_1, \ldots, m_r$ generates $\mathrm{Gr}_P M$ over $\mathrm{Gr}_P B$. Moreover, we have $r = \dim_{\kappa(P)} M_P/PM_P = \mathrm{rank}_{B_P} M_P$. As we have $(\mathrm{Gr}_P M)_P \cong \mathrm{Gr}_{PB_P} M_P \cong (\mathrm{Gr}_{PB_P} B_P)^r$, we have

$$\begin{aligned}\mathrm{rank}_{B/P} P^n M/P^{n+1}M &= \dim_{\kappa(P)} P^n M_P/P^{n+1}M_P \\ &= r \dim_{\kappa(P)} P^n B_P/P^{n+1}B_P = r\, \mathrm{rank}_{B/P} P^n/P^{n+1}.\end{aligned}$$

This simply shows that the canonical surjective map $(\mathrm{Gr}_P B)^r \to \mathrm{Gr}_P M$ is an isomorphism. Now, a standard argument shows that $B^r \to M$ is an isomorphism, and M is B-free.

3 This has nothing to do with normal flatness, and is proved easily using the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{B/P}(B/\mathfrak{n}, \mathrm{Tor}_q^B(B/P, M)) \Rightarrow \mathrm{Tor}_{p+q}^B(B/\mathfrak{n}, M).$$

4 Consider the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{B/P}(B/\mathfrak{n}, \mathrm{Ext}^q_B(B/P, M)) \Rightarrow \mathrm{Ext}_B^{p+q}(B/\mathfrak{n}, M).$$

As $\mathrm{Ext}^q_B(B/P, M)$ is B/P-free by assumption, we have $E_2^{p,q} = 0$ if $q < \mathrm{depth}\, M_P$ or $p < \mathrm{depth}\, B/P$. Moreover, we have

$$\mathrm{Ext}_B^{p_0+q_0}(B/\mathfrak{n}, M) \cong E^\infty_{p_0,q_0} \cong E^2_{p_0,q_0} \cong \mathrm{Ext}^{p_0}_{B/P}(B/\mathfrak{n}, \mathrm{Ext}^{q_0}_B(B/P, M)) \neq 0,$$

where $p_0 = \mathrm{depth}\, B/P$ and $q_0 = \mathrm{depth}\, M_P$. This shows that $\mathrm{depth}\, M = p_0 + q_0$. The equality for the Cohen-Macaulay type also follows, and **4** is proved. **5** is proved easily using the same spectral sequence. □

Note, not only that the result on regularity is fairly well-known, but also that all, but the statements on complete intersection, of the lemmas on normal flatness above are already proved or essentially proved and used in [5]. Note also that some of the argument in our proof is much due to [15].

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Reflexive modules as a tool of representation theory

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Gorenstein 完備局所環 $(R, \mathfrak{m}, k)$ 上の有限生成加群 M に対して、R- 双対 $(\ )^*\mathrm{Hom}_R(\ , R)$ による自然な写像を考える。

$$\varphi_M : M \to M^{**}$$

φ_M が単射のとき、M は torsion free である、といい、同型であるとき、M は reflexive である、という。

注意 1 次の完全列は、良く知られている。

$$0 \to \mathrm{Ext}^1_R(tr(M), R) \to M \overset{\varphi_M}{\to} M^{**} \to \mathrm{Ext}^2_R(tr(M), R) \to 0.$$

但し M の最小自由分解

$$F_{M\bullet} : \cdots \to F_{M1} \overset{d_{F_{M1}}}{\to} F_{M0} \to M \to 0$$

に対して、$tr(M) := \mathrm{Coker}\, d_{F_{M1}}{}^*$ と定義する。

この公式は、M^* の最小自由分解

$$F_{M^*\bullet} : \cdots \to F_{M^*1} \overset{d_{F_{M^*1}}}{\to} F_{M^*0} \to M^* \to 0$$

と併せて、複体 $F_{M^*\bullet} \to F_{M\bullet}^*$ のホモロジーを考えることにより、得られる。
次は、吉野雄二先生に御指摘頂いた。

注意 2 環 R は、*Gorenstein* としたので、M^* は *reflexive* 、即ち $M^{***} \cong M^*$ である。

Cohen-Macaulay 近似について復習しておこう。有限生成加群 M の、R 上の Cohen-Macaulay 近似、有限射影被覆とは、それぞれ、次の完全列をいう [1]。

$$0 \to Y_M^R \to X_M^R \to M \to 0, \tag{1}$$

$$0 \to M \to Y_R^M \to X_R^M \to 0. \tag{2}$$

但し Y_M^R、Y_R^M は射影次元有限な R-加群、X_R^M、X_M^R は極大 Cohen-Macaulay$R$-加群である。ここでは更に、完全列 (1) ((2)) は極小であるとしておく。即ち、Y_M^R と X_R^M (Y_R^M と X_M^R) が共通な直和因子を含まないとしておく。

命題 3 M を有限生成 R-加群とする。M から、*reflexive* な加群への任意の写像 $f : M \to X$ は、φ_M を経由する。詳しく言えば、同型を除いて、$f = f^{**}\varphi = M$ が成り立つ。特に、写像 $g : M \to M^{**}$ が全射ならば、同型を除いて、$g = \varphi_M$ である。

証明) 次の図式が可換である。

$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_X} \\ M^{**} & \xrightarrow{f^{**}} & X^{**} \end{array}$$

X は、reflexive 即ち φ_X は同型写像である。M^{**} が reflexive であることに注意すると、$g : M \to M^{**}$ が全射のとき、$g \cong g^{**}\varphi_M$ より、$g^{**} : M^{**} \to M^{****} \cong M^{**}$ も全射、従って同型である。(証明終。)

系 4 *[2]* $M^* = 0$ なる有限生成加群 M に対し、有限生成加群 N が $X_R^N \cong X_R^M$, $Y_R^N \cong Y_R^M$ を充たすならば、$N \cong M$ である。

証明) 完全列 $0 \to M \to Y_R^M \overset{\eta_M}{\to} X_R^M \to 0$ の R-双対を取れば、$M^* = 0$ より、$(Y_R^M)^* \cong (X_R^M)^*$ が判る。$(Y_R^M)^{**} \cong (X_R^M)$ 故、命題 3 の後半を適用して、$\eta_M \cong \varphi_{Y_R^M}$, $M \cong \mathrm{Ker}\ \varphi_{Y_R^M}$ を得る。N の極小有限射影被覆 $0 \to M \to Y_R^N \overset{\eta_N}{\to} X_R^N \to 0$ においても、命題 3 の後半を適用して、$\eta_N \cong \varphi_{Y_R^N} = \varphi_{Y_R^M}$, 従って $N \cong \mathrm{Ker}\ \varphi_{Y_R^M} \cong M$ である。証明終。)

Cohen-Macaulay 近似に関連して次が成り立つ。

定理 5 M の有限射影被覆 $0 \to M \to Y_R^M \to X_R^M \to 0$ に対して、次は完全列である。

$$0 \to M^{**} \to (Y_R^M)^{**} \to (X_R^M)^{**} \to 0.$$

従って、もし $(Y_R^M)^*$ が *Cohen-Macaulay* 加群ならば、M^* も *Cohen-Macaulay* 加群である。

証明） 与えられた M の有限射影被覆に $\mathrm{Hom}_R(\ ,R)$ を施すと、$0 \to M^* \to (Y_R^M)^* \to (X_R^M)^* \to 0$ は、完全列である。理由は、$\mathrm{Ext}_R^1(X_R^M, R) = 0$ による。この完全列に再び $\mathrm{Hom}_R(\ ,R)$ を施して得られる複体 $0 \to M^{**} \to (Y_R^M)^{**} \to (X_R^M)^{**} \to 0$ も完全列になる。もとより $0 \to M^{**} \to (Y_R^M)^{**} \to (X_R^M)^{**}$ は完全列ゆえ、右端の写像が全射であることを示せばよい。可換図式

$$
\begin{array}{ccccccccc}
0 & \to & M & \to & Y_R^M & \overset{\eta_M}{\to} & X_R^M & \to & 0 \\
 & & \downarrow \varphi_M & & \downarrow \varphi_{Y_R^M} & & \downarrow \varphi_{X_R^M} & & \\
0 & \to & M^{**} & \to & (Y_R^M)^{**} & \overset{\eta_M{}^{**}}{\to} & (X_R^M)^{**} & \to & 0
\end{array}
$$

より、$\eta_M{}^{**} \circ \varphi_{Y_R^M} = \varphi_{X_R^M} \circ \eta_M$ である。$\varphi_{X_R^M}$ は同型写像なので、$\eta_M{}^{**}$ は全射である。証明終。）

射影次元有限な加群 Y を定めて、mod R の部分圏 $\mathcal{M}_Y := \{M | Y_R^M \cong Y \quad up\ to\ free\ modules\}$ の socle element を求めることが出来る。

命題 **6** 加群 Y は、射影次元有限で、自由因子を含まないものとする。加群 $M \in \mathcal{M}_Y$ について、次の同型が成り立つ。

$$\mathrm{Ker}\ \varphi_Y \cong \mathrm{Ker}\ \varphi_M.$$

$$\mathrm{Coker}\ \varphi_Y \cong \mathrm{Ker}\ \varphi_M.$$

証明）加群 M の有限射影被覆 $0 \to M \to Y_R^M \overset{\eta_M}{\to} X_R^M \to 0$ において、写像 η_M は、直和分解 $Y_R^M = Y \oplus P$ （P：自由加群）に応じて $\eta_M = (f \quad \pi_f)$ と書ける。（但し、$f \in \mathrm{Hom}_R(Y, X_R^M)$、$\pi_f \in \mathrm{Hom}_R(P, X_R^M)$ は minimal projective cover $P \to \mathrm{Coker}\ f$ から 導かれるものとする。）

次の図式が可換である。

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & M^{**} & = & M^{**} \\
 & & & & & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Ker}\ \varphi_Y & \to & Y \oplus P & \overset{\left(\begin{smallmatrix}\varphi_Y & 0 \\ 0 & 1\end{smallmatrix}\right)}{\to} & Y^{**} \oplus P & \to & \mathrm{Coker}\ \varphi_Y \\
 & & \downarrow & & \| & & \downarrow & & \downarrow \\
0 & \to & M & \to & Y \oplus P & \overset{\eta_M = (f \pi_f)}{\to} & X_R^M & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

ここから、長完全列 $0 \to \mathrm{Ker}\ \varphi_Y \to M \to M^{**} \to \mathrm{Coker}\ \varphi_Y \to 0$ を得る。中央の写像 $M \to M^{**}$ が φ_M と同値であることも、図式から判る。証明終。)

$M \in \mathcal{M}_{\mathcal{Y}}$ は、写像 $Y \to X$ (X は極大 Cohen-Macaulay 加群)の(必要なら自由加群を付け加えて)核として捉えることが出来る。上記の命題は、φ_Y が、この意味で $\mathcal{M}_{\mathcal{Y}}$ の socle であることを示している。

特に、$\dim R = 1$ のときは、φ_Y は常に全射であって、$N_Y := \mathrm{Ker}\ \varphi_Y$ とおけば、任意の加群 $M \in \mathcal{M}_{\mathcal{Y}}$ に対して、次は完全列である。

$$0 \to N_Y \to M \to M^{**} \to 0.$$

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Almost minimal embeddings of quotient singular points into rational surfaces

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In this article, we shall state some results in [2].

0 Introduction

Let k be an algebraically closed field of characteristic zero. Let $\overline{X}$ be a normal algebraic surface with only one quotient singular point P. Let $f : X \to \overline{X}$ be a minimal resolution of $\overline{X}$ and let $D = \sum_{i=1}^{n} D_i$ be the reduced exceptional divisor with respect to f. We define the rational numbers $d_1, \ldots, d_n$ by the condition

$$(\sum_{i=1}^{n} d_i D_i + K_X \cdot D_j) = 0 \qquad 1 \leq \forall j \leq n,$$

and put $D^{\#} = \sum_{i=1}^{n} d_i D_i$ and $\mathrm{Bk}\,(D) = D - D^{\#}$.

Definition 0.1 (cf. Miyanishi-Tsunoda [4, p. 226]). The above pair (X, D) is called *almost minimal* if, for every irreducible curve C on X, either $(D^{\#} + K_X \cdot C) \geq 0$ or the intersection matrix of $C + \mathrm{Bk}\,(D)$ is not negative definite. And we say that the singular point P is *almost minimal* in $\overline{X}$ if the pair (X, D) is almost minimal.

By virtue of [4, 1.11], we can construct the almost minimal singular points from any quotient singular points which might be changed from the original singularities. We shall classify the above pair (X, D) when the logarithmic Kodaira dimension $\overline{\kappa}(X - D) \leq 1$ and X is a rational surface.

1 The case $\overline{\kappa}(X - D) = -\infty$

Let (X, D) be the same pair as in the introduction. In this section, we assume that $\overline{\kappa}(X - D) = -\infty$ and (X, D) is almost minimal. Since $K_{\overline{X}}$ is not numerically effective, there exists an extremal rational curve $\overline{\ell}$ on $\overline{X}$. Let ℓ be the proper transform of $\overline{\ell}$ on X. Then, by [4, Lemma 2.7], one of the following two cases takes place:

(A) The intersection matrix of $\ell + \mathrm{Bk}\,(D)$ is negative semidefinite, but not negative definite. Furthermore, $(\overline{\ell}^2) = 0$.

(B) The Picard number $\rho(\overline{X})$ equals to 1, and $-K_{\overline{X}}$ is ample.

In the subsequent arguments, we consider the case (A) only, leaving the case (B) to a forthcoming paper.

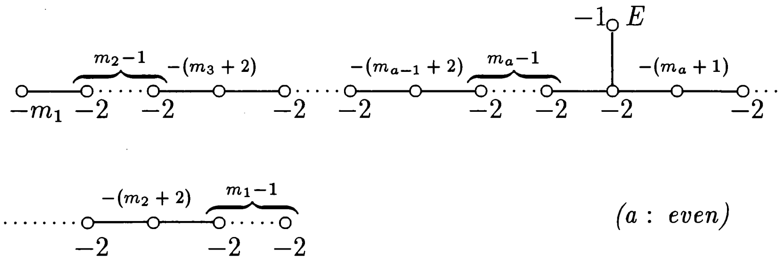
Since $\overline{X}$ is a normal projective surface at worst (only one) quotient singular point, there exists an integer $N > 0$ such that, for every Weil divisor $\overline{G}$ on $\overline{X}$, $N\overline{G}$ is a Cartier divisor on $\overline{X}$. By [4, Lemma 2.8], for a sufficiently large n, the linear system $|nNf^*(\overline{\ell})|$ is composed of an irreducible pencil, free from base points, whose general members are isomorphic to $\mathbf{P}^1$. Then we have the following result:

Theorem 1.1 *Let the notations and the assumptions be the same as above. Then the following assertions hold:*

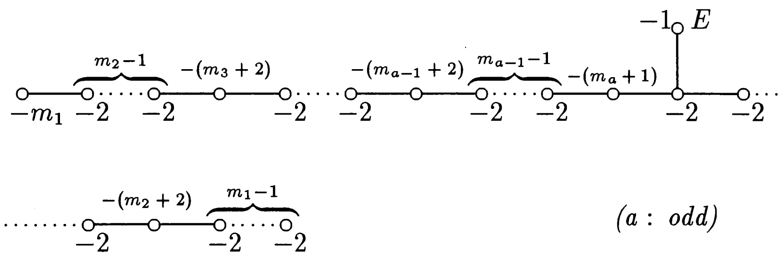
(1) *Let h be the $\mathbf{P}^1$-fibration of X over a curve C defined by the linear system $|nNf^*(\overline{\ell})|$. Then* $\mathrm{Supp}(D)$ *is contained in* $\mathrm{Supp}\,F_0$*, where F_0 is a fiber $h^{-1}(a)$ for some $a \in C$. Furthermore, there exists a unique (-1)-curve E on X such that* $\mathrm{Supp}\,(E + D)$ *coincides with* $\mathrm{Supp}\,(F_0)$.

(2) *The weighted dual graph of $E + D$ is one of the following:*

(i) *Case:* $\mathrm{Supp}\,D$ *is a linear chain.*



(a : even)

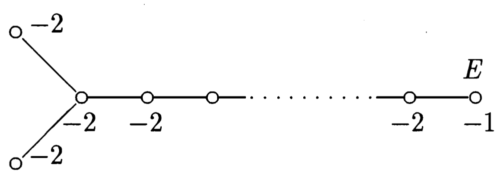


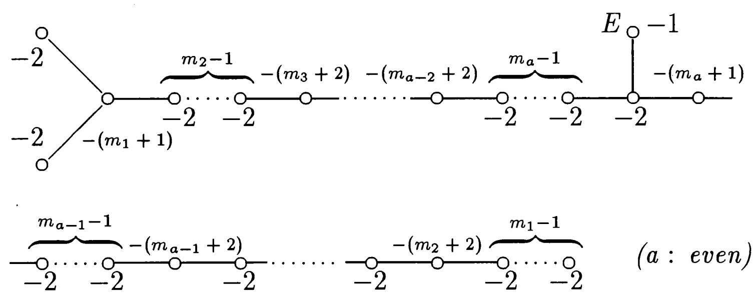
(a : odd)

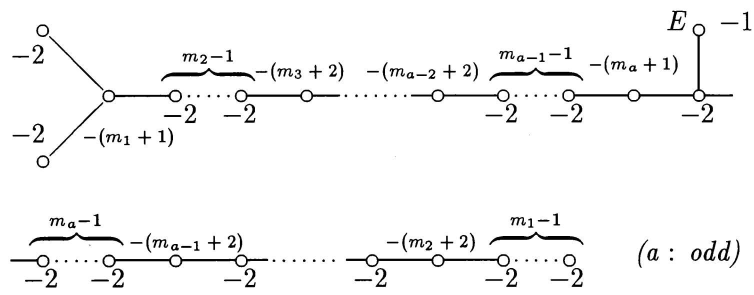
where $m_1 \geq 2$ *and* $m_i \geq 1$ *for* $2 \leq i \leq a$.

(ii) *Case*: Supp D *is not a linear chain.*

(Type D)

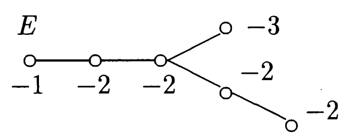






where $m_1 \geq 2$ *and* $m_i \geq 1$ *for* $2 \leq i \leq a$.

(Type E_6)



Remark. (1) In the above Theorem 1.1, we may not assume that X is rational.
(2) Let $(\overline{X}, P)$ (or (X, D)) be the same pair as in the introduction. Suppose that the case (B) occurs. If P is a rational double or triple singular point,

then such pairs have been classified completely. See Miyanishi-Zhang [5] and Zhang [9].

Corollary 1.2 *Assume that D is irreducible and $\overline{\kappa}(X - D) = -\infty$. Then X is a Hirzebruch surface F_n of degree n ($n \geq 2$) and D is the minimal section M_n of X.*

2 The case $\overline{\kappa}(X - D) \geq 0$

Let (X, D) be the same pair as in the introduction. Then we can easily show the following lemma:

Lemma 2.1 *Suppose that $\overline{\kappa}(X - D) \geq 0$ and every irreducible component of D is a (-2)-curve. Then, $\kappa(X) \geq 0$.*

In the subsequent arguments, we assume that X is a nonsingular rational surface and $\overline{\kappa}(X - D) \geq 0$. Furthermore, we assume that (X, D) is an almost minimal pair. Then, by [4, Theorem 2.1], it follows that $D^{\#} + K_X$ is numerically effective.

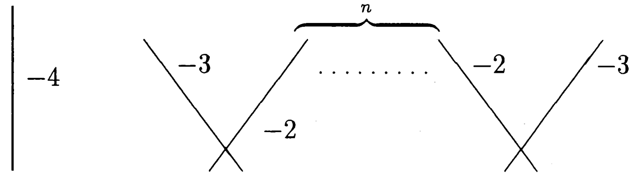
Lemma 2.2 *Suppose that $\overline{\kappa}(X - D) \geq 0$. Then we have*

$$(K_X^2) \leq -1.$$

In the subsequent arguments, we assume that $\overline{\kappa}(X - D) = 0$ or 1. We will give the configuration of such a divisor D.

Since X is a rational surface and the dual graph of D is a tree, we have $|D + K| = \emptyset$ (cf. [3, Lemma 2.1.3]). Hence, by [8, Proposition 2.2], we have the following result:

Theorem 2.3 *Let (X, D) be as above. Suppose that $\overline{\kappa}(X - D) = 0$. Then we have $D + 2K \sim 0$ and $h^0(2(D + K)) = 1$. Furthermore, the configuration of D is one of the following where $0 \leq n \leq 8$:*

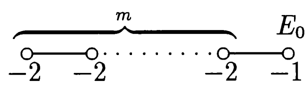


We consider the case $\overline{\kappa}(X - D) = 1$. Then $|j(D^{\#} + K)|$ gives rise to an irreducible pencil of elliptic curves or rational curves $h : X \to \mathbf{P}^1$ for a sufficientry large j by taking, if necessary, the Stein factorization of $\Phi_{|j(D^{\#}+K)|}$ (cf. Kawamata [1, Theorem 2.3]). More precisely, the following assertion holds:

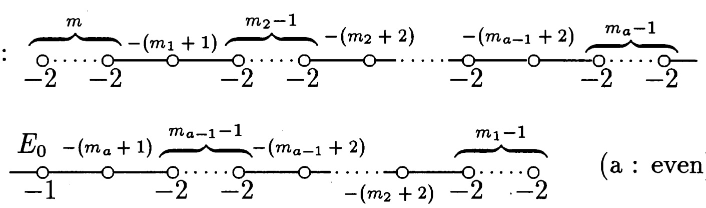
Lemma 2.4 *h is an elliptic fibration. Furthermore,* Supp (D) *is contained in some fiber F_0 of h.*

In order to state the folloing theorem, we define linear chains A_m and $A_{a,m}$ ($a \geq 1$, $m \geq 0$) as follows:

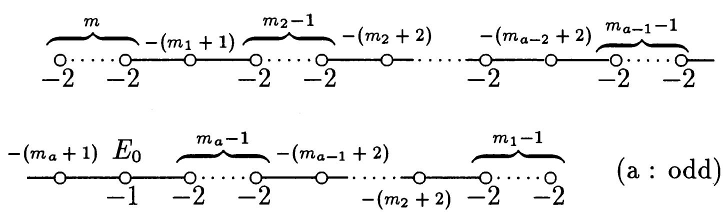
A_m :



$A_{a,m}$:



(a : even)



(a : odd)

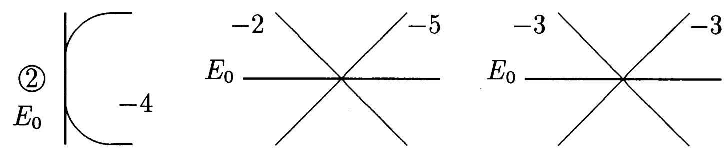
where $m_i \geq 1$ for $1 \leq i \leq a$.

Theorem 2.5 *Let (X, D) be as above. Suppose that $\overline{\kappa}(X - D) = 1$. Then the following assertions hold:*

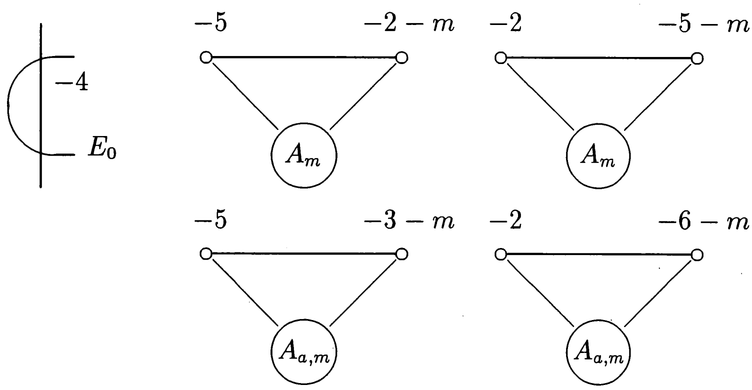
(1) *Let* h *be as in Lemma* 2.4. *And let* F_0 *be the fiber of* h *which contains* D. *Then there exists a unique* (-1)*-curve* E_0 *such that* $\mathrm{Supp}\,(D+E_0) = \mathrm{Supp}\,(F_0)$. *Furthermore, all the fibers of* h *except for* F_0 *contain no* (-1)*-curves.*

(2) *The configuration or the weighted dual graph of* $D + E_0$ *is one of the following:*

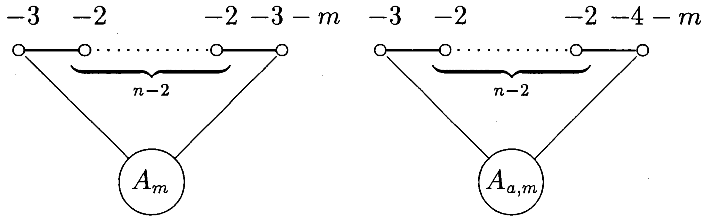
(a) *Cases* B_2 *and* B_3



(b) *Case* A_1'



(c) *Case* A_n' $(n \geq 2)$



By [7], we have the following:

Corollary 2.6 *With the notations as above, suppose that* (X, D) *is almost minimal and* D *is irreducible. Then the following assertions hold*:

(1) $\overline{\kappa}(X - D) = 0$ *if and only if* $n = 4$ *and* $D + 2K_X$ *is linearly equivalent to zero. Furthermore, if* E *is any* (-1)*-curve, the linear system* $|D + 2E|$ *is an irreducible pencil of elliptic curves. We have also a birational morphism* $f : X \to \mathbf{P}^2$ *such that* $f(D)$ *is a sextic with ten double points (possibly including infinitely near points).*

(2) $\overline{\kappa}(X - D) = 1$ *if and only if* $n = 4$ *and* $|D + 3K| \neq \emptyset$. *There exists a unique* (-1)*-curve* E_0 *such that* $(E_0 \cdot D) = 2$. *Furthermore, the linear system* $|D + 2E_0|$ *is an irreducible pencil of elliptic curves. There also exists a birational morphism* $f : X \to \mathbf{P}^2$ *such that* $f(D)$ *is a curve of degree* $3m$, $m \geq 3$ *with nine* m*-tuple points and one double point (possibly including infinitely near points).*

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Finitely generated algebras associated with rational vector fields

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0 Introduction

Let k be an algebraically closed field of characteristic zero and R a polynomial ring in n variables over k. A normal k-subalgebra S of R which is finitely generated over k is called *cofinite* if R is integral over S. It is one of central problems in affine algebraic geometry to consider the structures of cofinite normal k-subalgebras of R. If $n = 2$, it is known (cf. [2]) that $\operatorname{Spec} S$ is isomorphic to the quotient of the affine plane $\mathbf{A}_k^2$ modulo a finite subgroup G of $\mathrm{GL}(2, k)$ acting linearly on $\mathbf{A}_k^2$. In the case $n \geq 3$, nothing essential is known. In the present article, we propose one concrete method to construct a k-subalgebra of R.

For $R = k[x_1, \ldots, x_n]$, we consider a rational vector field

$$\delta = \frac{1}{f_1}\frac{\partial}{\partial x_1} + \cdots + \frac{1}{f_n}\frac{\partial}{\partial x_n},$$

where $f_1, \ldots, f_n$ are homogeneous polynomials of degree $m_1, \ldots, m_n$, respectively, such that the set $\{f_1, \ldots, f_n\}$ is a system of parameters of the maximal ideal $(x_1, \ldots, x_n)$ of R. If $n = 2$ this is equivalent to saying that $\gcd(f_1, f_2) = 1$. Let A be the k-subalgebra of R which is generated by all elements φ of R such that $\delta(\varphi) \in R$, that is,

$$A = k[\varphi \in R \mid \delta(\varphi) \in R].$$

In the present paper, we consider when A is finitely generated over k. Since most results will be stated in the case $n = 2$, we denote, for the sake of simplifying the notations, two variables x_1, x_2 by x, y, f_1, f_2 by f, g and m_1, m_2 by m, n, respectively.

Our main results are the following:

Theorem 1.10 *Let $\tilde{A}$ be the integral closure of A in its quotient field. Then the following conditions are equivalent:*

(1) *A is finitely generated algebra over k.*

(2) *$f, g \in \tilde{A}$.*

(3) *$\tilde{A} = k[x, y]$.*

Theorem 3.1 *Let $A_0 = k[f^i g^i \mid i \geq 2, j \geq 2]$. Assume that $A \cap k[f,g] \supsetneqq A_0$ and that $\deg f \not| \deg g$ and $\deg g \not| \deg f$. If A is finitely generated over k then $f^M \in A$ or $g^N \in A$ for some positive integers $M, N \geq 2$.*

For polynomials φ, ψ of R, we write $\varphi \sim \psi$ if $\psi = c\varphi$ with a nonzero element c of k. In order to indicate that a polynomial φ is the zero polynomial we denote $\varphi \equiv 0$ and distinguish it from the equation $\varphi = 0$. For relevant results on regular vector fields on $\mathbf{A}^2 = \mathrm{Spec}\, k[x,y]$, we refer to [3].

1 The integral closure $\widetilde{A}$ of A

Note that the hypothesis that the set $\{f_1, \ldots, f_n\}$ is a system of parameters of $(x_1, \ldots, x_n)$ implies that any two of the f_i have no common irreducible factors. We shall begin with the following result.

Lemma 1. 1 *For an element φ of R, φ belongs to A if and only if $f_i \mid \varphi_{x_i}$ for every $1 \leq i \leq n$, where $\varphi_{x_i} = \partial\varphi/\partial x_i$.*

Proof. If $\varphi \in A$, we write $(1/f_1)\varphi_{x_1} + \cdots + (1/f_n)\varphi_{x_n} = h$ with $h \in R$. Then we have

$$f_2 \cdots f_n \varphi_{x_1} = f_1 \left(f_2 \cdots f_n h - \sum_{i=2}^{n} f_2 \cdots \overset{\vee}{f_i} \cdots f_n \varphi_{x_i} \right) .$$

Since $\gcd(f_1, f_2 \cdots f_n) = 1$, we obtain $f_1 \mid \varphi_{x_1}$. Similarly, $f_i \mid \varphi_{x_i}$ for $2 \leq i \leq n$. It is clear that φ belongs to A if $f_i \mid \varphi_{x_i}$ for every $1 \leq i \leq n$. Q.E.D.

Lemma 1. 2 (1) *A is a graded ring.*

(2) *If $n = 2$ the quotient field $Q(A)$ of A is $k(x,y)$.*

Proof. (1) For $\varphi \in A$, we write φ as $\varphi = \sum_{r=1}^{s} \varphi_r$, where φ_r is the r-th homogeneous part of φ. Then $f_i \mid \varphi_{x_i}$ if and only if $f_i \mid (\varphi_r)_{x_i}$ for any i, where $1 \leq i \leq n$. Hence $\varphi \in A$ if and only if $\varphi_r \in A$ for any r. So A is a graded ring.

(2) Write $f = \ell^\alpha f_1, g = \ell'^\beta g_1$, where ℓ, ℓ' are linear polynomials, $\gcd(\ell, f_1) = 1$ and $\gcd(\ell', g_1) = 1$. Since $\gcd(f,g) = 1$, it follows that $k\ell + k\ell' = kx + ky$ as k-vector spaces. Let

$$\varphi_{ij} = \ell^i f_1^2 \ell'^j g_1^2 \quad \text{with} \quad i \geq \alpha + 1 \quad \text{and} \quad j \geq \beta + 1 .$$

Then it is straightforward to see that φ_{ij} is an element of A. Since

$$\ell = \varphi_{i+1,j}/\varphi_{ij} \quad \text{and} \quad \ell' = \varphi_{i,j+1}/\varphi_{ij} ,$$

it follows that $Q(A) = k(x,y)$. Q.E.D.

The following result gives a sufficient condition that A be a finitely generated k-algebra.

Lemma 1. 3 *The following assertions hold:*

(1) *There are some positive integers* $s_i \geq 2$ $(1 \leq i \leq n)$ *with* $f_i^{s_i} \in A$ *if and only if, after a change of indices* $\{1, 2, \ldots, n\}$ *which allows us to assume* $m_1 \leq m_2 \leq \cdots \leq m_n$*, the* f_i *are written in the following forms:*

$$
\begin{aligned}
f_1 &\sim x_1^{m_1} \\
f_2 &\sim x_1^{m_1+1}\sigma_{21}(x_1, x_2) + c_2 x_2^{m_2} \\
f_3 &\sim x_1^{m_1+1}\sigma_{31}(x_1, x_2, x_3) + x_2^{m_2+1}\sigma_{32}(x_2, x_3) + c_3 x_3^{m_3} \\
&\cdots\cdots\cdots \\
f_n &\sim x_1^{m_1+1}\sigma_{n1}(x_1, \ldots, x_n) + x_2^{m_2+1}\sigma_{n2}(x_2, \ldots, x_n) \\
&\quad + \cdots + x_{n-1}^{m_{n-1}+1}\sigma_{n,n-1}(x_{n-1}, x_n) + c_n x_n^{m_n} ,
\end{aligned}
$$

where $c_i \in k^*$ *and the* $\sigma_{ij}(x_j, \ldots, x_i)$ *are the homogeneous polynomials of degree* $m_i - (m_j + 1)$ *satisfying the conditions derived from the conditions*

$$f_j \mid (f_i)_{x_j} \quad \textit{for all pairs } (i, j) \textit{ with } \quad 1 \leq j < i,\ 2 \leq i \leq n .$$

(2) *If the* f_i *are written in the above forms in* (1), *then* A *is a finitely generated* k*-algebra.*

Proof. (1) Suppose $f_i^{s_i} \in A$ with $s_i \geq 2$ for all $1 \leq i \leq n$. Then, by Lemma 1.1, $f_j \mid (f_i)_{x_j}$ whenever $i \neq j$ because $\gcd(f_i, f_j) = 1$. This implies that $m_j \leq m_i - 1$ provided $(f_i)_{x_j} \not\equiv 0$. If the indices $\{1, \ldots, n\}$ are changed so that $m_1 \leq m_2 \leq \cdots \leq m_n$, this gives rise to the condition that $(f_i)_{x_j} \equiv 0$ whenever $j > i$. Namely, f_i is a polynomial in $x_1, \ldots, x_i$. So, $f_1 \sim x_1^{m_1}$. Since $f_1 \mid (f_2)_{x_1}$, we have

$$
\begin{aligned}
f_2 &= \int b x_1^{m_1}\rho(x_1, x_2)dx_1 + c x_2^{m_2} \\
&\sim x_1^{m_1+1}\sigma_{21}(x_1, x_2) + c_2 x_2^{m_2} .
\end{aligned}
$$

Suppose f_{i-1} is written as

$$f_{i-1} \sim \sum_{j=1}^{i-2} x_j^{m_j+1}\sigma_{i-1,j}(x_j, \ldots, x_{i-1}) + c_{i-1}x_{i-1}^{m_{i-1}} .$$

Since $f_{i-1} \mid (f_i)_{x_{i-1}}$, we can write

$$f_i \sim \sum_{j=1}^{i-1} x_j^{m_j+1}\sigma_{ij}(x_1, \ldots, x_i) + \sigma_{ii}(x_1, \ldots, x_{i-2}, x_i) ,$$

where we may assume that the following condition is satisfied:

(∗) σ_{ij} does not contain monomial terms whose x_k-exponent is greater than or equal to $m_k + 1$ for any k with $1 \leq k < j$.

We show by induction on j that $\sigma_{ij}(x_1, \ldots, x_i)$ is in fact a polynomial in $x_j, \ldots, x_i$. It is easy to see that the condition $x_1^{m_1} \mid (f_i)_{x_1}$ implies

$$x_1^{m_1} \mid \left(\sum_{j=2}^{i-1} x_j^{m_j+1}(\sigma_{ij})_{x_1} + (\sigma_{ii})_{x_1} \right) .$$

Since $\sigma_{ij}(x_1,\ldots,x_i)$ $(2\le j<i)$ contains no monomial terms whose x_1-exponent is greater than or equal to m_1+1, it follows that $(\sigma_{ij})_{x_1}\equiv 0$ for $2\le j\le i$. Suppose $\sigma_{ik},\ldots,\sigma_{ii}$ are polynomials in $x_k,\ldots,x_i$ for $1\le k<j$. Write

$$\begin{aligned}(f_i)_{x_{j-1}}\sim\ & x_1^{m_1+1}(\sigma_{i1})_{x_{j-1}}+\cdots+x_{j-2}^{m_{j-2}+1}(\sigma_{i,j-2})_{x_{j-1}}\\ &+x_{j-1}^{m_{j-1}}\left((m_{j-1}+1)\sigma_{i,j-1}+x_j(\sigma_{i,j-1})_{x_{j-1}}\right)\\ &+\textstyle\sum_{k=j}^{i-1}x_k^{m_k+1}(\sigma_{ik})_{x_{j-1}}+(\sigma_{ii})_{x_{j-1}}\end{aligned}$$

Note that $f_{j-1}\mid(f_i)_{x_{j-1}}$. Write $(f_i)_{x_{j-1}}=f_{j-1}h$ with $h\in R$. In both sides of this equation, we equate, by making use of the condition $(*)$, the terms divisible by $x_1^{m_1+1}$ first, the terms divisible by $x_2^{m_2+1}$ next in the remaining terms, and continue this way untill the terms divisible by $x_{j-2}^{m_{j-2}+1}$. Looking at the remaining terms, we then know that

$$x_{j-1}^{m_{j-1}}\mid\left(\sum_{k=j}^{i-1}x_k^{m_k+1}(\sigma_{ik})_{x_{j-1}}+(\sigma_{ii})_{x_{j-1}}\right).$$

It then follows by virtue of the condition $(*)$ that $\sigma_{ij},\ldots,\sigma_{ii}$ are polynomials in $x_j,\ldots,x_i$.

We show that $c_i\in k^*$. Indeed, if $c_i=0$ then $(f_1,\ldots,f_i)\subseteq(x_1,\ldots,x_{i-1})$, which is impossible because $\{f_1,\ldots,f_n\}$ is a system of parameters of the ideal $(x_1,\ldots,x_n)$. Hence $c_i\in k^*$.

Conversely, if $f_1,\ldots,f_n$ are written in the above forms and satisfy the condition that $f_j\mid(f_i)_{x_j}$ for $1\le j<i$ and $2\le i\le n$, one can readily show that $f_1^2,\ldots,f_n^2\in A$.

(2) With the expressions of $f_1,\ldots,f_n$ in (1) above, define a k-subalgebra B of A by

$$B=k[f_1^2,f_2^2,\ldots,f_n^2]$$

Let $\widetilde{B}$ be the integral closure of B in $k(x_1,\ldots,x_n)$ and let $\widehat{B}=k[x_1,f_2,\ldots,f_n]$. Then $\widehat{B}\supset B$ and $\widehat{B}$ is integral over B. Furthermore, it is easy to show that $x_1,\ldots,x_n$ are integral over $\widehat{B}$ because $c_i\in k^*$ for $2\le i\le n$. Hence $x_1,\ldots,x_n$ are integral over B and $\widetilde{B}=R$. Note that $\widetilde{B}$ is a finitely generated B-module and $\widetilde{B}$ contains A as a B-submodule. Since B is a Noetherian ring, A is a finitely generated B-module. Hence A is a finitely generated k-algebra. Q.E.D.

We define a k-subalgebra A_0 of A by

$$A_0=k[f_1^{i_1}f_2^{i_2}\cdots f_n^{i_n}\mid i_j\ge 2,1\le j\le n].$$

In conection with Lemma 1.3, we wish to show that $f_i^{s_i}\in A$ for some positive integer $s_i\ge 2$ $(1\le i\le n)$ provided A contains A_0 and A is a finitely generated k-algebra.

Modifying the settings a bit, we consider the following problem.

PROBLEM 1.4 *Let $p_1,\ldots,p_n$ be homogeneous polynomials of R of respective degrees $m_1,\ldots,m_n$ such that $\{p_1,\ldots,p_n\}$ is a system of parameters of the ideal $(x_1,\ldots,x_n)$. Let C be a finitely generated graded k-subalgebra of R which contains the following subalgebra*

$$C_0=k[p_1^{i_1}p_2^{i_2}\cdots p_n^{i_n}\mid i_j\ge 1,1\le j\le n].$$

Then, does it follow that $p_i^{s_i}\in C$ for some positive integers s_i, $1\le i\le n$?

With the notations of Lemma 1.3, if we take $C=A$ and $p_i=f_i^2$ $(1\le i\le n)$, then $A\supset k[p_1^{i_1}\cdots p_n^{i_n}\mid i_j\ge 1,1\le j\le n]$. If we can answer Problem 1.4 affirmatively, we obtain $f_i^{s_i}\in A$ and the converse of the assertion (2) of Lemma 1.3 will hold. Furtheremore, we note that $\text{tr.deg}_k k(f_1,\ldots,f_n)=n$ since $\{f_1,\ldots,f_n\}$is a system of parameters of $(x_1,\ldots,x_n)$.

Theorem 1. 5 *Let $\bar{C}$ be the integral closure of C in its qoutient field $Q(C)$. Then we have $p_1, \ldots, p_n \in \bar{C}$ provided $\bar{C} \cap k(p_1, \ldots, p_n)$ is finitely generated over k.*

Note that the condition in Theorem 1.5 is satisfied when $n = 2$ (cf. [4]). In order to prove Theorem 1.5, we may and shall replace, if necessary, $p_1, \ldots, p_n$ by their powers $p_1^{u_1}, \ldots, p_n^{u_n}$ and assume $m_1 = \cdots = m_n$. Our proof consists of several lemmas. The following result is well-known. (See, for example, Gurjar [1, Lemma 3] for a geometric proof in the case $n = 2$.)

Lemma 1. 6 $R = k[x_1, \ldots, x_n]$ *is integral over* $k[p_1, \ldots, p_n]$.

Proof. Set $S = k[p_1, \ldots, p_n]$. By the hypothesis that $\{f_1, \ldots, f_n\}$ is a system of parameters of $(x_1, \ldots, x_n)$, $R/(p_1, \ldots, p_n)R$ is a finite-dimensional k-vector space. Let $\{\varphi_1, \ldots, \varphi_N\}$ be a set of elements of R such that the set $\{\overline{\varphi}_1, \ldots, \overline{\varphi}_n\}$ of the residue classes of $\varphi_1, \ldots, \varphi_n$ modulo $(p_1, \ldots, p_n)R$ is a k-basis of $R/(p_1, \ldots, p_n)R$. We shall show that R is generated by $\varphi_1, \ldots, \varphi_n$ as an S-module.

Let ψ be a homogeneous polynomial of degree, say r, in R. Then one can write

$$\psi = \sum_{i=1}^{N} a_i \varphi_i + \sum_{j=1}^{n} p_j \psi_j \ ,$$

where $a_i \in k$ $(1 \leq i \leq N)$ and the ψ_j $(1 \leq j \leq n)$ are homogeneous polynomials of degree less than $r = \deg \psi$. By induction on r, every ψ_j is written as

$$\psi_j = \sum_{\ell=1}^{N} h_{j\ell} \varphi_\ell \ , \quad \text{where} \quad h_{j\ell} \in S \ .$$

Then we have

$$\psi = \sum_{\ell=1}^{N} \left\{ a_\ell + \sum_{j=1}^{n} p_j h_{j\ell} \right\} \varphi_\ell \ .$$

Hence R is a finite S-module generated by $\varphi_1, \ldots, \varphi_n$. So, R is integral over S. Q.E.D.

Lemma 1. 7 *Let φ, ψ be homogeneous polynomials of R such that* $\gcd(\varphi, \psi) = 1$ *in R. Then* $\gcd(\varphi(p_1, \ldots, p_n), \psi(p_1, \ldots, p_n)) = 1$ *in R.*

Proof. Write $\varphi(p) = \varphi(p_1, \ldots, p_n)$ and $\psi(p) = \psi(p_1, \ldots, p_n)$. Suppose to the contrary that $\gcd(\varphi(p), \psi(p)) \neq 1$. Then there exists a prime ideal $\underline{P}$ of height 1 of R such that $\varphi(p) \in \underline{P}$ and $\psi(p) \in \underline{P}$. Let $\underline{p} = \underline{P} \cap S$, where $S = k[p_1, \ldots, p_n]$. Since R is integral over S by Lemma 1.6, the going-down theorem implies that $\underline{p}$ is a prime ideal of height 1. Furthermore, $\underline{p}$ contains φ' and ψ', which are the copies of φ and ψ, respectively, and considered as elements of S. This contradicts the hypothesis that $\gcd(\varphi, \psi) = 1$. Q.E.D.

Lemma 1. 8 *The following assertions hold:*

(1) $\bar{C} \cap k(p_1, \ldots, p_n) = \bar{C} \cap k[p_1, \ldots, p_n]$.

(2) $C \cap k(p_1, \ldots, p_n) = C \cap k[p_1, \ldots, p_n]$.

Proof. We shall prove only the assertion (2). The assertion (1) can be proved in a similar fashion. Note first that $C \cap k(p_1, \ldots, p_n)$ is a graded ring since C is a graded ring. In fact, let ξ be any element of $C \cap k(p_1, \ldots, p_n)$ and let

$$\xi = \xi_n + \xi_{n+1} + \cdots + \xi_l \quad \text{with} \quad \xi_i \in C$$

be the homogeneous decomposition. Then we have $\xi = \psi(p)/\varphi(p)$, where $\varphi(p), \psi(p) \in k[p_1, \ldots, p_n]$. Let the homogeneous decompositions of $\varphi(p)$ and $\psi(p)$ be as follows:

$$\begin{aligned} \psi(p) &= \psi_c(p) + \psi_{c+1}(p) + \cdots + \psi_d(p) \\ \varphi(p) &= \varphi_e(p) + \varphi_{e+1}(p) + \cdots + \varphi_f(p) \end{aligned}$$

Since

$$(\xi_n + \xi_{n+1} + \cdots + \xi_l)(\varphi_e + \varphi_{e+1} + \cdots + \varphi_f) = \psi_c + \psi_{c+1} + \cdots + \psi_d,$$

we obtain the following relations:

$$\xi_n \varphi_e = \psi_c, \xi_{n+1}\varphi_e + \xi_n \varphi_{e+1} = \psi_{c+1}, \cdots, \xi_l \varphi_f = \psi_d$$

Then it follows that $\xi_n = \psi_c/\varphi_e, \xi_{n+1} = (\psi_{c+1}\varphi_e - \psi_c\varphi_{e+1})/\varphi_e^2, \ldots, \xi_\ell = \psi_d/\varphi_f$. They are elements of $C \cap k(p_1, \ldots, p_n)$. Hence $C \cap k(p_1, \ldots, p_n)$ is a graded ring.

We shall show the assertion (2). Let $\xi = \psi(p)/\varphi(p)$ be any homogeneous elelment of $C \cap k(p_1, \ldots, p_n)$, where φ, ψ are homogeneous polynomials in $k[p_1, \ldots, p_n]$ such that $\gcd(\varphi, \psi) = 1$. Since ξ is an element of $k[x_1, \ldots, x_n]$ and $\gcd(\varphi(p), \psi(p)) = 1$ in $k[x_1, \ldots, x_n]$ by Lemma 1.7, $\varphi(p)$ must be a constant. Hence ξ is an element of $C \cap k[p_1, \ldots, p_n]$. Q.E.D.

Note that $\text{tr.deg}_k k(p_1, \ldots, p_n) = n$. *Now we assume that $\tilde{C} \cap k(p_1, \ldots, p_n)$ is a finitely generated k-algebra.* If $n = 2$, a theorem of Zariski [4] then says that $\tilde{C} \cap k(p_1, p_2)$ is finitely generated over k. We replace $\tilde{C}$ by $\tilde{C} \cap k(p_1, \ldots, p_n)$ and assume that

$$k[p_1, \ldots, p_n] \supset \tilde{C} \supset C_0 := k[p_1^{i_1} \cdots p_n^{i_n} \mid i_j > 0, 1 \le j \le n].$$

On the other hand, it is clear that

$$Q(k[p_1, \ldots, p_n]) = Q(C_0) = k(p_1, \ldots, p_n).$$

Then, in order to prove Theorem 1.5, it suffices to show the following result.

Lemma 1.9 $\tilde{C} = k[p_1, \ldots, p_n]$.

Proof. Since $\tilde{C}$ is finitely generated over k, let $\varphi_1, \cdots, \varphi_r$ be homogeneous polynomials of degree d_i which generate $\tilde{C}$ over k. We need the following auxiliary result.

Claim. $\sqrt{(\varphi_1(p), \ldots, \varphi_r(p))R} = (x_1, \ldots, x_n)$.

Proof. Let $\underline{P}$ be a prime divisor of $(\varphi_1(p), \ldots, \varphi_r(p))R$ and let $\underline{p} = \underline{P} \cap S$, where $S = k[p_1, \ldots, p_n]$. Then $\varphi_1(p), \ldots, \varphi_r(p) \in \underline{p}$. We have only to show that $p_1, \ldots, p_n \in \underline{p}$. In fact, $\underline{p}$ has then height n and $\underline{P}$ has therefore height n because R is integral over S. Let $(\mathcal{O}, \underline{m})$ be a discrete valuation ring of the quotient field $k(p_1, \dots, p_n)$ such that $(\mathcal{O}, \underline{m})$ dominates $(S_{\underline{p}}, \underline{p}S_{\underline{p}})$, and let v be the associated valuation. Then $v(\varphi_1(p)) > 0, \dots, v(\varphi_r(p)) > 0$. Since $p_1^N p_2 \cdots p_n$ is an element of C_0, we can write

$$\begin{aligned} p_1^N p_2 \cdots p_n &= \Phi_N(\varphi_1(p), \dots, \varphi_r(p)) \\ &= \sum_{\alpha_1 d_1 + \cdots + \alpha_r d_r = M} c_\alpha \varphi_1(p)^{\alpha_1} \cdots \varphi_r(p)^{\alpha_r} \end{aligned}$$

where Φ_N is a weighted homogeneous polynomial of degree M. If N tends to the infinity, at least one of $\alpha_1, \dots, \alpha_r$ should tend to the infinity. Hence $v(p_1^N p_2 \cdots p_n)$ tends to the infinity as N tends to the infinity. This implies that $v(p_1) > 0$. Hence $p_1 \in \underline{p}$. Similarly, $p_2, \dots, p_n \in \underline{p}$. Q.E.D.

Now let us return to the proof of Lemma 1.9. By the above claim, we know that $R/(\varphi_1(p), \dots, \varphi_r(p))I$ is a finite-dimensional k-algebra. Since $\tilde{C}$ is a graded subalgebra of $k[x_1, \dots, x_n]$, we conclude by the same argument as in the proof of Lemma 1.6 that $k[x_1, \dots, x_n]$ is a finitely generated $\tilde{C}$-module. Since $k[x_1, \dots, x_n] \supset k[p_1, \dots, p_n] \supset \tilde{C}$, $k[x_1, \dots, x_n]$ contains $k[p_1, \dots, p_n]$ as a $\tilde{C}$-submodule. Since $\tilde{C}$ is a Neotherian ring, $k[p_1, \dots, p_n]$ is a finitely generated $\tilde{C}$-module. Hence $k[p_1, \dots, p_n]$ is integral over $\tilde{C}$. Since $Q(k[p_1, \dots, p_n]) = Q(\tilde{C})$, we have $\tilde{C} = k[p_1, \dots, p_n]$. This implies that $p_1, \dots, p_n \in \tilde{C}$, and Theorem 1.5 is thus proved.

Now we shall state and prove one of main results of the present paper.

Theorem 1. 10 *Suppose $n = 2$. With the same notations and assumptions as in the introduction, let $\bar{A}$ be the integral closure of A in its quotient field. Then the following conditions are equivalent:*

(1) *A is finitely generated over k.*

(2) *$f, g \in \bar{A}$.*

(3) *$\bar{A} = k[x, y]$.*

Proof. The condition (1) implies the condition (2) by Theorem 1.5. Suppose that $f, g \in \bar{A}$. Note that the condition $\gcd(p_1, p_2) = 1$ is the only condition which is necessary in the proof of Lemma 1.6. Hence the hypothesis $\gcd(f, g) = 1$ implies that $k[x, y]$ is integral over $k[f, g]$. Hence $k[x, y]$ is integral over A. In view of Lemma 1.2, this implies that $\bar{A} = k[x, y]$. Conversely, if $\bar{A} = k[x, y]$ it is clear that $f, g \in \bar{A}$. Thus the conditions (2) and (3) are equivalent. We shall show that the condition (2) implies the condition (1). Let

$$\Phi(x) = x^M + c_1 x^{M-1} + \cdots + c_{M-1} x + c_M = 0$$

$$\Psi(y) = y^N + d_1 y^{N-1} + \cdots + d_{N-1} y + d_N = 0$$

be the integral relations of x and y over A, where $c_1, \dots, c_M, d_1, \dots, d_N \in A$. Let $C = k[c_1, \dots, c_M, d_1, \dots, d_N]$. Then $k[x, y]$ is a finitely generated C-module and A is its C-submodule. Hence A is a finitely generated k-algebra. Q.E.D.

Problem 1.4 itself has the following counter-example in dimension 2.

Proposition 1. 11 *Let C be a subalgebra of $k[x, y]$ defined as $C = k[x + y, x^i y^j \mid \forall i > 0, \forall j > 0]$. Then C has the following properties,*

(1) $C \supset C_0 := k[x^i y^j \mid i > 0, j > 0]$.

(2) *C is finitely generated over k.*

(3) *C does not contain x^s, y^t for every $s > 0$ and every $t > 0$.*

Proof. (1) It is clear.

(2) Define a subalgebra C_1 of C as $C_1 = k[x+y, xy, x^2y]$. We shall show that $C = C_1$. First we prove $x^n y \in C_1$ by induction on n. If $n = 1, 2$, this holds clearly. Suppose that this holds for $x^i y$ with $i \leq n-1$, that is, $x^{n-1}y, x^{n-2}y, \cdots, xy \in C_1$. Since $x^n y = x^{n-1}y(x+y) - xy \cdot x^{n-2}y$, we then have $x^n y \in C_1$, where $n \geq 3$. Suppose $n \geq r \geq 2$. Since

$$x^n y^r = \begin{cases} (xy)^n & n = r \\ (xy)(x^{n-1}y^{r-1}) & n > r \end{cases}$$

we have $x^n y^r \in C_1$ by induction on r. Finally, when $n \leq r$, we suppose that $x^n y^{r'} \in C_1$ for r' with $n \leq r' < r$. Since $x^n y^r = x^n y^{r-1}(x+y) - x^{n+1}y^{r-1} \in C_1$, we have $x^n y^r \in C_1$ by induction on r. Hence $C = C_1$.

(3) Note that C is a graded subring of $k[x, y]$. If $x^s \in C$, we may write

$$x^s = a_{s00}(x+y)^s + \sum_{\substack{\alpha+2\beta+3\gamma=s \\ \beta>0 \text{ or } \gamma>0}} a_{\alpha\beta\gamma}(x+y)^\alpha (xy)^\beta (x^2y)^\gamma,$$

where $a_{s00} \neq 0$. Then we have

$$a_{s00}y^s = x\rho \quad \text{with} \quad \rho \in k[x, y].$$

This is a contradiction. Similarly, if $y^t \in C$, write

$$y^t = a_{t00}(x+y)^t + \sum_{\substack{\alpha+2\beta+3\gamma=t \\ \beta>0 \text{ or } \gamma>0}} a_{\alpha\beta\gamma}(x+y)^\alpha (xy)^\beta (x^2y)^\gamma$$

where $a_{t00} \neq 0$. Then we have

$$a_{t00}x^t = y\theta \quad \text{with} \quad \theta \in k[x, y].$$

This is a contradiction as well. Q.E.D.

2 Criteria for the finite generation of A

In this section, we treat the case $n = 2$ and consider whether or not A is finitely generated over k for some specific pairs of f, g. These results will be used in the next section. Here we write A as $A(f, g)$ whenever it is necessary to recall the vector field $\delta = 1/f\partial/\partial x + 1/g\partial/\partial y$. We shall begin with the following result.

Lemma 2. 1 (1) *For positive integers m, n, we have*

$$A(x^m, y^n) = k[x^{m+i}, y^{n+j} \mid 1 \leq i \leq m+1, 1 \leq j \leq n+1].$$

Hence $A(x^m, y^n)$ is finitely generated over k.

(2) *We have*

$$A(y,x) = k[x^iy^j \mid i \geq 1, j \geq 1].$$

Then A is not finitely generated over k.

Proof. (1) A homogeneous polynomial φ belongs to A if and only if $x^m \mid \varphi_x$ and $y^n \mid \varphi_y$. So, $\varphi \in A$ if and only if φ is a sum of monomials ax^sy^t with $a \in k$ such that $s = 0$ or $s \geq m+1$ as well as $t = 0$ or $t \geq n+1$. Then it is easy to see that A is generated by the x^{m+i} and the y^{n+j} with $1 \leq i \leq m+1$ and $1 \leq j \leq n+1$.

(2) It is straightforward to show that A is generated by the elements x^iy^j with $i \geq 1$ and $j \geq 1$. Then A is not finitely generated because A contains x^iy with $i \gg 0$. Q.E.D.

In the following two lemmas, we shall consider the case where f and g are linear homogeneous polynomials.

Lemma 2. 2 *The following assertions hold true:*

(1) *Let $f = y$ and $g = y - \alpha x$ with $\alpha \in k^*$. Then we have*

$$A(y, y-\alpha x) = k[u^iv^j \ (i \geq 1, j \geq 2), u^n - nu^{n-1}v \ (n \geq 2)],$$

where $u = y$ and $v = y - \alpha x$. The subalgebra A is not finitely generated over k.

(2) *Let $f = x$ and $g = y - \alpha x$ with $\alpha \in k^*$. Then we have*

$$A(x, y-\alpha x) = k[u^iv^j \ (i \geq 2, j \geq 2), v^n + n\alpha uv^{n-1} \ (n \geq 3), u^n \ (n \geq 2)],$$

where $u = x$ and $v = y - \alpha x$. The subalgebra A is then finitely generated over k.

(3) *Let $f = y - \alpha x$ and $g = x$. Then we have*

$$A(y-\alpha x, x) = k[u^iv^j \ (i \geq 2, j \geq 1), v^n - nuv^{n-1} \ (n \geq 2)],$$

where $u = y - \alpha x$ and $v = x$. The subalgebra A is not finitely generated over k.

(4) *Let $f = y - \alpha x$ and $g = y$. Then we have*

$$A(y-\alpha x, y) = k[u^iv^j \ (i \geq 2, j \geq 2), u^n - nu^{n-1}v \ (n \geq 3), v^n \ (n \geq 2)],$$

where $u = y - \alpha x$ and $v = y$. The subalgebra A is finitely generated over k.

Proof. We prove the assertions (1) and (2). The assertions (3) and (4) are verified in the same fashion as for (1) and (2), respectively.

(1) By the change of variables $u = y, v = y - \alpha x$, the vector field δ is written as

$$\delta = \frac{1}{v}\frac{\partial}{\partial u} + \frac{u - \alpha v}{uv}\frac{\partial}{\partial v}.$$

Let $\varphi \in R := k[x,y] = k[u,v]$ be a homogeneous polynomial of degree n and write it as

$$\varphi = \sum_{i+j=n} a_{ij}u^iv^j.$$

Then we have

$$\delta(\varphi) = \frac{1}{uv}\left\{(na_{n0} + a_{n-1,1})u^n + uv\psi - n\alpha a_{0n}v^n\right\},$$

where ψ is a homogeneous polynomial of degree $n-2$. Hence $\delta(\varphi) \in R$ if and only if $na_{0n}+a_{n-1,1} = 0$ and $a_{0n} = 0$. Namely, $\varphi = a_{20}(u^2 - 2uv)$ if $n = 2$ and

$$\varphi = a_{n0}(u^n - nu^{n-1}v) + \sum_{\substack{i+j=n \\ i,j\geq 2}} a_{ij}u^iv^j + a_{1,n-1}uv^{n-1}$$

if $n \geq 3$. Hence A is generated by the elements as in the statement. Since A contains uv^j $(j \gg 0)$, the subalgebra A is not finitely generated over k.

(2) By the change of variables $u = x$ and $v = y - \alpha x$, the vector field δ is written as

$$\delta = \frac{1}{u}\frac{\partial}{\partial u} + \frac{u-\alpha v}{uv}\frac{\partial}{\partial v}.$$

So, if $\varphi = \sum_{i+j=n} a_{ij}u^iv^j$ is a homogeneous polynomial of degree n, $\delta(\varphi)$ is computed as

$$\delta(\varphi) = \frac{1}{uv}\left\{a_{n-1,1}u^n + uv\psi + (a_{1,n-1} - n\alpha a_{0n})v^n\right\},$$

where ψ is a homogeneous polynomial of degree $n-2$. Hence $\delta(\varphi) \in R$ if and only if $a_{n-1,1} = 0$ and $a_{1,n-1} - n\alpha a_{0n} = 0$. The last condition implies that the subalgebra A is generated by the elements as given in the statement. In order to prove that the subalgebra A is finitely generated over k, we first show that u, v are integral over A. It is clear that u is integral over A. The element v is integral over A since there is the following integral relation with coefficients in A:

$$v^7 - (v^3 + 3\alpha uv^2)v^4 - (v^4 + 4\alpha uv^3)v^3 + (v^7 + 7\alpha uv^6) = 0.$$

Then R is a finite B-module, where $B = k[u^2, v^3+3\alpha uv^2, v^4+4\alpha uv^3, v^7-7\alpha uv^6]$ which is a subalgebra of A. Then A is a finite B-module, and A is therefore finitely generated over k. Q.E.D.

The following result will complement Lemma 2.2.

Lemma 2. 3 *Let $f = y - \alpha x$ and $g = y - \beta x$ with $\alpha, \beta \in k^*$ and $\alpha \neq \beta$. Then we have*

$$A(y-\alpha x, y-\beta x) = k[\ u^2 - 2uv + \tfrac{\alpha}{\beta}v^2, u^n - nu^{n-1}v, v^n - \tfrac{\beta}{\alpha}nv^{n-1}u\ (n \geq 3), \\ u^iv^j\ (i \geq 2, j \geq 2)],$$

where $u = y - \alpha x$ and $v = y - \beta x$. The subalgebra A is then finitely generated over k.

Proof. By the change of variables, the vector field δ is written as

$$\delta = \left(\frac{u-\alpha v}{uv}\right)\frac{\partial}{\partial u} + \left(\frac{u-\beta v}{uv}\right)\frac{\partial}{\partial v}.$$

Let $\varphi = \sum_{i+j=n} a_{ij}u^iv^j$ be a homogeneous polynomial of degree $n \geq 2$ in $R := k[x,y] = k[u,v]$. We then compute

$$\delta(\varphi) = \frac{1}{uv}\left\{(na_{n0} + a_{n-1,1})u^n + uv\psi - (\alpha a_{1,n-1} + n\beta a_{0n})v^n\right\},$$

where ψ is a homogeneous polynomial of degree $n-2$. Hence $\delta(\varphi) \in R$ if and only if $na_{n0}+a_{n-1,1}=0$ and $\alpha a_{1,n-1} + n\beta a_{0n} = 0$. The last condition implies that the subalgebra A is generated by the elements as given in the statement. The element v is integral over A since there is a monic relation in v with coefficients in A:

$$v^{11} - \left(v^5 - \frac{5\beta}{\alpha}v^4u\right)v^6 - \left(v^6 - \frac{6\beta}{\alpha}v^5u\right)v^5 + \left(v^{11} - \frac{11\beta}{\alpha}v^{10}u\right) = 0.$$

Then the element u is integral over $A[v]$, hence over A because there is a monic relation

$$u^2 - 2vu + \frac{\alpha}{\beta}v^2 - \left(u^2 - 2uv + \frac{\alpha}{\beta}v^2\right) = 0.$$

Then it follows that A is finitely generated over k. Q.E.D.

Lemma 2.3 implies that the polynomial ring contains a one-parameter family of two-dimensional subalgebras $\{A_t\}_{t\in k}$ such that A_t is finitely generated over k for each value of $t \neq 0, \beta$ and A_0 is not finitely generated over k, where β is some fixed element of k. In fact, we have only to take A_t to be $A(y-tx, y-\beta x)$. Then the assertion will follow from Lemmas 2.2 and 2.3.

In the section 3, we need the following result which is a variant of Lemma 2.2.

Lemma 2. 4 *The algebra $A = A(f,g)$ is finitely generated over k in each of the following cases, where $s \geq 2$ and $t \geq 2$ are integers:*

(1) $f = x^s$ *and* $g = (y-\alpha x)^t$ *with* $\alpha \in k^*$.

(2) $f = (y-\alpha x)^s$ *and* $g = y^t$ *with* $\alpha \in k^*$.

(3) $f = (y-\alpha x)^s$ *and* $g = (y-\beta x)^t$ *with* $\alpha, \beta \in k^*$ *and* $\alpha \neq \beta$.

Proof. We prove only the case (3), as the other cases can be treated similarly. Let $u = y-\alpha x$ and $v = y - \beta x$. The vector field δ is then written as

$$\delta = \left(\frac{u^s - \alpha v^t}{u^s v^t}\right)\frac{\partial}{\partial u} + \left(\frac{u^s - \beta v^t}{u^s v^t}\right)\frac{\partial}{\partial v}.$$

A computation as in the proof of Lemma 2.2 shows that the following elements are contained in A:

$$\varphi(m) = u^{m-t}\left\{u^t - {}_mC_1 u^{t-1}v + {}_mC_2 u^{t-2}v^2 + \cdots + (-1)^t {}_mC_t v^t\right\}$$

$$\psi(n) = v^{n-s}\left\{v^s - {}_nC_1\left(\frac{\beta}{\alpha}\right)v^{s-1}u + {}_nC_2\left(\frac{\beta}{\alpha}\right)^2 v^{s-2}u^2 + \cdots + (-1)^s {}_nC_s\left(\frac{\beta}{\alpha}\right)^s u^s\right\}$$

where ${}_mC_i$ and ${}_nC_j$ are the binomial coefficients and $m, n \geq s+t+2$. In order to show that A is finitely generated over k, it suffices to find two elements $\varphi(m)$ and $\psi(n)$ which are prime to each other. Since $\varphi(m)$ and $\psi(n)$ are homogeneous in u, v, it suffices to show that there are no elements ξ in k satisfying the equations

$$\xi^t - {}_mC_1\xi^{t-1} + {}_mC_2\xi^{t-2} + \cdots + (-1)^t {}_mC_t = 0$$

$$1 - {}_nC_1\left(\frac{\beta}{\alpha}\right)\xi + {}_nC_2\left(\frac{\beta}{\alpha}\right)^2\xi^2 + \cdots + (-1)^s {}_nC_s\left(\frac{\beta}{\alpha}\right)^s \xi^s = 0$$

provided m and n are sufficiently large and mutually independent. This assertion can be easily verified. Q.E.D.

Suppose that $f = f_1 f_2$ with $\deg f_1 > 0, \deg f_2 > 0$ and $\gcd(f_1, f_2) = 1$. Then it is easy to verify that $A(f_1, g) \cap A(f_2, g) = A(f, g)$. Similarly, $A(f, g_1 g_2) = A(f, g_1) \cap A(f, g_2)$ if $g = g_1 g_2$ with $\deg g_1 > 0, \deg g_2 > 0$ and $\gcd(g_1, g_2) = 1$. This easy remark entails the following result.

Lemma 2. 5 *Suppose that either $y \mid f$ or $x \mid g$. Then $A = A(f, g)$ is not finitely generated over k.*

Proof. Suppose, to the contrary, that A is finitely generated over k. Then the integral closure $\widetilde{A}$ of A in its quotient field is finitely generated and contains f, g by Theorem 1.10. Hence $k[x, y]$ is integral over A since $k[x, y]$ is integral over $k[f, g]$ by Lemma 1.6. Suppose $y \mid f$. Then we have

$$A \subset A(y, y - \alpha x) \subset k[x, y],$$

where $y - \alpha x$ is a linear factor of g and hence $\alpha \in k^*$. Hence $A(y, y - \alpha x)$ is finitely generated over k. This is, however, a contradiction because $A(y, y - \alpha x)$ is not finitely generated over k by Lemma 2.2, (1). The case $x \mid g$ is proved in a similar way. Q.E.D.

This lemma has immediate applications.

Corollary 2. 6 *For positive integers m, n, the following assertions hold true:*

(1) *$A(y^m, x^n)$ is not finitely generated over k.*

(2) *$A(y^m, y^n + a x^n)$ is not finitely generated over k, where $a \in k^*$.*

(3) *$A(y^m + b x^m, x^n)$ is not finitely generated over k, where $b \in k^*$.*

In connection with these observations, we pose the following problem.

PROBLEM **2. 7** *Let f, g be as above. Suppose $f = f_1 f_2$ with $\deg f_1 > 0$ and $\deg f_2 > 0$. Suppose, furthermore, that $A(f_1, g)$ and $A(f_2, g)$ are finitely generated over k. Is $A(f, g)$ then finitely generated over k ?*

3 A sufficient condition for powers of f and g are contained in A

Let f, g be as above and let A_0 be the k-subalgebra defined by all elements $f^i g^j$ with $i \geq 2$ and $g \geq 2$. Namely,

$$A_0 = k[f^i g^j \mid i \geq 2, j \geq 2].$$

Our objective is to prove the following result.

Theorem 3. 1 *Let $A_0 = k[f^i g^j \mid i \geq 2, j \geq 2]$. Assume that $A \cap k[f, g] \supsetneq A_0$ and that $\deg f \not| \deg g$ and $\deg g \not| \deg f$. Then if A is finitely generated over k then $f^M \in A$ or $g^N \in A$ for some positive integers $M, N \geq 2$.*

The proof will be given in this section.

In what follows, we assume that A is finitely generated, $A \cap k[f,g] \supsetneq A_0$ and $m \not| n$ and $n \not| m$, where $m = \deg f$ and $n = \deg g$. Furthermore, we assume that $f^M \notin A$ and $g^N \notin A$ for all $M \geq 2$ and all $N \geq 2$. We shall show that this assumption leads to a contradiction. Write $m = m'd$ and $n = n'd$ with $d = \gcd(m,n)$. Then the hypothesis $A \cap k[f,g] \supsetneq A_0$ implies that A contains a nonzero homogeneous element F of the form

$$F = af^\alpha + bf^\beta g + cfg^\gamma + dg^\delta \tag{1}$$

where $a, b, c, d \in k$ and $\alpha, \beta, \gamma, \delta$ are positive integers.

The following two lemmas will finish the proof of Theorem 3.1.

Lemma 3. 2 *With the above notations and assumptions, F is not of the form $F \sim f^\alpha, F \sim f^\beta g, F \sim fg^\gamma$ or $F \sim g^\delta$.*

Proof. By the hypotesis, it is clear that F is not of the form $F \sim f^\alpha$ or $F \sim g^\delta$. Assume that $F \sim f^\beta g$. Then we have

$$F_x \sim \beta f^{\beta-1} f_x g + f^\beta g_x \quad \text{and} \quad F_y \sim \beta f^{\beta-1} f_y g + f^\beta g_y$$

and $g \mid F_y$ implies $g \mid g_y$. Hence $g_y \equiv 0$. So $g \sim x^n$. Then A is not finitely generated by Lemma 2.5. Similarly, the case $F \sim fg^\gamma$ is excluded. Q.E.D.

Lemma 3. 3 *With the above notations and assumptions, F does not contain two or more terms among $f^\alpha, f^\beta g, fg^\gamma$ and g^γ.*

Proof. Our proof proceeds by the comparison of the degrees of these terms, which are given as follows:

$$\begin{aligned} &\deg f^\alpha = m\alpha = dm'\alpha, && \deg f^\beta g = m\beta + n = d(m'\beta + n'), \\ &\deg fg^\gamma = m + n\gamma = d(m' + n'\gamma), && \deg g^\delta = n\delta = dn'\delta. \end{aligned} \tag{2}$$

In view of the assumption that $m \not| n$ and $n \not| m$, it follows that $m' > 1$ and $n' > 1$. First we assume that $a \neq 0$.

(i) Suppose $ab \neq 0$. Note that F is a homogeneous polynomial. By comparison of degrees in (2), we have $m'\alpha = m'\beta + n'$, which yields $m'(\alpha - \beta) = n'$. Since $\gcd(m', n') = 1$, this is a contradiction. Hence $b = 0$ if $a \neq 0$. By exchanging the roles of a and d, we conclude that $cd = 0$, where we do not have to assume $a \neq 0$.

(ii) Suppose $ac \neq 0$. By comparison of degrees, we have $m'\alpha = m' + n'\gamma$. Hence $\alpha = n'\ell + 1$ and $\gamma = m'\ell$ for some integer $\ell \geq 0$. Then we can write

$$F \sim f^{n'\ell+1} + cfg^{m'\ell} \quad \text{with} \quad c \in k^*.$$

If $\ell = 0$ then $F \sim f$ and $f \in A$. This case is excluded by the hypothesis. So $\ell > 0$. Then we have

$$F_x \sim (n'\ell + 1) f^{n'\ell} f_x + cf_x g^{m'\ell} + cm'\ell f g^{m'\ell - 1} g_x \ .$$

Since $f \mid F_x$ (cf Lemma 1.1), we have $f \mid f_x$. So $f_x = 0$ and $f \sim y^m$. This is impossible by Lemma 2.5.

(iii) Suppose $ad \neq 0$. By comparison of degrees, we have $m'\alpha = n'\delta$. Hence we may write $\alpha = n'\ell$ and $\delta = m'\ell$ for some positive integer ℓ. Then F is written as

$$F \sim f^{n'\ell} + dg^{m'\ell} \quad \text{with} \quad d \in k^* \tag{3}$$

which gives rise to

$$F_x \sim n'\ell f^{n'\ell-1} f_x + dm'\ell g^{m'\ell-1} g_x,$$

$$F_y \sim n'\ell f^{n'\ell-1} f_y + dm'\ell g^{m'\ell-1} g_y \ ,$$

where $f \mid F_x$ implies $f \mid g_x$, and $g \mid F_y$ implies $g \mid f_y$. If $g_x \not\equiv 0$ and $f_y \not\equiv 0$, the comparison of degrees gives $m \leq n-1$ and $n \leq m-1$. This is a contradiction. So, $f_y \equiv 0$ or $g_x \equiv 0$. If $f_y \equiv 0$ then $f \sim x^m$ and $f^2 \in A$, which contradicts the assumption. Similarly, $g^2 \in A$ if $g_x \equiv 0$, which is a contradiction.

By (i) $\sim$ (iii) above, we conclude that $a = 0$. By exchanging the roles of f and g, we conclude also that $d = 0$. It then remains to consider the following case.

(iv) Suppose $bc \neq 0$. The comparison of degrees implies $m'\beta + n' = m' + n'\gamma$, which yields $m'(\beta - 1) = n'(\gamma - 1)$. Since $\gcd(m', n') = 1$, we can write $\beta = n'\ell + 1$ and $\gamma = m'\ell + 1$ for some integer $\ell \geq 0$. Then we have

$$F \sim f^{n'\ell+1} g + cfg^{m'\ell+1} \quad \text{with} \quad c \in k^* \ ,$$

$$F_x \sim (n'\ell + 1) f^{n'\ell} f_x g + f^{n'\ell+1} g_x + cf_x g^{m'\ell+1} + c(m'\ell + 1) f g^{m'\ell} g_x \ .$$

If $\ell = 0$ then $F \sim fg$, and this is impossible by Lemma 3.2. So, $\ell > 0$. Then $f \mid F_x$ implies $f \mid f_x$. Hence $f_x \equiv 0$ and $f \sim y^m$. This is impossible by Lemma 2.5. Q.E.D.

Thus we have proved Theorem 3.1.

REMARK 3.4 *(1) The assumption that* $\deg f \not| \deg g$ *and* $\deg g \not| \deg f$ *cannot be dropped in the assertion (1) of Theorem 3.1. Indeed, let* $f = y - \alpha x$ *and* $g = y - \beta x$ *with* $\alpha, \beta \in k^*$ *and* $\alpha \neq \beta$ *as in Lemma 2.3. Then* $A \cap k[f, g] \supsetneqq A_0$, *and* $f^M \notin A$ *and* $g^N \notin A$ *for any* $M, N \geq 2$.

(2) Notwithstanding the above remark, the arguments in the proof of Lemma 3.3 work also in the cases where (a) $m' > 1$ *and* $n' = 1$, *(b)* $m' = 1$ *and* $n' > 1$ *and (c)* $m' = n' = 1$ *except for the following point. Namely, one cannot conclude that the k-subalgebra A is not finitely generated if* $f \sim y^m + ag^{m/n}$ *or* $g \sim x^n + bf^{n/m}$ *with* $a, b \in k^*$. *We do not know, however, whether or not A is finitely generated if* $f \sim y^m + ag^{m/n}$ *or* $g \sim x^n + bf^{n/m}$ *with* $a, b \in k^*$.

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有限群の Rees 代数への作用とその不変部分環の Gorenstein 性について

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1 はじめに

以下 G は位数 N の有限群とし、可換環 A に対し環の自己同型として作用していると仮定する。

$$A^G = \{ a \in A \mid g(a) = a, \ ^{\forall} g \in G \}$$

とおき A の不変部分環という。A の環構造が如何に A^G に保たれるかという問いは、古くから人々を魅了する問題であり([H, N])、例えば標題にあるような Cohen-Macaulay 性あるいは Gorenstein 性というような環構造についても Hochster-Eagon [HE] や渡辺 [W1, W2]、後藤 [G1, G2]達によって、精密な結果が既に得られているところである。これに対し、Rees 代数の不変部分環の構造に関しては系統的な研究が見あたらず、知られるところも少いのではないかと思われる。本稿の目的は、先行する研究成果を踏まえつつ、群 G の作用を環 A 上の Rees 代数へと拡張し、その不変部分環について主に Gorenstein 性の解析を行い、多少なりともこの欠落を埋めようと試みることにある。

本稿の主結果を述べるには二、三 の記号と記述法を導入する必要がある。以下 $B = A[t]$ によって A 上 t を不定元とする多項式環を表す。A 内のイデアル I ($\neq A$) に対して

$\mathfrak{R}(I) := A[It] \subseteq A[t]$,

$\mathfrak{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}]$,

$\mathfrak{G}(I) := \mathfrak{R}'(I)/t^{-1}\mathfrak{R}'(I)$

と定め、これらをそれぞれイデアル I の Rees 代数、拡大 Rees 代数、随伴次数環

という。更に $C := A[t, t^{-1}]$ （A 上の Laurent 多項式環である）とおく。すると群 G の環 A への作用は

$$\sigma(t) = t, \ \forall \sigma \in G$$

と定めることによって環 C への作用に自然に拡張される。このとき、対象とするイデアル I が G-安定（すなわち $\sigma(I) \subseteq I, \forall \sigma \in G$）であれば、環 $\mathfrak{R}(I)$ と $\mathfrak{R}'(I)$ は群 G の環 C への作用に関して安定となり、群 G の環 $\mathfrak{R}(I)$ および $\mathfrak{R}'(I)$ への作用が誘導される。群 G の環 C への作用の定め方から $t^{-1} \in \mathfrak{R}'(I)^G$ となるので、群 G は随伴次数環 $\mathfrak{G}(I) = \mathfrak{R}'(I)/t^{-1}\mathfrak{R}'(I)$ へも作用する。以下本稿内で G-安定なイデアル I が与えられた場合には群 G の環 $\mathfrak{R}(I)$, $\mathfrak{R}'(I)$ および $\mathfrak{G}(I)$ への作用は上記のものを考える。

次の結果は本稿内で行われる議論の骨格をなす定理である。

定理(2.4) A は Cohen-Macaulay 局所環とし $I\ (\neq A)$ は環 A の G-安定なイデアルとし、群 G の位数 N は環 A 内で可逆であると仮定する。このとき $\mathrm{ht}_A I \geq 1$ (resp. $\mathrm{ht}_A I \geq 2$) であれば次の条件(1)と(2)は互いに同値である。

(1) $\mathfrak{R}(I)^G$ は Cohen-Macaulay (resp. Gorenstein) 環である。

(2) $\mathfrak{G}(I)^G$ は Cohen-Macaulay (resp. Gorenstein) 環であってかつ $a(\mathfrak{G}(I)^G) < 0$ (resp. $a(\mathfrak{G}(I)^G) = -2$) となっている。

ここで次数環の a-不変量の定義を与えておこう。一般に Noether 的次数環 $R = \oplus_{n \in \mathbb{Z}} R_n$ が与えられて

(i) $R_n = (0)$ $(n < 0)$

(ii) R_0 は局所環である

という二条件を満すと仮定する。このとき 環 R 内には次数イデアルで極大なものが唯一含まれているのでこれを $\mathfrak{M}$ で表す。このとき有限生成次数 R-加群 E に対しては、その i 次局所コホモロジー $H^i_{\mathfrak{M}}(E)$ の n 次斉次部分 $[H^i_{\mathfrak{M}}(E)]_n$ は n が十分大なるときは消滅するので、各 $i \in \mathbb{Z}$ に対して

$$a_i(E) = \sup\{ n \in \mathbb{Z} \mid [H^i_{\mathfrak{M}}(E)]_n \neq 0\}$$

と定め、これを E の i 次 a-不変量という。特に $\dim_R E = s$ のときには、$a(E) = a_s(E)$ とおき、これを単に E の a-不変量と呼ぶ([**GW**, (3.1.4)]).

上の定理(2.4)は $\mathfrak{R}(I)^G$ の環構造が（群 G の随伴次数環 $\mathfrak{G}(I)$ への作用を経由して）$\mathfrak{G}(I)^G$ の環構造とその a-不変量によって決定されることを保証するものであって、Rees 代数の不変式論の基本定理とみなされるべき主張であると考えられる。

云うまでもなく、環 A 内には多様な G-安定イデアルが含まれていて、これらすべてを制御しようとすることは労多くして得るところの少い企てであると思われるが、イデアル I が G-不変な元で生成されている場合や、或いは I が局所環 A 内の極大イデアルであるような場合には、群 G の作用も含めて比較的制御が容易な Rees 代数に出会うことがある。ここでは特に後者の場合に限定して、不変部分環の Gorenstein 性について考察を進めることにしたい。

以下 A は極大イデアル m を持つ Cohen-Macaulay 局所環であって $d = \dim A \geq 2$ なるものとし、群 G の位数 N は環 A 内で可逆であると仮定する。このとき A の極大イデアル m は G-安定であるので

$$\mathfrak{R} = \mathfrak{R}(m), \quad \mathfrak{B} = \mathfrak{G}(m)$$

とおき、群 G を極大イデアル m の Rees 代数 $\mathfrak{R}$ と随伴次数環 $\mathfrak{B}$ に作用させる。ここで議論を簡単にするため次の二つの仮定を設けたいと思う：

(i) 群 G は環 A の剰余体 $k = A/m$ には自明に作用する。

(ii) 環 $\mathfrak{B}$ は Gorenstein である。

この(i),(ii)の仮定の下に、群 G の環 $\mathfrak{B}$ への作用によって群 G の体 k 上の指標が自然に一つ定まる（正準指標と呼ぶ）のでこれを $\chi_{G,\mathfrak{B}}$ と書くことにすれば、定理(2.4)から次の主張を比較的容易に導くことができる。

定理(3.4) 次の三条件を考察しよう：

(1) $\mathfrak{R}^G$ は Gorenstein 環である。

(2) $\mathfrak{B}^G$ は Gorenstein 環であって $a(\mathfrak{B}^G) = -2$ である。

(3) $\chi_{G,\mathfrak{B}} = 1$ であって $a(\mathfrak{B}) = -2$ である。

このとき (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) が常に正しく、もし $a(\mathfrak{B}) \leqq -2$ であるか又は $\mathfrak{B}$ は正規環であってかつ環拡大 $\mathfrak{B}/\mathfrak{B}^G$ が divisorially unramified であれば、上記の三条件はすべて互いに同値である。

この定理(3.4)の系として次の二つの結果が得られることを記録しておきたい。

系(3.5) Rees 代数 $\mathfrak{R}$ が Gorenstein 環であれば次の二条件は互いに同値である。

(1) 不変部分環 $\mathfrak{R}^G$ は Gorenstein 環である。

(2) $\chi_{G,\mathfrak{B}} = 1$ である。

群 G の環 A への作用から誘導される k-ベクトル空間 $V = \mathfrak{m}/\mathfrak{m}^2$ 上での表現を $\rho : G \to GL(V)$ によって表す。

系(3.6) A は正則であると仮定せよ。このとき環 $\mathfrak{R}^G$ が Gorenstein であるための必要十分条件は $\dim A = 2$ であってかつ $\rho(G) \subseteq SL(V)$ となることである。

各論に入るため本稿の構成を説明しておく。定理(2.4) の証明は第2節で行う。近年 Cohen-Macaulay 局所環のイデアル族に随伴する Rees 環の構造研究が後藤・西田[GN]によって展開されている。我々の定理(2.4)も[GN]の結果に帰着して証明される。定理(3.4)の証明には正準指標 $\chi_{G,\mathfrak{B}}$ の解析が不可欠であるが、正準指標の理論は後藤[G3]によってすでに確立されているので、本稿では定理(3.4) とその系(3.5), (3.6)の証明に必要な部分のみを抜粋し概説するにとどめたいと思う。定理(3.4) とその系(3.5), (3.6)の証明は第3節で行う。第4節は定理(2.4)と(3.4)を用いて具体例を解析するのに費す予定である。

以下本稿においては、G は位数 N の有限群を表し、可換環 A に対して環の自己同型として作用しているものと仮定する。$B = A[t]$, $C = A[t, t^{-1}]$ とおき $\sigma(t) = t$, $\forall\sigma \in G$ と定めて群 G の環 A への作用を環 B, C への作用に拡大しておく。

2 主定理とその証明

以下 I を環 A の G-安定なイデアルとする。

$$\mathfrak{R} = \mathfrak{R}(I),\ \mathfrak{R}' = \mathfrak{R}'(I),\ \mathfrak{G}(I) = \mathfrak{G}$$

とおき、群 Gを環 $\mathfrak{R}$, $\mathfrak{R}'$, $\mathfrak{G}$ にそれぞれ環の自己同型として作用させる。各 $i \in \mathbb{Z}$ に対し $F_i = I^i \cap A^G$ と定めると、環 A^G のイデアル族 $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ が次の性質を持つことを容易に確かめることができる。

補題(2.1) (1) $F_i = A^G, \forall i \leq 0.$

(2) $F_i F_j \subseteq F_{i+j}, \forall i, j \in \mathbb{Z}.$

そこでこのイデアル族 $\mathcal{F} = \{I^i \cap A^G\}_{i \in \mathbb{Z}}$ に対して

$\mathfrak{R}(\mathcal{F}) := \Sigma_{i \geq 0} F_i t^i \subseteq B^G = A^G[t],$

$\mathfrak{R}'(\mathcal{F}) := \Sigma_{i \in \mathbb{Z}} F_i t^i \subseteq C^G = A^G[t, t^{-1}]$

と定めればそれぞれ、$\mathfrak{R}(\mathcal{F})$ は $A^G[t]$ の、$\mathfrak{R}'(\mathcal{F})$ は$A^G[t, t^{-1}]$ の次数 A^G-部分代数となる。更に

$\mathfrak{G}(\mathcal{F}) := \mathfrak{R}'(\mathcal{F}) / t^{-1}\mathfrak{R}'(\mathcal{F})$

とおき $\mathcal{F}$ に随伴する次数環と呼ぶ。

次の結果は基本的である。

命題(2.2) (1) 次数 A^G-代数として $\mathfrak{R}^G = \mathfrak{R}(\mathcal{F})$ であって $\mathfrak{R}'^G = \mathfrak{R}'(\mathcal{F})$ である。

(2) 群 G の位数 N が環 A 内で可逆であれば、次数 A^G-代数の自然な同型

$$\mathfrak{G}^G \cong \mathfrak{G}(\mathcal{F})$$

が得られる。

証明 (1) $\mathfrak{R}^G = \mathfrak{R} \cap A^G[t]$ であって $\mathfrak{R}'^G = \mathfrak{R}' \cap A^G[t, t^{-1}]$ であることに従う。

(2) N が A 内で可逆であるので、$\mathfrak{R}$-加群の完全列

$$0 \to \mathfrak{R}'(1) \xrightarrow{t^{-1}} \mathfrak{R}' \xrightarrow{\varphi} \mathfrak{G} \to 0$$

(但し φ は標準射である) から導かれる $\mathfrak{R}^G$-加群の列

$$0 \to \mathfrak{R}'^G(1) \xrightarrow{t^{-1}} \mathfrak{R}'^G \xrightarrow{\varphi|_{\mathfrak{R}'^G}} \mathfrak{G}^G \to 0$$

はやはり完全である (各 $x \in \mathfrak{R}'$ に対して $x^* = x \mod t^{-1}\mathfrak{R}'$ と定めると $x^* \in \mathfrak{G}^G$ のときには、x^* の φ による原像としては $(1/N)\Sigma_{\sigma \in G}\sigma(x)$ が採用される)。従って(1)を用いて同型

$$\mathfrak{G}^G \cong \mathfrak{R}'^G / t^{-1}\mathfrak{R}'^G = \mathfrak{G}(\mathcal{F})$$

が得られる。

補題(2.3) (1) 環拡大 A/A^G は整である。よって等式

$$\dim A = \dim A^G$$

が得られる。もし A が局所環であれば A^G も局所環である。

(2) A は Cohen-Macaulay 局所環であって群 G の位数 N は環 A 内で可逆であると仮定せよ。このとき等式

$$\mathrm{ht}_A I = \mathrm{ht}_{A^G} I^G$$

が成立つ。

証明 (1) 各 $a \in A$ に対して $f_a(t) = \Pi_{\sigma \in G}(t - \sigma(a))$ と定めれば、$f_a(t) \in B^G = A^G[t]$ であって $f_a(a) = 0$ である。よって環拡大 A/A^G は整であり等式 $\dim A = \dim A^G$ ([AM, (5.11)])が従う。最後の主張については例えば[AM, (5.8)]を見よ。

(2) Hochster-Eagon [HE, Proposition 13] より A^G は再び Cohen-Macaulay 環である。一方、イデアル I は G-安定であるから、群 G は環 A/I にも環の自己同型として作用するが、N が A 内で可逆であるので、自然な同型 $(A/I)^G \cong A^G/I^G$ が得られる。故に(1)より

$$\dim A = \dim A^G,$$
$$\dim A/I = \dim (A/I)^G = \dim A^G/I^G$$

が従う。ここで環 A と A^G がどちらも Cohen-Macaulay 局所環であるから等式

$$\dim A = \dim A/I + \mathrm{ht}_A I,$$
$$\dim A^G = \dim A^G/I^G + \mathrm{ht}_{A^G} I^G$$

が成立することに注意すれば、直ちに

$$\mathrm{ht}_A I = \mathrm{ht}_{A^G} I^G$$

が得られる。

さて本稿内で行われる議論の骨格をなす次の定理を証明しよう。

定理(2.4) A は Cohen-Macaulay 局所環とし $I\ (\neq A)$ は環 A の G-安定なイデアルとし、G の位数 N は A 内で可逆であると仮定する。このとき $\mathrm{ht}_A I \geq 1$ (resp. $\mathrm{ht}_A I \geq 2$) であれば次の条件(1)と(2)は互いに同値である。
(1) $\mathfrak{R}(I)^G$ は Cohen-Macaulay (resp. Gorenstein) 環である。
(2) $\mathfrak{G}(I)^G$ は Cohen-Macaulay (resp. Gorenstein) 環であってかつ $a(\mathfrak{G}(I)^G) < 0$ (resp. $a(\mathfrak{G}(I)^G) = -2$) となっている。

証明 [**HE**, Proposition 13]と(2.3)(1)により A^G は Cohen-Macaulay 局所環であって、(2.3)(2)により等式 $\mathrm{ht}_A I = \mathrm{ht}_{A^G} I^G$ が成立つ。よって $\mathrm{ht}_A I \geq 1$ (resp. $\mathrm{ht}_A I \geq 2$) ならば必ず $\mathrm{ht}_{A^G} I^G \geq 1$ (resp. $\mathrm{ht}_{A^G} I^G \geq 2$) である。一方で F_1 の定義により $F_1 = I^G$ である。従って[**GN**, Part II, (1.2), (1.4)]より $\mathrm{ht}_A I \geq 1$ (resp. $\mathrm{ht}_A I \geq 2$) のときは、環 $\mathfrak{R}(\mathcal{F})$ が Cohen-Macaulay (resp. Gorenstein) であるための必要十分条件は $\mathfrak{G}(\mathcal{F})$ が Cohen-Macaulay (resp. Gorenstein) であってかつ $a(\mathfrak{G}(\mathcal{F})) < 0$ (resp. $a(\mathfrak{G}(\mathcal{F})) = -2$) であることが導かれる。我々の状況下では $\mathfrak{R}(\mathcal{F}) = \mathfrak{R}^G$ であって $\mathfrak{G}(\mathcal{F}) \cong \mathfrak{G}^G$ であることが(2.2)によって既に確かめられているので、上記[**GN**]から定理(2.4)が従う。

3 $I = \mathfrak{m}$ の場合 - 正準指標による精密化

まず正準指標の定義とその基本的性質を述べることから始める。以下しばらくの間（命題(3.3)まで） $R = \oplus_{n \in \mathbb{Z}} R_n$ は Noether 的次数環であって次の三条件を満たすものと仮定する。
(i) R は Gorenstein 環であって、$R_n = (0)$ $(n < 0)$、$R_0 = k$ は体である。
(ii) 我々の群 G は R に k-代数の自己同型としてしかも次数環としての R の次数付けを保つように作用している。
(iii) N は k 内で 0 でない。
このとき不変部分環 R^G は仮定(iii)によって Cohen-Macaulay ([**HE**])である。また R^Gは $\{R^G \cap R_n\}_{n \in \mathbb{Z}}$ を次数付けとして R の次数 k-部分代数をなす。そして環拡大 R/R^G は加群として有限である ([**Hi**, **N**], (2.3)(1)を用いる)。そこで K_R と K_{R^G} とによってそれぞれ R と R^G の次数付正準加群([**HIO**, Ch. VII])を

表すことにすれば、次数 R-加群の同型

$$K_R \cong \mathrm{Hom}_{R^G}(R, K_{R^G})$$

が得られる([I, (1.10)])。上記の仮定(i)によって

$$K_R \cong R(a)$$

$(a = a(R))$ であるから、R-加群 $L := \mathrm{Hom}_{R^G}(R, K_{R^G})$ は次数 $-a$ のある一個の斉次元 ξ によって生成されることになる。一方で R-加群 L には

$$\sigma(f) = f \circ \sigma^{-1} \quad (\sigma \in G, f \in L)$$

によって群 G が自然に作用する。ここで仮定(i)と(ii)とにより $R_0 = k$ であってしかも群 G は体 k には自明に作用していることに注意すれば、元 $\xi \in L_{-a}$ への G の作用によって群 Gの体 k 上での指標 $\psi_{G,R}$ が一つ得られることがわかる：

$$\sigma(\xi) = \psi_{G,R}(\sigma)\xi \quad (\sigma \in G).$$

この指標 $\psi_{G,R}$ はR-加群 L の生成元である$\xi \in L_{-a}$の取り方には無関係に定まるので

定義(3.1) $\chi_{G,R} = \psi_{G,R}^{-1}$

とおきこれを群 G の(R への作用に関する) 正準指標と呼ぶ。

正準指標 $\chi_{G,R}$ については次の結果が基本的である。

命題(3.2) ([**G3**, (3.3), (3.5)]) $a = a(R)$とおく。このとき次数R^G-加群として

$$K_{R^G} \cong R_{\chi_{G,R}}(a)$$

である (但し $R_{\chi_{G,R}} = \{ f \in R \mid \sigma(f) = \chi_{G,R}(\sigma)f, \forall \sigma \in G \}$) 。

また、次が正しい。

(1) $a \geq a(R^G)$ である。

(2) $\chi_{G,R} = 1$ であるための必要十分条件は等式 $a(R^G) = a$ が成立つことである。このとき R^G は Gorenstein 環となる。

(3) $0 < n \in \mathbb{Z}$ とし $f \in [R^G]_n$ は R^G-正則と仮定する。このときは f は R-正則でもあってかつ等式

$$\chi_{G,R/fR} = \chi_{G,R}$$

が成立つ。

命題(3.3) ([**G3**, (3.6)]) $R = k[X_1, X_2, \dots, X_d]$ $(d \geq 1)$ は多項式環であると仮定する。このとき群 G の環 R への作用を k-部分空間 $V = R_1 = \Sigma_{1 \leq i \leq d} kX_i$ 上に制限して得られる表現を $\rho : G \to GL(V)$ とすれば等式

$$\chi_{G,R}(\sigma) = 1/\det(\rho(\sigma)) \quad (\sigma \in G)$$

が成立つ。

さて以上の結果を用いながら定理(3.4)とその系に証明を与えよう。

以下 A は極大イデアル m を持つ Cohen-Macaulay 局所環とし $d = \dim A \geq 2$ なるものとせよ。群 G は A に環の自己同型として作用しその位数 N は A 内で可逆であると仮定する。

$$\mathfrak{R} = \mathfrak{R}(m),\ \mathfrak{B} = \mathfrak{G}(m)$$

とおき、次の二条件を仮定する。

(i) 群 G は A の剰余体 $k = A/m$ には自明に作用する。

(ii) $\mathfrak{B}$ は Gorenstein 環である。

このとき定理(2.4)により下記の定理(3.4)内の条件(1)と(2)の同値性が直ちに得られる。

定理(3.4) 次の三条件を考えよう：

(1) $\mathfrak{R}^G$ は Gorenstein 環である。

(2) $\mathfrak{B}^G$ は Gorenstein 環であって $a(\mathfrak{B}^G) = -2$ である。

(3) $\chi_{G,\mathfrak{B}} = 1$ であって $a(\mathfrak{B}) = -2$ である。

このとき (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) が常に正しく、もし $a(\mathfrak{B}) \leqq -2$ であるか又は $\mathfrak{B}$ は正規環であってかつ環拡大 $\mathfrak{B}/\mathfrak{B}^G$が divisorially unramified であれば、上記の三条件はすべて互いに同値である。

証明 (3) $\Rightarrow$ (2) $\chi_{G,\mathfrak{B}} = 1$ であれば(3.2)(2)より $\mathfrak{B}^G$ は Gorenstein 環であって $a(\mathfrak{B}^G) = a(\mathfrak{B})$ が成立つ。仮定(3)によって $a(\mathfrak{B}) = -2$ であるからもちろん $a(\mathfrak{B}^G) = -2$ である。

次に $a(\mathfrak{B}) \leq -2$ であって(3.4)内の条件(2)が満たされていると仮定しよう。すると(3.2)(1)より $a(\mathfrak{B}) \geq a(\mathfrak{B}^G)$ であるから、等式 $a(\mathfrak{B}) = a(\mathfrak{B}^G) = -2$ が従い、

(3.2)(2)によって $\chi_{G,\mathfrak{B}} = 1$ が得られる。故に(3.4)内の条件(3)が満たされる。同様に、$\mathfrak{B}$ が正規環で環拡大 $\mathfrak{B}/\mathfrak{B}^G$ が divisorially unramified であってかつ(3.4)内の条件(2)が満たされているならば、環 $\mathfrak{B}^G$ の Gorenstein 性より $\chi_{G,\mathfrak{B}} = 1$ が導かれ（[W2]内の Theorem 2 の証明を参照せよ）、(3.2)(2)より等式 $a(\mathfrak{B}) = a(\mathfrak{B}^G) = -2$ が従う。

系(3.5) Rees 代数 $\mathfrak{R}$ が Gorenstein 環であれば次の二条件は互いに同値である。
(1) 不変部分環 $\mathfrak{R}^G$ は Gorenstein 環である。
(2) $\chi_{G,\mathfrak{B}} = 1$ である。

証明 $\mathfrak{R}$ が Gorenstein 環であるから [I, (3.6)]により（$\mathfrak{B}$ は Gorenstein 環であって） $a(\mathfrak{B}) = -2$ となる。従って $\mathfrak{R}^G$ がGorenstein 環であるための必要十分条件は、定理(3.4)の(1)と(3)の同値性より、$\chi_{G,\mathfrak{B}} = 1$ に他ならない。

群 G の環 A への作用から誘導される k-ベクトル空間 $V = m/m^2$ 上での表現を $\rho : G \to GL(V)$ によって表す。

系(3.6) A は正則であると仮定せよ。このとき環 $\mathfrak{R}^G$ が Gorenstein であるための必要十分条件は dim A = 2 であってかつ $\rho(G) \subseteq SL(V)$ となることである。

証明 A が正則局所環であれば $\mathfrak{B} = \mathfrak{G}(m)$ は体 $k = A/m$ 上 d （= dim A）変数の多項式環であるから、等式 $a(\mathfrak{B}) = -d \leq -2$ が成立つ（[GW, (3.1.6)]）。(3.4)に従えば、$\mathfrak{R}^G$ が Gorenstein 環であるための必要十分条件は $d = 2$ でかつ $\chi_{G,\mathfrak{B}} = 1$ が成立つことである。後者の条件の中で $\chi_{G,\mathfrak{B}} = 1$ の部分は、(3.3)により条件 $\rho(V) \subseteq SL(V)$ と同値である。

4 例と応用

以下 k は体とし $R = k[X_1, X_2, \dots, X_n]$ （$n \geq 1$）によって k 上 n 変数の多項式環を表す。 G は位数 N の有限群であって、環 R に対して次数環としての R の次数を保つような k-代数の自己同型として作用していると仮定せよ（もちろん R には

$R_0 = k$ で $\deg X_i = 1$ なる次数付を考えている)。群 G の位数 N は体 k 内で 0 でないと仮定する。さて $\mathfrak{a}\ (\neq R)$ は R の斉次イデアルであって G-安定なものとする。$R^* = R/\mathfrak{a}$ とおき群 G を次数環 R^* に自然に作用させる。すると $\mathfrak{M} = [R^*]_+$ とおけばイデアル $\mathfrak{M}$ は G-安定であるので

$$\sigma(a/s) = \sigma(a)/\sigma(s) \quad (a \in R,\ s \in R \setminus \mathfrak{M},\ \sigma \in G)$$

によって群 G は局所環 $A = R^*_{\mathfrak{M}}$ にも作用する。このとき $\mathfrak{m} = \mathfrak{M}R^*_{\mathfrak{M}}$ とおくと次が成立つことが容易に確かめられる。

補題(4.1) (1) 群 G は A の剰余体 $k = A/\mathfrak{m}$ に自明に作用する。
(2) 自然な同型 $R^* \cong \mathfrak{G}(\mathfrak{M}) \cong \mathfrak{G}(\mathfrak{m})$ は群 G の作用と可換であって次数環の同型

$$R^{*G} \cong \mathfrak{G}(\mathfrak{m})^G$$

が得られる。

この(4.1)により定理(2.4)を上記の局所環 $(A = R^*_{\mathfrak{M}},\ \mathfrak{m} = \mathfrak{M}R^*_{\mathfrak{M}})$ に適用する準備が整い次の結果が得られる。

命題(4.2) R^* が Cohen-Macaulay 環であって $\dim R^* = d$ であれば次の主張が正しい。
(1) $d \geq 1$ とせよ。$\mathfrak{R}(\mathfrak{m})^G$ (resp. $\mathfrak{R}(\mathfrak{m})$) が Cohen-Macaulay 環であるための必要十分条件は $a(R^{*G}) < 0$ (resp. $a(R^*) < 0$) なることである。
(2) $d \geq 2$ とせよ。$\mathfrak{R}(\mathfrak{m})^G$ (resp. $\mathfrak{R}(\mathfrak{m})$) が Gorenstein 環であるための必要十分条件は R^{*G} (resp. R^*) が Gorenstein 環であって等式 $a(R^{*G}) = -2$ (resp. $a(R^*) = -2$) が成立つことである。

証明 R^{*G} が Cohen-Macaulay であること ([**HE**, Proposition 13]) と A は Cohen-Macaulay 局所環であって $\mathrm{ht}_A \mathfrak{m} = d$ であることに注意すれば、$\mathfrak{R}(\mathfrak{m})^G$ に関する主張が(2.4)と(4.1)(2)に従う。$R^* \cong \mathfrak{G}(\mathfrak{m})$ であるので $\mathfrak{R}(\mathfrak{m})$ に関する主張は[**GS**, (1.1), (1.2)]に従う。

この命題(4.2)を用いて以下二、三の例を解析してみよう。

例(4.3) $G = \mathfrak{S}_n$ (n 次対称群) とし $1 \leq \ell \in \mathbb{Z}$ とする。基礎体 k の標数は 0 と仮定して G を多項式環 $R = k[X_1, X_2, \ldots, X_n]$ に変数の置換として作用させる：

$$\sigma(X_i) = X_{\sigma(i)} \ (\sigma \in G, \ 1 \leq i \leq n).$$

ここで $f = X_1^{\ell} + X_2^{\ell} + \ldots + X_n^{\ell}$ とおき $\mathfrak{a} = fR$ とせよ。すると $f \in R^G$ であるからイデアル $\mathfrak{a}$ は G-安定である。$R^* = R/\mathfrak{a}$ とし $A = R^*_{\mathfrak{M}}$ (但し $\mathfrak{M} = [R^*]_+$), $\mathfrak{m} = \mathfrak{M}R^*_{\mathfrak{M}}$ とすると次が正しい。

定理(4.4) (1) $n \geq 2$ とする。このとき

(a) $\mathfrak{R}(\mathfrak{m})$ が Cohen-Macaulay 環である $\Leftrightarrow \ell < n$.

(b) $\mathfrak{R}(\mathfrak{m})^G$ が Cohen-Macaulay 環である $\Leftrightarrow \ell < n(n+1)/2$.

(2) $n \geq 3$ とせよ。このとき

(a) $\mathfrak{R}(\mathfrak{m})$ が Gorenstein 環である $\Leftrightarrow \ell = n - 2$.

(b) $\mathfrak{R}(\mathfrak{m})^G$ が Gorenstein 環である $\Leftrightarrow \ell = \{n(n+1)-4\}/2$.

証明 R^* は Cohen-Macaulay 環であって $\dim R^* = n - 1$, $a(R^*) = \ell - n$ ([GW, (3.1.6)]) である。一方で $R^{*G} \cong R^G/fR^G$ であるから等式 $a(R^{*G}) = a(R^G) + \ell$ が従う。k-代数 R^G は基本対称式で生成され n 変数の多項式環と同型であるから、R^{*G} は Gorenstein 環であり、更に再び[GW, (3.1.6)]によって $a(R^G) = -\Sigma_{1 \leq i \leq n} i = -(n(n+1))/2$ が得られこれに上記の等式 $a(R^{*G}) = a(R^G) + \ell$ を併せて

$$a(R^{*G}) = \ell - n(n+1)/2$$

を得る。以上より(4.2)を用いて(4.4)内の主張(a), (b)が従う。

(4.4)によると特に $n = \ell \geq 2$ ととれば、$\mathfrak{R}(\mathfrak{m})$ は Cohen-Macaulay 環ではないが $\mathfrak{R}(\mathfrak{m})^G$ は Cohen-Macaulay 環であり、また $n \geq 3$ として $\ell = \{n(n+1)-4\}/2$ とすれば $\mathfrak{R}(\mathfrak{m})$ はやはり Cohen-Macaulay 環ではないが $\mathfrak{R}(\mathfrak{m})^G$ は Gorenstein 環となる局所環 Aと群 G の作用例が系列的に得られる。

例(4.5) $k = \mathbb{C}$, $n \geq 3$ としし $R = k[X_1, X_2, \ldots, X_n]$ は多項式環とする。ζ を 1の原始 n-2 乗根と k-代数の射 $\sigma : R \to R$ を

$$\sigma(X_i) = \zeta X_i \quad (1 \leq i \leq n-1),$$
$$\sigma(X_n) = \zeta^{-1} X_n$$

によって定める。G を σ によって生成された $\mathrm{Aut}_k R$ の巡回部分群とせよ。このとき $f = \sum_{1 \leq i \leq n} X_i^{n-2}$ とし $\mathfrak{a} = fR$ とおくと $\mathfrak{a}$ は G-安定な R のイデアルであり環 $\mathfrak{R}(\mathfrak{m})^G$ は Gorenstein となる。

証明 R* は Gorenstein で dim R* = n - 1, a(R*) = -2 である。よって(4.2)(2)より環 $\mathfrak{R}(\mathfrak{m})$ は Gorenstein である。一方で $f \in R^G$ であるから(3.2)(3)により $\chi_{G,R^*} = \chi_{G,R}$ を得る。ここで σ を $V = \sum_{1 \leq i \leq n} kX_i$ に制限して得られる V の線形変換を τ とすると $\det \tau = 1$ であるから(3.3)より $\chi_{G,R} = 1$ であることが分かる。よって $\chi_{G,R^*} = 1$ となり、群 G の作用も含めて $R^* \cong \mathfrak{G}(\mathfrak{m})$ である((4.1)(2))ので、(3.5)より $\mathfrak{R}(\mathfrak{m})^G$ が Gorenstein 環であることが従う。

例(4.6) 一般に A は Noether 局所環で dim A ≥ 2 のものとする。位数 N の有限群 G が環 A に環の自己同型として作用し、N は A 内で可逆であると仮定せよ。このときもし A^G が Gorenstein 環であれば、局所環 A^G の巴系の一部をなすような元 a,b をとって I = (a,b)A と定めると $\mathfrak{R}(I)^G$ は必ず Gorenstein 環である。

証明 環拡大 A/A^G は純粋であるので([**HE**, Proposition 13の証明])、$I^i \cap A^G = J^i$ がすべて $i \in \mathbb{Z}$ について成立つ(但し $J = (a,b)A^G$)。従って

$$\mathfrak{R}(I)^G = \mathfrak{R}(J) \cong A^G[X,Y]/(aX - bY)$$

(但し $A^G[X,Y]$ は A^G 上 2 変数の多項式環を表す)となり $\mathfrak{R}(I)^G$ は確かに Gorenstein 環である。

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THE BUCHSBAUM PROPERTY OF ASSOCIATED GRADED RINGS

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1. 序文

$(A, \mathfrak{m})$ を Neother 局所環, I を A のイデアルとする. I の associated graded ring $\mathrm{G}(I)$ とは, $\mathrm{G}(I) = \oplus_{n \geq 0} I^n / I^{n+1}$ で定まる A- 代数のこととする. M を $\mathrm{G}(I)$ の graded な極大イデアルとする. 本稿では, 局所環 $\mathrm{G}(I)_M$ の Buchsbaum 性について述べたいと思う. まず始めに Buchsbaum 環の定義を復習することから始める.

定義 (1.1) ([SV; Ch. I]). d 次元の局所環 A は, 次の同値な命題のいずれか一つを満たすとき, Buchsbaum 環と呼ばれる.

(1) 長さと重複度の差 $\ell_A(A/\mathfrak{q}) - \mathrm{e}_{\mathfrak{q}}(A)$ が A のパラメターイデアル $\mathfrak{q}$ の取り方によらずに一定.

(2) 任意の s.o.p. $a_1, a_2, ..., a_d$ が weak- 列 をなす, すなわち,

$$(a_1, a_2, ..., a_{i-1})A : a_i = (a_1, a_2, ..., a_{i-1})A : \mathfrak{m}$$

が, すべての $1 \leq i \leq d$ に対して成立.

(3) 任意の s.o.p. $a_1, a_2, ..., a_d$ が d- 列 をなす, すなわち

$$(a_1, a_2, ..., a_{i-1})A : a_i a_j = (a_1, a_2, ..., a_{i-1})A : a_j$$

が, すべての $1 \leq i \leq j \leq d$ に対して成立.

A が Buchsbaum 環であれば, FLC を持っており (i.e., 局所 cohomology 加群 $\mathrm{H}^i_{\mathfrak{m}}(A)$ が $i < d = \dim A$ で長さ有限), 従って $\mathrm{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ となる整数が定まる. 実は, 上で述べた長さと重複度の差はこの $\mathrm{I}(A)$ と等しくなっており, これを Buchsbaum 不変量と呼ぶ ([SV; Ch. I, (2.6)]).

$\mathrm{G}(I)_M$ の Buchsbaum 性について今までに判明されたことは, Cohen-Macaulay である場合を除けばそれほど多くはなく, 次の 3 つが代表として取り上げられる.

(1) ([G1; (1.1)]) A が Buchsbaum 環で, maximal embedding dimension を持つとき (この条件は $A/\mathfrak{m}$ が無限体なら, あるパラメターイデアル $\mathfrak{q}$ によって $\mathfrak{m}^2 = \mathfrak{m}\mathfrak{q}$ と書けることと同値), $\mathrm{G}(\mathfrak{m})_M$ は Buchsbaum 環である.

(2) ([G2; (1.2)]) A を Buchsbaum 環, $\mathfrak{q}$ をパラメターイデアルとするとき, $\mathrm{G}(\mathfrak{q})_M$ は Buchsbaum である.

(3) ([G3]) A が Cohen-Macaulay 環で, I が minimal multiplicity をもつ $\mathfrak{m}$- 準素イデアルのときの, $\mathrm{G}(I)_M$ の Buchsbaum 性の特徴づけ.

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そこで特に (1), (2) に注目してみる．基礎環 A は Buchsbaum, I は $\mathfrak{m}$- 準素イデアルと仮定して，あるパラメターイデアル $\mathfrak{q}$ により，$I^2 = \mathfrak{q}I$ と表現できるている場合に $\mathrm{G}(I)_M$ は Buchsbaum であろうと予想することは，それ程不自然でないと思う．実際，A が Cohen-Macaulay であれば [VV; (3.1)] より $\mathrm{G}(I)$ が Cohen-Macaulay となり，正しいことを述べている．しかしながら，後に述べる例 (2.2), (2.5) などによりこの推察は正しくない．それゆえ，上の仮定のもとで，何が $\mathrm{G}(I)_M$ の Buchsbaum 性を特徴づけているのかが問題になってくる．そうして得られた結果が次の定理である．

定理 **(1.2).** *$(A, \mathfrak{m})$ は d- 次元 Buchsbaum 局所環で，$A/\mathfrak{m}$ は無限体，I は $\mathfrak{m}$- 準素イデアルで，あるパラメターイデアル $\mathfrak{q} = (a_1, a_2, ..., a_d)A$ によって $I^2 = \mathfrak{q}I$ と書けるものとする．$f_i = a_i^* \in \mathrm{G}(I)$ を a_i の initial form とする．このとき次は同値．*

(1) *$\mathrm{G}(I)_M$ は Buchsbaum 環．*
(2) *$f_1, f_2, ..., f_d$ は USD (unconditioned strong d- 列)．すなわち，任意の整数 $n_1, n_2, ..., n_d > 0$ に対して，$f_1^{n_1}, f_2^{n_2}, ..., f_d^{n_d}$ が勝手な順序で d- 列をなす．*
(3) *$(a_1^2, a_2^2, ..., a_d^2) \cap I^n = (a_1^2, a_2^2, ..., a_d^2)I^{n-2}$ for all $3 \leq n \leq d+1$.*

このとき，$\mathrm{G}(I)$ の局所 cohomology の現れかたは

$$\mathrm{H}_M^i(\mathrm{G}(I)) = [\mathrm{H}_M^i(\mathrm{G}(I))]_{-i} + [\mathrm{H}_M^i(\mathrm{G}(I))]_{1-i} \quad \textit{for each } i < d$$

となり，a-invariant は $a(\mathrm{G}(I)) \leq 1-d$, Buchsbaum 不変量は $\mathrm{I}(\mathrm{G}(I)_M) = \mathrm{I}(A)$ となる．

少しこの定理について説明する．(3) の条件は (2) をイデアルの言葉に直したものだが，(2) と (3) の同値性を示すのにも A が Buchsbaum で $I^2 = \mathfrak{q}I$ の条件は必要である．また，ひとたび (3) が成立すれば，すべての $n \in \mathbb{Z}$ で $(a_1^2, ..., a_d^2) \cap I^n = (a_1^2, ..., a_d^2)I^{n-2}$ の等式は成立する．(1) $\Longrightarrow$ (2) は定義から従う．この定理の一番重要な所は (2) $\Longrightarrow$ (1) の部分である．その理由の一つとして，一般に USD をなすパラメターイデアルの存在は環の局所 cohomology 加群の有限性しかもたらさない (これは Buchsbaum よりかなり弱い条件) などが上げられるであろう．

定理 (1.2) によって $\mathrm{G}(I)_M$ の Buchsbaum 性の特徴づけが得られたわけだが，[G1; (1.1)] や [G2; (1.2)] の様な Buchsbaum 性の十分条件を与える命題も欲しいものである．それを述べたものが次の系である．

系 **(1.3).** *A, I および $\mathfrak{q} = (a_1, a_2, ..., a_d)A$ は，定理 (1.2) のものとする．もし，$I \supseteq \sum_{i=1}^d (a_1, ..., \check{a}_i, ..., a_d)A : a_i$ ならば，$\mathrm{G}(I)_M$ は Buchsbaum．このとき，$\mathrm{G}(I)$ の局所 cohomology の現れかたは $\mathrm{H}_M^i(\mathrm{G}(I)) = [\mathrm{H}_M^i(\mathrm{G}(I))]_{1-i}$ for each $i < d$ となる．*

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2. 例

以下特に断らない限り，$(A, \mathfrak{m})$ は d- 次元 Buchsbaum 環で $A/\mathfrak{m}$ は無限体，I は $\mathfrak{m}$- 準素イデアル，$\mathfrak{q} = (a_1, ..., a_d)A$ は I の minimal reduction とする．また $\widetilde{\sum}(a_1, ..., a_d)$ は $\sum_{i=1}^d (a_1, ..., \check{a}_i, ..., a_d)A : a_i$ の形のイデアルとする．k は 無限体とする．まずは，系 (1.3) を適用する例として，

例 **(2.1).** $d=2$ とする. $I=\widetilde{\sum}(a_1,a_2)$ に対して, $\mathrm{G}(I)_M$ は Buchsbaum である.

これは $I^2=\mathfrak{q}I$ を確かめれば (1.3) から直ちに得られる. $I^2=\mathfrak{q}I$ を確かめるには, 実際に計算してもよいし, または, 山岸氏の判定法「$A\times K_A$ (イデアル化) が Buchsbaum なら, $\widetilde{\sum}(\mathfrak{q})^2=\mathfrak{q}\widetilde{\sum}(\mathfrak{q})$」を用いてもよい (cf. [Y; (15.1), (15.5)]).

次の例は, $\dim A=2$ かつ $I^2=\mathfrak{q}I$ が必ずしも $\mathrm{G}(I)_M$ の Buchsbaum 性を導かない事を言っている.

例 **(2.2).** $A=k[[t,st,s^2,s^3]]$ を冪級数環 $k[[s,t]]$ の部分環とすると, A は 2- 次元 Buchsbaum 環となる. イデアル $I=(t,s^2,s^3)A$ をとると I は minimal reduction $\mathfrak{q}=(t,s^2)A$ をもち $I^2=\mathfrak{q}I$ を満たす. しかし $\mathrm{G}(I)_M$ は Buchsbaum でない.

この証明は $s^2A:t=(s^2,s^3)A$ と $tA:s^2=(t,st)A$ に注意して, 次の補題を適用することで得られる.

補題 **(2.3).** $\mathrm{G}(I)_M$ が *Buchsbaum* なら, 次は同値.

(1) $I\supseteq\widetilde{\sum}(a_1,a_2,...,a_d)$
(2) $I\supseteq(a_1,...,\check{a_j},...,a_d):a_j$ *for some* j.

以下, $\mathrm{U}(a_1,...,a_i)=(a_1,...,a_i)A:a_{i+1}$ とおく (cf. (1.1)(2)). 次元が 3 の場合は次の例がある.

例 **(2.4).** $\dim A=3$ とし, $I=\mathrm{U}(a_1)+\mathrm{U}(a_2)+\mathrm{U}(a_3)$ とおく. すると G_M は Buchsbaum 環.

この例について $I^2=\mathfrak{q}I$ となることは, 実際の計算で (2.1) を確かめるのと同じ方法でチェックできる. しかし, このイデアル I に対しては系 (1.3) を適用するわけにはいかず, 定理 (1.2)(3) の条件を調べることになる. この場合 (3) の条件とは $I^3\cap(a_1^2,a_2^2,a_3^2)A=(a_1^2,a_2^2,a_3^2)I$ と $I^4\cap(a_1^2,a_2^2,a_3^2)A=(a_1^2,a_2^2,a_3^2)I^2$ を確かめることである. 2 番目の等式は容易なのだが, 1 番めのそれは結構たいへんであり, 筆者は下田氏からその方法を教えて頂き, この例を得るに至った.

少しイデアルを変えてみる.

例 **(2.5).** $\dim A=3$ とし $I=\mathrm{U}(a_1,a_2)+\mathrm{U}(a_3)$ とおく. このとき次は同値である.

(1) G_M が Buchsbaum 環.
(2) $\mathrm{H}^2_{\mathfrak{m}}(A)=(0)$.

これは, $\mathrm{H}^2_{\mathfrak{m}}(A)=(0)$ の条件から $\mathrm{U}(a_1,a_2)=\mathrm{U}(a_1)+\mathrm{U}(a_2)$ が導かれ, よって (2.4) を適用し, (2) $\Longrightarrow$ (1) が得られる. 逆は, (2.3) を用いる事により $\mathrm{U}(a_2,a_3)=\mathrm{U}(a_2)+\mathrm{U}(a_3)$ を導き, それは $\mathrm{H}^2_{\mathfrak{m}}(A)=(0)$ 意味しており, よって (1) $\Longrightarrow$ (2) が示される.

A が 3 次元以上の場合, $I=\widetilde{\sum}(\mathfrak{q})$ に対して, 仮に $I^2=\mathfrak{q}I$ が確かめられれば系 (1.3) を用いて $\mathrm{G}(I)_M$ の Buchsbaum 性が直ちに従うのだが, 山岸氏の判定法 [Y; (15.1), (15.5)] を見たところ, 一般論としてそれは不成立である. しかしながら A として, 比較的構造の解りやすい Buchsbaum 環を持ってくれば, 任意次元での例が構成できる.

例 **(2.6).** $(B,\mathfrak{n})$ は d- 次元 Cohen-Macaulay 局所環, E を d- 次元 Buchsbaum B-加群とする. $A=B\times E$ (イデアル化) とし, $\mathfrak{m}=\mathfrak{n}\oplus E$ とおくと $(A,\mathfrak{m})$ は d- 次元 Buchsbaum 環となる. そこで B の s.o.p. $a_1,a_2,...,a_d$ をとれば, これは A の s.o.p.

をなし, A 内でのイデアル $I=\widetilde{\sum}(a_1,...,a_d;A)$ をとる. すると $\mathrm{G}(I)_M$ は Buchsbaum 環である.

説明は省略する. 最後に正標数のときの例をあげてこの節を終わりにする.

例 **(2.7).** A は正標数の excellent 正則局所環の準同形像で整閉整域, $d\geq 3$, $A/\mathfrak{m}$ は代数閉体であるものとし, パラメターイデアル $\mathfrak{q}=(a_1,...,a_d)A$ は, 各 a_i が test element であるものとする. $I=\mathfrak{q}^*$ を tight closure ととる. すると, [AHS; (3.1)] より, $I^2=\mathfrak{q}I$ が成り立ち, [K; (3.5)] より, $\widetilde{\sum}(\mathfrak{q})\subseteq I$ がいえる. よってさらにもし A が Buchsbaum であれば 系 (1.3) より G_M は Buchsbaum 環となる.

3. 定理の証明

この節では 定理 (1.2) と 系 (1.3) の証明の概略を述べる. 以下, $(A,\mathfrak{m})$ は d- 次元 Buchsbaum 環で $A/\mathfrak{m}$ は無限体, I は $\mathfrak{m}$- 準素イデアル, $\mathfrak{q}=(a_1,a_2,...,a_d)A$ は I の minimal reduction で $I^2=\mathfrak{q}I$ を満たすものとする. とくに $a_1,a_2,...,a_d$ は A 上の USD をなす. I の Rees 環を $\mathrm{R}(I)$ と書く. ここで Rees 環は多項式環 $A[t]$. の部分環とみなすこととする. $f_i=a_it\in\mathrm{R}(I)$ とおく. $\mathrm{G}(I)$ は $\mathrm{R}(I)$ の剰余環 $\mathrm{R}(I)/I\mathrm{R}(I)$ と同一視でき, f_i の $\mathrm{G}(I)$ での像が a_i^* である. $f_1,f_2,...,f_d$ は $\mathrm{G}(I)$ の graded な s.o.p. をなす. 簡略のため $\mathrm{R}=\mathrm{R}(I)$, $\mathrm{G}=\mathrm{G}(I)$ と書く. M を R の 極大な greded イデアルのこととする.

定理 *(1.2)* の証明. (2) $\Longleftrightarrow$ (3) は省略. (2) を仮定して (1) を導く. d に関する帰納法. M の生成系 $M=(\xi_1,\xi_2,...,\xi_k)\mathrm{R}$ を, どの d 個 $\xi_{i_1},\xi_{i_2},...,\xi_{i_d}$ も G_M の s.o.p. となるようにとる $(1\leq i_1<i_2<...<i_d\leq k)$. $\xi_i=x_i+b_it$ の形をしていると思って良い $(x\in\mathfrak{m},\ b_i\in I)$. このとき, すべての組 $\xi_{i_1},\xi_{i_2},...,\xi_{i_d}$ が G_M 上の USD になることを言えば証明は終わる. 簡単のため $i_1=1$, ..., $i_d=d$ とおく. また $g_i=b_it$ とおく.

Step 1. $d=1$ で *(1)* が正しい.

よって, $d>1$ とし, $d-1$ では正しいと仮定する. イデアル $(b_1,b_2,...,b_d)A$ は 必然的に I の minimal reduction となるのだがさらに,

Step 2. $I^2=(b_1,b_2,...,b_d)I$ *であり*, $g_1,g_2,...,g_d$ *は* G *上で USD をなす*.

このことから [GY; (2.5)] と [VV; (1.1)] より, $\mathrm{G}/g_d\mathrm{G}\cong\mathrm{G}(I/b_dA)$ が従う. $\overline{\mathrm{G}}=\mathrm{G}/g_d\mathrm{G}$ とおく. 帰納法の仮定より $\overline{\mathrm{G}}_M$ は Buchsbaum である. とくに $\xi_1,...,\xi_{d-1}$ は $\overline{\mathrm{G}}_M$ 上の USD となる. 一方で

Step 3. G *は quasi-Buchsbaum, i.e.,* $i<d$ *に対し* $M\cdot\mathrm{H}^i_M(\mathrm{G})=(0)$.

を保証しておく. すると,

$$\begin{aligned}\mathrm{I}(\xi_1,\xi_2,...,\xi_{d-1},g_d;\mathrm{G}_M)&=\mathrm{I}(\xi_1,\xi_2,...,\xi_{d-1};\overline{\mathrm{G}}_M)\\&=\mathrm{I}(\overline{\mathrm{G}}_M)\\&=\mathrm{I}(\mathrm{G}_M).\end{aligned}$$

となり, $\xi_1,...,\xi_{d-1},g_d$ は G_M 上の USD となることが従う (cf. [T; (2.1)], [GY; (6.18)]). ここで $\mathfrak{a}=(y_1,y_2,...,y_d)$ を 局所環 B のパラメターイデアルとするとき, $\mathrm{I}(\mathfrak{a};B)=\ell_B(B/\mathfrak{a})-\mathrm{e}_{\mathfrak{a}}(B)$ と定義した.

Step 4. $\xi_d \cdot \mathrm{H}^0_M(\mathrm{G}/(\xi_1, ..., \xi_{d-1})\mathrm{G}) = (0)$ を示す.

この主張が言えれば, USD の特徴づけのひとつである命題: $\xi_1, \xi_2, ..., \xi_d$ が G_M 上の USD $\Longleftrightarrow \xi_{j+1} \cdot \mathrm{H}^i_M(\mathrm{G}/(\xi_1, ..., \xi_j)\mathrm{G}) = (0)$ for all $i + j < d$ ([GY; (6.18)]) を用いて, $\xi_1, \xi_2, ..., \xi_d$ が USD となることが従い (すでに $\xi_1, ..., \xi_{d-1}, g_d$ が USD であることに注意), G_M は Buchsbaum となる.

このように定理の証明の概略を述べてきたが, 実際の証明では, 上の Step 4 を確かめるのはなかなかたいへんで, この部分の証明にいくつかの補題が必要となり, 証明の大半がここで費やされる.

系 *(1.3)* の証明. $\widetilde{\sum}(\mathfrak{q}) \subseteq I$ なので [K; Theorem 6 in Appendix] より, $a_1, a_2, ..., a_d$ は I 上で USD をなす. 従って [GY; (6.18)] より $f_1, f_2, ..., f_d$ は $\mathrm{G}_\mathfrak{q}(I)$ 上での USD となる. ここで $\mathrm{G}_\mathfrak{q}(I)$ は $\mathrm{G}_\mathfrak{q}(I) = \oplus_{n \geq 0} \mathfrak{q}^n I / \mathfrak{q}^{n+1} I$ の形をした graded $\mathrm{G}(\mathfrak{q})$- 加群のことである. 今 $I^2 = \mathfrak{q} I$ なので $\mathrm{G}_\mathfrak{q}(I) = \mathrm{G}_+(1)$ である. 完全列 $0 \to \mathrm{G}_+ \to \mathrm{G} \to A/I \to 0$ から生じる次の可換図式を考える.

$$\begin{array}{ccccc}
\mathrm{H}^i(\underline{f}; \mathrm{G}_+) & \longrightarrow & \mathrm{H}^i(\underline{f}; \mathrm{G}) & \longrightarrow & \mathrm{H}^i(\underline{f}; A/I) \\
{\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \beta_i} & & \downarrow \\
\mathrm{H}^i_M(\mathrm{G}_+) & \longrightarrow & \mathrm{H}^i_M(\mathrm{G}) & \longrightarrow & \mathrm{H}^i_M(A/I).
\end{array}$$

[T; (3.4)] より, $\forall i < d$ に対し α_i は全射である. 逆に $\forall i < d$ に対し β_i が全射なら再び [T; (3.4)] より, $f_1, f_2, ..., f_d$ は G 上の USD となるのでそれを示そう. $i > 0$ ならば $\mathrm{H}^i_M(A/I) = (0)$ となり, β_i は全射となる. $i = 0$ の場合も実は, $\underline{f} \cdot \mathrm{H}^0_M(\mathrm{G}) = (0)$ が言えて, $\mathrm{H}^0_M(\mathrm{G}) = \mathrm{H}(\underline{f}; \mathrm{G})$ となり β_0 の全射性が導かれる. よって $f_1, f_2, ..., f_d$ は G 上の USD. 定理 (1.2) より G_M は Buchsbaum となる.

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局所環の重複度の理論の一般化

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1 序

A は maximal ideal $\mathbf{m}$ をもつ d 次元の Noether 局所環とし I は A の $\mathbf{m}$-primary ideal とする。良く知られている様に整数 $e_0, e_1, \cdots, e_d$ を等式

$$\mathrm{length}_A\, A/I^{n+1} = e_0\binom{n+d}{d} + e_1\binom{n+d-1}{d-1} + \cdots + e_d$$

が $n \gg 0$ に対して常に成り立つ様に一意的にとれる。特に e_0 は "I に関する A の重複度" と呼ばれる重要な不変量で、可換環論における中心的な研究テーマのひとつである。この報告では、こうした従来の重複度の理論を自然に一般化し、A の任意の ideal に対して適用可能な理論を構築することを目的とする。

以下 $(A, \mathbf{m})$ は Norther 局所環で $|A/\mathbf{m}| = \infty$ と仮定する。A-mod は有限生成 A- 加群の category を表す。さらに A の ideal I を含む prime ideal 全体を $\mathrm{V}(I)$ と書く。

2 準備

Grothendieck 群 $\mathrm{K}_0(A)$ は有限生成 A- 加群全体を basis にもつ free Aberian group $\bigoplus_{M \in A\text{-mod}} \boldsymbol{Z} \cdot M$ の

$$\{M - L - N \mid 0 \to L \to M \to N \to 0 \text{ なる } A\text{-mod 内の完全列が存在する}\}$$

で生成される部分群による剰余群として定義される。$M \in A$-mod の $\mathrm{K}_0(A)$ での class を $[M]$ と書く。良く知られている様に、$M \in A$-mod ならば filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = (0)$ を $1 \le \forall i < r$ について $M_i/M_{i+1} \cong A/Q_i$ がある $Q_i \in \mathrm{Spec}\, A$ に対して成り立つ様にとれる。従って $\mathrm{K}_0(A)$ は $\{[A/Q] \mid Q \in \mathrm{Spec}\, A\}$ で生成される。特に A が Artin 環のときは $\mathrm{K}_0(A)$ は $[A/\mathbf{m}]$ で生成されるから、$\varphi(1) = [A/\mathbf{m}]$ なる $\varphi : \boldsymbol{Z} \to \mathrm{K}_0(A)$ は全射であるが、実はこれは同型射である。実際、A が Artin 環のときには $\sigma : \mathrm{K}_0(A) \to \boldsymbol{Z}$

を $\forall M \in A\text{-mod}$ について $\sigma([M]) = \mathrm{length}_A M$ となるように定めることができ、$\sigma \circ \varphi = \mathrm{id}_{\boldsymbol{Z}}$、$\varphi \circ \sigma = \mathrm{id}_{\mathrm{K}_0(A)}$ は容易に確かめられる（以降 σ を length function という）。一般に $f : A \to B$ が flat な環の射ならば $\mathrm{K}_0(A) \to \mathrm{K}_0(B)$ を $\forall M \in A\text{-mod}$ について $[M]$ を $[M \otimes_A B]$ に対応させる様に定められる。特に $Q \in \operatorname{Spec} A$ に対して $A \to A_Q$ から定まる $\mathrm{K}_0(A) \to \mathrm{K}_0(A_Q)$ （これは常に全射）による $\xi \in \mathrm{K}_0(A)$ の像を ξ_Q と書くことにする。$\operatorname{Min} A = \{Q_1, \cdots, Q_m\}$ とする。$1 \leq \forall i \leq m$ について A_{Q_i} は Artin 環なので $\mathrm{K}_0(A_{Q_i}) \cong \boldsymbol{Z}$ である。従って

$$\begin{array}{ccc} \mathrm{K}_0(A) & \longrightarrow & \oplus_{i=1}^m \mathrm{K}_0(A_{Q_i}) \\ \xi & \longmapsto & (\xi_{Q_i})_{Q_i} \end{array}$$

は split する。以上より

$$\mathrm{K}_0(A) \cong \underbrace{\boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}}_{m\text{ケ}} \oplus \widetilde{\mathrm{K}_0(A)}$$

となる。ここで $\widetilde{\mathrm{K}_0(A)}$ は $\{[A/Q] \mid Q \in \operatorname{Spec} A \setminus \operatorname{Min} A\}$ で生成される $\mathrm{K}_0(A)$ の部分群である。

次に I は A の ideal とし $\mathrm{K}_0(A/I)$ を考える。$L \in A\text{-mod}$ で $I \subseteq \sqrt{\mathrm{ann}_A L}$ なるものに対して

$$[L] := \sum_{i \geq 0} [I^i L / I^{i+1} L] \in \mathrm{K}_0(A/I)$$

と定める。このとき filtration $L = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_r = (0)$ で $1 \leq \forall i < r$ について $IL_i \subseteq L_{i+1}$ となるものが与えられれば、$L_i/L_{i+1} \in A/I\text{-mod}$ なので $[L_i/L_{i+1}] \in \mathrm{K}_0(A/I)$ が定まり、さらに

$$[L] = \sum_{i=0}^{r} [L_i/L_{i+1}]$$

となることが示せる。J が I に含まれる ideal ならば自然な全射 $A/J \to A/I$ を通して A/I-加群を A/J-加群と見ることにより $\mathrm{K}_0(A/I) \to \mathrm{K}_0(A/J)$ が定まる。$\sqrt{J} = \sqrt{I}$ のときはこの射は同型になる、。

最後に、加法群に値を持つ $\boldsymbol{Z}$ 上の関数についての一般論を簡単に述べておく。G を加法群とし $f : \boldsymbol{Z} \to G$ を考える。まず $\triangle f : \boldsymbol{Z} \to G$ を $\triangle f(n) = f(n) - f(n-1)$ で定義する。さらに $i > 0$ に対して $\triangle^i f = \triangle(\triangle^{i-1} f)$ として $\triangle^i f$ を帰納的に定める。又、$\triangle^0 f = f$ とする。もし $\forall n \gg 0$ について $f(n) = 0$ であれば $f \equiv 0$ と書くことにする。f

の degree を次の様に定める:

$$\deg f = \begin{cases} \max\{i \mid \triangle^i f \not\equiv 0\} & f \not\equiv 0 \text{ のとき} \\ -1 & f \equiv 0 \text{ のとき} \end{cases}$$

$\deg f < \infty$ となる f について次が成り立つ。

補題 2.1 整数 $r \geq 0$ について次の条件は同値である。

(1) $\deg f = r$

(2) $\xi_0, \xi_1, \cdots, \xi_r \in G$ で $\forall n \gg 0$ に対して

$$f(n) = \sum_{i=0}^{r} \binom{n+i}{i} \xi_i$$

となるものが存在する。

このとき $\xi_0, \xi_1, \cdots, \xi_r$ は f に対して一意的に定まる。

3 Hilbert-Samuel 関数

I は A の proper ideal とし $M \in A$-mod とする。$\forall n > 0$ に対して M/I^nM の class $[M/I^nM]$ が $\mathrm{K}_0(A/I)$ 内で定まる。そこで $\chi_I^M : \boldsymbol{Z} \to \mathrm{K}_0(A/I)$ を $\chi_I^M(n) = [M/I^{n+1}M]$ で定め、M の I に関する Hilbert-Samule 関数と言う。このとき次が成り立つ。

定理 3.1 $\max\{\dim_{A_Q} M_Q \mid Q \in \mathrm{Min}_A A/I\} \leq \deg \chi_I^M \leq \ell(I, M)$

ここで $\ell(I, M)$ は $\mathbf{G} = \mathrm{gr}_I A$, $X = \mathrm{gr}_I M$ としたとき $\ell(I, M) = \dim_{\mathbf{G}} X/\mathbf{m}X$ で定義される不変量である。特に $\ell(I, A)$ は $\ell(I)$ と書く。これは I の analytic spread と呼ばれ、$\mathrm{ht}_A I \leq \ell(I) \leq \max\{\mu_A(I), \dim A\}$ が成り立つ。又、明らかに $\ell(I, M) \leq \ell(I)$ である。以下 $\ell = \ell(I)$ とおく。2.1 と 3.1 より $\mathrm{e}_0(I, M), \mathrm{e}_1(I, M), \cdots, \mathrm{e}_\ell(I, M) \in \mathrm{K}_0(A/I)$ を $\forall n \gg 0$ に対して

$$(\sharp) \qquad \chi_I^M(n) = \sum_{i=0}^{\ell} \binom{n+i}{i} \mathrm{e}_i(I, M)$$

となる様に一意的にとれる(勿論 $i > \ell(I, M)$ ならば $\mathrm{e}_i(I, M) = 0$)。I が $\mathbf{m}$-primary のときは $\ell = \dim A$ であり、$\sigma : \mathrm{K}_0(A/I) \to \boldsymbol{Z}$ を length function とし $\mathrm{e'}_i = \sigma(\mathrm{e}_i(I, M))$ とおけば、$(\sharp)$ の両辺を σ で写すことにより

$$\mathrm{length}_A M/I^{n+1}M = \sum_{i=0}^{\dim A} \binom{n+i}{i} \mathrm{e'}_i$$

を得る。

4 局所環の重複度

この節では I は A の proper ideal とし $M \in A$-mod とする。 $\ell = \ell(I)$ とおく。

定義 4.1 $\mathrm{e}_I(M) := \mathrm{e}_\ell(I, M) \in \mathrm{K}_0(A/I)$ とおき、これを M の I に関する重複度と言う。

定理 4.2 $n \gg 0$ に対して $\mathrm{e}_I(M) = \triangle^\ell \chi_I^M(n)$. 従って $\ell(I, M) < \ell$ ならば $\mathrm{e}_I(M) = 0$.

定理 4.3 $m \geq 1$ とする。自然な全射 $A/I^m \to A/I$ から定まる $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/I^m)$ を通して $\mathrm{e}_{I^m}(M) = m \cdot \mathrm{e}_I(M)$.

定理 4.4 $I = (a_1, \cdots, a_\ell)A$ で $a_1, \cdots, a_\ell$ が M- 正則列ならば $\mathrm{e}_I(M) = [M/IM]$.

定理 4.5 $Q \in \mathrm{V}(I)$ とし $\ell(IA_Q) = m$ とすると $\mathrm{e}_{IA_Q}(M_Q) = \mathrm{e}_m(I, M)_Q$.

定理 4.6 J が I の *reduction* ならば $A/J \to A/I$ から定まる同型射 $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/J)$ を通して $\mathrm{e}_I(M) = \mathrm{e}_J(M)$ である。さらに $J = (a_1, \cdots, a_\ell)A$ が I の *minimal reduction* ならば $\mathrm{e}_I(M) = \chi_A(a_1, \cdots, a_\ell; M)$ となる。ここで $\chi_A(a_1, \cdots, a_\ell; M)$ は *Koszul complex* $\mathrm{K}.(a_1, \cdots, a_\ell; M)$ の *Euler-Poincaré characteristic* である。

定理 4.7 $0 \to L \to M \to N \to 0$ が *A-mod* 内の完全列ならば $\mathrm{e}_I(M) = \mathrm{e}_I(L) + \mathrm{e}_I(N)$.

4.7 は $\mathrm{e}_I : \mathrm{K}_0(A) \to \mathrm{K}_0(A/I)$ を $\forall M \in A$-mod に対して $\mathrm{e}_I([M]) = \mathrm{e}_I(M)$ となる様に定められることを意味する。

定理 4.8 $J = (a_1, \cdots, a_\ell)A$ は I の *minimal reduction* とする。 $0 \leq k \leq \ell$ に対して $K = (a_1, \cdots, a_k)A$ とおく。もし $\ell(I/K) = \ell - k$ ならば $\mathrm{e}_I(M) = \mathrm{e}_{I/K}(\mathrm{e}_K(M))$ となる。

系 4.9 J、 K は *4.8* の通りとし $\ell(I/K) = \ell - k$ とする。このとき

$$\mathrm{e}_K(M) = \sum_{Q \in \mathrm{V}(K)} m_Q \cdot [A/Q] \quad (m_Q \in \boldsymbol{Z})$$

と表現すると

$$\mathrm{e}_I(M) = \sum_{\substack{Q \in \mathrm{V}(K) \\ \ell(I+Q/Q) = \ell - k}} m_Q \cdot \mathrm{e}_{I/K}(A/Q)\,.$$

が成り立つ。

I が $\mathbf{m}$-primary のとき 4.9 は associativity formula を意味する。

定理 4.10 $\mathrm{K}_0(A/I)_+ := \{\xi \in \mathrm{K}_0(A/I) \mid \xi = [L]$ *for some* $L \in A/I$-*mod*$\}$ とおく。

(1) $\mathrm{e}_I(M) \in \mathrm{K}_0(A/I)_+$.

(2) J が I の *minimal reduction* ならば $[M/JM] - \mathrm{e}_I(M) \in \mathrm{K}_0(A/I)_+$.

定理 4.11 $I = (a_1, \cdots, a_\ell)A$ とする。 $n_1, \cdots, n_\ell > 0$ に対して $\ell((a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A) = \ell$ が成り立てば $\mathrm{K}_0(A/I) \to \mathrm{K}_0(A/(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A)$ を通して

$$\mathrm{e}_{(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A}(M) = n_1 n_2 \cdots n_\ell \cdot \mathrm{e}_I(M).$$

次に Lech の補題の一般化を述べる為に記号をひとつ定める。 G を加法群とし

$$f : \underbrace{\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}}_{m\ \text{ケ}} \longrightarrow G$$

を考える。このとき $1 \leq i \leq m$ に対して $\triangle_i f : \boldsymbol{Z} \times \cdots \times \boldsymbol{Z} \to G$ を

$$\triangle_i f(n_1, \cdots, n_i, \cdots, n_m) = f(n_1, \cdots, n_i, \cdots, n_m) - f(n_1, \cdots, n_i - 1, \cdots, n_m)$$

で定める。

定理 4.12 $I = (a_1, \cdots, a_\ell)A$ とし任意の $n_1, \cdots, n_\ell > 0$ と $0 \leq \forall k \leq \ell$ について

$$\ell((a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A/(a_1{}^{n_1}, \cdots, a_k{}^{n_k})A) = \ell - k$$

を仮定する。このとき

$$f : \underbrace{\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}}_{\ell\ \text{ケ}} \longrightarrow \mathrm{K}_0(A/I)$$

を $f(n_1, \cdots, n_\ell) = [M/(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})M]$ で定めれば $n_1, \cdots, n_\ell \gg 0$ に対して

$$\triangle_1 \triangle_2 \cdots \triangle_\ell f(n_1, \cdots, n_\ell) = \mathrm{e}_I(M)$$

が成り立つ。

定理 4.13 A が *Cohen-Macaulay* 環のとき次の *2* 条件は同値である。

(1) $\mathrm{e}_I(A) = [A/I]$.

(2) I は正則列で生成される。

定理 4.14 A/Q は正則局所環とし $\operatorname{Ass}\hat{A} = \operatorname{Assh}\hat{A}$ を仮定する。このとき次の *2* 条件は同値である。

(1) $\mathrm{e}_Q(A) = [A/Q]$.

(2) A は正則局所環である。

定理 4.15 A は *Gorenstein* 環とし $Q \in \operatorname{Spec} A$ で A/Q は *Cohen-Macaulay normal domain* とする。さらに $\mu_A(Q) = \mathrm{ht}_A Q + 1$ で A_Q は正則局所環と仮定する。

(1) $\mathrm{e}_Q(A) = [A/Q] - [K_{A/Q}]$ 但し $K_{A/Q}$ は A/Q の *canonical module* を表わす。

(2) A/Q が *Gorenstein* でなければ $\mathrm{e}_Q(A) \neq 0$ である。 $\dim A/Q = 2$ で $\mathrm{K}_0(A/Q)$ 内で $[A/\mathbf{m}] = 0$ ならば逆も正しい。

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UNCONDITIONED STRONG D-SEQUENCES AND ITS APPLICATIONS

KIKUMICHI YAMAGISHI

This note shall be devoted to discussing a few results on unconditioned strong d-sequences, which will soon appear as an appendix of the paper by Kazuhiko Kurano (Tokyo Metropolitan University) and the author [KY].

In 1985 Shiro Goto (Meiji University) and the author introduced a very useful notion of the sequence property, say an unconditioned strong d-sequence, and they studied its behaviours, especially, concerning local cohomology of the Rees modules and the associated graded modules with respect to ideals generated by them [GY]. This note contains a part of recent developments on unconditioned strong d-sequences after their works.

§1. MAIN RESULTS

Let A be a commutative ring, and E an A-module. A sequence $a_1, a_2, \dots, a_s$ $(s > 0)$ of elements in A is said to be a *d-sequence* on E, see [H], if the equality

$$\mathfrak{q}_{i-1}E : a_i a_j = \mathfrak{q}_{i-1}E : a_j$$

holds for $1 \leqq i \leqq j \leqq s$, here put $\mathfrak{q}_{i-1} = (a_1, a_2, \dots, a_{i-1})$ and $\mathfrak{q}_0 = (0)$, and moreover it is said to be an *unconditioned* d-sequence on E if it is still a d-sequence on E in any order.

Basic Definition [GY]. We will say that $a_1, a_2, \dots, a_s$ form an *unconditioned strong d-sequence* (abbrev. a u.s.d-sequence) on E if $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form a d-sequence on E in any order for every integer $n_1, n_2, \dots, n_s > 0$.

Our definition of a u.s.d-sequence apparently seems very *strong*, because of it requestes $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form an unconditioned d-sequence for *all* positive integers $n_1, n_2, \dots, n_s$. Concerning this, however, we mention that this requirement can be made a much weaker one than the above, see §5.

For each $p \in \mathbb{Z}$, the p-th local cohomology functor over A with respect to the system $\underline{b} = b_1, b_2, \dots, b_t$ is given as follows:

$$\mathrm{H}^p_{\underline{b}}(*) := \varinjlim_n \mathrm{H}^p({b_1}^n, \dots, {b_t}^n; *),$$

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where $\mathrm{H}^p(b_1{}^n, \ldots, b_t{}^n; *)$ denotes the p-th cohomology functor of the Koszul (co-)complex generated by $b_1{}^n, b_2{}^n, \ldots, b_t{}^n$ over an A-module $*$. When we set $I = (b_1, b_2, \ldots, b_t)A$, it is denoted by $\mathrm{H}_I^p(*)$ for the sake of convenience.

Let $R = \oplus_{n \geqq 0} R_n$ be a graded ring. Notice that if $b_1, b_2, \ldots, b_t$ is a sequence of homogeneous elements in R, then the local cohomology functors $H_{\underline{b}}^p(*)$, $p \in \mathbb{Z}$, are regarded as functors from the category of graded R-modules into itself. As usual, we put $R_+ = \oplus_{n \geqq 1} R_n$. Let $W = \oplus_{n \in \mathbb{Z}} W_n$ be a graded R-module. The homogeneous component of W of degree n is sometimes denoted by $[W]_n$ instead of W_n. We say that W is *finitely graded* if $W_n = (0)$ for all n except finitely many.

Let I be an ideal of A. The Rees module and the associated graded module of E with respect to I, writing $\mathrm{R}_I(E)$ and $\mathrm{G}_I(E)$, are defined as follows:

$$\mathrm{R}_I(E) := \bigoplus_{n \geqq 0} I^n E, \qquad \mathrm{G}_I(E) := \bigoplus_{n \geqq 0} I^n E / I^{n+1} E.$$

In the case $E = A$, we use $R(I)$ instead of $R_I(A)$, if no confusion can be expected, and notice that both graded modules $\mathrm{R}_I(E)$ and $\mathrm{G}_I(E)$ are regared as modules over $\mathrm{R}(I)$ as usual.

We denote by $[i, j]$ the set of integers n such that $i \leqq n \leqq j$. Of course, $[i, j] = \emptyset$ when $i > j$.

With these notations, our first result is stated as follows.

Theorem 1. *Suppose that there exist a sequence $a_1, a_2, \ldots, a_s$ $(s > 0)$ of elements in A and an integer $r > 0$ such that (i) $I \supset (a_1, a_2, \ldots, a_s)$, (ii) $I^{r+1} = (a_1, a_2, \ldots, a_s)I^r$, and (iii) $a_1, a_2, \ldots, a_s$ is a u.s.d-sequence on $I^r E$. Let $\mathfrak{N} = I\mathrm{R}(I) + \mathrm{R}(I)_+$. Then $\mathrm{H}_{\mathfrak{N}}^p(\mathrm{R}_I(E))$ (resp. $\mathrm{H}_{\mathfrak{N}}^p(\mathrm{G}_I(E))$) is finitely graded for every $0 \leqq p \leqq s$ (resp. $0 \leqq p < s$).*

When we can take $r = 1$ in Theorem 1, we have more explicit computations of them. To describe it, we need one more notation. For a system of elements in A, say $a_1, a_2, \ldots, a_s$, we define an A-submodule $\widetilde{\Sigma}(a_1, \ldots, a_s; E)$ of E as follows:

$$\widetilde{\Sigma}(a_1, \ldots, a_s; E) := \sum_{i=1}^{s} [(a_1, \ldots, \widehat{a_i}, \ldots, a_s)E : a_i],$$

where the hat $\widehat{}$ on a_i means to omit this element a_i from the system $a_1, a_2, \ldots, a_s$. With this notation we have the following main result.

Theorem 2. *Let I be a finitely generated ideal of A. Suppose that there exists a sequence of elements in A, say $a_1, a_2, \ldots, a_s$, which satisfies the following five conditions: (i) $s \geqq 2$; (ii) $a_1, a_2, \ldots, a_s$ is a u.s.d-sequence on E; (iii) $I \supset (a_1, a_2, \ldots, a_s)$; (iv) $I^2 = (a_1, a_2, \ldots, a_s)I$; and (v) $IE \supset \widetilde{\Sigma}(a_1, \ldots, a_s; E)$. Let $\mathfrak{N} = I\mathrm{R}(I) + \mathrm{R}(I)_+$. Then one has the following statements.*

(1) *For each $0 \leqq p \leqq s$,*

$$[\mathrm{H}_{\mathfrak{N}}^p(\mathrm{R}_I(E))]_n = \begin{cases} \mathrm{H}_I^0(E) & (p = 0,\ n = 0, 1) \\ \mathrm{H}_I^1(E) & (p = 1,\ n = 0) \\ \mathrm{H}_I^{p-1}(E) & (4 \leqq p \leqq s,\ n \in [3 - p, -1]) \\ (0) & (\text{else}), \end{cases}$$

and $[\mathrm{H}^{s+1}_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = (0)$ *for all* $n \geqq 0$. *In particular,* $\mathrm{H}^2_{\mathfrak{N}}(\mathrm{R}_I(E)) = (0)$, *and also* $\mathrm{H}^3_{\mathfrak{N}}(\mathrm{R}_I(E)) = (0)$ *when* $s \geqq 3$.

(2) *For each* $0 \leqq p < s$,

$$[\mathrm{H}^p_{\mathfrak{N}}(\mathrm{G}_I(E))]_n = \begin{cases} \mathrm{H}^p_I(E) & (n = 1-p) \\ (0) & (\textit{else}), \end{cases}$$

moreover $[\mathrm{H}^s_{\mathfrak{N}}(\mathrm{G}_I(E))]_n = (0)$ *for all* $n > 1-s$ *and*

$$[\mathrm{H}^s_{\mathfrak{N}}(\mathrm{G}_I(E))]_{1-s} = IE/\widetilde{\Sigma}(a_1, \ldots, a_s; E).$$

§2. Preliminaries

In this section we shall prepare several basic facts, which are needed later. Let A still be a commutative ring and E an A-module, and moreover let $a_1, a_2, \ldots, a_s$ be a sequence of elements in A of length $s > 0$. We set $\mathfrak{q} = (a_1, a_2, \ldots, a_s)$.

In 1964, C. Lech [L] introduced a very useful notion concerning sequence properties. Namely, a sequence $a_1, \ldots, a_s$ is said to be independent in A if $\widetilde{\Sigma}(a_1, \ldots, a_s) \subset \mathfrak{q}$, by our notation. He mentioned also that it is equivalent to saying $\mathfrak{q}/\mathfrak{q}^2$ is a free $A/\mathfrak{q}$-module in which $a_1, \ldots, a_s$ represents a basis. Here we also investigate the same situation.

Lemma 3. *Let F be an A-submodule of E. Then the following two conditions are equivalent.*

(1) $\widetilde{\Sigma}(a_1, \ldots, a_s; E) \subset F$.

(2) *The A-linear map* $(E/F)^s \longrightarrow \mathfrak{q}E/\mathfrak{q}F$ *induced by* $(e_1, \ldots, e_s) \mapsto \sum_{i=1}^s a_i e_i$ *is an isomorphism.*

When this is the case, one also has $\mathfrak{q}_{i-1}E : a_i = \mathfrak{q}_{i-1}F \underset{F}{:} a_i$ *for every* $1 \leqq i \leqq s$, *in particular* $\widetilde{\Sigma}(a_1, \ldots, a_s; E) = \sum_{i=1}^s [(a_1, \ldots, \widehat{a_i}, \ldots, a_s)F \underset{F}{:} a_i] = \widetilde{\Sigma}(a_1, \ldots, a_s; F)$.

Now we shall discuss how an u.s.d-sequence behaves on submodules. We begin with the following.

Lemma 4. *Let $a_1, a_2, \ldots, a_s$ be a u.s.d-sequence on E. Then*

$$\widetilde{\Sigma}({a_1}^{n_1}, \ldots, {a_s}^{n_s}; E) \subset \widetilde{\Sigma}({a_1}^{m_1}, \ldots, {a_s}^{m_s}; E)$$

for any integers $m_i \leqq n_i$ $(1 \leqq i \leqq s)$.

Proposition 5. *Let $a_1, a_2, \ldots, a_s$ be a u.s.d-sequence on E and let F be an A-submodule of E. Suppose that $\widetilde{\Sigma}(a_1, \ldots, a_s; E) \subset F$. Then $a_1, a_2, \ldots, a_s$ is a u.s.d-sequence on F too.*

To consider the converse of Proposition 5, we need more finiteness conditions at this moment.

Theorem 6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, E a finitely generated A-module of positive dimension s, F an A-submodule of E and $a_1, a_2, \dots, a_s$ a system of parameters for E. Suppose that $a_1, a_2, \dots, a_s$ is a u.s.d-sequence on E and $l_A(E/F) < \infty$. Then the following statements are equivalent.*

(1) $\widetilde{\Sigma}(a_1, \dots, a_s; E) \subset F$.

(2) $l_A(\mathfrak{q}E/\mathfrak{q}F) = s \cdot l_A(E/F)$.

(3) $a_1, \dots, a_s$ *is a u.s.d-sequence on* F *and* $\mathrm{H}^0_{\mathfrak{m}}(E) \subset F$.

Let $R = \oplus_{n \geqq 0} R_n$ still be a graded ring and $W = \oplus_{n \in Z} W_n$ a graded R-module. For an integer r, we denote by $W(r)$ the graded R-module defined by $[W(r)]_n = W_{r+n}$ for $n \in Z$. We also introduce the following notations:

$$W|_{\geqq r} := \bigoplus_{n \geqq r} W_n, \qquad W|_{<r} := \bigoplus_{n<r} W_n.$$

Then we find an exact sequence of graded R-modules as follows:

(i) $$0 \longrightarrow W|_{\geqq r} \longrightarrow W \longrightarrow W|_{<r} \longrightarrow 0.$$

Considering the exact sequence (i), this implies the next.

Proposition 7. *Let $\underline{f} = f_1, \dots, f_u$ (resp. $\underline{g} = g_1, \dots, g_v$) be a system of homogeneous elements in R of positive degree (resp. of degree 0). Then, there exists an isomorphism $\mathrm{H}^p_{\underline{f},\underline{g}}(W|_{<r}) \cong \mathrm{H}^p_{\underline{g}}(W|_{<r})$ of graded R-modules, and one also has the following statements.*

(1) *For any integer p,*

$$[\mathrm{H}^p_{\underline{f},\underline{g}}(W|_{<r})]_n = \begin{cases} \mathrm{H}^p_{\underline{g}}(W_n) & (n < r) \\ (0) & (n \geqq r). \end{cases}$$

(2) *Suppose that $W_n = (0)$ for all n small enough. Then $\mathrm{H}^p_{\underline{f},\underline{g}}(W|_{<r})$ is finitely graded for every p. Therefore, $H^p_{\underline{f},\underline{g}}(W)$ is finitely graded for every p if and only if so is $H^p_{\underline{f},\underline{g}}(W|_{\geqq r})$.*

§3. Outline of the proof of main theorems

Assume that $I \supset \mathfrak{q}$. Then there are canonical inclusions as follows:

$$\mathrm{R}(\mathfrak{q}) \hookrightarrow \mathrm{R}(I) \hookrightarrow A[t].$$

Hence any graded $\mathrm{R}(I)$-module is regarded as a graded module over $\mathrm{R}(\mathfrak{q})$ via the graded A-algebra map $\mathrm{R}(\mathfrak{q}) \longrightarrow \mathrm{R}(I)$.

Let us further assume there exists an integer $r > 0$ such that $I^{r+1} = \mathfrak{q}I^r$. Then it is clear that $I^{n+r}E = \mathfrak{q}^n I^r E$ for all $n \geqq 0$. This implies

$$\mathrm{R}_I(E)|_{\geqq r} = \mathrm{R}_{\mathfrak{q}}(I^r E)(-r), \qquad \mathrm{G}_I(E)|_{\geqq r} = \mathrm{G}_{\mathfrak{q}}(I^r E)(-r)$$

as graded $\mathrm{R}(\mathfrak{q})$-modules. This observation and the exact sequence (i) lead us to the following new exact sequences of graded $\mathrm{R}(\mathfrak{q})$-modules:

Proposition 8. *Suppose that $I \supset \mathfrak{q}$ and there exists an integer $r > 0$ such that $I^{r+1} = \mathfrak{q}I^r$. Then there are two exact sequences of graded* $\mathrm{R}(\mathfrak{q})$*-modules:*

$$\text{(ii)} \qquad 0 \longrightarrow \mathrm{R}_{\mathfrak{q}}(I^r E)(-r) \longrightarrow \mathrm{R}_I(E) \longrightarrow \mathrm{R}_I(E)|_{<r} \longrightarrow 0;$$

$$\text{(iii)} \qquad 0 \longrightarrow \mathrm{G}_{\mathfrak{q}}(I^r E)(-r) \longrightarrow \mathrm{G}_I(E) \longrightarrow \mathrm{G}_I(E)|_{<r} \longrightarrow 0.$$

Let $\mathfrak{N} = I\mathrm{R}(I) + \mathrm{R}(I)_+$. As is well-known, there exist isomorphisms

$$\mathrm{H}^p_I(*) \cong \mathrm{H}^p_{\mathfrak{q}}(*), \qquad \mathrm{H}^p_{\mathfrak{N}}(*) \cong \mathrm{H}^p_{\underline{at,a}}(*)$$

for every p, as connected sequences of covariant functors over A and graded covariant functors over $\mathrm{R}(I)$ (hence over $\mathrm{R}(\mathfrak{q})$ too), respectively. Therefore, to show our theorems we enoughly calculate $\mathrm{H}^p_{\underline{at,a}}(\mathrm{R}_I(E))$ and $\mathrm{H}^p_{\underline{at,a}}(\mathrm{G}_I(E))$ in terms of $\mathrm{H}^p_{\mathfrak{q}}(E)$'s.

Now we are ready to describe an outline of the proof of our theorems.

Proof of Theorem 1. Look at the exact sequence (ii) above. By Proposition 7, we have $\mathrm{H}^p_{\underline{at,a}}(\mathrm{R}_I(E)|_{<r})$ is finitely graded for all $0 \leqq p \leqq s$. By Theorem (4.1) of [GY], we know $\mathrm{H}^p_{\underline{at,a}}(\mathrm{R}_{\mathfrak{q}}(I^r E))$ is also finitely graded. So we obtain the first half of the assertion. Using the exact sequence (iii), the second half of our assertion is also shown in the same way.

Proof of Theorem 2. At first, our assumptions (*ii*) and (*v*) imply that $a_1, a_2, \ldots, a_s$ is a u.s.d-sequence on IE too, by Proposition 5. We have with the following claim.

Claim 9. $\mathrm{H}^p_{\mathfrak{q}}(IE) = \mathrm{H}^p_{\mathfrak{q}}(E)$ *for all* $p \neq 1$.

Note that $I\mathrm{R}_I(E) = \mathrm{R}_{\mathfrak{q}}(IE)$. Hence there are exact sequences

$$\text{(vi)} \qquad 0 \longrightarrow \mathrm{G}_{\mathfrak{q}}(IE)(-1) \longrightarrow \mathrm{G}_I(E) \longrightarrow \underline{E/IE} \longrightarrow 0;$$

$$\text{(vii)} \qquad 0 \longrightarrow \mathrm{R}_{\mathfrak{q}}(IE) \longrightarrow \mathrm{R}_I(E) \longrightarrow \mathrm{G}_I(E) \longrightarrow 0;$$

$$\text{(viii)} \qquad 0 \longrightarrow \mathrm{R}_{\mathfrak{q}}(IE)(-1) \longrightarrow \mathrm{R}_I(E) \longrightarrow \underline{E} \longrightarrow 0.$$

of graded $\mathrm{R}(\mathfrak{q})$-modules, cf. the exact sequences (ii), (iii). Then we claim also the next.

Claim 10.

(1) $\mathrm{H}^0_{\underline{at,a}}(\mathrm{G}_I(E)) = \underline{\mathrm{H}^0_{\mathfrak{q}}(E)}(-1)$.

(2) $\mathrm{H}^1_{\underline{at,a}}(\mathrm{G}_I(E)) = \mathrm{H}^1_{\underline{at,a}}(\mathrm{R}_I(E)) = \underline{\mathrm{H}^1_{\mathfrak{q}}(E)}$.

Now we finish our proof of Theorem 2. To (2): By Claim 10, we enoughly deal with the case $p \geqq 2$. Considering the exact sequence (vi),

$$\mathrm{H}^p_{\underline{at,a}}(\mathrm{G}_I(E)) = \mathrm{H}^p_{\underline{at,a}}(\mathrm{G}_{\mathfrak{q}}(IE))(-1)$$

for all $p \geqq 2$. Therefore, by Theorem (4.2) of [GY], all the rest of the assertion (2) follow immediately, in particular we get by Lemma 3 that

$$[\mathrm{H}^s_{\mathfrak{M}}(\mathrm{G}_I(E))]_{1-s} = IE/\sum_{i=1}^{s}[(a_1,\dots,\widehat{a_i},\dots,a_s)IE \underset{IE}{:} a_i] = IE/\widetilde{\Sigma}(a_1,\dots,a_s;E).$$

To (1): By Claim 10, it is enough to discuss the case $p \geqq 2$. The exact sequence (viii) leads us

$$[\mathrm{H}^p_{\underline{at},\underline{a}}(\mathrm{R}_I(E))]_n = [\mathrm{H}^p_{\underline{at},\underline{a}}(\mathrm{R}_\mathfrak{q}(IE))]_{n-1} \quad \text{for all } n \neq 0.$$

On the other hand, in the assertion (2) we have already shown $[\mathrm{H}^p_{\underline{at},\underline{a}}(\mathrm{G}_I(E))]_n = (0)$ for all $n > 1-p$, hence it yields from the exact sequence (vii) that

$$[\mathrm{H}^p_{\underline{at},\underline{a}}(\mathrm{R}_I(E))]_n = [\mathrm{H}^p_{\underline{at},\underline{a}}(\mathrm{R}_\mathfrak{q}(IE))]_n = (0) \quad \text{for all } n \geqq 0,$$

because of $p \geqq 2$. Comparing these observations and Theorem (4.1) of [GY], see also Claim 9 above, we finally get all the requirements.

Example 11. Here we check Theorem 2 once more, assuming that $(A,\mathfrak{m})$ is a complete Buchsbaum ring of dimension $s \geqq 2$, $E = A$, and I is an $\mathfrak{m}$-primary ideal. In this case, Theorem 2 requires that $I^2 = \mathfrak{q}I$ and $I \supset \widetilde{\Sigma}(a_1,\dots,a_s)$ for some parameter ideal $\mathfrak{q} = (a_1,\dots,a_s)$ of A.

In the case $I = \widetilde{\Sigma}(a_1,\dots,a_s)$, we can completely state the condition to $I^2 = \mathfrak{q}I$. Let K denote the canonical module of A, i.e., $K := \mathrm{Hom}_A(\mathrm{H}^s_\mathfrak{m}(A), E_A(A/\mathfrak{m}))$, where $E_A(A/\mathfrak{m})$ is the injective envelope of the residue field $A/\mathfrak{m}$. Then, by Theorem (3.3) of [Y], it follows that $I^2 = \mathfrak{q}I$ if and only if $IK = \mathfrak{q}K$. (Notice that the ideal $\Sigma(a_1,\dots,a_s)$ appeared in [Y] coincides with $\widetilde{\Sigma}(a_1,\dots,a_s)$, because of $s \geqq 2$.)

Therefore, if the idealization $A \ltimes K$ is a Buchsbaum ring, then any $\mathfrak{m}$-primary ideal $I = \widetilde{\Sigma}(a_1,\dots,a_s)$, where $a_1,a_2,\dots,a_s$ is a system of parameters for A, satisfies all the requirements in Theorem 2.

To close this section, we describe the relations between the length s of a given sequence $a_1,a_2,\dots,a_s$ and the reduction numeber r suth that $I^{r+1} = \mathfrak{q}I^r$, where $\mathfrak{q} = (a_1,a_2,\dots,a_s)$.

Reamrk 12. Let $(A,\mathfrak{m})$ be a Noetherian local ring of dimension $s \geqq 2$ and $\operatorname{depth} A > 0$. Let I be an $\mathfrak{m}$-primary ideal of A and $\mathfrak{q} = (a_1,a_2,\dots,a_s)$ a minimal reduction of I, i.e., $I^{r+1} = \mathfrak{q}I^r$ for some $r > 0$. Assume that $a_1,a_2,\dots,a_s$ form a u.s.d-sequence on the ideal I^r, here I^r is regarded as an A-module. Then the reduction number r of I with respect to $\mathfrak{q}$ must be one.

In fact, if $a_1,a_2,\dots,a_s$ form a u.s.d-sequence on I^r, then by Theorem 6 we have $I^r \supset \widetilde{\Sigma}(a_1,\dots,a_s) \supset \mathfrak{q}$. Since $\mathfrak{q}$ is a minimal reduction of I it implies that $r = 1$.

§4. The case $s = 1$

In this section we shall deal with the case $s = 1$. Through this section, let an element $a \in A$ be a d-sequence on E.

Since $0 : a^n = 0 : a$ for any $n > 0$, we see that this element a is also a u.s.d-sequence on E. Let F be an A-submodule of E. Then, we have $0 \underset{F}{:} a^2 = [0 : a^2] \cap F = [0 : a] \cap F = 0 \underset{F}{:} a$, thus the element a is a (u.s.)d-sequence on F too (with no other assumptions).

Let I be an finitely generated ideal of A and let further assume that $a \in I$ and $I^{r+1} = aI^r$ for some integer $r > 0$. Then we have

Lemma 13. *Let $\mathfrak{N} = I\mathrm{R}(I) + \mathrm{R}(I)_+$. Under the situation as above, the following statements are true.*

(1)
$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = \begin{cases} \mathrm{H}^0_I(E) \cap I^n E & (0 \leqq n \leqq r) \\ (0) & (else), \end{cases}$$

moreover $[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = (0)$ *for all* $n \notin [1, r-1]$ *and* $[\mathrm{H}^2_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = (0)$ *for all* $n \geqq r-1$.

(2)
$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{G}_I(E))]_n = \begin{cases} [(\mathrm{H}^0_I(E) + I^r E) : a^{r-n-1}] + I^{n+1}E/I^{n+1}E & (0 \leqq n \leqq r-2) \\ (\mathrm{H}^0_I(E) \cap I^n E) + I^{n+1}E/I^{n+1}E & (n = r-1, r) \\ (0) & (n > r) \end{cases}$$

moreover $[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}_I(E))]_n = (0)$ *for all* $n > r-1$ *and*
$$[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}_I(E))]_{r-1} = I^r E/([0 : a] \cap I^r E) + aI^r E.$$

If we can take $r = 1$ as Theorem 2, we have more explicit results as follws.

Proposition 14. *Let I be a finitely generated ideal of A. Suppose that there exists an element in A, say a, which satisfies the following four conditions: (i) a is a d-sequence on E; (ii) $a \in I$; (iii) $I^2 = aI$; and (iv) $IE \supset 0 : a$. Let $\mathfrak{N} = I\mathrm{R}(I) + \mathrm{R}(I)_+$. Then one has the following statements.*

(1)
$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = \begin{cases} \mathrm{H}^0_I(E) & (n = 0, 1) \\ (0) & (else), \end{cases}$$
$$[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = \begin{cases} IE/[0 : a] + aE & (n = 0) \\ (0) & (n \neq 0), \end{cases}$$

moreover $[\mathrm{H}^2_{\mathfrak{N}}(\mathrm{R}_I(E))]_n = (0)$ *for all* $n \geqq 0$.

(2)
$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{G}_I(E))]_n = \begin{cases} \mathrm{H}^0_I(E) & (n = 1) \\ (0) & (n \neq 1), \end{cases}$$

moreover $[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}_I(E))]_n = (0)$ *for all* $n > 0$ *and*
$$[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}_I(E))]_0 = IE/[0 : a] + aE$$

Finally, for the case $s = 1$ we give an example as follows.

Example 15. Let $k[[X, Y, Z]]$ be a formal power series ring over a field k and let $r \geqq 2$ be an integer. Put
$$A = k[[X, Y, Z]]/(X^{r+1}, X^r Y) = k[[x, y, z]], \quad I = (x, y).$$

Then it is easy to see that

(1) $I^{r+1} = yI^r$ and $I^r \supsetneqq yI^{r-1}$;
(2) $0 : y = 0 : y^2 = (x^r) \subset I^r$.

This implies the element y is a d-sequence on I^n for every $n \geqq 0$.

§5. $*$-SEQUENCES

In this section we discuss another sequence property. Let us still keep the following situations: A is a commutative ring, E is an A-module, and moreover $a_1, a_2, \dots, a_s$ is a sequence of elements in A of length $s > 0$. We set $\mathfrak{q} = (a_1, a_2, \dots, a_s)$.

We begin with the following definition of a new notion on the sequence property.

Definition 16. A sequence $a_1, a_2, \dots, a_s$ is called a *$*$-sequence* on E if the equality

$$\mathfrak{q}_{i-1}E : a_i^2 = \mathfrak{q}_{i-1}E : a_i$$

holds for all $1 \leqq i \leqq s$. We use the terminologies "unconditioned" and "strong" in the same meaning as above, an unconditioned $*$-sequence, an unconditioned strong $*$-sequence etc..

Using this definition we can describe some characterization concerning so called Goto's Lemma; see [SV] §4 of Chapter II.

Proposition 17. *The following statements are equivalent.*

(1) *$a_1, a_2, \dots, a_s$ is an unconditioned strong $*$-sequence on E.*
(2) *$a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form an unconditioned $*$-sequence on E for all $n_i = 1, 2$ $(1 \leqq i \leqq s)$.*
(3) *$a_1, a_2, \dots, a_s$ satisfy Goto's Lemma; namely the equality*

$$(a_\alpha^{n_\alpha} \mid \alpha \in \Lambda)E : a_\beta^2 = \sum_{\Gamma \subset \Lambda} a_\Gamma^{\underline{n}-1}[(a_\alpha \mid \alpha \in \Gamma)E : a_\beta]$$

holds for all $\Lambda \subsetneqq [1, s]$, $\beta \notin \Lambda$, $n_\alpha > 0$ $(\alpha \in \Lambda)$, where we put $a_\Gamma^{\underline{n}-1} = \prod_{\gamma \in \Gamma} a_\gamma^{n_\gamma - 1}$ if $\Gamma \neq \emptyset$ and $a_\emptyset^{\underline{n}-1} = 1$.

As an consequence of this lemma we get the following.

Theorem 18. *Let $a_1, a_2, \dots, a_s$ be an unconditioned d-sequence on E. Then the following statements are equivalent.*

(1) *$a_1, a_2, \dots, a_s$ is a u.s.d-sequence on E.*
(2) *$a_1, a_2, \dots, a_s$ is an unconditioned strong $*$-sequence on E.*
(3) *$a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form an unconditioned d-sequence on E for all $n_i = 1, 2$ $(1 \leqq i \leqq s)$.*
(4) *$a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form an unconditioned $*$-sequence on E for all $n_i = 1, 2$ $(1 \leqq i \leqq s)$.*
(5) *$a_1, a_2, \dots, a_s$ satisfy Goto's Lemma.*

Proof. By Lemma (2.11) of [GY], we already get the equivalence (1) $\Longleftrightarrow$ (2). Combining Proposition 17 and this observation we obtain our theorem at once.

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COHEN-MACAULAYNESS IN REES ALGEBRAS ASSOCIATED TO IDEALS OF MINIMAL MULTIPLICITY

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1. INTRODUCTION.

In this paper we are going to develop a theory of Cohen-Macaulayness in Rees algebras and graded rings associated to a certain class of ideals in Cohen-Macaulay rings.

Let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Assume the field $A/\mathfrak{m}$ is infinite. Let t be an indeterminate over A. For an ideal I ($\neq A$) in A we define

$$\begin{aligned} \mathrm{R}(I) &= A[It] \subseteq A[t], \\ \mathrm{R}'(I) &= A[It, t^{-1}] \subseteq A[t, t^{-1}], \quad \text{and} \\ \mathrm{G}(I) &= \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I) \end{aligned}$$

which we call the Rees algebra, the extended Rees algebra, and the associated graded ring of I. As is well-known, the canonical morphism $\operatorname{Proj} \mathrm{R}(I) \to \operatorname{Spec} A$, that is the blowing-up of A with center I plays a very important role in the analysis of singularities. In this paper we will also explore the Cohen-Macaulayness of $\operatorname{Proj} \mathrm{R}(I)$, but our main interests are located mostly in the analysis of properties of the algebra $\mathrm{R}(I)$.

Let I be an $\mathfrak{m}$-primary ideal in A and let Q be a minimal reduction of I. Hence Q is generated by d elements and $Q \subseteq I$ with $I^{n+1} = QI^n$ ([NR]). Let $\mathrm{e}_I(A)$ denote the multiplicity of A with respect to I. Then we have the inequality

$$\mu_A(I) \leq \mathrm{e}_I(A) + d - \ell_A(A/I)$$

(here for a given A-module E, $\mu_A(E)$ and $\ell_A(E)$ denote the number of elements in a minimal system of generators for E and the length of E, respectively), in which the equality

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is attained to if and only if $\mathfrak{m}I = \mathfrak{m}Q$, or equivalently $\mathfrak{m}I \subseteq Q$ ((2.1)). We say that the ideal I has minimal multiplicity if the equality $\mu_A(I) = \mathrm{e}_I(A) + d - \ell_A(A/I)$ holds true. Therefore a Cohen-Macaulay local ring A possesses maximal embedding dimension in the sense of Sally [S] if and only if the maximal ideal $\mathfrak{m}$ of A has mimimal multiplicity in our sense.

In this paper firstly we shall explore the Cohen-Macaulay and Gorenstein properties of graded algebras $\mathrm{R}(I)$ and $\mathrm{G}(I)$ associated to $\mathfrak{m}$-primary ideals I of minimal multiplicity, which we will perform in Section 2. Here let us summerize our main results on the Cohen-Macaulayness in $\mathrm{R}(I)$ into the following

Theorem (2.9). *Suppose $d = \dim A \geq 2$ and let I be an $\mathfrak{m}$-primary ideal in A possessing minimal multiplicity. Let Q be a minimal reduction of I. Then the following conditions are equivalent.*

(1) $\mathrm{R}(I)$ *is a Cohen-Macaulay ring.*
(2) $\mathrm{G}(I)$ *is a Cohen-Macaulay ring.*
(3) *The fibre cone* $\mathrm{S}(I) = A/\mathfrak{m} \otimes_A \mathrm{R}(I)$ *is a Cohen-Macaulay ring possessing maximal embedding dimension.*
(4) $I^2 = QI$.

When this is the case, for all integers $n \geq 0$ we have the equalities

$$\mu_A(I^n) = \binom{d+n-1}{d-1} + m\binom{d+n-2}{d-1} \qquad \text{and}$$
$$\ell_A(A/I^{n+1}) = \ell\binom{d+n}{d} + m\binom{d+n-1}{d},$$

where $\ell = \ell_A(A/I)$ and $m = \ell_A(I/Q) = \mu_A(I) - d$.

As is suggested by Korb and Nakamura in [KN], at least in the case where $\dim A$ is small, the negativity of the invariants $\mathrm{a}_i(\mathrm{R}(I))$'s of $\mathrm{R}(I)$ gives some influence on the Cohen-Macaulayness in $\mathrm{R}(I)$. Secondly, in Section 3 we shall explore this phenomenon in our context. So, let us briefly recall the definition of a-invariant below.

For a moment let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a Noetherian graded ring and assume that R contains a unique graded maximal ideal, say M. We denote by $\mathrm{H}^i_M(*)$ ($i \in \mathbb{Z}$) the $i^{\underline{th}}$ local cohomology functor of R relative to M. For a given graded R-module E and $n \in \mathbb{Z}$, let $[\mathrm{H}^i_M(E)]_n$ denote the homogeneous component of the graded R-module $\mathrm{H}^i_M(E)$ with degree n. Then if $R_n = (0)$ for $n < 0$ and E is a finitely generated graded R-module, we have $[\mathrm{H}^i_M(E)]_n = (0)$ for all $n \gg 0$ and $i \in \mathbb{Z}$; so we define

$$\mathrm{a}_i(E) = \sup\{n \in \mathbb{Z} \mid [\mathrm{H}^i_M(E)]_n \neq (0)\}$$

and call it the $i^{\underline{th}}$ a-invariant of E. When $\dim_R E = s$, we denote $\mathrm{a}_s(E)$ simply by $\mathrm{a}(E)$ and call it the a-invariant of E (cf. [GW, (3.1.4)]).

With the notation introduced above, among the others we will prove in Section 3 the following result, in which $\mathrm{r}(A) = \ell_A(\mathrm{Ext}_A^d(A/\mathfrak{m}, A))$ denotes the Cohen-Macaulay type of A.

Corollary (3.8). *Let I be an $\mathfrak{m}$-primary ideal in A possessing minimal multiplicity. Suppose $\dim A = 3$ and $\mathrm{r}(A) \leq 3$. Then $\mathrm{R}(I)$ is a Cohen-Macaulay ring if and only if $\mathrm{Proj}\,\mathrm{R}(I)$ is a Cohen-Macaulay scheme and $\mathrm{a}_i(\mathrm{R}(I)) < 0$ for all $i \in \mathbb{Z}$.*

In Section 4 we will explore an example (4.1), which shows the hypothesis in (3.8) that $\mathrm{r}(A) \leq 3$ is not superfluous. The example provides also the main conjecture of Korb and Nakamura [KN] with a counterexample; they asked if the ring $\mathrm{R}(I)$ is Cohen-Macaulay, once $\mathrm{Proj}\,\mathrm{R}(I)$ is a Cohen-Macaulay scheme and $\mathrm{a}_i(\mathrm{R}(I) < 0$ for all $i \in \mathbb{Z}$.

In what follows, let A denote a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. We assume the field $A/\mathfrak{m}$ is infinite. Let $\mathrm{H}_{\mathfrak{m}}^i(*)$ ($i \in \mathbb{Z}$) stand for the $i^{\underline{th}}$ local cohomology functor of A with respect to $\mathfrak{m}$. Otherwise specified, for a given Noetherian graded ring $R = \oplus_{n\in\mathbb{Z}} R_n$ with a unique graded maximal ideal M and a finitely generated graded R-module E, we shall simply denote $\dim_{R_M} E_M$ and $\mathrm{depth}_{R_M} E_M$ by $\dim_R E$ and $\mathrm{depth}_R E$, respectively.

2. Cohen-Macaulayness and Gorensteinness in $\mathrm{R}(I)$.

Let I be an $\mathfrak{m}$-primary ideal in A and let $Q = (a_1, a_2, ..., a_d)$ be a minimal reduction of I. Let $\mathcal{R} = \mathrm{R}(I)$, $\mathcal{R}' = \mathrm{R}'(I)$, $\mathcal{G} = \mathrm{G}(I)$, and $\mathcal{S} = A/\mathfrak{m} \otimes_{A/\mathfrak{m}} \mathcal{R}$. Hence $\dim \mathcal{R} = \dim \mathcal{R}' = d+1$ and $\dim \mathcal{G} = \dim \mathcal{S} = d$. Let M denote the unique graded maximal ideal in $\mathcal{R}$ and let $f_i = a_i t$ for $1 \leq i \leq d$. The purpose of this section is to explore the Cohen-Macaulayness and Gorensteinness in $\mathcal{R}$ and $\mathcal{G}$.

We begin with the following. This is known by Chuai [C] but let us give a brief proof for completeness.

Lemma (2.1) ([C]). *The following assertions hold true.*

(1) $\mu_A(I) \leq \mathrm{e}_I(A) + d - \ell_A(A/I)$.

(2) $\mu_A(I) = \mathrm{e}_I(A) + d - \ell_A(A/I)$ *if and only if* $\mathfrak{m}I = \mathfrak{m}Q$.

Proof. Recall that $\mathrm{e}_I(A) = \mathrm{e}_Q(A) = \ell_A(A/Q)$ and that $\mathfrak{m}I \cap Q = \mathfrak{m}Q$ ([NR]). Let $E = I/Q$. Then by the standard exact sequence $0 \to Q/\mathfrak{m}Q \to I/\mathfrak{m}I \to E/\mathfrak{m}E \to 0$ we have $\mu_A(I) = d + \mu_A(E)$. Hence $\mu_A(I) \leq \mathrm{e}_I(A) + d - \ell_A(A/I)$, because $\mu_A(E) \leq \ell_A(E)$ and

$\ell_A(E) = \ell_A(A/Q) - \ell_A(A/I) = e_I(A) - \ell_A(A/I)$. The equality $\mu_A(I) = e_I(A) + d - \ell_A(A/I)$ holds if and only if $\mu_A(E) = \ell_A(E)$, that is $\mathfrak{m}I \subseteq Q$, or equivalently $\mathfrak{m}I = \mathfrak{m}Q$.

Assume that the ideal I has minimal multiplicity. Then $\mathfrak{m}I = \mathfrak{m}Q$ by (2.1) and so we have $\mathfrak{m}I^n = \mathfrak{m}Q^n$ for all $n \in \mathbb{Z}$. Hence $\mathfrak{m}\mathcal{R} = \mathfrak{m}\mathrm{R}(Q)$. Let $\mathcal{C} = \mathcal{R}/\mathrm{R}(Q)$ (hence $\mathcal{C}_n = (0)$ if $n \leq 0$). Then as $\mathfrak{m}\mathcal{C} = (0)$, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{R}(Q) & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \end{array} \tag{2.2}$$

with exact rows, in which $\mathcal{P} = A/\mathfrak{m} \otimes_A \mathrm{R}(Q)$ and the vertical maps are canonical epimorphisms. Recall that $\mathcal{P}$ is a polynomial ring in d variables over the field $A/\mathfrak{m}$ and that $\mathcal{S}$ is a module-finite extension of $\mathcal{P}$. Then by a theorem of Hochster [Ho] the bottom row in (2.2) is split, so that we have

$$\mathcal{S} \cong \mathcal{P} \oplus \mathcal{C} \tag{2.3}$$

as graded $\mathcal{P}$-modules.

Let N be the unique graded maximal ideal in $\mathrm{R}(Q)$. Then as $\mathcal{P}$ is a polynomial ring, by (2.3) we get $\mathrm{H}^i_M(\mathcal{S}) \cong \mathrm{H}^i_N(\mathcal{C})$ for $i \leq d-1$ and $\mathrm{H}^d_M(\mathcal{S}) \cong \mathrm{H}^d_N(\mathcal{P}) \oplus \mathrm{H}^d_N(\mathcal{C})$. On the other hand, since $\mathrm{R}(Q)$ is a Cohen-Macaulay ring of $\dim \mathrm{R}(Q) = d+1$, from the top row in (2.2) the isomorphisms $\mathrm{H}^i_M(\mathcal{R}) \cong \mathrm{H}^i_N(\mathcal{C})$ for $i \leq d-1$ and the exact sequence $0 \to \mathrm{H}^d_M(\mathcal{R}) \to \mathrm{H}^d_N(\mathcal{C}) \to \mathrm{H}^{d+1}_N(\mathrm{R}(Q))$ follow. Because $\mathrm{a}(\mathrm{R}(Q)) = -1$ and $\mathrm{a}(\mathcal{P}) = -d$ (cf. [GN, Part II, (3.3)] and [GW, (3.1.6)]), summarizing these observations, we have

Proposition (2.4). (1) $[\mathrm{H}^i_M(\mathcal{R})]_n \cong [\mathrm{H}^i_N(\mathcal{C})]_n \cong [\mathrm{H}^i_M(\mathcal{S})]_n$ for all $i \leq d-1$ and $n \in \mathbb{Z}$.

(2) $\mathrm{a}_i(\mathcal{R}) = \mathrm{a}_i(\mathcal{C}) = \mathrm{a}_i(\mathcal{S})$ if $i \leq d-1$.

(3) $\mathrm{a}(\mathcal{S}) = \max\{\mathrm{a}_d(\mathcal{C}), -d\}$.

(4) $\mathrm{a}_d(\mathcal{R}) \leq \mathrm{a}_d(\mathcal{C}) \leq \max\{\mathrm{a}_d(\mathcal{R}), -1\}$.

Lemma (2.5). (1) Let $I \neq Q$. Then $\dim_{\mathcal{P}} \mathcal{C} = d$ and $\min\{\mathrm{depth}_{\mathcal{P}} \mathcal{C}, \mathrm{depth}\, \mathcal{S}\} \geq 1$.

(2) $\mathcal{G}$ is a Cohen-Macaulay ring if and only if $I^2 = QI$. When this is the case, $\mathrm{a}(\mathcal{G}) = 1-d$ if $I \neq Q$ and $\mathrm{a}(\mathcal{G}) = -d$ if $I = Q$.

(3) $\mathrm{depth}\, \mathcal{R} = \mathrm{depth}\, \mathcal{G} + 1$ if $d \geq 2$.

Proof. (1) We have $\mathcal{C} \neq (0)$ since $I \neq Q$. As $\mathrm{H}^0_M(\mathcal{R}) = (0)$, by (2.4)(1) we get $\mathrm{H}^0_N(\mathcal{C}) = \mathrm{H}^0_M(\mathcal{S}) = (0)$, whence $\min\{\mathrm{depth}_{\mathcal{P}} \mathcal{C}, \mathrm{depth}\, \mathcal{S}\} \geq 1$. The element f_d is actually $\mathcal{C}$-regular.

(In fact, let $x \in I^n$ with $n \geq 1$ and assume $f_d \cdot xt^n \in \mathrm{R}(Q)$. Then $a_d x \in a_d A \cap Q^{n+1} = a_d Q^n$ whence $x \in Q^n$.) To see that $\dim_{\mathcal{P}} \mathcal{C} = d$, it suffices to check $\dim_{\mathcal{P}} \mathcal{C} \geq d$. This is clear for $d = 1$. Let $d \geq 2$ and assume that our assertion is true for $d-1$. Let $\overline{A} = A/a_d A$, $\overline{\mathfrak{m}} = \mathfrak{m}/a_d A$, $\overline{I} = I/a_d A$, and $\overline{Q} = Q/a_d A$. Then $\overline{Q}$ is a reduction of $\overline{I}$ with $\overline{\mathfrak{m}}\overline{I} = \overline{\mathfrak{m}}\overline{Q}$. Hence the ideal $\overline{I}$ has minimal multiplicity so that from the hypothesis on d we see $\dim_{\overline{\mathcal{P}}} \mathrm{C}(\overline{I}) \geq d-1$, where $\mathrm{C}(\overline{I}) = \mathrm{R}(\overline{I})/\mathrm{R}(\overline{Q})$ and $\overline{\mathcal{P}} = \overline{A}/\overline{\mathfrak{m}} \otimes_{\overline{A}} \mathrm{R}(\overline{Q})$. As $\mathrm{C}(\overline{I})$ is naturally a homomorphic image of $\mathcal{C}/f_d\mathcal{C}$, we get $\dim_{\mathcal{P}} \mathcal{C}/f_d\mathcal{C} \geq d-1$. Thus $\dim_{\mathcal{P}} \mathcal{C} \geq d$ since f_d is $\mathcal{C}$-regular.
(2) The *if part* is due to [VV, (3.1)]. Let $\mathcal{G}$ be a Cohen-Macaulay ring. Then $Q \cap I^n = QI^{n-1}$ for all $n \in \mathbb{Z}$ by [VV, (2.7)], while $I^2 \subseteq Q$ as $I^2 \subseteq \mathfrak{m}I = \mathfrak{m}Q$. Hence $I^2 = QI$. The last assertion now follows from the equality $\mathrm{a}(\mathcal{G}) = \mathrm{a}(\mathcal{G}/(f_1, f_2, ..., f_d)\mathcal{G}) - d$ (cf. [GW, (3.1.6)]), because $\mathrm{a}(\mathcal{G}/(f_1, f_2, ..., f_d)\mathcal{G}) = 1$ (resp. $\mathrm{a}(\mathcal{G}/(f_1, f_2, ..., f_d)\mathcal{G}) = 0$) if $I \neq Q$ (resp. $I = Q$).
(3) If $\mathcal{G}$ is Cohen-Macaulay, then $I^2 = QI$ by (2) so that the ring $\mathcal{R}$ is Cohen-Macaulay (cf. [GS, (3.10)]). The equality $\operatorname{depth} \mathcal{R} = \operatorname{depth} \mathcal{G} + 1$ is due to [HM] in the case where $\mathcal{G}$ is not Cohen-Macaulay.

If the ring $\mathcal{S}$ is not Cohen-Macaulay, then $I \neq Q$ and $\operatorname{depth} \mathcal{R} = \operatorname{depth} \mathcal{S} = \operatorname{depth} \mathcal{C}$ by (2.4)(1). As $d \geq 2$ by (2.5)(1), by (2.5)(2) we get $\operatorname{depth} \mathcal{R} = \operatorname{depth} \mathcal{G} + 1$. Hence

Corollary (2.6). *Suppose $\mathcal{S}$ is not a Cohen-Macaulay ring. Then* $\operatorname{depth} \mathcal{R} = \operatorname{depth} \mathcal{S} = \operatorname{depth} \mathcal{C} = \operatorname{depth} \mathcal{G} + 1$.

Let $\mathrm{r}_Q(I) = \min\{n \geq 0 \mid I^{n+1} = QI^n\}$ and call it the reduction number of I with respect to Q.

Proposition (2.7). *The following four conditions are equivalent.*

(1) *$\mathcal{S}$ is a Cohen-Macaulay ring.*
(2) $\operatorname{depth} \mathcal{R} \geq d$.
(3) $\operatorname{depth} \mathcal{G} \geq d-1$.
(4) *$\mathcal{C}$ is $\mathcal{P}$-free.*

When this is the case, $\mathrm{a}(\mathcal{S}) = \mathrm{r}_Q(I) - d$.

Proof. (1)$\Longleftrightarrow$ (2) $\Longleftrightarrow$ (4) See (2.4)(1).
(2) $\Longleftrightarrow$ (3) This follows from (2.5)(3).
To check the last equality, note $\mathrm{a}(\mathcal{S}) = a(\mathcal{S}/(f_1, f_2, ..., f_d)\mathcal{S}) - d$ (cf. [GW, (3.1.6)]). Then as $\mathrm{a}(\mathcal{S}/(f_1, f_2, ..., f_d)\mathcal{S}) = \max\{n \geq 0 \mid I^n \not\subseteq QI^{n-1} + \mathfrak{m}I^n\}$, via Nakayama's lemma we get $\mathrm{a}(\mathcal{S}/(f_1, f_2, ..., f_d)\mathcal{S}) = \mathrm{r}_Q(I)$. Thus $\mathrm{a}(\mathcal{S}) = \mathrm{r}_Q(I) - d$.

Corollary (2.8). *$\mathcal{G}$ is a Cohen-Macaulay ring if and only if $\mathcal{S}$ is a Cohen-Macaulay ring and $\mathrm{a}(\mathcal{S}) \leq 1-d$.*

Proof. See (2.5)(2) and (2.7).

When $d=1$, $\mathcal{R}$ is a Cohen-Macaulay ring if and only if $I=Q$ (cf. [GS, (3.10)]). As for the case where $d \geq 2$ we note the following

Theorem (2.9). *Suppose $d \geq 2$. Then the following four conditions are equivalent.*

(1) *$\mathcal{R}$ is a Cohen-Macaulay ring.*
(2) *$\mathcal{G}$ is a Cohen-Macaulay ring.*
(3) *$\mathcal{S}$ is a Cohen-Macaulay ring possessing maximal embedding dimension.*
(4) *$I^2 = QI$.*

When this is the case, for all $n \geq 0$ we have the equalities

$$\mu_A(I^n) = \binom{d+n-1}{d-1} + m\binom{d+n-2}{d-1} \qquad \text{and}$$
$$\ell_A(A/I^{n+1}) = \ell\binom{d+n}{d} + m\binom{d+n-1}{d},$$

where $\ell = \ell_A(A/I)$ and $m = \ell_A(I/Q) = \mu_A(I) - d$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (4) This follows from (2.5)(2) and (3).
(2) $\Longleftrightarrow$ (3) By (2.7) we may assume $\mathcal{S}$ is Cohen-Macaulay. Let $\mathfrak{M} = \mathcal{S}_+$. Then $\mathcal{S}$ has maximal embedding dimension if and only if $\mathfrak{M}^2 = (f_1, f_2, ..., f_d)\mathfrak{M}$, and the latter condition is equivalent to saying that $\mathrm{a}(\mathcal{S}/(f_1, f_2, ..., f_d)\mathcal{S}) \leq 1$. So the equivalence (2) $\Longleftrightarrow$ (3) follows from (2.8), since $\mathrm{a}(\mathcal{S}/(f_1, f_2, ..., f_d)\mathcal{S}) = a(\mathcal{S}) + d$.
Let us check the last equalities. Let $\mathrm{H}(\mathcal{G}, \lambda) = \sum_{n=0}^{\infty} \ell_A(\mathcal{G}_n)\lambda^n$ be the Hilbert series of $\mathcal{G}$. Note $[\mathcal{G}/(f_1, f_2, ..., f_d)\mathcal{G})]_1 = I/Q$ and $[\mathcal{G}/(f_1, f_2, ..., f_d)\mathcal{G})]_n = (0)$ for all $n \geq 2$, as $I^2 = QI$. Then we get $\mathrm{H}(\mathcal{G}, \lambda) = (\ell + m\lambda)/(1-\lambda)^d$ (recall the sequence f_1, f_2,..., f_d is $\mathcal{G}$-regular). Hence $\ell_A(I^i/I^{i+1}) = \ell\binom{d+i-1}{d-1} + m\binom{d+i-2}{d-1}$ for $i \geq 0$ so that $\ell_A(A/I^{n+1}) = \sum_{i=1}^{n} \ell_A(I^i/I^{i+1}) = \ell\binom{d+n}{d} + m\binom{d+n-1}{d}$ if $n \geq 0$. Since $\mathrm{H}(\mathcal{S}, \lambda) = (1+m\lambda)/(1-\lambda)^d$, we have $\mu_A(I^n) = \binom{d+n-1}{d-1} + m\binom{d+n-2}{d-1}$ for $n \geq 0$.

Corollary (2.10). *Suppose A is a Gorenstein ring. Then $\mathcal{G}$ is a Cohen-Macaulay ring. If $d \geq 2$, $\mathcal{R}$ is a Cohen-Macaulay ring too.*

Proof. We may assume $I \neq Q$. Then $I = Q : \mathfrak{m}$, since $Q \subseteq I \subseteq Q : \mathfrak{m}$ and $\ell_A((Q:\mathfrak{m})/Q) = 1$. We will show $I^2 = QI$. Let $a, b \in I$ and write $ab = \sum_{i=1}^{d} a_i c_i$ with $c_i \in A$. Then for each $x \in \mathfrak{m}$ we get $xab \in Q^2$ (since $\mathfrak{m}I^2 = \mathfrak{m}Q^2 \subseteq Q^2$). Therefore $\sum_{i=1}^{d} a_i \cdot xc_i \in Q^2$ and so $xc_i \in Q$ for all $1 \leq i \leq d$. Hence $c_i \in Q : \mathfrak{m} = I$ so that $I^2 = QI$. The last assertion follows from (2.5)(2) and (2.9).

Let us add a few remarks on the Gorenstein property of $\mathcal{R}$ and $\mathcal{G}$. For a given Cohen-Macaulay local ring $(B, \mathfrak{n})$ of $\dim B = n$ we put $\mathrm{r}(B) = \ell_B(\mathrm{Ext}_B^n(B/\mathfrak{n}, B))$. If B is not necessarily local, we put $\mathrm{r}(B) = \sup_{\mathfrak{p} \in \mathrm{Spec}\, B} \mathrm{r}(B_{\mathfrak{p}})$ and call it the Cohen-Macaulay type of B.

Proposition (2.11). *Suppose $I^2 = QI$. Then the Cohen-Macaulay type $\mathrm{r}(\mathcal{G})$ of $\mathcal{G}$ is given by the following formula*

$$\mathrm{r}(\mathcal{G}) = \begin{cases} \mathrm{r}(A/I) + \mu_A(I) - d & \text{if } I \neq \mathfrak{m}, \\ \mu_A(I) - d & \text{if } I = \mathfrak{m} \neq Q, \\ 1 & \text{if } I = \mathfrak{m} = Q. \end{cases}$$

Proof. Recall $\mathrm{r}(\mathcal{G}) = \mathrm{r}(\mathcal{G}_{\mathfrak{M}})$ (cf. [AG]) where $\mathfrak{M}$ is the graded maximal ideal in $\mathcal{G}$. On the other hand, since the sequence $f_1, f_2, \dots, f_d$ is $\mathcal{G}$-regular (cf. [VV, (2.1)]), we have $\mathrm{r}(\mathcal{G}_{\mathfrak{M}}) = \mathrm{r}(\mathcal{G}/(f_1, f_2, \dots, f_d)\mathcal{G})$ and the isomorphism $\mathcal{G}/(f_1, f_2, \dots, f_d)\mathcal{G} = \mathrm{G}(I/Q)$ as well. Thus $\mathrm{r}(\mathcal{G}) = \mathrm{r}(\mathrm{G}(I/Q))$. Let V denote the socle of $\mathrm{G}(I/Q) = A/I \oplus I/Q$. Then $V = (I : \mathfrak{m})/I \oplus I/Q$ if $I \neq \mathfrak{m}$, $V = I/Q$ if $I = \mathfrak{m} \neq Q$, and $V = A/\mathfrak{m}$ if $I = \mathfrak{m} = Q$, from which the formula follows because $\ell_A(I/Q) = \mu_A(I) - d$.

Corollary (2.12). *$\mathcal{G}$ is a Gorenstein ring if and only if either (1) $I = Q$ and A is a Gorenastein ring or (2) $I = \mathfrak{m}$ and $\mu_A(\mathfrak{m}) = d + 1$.*

Proof. The assertion follows from (2.11). Note A is a Gorenstein ring if so is $\mathcal{G}$.

Theorem (2.13). *$\mathcal{R}$ is a Gorenstein ring if and only if either (1) $d \leq 2$, $I = Q$, and A is a Gorenstein ring or (2) $d = 3$, $I = \mathfrak{m}$, and $\mu_A(\mathfrak{m}) = 4$.*

Proof. We may assume $d \geq 2$ (recall that $I = Q$ if $d = 1$ and if $\mathcal{R}$ is a Cohen-Macaulay ring). Then thanks to Ikeda's theorem [I, (3.7)] $\mathcal{R}$ is a Gorenstein ring if and only if $\mathcal{G}$ is a Gorenstein ring of $\mathrm{a}(\mathcal{G}) = -2$, so that the assertion follows from (2.5)(2) and (2.12).

We close this section with the following examples. Let k be an infinite field.

Example (2.14). Let $A = k[[X^4, X^3Y, X^2Y^2, XY^3, Y^4]]$ be the subring of the formal power series ring $k[[X, Y]]$ over k in two variables X and Y. Let $I = (X^4, X^3Y, XY^3, Y^4)A$ and $Q = (X^4, Y^4)A$. Then $I^3 = QI^2$, $\mathfrak{m}I = \mathfrak{m}Q$, and $\operatorname{depth} \mathcal{G} = 0$. Hence $\operatorname{depth} \mathcal{R} = \operatorname{depth} \mathcal{S} = \operatorname{depth} \mathcal{C} = 1$, so that the ring $\mathcal{S}$ cannot be Cohen-Macaulay.

Proof. It is routine to check that $I^3 = QI^2$ and $\mathfrak{m}I \subseteq Q$. As $(X^4t, Y^4t)\cdot X^2Y^2 \subseteq t^{-1}\mathcal{R}'$, we see $\operatorname{depth}\mathcal{G} = 0$. Hence by (2.7) $\mathcal{S}$ is not a Cohen-Macaulay ring and by (2.6) $\operatorname{depth}\mathcal{R} = \operatorname{depth}\mathcal{S} = \operatorname{depth}\mathcal{C} = 1$.

Example (2.15). Let $A = k[[X^2, Y^2, Z^2, XY, YZ, ZX]]$ in the formal power series ring $k[[X, Y, Z]]$. Let $I = (X^2, Y^2, Z^2, XY, YZ)A$ and $Q = (X^2, Y^2, Z^2)A$. Then $I^3 = QI^2$ and $\mathfrak{m}I = \mathfrak{m}Q$. The ring $\mathcal{S}$ is Cohen-Macaulay and $\operatorname{depth}\mathcal{R} = \operatorname{depth}\mathcal{G} + 1 = 3$. Hence $\mathcal{R}$ is not Cohen-Macaulay. The scheme $\operatorname{Proj}\mathcal{R}$ is not Cohen-Macaulay, since $\mathrm{H}^2_M(\mathcal{G})$ is not a finitely generated $\mathcal{G}$-module.

Proof. It is routine to check that $I^3 = QI^2$ and $\mathfrak{m}I \subseteq Q$. We will show that X^2t, Z^2t form a $\mathcal{G}$-regular sequence. It is enough to see $(X^2, Z^2) \cap I^n = (X^2, Z^2)I^{n-1}$ for all $n \geq 2$ (cf. [VV, (2.7)]). Note $I^2 = (X^2, Z^2)I + Y^2\mathfrak{m}$. Then we have $(X^2, Z^2) \cap I^2 = (X^2, Z^2)I + (X^2, Z^2) \cap Y^2\mathfrak{m}$, whence $(X^2, Z^2) \cap I^2 = (X^2, Z^2)I$ as $(X^2, Z^2) \cap Y^2\mathfrak{m} \subseteq Y^2(X^2, Z^2)$. Let $n \geq 3$ and assume that our equality holds true for $n-1$. Then $I^n = QI^{n-1}$ so that $(X^2, Z^2) \cap I^n = (X^2, Z^2)I^{n-1} + (X^2, Z^2) \cap Y^2I^{n-1}$. As $(X^2, Z^2) \cap Y^2I^{n-1} = Y^2[(X^2, Z^2) \cap I^{n-1}]$, from the hypothesis on n we see $(X^2, Z^2) \cap Y^2I^{n-1} = Y^2(X^2, Z^2)I^{n-2}$. Thus $(X^2, Z^2) \cap I^n = (X^2, Z^2)I^{n-1}$ so that $\operatorname{depth}\mathcal{G} \geq 2$. Hence by (2.7) $\mathcal{S}$ is a Cohen-Macaulay ring. As $XY^2Z \in I^2$ but $XZ \notin I$, Y^2t is a zerodivisor in $\mathcal{G}$. Therefore $\operatorname{depth}\mathcal{G} = 2$ and $\operatorname{depth}\mathcal{R} = 3$ by (2.5)(3). If $\mathrm{H}^2_M(\mathcal{G})$ were a finitely generated $\mathcal{G}$-module, every subsystem f, g of homogeneous parameters for $\mathcal{G}$ must be a $\mathcal{G}$-regular sequence ([STC, (2.5) and (2.11)]), which is impossible because Y^2t is a zerodivisor in $\mathcal{G}$. Since the finite generation of $\mathrm{H}^2_M(\mathcal{G})$ is equivalent to the Cohen-Macaulayness of $\operatorname{Proj}\mathcal{G}$ ([STC, (2.5)], (2.11), and (3.8); recall that I is $\mathfrak{m}$-primary), we have $\operatorname{Proj}\mathcal{G}$ cannot be Cohen-Macaulay. Hence $\operatorname{Proj}\mathcal{R}$ is not Cohen-Macaulay as well.

Example (2.16). Let $n \geq d \geq 2$ be integers and let $R = k[X_1, X_2, ..., X_d]$ be the polynomial ring in d variables over k. Let $S = R^{(n)}$ denote the Veronesean subring of R with order n. We put $\mathfrak{M} = S_+$ and $A = S_{\mathfrak{M}}$. Let $Q = (X_1^n, X_2^n, ..., X_d^n)A$ and $I = Q + WA$, where $W = S_{d-1} = R_{n(d-1)}$. Then $I^2 = QI$ and $\mathfrak{m}I = \mathfrak{m}Q$. The ring $\mathcal{R}$ is Cohen-Macaulay, $d = \dim A$, and $\mu_A(I) = d + \binom{n-1}{d-1}$.

Proof. The ring S is Cohen-Macaulay with $\dim S = d$ and $\mathrm{a}(S) = -1$ (cf. [GW, (3.1.1)]). Hence $[S/(X_1^n, X_2^n, ..., X_d^n)S]_{d-1} \neq (0)$ but $[S/(X_1^n, X_2^n, ..., X_d^n)S]_i = (0)$ for $i \geq d$. Therefore $I^2 = QI$ and $\mathfrak{m}I \subseteq Q$ so that by (2.9) $\mathcal{R}$ is a Cohen-Macaulay ring. We get $\mu_A(I) = d + \binom{n-1}{d-1}$, because $\mu_A(I) = d + \mu_A(I/Q)$ and $\mu_A(I/Q) = \dim_k[S/(X_1^n, X_2^n, ..., X_d^n)S]_{d-1} = \dim_k[R/(X_1^n, X_2^n, ..., X_d^n)R]_{n(d-1)}$.

The typical example satisfying condition (2) in (2.13) is as follows.

Example (2.17). Let $R = k[[X,Y,Z,W]]$ be the formal power series ring and $A = R/(XY - ZW)$. Let x, y, z, and w denote respectively, the reductions of X, Y, Z, and W mod $(XY - ZW)$. Then $\dim A = 3$ and $\mathfrak{m}^2 = (x, y, z - w)\mathfrak{m}$. Hence the maximal ideal $\mathfrak{m}$ in A has minimal multiplicity with $\mu_A(\mathfrak{m}) = 4$, so that $\mathrm{R}(\mathfrak{m})$ is a Gorenstein ring.

3. Cohen-Macaulayness in $\mathrm{Proj}\,\mathrm{R}(I)$ and the negativity of $\mathrm{a}_i(\mathrm{R}(I))$'s.

As is explored in [KN], at least in the case where $\dim A$ is small, the negativity of $\mathrm{a}_i(\mathrm{R}(I))$'s gives some influence on the Cohen-Macaulayness in $\mathrm{R}(I)$. We shall also discuss this phenomenon in our context. Let I be an $\mathfrak{m}$-primary ideal in A possessing minimal multipilicity. We maintain the same notation as is given in Section 2. We begin with the following.

Theorem (3.1). *Suppose that $I^{n+1} = \mathfrak{m}I^n$ for some $n \geq 0$. Then $\mathrm{Proj}\,\mathcal{R}$ is a Cohen-Macaulay scheme.*

Proof. By [K, (2.13)] it suffices to check that a_1, a_2,..., a_d is a d-sequence on I^p for all $p \geq n+2$. Let $Q_i = (a_1, a_2, ..., a_i)$ for $0 \leq i \leq d$. Firstly we will show that $Q_i \cap I^p = Q_i I^{p-1}$. In fact, as $I^p = \mathfrak{m}I^{p-1} = \mathfrak{m}Q^{p-1}$ and $Q_i \cap Q^{p-1} = Q_i Q^{p-2}$, we see $Q_i \cap I^p \subseteq Q_i Q^{p-2} \cap \mathfrak{m}Q^{p-1} \subseteq \mathfrak{m}Q_i \cdot Q^{p-2} = Q_i I^{p-1}$ (note that $\mathrm{G}(Q)$ is a polynomial ring). Hence $Q_i \cap I^p = Q_i I^{p-1}$ for $0 \leq i \leq d$. Let $1 \leq i \leq j \leq d$ be integers and choose $x \in I^p$ so that $a_i a_j x \in Q_{i-1}I^p$. Then $x \in Q_{i-1} \cap I^p = Q_{i-1}I^{p-1}$ whence $a_j x \in Q_{i-1}I^p$, and thus a_1, a_2,..., a_d is a d-sequence on I^p.

Example (3.2). Let $k[[X,Y]]$ be the formal power series ring in two variables over an infinite field k and $A = k[[X^5, X^4Y, X^3Y^2, X^2Y^3, XY^4, Y^5]]$ in $k[[X,Y]]$. Let $I = (X^5, X^4Y, XY^4, Y^5)A$ and $Q = (X^5, Y^5)A$. Then $I^4 = QI^3$ and $\mathfrak{m}I = \mathfrak{m}Q$. The ring $\mathcal{R}$ is not Cohen-Macaulay by (2.9) since $I^3 \neq QI^2$, while $\mathrm{Proj}\,\mathcal{R}$ is a Cohen-Macaulay scheme by (3.1) because $I^3 = \mathfrak{m}I^2$. As $X^3Y^2 \notin I$ but $(X^{10}, Y^{10}) \cdot X^3Y^2 \subseteq I^3$, $\mathrm{depth}\,\mathcal{G} = 0$. Hence $\mathrm{depth}\,\mathcal{R} = \mathrm{depth}\,\mathcal{S} = 1$ by (2.6) and (2.7).

Let N be the unique graded maximal ideal in $\mathrm{R}(Q)$. We note $\mathrm{a}(\mathcal{R}) = -1$ (cf. [GN, Part II, (3.3)]).

Lemma (3.3). (1) $\mathrm{H}^1_M(\mathcal{R})$ is a finitely generated $\mathcal{R}$-module and $\mathfrak{m} \cdot \mathrm{H}^1_M(\mathcal{R}) = (0)$.

(2) $\mathrm{H}^0_N(\mathcal{C}) = \mathrm{H}^1_N(\mathcal{C}) = (0)$ if $\mathrm{a}_1(\mathcal{R}) < 0$.

(3) $I = Q$ if $d = 1$ and $a_1(\mathcal{R}) < 0$.

Proof. (1) The second assertion follows from the embedding $\mathrm{H}^1_M(\mathcal{R}) \subseteq \mathrm{H}^1_N(\mathcal{C})$ (cf. (2.2)). To see the first one we may assume A is complete. Let $\mathrm{K}_{\mathcal{R}}$ be the graded canonical module of $\mathcal{R}$. Then $(0) :_{\mathcal{R}} \mathrm{K}_{\mathcal{R}} = (0)$ as $\dim \mathcal{R}/P = d+1$ for all $P \in \operatorname{Ass}\mathcal{R}$ (cf. [V, (1.7)]). Let $E = \operatorname{End}_{\mathcal{R}} \mathrm{K}_{\mathcal{R}}$ and apply the functors $\mathrm{H}^i_M(*)$ to the exact sequence $0 \to \mathcal{R} \to E \to E/\mathcal{R} \to 0$. Then $\mathrm{H}^0_M(E/\mathcal{R}) \cong \mathrm{H}^1_M(\mathcal{R})$ as $\operatorname{depth}_{\mathcal{R}} E \geq 2$. Thus $\mathrm{H}^1_M(\mathcal{R})$ is a finitely generated $\mathcal{R}$-module.

(2) and (3). Let $f = a_d t$. Then f is a nonzerodivisor on $\mathcal{C}$ (cf. Proof of (2.5) (1)). Let $\overline{\mathcal{C}} = \mathcal{C}/f\mathcal{C}$. Then by the exact sequence $0 \to \mathcal{C}(-1) \xrightarrow{f} \mathcal{C} \to \overline{\mathcal{C}} \to 0$, we get the embedding $\mathrm{H}^0_N(\overline{\mathcal{C}}) \subseteq \mathrm{H}^1_N(\mathcal{C})(-1)$. Hence $\mathrm{a}_0(\overline{\mathcal{C}}) \leq \mathrm{a}_1(\mathcal{C}) + 1$. Note $\mathrm{a}_1(\mathcal{C}) = a_1(\mathcal{R})$ (resp. $\mathrm{a}_1(\mathcal{C}) \leq \max\{\mathrm{a}_1(\mathcal{R}), -1\}$) if $d \geq 2$ (resp. $d=1$) (cf. (2.4)). And we see $\mathrm{a}_1(\mathcal{C}) \leq -1$ so that $\mathrm{a}_0(\overline{\mathcal{C}}) \leq 0$. Because $\overline{\mathcal{C}}_n = (0)$ for $n \leq 0$, this forces $\mathrm{H}^0_N(\overline{\mathcal{C}}) = (0)$ whence $\mathrm{H}^1_N(\mathcal{C}) = (0)$. Assertion (3) is clear.

First we note the following result in the case where $\dim A = 2$.

Proposition (3.4). *Suppose $d = 2$. Then*

(1) *$\mathrm{H}^1_M(\mathcal{S})$ is a finitely generated $\mathcal{S}$-module.*
(2) *$\mathcal{S}$ is a Cohen-Macaulay ring if $\mathrm{a}_1(\mathcal{R}) < 0$.*
(3) *$\mathcal{R}$ is a Cohen-Macaulay ring if and only if $\mathrm{a}_i(\mathcal{R}) < 0$ for all $i \in \mathbb{Z}$.*

Proof. (1) This follows from (2.4)(1) and (3.3)(1).
(2) See (2.4)(1). Note that by (3.3)(2) $\mathrm{H}^i_N(\mathcal{C}) = (0)$ for $i \leq 1$.
(3) Assume $\mathrm{a}_i(\mathcal{R}) < 0$ for all $i \in \mathbb{Z}$. Then $\mathcal{S}$ is Cohen-Macaulay by (2). We have $\mathrm{a}(\mathcal{S}) \leq -1 = 1-d$, because $\mathrm{a}(\mathcal{S}) = \max\{\mathrm{a}_2(\mathcal{C}), -2\}$ and $\mathrm{a}_2(\mathcal{C}) \leq \max\{\mathrm{a}_2(\mathcal{R}), -1\}$ by (2.4)(3) and (4). Hence $\mathcal{R}$ is a Cohen-Macaualy ring by (2.8) and (2.9).

Theorem (3.5). *Suppose $d = 3$. Then $\mathcal{S}$ is a Cohen-Macaulay ring and $I^3 = QI^2$ if and only if $\mathrm{a}_i(\mathcal{R}) < 0$ for all $i \in \mathbb{Z}$.*

Proof. Assume that $\mathcal{S}$ is a Cohen-Macaulay ring and $I^3 = QI^2$. Then $\mathrm{a}(\mathcal{S}) \leq -1$ by (2.7) so that $\mathrm{a}_3(\mathcal{R}) \leq -1$ by (2.4)(3) and (4). Hence $\mathrm{a}_i(\mathcal{R}) < 0$ for all $i \in \mathbb{Z}$ (recall that $\mathrm{a}(\mathcal{R}) = -1$ and $\operatorname{depth}\mathcal{R} \geq 3$ by (2.4)(1)). Conversely assume that $\mathrm{a}_i(\mathcal{R}) < 0$ for all $i \in \mathbb{Z}$. Firstly we will show that $\mathcal{S}$ is a Cohen-Macaulay ring. Assume the contrary. Then $\operatorname{depth}\mathcal{S} = 2$ by (2.4)(1) because $\operatorname{depth}_{\mathcal{P}} \mathcal{C} \geq 2$ by (3.3)(2). Hence $\operatorname{depth}\mathcal{R} = 2$ and $\operatorname{depth}\mathcal{G} = 1$ by (2.6). Let $\mathfrak{a} = \mathcal{R}_+$ and consider the standerd exact sequences

(a) $$0 \longrightarrow \mathfrak{a} \longrightarrow \mathcal{R} \longrightarrow A \longrightarrow 0,$$

(b) $$0 \longrightarrow \mathfrak{a}(1) \longrightarrow \mathcal{R} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then applying the funtors $\mathrm{H}^i_M(*)$ to (a) and (b), we get an isomophism $\mathrm{H}^2_M(\mathfrak{a}) \cong \mathrm{H}^2_M(\mathcal{R})$ and the embedding $\mathrm{H}^1_M(\mathcal{G}) \subseteq \mathrm{H}^2_M(\mathfrak{a})(1)$. Hence $\mathrm{a}_1(\mathcal{G}) \leq -2$ because $\mathrm{a}_2(\mathfrak{a}) = \mathrm{a}_2(\mathcal{R}) < 0$ and

$a_1(\mathcal{G}) \leq a_2(\mathfrak{a}) - 1$. Choose an element $g \in \mathcal{G}_1$ so that g is $\mathcal{G}$-regular (this choice is possible, because $\operatorname{depth} \mathcal{G} > 0$ and $A/\mathfrak{m}$ is infinite). Let $\overline{\mathcal{G}} = \mathcal{G}/g\mathcal{G}$ and apply $H^i_M(*)$ to the exact sequence $0 \to \mathcal{G}(-1) \xrightarrow{g} \mathcal{G} \to \overline{\mathcal{G}} \to 0$. Then from the embedding $H^0_M(\overline{\mathcal{G}}) \subseteq H^1_M(\mathcal{G})(-1)$ we see $a_0(\overline{\mathcal{G}}) \leq a_1(\mathcal{G}) + 1$; hence $a_0(\overline{\mathcal{G}}) \leq -1$. Therefore $H^0_M(\overline{\mathcal{G}}) = (0)$ so that $\operatorname{depth} \mathcal{G} \geq 2$, which is absurd and thus $\mathcal{S}$ is a Cohen-Macaulay ring. Because $a(\mathcal{S}) = \max\{a_3(\mathcal{C}), -3\}$ and $a_3(\mathcal{C}) \leq \max\{a_3(\mathcal{R}), -1\}$ by (2.4), we get $a(\mathcal{S}) \leq -1$ whence $r_Q(I) \leq 2$ by (2.7). Thus $I^3 = QI^2$, which completes the proof of Theorem (3.5).

The scheme $\operatorname{Proj} \mathcal{R}$ (resp. $\operatorname{Proj} \mathcal{G}$) is Cohen-Macaulay if and only if $H^i_M(\mathcal{R})$ (resp. $H^i_M(\mathcal{G})$) is a finitely generated $\mathcal{R}$-module (resp. a finitely generated $\mathcal{G}$-module) for all $i \neq d+1$ (resp. $i \neq d$) (cf. [STC, (2.5), (2.11), and (3.8)]; note $\sqrt{I} = \mathfrak{m}$) and $\operatorname{Proj} \mathcal{R}$ is Cohen-Macaulay if and only if so is $\operatorname{Proj} G$. When this is the case, the sequence b_1t, b_2t, ..., b_st ($s = \operatorname{depth} \mathcal{G}$) is $\mathcal{G}$-regular for any system $b_1, b_2, ..., b_d$ of generators for Q (cf. [STC, (2.5) and (2.11)]).

Theorem (3.6). *Suppose $\mu_A(I) \geq r(A) + 1$ or $r(A/I) \leq d - 1$. Then $I^2 = QI$ if $\operatorname{Proj} \mathcal{R}$ is a Cohen-Macaulay scheme and $\mathcal{S}$ is a Cohen-Macaulay ring.*

Proof. Let $x \in I^2$ and write $x = \sum_{i=1}^d a_i x_i$ with $x_i \in A$. Then for the same reason as is in the proof of (2.10), we get $x_i \in Q : \mathfrak{m}$ for all $1 \leq i \leq d$. Let $J = Q : \mathfrak{m}$. Then $r(A/I) \geq \ell_A(J/I) = \ell_A(J/Q) - \ell_A(I/Q) = r(A) - \mu_A(I) + d$, because $\ell_A(J/Q) = r(A)$ and $\ell_A(I/Q) = \mu_A(I) - d$. Therefore if $r(A/I) \leq d - 1$ or more generally $\mu_A(I) \geq r(A) + 1$, we have $\ell_A(J/I) \leq d - 1$ so that the elements x_1, x_2, ..., x_d mod I cannot be $A/\mathfrak{m}$-linearly independent in J/I. Without loss of generality we may write $x_d = \sum_{i=1}^{d-1} c_i x_i + y$ with $c_i \in A$ and $y \in I$. Then since $x = \sum_{i=1}^d a_i x_i = \sum_{i=1}^{d-1} (a_i + a_d c_i) x_i + a_d y$, we have $x - a_d y \in (a_i + a_d c_i \mid 1 \leq i \leq d-1) \cap I^2$. Recall that $\operatorname{depth} \mathcal{G} \geq d - 1$ by (2.7) since $\mathcal{S}$ is a Cohen-Macaulay ring. And we get $(a_i + a_d c_i \mid 1 \leq i \leq d-1) \cap I^2 = (a_i + a_d c_i \mid 1 \leq i \leq d-1) I$ by [VV, 2.7], because $Q = (a_i + a_d c_i \mid 1 \leq i \leq d-1) + (a_d)$ and because $\operatorname{Proj} \mathcal{G}$ is a Cohen-Macaulay scheme with $\operatorname{depth} \mathcal{G} \geq d - 1$. Hence $x - a_d y \in (a_i + a_d c_i \mid 1 \leq i \leq d-1) I \subseteq QI$ so that $x \in QI$. Thus $I^2 = QI$.

Corollary (3.7). *Suppose $r(A) \leq d$. Then $I^2 = QI$ if $\operatorname{Proj} \mathcal{R}$ is a Cohen-Macaulay scheme and $\mathcal{S}$ is a Cohen-Macaulay ring.*

Proof. We may assume $I \neq Q$. Hence $\mu_A(I) \geq d + 1$ and the assertion follows from (3.6).

Corollary (3.8). *Suppose $d = 3$ and $r(A) \leq 3$. Then $\mathcal{R}$ is a Cohen-Macaulay ring if and only if $\operatorname{Proj} \mathcal{R}$ is a Cohen-Macaulay scheme and $a_i(\mathcal{R}) < 0$ for all $i \in \mathbb{Z}$.*

Proof. See (2.9), (3.5), and (3.7).

4. Example.

Let k be an algebraically closed field. Let $R = k[X,Y,Z,V,A,B,C]$ be the polynomial ring in 7 variables over k and let

$$\mathfrak{a} = (X,Y,Z)\cdot(X,Y,Z,V) + (V^2 - (AX + BY + CZ)).$$

We put $S = R/\mathfrak{a}$ and let x, y, z,..., c denote respectively, the reductions of X, Y, Z,..., C mod $\mathfrak{a}$. Let $M = S_+$, $\mathcal{O} = S_M$, and $\mathfrak{m} = MS_M$. We put $Q = (a,b,c)\mathcal{O}$ and $I = Q + v\mathcal{O}$. Then we have

Example (4.1). (1) $(\mathcal{O},\mathfrak{m})$ is a Cohen-Macaulay local ring of $\dim \mathcal{O} = 3$.

(2) $\mathfrak{m}^2 = Q\mathfrak{m}$, $I^3 = QI^2$, and $\mathfrak{m}I = \mathfrak{m}Q$. But $I^2 \neq QI$. Hence the rings $\mathrm{R}(I)$ and $\mathrm{G}(I)$ are not Cohen-Macaulay.

(3) $\mathrm{e}_{\mathfrak{m}}(\mathcal{O}) = 5$ and $\mathrm{r}(\mathcal{O}) = 4$.

(4) $\operatorname{depth}\mathrm{R}(I) = 3$ and $\mathrm{a}_3(\mathrm{R}(I)) < 0$. Hence $\mathrm{a}_i(\mathrm{R}(I)) < 0$ for all $i \in \mathbb{Z}$.

(5) $\operatorname{Proj}\mathrm{R}(I)$ is a Cohen-Macaulay scheme.

Proof. (1) (2) and (3). Let $\mathfrak{q} = (a,b,c)S$ and $J = \mathfrak{q} + vS$; hence $M = J + (x,y,z)S$ and $MJ = M\mathfrak{q}$. Let $P = (X,Y,Z,V)R$. Then $P = \sqrt{\mathfrak{a}}$ so that $\dim\mathcal{O} = \dim S = 3$. Since $v^2 = ax + by + cz$ and $v^3 = 0$, we get $M^2 = \mathfrak{q}M$ and $J^3 = \mathfrak{q}J^2$. Therefore a, b, c form a homogeneous system of parameters for S with $S/\mathfrak{q} \cong k[X,Y,Z,V]/(X,Y,Z,V)^2$, whence $\ell_S(S/\mathfrak{q}) = 5$. Consequently, to see that S is a Cohen-Macauly ring, it suffices to show

Claim (4.2). $\mathrm{e}_{\mathfrak{q}}(S) = 5$.

Proof of Claim (4.2). We have $\mathrm{e}_{\mathfrak{q}}(S) = \ell_{R_P}(R_P/\mathfrak{a}R_P)$, because $P = \sqrt{\mathfrak{a}}$ and $R/P \cong k[A,B,C]$. Let $\tilde{k} = k[C,1/C]$ and $\tilde{R} = R[1/C]$. Then $\tilde{R} = \tilde{k}[X_1,Y_1,Z_1,V_1,A_1,B_1]$ where $X_1 = X/C$, $Y_1 = Y/C$, $Z_1 = Z/C$,..., and $B_1 = B/C$. As $\mathfrak{a}\tilde{R} = (X_1,Y_1,Z_1)\cdot(X_1,Y_1,Z_1,V_1) + (V_1^2 - (A_1X_1 + B_1Y_1) - Z_1)$ and as X_1, Y_1, Z_1, V_1, A_1, and B_1 are algebraically independent over $\tilde{k}$, substituting Z_1 with $V_1^2 - (A_1X_1 + B_1Y_1)$, we get the identification

$$\tilde{R}/\mathfrak{a}\tilde{R} = \tilde{k}[X_1,Y_1,V_1,A_1,B_1]/(X_1,Y_1,V_1)(X_1,Y_1,V_1^2).$$

Let T denote the ring of the right hand side. Then the ideal $P\tilde{R}/\mathfrak{a}\tilde{R}$ corresponds, via the identification, to the prime ideal $\mathfrak{p} = (X_1,Y_1,V_1)T$ so that, counting the number of the surviving monomials in X_1, Y_1, and V_1, we readily get $\ell_{R_P}(R_P/\mathfrak{a}R_P) = \ell_{T_\mathfrak{p}}(T_\mathfrak{p}) = 5$. Hence $\mathrm{e}_{\mathfrak{q}}(S) = 5$ and S is a Cohen-Macaulay ring.

Suppose $v^2 \in \mathfrak{q}J$ and write $v^2 = av_1 + bv_2 + c_3v_3$ with $v_i \in J$. Then since $ax + by + cz = av_1 + bv_2 + c_3v_3$ and since a, b, c is an S-regular sequence, we have $z - v_3 \in (a,b)S$.

Consequently $Z \in (A, B, C, V)R + \mathfrak{a}$, which is impossible because the ideal $\mathfrak{a}$ is generated by forms of degree 2. Hence $v^2 \notin \mathfrak{q}J$ so that we have $I^2 \neq QI$. Therefore by (2.9) the rings $\mathrm{R}(I)$ and $\mathrm{G}(I)$ cannot be Cohen-Macaulay. As $\mathfrak{m}^2 = Q\mathfrak{m}$, we get $\mathrm{r}(\mathcal{O}) = \mathrm{r}(\mathcal{O}/Q) = \ell_{\mathcal{O}}(\mathfrak{m}/Q) = 4$. Of course $\mathrm{e}_{\mathfrak{m}}(\mathcal{O}) = \mathrm{e}_Q(\mathcal{O}) = 5$ by (4.2).
(4) We need the following.

Claim (4.3). *$aS \cap J^n = aJ^{n-1}$ and $(a,b) \cap J^n = (a,b)J^{n-1}$ for all $n \in \mathbb{Z}$.*

Proof of Claim (4.3). We may assume $n \geq 2$. Firstly we will check the second equality. Since $J^2 = \mathfrak{q}J + v^2S$, we have $(a,b)S \cap J^2 = (a,b)J + (a,b)S \cap (cJ + v^2S)$. Let $\varphi \in (a,b)S \cap (cJ + v^2S)$ and write $\varphi = ci + v^2\xi$ with $i \in J$ and $\xi \in S$. Then because $v^2 = ax + by + cz$ and $\varphi \in (a,b)S$, we see $c(i + z\xi) \in (a,b)S$ so that $i + z\xi \in (a,b)S \subseteq J$; hence $z\xi \in J$. As $z \notin J$, this forces $\xi \in M = J + (x,y,z)S$. Let $\xi = j + (\alpha x + \beta y + \gamma z)$ with $j \in J$ and $\alpha, \beta, \gamma \in S$. Then $\varphi = a(xj) + b((yj) + c(i + zj)$ because $v^2\xi = (ax + by + cz)j$. Consequently $i + zj \in (a,b)S$ as $\varphi \in (a,b)S$, whence $\varphi \in (a,b)J$. Thus $(a,b)S \cap (cJ + v^2S) \subseteq (a,b)J$ so that we have $(a,b)S \cap J^2 = (a,b)J$. Now let $n \geq 3$ and suppose that $(a,b)S \cap J^{n-1} = (a,b)J^{n-2}$. Then because $J^n = \mathfrak{q}J^{n-1}$, we see

$$\begin{aligned}
(a,b)S \cap J^n &= (a,b)J^{n-1} + (a,b)S \cap cJ^{n-1} \\
&= (a,b)J^{n-1} + c[(a,b)S \cap J^{n-1}] \\
&= (a,b)J^{n-1} + c \cdot (a,b)J^{n-2} \qquad \text{(by the hypothesis of induction on } n) \\
&= (a,b)J^{n-1}.
\end{aligned}$$

This proves the second equality. The first one easily follows, by induction on n, from the second.

By (4.3) and [VV; (2.7)] we get $\operatorname{depth} \mathrm{G}(I) = 2$, since by (2) $\mathrm{G}(I)$ is not Cohen-Macaulay. Therefore $\operatorname{depth} \mathrm{R}(I) = 3$ by (2.5)(3). On the other hand, by (2.7) $\mathrm{S}(I)$ is a Cohen-Macaulay ring of $\mathrm{a}(\mathrm{S}(I)) = \mathrm{r}_Q(I) - 3 = -1$. Hence $\mathrm{a}_3(\mathrm{R}(I)) < 0$ by (2.4)(3) and (4); so $a_i(\mathrm{R}(I)) < 0$ for all $i \in \mathbb{Z}$.
(5) By [K; (2.13)] this follows from (4.3) (recall that the field $k = \mathcal{O}/\mathfrak{m}$ is algebraically closed).

This example (4.1) shows the assumption in (3.7) and (3.8) that $\mathrm{r}(A) \leq d$ is not superfluous. It also provides with a counterexample the main conjecture explored by [KN].

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可換環上の単純拡大について

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この一連の研究に共通の記号，設定がなされていますので，まずそれからはじめましょう。なお，紙面のつごう上証明ははぶきます．証明は参考文献をみて下さい．

Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. Our special notations are indicated below, and our general reference for unexplained technical terms is [M1].

In what follows, we use the following notations unless otherwise specified :

R : a Noetherian integral domain,
$K := K(R)$: the quotient field of R,
$\overline{R}$: the integral closure of R in K,
L : an algebraic field extension of K,
α : a non-zero element of L,
$d = [K(\alpha) : K]$,
$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \ldots + \eta_d$, the minimal polynomial of α over K.
$\zeta_i := \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i \quad (1 \le i \le d-1)$,
$I_{[\alpha]} := \bigcap_{i=1}^{d}(R :_R \eta_i)$, which is an ideal of R.
$I_a := R :_R aR$ for $a \in K$.

It is clear that for $a \in K, I_{[a]} = I_a$ from definitions.

Let R be a Noetherian domain, K its quotient field. Take an element α in a field extension of K. Let $\pi : R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. The element α is called an *anti-integral* element of degree d over R if $Ker\,\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an *anti-integral extension* of R. (See [OSaY] for detail.)

$J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \cdots, \eta_d)$，これは $I_{[\alpha]}\varphi_\alpha(X)$ の係数イデアルです。

grade $J_{[\alpha]} > 1$ のとき，α は super-primitive element とよび，super-primitive のとき，anti-integral になります。

まず anti-integral の特徴付けをしましょう。

Lemma

i) α が R 上 anti-integral $\Leftrightarrow$ α^{-1} は R 上 anti-integral

ii) α は R 上 anti-integral $\Leftrightarrow$ $^\forall \mathfrak{p} \in Dp_1(R)$ に対して，α は $R_\mathfrak{p}$ 上 anti-integral.

anti-integral という名前をつけたのは，$\alpha \in K$ の場合，
α が integral かつ anti-integral $\Leftrightarrow$ $\alpha \in R$ だからです．
α が high degree のときは次が与えられます。

Proposition

i) α が integral かつ anti-integral $\Leftrightarrow$ $A = R[\alpha] = R + R\alpha + \cdots + R\alpha^{d-1}$ free R-module

そこで

ii) α が anti-integral のとき，

$$\{ \mathfrak{p} \in \operatorname{Spec} R \mid A_{\mathfrak{p}} \text{ は } R_{\mathfrak{p}} \text{ 上 integral} \} = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \not\supset I_{[\alpha]} \}$$

すなわち α が integral $\Leftrightarrow$ $I_{[\alpha]}$.

integrality の obstruction ideal は $I_{[\alpha]}$ で与えられます。

ideal $J_{[\alpha]}$ は重要な働きをします。

Theorem

α は R 上 anti-integral とすれば，$A = R[\alpha]$

i) $A/J_{[\alpha]}A \cong R/J_{[\alpha]}[T]$，$T$ は変数，そして

ii) $\{ \mathfrak{p} \in \operatorname{Spec} R \mid A_{\mathfrak{p}}/R_{\mathfrak{p}} \text{ は flat} \} = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \not\supset J_{[\alpha]} \}$

$= \{ \mathfrak{p} \in \operatorname{Spec} R \mid \dim_{k(\mathfrak{p})} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} < +\infty \}$ （dim は ベクトル space の次元）

従って

iii) A/R は flat $\Leftrightarrow$ $J_{[\alpha]} = R$

$J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1})$ なので，このとき $I_{[\alpha]}$ は invertible ideal.

anti-integral の概念を得たのは Mirbagheri and Ratlif さんの論文を読んで与えられたインスピレーションでした。それは拡大次数 $d=1$, すなわち $\alpha \in K$ のとき,

$$R[\alpha] \cap R[\alpha^{-1}] = R \iff \mathrm{Ker}\,\pi = I_\alpha (X-\alpha) R[X]$$

という結果ですが, はじめ, 左側にばかり気がいっていたのですが発想の逆転といいますか, 右側の定義を High degree にもっていったのがよかったわけです。

そこで $d \geq 1$ について $B = R[\alpha] \cap R[\alpha^{-1}]$ はどうなっているかというと, 1ページの記号を使って, α が anti-integral のとき $B = R + I_{[\alpha]}\zeta_1 + I_{[\alpha]}\zeta_2 + \cdots + I_{[\alpha]}\zeta_{d-1}$ (直和)

ですから, B は flat R-module $\iff I_{[\alpha]}$: invertible ideal

従って A/R が flat なら B/R は flat です。

$\varphi : \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ の image について考えてみましょう。

Proposition

α: anti-integral のとき

$$\mathrm{Im}\,\varphi = D(I_{[\alpha]}) \cup V(J_{[\alpha]})$$

そして,

$$\tilde{J}_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_{d-1}) \supset J_{[\alpha]}$$

とすれば

$$\{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p}A = A\} = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p} \supset \tilde{J}_{[\alpha]} \text{ で } \mathfrak{p} + J_{[\alpha]} = R\}$$

従って A/R が flat のときは

$$\{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p}A = A\} = V(\tilde{J}_{[\alpha]})$$

unramified について考えてみましょう.

$\Omega_R(A) \cong A/I_{[\alpha]}\varphi_\alpha'(\alpha)A$　（$\varphi_\alpha'(X)$ は $\varphi_\alpha(X)$ の微分）ですから

A/R : unramified $\Leftrightarrow$ $I_{[\alpha]}\varphi_\alpha'(\alpha)A = A$

このとき A/R は flat であることが出ますので étale になります。

又少し条件をつければ Purity of blanch locus (i.e., $^\forall \mathfrak{z} \in D_{h_1}(R)$ 上で unramified なら A/R は unramified) が成り立ちます。

B/R は通常 unramified になりません，というのは $I_{[\alpha]}$ の部分が確実に ramify するからです.

B/R の ramification locus は $I_{[\alpha]}^d \varphi_\alpha'(\alpha)B$ で与えられます。

$A \cap K$ について考えてみましょう，このときは $d>1$ のときでないと面白くない。

Proposition

α が R 上 super-primitive element のとき次は同値

(i)　$A \cap K = R$

(ii)　$\bigcap_{i=1}^{d-1} I_{\eta_i} \subsetneqq I_{\eta_d}$

(iii)　grade $J_{[\alpha]} > 1$

$B = R[\alpha] \cap R[\alpha^{-1}]$ は R 上 integral ですが、B が $A = R[\alpha]$ の中で integrally closed である条件を与えることが出来ました。従って下で述べる条件のときには R の A の中での integral closure は B であるといえます。

まず次の結果を述べておきましょう。

Lemma

$\overline{R}$ を R の K の中での integral closure とし、finite R-module とする。$\mathfrak{c}(\overline{R}/R) = R :_R \overline{R}$.

α が super-primitive のとき

$I_{[\alpha]}$ は ideal として、integrally closed $\Leftrightarrow$ $\mathrm{grade}(I_{[\alpha]} + \mathfrak{c}(\overline{R}/R)) > 1$.

このとき

$$\mathrm{Ass}_R(R/I_{[\alpha]}) \ni \mathfrak{p} \Leftrightarrow R_{\mathfrak{p}} \text{ は DVR で } \mathfrak{p} \supset I_{[\alpha]}$$

Theorem

$\overline{R}$ は finite R-module で α は super-primitive element

このとき

B は A の中で integrally closed $\Leftrightarrow$ $\mathfrak{p} \in \mathrm{Ass}_R(R/I_{[\alpha]})$ について次の2つの内どちらかが成り立つ

a) $I_{[\alpha]}$ は integrally closed as ideal で $(I_{[\alpha]})_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$

b) $I_{[\alpha]}$ は integrally closed as ideal で $(I_{[\alpha]})_{\mathfrak{p}} = (I_{\eta_1})_{\mathfrak{p}}$

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Generic hyperplane section of complete intersections of height three

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Let (A, m, k) be an Artinian ring. In [3] I proved that the ineqality $\mu(I) \leq$ length $(A/(\ell))$ holds for any element $\ell \in m$ and for any ideal $I \subset A$. This led me to the consideration of the problem: For which Artinian rings (A, m, k) is it true that

$$\mathrm{Max}\{\mu(I) | I \subset A\} = \mathrm{Min}\{\mathrm{length}(A/(\ell)) | \ell \in m\} \tag{1}$$

For simplicity we will consider only homogeneously graded Artinian rings (A, m, k), i.e., $A = \bigoplus_{i=0}^{c} A_i$, $m = \bigoplus_{i=1}^{c} A_i$, with $A_c \neq 0$ and $A = k[A_1]$ where $k := A_0$ is a field of characteristic 0. We will say that (A, m, k) satisfies the weak Lefschetz condition (WLC) if there is an element $\ell \in A_1$ such that the map induced by the multiplication by ℓ

$$\ell : A_i \to A_{i+1}$$

is either injective or surjective for each i. Note that the WLC on A implies A has a unimodal Hilbert function.

The WLC seems to be the quickest way to prove the equlity (1), since we have

$$s(A) \leq \mathrm{Max}\{\mu(I)\} \leq \mathrm{Min}\{\mathrm{length}(A/(\ell))\} \leq s(A),$$

where $s(A) = \mathrm{Max}\{\dim_k A_i\}$.

Here are some known results.

- $A = k[x,y]/I$, I any Gorenstein ideal. (A stronger statement is proved in Iarrobino [1].)
- $A = k[x_1, x_2, \cdots, x_n]/I$, $I = (x_1^{d_1}, x_2^{d_2}, \cdots, x_n^{d_n})$ (We will call this a monomial complete intersection.)
- $A = k[x_1, x_2, \cdots, x_n]/J$, $J = (x_1^{d_1}, x_2^{d_2}, \cdots, x_n^{d_n}) : f$,
  where f is a homogeneous element of any degree, which is general enough. (We will call such an ideal a Gorenstein ideal of general type.) For details see [2].)

Hidemi Ikeda has constructed a Gorenstein ring of embedding codimenstion of 4 which does not satisfy WLC and which has a unimodal Hilbert function. (Her example is not a complete intersection.)

Here are some open problems:

Problem 1. Does any complete intersection of any embedding codimension satisfy WLC?

Problem 2. Does any height three Gorenstein ideal satisfy WLC?

Problem 3. Is the Hilbert vector of any Gorenstein ring of embedding codimension 4 a unimodal sequence?

Recently I proved the following

Theorem 1 *Let $R = k[x,y,z]$ be the polynomial ring over a field k of characteristic 0. Let I be a complete intersection ideal of R generated by homegeneous elements $f_1, f_2, f_3 \in R$ of degrees d_1, d_2, d_3 respectively, where we assume that $2 \leq d_1 \leq d_2 \leq d_3$. Then the following conditions are equivalent.*

(i) $\mu(I + \ell R/\ell R) = 3$ *for any generic linear form $\ell \in R$.*

(ii) $d_3 \leq d_1 + d_2 - 2$.

Here the meaning of a generic linear form is this: Let ξ, η, ζ be indeterminates over R and $k' = k(\xi, \eta, \zeta), R' = k'[x, y, z]$. Since $R/J \to R'/JR'$ is faithfully flat we may replace R by R' without affecting our situation. The element $\ell = \xi x + \eta y + \zeta z$ is called a generic linear form of R. Actually it is an element of R' but we treat it as though it is an element of R. For proof of the Theorem see [4]. Below I will explain how this theorem is used to prove the WLC for certain cases for complete intersections in $k[x, y, z]$.

Proposition 2 *As in Theorem 1 let $R = k[x, y, z]$ be the polynomial ring over a field k of characteristic 0. Let I be a complete intersection ideal of R generated by homogeneous elements $f_1, f_2, f_3 \in R$ of degrees d_1, d_2, d_3 respectively, where we assume that $2 \le d_1 \le d_2 \le d_3$. Put $A = R/I$. Let $h_i = dim_k A_i$. Let "$^-$" denote the reduction by a generic linear form. Then the following conditions are equivalent.*

(i) *A satisfies the WLC.*

(ii) *$s(A) = length(\bar{A})$.*

(iii) *The sequence $h_0, h_1 - h_0, h_2 - h_1, \cdots$ (with non positive parts ignored) is the Hilbert function of $\bar{A}$.*

(iv) *Either $d_3 > d_1 + d_2 - 2$ or if e_1 and e_2 are the relation degrees of $\bar{f}_1, \bar{f}_2, \bar{f}_3$ over $\bar{R}$ (which is a two dimensional polynomial ring), then $|e_1 - e_2| \le 1$.*

Proof. The equivalence of (i), (ii) and (iii) are straightforward. Let us prove that (i) implies (iv). If $d_3 > d_1 + d_2 - 2$, then there is nothing to prove. Assume $d_3 \le d_1 + d_2 - 2$. By Theorem 1 we have that $\mu(\bar{I}) = 3$. Let

$$0 \to \bar{R}(-e_1) \oplus \bar{R}(-e_2) \to \bar{R}(-d_1) \oplus \bar{R}(-d_2) \oplus \bar{R}(-d_3) \to \bar{I}$$

be a minimal free resolution of $\bar{I}$. Since $^-$ is the reduction by a generic linear form the ideal $\bar{I}$ is a height 2 perfect ideal. The WLC implies that length$(\bar{A}) = s(A)$, which is the least possible number for length$(\bar{A})$ expected only from the Hilbert function of A. Thus it implies that $|e_1 - e_2| \le 1$. (Note that the greater the value $|e_1 - e_2|$ is the length $\bar{A}$ is the greater. Also note that since $\sum e_i = \sum d_j$, $\sum d_j =$ even $\Rightarrow |e_1 - e_2| = 0$ and $\sum d_j =$ odd $\Rightarrow |e_1 - e_2| = 1$.)

Now we prove that (iv) implies (i). Assume that $d_3 \leq d_1 + d_2 - 2$. Let $J = (x^{d_1}, y^{d_2}, z^{d_3})$ and $B = R/J$. We know that B satisfies the WLC, hence the relation degrees e_1, e_2 for $\bar{J}$ satisfy $|e_1 - e_2| \leq 1$ by the implication (i) $\Rightarrow$ (iv). Since the Hilbert function of $\bar{A}$ is the same as that of $\bar{B}$ it implies that A satisfies the WLC.

It remains to prove that if $d_3 > d_1 + d_2 - 2$ then A satisfies the WLC. Let $A' = R/(f_1, f_2)$. For $i < d_3$, we may identify $A'_i = A_i$, and the multiplication $\ell : A_{i-1} \longrightarrow A_i$ may be regarded the same as that for A'. Since A' is a complete intersection of dimension 1, we have that the multiplication $\ell : A_{i-1} \longrightarrow A_i$ is injective for $i < d_3$. Note $d_3 \geq d_1 + d_2 - 3 \Rightarrow d_3 \geq (d_1 + d_2 + d_3 - 3)/2$, which is the socle degree of A. It follows that $\ell : A_{i-1} \longrightarrow A_i$ is injective for at least first half of the graded pieces. For the rest it is surjective by the duality of a Gorenstein algebra. Hence the WLC follows.

vspace2ex Now we can use Theorem 1 and Propsition 2 to prove WLC for certain cases in $R = k[x, y, z]$.

Corollary 3 *Let* $I = (f_1, f_2, f_3) \subset R$ *be a regular sequence with degrees* d_1, d_2, d_3. *Assume that* $d_3 \geq Max\{d_1, d_2\}$. *If* $d_3 \geq d_1 + d_2 - 3$ *then the Weak Lefschetz condition holds on the ring* R/I.

Proof. (i) The case $d_3 > d_1 + d_2 - 2$ was explained in Proposition 2.

(ii) Assume that $d_3 = d_1 + d_2 - 2$. Denote by "$-$" the reduction by a generic linear element. One sees easily that $\bar{f}_3$ is a generator of the socle of $\bar{R}/(\bar{f}_1, \bar{f}_2)$. Hence we have $\bar{x}\bar{f}_3 \in (\bar{f}_1, \bar{f}_2)$ and $\bar{y}\bar{f}_3 \in (\bar{f}_1, \bar{f}_2)$, which gives two (independent) syzygies of the same degree. Thus by Proposition 2 the WLC follows.

(iii) Assume that $d_3 = d_1 + d_2 - 3$. We may assume that $\bar{f}_1, \bar{f}_2$ are a regular sequence. Then we have that $\bar{x}\bar{f}_3$ and $\bar{y}\bar{f}_3$ are linearly dependent modulo $(\bar{f}_1, \bar{f}_2)$ as they are in the socle of $\bar{R}/(\bar{f}_1, \bar{f}_2)$. This gives a syzygy of degree $d_3 + 1$. The degree of another basic syzygy is automatically $d_3 + 2$. Hence by Proposition 2 we have the WLC.

Remark 4 *By the Corollary above we see that if* $deg(f_1) = deg(f_2) = 3$ *then WLC holds with any* f_3 *(i.e. independent of* $deg(f_3)$*), provided that they are a regular sequence. Also if* $deg(f_1) = deg(f_2) = 4$ *then WLC holds with any* $d_3 = deg(f_3)$ *except for* $d_3 = 4$. *Thus the unknown easiest case for WLC is the case* $d_1 = d_2 = d_3 = 4$.

Here is another consequence of Thoerem 1.

Corollary 5 *Let $R = k[x, y, z]$ be the polynomial ring over a field k of characteristic 0. Let I be a complete intersection ideal of R generated by homogeneous elements $f_1, f_2, f_3 \in R$ of degrees d_1, d_2, d_3 respcetively, where we assume that $2 \leq d_1 \leq d_2 \leq d_3$. We have*

(i) $d_3 > d_1 + d_2 - 2 \leq d_3 \Rightarrow I : \ell$ *is generated by 3 elements.*

(ii) $d_3 \leq d_1 + d_2 - 2 \Rightarrow I : \ell$ *is generated by 5 elements.*

For proof see [4].

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Zariski問題に関連した永田の問題について

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歴史的な経緯の概略を述べたいのだが、先ずHilbertの第14問題としてよく知られている次のことから始めよう。

$x_1, \ldots, x_n$ は体k上代数的独立な元，Lは$k(x_1, \ldots, x_n)$のkを含む部分体としたとき，共通部分 $k[x_1, \ldots, x_n] \cap L$ はk上のアフィン環となるか。

ここで，環AがB上アフィン環であるとは，AはBを部分環として含みかつAはB上環として有限生成であることを意味する。

Zariskiはこの問題を次の形で取扱った。

体k上のアフィン正規環Aと，k上の函数体Lについて，共通部分 $A \cap L$ はk上のアフィン環となるか。

永田はこれに関して肯定的であろうという期待の下に，次の結果を得た。[2]

Dedekind環D上のアフィン正規環AとD上の函数体Lとの共通部分 $A \cap L$ の形で表わされる必要十分条件は，D上のアファイン正規環CのあるイデアルIによるイデアルトランスフォーム $(C, I)^{\sim}$ の形で表わされることである。

ここで，D上の函数体とは，D上のアファイン環の商体を意味する。

（[2] の命題の述べ方は，「Dは基礎環」で「一般に $A \cap L$ がアファイン環になるということと，一般に $(C, I)^{\sim}$ がアファイン環になるということが同値」と読める言い方ですが，証明している内容は永田先生の御指摘の通り上で述べていることのようです。）

ところが，Reesはこの永田の結果を利用して，代数幾何学的考察から次の結果を得た。[4]

Zariski 問題は、一般には否定的である。

そして、これを契機に Hilbert の第14問題の反例が永田によって構成されてゆくのである。

以上、先の永田の結果は歴史的な役割を十分果たしているのであるが、Zariski 問題が否定的であることが解かった立場から、できるだけ一般の形で証明し直そうとして次の結果を得た。

定理（永田[3]） ネーター整域 B が、後に述べる条件*)をみたせば，次は同値である。

1) B 上のアファイン環 A の正規化 $\hat{A}$ と，A の商体の部分体 L で B 上の函数体となっているものとの共通部分 $\hat{A}\cap L$ の形で表される。

2) B 上のアファイン環 C の正規化 $\tilde{C}$ のあるイデアル I によるイデアルトランスフォーム $(\tilde{C}, I)^{\sim}$ の形で表される。

条件*)とは次である。

*) B 上の任意の因子的付値環 D について，B の商体 K との共通部分 $D\cap K$ は B 上の因子的付値環となる。

ここで，D が B 上の因子的付値環とは，D が B 上有限生成な整域 C の正規化 $\tilde{C}$ の高さ 1 の素イデアル $\mathfrak{P}$ による局所化 $D=\tilde{C}_{\mathfrak{P}}$ の形をしていることである。

上定理で，証明に*)を使っているところは 1)⇒2) で，2)⇒1)には不要である。そして永田は次の問題を残した。[3]

問題　条件*)をみたす環のクラスは何か。

ネーター整域は一般に*)をみたすことを示すのが，この小講演の目的である。次の補題から始めたいが，ここで unibranched 局所整域とは，ネーター整域で

その商体での正規化が，極大イデアルを唯一つしかもたないものである。

補題　$\dim A \geq 2$ である unibranched 局所整域 $(A, \mathfrak{m})$ は，その完備化 $\hat{A}$ のどの極小素イデアル P についても，$\dim \hat{A}/P \geq 2$ となる。

証明）．A の正規化 $\bar{A}$ は擬局所環となり，長さ２以上の正則列が取れるという意味で $\operatorname{depth} \hat{\bar{A}} \geq 2$ となる。

一方，$C = \bar{A} \otimes_A \hat{A}$ とおけば，これは局所環の帰納的極限と表わされるので擬局所環となり，$\operatorname{depth} C \geq 2$ もわかる。

さて，$\hat{A}$ の極小素イデアル P について，$P \cap A = 0$ となるので，A の商体 K について，P は $K \otimes \hat{A}$ の素イデアルともみなせるが，ネーター環 $K \otimes_A \hat{A}$ の零イデアルを $0 = \mathfrak{a}' \cap \mathfrak{b}'$ と分解する。ここで，$\mathfrak{a}'$ は P に属する準素成分であり，$\mathfrak{b}'$ は その他に属する準素成分の共通部分である。

$\mathfrak{a} = \mathfrak{a}' \cap C$，$\mathfrak{b} = \mathfrak{b}' \cap C$ とおけば，C における 0 の分解，$0 = \mathfrak{a} \cap \mathfrak{b}$ が得られ，A－加群の完全列

$$0 \longrightarrow C \longrightarrow C/\mathfrak{a} \oplus C/\mathfrak{b} \longrightarrow C/\mathfrak{a}+\mathfrak{b} \longrightarrow 0$$

が得られる。

もし，$\dim \hat{A}/P = 1$ であれば，P が極小で $P \not\supset \mathfrak{b}$ より $\dim C/\mathfrak{a}+\mathfrak{b} = 0$ となり，$\operatorname{Ext}^1_A(A/\mathfrak{m}, C) \neq 0$ だが，これは $\operatorname{depth} C = 1$ を導いて矛盾を生ずる。

命題　$\dim A \geq 2$ となる unibranched な局所整域 $(A, \mathfrak{m})$ について，A 上のアファイン環 C の高さ１の素イデアル P で $\mathfrak{m}$ に乗っているものについて，

$$\operatorname{tr.deg.}_{k(\mathfrak{m})} k(P) > \operatorname{tr.deg.}_K L$$

である。但し，$k(P)$，$k(\mathfrak{m})$ はそれぞれ P，$\mathfrak{m}$ での剰余体，L, K はそれぞれ C，A の商体である。

証明）Aの完備化$\hat{A}$について，$P' = P(\hat{A} \otimes_A C)$ は $\hat{A} \otimes_A C$ の素イデアルで高さ1であることが容易にわかるから，極小素イデアル $Q' \subset P'$ で $ht\, P'/Q' = 1$ となるものをとる。

$\mathfrak{q} = Q' \cap \hat{A}$ とおけば，これは$\hat{A}$の極小素イデアルである。

実際，$Q' \cap C = 0$ となるから，Q' は $\hat{A} \otimes_A L$ の，$\mathfrak{q}$ は $\hat{A} \otimes_A K$ の素イデアルにそれぞれ対応しているが，平坦射 $\hat{A} \otimes_A K \longrightarrow \hat{A} \otimes_A L$ に going down が成り立つので，もし$\mathfrak{q}$が極小でなければ，Q'も極小でないことになり矛盾を生ずるのである。

よって補題より $\dim \hat{A}/\mathfrak{q} \geq 2$ となるが，完備環$\hat{A}/\mathfrak{q}$は universally catenary となるので，$\hat{A}/\mathfrak{q} \longrightarrow \hat{A} \otimes C/Q'$ に dimension formula を使うと（このあたりの言葉の定義や定理は［1］を参照）

$$ht\, P'/Q' = \dim \hat{A}/\mathfrak{q} + \mathrm{tr.deg}_{\kappa(\mathfrak{q})} \kappa(Q') - \mathrm{tr.deg}_{\kappa(\mathfrak{m})} \kappa(P)$$

が得られるが，Q'は $\hat{A}/\mathfrak{q} \otimes_A L$ の極小素イデアルに対応しているから，

$$\mathrm{tr.deg}_{\kappa(\mathfrak{q})} \kappa(Q') = \mathrm{tr.deg}_K L$$

を得る。よって

$$\mathrm{tr.deg}_{\kappa(\mathfrak{m})} \kappa(P) = \mathrm{tr.deg}_K L + \dim \hat{A}/\mathfrak{q} - 1 > \mathrm{tr.deg}_K L$$

となる。

定理　ネーター整域は，条件＊）をみたす。

証明）ネーター整域B上の因子的付値環$(D, \mathfrak{n})$について，$D \cap K$がB上因子的であることを示したいのだが，$D \cap K \neq K$としてよい。ここでKはBの商体である。

$D \cap K$の元をいくつかつけ加えることにより，B上のアフィン環Aで $\mathfrak{m} = A \cap \mathfrak{n}$ とおいたとき，$D \cap K / \mathfrak{n} \cap K$ は $A/\mathfrak{m}$上代数的とすることができる。必要なら，更に元をつけ加えておいて，$A_\mathfrak{m}$は unibranched 局所整域であるとしておいてよい。ここで，$\dim A_\mathfrak{m} = 1$ が示せれば $D \cap K$ は $A_\mathfrak{m}$の正規化の局所化とな

り，B上因子的になって証明が終る。

そこで，$\dim A_{\mathfrak{m}} \geq 2$ として矛盾を導こう。D は B上因子的なので，A上のアファイン環 C をうまくとれば，C の正規化 $\tilde{C}$ とその高さ1の素イデアル $\tilde{\mathfrak{p}}$ により，$D = \tilde{C}_{\tilde{\mathfrak{p}}}$ の形となる。必要なら，$\tilde{C}$ の元をいくつかつけ加えておくことにより，C_P は unibranched 局所整域であるとしておいてよい。ここで $P = \mathfrak{R} \cap C$ である。すると $ht\, P = 1$ であり $P \cap A = \mathfrak{R} \cap A = \mathfrak{m}$ である。

一方，D は $D \cap K$ 上因子的であり，離散付値環 $D \cap K$ と D には dimension formula が成り立つから，D の商体 L について

$$\mathrm{tr.deg}_K L = \mathrm{tr.deg}_{D\cap K/\mathfrak{R}\cap K} D/\mathfrak{R}$$

だが，命題を適用して $D\cap K/\mathfrak{R}\cap K$ が $\kappa(\mathfrak{m})$ 上代数的となっていることに注意すれば

$$\mathrm{tr.deg}_{\kappa(\mathfrak{m})} \kappa(P) > \mathrm{tr.deg}_K L = \mathrm{tr.deg}_{D\cap K/\mathfrak{R}\cap K} D/\mathfrak{R} \geq \mathrm{tr.deg}_{\kappa(\mathfrak{m})} \kappa(P)$$

となって矛盾を生ずる。

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与えられたベッチ数列をもつ高さ３の Gorenstein イデアルの構成

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§0 序文: $A = \oplus_{i\geq 0} A_i$ を体 k 上の標準的次数付き環とする．すなわち, $A_0 = k$, $A = k[A_1]$, $\dim_k A_1 < \infty$ である．$H(A,i) = \dim_k A_i$, $i = 0,1,2,\ldots$, を A の Hilbert 関数, $F(A,\lambda) = \sum_{i\geq 0} H(A,i)\lambda^i$ を Hilbert 級数と言う．A を Artin とする．このとき, $c(A) = \mathrm{Max}\{i \mid A_i \neq (0)\}$ とおいて, A の socle degree と言う．当然, A の Hilbert 関数は, $i > c(A)$ に対して $H(A,i) = 0$ となる．整数からなる数列 $h = \{h_0 = 1, h_1, \ldots, h_c, 0, 0, \ldots\}$ (ただし, $h_i > 0, 0 \leq i \leq c$) が, ある Gorenstein Artin 環の Hilbert 関数になっているとき, この数列 h を Gorenstein 数列と呼ぶことにする．

これまでに, Gorenstein 数列の特徴付け問題に関する仕事が, 多くの人たちによってなされてきた．次の Stanley 氏による $h_1 \leq 3$ の場合の特徴付けを与えた結果は, この問題において基本的であり, もちろんこれまでの多くの仕事に影響を与えてきたという事実からもその重要性がわかる．

Stanley's characterization ([10, Theorem 4.2]). $h = \{h_0 = 1, h_1 \leq 3, h_2, \ldots, h_c \neq 0, 0, \ldots\}$ が, Gorenstein 数列であるための必要充分条件は次の (S1) かつ (S2) である:

(S1) h : symmetric, i.e., $h_i = h_{c-i}$ for all $i = 0, 1, \ldots, [c/2]$;

(S2) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[c/2]} - h_{[c/2]-1}, 0, 0, \ldots)$ は O-sequence.

その後２０年あまり経過したが, $h_1 \geq 4$ の場合の Gorenstein 数列の特徴付け問題は未だ未解決である．

この条件 (S1) と (S2) の必要性の証明を考えるとき, 条件 (S1) は,（一般の）Gorenstein 環の self-duality から説明がつくが, 条件 (S2) に対しては, いったい Gorenstein 環のどんな性質から導けるのか, すぐには見当がつかないと思う．

前々回のシンポジウムでは, これらの２つの条件 (S1), (S2) の充分性の証明について報告した．つまり, linkage 理論の基本的事実 [9, Remark 1.4]（幾何的に link する２つの高さ n の Cohen-Macaulay イデアルの和は, 高さ $n+1$ の Gorenstein イデアルである）を使って, 与えられた (S1), (S2) をみたす数列を Hilbert 関数にもつ Gorenstein Artin 環を具体的に構成した [4].

そこで今回は, その条件の必要性の証明を, Stanley 氏とは別のアプローチによる方法（同じ Hilbert 関数をもつ Betti 数列の中で maximum なものに注目して）で与え, それらの証明を通して, Stanley's formulation （とくに条件 (S2)）に隠されている代数的な解釈（Gorenstein 環, Gorenstein イデアルが持っているかもしれない性質）を考察する．

§1 Gorenstein Artin 次数付き環 $k[x,y,z]/I$ の diagonal degrees: 以下, 簡単のために体 k は標数 0 の代数的閉体とする．I を体 k 上３変数多項式環 $R = k[x,y,z] = \oplus_{i\geq 0} R_i$, $\deg x = \deg y = \deg z = 1$, の高さ３の Gorenstein 斉次イデアルとする, i.e., $A = R/I = \oplus_{i\geq 0} A_i$ は Gorenstein Artin 次数付き環である．[11, Corollary] から I の生成元の個数 $\mu(I)$

は3以上の奇数であるので, $\mu(I) = 2m+1$ $(m \geq 1)$ とおく. A の次数付き R 加群としての極小自由分解

$$0 \longrightarrow R(-s) \longrightarrow \oplus_{i=1}^{2m+1} R(-p_i) \xrightarrow{[f_{ij}]} \oplus_{i=1}^{2m+1} R(-q_i) \longrightarrow R(0) \longrightarrow A \longrightarrow 0$$

（$q_1 \leq \ldots \leq q_{2m+1}, p_1 \geq \ldots \geq p_{2m+1}$）に登場する正の整数 (numerical characters) の列

$$\{q_1, \ldots, q_{2m+1}; p_1, \ldots, p_{2m+1}; s\}$$

を考える. さらに, 行列 $[f_{ij}]$ の対角成分 f_{ii} の次数 (diagonal degrees) を r_i とする, i.e.,

$$r_i = p_i - q_i \ (1 \leq i \leq 2m+1).$$

[1, page 466] から, numerical characters と diagonal degrees の関係は,

$$\text{(BE1)}\ q_i = \frac{1}{2}\sum_{j \neq i} r_j, \quad \text{(BE2)}\ p_i = \frac{1}{2}(r_i + \sum_{j=1}^{2m+1} r_j), \quad \text{(BE3)}\ s = \sum_{j=1}^{2m+1} r_j$$

であることが分かっている. ゆえに, A の diagonal degrees, numerical characters, Betti 数列は, 1つが分かれば他の2つも分かる. このとき, A の Hilbert 関数も当然計算できる.

diagonal degrees の特徴付け ([2, Proposition 3.1], [3, Theorem 2.1], [7, page 62-63]). 整数からなる数列 $\{r_1, \ldots, r_{2m+1}\}$ が, ある Gorenstein 環の diagonal degrees であるための必要充分条件は次の (D1), (D2) かつ (D3) である:

(D1) $r_1 \geq \ldots \geq r_{2m+1}$;
(D2) r_i はすべて偶数または奇数;
(D3) $r_1 > 0, r_2 + r_{2m+1} > 0, r_3 + r_{2m} > 0, \ldots, r_{m+1} + r_{m+2} > 0$.

次節では, 与えられた (D1-3) をみたす数列 $\{r_i\}$ を diagonal degrees にもつ Gorenstein Artin 環の具体的な構成方法を与える. この構成は, [4, Theorem 3.3] の構成方法を, より分析することによって得られる.

§2 Gorenstein イデアルの構成: [9, Remark 1.4] の系として, $\mathbf{P}^2$ の点を使って, 簡単に高さ 3 の Gorenstein イデアルの例が作れる.

補題 1. X, Y を $\mathbf{P}^2$ の有限個の点からなる集合で, $X \cap Y = \phi$, $X \cup Y$ は完全交叉とする. このとき, $A = R/I(X) + I(Y)$ は Gorenstein Artin 環である.

定義 2. 例えば, 有限個の正の整数の組 $(5,4), (3,3), (1,2)$ に対して, 次のような配置にある $\mathbf{P}^2$ の 31 個の点の集合 X たちの1つを $X = B(5,4) \cup B(3,3) \cup B(1,2)$ で表す.

X ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○ ○ ○ ○

同じようにして, $X = \cup_{i=1}^{m} B(d_i, e_i)$ を定義する. ただし, いつも $e_1 > \cdots > e_m$ とする. このような配置にある有限個の点の集合 X を pure configuration と呼ぶことにする.

定義 3. (D1-3) をみたす $\{r_1, \ldots, r_{2m+1}\}$ に対して,

$$\begin{aligned} d_i &= \tfrac{1}{2}(r_{m+2-i} + r_{m+1+i})\ (1 \le i \le m), & e_m &= \tfrac{1}{2}(r_1 + r_{2m+1}), \\ e_i - e_{i+1} &= \tfrac{1}{2}(r_{m+1-i} + r_{m+1+i})\ (1 \le i \le m-1), & d &= (\textstyle\sum_{i=1}^m d_i) + \tfrac{1}{2}(r_1 + r_{m+1}) \end{aligned}$$

なる正の整数を定義し, $X = \cup_{i=1}^m B(d_i, e_i)$ をとってきて, さらに X を含む $B = B(d, e_1)$ をとる. $Y = B \backslash X$ とおく. このような (X, Y) を $\{r_i\}$ の G-pair と呼ぶことにする.

定理 4 ([6]). (X, Y) を $\{r_i\}$ の G-pair とする. このとき, $A = R/I(X) + I(Y)$ の diagonal degrees は $\{r_i\}$ に一致する.

この定理を証明するために, 必要な補題を証明なしで（詳しくは [4,6] をご覧下さい）述べておく.

定義 3 の記号の下で, pure configuration $X = \bigcup_{i=1}^m B(d_i, e_i) \subset B = B(d, e_1)$ に対して,

$$I(B(d_i, e_i)) = (\prod_{j=v_{i-1}+1}^{v_i} (x - b_j z),\ \prod_{j=1}^{e_i} (y - c_j z)),$$

ただし $v_0 = 0$, $v_i = d_1 + \cdots + d_i (1 \le i \le m)$, をみたす b_j, $c_j \in k$ がとれる.

$$g_i = \prod_{j=v_{i-1}+1}^{v_i} (x - b_j z),\ h_i = \prod_{j=e_{i+1}+1}^{e_i} (y - c_j z)$$

$1 \le i \le m$, ただし $e_{m+1} = 0$, とおく. さらに,

$$g_{m+1} = \prod_{j=v_m+1}^{d} (x - b_j z)$$

とおく. また,

$$\begin{aligned} &G_1 = g_1 g_2 \cdots g_m, G_2 = g_1 \cdots g_{m-1} h_m, \ldots, G_m = g_1 h_2 \cdots h_m, \\ &G_{m+1} = h_1 \cdots h_m, \\ &G_{m+2} = g_2 g_3 \cdots g_{m+1}, G_{m+3} = h_1 g_3 \cdots g_{m+1}, \ldots, G_{2m+1} = h_1 \cdots h_{m-1} g_{m+1} \end{aligned}$$

とおく. このとき, 初等的な方法で次が示せる.

補題 5. $\{G_1, \ldots, G_{2m+1}\}$ は, $I(X) + I(Y)$ の極小生成系であり, $\deg G_1 \le \cdots \le \deg G_{2m+1}$ となる.

X を $\mathbf{P}^n$ の有限個の点（点の個数を $\#X$ で表す）の集合とする. X の斉次座標環 C は 1 次元 Cohen-Macaulay であるので, linear な non-zero divisor l がある. このことを使って, 以下のことが考察できる. X の Hilbert 関数（すなわち $H(X, i) = H(C, i)$）は, 充分大きな i に対して, $H(X, i) = \#X$ である. そこで,

$$\beta(X) = \mathrm{Min}\{i \mid H(X, i) = \#X\}$$

とおく. このとき,

$$H(X,0) < H(X,1) < \cdots < H(X,\beta(X)) = H(X,\beta(X)+1) = \cdots = \#X$$

となる. さらに, X の Hilbert 関数の 1 回差分

$$\Delta H(X,0), \Delta H(X,1), \cdots \Delta H(X,\beta(X)), 0, 0, \ldots$$

は, C/lC の Hilbert 関数である, すなわち O-sequence である.

X, Y を $\mathbf{P}^n$ の有限個の点の集合で, $X \cap Y = \phi$, $X \cup Y$ は完全交叉とする. $A = k[x_0, x_1, \ldots, x_n]/I(X) + I(Y)$ とおく. このとき,

補題 6 ([4]). $0 \le i \le \beta(X \cup Y)$ に対して,

$$H(A,i) = H(X,i) + H(X,\beta(X \cup Y) - 1 - i) - \#X$$

が成立する.

さらに, この補題を使って次がわかる.

補題 7. $c(A) = \beta(X \cup Y) - 1$.

定理 4 の証明の概略. まず, 次に注意する:

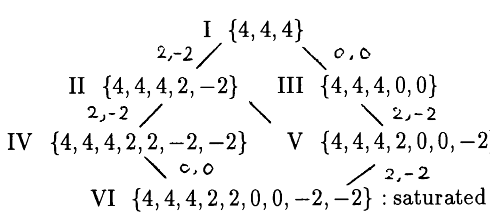
$$\deg G_i = \frac{1}{2}(r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_{2m+1});$$
$$\beta(X \cup Y) = (\sum_{i=1}^{2m+1} r_i) - 2.$$

これらと補題 5, 7 から, A の diagonal degrees が $\{r_i\}$ と一致することがわかる. Q.E.D.

例 8 (同じ Hilbert 関数をもつ diagonal degrees ([2, page 379-381])). 具体例を使って説明する. $R/(x^4, y^4, z^4)$ の Hilbert 関数 h :

h :	1,	3,	6,	10,	12,	12,	10,	6,	3,	1,	0,	0,	0,	0,	...
Δh :	1,	2,	3,	4,	2,	0,	−2,	−4,	−3,	−2,	−1,	0,	0,	0,	...
$\Delta^2 h$:	1,	1,	1,	1,	−2,	−2,	−2,	−2,	1,	1,	1,	1,	0,	0,	...
$\Delta^3 h$:	1,	0,	0,	0,	−3,	0,	0,	0,	3,	0,	0,	0,	−1,	0,	....

numerical characters : $\{4,4,4;8,8,8;12\}$.
diagonal degrees : $\{4,4,4\}$.
同じ Hilbert 関数をもつ $\{r_i\}$ は, $\{4,4,4\}$ に順次 $\{d,-d\}$ $(d \ge 0)$ を付加して, その中で, かつ (D1-3) をみたすものを見つければよい :



対応する numerical characters は：

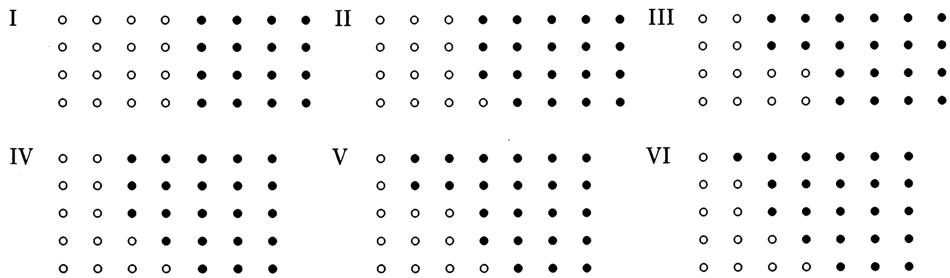
I $\{4,4,4;8,8,8;12\}$: minimum

II $\{4,4,4,5,7;8,8,8,7,5;12\}$ III $\{4,4,4,6,6;8,8,8,6,6;12\}$

IV $\{4,4,4,5,5,7,7;8,8,8,7,7,5,5;12\}$ V $\{4,4,4,5,6,6,7;8,8,8,7,6,6,5;12\}$

VI $\{4,4,4,5,5,6,6,7,7;8,8,8,7,7,6,6,5,5;12\}$: maximum

それらを実現する G-pair たちは：



定義 9 ([2]). (D1-3) をみたす $\{r_i\}$ が saturated であるとは, この数列にどんな $\{d,-d\}$ を付加しても, (D1-3) をみたし, かつ同じ Hilbert 関数をもつ数列は存在しないときに言う.

補題 10 ([2, Theorem 3.2]). $\{r_i\}$ が saturated であるための必要充分条件は, $r_i + r_{2m+3-i} = 2$ $(2 \le i \le m+1)$ である. また, どんな diagonal degrees もただ 1 つの saturation をもつ.

定理 11 ([6]). (X,Y) を saturated diagonal degrees $\{r_i\}$ の G-pair とする. $A = R/I(X)+I(Y)$ とおく. このとき, 次がわかる.

(1) $\beta(X) = (\frac{1}{2}\sum_{i\neq m+1} r_i) - 1.$

(2) $\beta(X\cup Y) = (\sum_{i=1}^{2m+1} r_i) - 2.$

(3) $c(A) = (\sum_{i=1}^{2m+1} r_i) - 3 = \beta(X\cup Y) - 1.$

(4) $\beta(X) \le \beta(X\cup Y) - 1 - \beta(X).$

(5) A の Hilbert 関数は, 次のように X の Hilbert 関数で表すことができる:

$$H(A,i) = \begin{cases} H(X,i) & 0 \le i \le \beta(X)-1; \\ \#X & \beta(X) \le i \le \beta(X\cup Y) - 1 - \beta(X); \\ H(X, c(A)-i) & \beta(X\cup Y) - \beta(X) \le i \le c(A). \end{cases}$$

(6) A は weak Lefshetz condition を充たす. すなわち, $\exists g \in A_1$ s.t $A_i \xrightarrow{g} A_{i+1}$ $(0 \leq i < c)$ は単射または全射である.

(7) A の Hilbert 関数は条件 (S1), (S2) を充たす.

注意 12. I, II, IV の例では定理 11 (5) は成立していないが, III, V の例では成立している.

補題 13. pure configuration $X = \bigcup_{i=1}^{m} B(d_i, e_i)$ に対して, 次が成立する.

(1) $F(X, \lambda) = \sum_{i=1}^{m} \lambda^{v_{i-1}} \dfrac{(1-\lambda^{d_i})(1-\lambda^{e_i})}{(1-\lambda)^3}$.

(2) $H(X, j) = \sum_{i=1}^{m} H(B(d_i, e_i), j - v_{i-1})$.

(3) $\beta(X) = \mathrm{Max}\{e_i + v_i - 2 \mid 1 \leq i \leq m\}$.

定理 11 の証明の概略. (1), (2) は補題 13 (3) からすぐ. (3) は (2) と補題 7 からすぐ. (4) は (1), (2) と "saturated" の定義からすぐ（ここに saturated が効く）. (5) は補題 6 を使う. (6) は [9] を参照. (7) は (6) からすぐ. Q.E.D.

Stanley's characterization の証明. 充分性については前々回のシンポウムで報告した ([4]). 必要性: $B = k[x, y, z]/J$ を勝手な Gorenstein Artin 次数付き環とする. B の diagonal degrees の saturation の G-pair (X, Y) をとってくる. このとき, $A = k[x, y, z]/I(X) + I(Y)$ と B の Hilbert 関数は一致するので, 定理 11 (7) から, B の Hilbert 関数は (S1), (S2) をみたすことがわかる. Q.E.D.

§3 問題. 定理 4 で構成した Gorenstein イデアルは, 以下の（問題の中で述べられる）性質を持つことが確認できる.

問題 1 ([3]). Gorenstein イデアルは, いつ幾何的に link する 2 つの Cohen-Macaulay イデアルの和として表されるか？

問題 2. saturated diagonal degrees をもつ $R = k[x, y, z]$ の高さ 3 の Gorenstein イデアルは, 幾何的に link する高さ 2 の Cohen-Macaulay イデアルの和で表されるか？

問題 3 ([8], [12,13]). Gorenstein Artin 環 $A = \oplus_{i=0}^{c} A_i$ は, いつ weak Lefschetz condition をみたすか？

問題 4. I は saturated diagoal degrees をもつ $R = k[x, y, z]$ の高さ 3 の Gorenstein イデアルとする. このとき, R/I はいつ WLC をみたすか？

定義 ([8]). $R = k[x_1, \ldots, x_n]$ の高さ n の Gorenstein イデアル I $(A = R/I = \oplus A_i)$ に対して, 次の条件をみたす R の 高さ $n-1$ の Cohen-Macaulay イデアル J が存在するとき, I は J の tight divisor と言う: $I \supset J$ であって, A の Sperner number と R/J の multiplicity が一致する. $\mathrm{Max}\{\dim A_i\}$ を A の Sperner number と言う ([12]).

問題 5. Gorenstein イデアルは, いつ Cohen-Macaulay イデアルの tight divisor になるか？

問題 6. saturated diagonal degrees をもつ $R = k[x, y, z]$ の高さ 3 の Gorenstein イデアルは, いつ tight divisor になるか？

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“悪い” 特異点の FROBENIUS 写像による特徴付け

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今まで F-terminal, F-canonical, F-regular, F-pure, F-rational などの環を扱って来た．これらの環は標数 0 のそれぞれ terminal, canonical, log terminal, log canonical, rational singularity に対応する (対応するように定義した) ものだが，これらは全体の中で見るとかなり「良い」環であり,「普通」の環はもっと「悪い」環なので以上のような環はかなり特殊なものと言える．本稿は環 (特異点) の「悪さ」を Frobenius 写像に関わるいろいろな概念でどのように表現するかの試みである．

ここでは特に，特異点の解消に於ける例外因子の “discrepancy” と Frobenius 写像の splitting に対する obstruction との関係を調べ，それを媒介にして，tight closure の理論から得られる test ideal の概念の “標数 0” での意味付けも試みる．

以下 A は標数 $p>0$ の環，Frobenius 写像 $F: A \to A$, $F(a)=a^p$ は finite map とする．また，以下常に A は reduced とする．以下 $q=p^e$ は必ず p の巾とし，次の 3 つの写像

$$F^e : A \to A, \quad A^q \hookrightarrow A, \quad A \hookrightarrow A^{1/q}$$

を同一視する．

また，イデアル I に対して $I^{[q]} = (a^q | a \in I)$ とし，I の tight closure I^* を $x \in I^* \iff \exists c \in A^0, \forall q \gg 1, cx^q \in I^{[q]}$ と定義する．(但し，A^0 を A の minimal prime に含まれない元の集合とする – A が整域のとき，$A^0 = A \setminus \{0\}$．)

1. イデアル $\mathcal{F}(A)$

$q=p^e$ に対して injection $A \to A^{1/q}$ は

$$\mathrm{Hom}_A(A^{1/q}, A) \to A \cong \mathrm{Hom}_A(A, A)$$

をひきおこす．この像を $\mathcal{F}_q(A)$ とおく．$\mathcal{F}_q(A)$ は q に関して減少列をなし，

$$\mathcal{F}_q(A) = A \iff A \to A^{1/q} \text{ が split} \iff A \text{ が F-pure}$$

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である．また，$c \in A$ に対し，

(1.2) $$c \in \mathcal{F}_q(A) \iff c : A \to A \text{ が } A \to A^{1/q} \text{ を経由して分解}$$

が言える．$\mathcal{F}_q(A)$ は A が F-pure とならない obstruction を記述している．

現在 q が増加するとき $\mathcal{F}_q(A)$ が無限減少列となる具体例は作っていないが，後述するように $\mathcal{F}_q(A)$ は A の "test ideal" を含む．従って，例えば local ring $(A, \mathfrak{m})$ に於て $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ が F-regular なら，$\mathcal{F}_q(A)$ は $q \gg 1$ で stable になる．このイデアルを $\mathcal{F}(A)$ と書く．(一般に $\mathcal{F}(A) = \cap_q \mathcal{F}_q(A)$ とおく．)

定義より，$\mathcal{F}_q(A)$ は局所化と可換である．

2. 剰余体の injective envelope を用いた $\mathcal{F}(A)$ の判定.

$(A, \mathfrak{m})$ を標数 $p > 0$ の Noetherian local ring, $E := E_A(A/\mathfrak{m})$ を剰余体の injective envelope とする．F-pure の判定と同様にして次が示せる ([Fe] Lemma 1.2 参照).

Lemma 2.1. 次の条件は同値

(1) $c \in \mathcal{F}_q(A)$

(2) $c.\mathrm{Ker}\,(E \to E \otimes_A A^{1/q}) = (0)$

$\mathrm{Ker}\,(E \to E \otimes_A A^{1/q})$ の (q に関する) 和集合を E に於ける (0) の Frobenius closure と言い，$(0)^F_E$ と書く．

一方，イデアルの tight closure に関連して test ideal の概念がある．

定義. $c \in A$ が *test element* $\iff$ 任意のイデアル I と任意の $x \in I^*$ に対して $cx \in I$. A の test elements で生成されたイデアルを *test ideal* と言い，$\tau(A)$ と書く．

$\tau(A) = A \iff A$ のすべてのイデアルが tightly closed $\iff A$ は weakly F-regular.

test ideal は E での (0) の tight closure $(0)^*$ の annhilator であるから ([HH], (8.23) – 正確には $(0)^{*f.g.}_E$) $(0)^* \subset (0)^F$ より $\mathcal{F}(A) \supset \tau(A)$ がわかる．この稿では $\tau'(A) = \mathrm{Ann}_A(0)^*_E$ と定義しておく．従って $\tau'(A)$ は$\tau(A)$ より真に小さい可能性がある (まだそういう例はないし，多分弱い条件の下で一致すると思われるが). A の canonical class が有限位数をもつとき $\tau'(A) = \tau(A)$ が証明されたという情報がある．

(2.2) $\tau'(A)$ が $\mathfrak{m}$-primary のとき，$\tau'(A) \supset \mathfrak{m}\mathcal{F}(A)$. 従って $\mathcal{F}(A) \supset \tau'(A) \supset \mathfrak{m}\mathcal{F}(A)$.

[証明] $x \in (0)_E$ と $a \in \mathfrak{m}$ に対して $ax \in (0)^F$ が言えれば良い．$a^q \in \tau(A)$ となる $q = p^e$ を取ると，$F^e(ax) = a^q F^e(x) = 0$ となり $ax \in (0)^F$ が示せる．

次の事実は Huneke の今年の Barcelona conference での lecture で注意されている ([Hu]).

系 2.3. $\tau(A)$ が $\mathfrak{m}$-primary で (即ち，素イデアル $\mathfrak{p} \neq \mathfrak{m}$ に対して $A_{\mathfrak{p}}$ が F-regular) もし A が F-pure なら $\tau(A) = \mathfrak{m}$.

$\tau(A) = \mathfrak{m}$ が成立すると，A のパラメーター系 $(x_1, \dots, x_d)$ に対し，"tight closure はコロンを含む" という事から $(x_1, \dots, x_{i-1}) : x_i \subset (x_1, \dots, x_{i-1})^*$ で，$(\quad)^*$ は test ideal で消せるので，任意のパラメーター系が weakly regular sequence になり，A は Buchsbaum ring である．

例えば，"土橋カスプ特異点" はこの性質を持つと思われる (Buchsbaum 性は石田正典氏によって示されている)．標数 0 のアーベル多様体の cone, "simple K3 特異点" 等の mod p reduction なども $p \gg 0$ のとき $\tau(A) = \mathfrak{m}$ が成り立つと予想される．これらの特異点は "無限個の p に対して" F-pure であると思われるが，こちらの証明は難しそうだ．

3. Discrepancy との関係.

A が normal local, $(A, \mathfrak{m})$ に於て $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ が F-regular と仮定する．

$$f : X \to Y := \mathrm{Spec}(A)$$

が projective birational map, X は normal, Gorenstein とする．更に A の canonical class $cl(K_A)$ が $Cl(A)$ で位数 r の torsion とすると，$K_X = f^*(K_Y) + \sum a_i E_i$ となる $a_i \in \mathbb{Q}$ $(ra_i \in \mathbb{Z})$ が定義できる．(ここで E_i は exceptional divisor をすべて動くとする．) この a_i は E_i の *discrepancy* と呼ばれている．

定理 (3.1) 各 E_i に対し，

$$b_i := \min\{v_{E_i}(x) \mid x \in \mathcal{F}(A)\}$$

とおくと，各 i に対し，$a_i \geq -b_i - 1$ である．

[証明] この証明は本質的に A が F-regular (resp. F-pure, F-terminal) ならば各 i に対し，$a_i > -1$ (resp. $\geq -1, > 0$)，即ち A が log-terminal (resp. log-canonical, terminal) を示したのと同一である．

$c \in \mathcal{F}(A), v_{E_i}(c) = b_i$ を取る．$q \gg 1$ とし，埋め込み $i : A \to A^{1/q}$ と $\phi : A^{1/q} \to A$ との合成が c による乗法とする．ϕ を商体に拡大したと考えておく．さて，adjunction formula より

$$\mathrm{Hom}_{O_X}(O_X^{1/q}, O_X) \cong \mathrm{Hom}_{O_X}(O_X^{1/q}, O_X(K_X)) \otimes_{O_X} O_X(-K_X)$$
$$\cong O_X(K_X)1/q \otimes_{O_X} O_X(-K_X) \cong O_X((1-q)K_X)^{1/q}$$

である．一方，同様に $\phi \in O_Y((1-q)K_Y)^{1/q}$ だから，$\phi \in O_X((q-1)(\sum a_i E_i))^{1/q})$．(本当は a_i は分数だから，切り捨て等の議論をする必要があるが，$q \gg 1$ のときは結果に影響しない．)

ξ を E_i の generic point とする．ξ での $\mathrm{Hom}_{O_X}(O_X^{1/q}, O_X)$ の生成元を β とすると，ξ に於て $\phi = s\beta$ と書くと $s \in \mathfrak{m}^{(q-1)(-a_i)}$ である．$a_i < -b_i - 1$ と仮定すると，$(q-1)(-a_i) > q(b_i + 1)$ となり，$c = \phi i$ の E_i での value が b_i であることに反する．

4.　種々のイデアルの間の関係.

(4.1) $(A, \mathfrak{m})$ が normal, $K_A \cong A$, $\tau(A)$ が $\mathfrak{m}$-primary，$\dim A = d$ のとき次の 3 つのイデアルは次の意味で「同じ」である．(但し (1), (2) は "標数 0" のとき，(3), (4) は標数 $p > 0$ のとき．)

(1)　特異点の解消 $f : X \to Y := \mathrm{Spec}(A)$ に対し，$\mathrm{Ann}_A(H^{d-1}(X, O_X)$ (2)　特異点の解消 $f : X \to Y := \mathrm{Spec}(A)$ に対し，$f_*(\omega_X) = H^0(X, \omega_X) \subset K_A \cong A$.

(3)　A の test ideal $\tau(A)$ ($c \in A$ が $c \in \tau(A)$ とは任意のイデアル I と任意の $x \in I^*$ (tight closure of I) に対し $cx \in I$ となること).

まず Grauert-Riemenshneider 消滅定理の下に (1) = (2)，A が標数 0 のとき，reduction mod p, $p \gg 0$ に対して [Ha] により (1) = (3) が云える．

一般の A に対しては (1), (2) と (3) の関係は (1), (2) が (3) に含まれることしかわからないが (例えば A が rational singularity のとき (1) のイデアルは A だが，A は F-regular とは限らないから一般に $\tau(A) \neq A$)，A が normal で canonical class の位数が有限のとき，$r = \mathrm{ord}(cl(K_A))$ とおき，canonical cover

$$B = \oplus_{n \in \mathbb{Z}_r} B_n = \oplus_{n \in \mathbb{Z}_r} K_A^{(n)}$$

を考える ($K_A^{(n)}$ は K_A^n の divisorial hull).

$$K_B = \mathrm{Hom}_A(B, K_A) \cong \oplus_{n\in\mathbb{Z}_r}\mathrm{Hom}_A(K_A^{(-n)}, A),$$

$$E_B = H^d_{\mathfrak{m}}(K_B) = \oplus_{n\in\mathbb{Z}_r} H^d_{\mathfrak{m}}(K_A^{(n)})$$

となる．$(r, p) = 1$ のとき（この仮定は不要だと思うのだが現在は証明に必要）E_B の中での $(0)^*$ は各 degree の $(0)^*$ に分解し，E_B の socle は degree 0 にあるので（[NW]）次が成立する．

(4.2) $\tau'(A) = \tau'(B) \cap A$

後記　この講演の主題は $\mathcal{F}(A)$ を軸にして A の test ideal と，標数 0 の特異点の解消から得られるいろいろの不変量（discrepancy など）の間を結ぶ事であった，しかし最近の 原氏の仕事により，test ideal と discrepancy との関係がより明らかになって来たことを付記させて頂く．

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2次元Jacobian Conjecture に関連して考えたこと
－ 等式 J(h,k) = h について －

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Abhyankar[A]は２次元 Jacobian Conjectureを、いろいろな角度から検討しているが、その一つの方向に Newton polygon を利用しているものがある。すなわち、２変数x，yの多項式を考える際に、x，yに weight p，q（p，qは自然数で共通因数なし）を与えて、その意味での斉次式を(p,q)-formと呼び、多項式 f(x,y) のこの意味での leading form を f(x,y)のleading (p,q)-formと呼ぶ。Jacobian J(f(x,y),g(x,y)) が 1に等しく、f，gの普通の意味の次数が共に＞ 1であるような多項式 f，g があるとき、任意の自然数の組p，q について、f，gのleading (p,q)-forms は同一の (p,q)-form h(x,y)のベキに定数をかけた形であって、このh(x,y)に対して、適当な(p,q)-form k(x,y) をとれば、Jacobian J(h,k)が hと一致するのである。

このようなh，kの存在は、p，qについての条件をゆるめて (1) p，q は有理整数であって、その少なくとも一方は正で、(2) p + q ＞ 0であればよいことが永田[N] によって知られている。そこで、そのような場合のh，kについての情報が詳しくわかれば、Jacobian Conjecture の解決に貢献する可能性があるかも知れないので、Abhyankar の考えなかった場合 (p，qの一方が 0または負の場合）を主としてについて考えてみた。

今のところ、役に立つ可能性は見いだせていないが、J(h,k) = hの解がどれだけあるかは、興味の沸く問題であるように思われる。

１．諸注意。

(1) h，kが (p,q)-formsで、J(h,k) = hをみたし、c が 0でない定数であれば、ch，k の組もこの関係式の解であるから、h の定数因子は無視してよい。

(2) h，kが (p,q)-formsで、J(h,k) = ch をみたし、c が 0でない定数であれば、h と $c^{-1}k$の組は関係式J(h,k) = hの解になるので、J(h,k) = ch ($0\neq c\in\mathbb{C}$) の解も同等と考えられる。

(3) $h = x^a y^b$,　$k = (a-b)^{-1}xy$ ($a, b\in \{0\}\cup\mathbb{N}$, $a\neq b$) とすれば J(h,k) = h であるが、これはp，qの値に無関係に、この問題の解になる。さらに、 p，q がともに正である場合については、この解をもとにして、一般な解がすべて得られることが知られている (Abhyankar [A])。すなわち、

(i) (p,q) = (1,1) の場合は、x，yの１次変換($x\to cx+dy$, $y\to c'x+d'y$ ($cd'\neq c'd$))によって、上の解から得られる。

(ii) p = 1, q＞ 1の場合は、変換$x\to x$, $y\to y+cx^q$ ($0\neq c\in\mathbb{C}$) によって得られる。p ＞ 1, q = 1 の場合は同様である。

(iii) p ＞ 1, q ＞ 1の場合は、本質的には、上の場合に限られる。

２．p = 0 の場合

この場合 p = 0, q = 1 である。一般に、(0,1)-formは $y^dF(x)$ の形である。そこで、

h, kが(0,1)-forms のとき、$h = y^d H(x)$, $k = y^e K(x)$と表せば、

$$J(h,k) = y^{d+e-1}(eH'K - dHK')$$

は容易に得られる。したがって、次の定理が得られる。

定理1.1. 上の記号のもとで、$J(h,k) = h$は、$e = 1$ かつ、$H = H'K - dHK'$と同値。

実際にどれだけの解があるかは、一般的にはむつかしい。しかし、$d = 0$ の場合と、$\deg K = 1$ の場合には、解は簡単にわかる。すなわち:

定理1.2. (1) $d = 0$ の場合は

$$K = t^{-1}(x - b),\ H = c(x - b)^t \quad (b,\ c \in \mathbb{C},\ c \neq 0,\ t \in \mathbb{N})$$

(2) $d \neq 0$, $\deg K = 1$ の場合は、$K = x - b$ $(b \in \mathbb{C})$ とすると、

$$H = c(x - b)^{d+1} \quad (0 \neq c \in \mathbb{C})$$

証明 (1) $d = 0$ であれば、条件は $H = H'K$である。次数を比べて、$\deg K = 1$ がわかり、K がH の因子であるから、$t = \deg H$ として、上の解が得られる。

(2) この場合、条件は $H = (x - b)H' - dH$、すなわち、$(1 + d)H = (x - b)H'$になる。したがって, H は $x - b$のベキ K^j としてよい。すると、$H' = jK^{j-1}$から、$(1 + d)K^j = jK^j$. したがって、$j = 1 + d$. (証明終わり)

$\deg K > 1$の場合は、解がどれだけあるのかを知ることは、むつかしい問題である。

定理1.3. H, K が定理1.1 の条件をみたせば、(1) K には重根はなく、(2) H の1次因子は Kの因子である。

証明 条件は $H = H'K - dHK'$ であった。(2) の証明: f が Hの1次因子で、$H = f^n g$ (g は fで割り切れない) であるとき、H'はf^{n-1}で割り切れ、f^nでは割り切れないから、f は Kの因子である。(1) の証明: f が Kの1次因子で、$K = f^m g^*$ (g^* はf で割り切れない、$m > 1$) とすると、K'はf^{m-1}で割り切れるから、条件の右辺は fで割り切れる。したがって、f は Hの因子である。(2) の証明の記号 $f^n g$を用いると、条件の両辺をf^nで割ると、右辺はさらにf で割れることになって、矛盾を知る。(証明終わり)

したがって、K の次数を nと定めると、

$$K = a(x - b_1)\cdots(x - b_n),\ H = a'(x - b_1)^{e1}\cdots(x - b_n)^{en}$$

とおくことができ、a, b_i, e_j についての条件を書き下すことができる。a'については、$\neq 0$だけが条件であり、無視してもよい。たとえば、$n = 2$ のときは、

$$a^{-1}(b_1 - b_2)^{-1} \in \mathbf{Z} \quad \text{かつ} \quad e_1 = d + a^{-1}(b_1 - b_2)^{-1},\ e_2 = d - a^{-1}(b_1 - b_2)^{-1}$$

$\geqq 0$ が条件になる。

その証明: $H = (x - b_1)^{e1}(x - b_2)^{e2}$, $K = a(x - b_1)(x - b_2)$ ゆえ、条件式の右辺は $a(x - b_1)^{e1}(x - b_2)^{e2}[(e_1 - d)(x - b_2) + (e_2 - d)(x - b_1)]$ すなわち、

$$1 = a(e_1 + e_2 - 2d)x + a(d - e_1)b_2 + a(d - e_2)b_1$$

ゆえに、$e_1 + e_2 = 2d$, $e_1 b_2 + e_2 b_1 = d(b_1 + b_2) - a^{-1}$

これを e_1, e_2 について解いて、上記の結果が得られる。(証明終わり)

3. $p = -1$, $q = 1$ の場合

この場合、$p + q = 0$ で、出発した問題にはそぐわないが、考えることにする。

一般に、(-1,1)-form は1変数の多項式 F(T) を用いて、$x^a y^b F(xy)$ $(F(0) \neq 0)$ の形に表される。$J(h,k) = h$であるとき、まず、k の定数項は変更しても成り立つので、定数

項なしの場合を計算する。$h = x^a y^b H(xy)$, $k = x^s y^t K(xy)$ $(H(0) \neq 0,\ K(0) \neq 0)$ とおく。(-1,1)-degree $d_{(-1,1)}$ を考えると、

$$d_{(-1,1)}(J(h,k)) = d_{(-1,1)}(hk) - d_{(-1,1)}(xy) = d_{(-1,1)}(hk)$$

であるから、$d_{(-1,1)}(k) = 0$, すなわち、$s = t$ であって、k に定数項がないと仮定したから、$s = t > 0$. これを考慮して計算すると、

$$J(h,k) = x^{a+s-1}y^{b+s-1}[s(a - b)H(xy)K(xy) + (a - b)xyH(xy)K'(xy)]$$

したがって、$s = 1$ $(= t)$ かつ、$1 = (a - b)[K(xy) + xyK'(xy)]$

この最後に式から、$\deg K(T) = 0$ となり

$$k = (a - b)^{-1}xy$$

が得られ, h については $a \neq b$が必要になる。k に定数項を加えてもよいから、

定理2.1. (-1,1)-forms h, kであって、$J(h,k) = k$となるのは、次の形の組である。

$h = x^a y^b H(xy)$ $(H(T) \in \mathbb{C}[T]$; a, b は負でない有理整数で$a \neq b)$

$k = (a - b)^{-1}xy + c$ $(c \in \mathbb{C})$

4．$p < 0,\ q > 0,\ p + q \neq 0$の場合

$s = -p$とすると、p, qに共通因数がないのだから、(p,q)-degreeが 0の単項式は $x^q y^s$ のベキである。したがって、(p,q)-formは１変数の多項式F(T)を用いて $x^a y^b F(x^q y^s)$の形に表される。そこで、関係式$J(h,k) = h$において、$h = x^a y^b H(x^q y^s)$, $k = x^c y^d K(x^q y^s)$として計算すると、まず、$d_{(p,q)}(J(h,k)) = d_{(p,q)}(h) + d_{(p,q)}(k) - d_{(p,q)}(xy)$ であるから、$c = d = 1$ としてよい。他方、$H(0) \neq 0$としてよい. すると、

$$h = J(h,k) = (a - b)x^a y^b H(x^q y^s)K(x^q y^s) + (as - bq)x^{a+q}y^{b+s}H(x^q y^s)K'(x^q y^s) + (q - s)x^{a+q}y^{b+s}H'(x^q y^s)K(x^q y^s)$$

$$\therefore H(T) = (a - b)H(T)K(T) + (as - bq)TH(T)K'(T) + (q - s)TH'(T)K(T) \cdots ①$$

①に $T = 0$を代入して、 $(a - b)K(0) = 1$ $\cdots$ ②

定理3.1. 上の状況のもとで、$J(h,k) = h$ならば、①, ②の他に、次が言える。

(1) r が H(T) の根であれば、r は K(T) の根でもある。したがって、$\deg H(T) \neq 0$であれば $\deg K(T) \geq 1$.

(2) K(T)には重根はない。

(3) (i) H(T), K(T)が共に定数ならば、$a \neq b$であって、$k = (a - b)^{-1}xy$

(ii) そうでない場合は $(a-b) + (as-bq)(\deg K(T)) + (q-s)(\deg H(T)) = 0$

証明 (1): $f = T - c$ が H(T) の１次因子で$H(T) = f^n g(T)$ $(g(c) \neq 0)$ であれば、①の最後の項がf^nで割り切れることから、f はK(T)の因子である。

(2): K(T)が$f = T - c$ の平方で割り切れれば、①の右辺がf で割り切れるので、f はH(T)の１次因子である。$H(T) = f^n g(T)$ $(g(c) \neq 0)$ とすると、①の右辺はf^{n+1}で割り切れるので、矛盾である。

(3): (i) は①からわかる。(ii)は、(1) により、$\deg H \neq 0$ならば $\deg K \neq 0$に注意して、①の Tについての最高次の係数の比較でわかる。（証明終わり）

注意 (3),(i) の場合以外では、これらの条件は、$J(h,k) = h$の充分条件ではない。

例. $\deg H(T) = 0$, $\deg K(T) > 0$の場合は

$K(T) = rT^u + (a-b)^{-1}$, $a(us + 1) = b(uq + 1)$, ただし$u \in \mathbb{N}$, $0 \neq r \in \mathbb{C}$

証明　$H(T) = 1$としてよい。①は $1 = (a - b)K(T) + (as - bq)TK'(T)$ になる。
$\deg K(T) = u$ として、$K(T) = c_0T^u + c_1T^{u-1} + \cdots + c_u$ とおくとこの条件式で
T^uの係数から　$0 = (a - b)c_0 + (as - bq)uc_0$　$\therefore\ a - b = -u(as - bq)$
これから、上の第２式が出る。
定数項から、$1 = (a - b)c_u$　$\therefore\ c_u = (a - b)^{-1}$
したがって、$u = 1$ ならば、上記のようになっている。$u > 1$としよう。T^i $(0 < i < u)$の係数から、$0 = (a - b)c_{u-i} + (as - bq)ic_{u-i}$
$c_{u-1} \neq 0$と仮定すると $a - b + i(as - bq) = 0$ となるが、これは上で証明した第２式に矛盾する。ゆえに、$c_{u-i} = 0$. これで、H, Kは上記の条件をみたすことがわかった。
逆に、H, Kが上の条件をみたしたとしよう。すると、上の計算と同様にして、$J(h,k) = h$ をみたすことがわかる。（証明終わり）
一般に、$\deg K(T) = 1$と仮定すると、$|q - s| = |p + q|$ の値を利用して、条件を書き下すことができて、いろいろな例が得られるが、うまくまとめるよい方法はまだわからない。$\deg K(T)$を定めれば、同様に条件を書き下すことができるが、条件は大分煩雑になってしまう。

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3 次元 Gorenstein Stanley-Reisner rings の Betti 数について

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有限集合 $V = \{x_1, x_2, \ldots, x_v\}$ に対して、頂点集合 V 上の単体的複体 (simplicial complex) Δ を次の条件 (1)、(2) を満たす $\mathbf{2}^V$ の部分集合とする。但し、$\mathbf{2}^V$ は V の部分集合全体からなる集合とする。

(1) $1 \leq i \leq v$ に対して、$\{x_i\} \in \Delta$。

(2) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$。

$\sharp(\sigma)$ で 有限集合 σ の濃度を表すことにする。Δ の元 σ を Δ の面 (face) という。特に、$\sharp(\sigma) = i+1$ のとき、i-face という。$d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ とおき、Δ の次元 (dimension) を $\dim \Delta = d-1$ で定義する。

$A = k[x_1, x_2, \ldots, x_v]$ を体上の v 変数多項式環とする。$V = \{x_1, x_2, \ldots, x_v\}$ 上の単体的複体 Δ に対して A のイデアル I_Δ を次のように定義する。

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq v, \ \{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta)$$

$k[\Delta] := A/I_\Delta$ を Δ の Stanley-Reisner 環という。

以後、A を $\deg x_i = 1$ として 次数付き環 $A = \bigoplus_{n \geq 0} A_n$ とみなす。すると、$k[\Delta]$ もまた、自然に A 上の次数付き加群 $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ とみなせる。

$k[\Delta]$ の A 上の次数付き極小自由分解を

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h,j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1,j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0$$

とする。ここで、h を $k[\Delta]$ の ホモロジー次元 (homological dimension) といい、$h = \mathrm{hd}_A(k[\Delta])$ とあらわす。このとき $v-d \leq h \leq v$ が成り立つことが知られている。各 $\beta_{i,j}$ を $k[\Delta]$ の (i,j) ベッチ数 ((i,j)-th Betti number) といい、また、$\beta_i := \sum_{j \in \mathbf{Z}} \beta_{i,j}$ を $k[\Delta]$ の第 i ベッチ数 (i-th Betti number) という。Δ が $d-1$ 次元の Gorenstein complex ならば、$h = v-d$, $\beta_h = 1$ となる。

このとき、次の定理がえられた。

定理 単体的複体 Δ を v 個の頂点をもつ 2 次元の *Gorenstein complex* とする。このとき

$$\beta_i(k[\Delta]) \leq \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1}, \quad 1 \leq i \leq v-1.$$

が成り立つ。

証明には、Hochster の公式と Brücker-Eberhard の帰納定理を用いる。

§1. Preliminaries

Let Δ be a simplicial complex on a vertex set V. We explain Hochster's formula. Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Hochster's formula [Hoc, Theorem 5.1] is that

$$\beta_{i,j} = \sum_{W \subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k).$$

Thus, in particular,

$$\beta_i(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k).$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. See, e.g., [B–H$_1$], [B–H$_2$], [H$_2$], [H$_3$], [H$_4$], and [T–H$_1$].

We define the Gorenstein complexes. Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set V with v vertices. We define

$$\mathrm{core}V := \{x \in V \mid \mathrm{star}_\Delta\{x\} \neq V\},$$

and

$$\mathrm{core}\Delta := \Delta_{\mathrm{core}V}.$$

We call Δ *Gorenstein* over k (or *k-Gorenstein*) if it satisfies one of the following equivalent conditions;
(1)For all faces $\sigma \in \mathrm{core}\Delta$ (including $\sigma = \emptyset$) we have

$$\tilde{H}_i(\ \mathrm{link}_{\ \mathrm{core}\Delta}(\sigma);\ k) \cong \begin{cases} k, & \text{if } i = \dim\ \mathrm{link}_{\ \mathrm{core}\Delta}(\sigma). \\ 0, & \text{otherwise.} \end{cases}$$

(2)$k[\Delta]$ is Gorenstein ring.

As for Betti numbers of $(d-1)$-dimensional Gorenstein complex Δ with v vertices, it is well known that $h = v - d$, and $\beta_h(k[\Delta]) = 1$ hold. where h is the homological dimension of $k[\Delta]$.

A Gorenstein complex Δ is said to be *Gorenstein** over k (or *k-Gorenstein**) if it satisfies one of the following equivalent conditions;
(1)core$\Delta = \Delta$.
(2)For all $1 \le i \le v$. x_i are zero-divisors in $k[\Delta]$.
(3)Δ is non-acyclic. where Δ is acyclic if and only if $\tilde{H}_i(\Delta;\ k) = 0$ for all i.

We have the following hierarchy;

{Boundary complexes of simplicial polytopes}
$\subset$ {Triangulations of a sphere}
$\subset$ {k-Gorenstein* complexes}

Remark. (1) All the inclusions above are strict. Non-shellable triangulation of a sphere (a Poincaré sphere respectively) gives the first (second respectively) inclusion strictness.
(2) If we consider all the classes restricted to $v - d \le 3$, then they are all equal ([Bru–Her$_2$]).
(3) If we consider all the classes restricted to $d \le 3$, then they are all equal.

§2. Main result

Now we state our main result.

Theorem 2.1. *Let k be a field. and let Δ be a 2-dimensional Gorenstein complex vith v (≥ 5) vertices. Then we have*

$$\beta_i(k[\Delta]) \le \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1}, \quad 1 \le i \le v-1.$$

Remark. If Δ is the boundary complex of a 3-dimensional stacked polytope, then equality in Theorem 2.1 holds. See [T H3] for the definition and the proof.

From now on we fix a field k. We first consider the acyclic Gorenstein case. Let Δ be a 2-dimensional acyclic k-Gorenstein complex with v vertices. Then Δ is known to be of the form

$$\Delta = \Gamma(v-1) * \{o\},$$

where $\Gamma(v-1)$ is the boundary complex of $(v-1)$-gon, $\{o\}$ is the simplicial complex with one vertex, and $\Delta_1 * \Delta_2$ is a *simplicial join* of Δ_1 and Δ_2, which is defined to be

$$\Delta_1 * \Delta_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}.$$

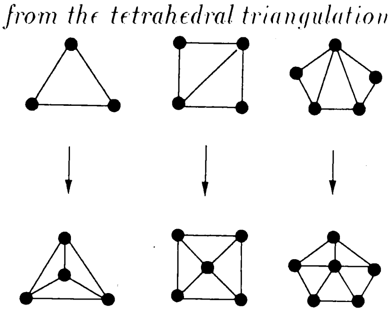
By [De] we have

$$\begin{aligned} \beta_i(k[\Delta]) &= \beta_i(k[\Gamma(v-1)]) \\ &= \binom{v-3}{i+1} i + \binom{v-3}{i-1}(v-i-3) \\ &= \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1} \end{aligned}$$

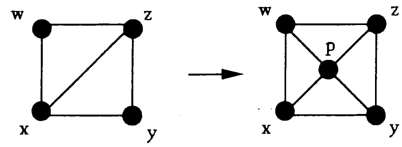
for $1 \le i \le v-4$, as desired.

From now on we concentrate on the Gorenstein* case. By Remark (3) in the previous section, 2-dimensional Gorenstein* complexes are nothing but triangulations of a sphere. Then we treat the problem in a combinatorial view point. To prove the theorem, we use:

THE INDUCTION THEOREM OF BRÜCKER-EBERHARD (cf. [Oda, p190]). *Suppose a finite triangulation Δ of $\mathbf{S}^2$ is given. We get a triangulation Δ' of $\mathbf{S}^2$ with one more vertex, if a vertex of Δ is " split into two " by one of the three steps (1), (2), (3) shown in the figures below. We can obtain any given finite triangulation of $\mathbf{S}^2$ from the tetrahedral triangulation by splitting vertices finitely many times.*



LEMMA 2.2. *Let Δ be a triangulation of $\mathbf{S}^2$ on a vertex set V with v vertices. And let Δ' be a triangulation obtained from Δ by (2) in the Induction Theorem, which is indicated as below.*



Put $V' := V \cup \{p\}$ and $W := W' - \{p\}$ for $W' \subset V'$.

(1)We have $|\dim_k \tilde{H}_0(\Delta'_{W'}; k) - \dim_k \tilde{H}_0(\Delta_W; k)| \leq 1$ for $W' \subset V'$.

(2)$\dim_k \tilde{H}_0(\Delta'_{W'}; k) = \dim_k \tilde{H}_0(\Delta_W; k) + 1$ holds if and only if W' is one of following cases;

(a) $p \in W'$, $w, x, y, z \notin W'$.

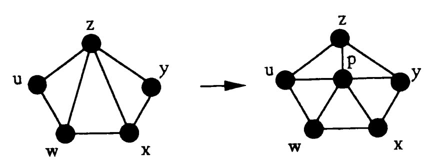
(b) $w, y \in W'$, $p, x, z \notin W'$, and w and y are disconnected in $\Delta'_{W'}$.

(3)Let $n(a)_j$ (resp. $n(b)_j$) be the number of j-element subsets W' of V' which satisfy the condition (a) (resp. (b)). Then we have $n(a)_j = \binom{v-4}{j-1}$ and $n(b)_j \leq \binom{v-4}{j-2}$ for $j \geq 2$.

Proof. (1) and (2) can be proved by one by one checking.

(3) As j-element subset W' satisfying (a) we can freely choose $(j-1)$ elements from $V - \{w, x, y, z\}$, which has just $(v-4)$ elements. We use similar arguement for (b). Q.E.D.

LEMMA 2.3. *Let Δ be a triangulation of $\mathbf{S}^2$ on a vertex set V with v vertices. And let Δ' be a triangulation obtained from Δ by (3) in the Induction Theorem, which is indicated as below.*



Put $V' := V \cup \{p\}$ *and* $W := W' - \{p\}$ *for* $W' \subset V'$.

(1) We have $|\dim_k \tilde{H}_0(\Delta'_{W'}; k) - \dim_k \tilde{H}_0(\Delta_W; k)| \leq 1$ *for* $W' \subset V'$

(2) $\dim_k \tilde{H}_0(\Delta'_{W'}; k) = \dim_k \tilde{H}_0(\Delta_W; k) + 1$ *holds if and only if* W' *is one of following cases;*

(a_1) $p \in W'$, $u, w, x, y, z \notin W'$.

(a_2) $w, z \in W'$, $p, u, x, y, \notin W'$, *and* w *and* z *are disconnected in* $\Delta'_{W'}$.

(a_3) $x, z \in W'$, $p, u, w, y \notin W'$ *and* x *and* z *are disconnected in* $\Delta'_{W'}$.

(a_4) $u, x, z \in W'$, $p, w, y \notin W'$ *and* u *and* x *are disconnected in* $\Delta'_{W'}$.

(a_5) $w, x, z \in W'$, $p, u, y \notin W'$ *and* w *and* z *are disconnected in* $\Delta'_{W'}$.

(a_6) $w, y, z \in W'$, $p, u, x \notin W'$ *and* w *and* y *are disconnected in* $\Delta'_{W'}$.

(3) If $W \in V$ *satisfies one of the following* (b_1) *or* (b_2), *then* $\dim_k \tilde{H}_0(\Delta'_{W'}; k) = \dim_k \tilde{H}_0(\Delta_W; k) - 1$ *holds;*

(b_1) $p, u, x \in W'$, $w, y, z \notin W'$ *and* v *and* x *are disconnected in* Δ_W.

(b_2) $p, w, y \in W'$, $u, x, z \notin W'$ *and* w *and* y *are disconnected in* Δ_W.

(4) Let $n(a_i)_j$, $1 \le i \le 8$ *(resp.* $n(b_i)_j$, $1 \le i \le 2$*) be the number of* j*-element subsets* W' *of* V' *which satisfy the condition* (a_i) *(resp.* (b_i)*). Then we have* $n(a_1)_j = \binom{v-5}{j-1}$, $n(a_2)_j \le \binom{v-5}{j-2}$, $n(a_3)_j \le \binom{v-5}{j-2}$, $n(a_5)_j \le \binom{v-5}{j-3}$, $n(a_4)_j \le n(b_1)_j$ *and* $n(a_6)_j \le n(b_2)_j$ *for* $j \ge 3$.

Proof. (1),(2), and (3) follow from one by one checking.

(4)For $n(a_1)_j$, $n(a_2)_j$, $n(a_3)_j$, and $n(a_5)_j$ we can see the assertion as in Lemma 2.2 (3).

Let $A_{i,j}$, $1 \le i \le 6$ (resp. $B_{i,j}$, $1 \le i \le 2$), be the set of all j-element subsets W' of V' which satisfy the condition (a$_i$) (resp. (b$_i$)). We define the map $A_{4,j} \to B_{1,j}$ ($W' \mapsto W' \cup \{p\} - \{z\}$), which is easily seen to be well-defined and injective. Then we have $n(a_4)_j \le n(b_1)_j$ for $j \ge 3$. We can prove $n(a_6)_j \le n(b_2)_j$ for $j \ge 3$ in the same way. Q.E.D.

LEMMA 2.4. *Let* Δ *be a triangulation of* $\mathbf{S}^2$ *with* v *vertices. And let* Δ' *be a triangulation obtained from* Δ *by (1),(2), or (3) in the Induction Theorem above. Then we have*

$$\beta_{i,i+1}(k[\Delta']) \le \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{v-3}{i},$$

for $i \ge 1$.

Proof. In the case of (1), the assertion is proved in [T-H$_3$, Lemma 2.3.1] with equality. By Hochster's formula we have

$$\begin{aligned} \beta(k[\Delta'])_{i,i+1} &= \sum_{W' \subset V',\ \sharp(W') = i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k) \\ &= \sum_{v \notin W' \subset V',\ \sharp(W') = i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k) \\ &\quad + \sum_{v \in W' \subset V',\ \sharp(W') = i+1} \dim_k \tilde{H}_0(\Delta'_{W'}; k). \end{aligned}$$

Hence, for the case (2) by Lemma 2.2 we have

$$\begin{aligned}
\beta_{i,i+1}(k[\Delta']) &\leq \sum_{W\subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W;k) \\
&\quad + \sum_{W\subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W;k) + \binom{v-4}{i} + \binom{v-4}{i-1} \\
&= \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{v-3}{i}
\end{aligned}$$

as desired.

For the case (3), similarly, by Lemma 2.3 we have

$$\begin{aligned}
\beta_{i,i+1}(k[\Delta']) &\leq \sum_{W\subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W;k) \\
&\quad + \sum_{W\subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W;k) + \binom{v-5}{i} + 2\binom{v-5}{i-1} + \binom{v-5}{i-2} \\
&= \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{v-3}{i}.
\end{aligned}$$

Q. E. D.

THEOREM 2.5. *Let Δ be a triangulation of $\mathbf{S}^2$ with v vertices. Then we have*

$$\beta_{i,i+1}(k[\Delta]) \leq i\binom{v-3}{i+1}.$$

Proof. We give a proof by induction v. Thanks to Lemma 2.4, we have

$$\begin{aligned}
\beta_{i,i+1}(k[\Delta]) &\leq i\binom{v-4}{i+1} + (i-1)\binom{v-4}{i} + \binom{v-4}{i} \\
&= i(\binom{v-4}{i+1} + \binom{v-4}{i}) \\
&= i\binom{v-3}{i+1}
\end{aligned}$$

as required. Q. E. D.

Now we give the proof of Theorem 2.1 in Gorenstein* case. First note that non-zero Betti numbers $\beta_{i,j} := \beta_{i,j}(k[\Delta])$ only appear in the 2-linear part $(\beta_{1,2},\ldots,\beta_{v-4,v-3})$ and in the 3-linear part $(\beta_{1,3},\ldots,\beta_{v-4,v-2})$ for

$1 \le i \le v-4$. Since Δ is Gorenstein*, we have $\beta_{i,j}(k[\Delta]) = \beta_{v-i-4,v-j}(k[\Delta])$ for every i and j. Put $j := i+2$. We have $\beta_{i,i+2}(k[\Delta]) = \beta_{v-i-3,v-i-2}(k[\Delta])$. Then we have

$$\begin{aligned}\beta_i(k[\Delta]) &= \beta_{i,i+1}(k[\Delta]) + \beta_{i,i+2}(k[\Delta]) \\ &= \beta_{i,i+1}(k[\Delta]) + \beta_{v-i-3,v-i-2}(k[\Delta]) \\ &\le i\binom{v-3}{i+1} + (v-i-3)\binom{v-3}{v-i-2} \\ &= \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1}.\end{aligned}$$

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Gorenstein algebras of Veronese type

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序．本稿は Emanuela De Negri との共同研究の報告である．一般に，n 次元ユークリッド空間に格子点の有限集合 $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_v\}$ があったとき，体 K 上の n 変数 Laurent 多項式環 $K[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$ において，v 個の単項式

$$t^{\mathbf{a}_i} = t_1^{a_i(1)} \cdots t_n^{a_i(n)}, \qquad \mathbf{a}_i = (a_i(1),\ldots,a_i(n)), \qquad 1 \leqq i \leqq v$$

が生成する部分環 $K[\mathcal{A}]$ を考え，また，v 変数多項式環 $K[y_1,\ldots,y_v]$ から $K[\mathcal{A}]$ への自然な全射 $y_i \to t^{\mathbf{a}_i}$ の核を $I_{\mathcal{A}}$ で表す．このとき，$I_{\mathcal{A}}$ は binomial によって生成され，$\mathcal{A}$ の toric ideal と呼ばれる．イデアル $I_{\mathcal{A}}$ の initial ideal $\mathrm{in}_<(I_{\mathcal{A}})$，グレブナー基底は，昨今の Computational Commutative Algebra における重要な研究対象である．他方，有限集合 $\mathcal{A}$ の凸閉包 $\mathcal{A}_{凸}$ の regular な三角形分割，unimodular な三角形分割などは，超幾何函数の理論とも関連し，計算幾何の分野において盛んに研究されている．このような背景については，たとえば，［B. Sturmfels, ``Gröbner Bases and Convex Polytopes,'' Amer. Math. Soc., Providence, RI, 1995］が詳しい．可換環論に有益な事実を列挙すると，第 1 点は，凸多面体 $\mathcal{A}_{凸}$ の regular な三角形分割と $I_{\mathcal{A}}$ の initial ideal $\mathrm{in}_<(I_{\mathcal{A}})$ が一対一に対応していること，第 2 点として，$\mathcal{A}_{凸}$ の regular な三角形分割が unimodular であるための必要十分条件は対応する initial ideal $\mathrm{in}_<(I_{\mathcal{A}})$ が square-free であること，第 3 点として，$\mathcal{A}_{凸}$ に unimodular な三角形分割が存在すれば，可換環 $K[\mathcal{A}]$ は正規環であって，従って，Cohen-Macaulay 環であること，等である．特に，$I_{\mathcal{A}}$ が square-free な initial ideal を持てば，$K[\mathcal{A}]$ は正規環であって，この結果は，具体的な affine semigroup ring が正規環であることを判定するのに，時として，極めて効果的である．本稿では，Veronese 型代数と呼ばれる正規な semigroup ring で Gorenstein 環となるものを分類するが，Veronese 型代数も，多項式環の Veronese 部分環の単なる一般化として可換環論の枠内で捕えるだけではいささか不十分であって，計算幾何及び計算代数の然るべき背景を踏まえて理解する必要がある．

Let K be a field and $K[t_1, t_2, \ldots, t_n]$ the polynomial ring in n variables over K. We fix an integer d and a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of integers with $1 \leq a_1 \leq a_2 \leq \cdots a_n \leq d$ and $d < \sum_{i=1}^{n} a_i$. Let $A(\mathbf{a}; d)$ denote the K-subalgebra of $K[t_1, t_2, \ldots, t_n]$ generated by all monomials of the form $t_1^{x_1} t_2^{x_2} \cdots t_n^{x_n}$ with $x_1 + x_2 + \cdots + x_n = d$ and with $x_i \leq a_i$ for each $1 \leq i \leq n$. Such an algebra is called an algebra of Veronese type. Note that if $a_i = d$ for every i then $A(\mathbf{a}; d)$ is the d-th Veronese subring of $K[t_1, t_2, \ldots, t_n]$, and that if each $a_i = 1$ then $A(\mathbf{a}; d)$ is generated by all the square-free monomials of degree d in $K[t_1, t_2, \ldots, t_n]$. The algebra of Veronese type has rich geometric and computational background and has been studied from the view point of geometry of toric varieties as well as Gröbner bases. See, e.g., [Stu].

The purpose of the present paper is to classify all the Gorenstein algebras of Veronese type. We first observe that an algebra $A(\mathbf{a}; d)$ of Veronese type coincides with the Ehrhart ring $A(\mathcal{P}(\mathbf{a}; d))$ of an integral convex polytope $\mathcal{P}(\mathbf{a}; d)$, which guarantees that $A(\mathbf{a}; d)$ is a Cohen–Macaulay normal domain. We then apply a certain combinatorial criterion for $A(\mathcal{P}(\mathbf{a}; d))$ to be Gorenstein and determine all the sequences $\mathbf{a}$ and integers d for which $A(\mathbf{a}; d)$ is Gorenstein. Our approach is rather geometric and relies on finding the equations of facets of a convex polytope.

We refer the reader to, e.g., [Grü] for fundamental results on convex polytopes. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{R}$ the set of real numbers. We write $\sharp(X)$ for the cardinality of a finite set X.

§1. Ehrhart rings of rational convex polytopes

Let $\mathcal{P} \subset \mathbb{R}^N$ denote a convex porytope of dimension d and suppose that $\mathcal{P}$ is rational, i.e., each vertex of $\mathcal{P}$ has rational coordinates. Let $Y_1, Y_2, \ldots, Y_N$ and T be indeterminates over a field K. Given an integer $q \geq 1$, we write $A(\mathcal{P})_q$ for the vector space over K which is spanned by those monomials $Y_1^{\alpha_1} Y_2^{\alpha_2} \cdots Y_N^{\alpha_N} T^q$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in q\mathcal{P} \bigcap \mathbb{Z}^N$. Here $q\mathcal{P} := \{q\alpha \,;\, \alpha \in \mathcal{P}\}$. Since $\mathcal{P}$ is convex, $A(\mathcal{P})_p A(\mathcal{P})_q \subset A(\mathcal{P})_{p+q}$ for all p and q. It follows easily that the graded algebra $A(\mathcal{P}) := \bigoplus_{q=0}^{\infty} A(\mathcal{P})_q$ is finitely generated over $K = A(\mathcal{P})_0$ with Krull-dim $A(\mathcal{P}) = d + 1$. Moreover, A(P) is normal; hence Cohen–Macaulay ([Hoc]). We say that $A(\mathcal{P})$ is the *Ehrhart ring* associated with a rational convex polytope $\mathcal{P} \subset \mathbb{R}^N$. Consult

[B–H] and [H_3] for the detailed information about algebra and combinatorics on Ehrhart rings.

A combinatorial criterion for $A(\mathcal{P})$ to be Gorenstein is obtained in [H_2]. Let $\mathcal{P} \subset \mathbb{R}^N$ be a convex polytope of dimension d and $\partial\mathcal{P}$ the boundary of $\mathcal{P}$. Then, $\mathcal{P}$ is called of standard type if (i) $d = N$ and (ii) the origin of $\mathbb{R}^N$ is contained in the interior $\mathcal{P} - \partial\mathcal{P}$ of $\mathcal{P}$. When $\mathcal{P} \subset \mathbb{R}^d$ is of standard type, the polar set

$$\mathcal{P}^\star := \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \ ; \ \sum_{i=1}^{d} \alpha_i\beta_i \leq 1 \ \text{ for every } \ (\beta_1, \ldots, \beta_d) \in \mathcal{P}\}$$

is again a convex polytope of standard type and $(\mathcal{P}^\star)^\star = \mathcal{P}$. We say that $\mathcal{P}^\star$ is the *dual polytope* of $\mathcal{P}$. A basic fact is the existence of an inclusion-reversing bijection between the set of all faces of $\mathcal{P}$ and that of $\mathcal{P}^\star$. In particular, if $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$ and if $\mathcal{H} \subset \mathbb{R}^d$ is the hyperplane defined by the equation $\sum_{i=1}^{d} \alpha_i x_i = 1$, then $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a vertex of $\mathcal{P}^\star$ if and only if $\mathcal{H} \bigcap \mathcal{P}$ is a facet (i.e., $(d-1)$-dimensional face) of $\mathcal{P}$. Hence, the dual polytope of a rational convex polytope is rational.

(1.1) THEOREM ([H_2]). *Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational convex polytope of dimension d and let $\delta \geq 1$ denote the smallest integer for which $\delta(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^d \neq \emptyset$. Fix $\alpha \in \delta(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^d$ and write $\mathcal{Q}$ for the rational convex polytope $\delta\mathcal{P} - \alpha \subset \mathbb{R}^d$ of standard type. Then, the Ehrhart ring $A(\mathcal{P})$ of $\mathcal{P}$ is Gorenstein if and only if the following conditions are satisfied:*

(i) *The dual polytope $\mathcal{Q}^\star$ of $\mathcal{Q}$ is integral, i.e., every vertex of $\mathcal{Q}^\star$ has integer coordinates ;*

(ii) *Let $\tilde{\mathcal{P}} \subset \mathbb{R}^{d+1}$ denote the rational convex polytope which is the convex hull of the subset $\{(\beta, 0) \in \mathbb{R}^{d+1} \ ; \ \beta \in \mathcal{P}\} \bigcup \{(0, \ldots, 0, 1/\delta)\}$ in $\mathbb{R}^{d+1}$. Then, $\tilde{\mathcal{P}}$ is facet-reticular, that is to say, if $\mathcal{H}$ is a hyperplane in $\mathbb{R}^{d+1}$ and if $\mathcal{H} \bigcap \tilde{\mathcal{P}}$ is a facet of $\tilde{\mathcal{P}}$, then $\mathcal{H} \bigcap \mathbb{Z}^{d+1} \neq \emptyset$.*

The original proof of Theorem (1.1) obtained in [H_2] is combinatorial and is based on the fact ([Sta_1] and [Dan]) that the canonical module of $A(\mathcal{P})$ is generated by those monomials $Y_1^{\alpha_1} \cdots Y_d^{\alpha_d} T^q$ with $(\alpha_1, \ldots, \alpha_d) \in q(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^d$. An algebraic proof of Theorem (1.1) related with the geometry of toric varieties also appears in [Nom].

(1.2) COROLLARY. (1) ([H$_1$]) *Suppose that* $\mathcal{P} \subset \mathbb{R}^d$ *is a rational convex polytope of standard type. Then, the Ehrhart ring* $A(\mathcal{P})$ *is Gorenstein if and only if the dual polytope* $\mathcal{P}^\star$ *of* $\mathcal{P}$ *is integral.*

(2) *Work with the same notation* $\mathcal{P}$, δ, α *and* $\mathcal{Q}$ *as in* (1.1) *and, in addition, suppose that* $\mathcal{P}$ *is integral. Then, the Ehrhart ring* $A(\mathcal{P})$ *is Gorenstein if and only if the dual polytope* $\mathcal{Q}^\star$ *of* $\mathcal{Q}$ *is integral.*

Proof. In fact, the condition (ii) on $\tilde{\mathcal{P}}$ of Theorem (1.1) is guaranteed if either $\delta = 1$ or $\mathcal{P}$ is integral. Q. E. D.

Let $a(A(\mathcal{P}))$ denote the a-invariant (e.g., [B–H, p. 139]) of $A(\mathcal{P})$. Then, the integer δ in (1.1) coincides with $-a(A(\mathcal{P}))$. Thus, the above Corollary (1.2) says that, if $\mathcal{P}$ is integral, then $A(\mathcal{P})$ is Gorenstein if and only if the Veronese subring $A(\mathcal{P})^{(\delta)}$ of $A(\mathcal{P})$ with $\delta = -a(A(\mathcal{P}))$ is Gorenstein.

§2. Algebras of Veronese type

We now study the classification problem of finding all the Gorenstein algebras of Veronese type. Let $n \geq 2$. Fix a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ with $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq d$ and $d < \sum_{i=1}^n a_i$. First, it is required to state a numerical result which enables us to see that every algebra of Veronese type coincides with the Ehrhart ring of an integral convex polytope.

(2.1) LEMMA. *Let* I_q *denote the set of all sequences* $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ *such that* (i) $0 \leq x_i \leq qa_i$ *for each* $1 \leq i \leq n$ *and* (ii) $\sum_{i=1}^n x_i = qd$. *Then, every element belonging to* I_q *is the sum of* q *elements in* I_1.

Let $\mathcal{P}(\mathbf{a}; d) \subset \mathbb{R}^n$ denote the rational convex polytope

$$\mathcal{P}(\mathbf{a}; d) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \; ; \; 0 \leq x_i \leq a_i \text{ for each } 1 \leq i \leq n, \; x_1 + \cdots + x_n = d\}.$$

Then, Lemma (2.1) guarantees that the above rational polytope $\mathcal{P}(\mathbf{a}; d)$ is, in fact, integral and that the algebra $A(\mathbf{a}; d)$ of Veronese type coincides with the Ehrhart ring $A(\mathcal{P}(\mathbf{a}; d))$ of $\mathcal{P}(\mathbf{a}; d)$. That is to say,

(2.2) COROLLARY. (1) *The convex polytope* $\mathcal{P}(\mathbf{a}; d)$ *is an integral convex polytope of dimension* $n - 1$.

(2) *The Ehrhart ring* $A(\mathcal{P}(\mathbf{a};d))$ *of* $\mathcal{P}(\mathbf{a};d)$ *is generated by* $A(\mathcal{P}(\mathbf{a};d))_1$ *as an algebra over* $K = A(\mathcal{P}(\mathbf{a};d))_0$.

(3) *The algebra* $A(\mathbf{a};d)$ *of Veronese type is isomorphic to the Ehrhart ring* $A(\mathcal{P}(\mathbf{a};d))$ *as graded algebras over* K.

(4) *The algebra* $A(\mathbf{a};d)$ *of Veronese type is a Cohen–Macaulay normal domain with* Krull-dim $A(\mathbf{a};d) = n$.

Proof. (1) Let $I_q \subset \mathbb{Z}^n$ be the same finite set as in Lemma (2.1). Then, $I_q = q\mathcal{P}(\mathbf{a};d)\bigcap \mathbb{Z}^n$ for every $q \geq 1$. Let $\mathrm{CONV}(I_1)$ denote the convex hull of I_1 in $\mathbb{R}^n$. Since $I_1 = \mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$, we have $\mathrm{CONV}(I_1) \subset \mathcal{P}(\mathbf{a};d)$. By Lemma (2.1), if $\mathbf{x} = (x_1,\dots,x_n) \in I_q$, then $\mathbf{x}$ has an expression of the form $\mathbf{x} = \mathbf{x}^{(1)} + \cdots + \mathbf{x}^{(q)}$ with each $\mathbf{x}^{(j)} \in I_1$. Since $(1/q)\mathbf{x}^{(1)} + \cdots + (1/q)\mathbf{x}^{(q)} \in \mathrm{CONV}(I_1)$, $\mathbf{x} = q((1/q)\mathbf{x}^{(1)} + \cdots + (1/q)\mathbf{x}^{(q)})$ belongs to $q\mathrm{CONV}(I_1)\bigcap\mathbb{Z}^n$. Hence, $I_q \subset q\mathrm{CONV}(I_1)\bigcap\mathbb{Z}^n$. Since $I_q = q\mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$ and $\mathrm{CONV}(I_1) \subset \mathcal{P}(\mathbf{a},d)$, we have $q\mathrm{CONV}(I_1)\bigcap\mathbb{Z}^n = q\mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n = I_q$ for every $q \geq 1$. Hence, $\mathrm{CONV}(I_1) = \mathcal{P}(\mathbf{a};d)$. Thus, $\mathcal{P}(\mathbf{a};d)$ is an integral convex polytope. It is a fundamental fact on convex polytopes that if $\mathcal{X} \subset \mathbb{R}^N$ is a convex polytope of dimension d and if $\mathcal{H} \subset \mathbb{R}^N$ is a hyperplane with $\mathcal{H}\bigcap(\mathcal{X} - \partial\mathcal{X}) \neq \emptyset$, then $\mathcal{H}\bigcap\mathcal{X}$ is a convex polytope of dimension $d-1$. Hence, the convex polytope $\mathcal{P}(\mathbf{a};d) \subset \mathbb{R}^n$ is of dimension $n-1$ since $d < \sum_{i=1}^n a_i$.

(2) The Ehrhart ring $A(\mathcal{P}(\mathbf{a};d))$ is generated by $A(\mathcal{P}(\mathbf{a};d))_1$ if and only if every $\mathbf{x} = (x_1,\dots,x_n) \in q\mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$ has the expression of the form $\mathbf{x} = \mathbf{x}^{(1)} + \cdots + \mathbf{x}^{(q)}$ with each $\mathbf{x}^{(j)} \in \mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$. Hence, by Lemma (2.1) together with $I_q = q\mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$, $A(\mathcal{P}(\mathbf{a};d))$ is generated by $A(\mathcal{P}(\mathbf{a};d))_1$ as required.

(3) Since $A(\mathcal{P}(\mathbf{a};d))$ is generated by $A(\mathcal{P}(\mathbf{a};d))_1$, $A(\mathcal{P}(\mathbf{a};d))$ is generated by those monomials $Y_1^{\alpha_1}\cdots Y_n^{\alpha_n}T$ with $(\alpha_1,\dots,\alpha_n) \in \mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$ as an algebra over $K = A(\mathcal{P}(\mathbf{a};d))_0$. On the other hand, the algebra $A(\mathbf{a};d)$ of Veronese type is the subalgebra of $K[t_1,t_2,\dots,t_n]$ generated by those monomials $t_1^{x_1}t_2^{x_2}\cdots t_n^{x_n}$ with each $0 \leq x_i \leq a_i$ and $\sum_{i=1}^n x_i = d$, i.e., $(x_1,x_2,\dots,x_n) \in \mathcal{P}(\mathbf{a};d)\bigcap\mathbb{Z}^n$. Hence, $A(\mathcal{P}(\mathbf{a};d)) \cong A(\mathbf{a};d)$.

(4) Now, the Ehrhart ring associated with a rational convex polytope of dimension $n-1$ is always a Cohen–Macaulay normal domain of Krull-dimension n. Hence, the algebra $A(\mathbf{a};d)$ is a Cohen–Macaulay normal domain of Krull-dimension n as desired. Q. E. D.

We are now in the position to state our main result of the present paper.

(2.3) THEOREM. *Let $n \geq 2$. Fix a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ with $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq d$ and $d < \sum_{i=1}^{n} a_i$. Let $K[t_1, t_2, \ldots, t_n]$ denote the polynomial ring in n variables over a field K and suppose that $A(\mathbf{a}; d)$ is the K-subalgebra of $K[t_1, t_2, \ldots, t_n]$ generated by all monomials of the form $t_1^{x_1} t_2^{x_2} \cdots t_n^{x_n}$ with $x_1 + x_2 + \cdots + x_n = d$ and with $x_i \leq a_i$ for each $1 \leq i \leq n$. Then, the algebra $A(\mathbf{a}; d)$ of Veronese type is Gorenstein if and only if one of the following conditions is satisfied:*

(a) *d divides n, and $a_i = d$ for every $i = 1, \ldots, n$;*
(b) *$n = d$, and $a_i \in \{2, d\}$ for every $i = 1, \ldots, n$;*
(c) *$n = 2d$, and $a_i \in \{1, d\}$ for every $i = 1, \ldots, n$;*
(d) *$a_1 + \cdots + a_n - d$ divides n, and $d \geq a_2 + \cdots + a_n$;*
(e) *$d < a_2 + \cdots + a_n$, $n = a_1 + \cdots + a_n - d$, and $a_i = 2$ if $d < \sum_{j=1}^{n} a_j - a_i$;*
(f) *$d < a_2 + \cdots + a_n$, $n = 2(a_1 + \cdots + a_n - d)$, and $a_i = 1$ if $d < \sum_{j=1}^{n} a_j - a_i$;*
(g) *$a_1 = \cdots = a_{n-1} = 1$, $a_n = d$, and $d \geq n - 1$;*
(h) *$a_1 = \cdots = a_{n-1} = 2$, $a_n = d$, and $d \geq 2(n-1)$.*

Proof. Let $\mathcal{H}$ be the hyperplane in $\mathbb{R}^n$ defined by the equation $x_1 + \ldots + x_n = d$, and let $\psi : \mathbb{R}^{n-1} \to \mathcal{H}$ denote the affine map defined by $\psi(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, d - (x_1 + \cdots + x_{n-1}))$ if $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Then, ψ is an affine isomorphism with $\psi(\mathbb{Z}^{n-1}) = \mathcal{H} \bigcap \mathbb{Z}^n$. Hence, $\psi^{-1}(\mathcal{P}(\mathbf{a}; d)) \subset \mathbb{R}^{n-1}$ is an integral convex polytope of dimension $d-1$ and the Ehrhart ring $A(\psi^{-1}(\mathcal{P}(\mathbf{a}; d)))$ is isomorphic to $A(\mathcal{P}(\mathbf{a}; d))$ as graded algebras over K. Recall that the algebra $A(\mathbf{a}; d)$ of Veronese type is isomorphic to $A(\mathcal{P}(\mathbf{a}; d))$. Hence, our work is to study the problem when $A(\psi^{-1}(\mathcal{P}(\mathbf{a}; d)))$ is Gorenstein. In this proof, to avoid difficult notation, we write $\mathcal{P}$ instead of $\psi^{-1}(\mathcal{P}(\mathbf{a}, d))$; that is to say,

$$\mathcal{P} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \; ; \; 0 \leq x_i \leq a_i \text{ for each } 1 \leq i \leq n-1, \quad d - a_n \leq x_1 + \cdots + x_{n-1} \leq d\}.$$

We now apply Corollary (1.2) to the integral convex polytope $\mathcal{P} \subset \mathbb{R}^{n-1}$ of dimension $n-1$ and determine all the sequences $\mathbf{a}$ and integers d for which the Ehrhart ring $A(\mathcal{P})$ is Gorenstein. First, let $\delta \geq 1$ denote the smallest integer with $\delta(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1} \neq \emptyset$. If $q > \delta$, then $q(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1}$ contains $\delta(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1} + (q - \delta)(\mathcal{P} \bigcap \mathbb{Z}^{n-1})$. Hence, $\sharp[q(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1}] > 1$ since $\mathcal{P}$ is integral. Moreover, if $A(\mathcal{P})$ is Gorenstein, then $\sharp[\delta(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1}] = 1$. Thus, a basic step for our classification is to determine when $\sharp[\delta(\mathcal{P} -$

$\partial\mathcal{P})\bigcap\mathbb{Z}^{n-1}] = 1$. Note that $(z_1,\ldots,z_{n-1}) \in \mathbb{Z}^{n-1}$ belongs to $\delta(\mathcal{P}-\partial\mathcal{P})$ if and only if

(i) $1 \leq z_i \leq \delta a_i - 1$ for each $1 \leq i \leq n-1$;

(ii) $\delta(d-a_n)+1 \leq z_1 + \ldots + z_{n-1} \leq \delta d - 1$.

Thus, since $\{z_1+\ldots+z_{n-1}\ ;\ 1 \leq z_i \leq \delta a_i - 1$ for each $1 \leq i \leq n-1\}$ coincides with $\{n-1, n, n+1, \ldots, \delta(a_1+\ldots+a_{n-1})-(n-1)\}$, if $\sharp[\delta(\mathcal{P}-\partial\mathcal{P})\bigcap\mathbb{Z}^{n-1}] = 1$, then the following four possible cases arise:

[1] $n-1 = \delta d - 1$, i.e., $n = \delta d$;

[2] $\delta(a_1+\cdots+a_{n-1})-(n-1) = \delta(d-a_n)+1$, i.e., $n = \delta(a_1+\cdots+a_n-d)$;

[3] $\delta(d-a_n)+1 = \delta d - 1$, i.e., $\delta a_n = 2$;

[4] $\delta a_i = 2$ for every $1 \leq i \leq n-1$.

If none of the above conditions **[1]**, **[2]**, **[3]** and **[4]** is satisfied, then $A(\mathbf{a}; d)$ is not Gorenstein. We analyze combinatorics on $\delta\mathcal{P}$ for each of the above **[1]**, **[2]**, **[3]** and **[4]** in what follows.

Case [1]: Let $n = \delta d$. Then, $(1,\ldots,1) \in \delta(\mathcal{P}-\partial\mathcal{P})$. Let $\mathcal{Q} = \delta\mathcal{P} - (1,\ldots,1)$. Since $\delta(d-a_n)-(n-1) = 1-\delta a_n$ and $\delta d-(n-1) = 1$,

$$\mathcal{Q} = \{(x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}\ ;\ -1 \leq x_i \leq \delta a_i - 1 \text{ for each } 1 \leq i \leq n-1, \quad 1-\delta a_n \leq x_1+\cdots+x_{n-1} \leq 1\}.$$

Corollary (1.2) guarantees that $A(\mathcal{P})$ is Gorenstein if and only if the dual polytope $\mathcal{Q}^\star$ of $\mathcal{Q}$ is integral. To see when $\mathcal{Q}^\star$ is integral, we must find all the facets of $\mathcal{Q}$. Let $\mathcal{H}_i$ denote the hyperplane in $\mathbb{R}^{n-1}$ defined by the equation $x_i = \delta a_i - 1$ for $i = 1,\ldots,n-1$. Then, $\mathcal{H}_i\bigcap\mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if the closed halfspace $\mathcal{H}_i^{(-)}\ :\ x_i \leq \delta a_i - 1$ appears in the irredundant representation of $\mathcal{Q}$ as the intersection of finite closed halfspaces. Let, say, $i = 1$. It follows from the linear inequalities which define $\mathcal{Q}$ that $x_1 \leq 1-(x_2+\cdots+x_{n-1}) \leq n-1$. Hence, $\mathcal{H}_1\bigcap\mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $\delta a_1 - 1 < n-1 = \delta d - 1$, in other words, if and only if $a_1 < d$. Similarly, for each $2 \leq i \leq n-1$, $\mathcal{H}_i\bigcap\mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $a_i < d$. Let $\mathcal{H}_n$ denote the hyperplane in $\mathbb{R}^{n-1}$ defined by the equation $x_1+\cdots+x_{n-1} = 1-\delta a_n$. Since $1-n \leq x_1+\cdots+x_{n-1}$, the same technique as above enables us to show that $\mathcal{H}_n\bigcap\mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $1-n < 1-\delta a_n$, i.e., $a_n < d$. Thus, for every $1 \leq i \leq n$, we know that $\mathcal{H}_i \cap \mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $a_i < d$.

By virtue of Corollary (1.2), if $n = \delta d$ with $\delta \geq 1$, then $A(\mathcal{P})$ is Gorenstein if and only if $\delta a_i = 2$ for every i with $a_i < d$. Thus, if

(a) $\quad d$ divides n and $a_i = d$ for every $i = 1, \ldots, n$,

then $A(\mathcal{P})$ is Gorenstein. Suppose now that we have $a_j < d$ for some $1 \leq j \leq n$. Then, $a_1 < d$. Hence, $\delta a_1 = 2$. Thus, two possible cases occur. If $\delta = 1$ and $a_1 = 2$, then $a_i = 2$ for every i with $a_i < d$. Hence,

(b) $\quad n = d$ and $a_i \in \{2, d\}$ for every $i = 1, \ldots, n$.

If $\delta = 2$ and $a_1 = 1$, then $a_i = 1$ for every i with $a_i < d$. Hence,

(c) $\quad n = 2d$ and $a_i \in \{1, d\}$ for every $i = 1, \ldots, n$.

Thus, if $n = \delta d$ with $\delta \geq 1$, then the algebra $A(\mathbf{a}; d)$ is Gorenstein if and only if one of the above conditions (a), (b) and (c) is satisfied.

Case [2]: Let $n = \delta(a_1 + \cdots + a_n - d)$. Then, $(\delta a_1 - 1, \ldots, \delta a_{n-1} - 1) \in \delta(\mathcal{P} - \partial\mathcal{P})$. Let $\mathcal{Q} = \delta\mathcal{P} - (\delta a_1 - 1, \ldots, \delta a_{n-1} - 1)$. Then, since $\delta(d - a_n) - \delta(a_1 + \cdots + a_{n-1}) + (n-1) = -1$, and $\delta d - \delta(a_1 + \cdots + a_{n-1}) + (n-1) = \delta a_n - 1$,

$$\begin{aligned} \mathcal{Q} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \ ; \ 1 - \delta a_i \leq x_i \leq 1 \text{ for each } 1 \leq i \leq n-1, \\ -1 \leq x_1 + \cdots + x_{n-1} \leq \delta a_n - 1\}. \end{aligned}$$

Let $\mathcal{H}_i$ be the hyperplane in $\mathbb{R}^{n-1}$ defined by the equation $x_i = 1 - \delta a_i$ for $i = 1, \ldots, n-1$. Let, say, $i = 1$. It follows from the linear inequalities which define $\mathcal{Q}$ that $x_1 \geq -1 - (x_2 + \cdots + x_{n-1}) \geq 1 - n$. Hence, $\mathcal{H}_1 \cap \mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $1 - \delta a_1 > 1 - n$, i.e., $\delta a_1 < n$. Similarly, for each $2 \leq i \leq n-1$, $\mathcal{H}_i \cap \mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $\delta a_i < n$. Let $\mathcal{H}_n$ denote the hyperplane in $\mathbb{R}^{n-1}$ defined by the equation $x_1 + \cdots + x_{n-1} = \delta a_n - 1$. Since $x_1 + \cdots + x_{n-1} \leq n - 1$, $\mathcal{H}_n \cap \mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $\delta a_n - 1 < n - 1$, i.e., $\delta a_n < n$. Since $n = \delta(a_1 + \cdots + a_n - d)$, we have $\delta a_i < n$ if and only if $d < \sum_{j=1}^{n} a_j - a_i$. Hence, for every $1 \leq i \leq n$, $\mathcal{H}_i \cap \mathcal{Q}$ is a facet of $\mathcal{Q}$ if and only if $d < \sum_{j=1}^{n} a_j - a_i$. Since $a_1 \leq a_2 \leq \cdots \leq a_n$, if $d \geq a_2 + \cdots + a_n$, then $d \geq \sum_{j=1}^{n} a_j - a_i$ for every $1 \leq i \leq n$. Thus, $A(\mathcal{P})$ is Gorenstein if

(d) $\quad a_1 + \cdots + a_n - d$ divides n, and $d \geq a_2 + \cdots + a_n$.

Suppose now that $d < a_2 + \cdots + a_n$ and let $k \in \{1, 2, \ldots, n\}$ denote the largest index with $d < \sum_{j=1}^{n} a_j - a_k$. Then, $A(\mathcal{P})$ is Gorenstein if and only if $\delta a_1 = \delta a_2 = \cdots = \delta a_k = 2$. Note that, since $d \geq \sum_{j=1}^{n} a_j - a_i$ for $i = k+1, k+2, \ldots, n$, no restriction is required for each of $a_{k+1}, a_{k+2}, \ldots, a_n$. Since $\delta a_1 = 2$, we have either $\delta = 1$ or $\delta = 2$. Hence,

(e) $$n = a_1 + \cdots + a_n - d, \text{ and } a_i = 2 \text{ for } i = 1, ..., k;$$

or

(f) $$n = 2(a_1 + \cdots + a_n - d), \text{ and } a_i = 1 \text{ for } i = 1, ..., k.$$

Thus, if $n = \delta(a_1 + \cdots + a_n - d)$ with $\delta \geq 1$, then the algebra $A(\mathbf{a}; d)$ is Gorenstein if and only if one of the above conditions (d), (e) and (f) is satisfied.

Case [3]: Let $\delta a_n = 2$. If $\delta = 2$ and $a_n = 1$, then $a_1 = \cdots = a_n = 1$. Since $2(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1} \neq \emptyset$, we have $(1, ..., 1) \in 2(\mathcal{P} - \partial\mathcal{P})$. Hence, $2(d-1) < n-1 < 2d$, i.e., $n = 2d$, which is a special case of (c) as above. On the other hand, if $\delta = 1$ and $a_n = 2$, then $a_i \leq 2$ for every i. Since $(\mathcal{P} - \partial\mathcal{P}) \bigcap \mathbb{Z}^{n-1} \neq \emptyset$, we have $a_i = 2$ for every $1 \leq i \leq n$ and $(1, ..., 1) \in \mathcal{P} - \partial\mathcal{P}$. Hence, $d - 2 < n - 1 < d$, i.e., $n = d$, which is a special case of (b) as above.

Case [4]: Let $\delta a_i = 2$ for every $1 \leq i \leq n-1$. First, if $\delta = 2$, then $a_i = 1$ for every $1 \leq i \leq n-1$. Since $\mathcal{Q} = 2\mathcal{P} - (1, \ldots, 1)$ is defined by the linear inequalities $-1 \leq x_i \leq 1$ for $i = 1, \ldots, n-1$ together with $2(d - a_n) - (n-1) \leq x_1 + \cdots + x_{n-1} \leq 2d - (n-1)$, it follows that $A(\mathcal{P})$ is Gorenstein if and only if (i) $n = 2d$ if $d < n-1$, and (ii) $n = 2(d - a_n + 1)$ if $d > a_n$. Let $d < n-1$ and $d > a_n$. Then, $n = 2d$ and $a_n = 1$, which is a special case of (c) as above. Let $d < n-1$ and $d = a_n$. Then, $n = 2d$ and $a_n = d$, which is again a special case of (c) as above. Let $d \geq n-1$ and $d > a_n$. Then, $2(a_1 + \cdots + a_n - d) = 2(n - 1 + a_n - d) = 2n - 2(d - a_n + 1) = n$ and $d \geq n - 1 = \sum_{j=1}^{n} a_j - a_n$. Moreover, if $d \geq a_2 + \cdots + a_n = n - 2 + a_n$, then $d \geq 2(d - a_n + 1) - 2 + a_n = 2d - a_n$, i.e., $d \leq a_n$, a contradiction. Hence, $d < a_2 + \cdots + a_n$; thus we have a special case of (f) as above. Let $d \geq n-1$ and $d = a_n$. Then,

(g) $$a_1 = \cdots = a_{n-1} = 1, \; a_n = d, \text{ and } d \geq n-1.$$

Secondly, if $\delta = 1$, then $a_i = 2$ for every $1 \leq i \leq n-1$. Since $\mathcal{Q} = \mathcal{P} - (1, \ldots, 1)$ is defined by the linear inequalities $-1 \leq x_i \leq 1$ for $i = 1, \ldots, n-1$

together with $d-a_n-(n-1) \leq x_1+\cdots+x_{n-1} \leq d-(n-1)$, it follows that $A(\mathcal{P})$ is Gorenstein if and only if (i) $n = d$ if $d < 2(n-1)$, and (ii) $n = d - a_n + 2$ if $d > a_n$. Let $d < 2(n-1)$ and $d > a_n$. Then, $n = d$ and $a_n = 2$, which is a special case of (b) as above. Let $d < 2(n-1)$ and $d = a_n$. Then, $n = d$ and $a_n = d$, which is again a special case of (b) as above. Let $d \geq 2(n-1)$ and $d > a_n$. Then, $a_1+\cdots+a_n-d = 2(n-1)+a_n-d = 2n-(d-a_n+2) = n$ and $d \geq 2(n-1) = \sum_{j=1}^{n} a_j - a_n$. Moreover, if $d \geq a_2+\cdots+a_n = 2(n-2)+a_n$, then $d \geq 2(d-a_n+2)-4+a_n = 2d-a_n$, i.e., $d \leq a_n$, a contradiction. Hence, $d < a_2+\cdots+a_n$; thus we have a special case of (e) as above. Let $d \geq 2(n-1)$ and $d = a_n$. Then,

(h) $$a_1 = \cdots = a_{n-1} = 2,\ a_n = d, \text{ and } d \geq 2(n-1).$$

We now complete our classification of all the Gorenstein algebras $A(\mathbf{a}; d)$ of Veronese type. Q. E. D.

(2.4) EXAMPLE. (1) Let $n \geq 2$ and each $a_i = d$. Thus, $A(\mathbf{a}; d)$ is the d-th Veronese subring of $K[t_1, t_2, \ldots, t_n]$. Then, $A(\mathbf{a}; d)$ is Gorenstein if and only if d divides n. This result is obtained in [Mat]. See also [Got].

(2) Let $n \geq 2$ and each $a_i = 1$. Thus, $A(\mathbf{a}; d)$ is the K-subalgebra of $K[t_1, t_2, \ldots, t_n]$ generated by all square-free monomials of degree d. Then, $A(\mathbf{a}; d)$ is Gorenstein if and only if (i) $d = 1$, or (ii) $d = n-1$, or (iii) $n = 2d$.

(3) An example of each of the cases (d), (e) and (f) of (2.3) is as follows: (d) $n = 3$, $\mathbf{a} = (3, 5, 7)$, $d = 12$; (e) $n = 3$, $\mathbf{a} = (2, 3, 5)$, $d = 7$; (f) $n = 4$, $\mathbf{a} = (1, 1, 3, 5)$, $d = 8$.

(4) Let $n = 3$. Then, $\mathbf{a} = (1, 1, d)$ with $d \geq 2$ is an example of the case (g) of (2.3), and $\mathbf{a} = (2, 2, d)$ with $d \geq 4$ is an example of the case (h) of (2.3). We should remark that, if $\mathcal{P} \subset \mathbb{R}^{n-1}$ is the unit cube, i.e., the convex polytope defined by the linear inequalities $0 \leq x_i \leq 1$ for $i = 1, \ldots, n-1$, then every Gorenstein algebra of the case (g) of (2.3) is isomorphic to the Ehrhart ring $A(\mathcal{P})$ of $\mathcal{P}$ and, moreover, every Gorenstein algebra of the case (h) of (2.3) is isomorphic to the Veronese subring $A(\mathcal{P})^{(2)}$ of $A(\mathcal{P})$.

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