研 究 集 会

第20回可換環論シンポジウム

(The 20th Symposium on Commutative Algebra in Japan)

1998年11月24日～27日

於 大阪商工会議所・賢島研修センター

(Kashikojima, Japan)

平成10年度文部省科学研究費基盤(A)(1)
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序 (Preface)

この報告集に収録されている原稿は、第 20 回可換環論シンポジウムの講演の記録です。本研究集会は 1998 年 11 月 24 日から 11 月 27 日にかけて、大阪商工会議所・賢島研修センター（三重県志摩郡）において開催されました。この研究集会には、国内 (約 60 名) の研究者・大学院生の他、海外からも 3 名が参加し、合計 19 もの興味深い講演が行われました。特に, Karen E. Smith 氏, Sunsook Noh 氏, 及び 兼田 正治氏にはお忙しい中, 快く招待講演を引き受けて下さり, 可換環論の今後の方向性を探る上でも意義のある講演を聴くことができ, 大変感謝致しております。また, 種々の注文にも関わらず, 素晴らしい講演をしていただいた皆様には, この場を借りて感謝したいと思っております。

シンポジウム開催にあたり、旅費・会場費などは、東北大学の石田正則氏を研究代表者とする文部省科学研究費基盤 (A)(1) から援助を受けました。ここにあらためて感謝致します。

1999 年 2 月

名古屋大学　橋本光靖
吉田健一

第２０回可換環論シンポジウム・プログラム

１１月２４日（火）

19:00～19:05　あいさつと諸注意

19:05～20:05　柳川 浩二（阪大・理）
Associated primes of (co-)generic monomial ideals

20:20～21:00　大杉 英史（阪大・理）
Combinatorial pure subrings

１１月２５日（水）

9:00～9:40　尼崎 睦実（広島大・学校教育）
Remark on free resolutions of maximal Buchsbaum modules over Gorenstein local rings

10:00～10:40　宮崎 誓（長野高専・一般科）
Zero-dimensional schemes and the Castelnuovo-Mumford regularity in positive characteristic (joint work with E. Ballico)

11:00～12:00　Karen E. Smith（Univ. of Michigan）
Tight closure, I

13:20～14:20　蔵野 和彦（都立大・理）
標数 0 での Dutta multiplicity の正値性

14:40～15:40　Karen E. Smith（Univ. of Michigan）
Tight closure, II

16:00～17:00　渡辺 敬一 (日大・文理)
Hilbert-Kunz multiplicity — Many questions and very few answers

19:00～20:00　兼田 正治（大阪市大・理）
$\mathcal{D}$-modules in positive characteristic, I

20:20～21:00　下田 保博（北里大・一般教育）
Relation type に関連したある sequence について

１１月２６日（木）

9:00～10:00　兼田 正治（大阪市大・理）
$\mathcal{D}$-modules in positive characteristic, II

10:20～11:00　山岸 規久道 (姫路獨協大・一般教育)
On the I-invariant of the associated graded rings
of powers of m-primary ideals

11:20～12:00　後藤 四郎 (明治大・理工) 居相 真一郎 (明治大・理工)
On the Gorensteinness of graded rings associated to ideals

14:00～14:40　Sunsook Noh (Ewha Womans Univ.)
Prime divisors of 2-dimensional regular local rings and their valuation ideals

15:00～15:40　岸本 崇 (阪大・理)
On a projective plane curve whose complement has
logarithmic Kodaira dimension 1

16:00～17:00　浅沼 照雄 （富山大・教育）
On automorphism group of $A^{[1]}$

18:00～20:00　懇親会

１１月２７日（金）

9:00～9:40　寺井 直樹 （佐賀大・文化教育）
Monomial ideal サポートの local cohomology module について

10:00～10:40　加藤 希理子 （立命館大・理工）
Modules with certain homology

11:00～12:00　宮崎 充弘 (京都教育大) 吉野 雄二（京大・総人）
On heights and grades of determinantal ideals

Schedule of Talks

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19:00 ~ 19:05 Opening

19:05 ~ 20:05 Kohji Yanagawa (Osaka Univ.)
Associated primes of (co-)generic monomial ideals

20:20 ~ 21:00 Hidefumi Ohsugi (Osaka Univ.)
Combinatorial pure subrings

Wednesday, November 25

9:00 ~ 9:40 Mutsumi Amasaki (Hiroshima Univ.)
Remark on free resolutions of maximal Buchsbaum modules over Gorenstein local rings

10:00 ~ 10:40 Chikashi Miyazaki (Nagano National College of Technology)
Zero-dimensional schemes and the Castelnuovo-Mumford regularity in positive characteristic (joint work with E. Ballico)

11:00 ~ 12:00 Karen E. Smith (Univ. of Michigan)
Tight closure, I

13:20 ~ 14:20 Kazuhiko Kurano (Tokyo Metropolitan Univ.)
Dutta multiplicity in characteristic zero

14:40 ~ 15:40 Karen E. Smith (Univ. of Michigan)
Tight closure, II

16:00 ~ 17:00 Kei-ichi Watanabe (Nihon Univ.)
Hilbert-Kunz multiplicity — Many questions and very few answers

19:00 ~ 20:00 Masaharu Kaneda (Osaka City Univ.)
$\mathcal{D}$-modules in positive characteristic, I

20:20 ~ 21:00 Yasuhiro Shimoda (Kitasato Univ.)
On the sequences relevant of the relation type

Thursday, November 26

9:00 ~ 10:00	Masaharu Kaneda (Osaka City Univ.) $\mathcal{D}$-modules in positive characteristic, II
10:20 ~ 11:00	Kikumichi Yamagishi (Himeji Dokkyo Univ.) On the I-invariant of the associated graded rings of powers of m-primary ideals
11:20 ~ 12:00	Shiro Goto (Meiji Univ.) Shin-ichiro Iai (Meiji Univ.) On the Gorensteinness of graded rings associated to ideals
14:00 ~ 14:40	Sunsook Noh (Ewha Womans Univ.) Prime divisors of 2-dimensional regular local rings and their valuation ideals
15:00 ~ 15:40	Takashi Kishimoto (Osaka Univ.) On a projective plane curve whose complement has logarithmic Kodaira dimension 1
16:00 ~ 17:00	Teruo Asanuma (Toyama Univ.) On automorphism group of $A^{[1]}$
18:00 ~ 20:00	Conference Dinner

Friday, November 27

9:00 ~ 9:40	Naoki Terai (Saga Univ.) On local cohomology modules with respect to monomial ideals
10:00 ~ 10:40	Kiriko Kato (Ritsumeikan Univ.) Modules with certain homology
11:00 ~ 12:00	Mitsuhiro Miyazaki (Kyoto Univ. of Education) Yuji Yoshino (Kyoto Univ.) On heights and grades of determinantal ideals

参加者リスト (List of Participants)

青木 尚代	(大阪大学)	佐藤 眞久	(山梨大学)
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浅沼 照雄	(富山大学)	鈴木 直義	(静岡県立大学)
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池田 ひでみ	(中京学院大学)	中村 幸男	(明治大学)
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五十川 読	(八代高専)	西田 康二	(千葉大学)
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後藤 四郎	(明治大学)	Hei Sook Lee	(Ewha Womans Univ.)

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２０１３年６月

橋本光靖

LATTICE IDEALS, THEIR INITIAL IDEALS AND (CO-)GENERIC MONOMIAL IDEALS

KOHJI YANAGAWA

本稿は, 基本的に E. Miller, B. Sturmfels 両氏との共同研究 [9] の短縮版です. 多くの結果について, 証明を省略してあります. 反面, [9] には含まれない結果や話題も, 幾つか述べられています (特に §3). また, 今回のシンポジウムでの講演中に頂いた質問に答える形で, 例を少し追加しました. 貴重なご意見を下さった参加者の皆様に, 感謝いたします.

Abstract. Monomial ideals which are generic with respect to either their generators or irreducible components have minimal free resolutions derived from simplicial complexes. For a generic monomial ideal, the associated primes satisfy a saturated chain condition, and the Cohen-Macaulay property implies shellability for both the Scarf complex and the Stanley-Reisner complex. Reverse lexicographic initial ideals of generic lattice ideals are generic. Cohen-Macaulayness for cogeneric ideals is characterized combinatorially; in the cogeneric case the Cohen-Macaulay type is greater than or equal to the number of irreducible components. Methods of proof include Alexander duality and Stanley's theory of local h-vectors.

1. Genericity of Monomial Ideals Revisited

Let M be a monomial ideal minimally generated by monomials $m_1, \dots, m_r$ in a polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. For a subset $\sigma \subseteq \{1, \dots, r\}$, we set $m_\sigma := \operatorname{lcm}(m_i \,|\, i \in \sigma)$, and $\mathbf{a}_\sigma := \deg m_\sigma \in \mathbb{N}^n$ the exponent vector of m_σ. Here $m_\emptyset = 1$. For a monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, we set $\deg_{x_i}(\mathbf{x}^{\mathbf{a}}) := a_i$, and we call $\operatorname{supp}(\mathbf{x}^{\mathbf{a}}) := \{i \,|\, a_i \neq 0\} \subseteq \{1, \dots, n\}$ the *support* of $\mathbf{x}^{\mathbf{a}}$.

Definition 1.1. A monomial ideal $M = \langle m_1, \dots, m_r \rangle$ is called *generic* if for any two distinct generators m_i, m_j of M which have the same positive degree in some variable x_s there exists a third monomial generator $m_l \in M$ which divides $m_{\{i,j\}} = \operatorname{lcm}(m_i, m_j)$ and satisfies $\operatorname{supp}(m_{\{i,j\}}/m_l) = \operatorname{supp}(m_{\{i,j\}})$.

The above definition of genericity is more inclusive than the one given by Bayer-Peeva-Sturmfels [1]. In [1], M is called generic if no variable x_s appears with the same non-zero exponent in m_i and m_j for any $i \neq j$. But we will see that this definition permits the same algebraic conclusions as the one in [1]. There are important families of monomial ideals which are generic in the sense of Definition 1.1 but not in the sense of [1]. One such family is the initial ideals of generic lattice ideals as in Theorem 3.2. Here is another one:

Example 1.2. The *tree ideal* $M = \langle \left(\prod_{s \in I} x_s\right)^{n-|I|+1} \mid \emptyset \neq I \subseteq \{1, \dots, n\} \rangle$ is generic in the new sense but very far from generic in the old sense. This ideal is Artinian of colength $(n+1)^{n-1}$, the number of trees on $n+1$ labelled vertices.

Recall that a monomial ideal $M \subset S$ can be uniquely written as a finite irredundant intersection $M = \bigcap_{i=1}^{r} M_i$ of irreducible monomial ideals (i.e., ideals generated by powers of variables). We say M_i is an *irreducible component* of M.

Definition 1.3. A monomial ideal with irreducible decomposition $M = \bigcap_{i=1}^{r} M_i$ is called *cogeneric* if the following condition holds: if distinct irreducible components M_i and M_j have a minimal generator in common, there is an irreducible component $M_l \subset M_i + M_j$ such that M_l and $M_i + M_j$ do not have a minimal generator in common.

A monomial ideal M is cogeneric if and only if its *Alexander dual ideal* $M^{\mathbf{a}}$ is generic. See [8] or Section 4 for the relevant definitions. Cogeneric monomial ideals will be studied in detail in Section 4. The remainder of this section is devoted to basic properties of generic monomial ideals.

Let $M \subset S$ be a monomial ideal minimally generated by monomials $m_1, \dots, m_r$ again. The following simplicial complex on r vertices, called the *Scarf complex* of M, was introduced by Bayer, Peeva and Sturmfels in [1]:

$$\Delta_M \quad := \quad \{\sigma \subseteq \{1, \dots, r\} \mid m_\tau \neq m_\sigma \text{ for all } \tau \neq \sigma\}.$$

Let $S(-\mathbf{a}_\sigma)$ denote the free S-module with one generator e_σ in multidegree $\mathbf{a}_\sigma$. The *algebraic Scarf complex* F_{Δ_M} is the free S-module $\bigoplus_{\sigma \in \Delta_M} S(-\mathbf{a}_\sigma)$ with the differential

$$d(e_\sigma) \quad = \quad \sum_{i \in \sigma} \operatorname{sign}(i, \sigma) \cdot \frac{m_\sigma}{m_{\sigma \setminus \{i\}}} \cdot e_{\sigma \setminus \{i\}}$$

where $\operatorname{sign}(i, \sigma)$ is $(-1)^{j+1}$ if i is the j-th element in the ordering of σ. It is known that F_{Δ_M} is always contained in the minimal free resolution of S/M as a subcomplex [1, §3], although F_{Δ_M} need not be acyclic in general. However we will see in Theorem 1.4 that it is acyclic if M is generic, as was the case under the old definition.

Theorem 1.4. *If M is a generic monomial ideal, then the algebraic Scarf complex F_{Δ_M} equals the minimal free resolution of S/M.*

For an arbitrary monomial ideal M, Bayer and Sturmfels [2, §2] constructed a polyhedral complex hull(M) supporting a (not necessarily minimal) free resolution of M. Definition 1.1 suffices to imply that the hull complex equals the Scarf complex:

Proposition 1.5. *If M is a generic monomial ideal, then the hull complex* hull(M) *coincides with Δ_M, and in this case the hull resolution $F_{\mathrm{hull}(M)} = F_{\Delta_M}$ is minimal.*

Example 1.2 *(continued)* The Scarf complex Δ_M of M is the first barycentric subdivision of the $(n-1)$-simplex. By Theorem 1.4, F_{Δ_M} gives a minimal free resolution of S/M. Miller [8] also constructed a minimal free resolution of S/M as a *cohull resolution*, derived essentially from the coboundary complex of a permutahedron.

2. Associated Primes and Irreducible Components

In this section we study the primary decomposition of a generic monomial ideal M.

Let $M = \bigcap_{i=1}^{r} M_i$ be the irreducible decomposition of a monomial ideal M. Then we have $\{\mathrm{rad}(M_i) \mid 1 \leq i \leq r\} = \mathrm{Ass}(S/M)$. Note that distinct irreducible components may have the same radical. Bayer, Peeva and Sturmfels [1, §3] give a method for computing the irreducible decomposition of a generic monomial ideal (in the old definition). The generalization of this method by Miller [8, Theorem 5.12] shows that [1, Theorem 3.7] remains valid here, as we will show in Theorem 2.2 below.

Recall that $\mathrm{codim}(I) \leq \mathrm{codim}(P) \leq \mathrm{proj\text{-}dim}_S(S/I) \leq n$ for any graded ideal $I \subset S$ and any associated prime $P \in \mathrm{Ass}(S/I)$. Of course, there always exists a minimal prime $P \in \mathrm{Ass}(S/I)$ with $\mathrm{codim}(P) = \mathrm{codim}(I)$, but there is no $P \in \mathrm{Ass}(S/I)$ with $\mathrm{codim}(P) = \mathrm{proj\text{-}dim}_S(S/I)$ in general.

Theorem 2.1. *Let $M \subset S$ be a generic monomial ideal. Then*

(a) For each integer i with $\mathrm{codim}(M) < i \leq \mathrm{proj\text{-}dim}_S(S/M)$, there is an embedded associated prime $P \in \mathrm{Ass}(S/M)$ with $\mathrm{codim}(P) = i$.

(b) For all $P \in \mathrm{Ass}(S/M)$ there is a chain of associated primes $P = P_0 \supset P_1 \supset \cdots \supset P_t$ with $\mathrm{codim}(P_i) = \mathrm{codim}(P_{i-1}) - 1$ for all i and P_t is a minimal prime of M.

Remark 2.2. Let $M \subset S$ be a generic monomial ideal, and $P, P' \in \mathrm{Ass}(S/M)$ such that $P \supset P'$ and $\mathrm{codim}\, P \geq \mathrm{codim}\, P' + 2$. Theorem 2.1 does *not* state that there is an associated prime between P and P'. For example, set $M = \langle ac, bd, a^3b^2, a^2b^3\rangle$. Then $\langle a, b\rangle, \langle a, b, c, d\rangle \in \mathrm{Ass}(S/M)$, but there is no associated prime between them.

Following [1, §3], we next define the *extended Scarf complex* Δ_{M^*} of M. Let

$$M^* \;:=\; M + \langle x_1^D, \ldots, x_n^D\rangle \qquad (1)$$

with D larger than any exponent on any minimal generator of M. We index the new monomials x_s^D just by their variables x_s; so the vertex set of Δ_{M^*} is a subset of $\{1, \ldots, r\} \cup \{x_1, \ldots, x_n\}$. This subset is proper if M contains a power of a variable. Recall ([1, Corollary 5.5] for the old genericity or [8, Proposition 5.16] for the new) that Δ_{M^*} is a regular triangulation of an $(n-1)$-simplex Δ. The vertex set of Δ equals $\{x_1, \ldots, x_n\}$ unless M contains a power of a variable. The restriction of Δ_{M^*} to $\{1, \ldots, r\}$ equals the Scarf complex Δ_M of M. We next determine the restriction of Δ_{M^*} to $\{x_1, \ldots, x_n\}$.

The radical $\mathrm{rad}(M)$ of M is a square-free monomial ideal. Let $V(M)$ denote the corresponding *Stanley-Reisner complex*, which consists of all subsets of $\{x_1, \ldots, x_n\}$ which are not support sets of monomials in M. Then we have the following:

Lemma 2.3. *For a generic monomial ideal M, the restriction of the extended Scarf complex Δ_{M^*} to $\{x_1, \ldots, x_n\}$ coincides with the Stanley-Reisner complex $V(M)$.*

The following theorem generalizes [17, Corollary 2.4]. For the definition of shellability, see [12, §III.2] or [18, Lecture 8].

Theorem 2.4. *Let M be a generic monomial ideal. If M has no embedded associated primes, then M is Cohen-Macaulay. In this case, both Δ_M and $V(M)$ are shellable.*

Proof. The first statement immediately follows from Theorem 2.1. For the second statement we note that all facets σ of Δ_{M^*} have the following property:

$$|\sigma \cap \{1, \ldots, r\}| = \mathrm{codim}\, M \quad \text{and} \quad |\sigma \cap \{x_1, \ldots, x_n\}| = \dim S/M. \qquad (2)$$

In particular, both cardinalities in (2) are independent of the facet σ. On the other hand, Δ_{M^*} is shellable since it is a regular triangulation of a simplex. A theorem of Björner [3, Theorem 11.13] implies that the restrictions of Δ_{M^*} to $\{1, 2, \ldots, r\}$ and to $\{x_1, \ldots, x_n\}$ are both shellable. We are done in view of Lemma 2.3. □

If we put further restrictions on the generators of a generic monomial ideal M, then, since the extended Scarf complex Δ_{M^*} is a triangulation of a simplex, we can apply Stanley's theory of local h-vectors [12]. The next result will be reinterpreted in Section 4 in terms of cogeneric ideals using Alexander duality [8].

Again let M^* be as in (1), and define the *excess* of a face $\sigma \in \Delta_{M^*}$ to be $e(\sigma) := \#\operatorname{supp}(m_\sigma) - \#\sigma$. This agrees, in our situation, with the definition of excess in [12].

Theorem 2.5. *If M is generic and all r generators $m_1, \ldots, m_r$ have support of size c, i.e. $\#\operatorname{supp}(m_i) = c$ for all i, then M has at least $(c-1) \cdot r + 1$ irreducible components.*

Example 2.6. This is false without the assumption that M is generic. For instance, the non-generic monomial ideal $M = \langle x_1, y_1 \rangle \cap \ldots \cap \langle x_n, y_n \rangle$ has $r = 2^n$ generators, and each generator has support of size $c = n$, but M has only n irreducible components.

Proof. If $c = 1$, there is nothing to prove, so we may assume that $c \geq 2$. Set $\Gamma = \Delta_{M^*}$. The hypothesis on the generators of M means that Γ has n vertices of excess 0 and r vertices of excess $c-1$. To prove the assertion, we use the decomposition

$$h(\Gamma, x) \;=\; \sum_{W \in \Delta} \ell_W(\Gamma_W, x) \tag{3}$$

of the h-polynomial of Γ into local h-polynomials [12, eqn. (3)]. Here Δ denotes the simplex on $\{x_1, \ldots, x_n\}$ and Γ_W the restriction of Γ to a face W of Δ. We have

$$\ell_W(\Gamma_W, x) = 1 \qquad \text{if } W = \emptyset. \tag{4}$$

Next, we consider the case $\#W = c$. In the Γ_W, the vertices corresponding to generators of M have excess $c-1$, and all other faces have excess less than $c-1$. So we have

$$\ell_W(\Gamma_W, x) = \ell_1(\Gamma_W)x + \ell_2(\Gamma_W)x^2 + \cdots + \ell_{c-1}(\Gamma_W)x^{c-1} \qquad \text{if } \#W = c, \tag{5}$$

where $\ell_1(\Gamma_W)$ is the number of generators of M whose support corresponds to the face W of Δ by [12, Example 2.3(f)]. Moreover $\ell_i(\Gamma_W) \geq \ell_1(\Gamma_W)$ for all $1 \leq i \leq c-1$ by [12, Theorem 5.2 and Theorem 3.3].

The coefficients of $\ell_W(\Gamma_W, x)$ are non-negative for all $W \in \Delta$ by [12, Corollary 4.7]. We now substitute the expressions in (4) and (5) into the sum on the right hand side of (3), and then we evaluate at $x = 1$. The number of irreducible components of M equals the number $f_{n-1}(\Gamma) = h(\Gamma, 1)$ of facets of Γ by [8, Theorem 5.12], hence

$$h(\Gamma, 1) \;\geq\; 1 + \sum_{\#W=c} \Bigl(\sum_{i=1}^{c-1} \ell_i(\Gamma_W)\Bigr) \;\geq\; 1 + \sum_{\#W=c} (c-1) \cdot \ell_1(\Gamma_W) \;=\; (c-1) \cdot r + 1.$$

This yields the desired inequality. □

The inequality in Theorem 2.5 is sharp for all c and r; see Example 4.15 below.

3. Initial Ideals of Lattice Ideals

We fix a sublattice $\mathcal{L}$ of $\mathbb{Z}^n$ which contains no nonnegative vectors. The *lattice ideal* $I_{\mathcal{L}}$ associated to $\mathcal{L}$ is defined by

$$I_{\mathcal{L}} := \langle \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \text{ and } \mathbf{a} - \mathbf{b} \in \mathcal{L} \rangle \subset S,$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The ideal $I_{\mathcal{L}}$ is homogeneous with respect to some grading where $\deg(x_s)$ is a positive integer for each s. In this section, the word "reverse lexicographic term order" means a *degree* reverse lexicographic term order with respect to this grading. We have $\operatorname{codim}(I_{\mathcal{L}}) = \operatorname{rank}(\mathcal{L})$. The ring $S/I_{\mathcal{L}}$ also has a fine grading by $\mathbb{Z}^n/\mathcal{L}$ (cf. [10]).

The following three conditions are equivalent: (a) The abelian group $\mathbb{Z}^n/\mathcal{L}$ is torsion free, (b) $I_{\mathcal{L}}$ is a prime ideal, and (c) $I_{\mathcal{L}}$ is a toric ideal (i.e., $S/I_{\mathcal{L}}$ is an affine semigroup ring). Even if $I_{\mathcal{L}}$ is not prime, all monomials are non-zero divisors of $S/I_{\mathcal{L}}$, and all associated primes of $I_{\mathcal{L}}$ have the same codimension. If I_A is the toric ideal of an integer matrix A, as defined in [15], then I_A coincides with the lattice ideal $I_{\mathcal{L}}$ where $\mathcal{L} \subset \mathbb{Z}^n$ is the kernel of A.

Following Peeva and Sturmfels [11], we call a lattice ideal $I_{\mathcal{L}}$ *generic* if it is generated by binomials with full support, i.e.,

$$I_{\mathcal{L}} = \langle \mathbf{x}^{\mathbf{a}_1} - \mathbf{x}^{\mathbf{b}_1}, \mathbf{x}^{\mathbf{a}_2} - \mathbf{x}^{\mathbf{b}_2}, \ldots, \mathbf{x}^{\mathbf{a}_r} - \mathbf{x}^{\mathbf{b}_r} \rangle$$

where none of the r vectors $\mathbf{a}_i - \mathbf{b}_i \in \mathbb{Z}^n$ has a zero coordinate.

Peeva-Sturmfels [10] also constructed the *algebraic Scarf complex* $F_{\mathcal{L}}$ of a lattice ideal $I_{\mathcal{L}}$. $F_{\mathcal{L}}$ is not acyclic in general, but it is always contained in the minimal free resolution as a subcomplex. The algebraic Scarf complex $F_{\mathcal{L}}$ of a lattice ideal $I_{\mathcal{L}}$ is acyclic (hence $F_{\mathcal{L}}$ is the minimal free resolution of $S/I_{\mathcal{L}}$) in the following cases.

- $I_{\mathcal{L}}$ is not a complete intersection and $\operatorname{codim} I_{\mathcal{L}} = 2$ ([10]).
- $I_{\mathcal{L}}$ is generic ([11]).

Proposition 3.1. *Suppose that the minimal free resolution of a lattice ideal $I_{\mathcal{L}}$ is given by the algebraic Scarf complex (e.g. $I_{\mathcal{L}}$ is generic). Then we have the following.*

(a) The Betti numbers of $S/I_{\mathcal{L}}$ do not depend on the characteristic of k.

(b) If $S/I_{\mathcal{L}}$ satisfies the Serre's condition (S_2), then $S/I_{\mathcal{L}}$ is Cohen-Macaulay.

(c) If $S/I_{\mathcal{L}}$ is Gorenstein, then $I_{\mathcal{L}}$ is a principal ideal.

Next, we will study the initial ideals of these lattice ideals.

Theorem 3.2. *Let $I_{\mathcal{L}}$ be a generic lattice ideal, and M the initial ideal of $I_{\mathcal{L}}$ with respect to a reverse lexicographic term order. Then M is a generic monomial ideal.*

Example 3.3. Theorem 3.2 is false for the old definition of "generic monomial ideal" given in [1]. For example, consider the following generic lattice ideal in $k[a, b, c, d]$:

$$I_{\mathcal{L}} = \langle a^4 - bcd, a^3c^2 - b^2d^2, a^2b^3 - c^2d^2, ab^2c - d^3, b^4 - a^2cd, b^3c^2 - a^3d^2, c^3 - abd \rangle$$

This ideal was featured in [11, Example 4.5]; it defines the toric curve $(t^{20}, t^{24}, t^{25}, t^{31})$. Consider a reverse lexicographic term order with $a > b > c > d$. Then $M = \langle a^4, a^3c^2, a^2b^3, ab^2c, b^4, b^3c^2, c^3 \rangle$. Since a^3c^2 and b^3c^2 are minimal generators of M, it is not generic in the sense of [1]. But M satisfies Definition 1.1 since $ab^2c \in M$. □

An important problem in combinatorial commutative algebra is to characterize those monomial ideals which are initial ideals of lattice ideals. The recent "Chain Theorem" of Hoşten and Thomas [7] provides a remarkable necessary condition.

Theorem 3.4 (Hoşten–Thomas [7]). *Let M be the initial ideal of a lattice ideal $I_{\mathcal{L}}$ with respect to any term order. For each $P \in \mathrm{Ass}(S/M)$, there is a chain of associated primes $P = P_0 \supset P_1 \supset \cdots \supset P_t$ of M such that P_t is a minimal prime and* $\mathrm{codim}(P_i) = \mathrm{codim}(P_{i-1}) - 1$ *for all i.*

In other words, initial ideals of lattice ideals satisfy conclusion (b) of Theorem 2.1, even if they are not generic. We do not know whether part (a) holds as well.

Conjecture 3.5. *Let M be the initial ideal of $I_{\mathcal{L}}$ with respect to some term order. Then there is an associated prime $P \in \mathrm{Ass}(S/M)$ with* $\mathrm{codim}(P) = \text{proj-dim}_S(S/M)$.

Note that all *minimal* primes of an initial ideal M have the same codimension.

Corollary 3.6. *Conjecture 3.5 holds for the reverse lexicographic term order if the lattice ideal $I_{\mathcal{L}}$ is generic.*

Proof. Immediate from Theorem 2.1 and Theorem 3.2. ☐

The following result appears implicitly in the work of Hoşten-Thomas [7] and Peeva-Sturmfels [10].

Lemma 3.7. *Let M be the initial ideal of a lattice ideal $I_{\mathcal{L}}$ with respect to any term order. Then we have* $\text{proj-dim}_S(S/M) \leq 2^c - 1$ *where* $c := \mathrm{codim}\, I_{\mathcal{L}} = \mathrm{codim}\, M$.

Proof. Following [10, Algorithm 8.2], we construct a lattice ideal $I_{\mathcal{L}'}$ in $S[t] = k[x_1, \ldots, x_n, t]$ whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and M respectively. Moreover t is a non-zero divisor of $S[t]/I_{\mathcal{L}'}$, and the codimension of $I_{\mathcal{L}'}$ in $S[t]$ is equal to $\mathrm{codim}(I_{\mathcal{L}})$. Since $S/M = S[t]/(I_{\mathcal{L}'} + \langle t \rangle)$, we have $\text{proj-dim}_S(S/M) = \text{proj-dim}_{S[t]}(S[t]/I_{\mathcal{L}'}) \leq 2^c - 1$. The last inequality follows from [10, Theorem 2.3]. ☐

We note that Conjecture 3.5 is also true in codimension 2. In fact, we can prove more.

Theorem 3.8. *Let M be an initial ideal of a lattice ideal $I_{\mathcal{L}} \subset S$ of codimension 2. Then the minimal free resolution of M is given by the algebraic Scarf complex F_{Δ_M}.*

Proof. If M is a complete intersection, then the assertion is obvious. So we may assume that M is not a complete intersection. Let $I_{\mathcal{L}'} \subset S[t] = k[x_1, \ldots, x_n, t]$ be a lattice ideal whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and M respectively. Since $I_{\mathcal{L}'}$ is not a complete intersection, the algebraic Scarf complex $F_{\mathcal{L}'}$ is the minimal free resolution. Hence the i-faces of Δ_M are in bijection to the $i+1$ faces of $\Delta_{\mathcal{L}'}/\mathcal{L}'$ (see [11] for the definition) for all i by the argument same to [11, Theorem 5.2]. Since t is a non-zero divisor on $S[t]/I_{\mathcal{L}'}$ and $S[t]/(I_{\mathcal{L}} + t) \simeq S/M$, the multi-graded Betti numbers of S/M (over S) coincide with those of $S[t]/I_{\mathcal{L}'}$ (over $S[t]$). By the construction of $F_{\mathcal{L}'}$ and the correspondence between the faces of $\Delta_{\mathcal{L}'}/\mathcal{L}'$ and Δ_M, the multi-graded Betti numbers of S/M are concentrated in Δ_M parts. Thus F_{Δ_M} is the minimal free resolution. ☐

An initial ideal of a codimension 2 lattice ideal may not be generic. Set $I_{\mathcal{L}} := \langle ac - b^2, ad - bc, bd - c^2 \rangle \subset S = k[a,b,c,d]$ be the defining ideal of the twisted cubic curve in $\mathbb{P}^3$. $S/I_{\mathcal{L}}$ is normal and Cohen-Macaulay. It is known that $I_{\mathcal{L}}$ has eight distinct initial ideals, when we consider all possible term orders (see §4 of [14]), but seven of them are *not* generic. We also remark that four of the eight initial ideals are not Cohen-Macaulay and have embedded associated primes of codimension 3.

Corollary 3.9. *Conjecture 3.5 holds for any term order if* $\operatorname{codim}(I_{\mathcal{L}}) = 2$.

Proof. The assertion follows from Theorem 3.8 and [17, Corollary 2.7]. □

The above result also holds for the initial ideal $\mathrm{in}_\omega(I_{\mathcal{L}})$ with respect to a weight vector $\omega \in \mathbb{R}^n$ (c.f. [15]). Note that $\mathrm{in}_\omega(I_{\mathcal{L}})$ is not a monomial ideal in general. For any term order $\prec$, there is a weight vector $\omega \in \mathbb{R}^n$ such that $\mathrm{in}_\prec(I_{\mathcal{L}}) = \mathrm{in}_\omega(I_{\mathcal{L}})$ (c.f. [5]). As the usual term order case, we can construct a lattice ideal $I_{\mathcal{L}'}$ in $S[t] = k[x_1, \dots, x_n, t]$ whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and $\mathrm{in}_\omega(I_{\mathcal{L}})$ respectively. So Proposition 3.7 also holds for $\mathrm{in}_\omega(I_{\mathcal{L}})$.

Theorem 3.10. *Let $I_{\mathcal{L}} \subset S$ be a lattice ideal of codimension 2, and* $\mathrm{in}_\omega(I_{\mathcal{L}})$ *the initial ideal with respect to a weight vector $\omega \in \mathbb{R}^n$. If* $\text{proj-dim}_S(S/\mathrm{in}_\omega(I_{\mathcal{L}})) = 3$ *(equivalently, $S/\mathrm{in}_\omega(I_{\mathcal{L}})$ is not Cohen-Macaulay) and $\mathrm{in}_\omega(I_{\mathcal{L}}) \neq I_{\mathcal{L}}$, then there is a codimension 3 embedded associated prime of* $\mathrm{in}_\omega(I_{\mathcal{L}})$.

4. A Study of Cogeneric Monomial Ideals

Cogeneric monomial ideals were introduced in Definition 1.3. As with genericity, our definition of cogenericity is slightly different from the original one of [16]. In Theorem 4.6 we shall see that the result of [16], an explicit description of the minimal free resolution of a cogeneric monomial ideal, is still true here. In fact, Alexander duality for arbitrary monomial ideals [8] allows us to shorten the construction of this resolution and clarify its relation to Theorem 1.4. For the reader's convenience, we briefly recall the definitions pertaining to Alexander duality. For details see [8].

The maximal $\mathbb{N}^n$-graded ideal $\langle x_1, \dots, x_n \rangle \subset S$ will be denoted by $\mathfrak{m}$. Monomials and irreducible monomial ideals may each be specified by a single vector $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{N}^n$, so we will write $\mathbf{x}^\mathbf{b} = x_1^{b_1} \cdots x_n^{b_n}$ and $\mathfrak{m}^\mathbf{b} = \langle x_s^{b_s} \mid b_s \geq 1 \rangle$. Given a vector $\mathbf{a} = (a_1, \dots, a_n)$ such that $b_s \leq a_s$ for all s, we define the Alexander dual vector $\mathbf{b}^\mathbf{a}$ with respect to $\mathbf{a}$ by setting its s^{th} coordinate to be

$$(\mathbf{b}^\mathbf{a})_s = \begin{cases} a_s + 1 - b_s & \text{if } b_s \geq 1 \\ 0 & \text{if } b_s = 0. \end{cases}$$

Whenever we deal with Alexander duality, we assume that we are given a vector $\mathbf{a}$ such that for each s, the integer a_s is larger than or equal to the s^{th} coordinate of any minimal monomial generator of M. This implies that a_s is also larger than or equal to the s^{th} coordinate of any irreducible component of M, and vice versa. The *Alexander dual ideal* $M^\mathbf{a}$ of M with respect to $\mathbf{a}$ is defined by

$$\begin{aligned} M^\mathbf{a} &= \langle \mathbf{x}^{\mathbf{b}^\mathbf{a}} \mid \mathfrak{m}^\mathbf{b} \text{ is an irreducible component of } M \rangle \\ &= \bigcap \{ \mathfrak{m}^{\mathbf{c}^\mathbf{a}} \mid \mathbf{x}^\mathbf{c} \text{ is a minimal generator of } M \}. \end{aligned}$$

That these two formulas give the same ideal is not obvious; it is equivalent to $(M^{\mathbf{a}})^{\mathbf{a}} = M$. It follows from these statements that M is generic if and only if $M^{\mathbf{a}}$ is cogeneric.

Example 4.1. The following monomial ideal in $S = k[x, y, z]$ is cogeneric:

$$M = \langle yz^2, xz^2, y^2z, xy^2, x^2\rangle = \langle x, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z\rangle.$$

Its Alexander dual with respect to $\mathbf{a} = (2, 2, 2)$ is generic:

$$M^{\mathbf{a}} = \langle x^2y^2, xyz, x^2z^2\rangle = \langle y^2, z\rangle \cap \langle x^2, z\rangle \cap \langle y, z^2\rangle \cap \langle x^2, y\rangle \cap \langle x\rangle.$$

Example 4.2 ([8, Examples 1.9, 5.22]). If M is the tree ideal of Example 1.2 and $\mathbf{a} = (n, \ldots, n)$, then its Alexander dual $M^{\mathbf{a}}$ is the *permutahedron ideal*:

$$M^{\mathbf{a}} = \langle\, x_1^{\pi(1)} x_2^{\pi(2)} \cdots x_n^{\pi(n)} \;:\; \pi \text{ is a permutation of } \{1, 2, \ldots, n\}\,\rangle.$$

Thus the permutahedron ideal is cogeneric. Its minimal free resolution is the *hull resolution*, which is cellular and supported on a permutahedron [2, Example 1.9]. The following discussion reinterprets this resolution as a co-Scarf complex. □

Definition 4.3. Let $M = \bigcap_{i=1}^{r} M_i$ be a cogeneric monomial ideal. Set $\mathbf{a} = (D - 1, \ldots, D-1)$ with D larger than any exponent on any minimal generator of M. The Alexander dual ideal $M^{\mathbf{a}}$ is minimally generated by monomials $m_1, \ldots, m_r$, where $m_i = \mathbf{x}^{\mathbf{b}_i{}^{\mathbf{a}}}$ for $M_i = \mathfrak{m}^{\mathbf{b}_i}$. We define the *co-Scarf complex* $\Delta_M^{\mathbf{a}}$ to be the extended Scarf complex of $M^{\mathbf{a}}$. More precisely, we set $(M^{\mathbf{a}})^* := M^{\mathbf{a}} + \langle x_1^D, \ldots, x_n^D\rangle$ and $\Delta_M^{\mathbf{a}}$ the Scarf complex of $(M^{\mathbf{a}})^*$. Since we index a new monomial x_s^D just by x_s, we see that $\Delta_M^{\mathbf{a}}$ is a simplicial complex on (a subset of) $\{1, \ldots, r, x_1, \ldots, x_n\}$.

Remark 4.4. (a) There is nothing special about our choice of $\mathbf{a}$, except that it makes for convenient notation. Everything we do with $\Delta_M^{\mathbf{a}}$ is independent of which sufficiently large $\mathbf{a}$ is chosen. In particular, the regular triangulation of the $(n-1)$-simplex is independent of $\mathbf{a}$, as is the algebraic co-Scarf complex (Definition 4.5) it determines. We therefore set $\mathbf{a} = (D - 1, \ldots, D - 1)$ for the remainder of this section.

(b) For $\sigma \subseteq \{1, \ldots, r\}$, let M_σ be the irreducible monomial ideal $\sum_{i\in\sigma} M_i$. Then $m_\sigma = \mathbf{x}^{\mathbf{b}^{\mathbf{a}}}$ if $M_\sigma = \mathfrak{m}^{\mathbf{b}}$, and $\Delta_M^{\mathbf{a}} \cap \{1, \ldots, r\} = \{\sigma \subset \{1, \ldots, r\} \mid M_\tau \neq M_\sigma \text{ for all } \tau \neq \sigma\}$ is just the Scarf complex of $M^{\mathbf{a}}$.

A face σ of the co-Scarf complex $\Delta_M^{\mathbf{a}}$ fails to be in the (topological) boundary $\partial\Delta_M^{\mathbf{a}}$ of $\Delta_M^{\mathbf{a}}$ if and only if the monomial m_σ has full support, where m_σ is $\operatorname{lcm}(m_i \mid i \in \sigma)$ under the notation of Definition 4.3. Such a face will be called an *interior face* of $\Delta_M^{\mathbf{a}}$. The set $\operatorname{int}(\Delta_M^{\mathbf{a}})$ of interior faces is closed under taking supersets; that is, $\operatorname{int}(\Delta_M^{\mathbf{a}})$ is a *simplicial cocomplex*. Just as the algebraic Scarf complex is constructed from Δ_M for generic M, we construct an algebraic free complex from $\operatorname{int}(\Delta_M^{\mathbf{a}})$, but this time we use the coboundary map instead of the boundary map. The following is a special kind of *relative cocellular resolution* (in fact a cohull resolution) [8, §5].

Definition 4.5. Let $\mathbf{D} = (D, \ldots, D) \in \mathbb{N}^n$ and $S(\mathbf{a}_\sigma - \mathbf{D})$ be the free S-module with one generator e_σ^* in multidegree $\mathbf{D} - \mathbf{a}_\sigma$. The *algebraic co-Scarf complex* $F^{\Delta_M^{\mathbf{a}}}$

of M is the free S-module

$$\bigoplus_{\sigma \in \mathrm{int}(\Delta_M^{\mathbf{a}})} S(\mathbf{a}_\sigma - \mathbf{D}) \quad \text{with differential} \quad d^*(e_\sigma^*) = \sum_{\substack{i \notin \sigma \\ \sigma \cup \{i\} \in \mathrm{int}(\Delta_M^{\mathbf{a}})}} \mathrm{sign}(i, \sigma \cup \{i\}) \cdot \frac{m_{\sigma \cup \{i\}}}{m_\sigma} \cdot e_{\sigma \cup \{i\}}^*$$

where $\mathrm{sign}(i, \sigma \cup \{i\})$ is $(-1)^{j+1}$ if i is the j-th element in the ordering of $\sigma \cup \{i\}$. Put the summand $S(\mathbf{a}_\sigma - \mathbf{D})$ in homological degree $n - \#\sigma = n - \dim(\sigma) - 1$.

Theorem 4.6. *If M is a cogeneric monomial ideal, then the algebraic co-Scarf complex $F^{\Delta_M^{\mathbf{a}}}$ equals the minimal free resolution of M over S. In particular, M is minimally generated by the set of monomials $\{\mathbf{x}^{\mathbf{D}-\mathbf{a}_\sigma} \mid \sigma$ is a facet of $\Delta_M^{\mathbf{a}}\}$.*

Example 4.1 *(continued)* For the cogeneric ideal $M = \langle x, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z\rangle$, the interior faces of $\Delta_M^{\mathbf{a}}$ are $\{2\}$, $\{1,2\}$, $\{2,3\}$, $\{2,x\}$, $\{2,y\}$, $\{2,z\}$, $\{1,2,x\}$, $\{1,2,y\}$, $\{2,3,x\}$, $\{2,3,z\}$ and $\{2,y,z\}$. The co-Scarf resolution is $0 \to S \to S^5 \to S^5 \to M \to 0$. The generators of M have exponent vectors $\mathbf{D} - \mathbf{a}_{\{1,2,x\}} = (0,1,2)$, $\mathbf{D} - \mathbf{a}_{\{1,2,y\}} = (1,0,2)$, $\mathbf{D} - \mathbf{a}_{\{2,3,x\}} = (0,2,1)$, $\mathbf{D} - \mathbf{a}_{\{2,3,z\}} = (1,2,0)$ and $\mathbf{D} - \mathbf{a}_{\{2,y,z\}} = (2,0,0)$.

Theorem 4.7. *Let $M \subset S$ be a cogeneric monomial ideal of codimension c with the irreducible decomposition $M = \bigcap_{i=1}^r M_i$. Then the following conditions are equivalent.*
(a) S/M is Cohen-Macaulay.
(b) S/M satisfies Serre's condition (S_2).
(c) $\mathrm{codim}\, M_i = c$ *for all* i, *and* $\mathrm{codim}(M_i + M_j) \le c + 1$ *for all edges* $\{i, j\} \in \Delta_M^{\mathbf{a}}$.
(d) Every face of $\Delta_M^{\mathbf{a}}$ has excess $< c$.
(e) $\Delta_M^{\mathbf{a}}$ has no interior faces of dimension $< n - c$.

Remark 4.8. Hartshorne [6] proved that a catenary local ring satisfying Serre's condition (S_2) is pure and connected in codimension 1. The converse is not true even for cogeneric monomial ideals. If we take $M = \langle x, y^2\rangle \cap \langle y, z\rangle \cap \langle z^2, w\rangle$ then S/M is pure and connected in codimension 1, but does not satisfy the condition (S_2); in fact, $\mathrm{depth}(S/M) = 1$. On the other hand, $M' = \langle x, y\rangle \cap \langle y^2, z^2\rangle \cap \langle z, w\rangle$ is Cohen-Macaulay, although $\mathrm{Ass}(M) = \mathrm{Ass}(M')$.

The above theorem and remark leads to a natural question.

Problem 4.9. Which Cohen-Macaulay simplicial complexes have Stanley-Reisner ideal $\mathrm{rad}(M)$ for some Cohen-Macaulay cogeneric monomial ideal M?

Recall that the *type* of a Cohen-Macaulay quotient S/M is the nonzero total Betti number of highest homological degree; if M is cogeneric then this Betti number equals the number of interior faces of minimal dimension in $\Delta_M^{\mathbf{a}}$ by Theorem 4.6.

Theorem 4.10. *Let M be a Cohen-Macaulay cogeneric monomial ideal of codimension ≥ 2. The type of S/M is at least the number of irreducible components of M.*

Recall that S/M is *Gorenstein* if its Cohen-Macaulay type equals 1. This implies:

Corollary 4.11. *Let M be a cogeneric monomial ideal. Then S/M is Gorenstein if and only if M is either a principal ideal or an irreducible ideal.*

Remark 4.12. In the generic monomial ideal case, we have the opposite inequality to the one in Theorem 4.10. More precisely, if M is Cohen-Macaulay and generic then

$$\begin{array}{rcl}\text{Cohen-Macaulay type of } S/M & = & \#\{\text{facets of the Scarf complex } \Delta_M\} \\ \leq \#\{\text{facets of } \Delta_{M^*}\} & = & \#\{\text{irreducible components of } M\},\end{array}$$

because the map $\Delta_{M^*} \to \Delta_M$, $\sigma \mapsto \sigma \cap \{1, \ldots, r\}$ is surjective on facets. Also here, S/M is Gorenstein if and only if it is complete intersection [17, Corollary 2.11].

After we had gotten an algebraic proof of Theorem 4.10, we conjectured the following more general result about arbitrary triangulations of a simplex. Margaret Bayer proved our conjecture for *quasigeometric triangulations*, using local h-vectors [12]. Since the co-Scarf complex is a quasigeometric triangulation, Theorem 4.13 provides a proof of Theorem 4.10.

Theorem 4.13 (M. Bayer, personal communication). *Let $p_1, p_2, \ldots, p_r$ be points which lie in the relative interior of $(c-1)$-faces of a $(n-1)$-simplex Δ. Let Γ be a quasigeometric triangulation of Δ having the p_i among its vertices and having no interior $(n-c-1)$-face. Then the number of interior $(n-c)$-faces is at least r.*

Proof. According to the hypothesis, we have $\sum_{F\in\Delta, \#F=c} f_0(\mathrm{int}(\Gamma_F)) \geq r$, and $f_i(\mathrm{int}(\Gamma)) = 0$ for all $-1 \leq i \leq n-c-1$. By the decomposition of the h-polynomial of Γ into local h-polynomials and the positivity of local h-vectors [12, Theorem 4.6], we have

$$h_{c-1}(\Gamma) \quad = \quad \sum_{F\in\Delta} \ell_{c-1}(\Gamma_F) \quad \geq \quad \sum_{\substack{F\in\Delta \\ \#F=c}} \ell_{c-1}(\Gamma_F).$$

On the other hand, we have seen that $\ell_1(\Gamma_F) = f_0(\mathrm{int}(\Gamma_F))$ in the proof of Theorem 2.5. Since a local h-vector is symmetric [12, Theorem 3.3], we have $\ell_{c-1}(\Gamma_F) = \ell_1(\Gamma_F) = f_0(\mathrm{int}(\Gamma_F))$ for $F \in \Delta$ with $\#F = c$. So

$$h_{c-1}(\Gamma) \quad \geq \quad \sum_{\substack{F\in\Delta \\ \#F=c}} \ell_{c-1}(\Gamma_F) \quad = \quad \sum_{\substack{F\in\Delta \\ \#F=c}} f_0(\mathrm{int}(\Gamma_F)) \geq r.$$

Since the h-vector of $\mathrm{int}(\Gamma)$ is the reverse of the h-vector of Γ (see the comment preceding [13, Theorem 10.5]), we have

$$\begin{aligned} h_{c-1}(\Gamma) &= h_{n+1-c}(\mathrm{int}(\Gamma)) \\ &= \sum_{i=0}^{n-c+1} (-1)^{n+1-c-i} \binom{n-i}{c-1} (f_{i-1}(\mathrm{int}(\Gamma))) \\ &= f_{n-c}(\mathrm{int}(\Gamma)). \end{aligned}$$

Thus, the number of interior $(n-c)$-faces of Γ is at least r. □

Theorem 4.14. *Let M be a cogeneric monomial ideal with r irreducible components, each having the same codimension c. Then M has at least $(c-1) \cdot r + 1$ minimal generators. If M has exactly $(c-1) \cdot r + 1$ generators then S/M is Cohen-Macaulay.*

Proof. The former statement is Alexander dual to Theorem 2.5. To prove the latter statement, we recall the proof of Theorem 2.5. Assume that S/M is not Cohen-Macaulay. Then $\Gamma := \Delta_M^{\mathbf{a}}$ has an edge $\{i,j\}$ whose excess e satisfies $e \geq c$, by Theorem 4.7. Let $W \in \Delta$ be the support of $m_{\{i,j\}}$. Then $\#W = e+2$. By [12, Proposition 2.2],

$$\ell_W(\Gamma_W, x) = \ell_2(\Gamma_W)x^2 + \ell_3(\Gamma_W)x^3 + \cdots,$$

where $\ell_2(\Gamma_W)$ is the number of edges of Γ whose supports are W. So we have $f_{n-1}(\Gamma) = h(\Gamma, 1) \geq (c-1)\cdot r + 1 + \ell_2(\Gamma_W) > (c-1)\cdot r + 1$ by an argument similar to the proof of Theorem 2.5. Since $f_{n-1}(\Gamma)$ is equal to the number of generators of M, the proof is done. □

Let $M = \bigcap_{i=1}^r M_i$ be a cogeneric monomial ideal without codimension 1 component, and $\Gamma := \Delta_M^{\mathbf{a}}$ its co-Scarf complex. Since Γ is shellable, the Stanley-Reisner ring $k[\Gamma]$ is always Cohen-Macaulay. Let $(h_0, h_1, \ldots, h_n)$ be the h-vector of Γ. Since Γ is Cohen-Macaulay, $h_i \geq 0$ for all i. Moreover we have $h_0 = 1$, $h_1 = r$, $\text{proj-dim}_S(S/M) = \min\{i \geq 0 \mid h_i = 0\}$, and the number of minimal generators of M is equal to $f_{n-1}(\Gamma) = \sum_{i=0}^n h_i$. In particular, when M has pure codimension c, then M is Cohen-Macaulay if and only if $h_c = h_{c+1} = \cdots = h_n = 0$. In this case, $k[\Gamma]$ is a level ring (see [4] for the definition), and the Cohen-Macaulay types of both S/M and $k[\Gamma]$ are equal to h_{c-1}. Note that $k[\Gamma]$ can be level, even if M is not Cohen-Macaulay. The essential part of the proof of Theorem 2.5 is to show $h_i \geq h_1$ for all $1 \leq i \leq c-1$, when M has pure codimension c. We can understand Theorem 4.14 more clearly from this point of view.

Example 4.15. (a) The ideal $M = \bigcap_{i=1}^r \langle x_1^i, x_2^i, \cdots, x_{c-1}^i, x_{c-1+i}\rangle$ is cogeneric and has $(c-1)\cdot r + 1$ minimal generators. Thus the inequality in Theorem 4.14 is tight.

(b) The converse of the latter statement of Theorem 4.14 is false. For instance, $M = \langle a^4, b, c\rangle \cap \langle a^2, b^4, d\rangle \cap \langle a, b^3, e\rangle \cap \langle a^3, b^2, e^2\rangle \subset k[a, \ldots, e]$ is a Cohen-Macaulay cogeneric monomial ideal with 4 irreducible components, but M needs 12 generators. We also note that the Cohen-Macaulay type of S/M is 7, this is larger than the number of irreducible components.

But in the codimension 2 case, we can prove the converse of Theorem 4.14.

Proposition 4.16. *Let M be a cogeneric monomial ideal with r irreducible components, all of codimension 2. Then S/M is Cohen-Macaulay if and only if M has exactly $r+1$ generators.*

Proof. Let $(h_0, \cdots, h_n)$ be the h-vector of $\Delta_M^{\mathbf{a}}$. We always have $h_0 = 1$, $h_1 = r$ and $h_i \geq 0$ for all $0 \leq i \leq n$. By the remark before Example 4.15, M needs $\sum_{i=0}^n h_i$ generators, and M is Cohen-Macaulay if and only if $h_2 = h_3 = \cdots = 0$. □

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Combinatorial pure subrings

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Abstract

This manuscript is a brief summary of the paper [8] with Jürgen Herzog and Takayuki Hibi. Based on some fundamental results on combinatorial pure subrings of affine semigroup rings, the following two theorems will be proved: (i) The q-th squarefree Veronese subring of order d, where $2 \leq q < d$, comes from a poset if and only if either $q = 2$ and $3 \leq d \leq 4$, or $q \geq 3$ and $d = q + 1$; (ii) The Lawrence lifting of a homogeneous semigroup ring A is normal if and only if A is unimodular, i.e., all initial ideals of the defining ideal of A are squarefree.

1 Basic results

Let K be a field and $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ the polynomial ring in d variables over K. Let $\mathcal{A} = \{f_1, \ldots, f_n\}$ be a set of monomials belonging to $K[\mathbf{t}]$. Suppose that the affine semigroup ring $K[\mathcal{A}] = K[f_1, \ldots, f_n]$ is a homogeneous K-algebra, i.e., $K[\mathcal{A}]$ is a graded algebra $K[\mathcal{A}] = (K[\mathcal{A}])_0 \bigoplus (K[\mathcal{A}])_1 \bigoplus \cdots$ with $(K[\mathcal{A}])_0 = K$ and with each $f_i \in (K[\mathcal{A}])_1$. Such a semigroup ring $K[\mathcal{A}]$ is called a *homogeneous* semigroup ring. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K with each $\deg x_i = 1$ and let $I_{\mathcal{A}}$ denote the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ defined by $\pi(x_i) = f_i$ for all $1 \leq i \leq n$. We call $I_{\mathcal{A}}$ the *defining ideal* of $K[\mathcal{A}]$.

Let $[d] = \{1, \ldots, d\}$. If T is a nonempty subset of $[d]$, then we write $\mathcal{A}_T$ for the subset $\mathcal{A} \cap K[\{t_j\,;\, j \in T\}]$ of $\mathcal{A}$. A subring of $K[\mathcal{A}]$ of the form $K[\mathcal{A}_T]$ with $\emptyset \neq T \subset [d]$ is called a *combinatorial pure subring* of $K[\mathcal{A}]$. If $\mathcal{A}_T = \{f_{i_1}, f_{i_2}, \ldots, f_{i_r}\}$, then we set $K[\mathbf{x}_T] = K[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$. Thus $I_{\mathcal{A}_T} = I_{\mathcal{A}} \cap K[\mathbf{x}_T]$.

Let $<$ be an arbitrary term order on $K[\mathbf{x}]$ and $g \in I_{\mathcal{A}}$ a binomial of $K[\mathbf{x}]$. If the initial monomial $in_<(g)$ of g belongs to $K[\mathbf{x}_T]$, then g must belong to $K[\mathbf{x}_T]$. In fact, if $g = u - v$ where u and v are monomials of $K[\mathbf{x}]$, then $\pi(u) = \pi(v)$ since $g \in I_{\mathcal{A}}$. Thus $\pi(u) \in K[\{t_j\,;\, j \in T\}]$ if and only if $\pi(v) \in K[\{t_j\,;\, j \in T\}]$. Since $\pi(x_i) \in K[\{t_j\,;\, j \in T\}]$ if and only if $i \in \{i_1, i_2, \ldots, i_r\}$, it follows that $\pi(u) \in K[\{t_j\,;\, j \in T\}]$ if and only if $u \in K[\mathbf{x}_T]$.

This simple observation yields the fundamental result on elimination of Gröbner bases for combinatorial pure subrings.

Proposition 1.1. *If G is the reduced Gröbner basis of $I_{\mathcal{A}}$ with respect to a term order $<$ on $K[\mathbf{x}]$, then $G \cap K[\mathbf{x}_T]$ is the reduced Gröbner basis of $I_{\mathcal{A}_T}$ (with respect to the term order on $K[\mathbf{x}_T]$ induced by $<$).*

Proposition 1.2. *If $K[\mathcal{A}]$ is normal, then any combinatorial pure subring of $K[\mathcal{A}]$ is normal.*

Proposition 1.3. *If $K[\mathcal{A}]$ is Koszul, then any combinatorial pure subrings of $K[\mathcal{A}]$ is Koszul.*

2 Squarefree Veronese subrings

Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K and suppose that I is an ideal of $K[\mathbf{x}]$ which is generated by squarefree quadratic monomials. We say that I is the Stanley–Reisner ideal of the order complex of a finite poset if there exists a partial order on $[n]$ such that I is generated by those squarefree quadratic monomials $x_i x_j$ such that i and j are incomparable in the partial order. A combinatorial criterion for a squarefree quadratic monomial ideal to be the Stanley–Reisner ideal of a finite poset is known in, e.g., [3].

We say that a homogeneous semigroup ring $K[\mathcal{A}]$ *comes from a poset* if $I_{\mathcal{A}}$ possesses an initial ideal which is the Stanley–Reisner ideal of the order complex of a finite poset. For example, every monomial ASL (algebra with straightening laws) discussed in, e.g., [2] comes from a poset. It is shown in [9] that if $K[\mathcal{A}]$ comes from a poset, then the infinite divisor poset of $K[\mathcal{A}]$ is shellable. Here, the infinite divisor poset of $K[\mathcal{A}]$ is the infinite poset consisting of all monomials of $K[\mathcal{A}]$, ordered by divisibility.

Proposition 2.1. *If a homogeneous semigroup ring $K[\mathcal{A}]$ comes from a poset, then any combinatorial pure subring of $K[\mathcal{A}]$ comes from a poset.*

Let $K[t_1, \ldots, t_d]$ be the polynomial ring in d variables over a field K with each $\deg t_j = 1$. Let $2 \leq q < d$. The q-th *squarefree Veronese subring* of order d is the affine semigroup ring $\mathcal{R}_d^{(q)}$ which is generated by all squarefree monomials of degree q belonging to $K[t_1, \ldots, t_d]$. It is known [10] that each $\mathcal{R}_d^{(q)}$ has an initial ideal generated by squarefree quadratic monomials. However, it seems to be unknown if each $\mathcal{R}_d^{(q)}$ comes from a poset.

Theorem 2.2. *Let $2 \leq q < d$. The q-th squarefree Veronese subring of order d comes from a poset if and only if either* (i) $q = 2$ *and* $3 \leq d \leq 4$, *or* (ii) $q \geq 3$ *and* $d = q + 1$.

In [2], it is proved that the infinite divisor posets of the second squarefree Veronese subrings $\mathcal{R}_d^{(2)}$ are shellable ([2, Theorem 4.1]). However, the shellability of the infinite divisor posets of $\mathcal{R}_d^{(2)}$ cannot follow from [9] if $d \geq 5$. It remains open if the infinite divisor posets of *all* squarefree Veronese subrings $\mathcal{R}_d^{(q)}$ with $q \geq 3$ are shellable.

3 Lawrence liftings of semigroup rings

Let, as before, $\mathcal{A} = \{f_1, \ldots, f_n\}$ be a set of monomials of $K[t_1, \ldots, t_d]$ and suppose that the affine semigroup ring $K[\mathcal{A}] = K[f_1, \ldots, f_n]$ is a homogeneous semigroup ring. Let $I_{\mathcal{A}} \subset K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the defining ideal of $K[\mathcal{A}]$.

If $u \in K[\mathbf{x}]$ is a monomial, then we write $\text{supp}(u)$ for the support of u, i.e., $\text{supp}(u)$ is the set of variables x_i which divide u. If $g = u - v$ is a binomial of $K[\mathbf{x}]$, where u and v are monomials of $K[\mathbf{x}]$, then the support of g is $\text{supp}(g) = \text{supp}(u) \cup \text{supp}(v)$.

A binomial $g = u - v \in I_{\mathcal{A}}$ is called *primitive* if there exists no binomial $g' = u' - v' \in I_{\mathcal{A}}$ with $g' \neq g$ such that u' divides u and v' divides v. The set of all primitive binomials of $I_{\mathcal{A}}$ is called the *Graver basis* of $I_{\mathcal{A}}$.

A binomial $g = u - v \in I_{\mathcal{A}}$ is called *circuit* if g is irreducible and if there exists no binomial $g' = u' - v' \in I_{\mathcal{A}}$ with $\text{supp}(g') \subset \text{supp}(g)$ and with $\text{supp}(g') \neq \text{supp}(g)$.

The *universal Gröbner basis* of $I_{\mathcal{A}}$ is the union of all reduced Gröbner bases of $I_{\mathcal{A}}$. Every circuit of $I_{\mathcal{A}}$ belongs to the universal Gröbner basis of $I_{\mathcal{A}}$, and the universal Gröbner basis of $I_{\mathcal{A}}$ is a subset of the Graver basis of $I_{\mathcal{A}}$. See [10, Proposition 4.11].

Let $\Lambda(\mathcal{A}) = \{f_1 z_1, \ldots, f_n z_n, z_1, \ldots, z_n\}$, where $z_1, \ldots, z_n$ are variables over K. The homogeneous semigroup ring

$$K[\Lambda(\mathcal{A})] = K[f_1 z_1, \ldots, f_n z_n, z_1, \ldots, z_n]$$

is called the *Lawrence lifting* of $K[\mathcal{A}]$.

Let $K[\mathbf{x}, \mathbf{y}] = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in $2n$ variables over K. If $u = x_{i_1} x_{i_2} \cdots x_{i_k}$ is a monomial of $K[\mathbf{x}]$, then we write $\bar{u}$ for the monomial $y_{i_1} y_{i_2} \cdots y_{i_k}$ of $K[\mathbf{y}]$. Moreover, if $g = u - v$ is a binomial of $K[\mathbf{x}]$, then we define the binomial $\bar{g}$ of $K[\mathbf{x}, \mathbf{y}]$ by $\bar{g} = u\bar{v} - v\bar{u}$. It then follows that the defining ideal $I_{\Lambda(\mathcal{A})}$ of the Lawrence lifting $K[\Lambda(\mathcal{A})]$ of $K[\mathcal{A}]$ is generated by all binomials $\bar{g}$ with $g \in I_{\mathcal{A}}$. Moreover, the Graver basis of $I_{\Lambda(\mathcal{A})}$ coincides with the set of those binomials $\bar{g}$ such that g belongs to the Graver basis of $I_{\mathcal{A}}$, and the set of circuits of $I_{\Lambda(\mathcal{A})}$ coincides with the set of those binomials $\bar{g}$ such that g is a circuit of $I_{\mathcal{A}}$.

In the present section, we are interested in the question when the Lawrence lifting $K[\Lambda(\mathcal{A})]$ of $K[\mathcal{A}]$ is normal.

We say that a homogeneous semigroup ring $K[\mathcal{A}]$ is *unimodular* if all initial ideals of $I_{\mathcal{A}}$ are squarefree. It follows from [10, Remark 8.10] that $K[\mathcal{A}]$ is unimodular if and only if all triangulations of the configuration associated with $\mathcal{A}$ are unimodular. In addition, $K[\mathcal{A}]$ is unimodular if and only if all lexicographic initial ideals of $K[\mathcal{A}]$ are squarefree.

A quite effective criterion for a homogeneous semigroup ring $K[\mathcal{A}]$ to be unimodular is known as follows. A binomial $g = u - v$ is called *squarefree* if both the monomials u and v are squarefree.

Proposition 3.1. *A homogeneous semigroup ring $K[\mathcal{A}]$ is unimodular if and only if every circuit of $I_{\mathcal{A}}$ is squarefree.*

We are now in the position to give a main result of this section.

Theorem 3.2. *Let $K[\mathcal{A}]$ be a homogeneous semigroup ring and $K[\Lambda(\mathcal{A})]$ its Lawrence lifting. Then, the following conditions are equivalent:*

(i) *$K[\mathcal{A}]$ is unimodular;*

(ii) $K[\Lambda(\mathcal{A})]$ *is unimodular*;
(iii) $K[\Lambda(\mathcal{A})]$ *is normal.*

We conclude this paper with some examples of homogeneous semigroup rings which are unimodular.

Example 3.3. (a) Let $\mathcal{R}_K[L]$ denote the monomial ASL (algebra with straightening laws) associated with a finite distributive lattice L discussed in [5]. Then, $\mathcal{R}_K[L]$ is unimodular if and only if L is planar. See also [1].

(b) Let $K[G]$ denote the homogeneous semigroup ring arising from a finite connected graph G studied in, e.g., [6] and [7]. Then, $K[G]$ is unimodular if and only if any two cycles of odd length of G possess a common vertex. In particular, $K[G]$ is unimodular if G is bipartite.

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Remark on free resolutions of maximal Buchsbaum modules over Gorenstein local rings

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Abstract

Let M be a maximal quasi-Buchsbaum module over a Gorenstein local ring $(A, \mathfrak{m})$ of dimension d. Assume that M has no free direct summand and let

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

be a minimal free resolution of M over A and

$$\cdots \xrightarrow{\delta_{-2}^{\vee}} P_{-2}^{\vee} \xrightarrow{\delta_{-1}^{\vee}} P_{-1}^{\vee} \xrightarrow{\delta_{0}^{\vee}} P_{0}^{\vee} \xrightarrow{\delta_{1}^{\vee}} \mathrm{Im}(\delta_1^{\vee}) \longrightarrow 0,$$

a minimal free resolution of $\mathrm{Im}(\delta_1^{\vee})$ over A. We put $G_i = P_{d-i}^{\vee}$, $\gamma_i = \delta_{d-i+1}^{\vee}$ for $i \in \mathbf{Z}$ and let $(G_\bullet, \gamma_\bullet)$ be the minimal complex thus obtained (cf. [2, (4.2)]). Then M is quasi-Buchsbaum if and only if $G_\bullet$ is the minimal part of a complex $E_\bullet$ constructed in the following manner (see [2, (1.1)]). Define chain maps

$$\lambda_{0,\bullet} : F_\bullet \longrightarrow (L_{\bullet-1})^{p_0},$$
$$\lambda_{j,\bullet} : \mathrm{con}(\lambda_{j-1,\bullet})_\bullet \longrightarrow (L_{\bullet-j-1})^{p_j} \quad (1 \le j \le d-1)$$

inductively and let $E_\bullet := \mathrm{con}(\lambda_{d-1,\bullet})_\bullet$, where

$$\cdots \longrightarrow L_d \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow k \longrightarrow 0$$

is a minimal free resolution of the residue field k over A, $L_i = 0$ for $i < 0$, $F_\bullet$ is a minimal free exact complex, and $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ denotes the mapping cone of $\lambda_{j-1,\bullet}$ (see [2, (1.6)]). When is M Buchsbaum? We give an answer to this question. Let $K_\bullet$ denote the Koszul complex of $x_1, \ldots, x_r$ with respect to A, where $x_1, \ldots, x_r$ are minimal generators of $\mathfrak{m}$. Observe that $K_\bullet$ is a subcomplex of $L_\bullet$ such that $L_\bullet/K_\bullet$ is free.

Main Theorem. *With the above notation, M is Buchsbaum if and only if we can choose $\lambda_{j,\bullet}$ $(0 \le j \le d-1)$ so that $U_{j-1,\bullet} := \bigoplus_{l=0}^{j-1}(K_{\bullet-l})^{p_l}$ is a subcomplex of $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ with $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet/U_{j-1,\bullet}$ free for each $1 \le j \le d$ and $\lambda_{j,\bullet}|_{U_{j-1,\bullet}} = 0$ for all $1 \le j \le d-1$.*

In this paper, technical results are mostly omitted and proofs are given only to some major parts of the theory. See [3] for the detail.

Buchsbaum cones and maximal Buchsbaum modules

Throughout this paper we denote by $(A, \mathfrak{m})$ a local Gorenstein ring of dimension $d > 0$ with $k := A/\mathfrak{m}$ and $r := \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Let

$$\cdots \longrightarrow L_d \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow k \longrightarrow 0$$

be a minimal free resolution of the residue field k over A. We extend this resolution by setting $L_i = 0$ for $i < 0$ and denote the resulting complex by $L_\bullet$. For minimal generators $x_1, \ldots, x_r$ of $\mathfrak{m}$, let $K(x_1, \ldots, x_l)_\bullet$ denote the Koszul complex of $x_1, \ldots, x_l$ with respect to A for $0 \leq l \leq r$ and $K_\bullet := K(x_1, \ldots, x_r)_\bullet$.

Lemma 1. *With the notation above, we may think of $K_\bullet$ as a subcomplex of $L_\bullet$ such that $L_\bullet/K_\bullet$ is free.*

Let $z_1, \ldots, z_r$ be an A-basis of $\mathfrak{m}$, namely a system of minimal generators of $\mathfrak{m}$ such that $z_{i_1}, \ldots, z_{i_d}$ form a system of parameters of A for every sequence $i_1, \ldots, i_d$ of integers with $1 \leq i_1 < \cdots < i_d \leq r$ (see [12, Chapter I:Definition 1.7]). Recall that such $z_1, \ldots, z_r$ always exist by Proposition 1.9 loc.cit. In the following argument, we set $\mathfrak{B} := \{ (z_{\sigma(1)}, \ldots, z_{\sigma(r)}) \mid \sigma \in \mathfrak{S}_r \} \subset \mathfrak{m}^{\oplus r}$ and assume that $x_1, \ldots, x_r$ are minimal generators of $\mathfrak{m}$ obtained by permuting $z_1, \ldots, z_r$, i.e. $(x_1, \ldots, x_r) \in \mathfrak{B}$, where $\mathfrak{S}_r$ denotes the symmetric group on r letters.

Given an A-module E and an integer n with $0 \leq n \leq d$, we will denote by ${}_{(n)}E$ the module $E/(x_{r-n+1}, \ldots, x_r)E$. Further, for a complex $(S_\bullet, \varphi_\bullet)$, we will denote by $({}_{(n)}S_\bullet, {}_{(n)}\varphi_\bullet)$ the complex obtained by tensoring S_i and φ_i with ${}_{(n)}A$ over A for all $i \in \mathbf{Z}$. By the property of the differential of a Koszul complex, we have

$${}_{(n)}K_\bullet = \bigoplus_{j=0}^{n} \bigoplus_{r-n+1 \leq s_1 < \cdots < s_j \leq r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet - j} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_j},$$

where $\{\chi_1, \ldots, \chi_r\}$ is the free basis of K_1 corresponding to $x_1, \ldots, x_r$. Let

$$K''^{i,n}_\bullet := \bigoplus_{r-n+1 \leq s_1 < \cdots < s_{i-1} \leq r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet-(i-1)} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_{i-1}},$$

$$K'''^{i,n}_\bullet := \bigoplus_{r-n+1 \leq s_1 < \cdots < s_i \leq r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet - i} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_i}$$

for $0 \leq n \leq d$, $i \in \mathbf{Z}$, $l \geq 0$, where we understand $K''^{i,n}_\bullet = 0$ (resp. $K'''^{i,n}_\bullet = 0$) if $n < i - 1$ or $i < 1$ (resp. $n < i$ or $i < 0$). These complexes are subcomplexes of ${}_{(n)}K_\bullet$. Further we denote the direct sum of the remaining summands of ${}_{(n)}K_\bullet$ by $K'^{i,n}_\bullet$, more precisely,

$$K'^{i,n}_\bullet := \bigoplus_{0 \leq j < i-1,\ i < j \leq n} \bigoplus_{r-n+1 \leq s_1 < \cdots < s_j \leq r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet - j} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_j}.$$

We have

$$K_\bullet''^{i,n} = K_\bullet'''^{i-1,n}, \quad {}_{(n)}K_\bullet = K_\bullet'^{i,n} \oplus K_\bullet''^{i,n} \oplus K_\bullet'''^{i,n}.$$

Since $K_\bullet$ is a subcomplex of $L_\bullet$ by Lemma 1, there are natural inclusions of $K_\bullet''^{i,n}$ and $K_\bullet'''^{i,n}$ into ${}_{(n)}L_\bullet$.

Let $F_\bullet$ be a minimal free exact complex, namely a free complex satisfying $\mathrm{Im}(\partial_i) \subset \mathfrak{m}F_{i-1}$ and $H_i(F_\bullet) = 0$ for all $i \in \mathbf{Z}$, where $\partial_\bullet$ is the differential of $F_\bullet$. Let further m be an integer with $1 \leq m \leq d$ and p_i $(0 \leq i \leq m-1)$ nonnegative integers. We say that a complex $E_\bullet$ is a *Buchsbaum cone* with base $(F_\bullet, p_0, \dots, p_{m-1})$, if it satisfies the following conditions (i) and (ii).

(i) There are inductively defined chain maps

$$\lambda_{0,\bullet} : F_\bullet \longrightarrow (L_{\bullet-1})^{p_0},$$
$$\lambda_{j,\bullet} : \mathrm{con}(\lambda_{j-1,\bullet})_\bullet \longrightarrow (L_{\bullet-j-1})^{p_j} \quad (1 \leq j \leq m-1)$$

and $E_\bullet = \mathrm{con}(\lambda_{m-1,\bullet})_\bullet$, where $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ denotes the mapping cone of $\lambda_{j-1,\bullet}$.

(ii) Let $U_{j,i} := \bigoplus_{l=0}^{j}(K_{i-l})^{p_l} \subset F_i \oplus \left(\bigoplus_{l=0}^{j}(L_{i-l})^{p_l}\right) = \mathrm{con}(\lambda_{j,\bullet})_i$ for $0 \leq j \leq m-1$. Then $\lambda_{j,i}|_{U_{j-1,i}} = 0$ for all $1 \leq j \leq m-1$, $i \in \mathbf{Z}$. In other words, $U_{j-1,\bullet} := \bigoplus_{l=0}^{j-1}(K_{\bullet-l})^{p_l}$ is a subcomplex of $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ such that $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet/U_{j-1,\bullet}$ is free for each $1 \leq j \leq m$ and $\lambda_{j,\bullet}|_{U_{j-1,\bullet}} = 0$ for all $1 \leq j \leq m-1$.

If a complex $E_\bullet$ satisfies the condition (i) above, we call it a *quasi-Buchsbaum cone* with base $(F_\bullet, p_0, \dots, p_{m-1})$.

We investigate necessary and sufficient conditions for a quasi-Buchsbaum cone to be a Buchsbaum cone, and by doing so, characterize complexes defining maximal Buchsbaum modules.

Given an integer m with $1 \leq m \leq d$, nonnegative integers p_l $(0 \leq l \leq m-1)$ and minimal generators $x_1, \dots, x_r$ of $\mathfrak{m}$, we define

$$(*) \quad \begin{cases} U_\bullet := \bigoplus_{l=0}^{m-2}(K_{\bullet-l})^{p_l}, & U_\bullet''^{i,n} := \bigoplus_{l=0}^{m-2}(K''^{\,i-l,n}_{\bullet-l})^{p_l}, & U_\bullet'''^{i,n} := \bigoplus_{l=0}^{m-2}(K'''^{\,i-l,n}_{\bullet-l})^{p_l}, \\ W_\bullet := \bigoplus_{l=0}^{m-1}(K_{\bullet-l})^{p_l}, & W_\bullet''^{i,n} := \bigoplus_{l=0}^{m-1}(K''^{\,i-l,n}_{\bullet-l})^{p_l}, & W_\bullet'''^{i,n} := \bigoplus_{l=0}^{m-1}(K'''^{\,i-l,n}_{\bullet-l})^{p_l}, \end{cases}$$

where $U_\bullet = U_\bullet''^{i,n} = U_\bullet'''^{i,n} = 0$ in case $m = 1$. For a while, we fix such m and $p_0, \dots, p_{m-1}$.

Lemma 2. *Let $S_\bullet$ be a free complex which contains $U_\bullet$ as a free subcomplex such that $S_\bullet/U_\bullet$ is free. Let further $\lambda_\bullet : S_\bullet \longrightarrow (L_{\bullet-m})^{p_{m-1}}$ be a chain map and $C_\bullet := \mathrm{con}(\lambda_\bullet)_\bullet$ its mapping cone. If $H_m({}_{(n)}S_\bullet/(U_\bullet''^{m,n} \oplus U_\bullet'''^{m,n})) = 0$ and $\mathfrak{m}H_{m-1}({}_{(n)}C_\bullet) = 0$ for all $0 \leq n < d$ and $(x_1, \dots, x_r) \in \mathfrak{B}$, then $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$.*

Lemma 3. *Let $(S_\bullet, \varphi_\bullet)$ be a free complex, $U_\bullet$ its free subcomplex with $S_\bullet/U_\bullet$ free, $(T_\bullet, \psi_\bullet)$ a minimal free complex with $T_i = 0$ for $i < m-1$ satisfying $\mathfrak{m}H_{m-1}(T_\bullet) = 0$, $H_i(T_\bullet) = 0$ for $i > m-1$ and $\lambda_\bullet : S_\bullet \longrightarrow T_{\bullet-1}$ a chain map. If $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$, then there is a chain map $\lambda'_\bullet : S_\bullet \longrightarrow T_{\bullet-1}$ satisfying $\lambda'_\bullet|_{U_\bullet} = 0$ which is chain equivalent to $\lambda_\bullet$.*

Proposition 4. *Let $E_\bullet$ be a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$. Let further $W_\bullet$, $W''^{i,n}_\bullet$ and $W'''^{i,n}_\bullet$ be subcomplexes of $E_\bullet$ as in $(*)$. Then the natural homomorphism $H_i(W'''^{i,n}_\bullet) \longrightarrow H_i({}_{(n)}E_\bullet)$ is surjective and $H_i({}_{(n)}E_\bullet/(W''^{i,n}_\bullet \oplus W'''^{i,n}_\bullet)) = 0$ for all $0 \leq n \leq d$, $i \in \mathbf{Z}$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$.*

Corollary 5. *Let $E_\bullet$ be a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$. Then $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ for all $0 \leq n \leq d$, $i \in \mathbf{Z}$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$.*

Proof. Since $W'''^{i,n}_j = 0$ for $j < i$ and

$$W'''^{i,n}_\bullet = \bigoplus_{l=0}^{m-1} \left(\bigoplus_{r-n+1 \leq s_1 < \cdots < s_{i-l} \leq r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet - i} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_{i-l}} \right)^{p_l},$$

we see $\mathfrak{m}H_i(W'''^{i,n}_\bullet) = 0$. Hence $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ by the preceding proposition.

Proposition 6. *Let $E_\bullet$ be a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$. If $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ for all $0 \leq i < m$, $0 \leq n < d$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$, then $E_\bullet$ is chain isomorphic to a Buchsbaum cone with the same base.*

Proof. If $m = 1$ there is nothing to prove. Suppose that $m \geq 2$ and that our assertion is true for $m-1$. Denote $\mathrm{con}(\lambda_{m-2,\bullet})_\bullet$ by $D_\bullet$. Since ${}_{(n)}E_\bullet$ is the mapping cone of ${}_{(n)}\lambda_{m-1,\bullet}$ for $0 \leq n < d$, there is an exact sequence

$$0 \longrightarrow {}_{(n)}L_{\bullet-(m-1)} \longrightarrow {}_{(n)}E_\bullet \longrightarrow {}_{(n)}D_\bullet \longrightarrow 0$$

that yields a long exact sequence

$$\cdots \longrightarrow H_i({}_{(n)}E_\bullet) \longrightarrow H_i({}_{(n)}D_\bullet) \longrightarrow H_{i-1}({}_{(n)}L_{\bullet-(m-1)}) \longrightarrow \cdots.$$

Here $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ and ${}_{(n)}L_{(i-1)-(m-1)} = 0$ for all $0 \leq i < m$, so that $\mathfrak{m}H_i({}_{(n)}D_\bullet) = 0$ for all $0 \leq i < m-1$ and $(x_1, \ldots, x_r) \in \mathfrak{B}$. Since $D_\bullet$ is a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-2})$, it is chain isomorphic to a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-2})$ by the induction hypothesis, say $S_\bullet$. By Proposition 4, this $S_\bullet$ satisfies $H_m({}_{(n)}S_\bullet/(U''^{m,n}_\bullet \oplus U'''^{m,n}_\bullet)) = 0$ with the notation of $(*)$. Let $\nu_\bullet : S_\bullet \longrightarrow D_\bullet$ be the isomorphism mentioned above, $\lambda_\bullet := \lambda_{m-1,\bullet} \circ \nu_\bullet$, and $C_\bullet := \mathrm{con}(\lambda_\bullet)_\bullet$. Since $C_\bullet = \mathrm{con}(\lambda_\bullet)_\bullet \cong \mathrm{con}(\lambda_{m-1,\bullet})_\bullet = E_\bullet$, we have $\mathfrak{m}H_{m-1}({}_{(n)}C_\bullet) = 0$ for all $0 \leq n < d$ and $(x_1, \ldots, x_r) \in \mathfrak{B}$ by hypothesis. Hence $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$ by Lemma 2. Finally, let $\lambda'_\bullet : S_\bullet \longrightarrow (L_{\bullet-m})^{p_{m-1}}$ be the chain map satisfying $\lambda'_\bullet \simeq \lambda_\bullet$ and $\lambda'_\bullet|_{U_\bullet} = 0$ as in Lemma 3. Then the mapping cone $\mathrm{con}(\lambda'_\bullet)_\bullet$ is a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$ which is isomorphic to $C_\bullet \cong E_\bullet$.

Theorem 7. *Let m be an integer with $1 \le m \le d$ and let $G_\bullet$ be a minimal free complex satisfying $H^i(G_\bullet^\vee) = 0$ for $i < d$, $H_i(G_\bullet) = 0$ for $i < 0$, $i \ge m$, and $\mathfrak{m}H_i({}_{(n)}G_\bullet) = 0$ for all $0 \le i < m$, $0 \le n < d$, $(x_1, \dots, x_r) \in \mathfrak{B}$. Then there is a Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ such that $G_\bullet = \min(E_\bullet)_\bullet$, where $p_i = l_R(H_i(G_\bullet))$ $(0 \le i \le m-1)$.*

Proof. If $p_i = 0$ for all $0 \le i \le m-1$, then $G_\bullet$ is a Buchsbaum cone with base $(G_\bullet, 0, \dots, 0)$. Suppose that $p_i \ne 0$ for some $0 \le i \le m-1$. With the notation of [2, Section 1], let $F_\bullet := \sigma_0(G_\bullet)$. Then by [2, (1.6)] there is a quasi-Buchsbaum cone $E'_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ such that $G_\bullet = \min(E'_\bullet)_\bullet$. Since $H_i({}_{(n)}G_\bullet) \cong H_i({}_{(n)}E'_\bullet)$ for all $0 \le n < d$, $i \in \mathbf{Z}$, and $(x_1, \dots, x_r) \in \mathfrak{B}$, we find by Proposition 6, that $E'_\bullet$ is isomorphic to a Buchsbaum cone $E_\bullet$ with the same base. Our assertion follows from the uniqueness of the minimal part of a free complex (see [2, (1.1)]). □

Applying the results obtained so far, we now prove our main theorem which generalizes Goto's structure theorem for maximal Buchsbaum modules over regular local rings (see [6]).

The next theorem can be proved in a more or less standard manner.

Theorem 8. *Let M be a maximal quasi-Buchsbaum module over A. Then M is Buchsbaum if and only if $\mathfrak{m}\operatorname{Ext}^i_A(M, A/(x_{r-n+1}, \dots, x_r)) = 0$ for all $i > 0$, $0 \le n < d$, and $(x_1, \dots, x_r) \in \mathfrak{B}$.*

Theorem 9. *Let M be a finitely generated maximal module over A having no free direct summand,*

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

a minimal free resolution of M over A and

$$\cdots \xrightarrow{\delta^\vee_{-2}} P^\vee_{-2} \xrightarrow{\delta^\vee_{-1}} P^\vee_{-1} \xrightarrow{\delta^\vee_0} P^\vee_0 \xrightarrow{\delta^\vee_1} \operatorname{Im}(\delta^\vee_1) \longrightarrow 0,$$

a minimal free resolution of $\operatorname{Im}(\delta^\vee_1)$ over A. We put $G_i = P^\vee_{d-i}$, $\gamma_i = \delta^\vee_{d-i+1}$ for $i \in \mathbf{Z}$ and let $(G_\bullet, \gamma_\bullet)$ be the minimal complex thus obtained (cf. [2, (4.2)]). Then M is Buchsbaum if and only if $G_\bullet$ is the minimal part of a Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{d-1})$ for some minimal free exact complex $F_\bullet$ and nonnegative integers $p_0, \dots, p_{d-1}$. Moreover when this is the case, we have $p_i = l_A(H^i_\mathfrak{m}(M))$ for all $0 \le i < d$.

Proof. First of all, by definition, $H^i(G^\vee_\bullet) = 0$ for $i < d$, $H_i(G_\bullet) = \operatorname{Ext}^{d-i}_A(M, A) = 0$ for $i < 0$ and $H_i(G_\bullet) = H^{d-i}(P^\vee_\bullet) = 0$ for $i \ge d$. Let $(x_1, \dots, x_r) \in \mathfrak{B}$. We have $\mathfrak{m}H^i_\mathfrak{m}(M) = 0$ if and only if $\mathfrak{m}\operatorname{Ext}^{d-i}_A(M, A) = \mathfrak{m}H_i(G_\bullet) = 0$ for $i < d$ by local duality and $\operatorname{Ext}^{d-i}_A(M, {}_{(n)}A) = H_i({}_{(n)}G_\bullet)$ for $0 \le i < d$, $0 \le n < d$. Suppose that M is quasi-Buchsbaum. Then, since $\mathfrak{m}H_i(G_\bullet) = \mathfrak{m}\operatorname{Ext}^{d-i}_A(M, A) = 0$ for $i < d$ and $H_i(G_\bullet) = 0$ for $i < 0$, $i \ge d$ as we have already mentioned above, the complex $G_\bullet$ is the minimal part of a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{d-1})$ for some minimal free exact complex $F_\bullet$ and nonnegative integers $p_i := l_A(H_i(G_\bullet)) = l_A(H^i_{\mathfrak{m}}(M))$ $(0 \le i < d)$ by [2, (1.6)]. If further M is Buchsbaum, then $\mathfrak{m}H_i(_{(n)}G_\bullet) = \mathfrak{m}\operatorname{Ext}_A^{d-i}(M, {}_{(n)}A) = 0$ for all $0 \le i < d$, $0 \le n < d$ by Theorem 8. Hence $G_\bullet$ is the minimal part of a Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{d-1})$ by Theorem 7 as desired. Since a Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \dots, p_{d-1})$ satisfies $\mathfrak{m}H_i(_{(n)}E_\bullet) = 0$ for all $0 \le n \le d$, $i \in \mathbf{Z}$, and $(x_1, \dots, x_r) \in \mathfrak{B}$ by Corollary 5, the converse also holds by Theorem 8. □

Remark 10. If one prefers derived category argument, then assuming that A is a residue class ring of a regular local ring, he/she will be able to give another proof of the above theorem with the use of Yoshino's corrected version of Schenzel's theorem [14, (2.3)].

We close this paper by giving a formulation in our language to some of Kawasaki's results on maximal surjective-Buchsbaum modules (see [7] and [8]) in the case where the base ring is Gorenstein.

In what follows, we will denote by $O_\bullet$ the complex such that $O_i = 0$ for all $i \in \mathbf{Z}$. Given a quasi-Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ and chain maps $\lambda_{0,\bullet} : F_\bullet \longrightarrow (L_{\bullet-1})^{p_0}$, $\lambda_{j,\bullet} : \operatorname{con}(\lambda_{j-1,\bullet})_\bullet \longrightarrow (L_{\bullet-j-1})^{p_j}$ $(1 \le j \le m-1)$, we define chain maps

$$\lambda^\star_{0,\bullet} : O_\bullet \longrightarrow (L_{\bullet-1})^{p_0}, \quad \lambda^\star_{j,\bullet} : \operatorname{con}(\lambda^\star_{j-1,\bullet})_\bullet \longrightarrow (L_{\bullet-j-1})^{p_j} \ (1 \le j \le m-1)$$

inductively so that $\operatorname{con}(\lambda^\star_{j,\bullet})_\bullet$ is a subcomplex of $\operatorname{con}(\lambda_{j,\bullet})_\bullet$ satisfying

$$\operatorname{con}(\lambda^\star_{j,\bullet})_i = \bigoplus_{l=0}^{j}(L_{i-l})^{p_l} \subset F_i \oplus \left(\bigoplus_{l=0}^{j}(L_{i-l})^{p_l}\right) \quad \text{and}$$
$$\operatorname{con}(\lambda_{j,\bullet})_\bullet/\operatorname{con}(\lambda^\star_{j,\bullet})_\bullet \cong F_\bullet \quad \text{for} \quad 0 \le j \le m-1,$$

and $\lambda^\star_{j,\bullet} := \lambda_{j,\bullet}|_{\operatorname{con}(\lambda^\star_{j-1,\bullet})_\bullet}$ for $1 \le j \le m-1$. With this notation, we say that a quasi-Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ and chain maps $\lambda_{j,\bullet}$ $(0 \le j \le m-1)$ is a *surjective-Buchsbaum cone* if it satisfies the condition

(ii)′ $\lambda^\star_{j,\bullet} = 0$ for all $0 \le j \le m-1$.

Surjective Buchsbaum modules over Gorenstein local rings can be characterized in the following way.

Theorem 11. *Let M be a finitely generated maximal module over A having no free direct summand and $(G_\bullet, \gamma_\bullet)$ the minimal complex defined as in Theorem 9. Then M is surjective-Buchsbaum if and only if $G_\bullet$ is the minimal part of a surjective-Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{d-1})$ for some minimal free exact complex $F_\bullet$ and nonnegative integers $p_0, \dots, p_{d-1}$. Moreover when this is the case, we have $p_i = l_A(H^i_{\mathfrak{m}}(M))$ for all $0 \le i < d$.*

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Zero-dimensional schemes and the Castelnuovo-Mumford regularity (a joint work with E. Ballico)

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The purpose of this short note is to give a survey on a joint work [2] with Edoardo Ballico. We study an upper bound of the Castelnuovo-Mumford regularity of zero-dimensional schemes, especially of a generic hyperplane section of projective curves in positive characteristic.

For a projective scheme $X \subset \mathbf{P}_K^N$, the Castelnuovo-Mumford regularity $\operatorname{reg}(X)$ is defined as the smallest integer m such that $\mathrm{H}^i(\mathbf{P}_K^N, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$, see, e.g., [3]. The interest in this concept stems partly from the well-known fact: The regularity $\operatorname{reg}(X)$ is the smallest integer m such that the minimal generators of the n-th syzygy module of the defining ideal I of X occur in degree $\leq m+n$ for all $n \geq 0$.

In particular, for a zero-dimensional scheme $S \subset \mathbf{P}_K^N$, the index of regularity $i(S)$ of S is defined as the smallest integer t such that $\mathrm{H}^1(\mathbf{P}_K^N, \mathcal{I}_S(t)) = 0$. Remark that $\operatorname{reg}(S) = i(S) + 1$.

Let $S \subset \mathbf{P}_K^N$ be a generic hyperplane section of a nondegenerate projective curve $C \subset \mathbf{P}_K^{N+1}$ over an algebraically closed field K. Then S has the

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uniform position property in case $\text{char}(K) = 0$, see [5], while the property does not necessarily hold in case $\text{char}(K) > 0$, see [14]. Instead, even for the positive characteristic case, S has the linear semi-uniform position property introduced in [1].

A study of the h-vectors of a zero-dimensional scheme S in linear semi-uniform position gives an upper bound on the index of regularity, that is, $i(S) \leq \lceil(\deg(S) - 1)/N\rceil$. There are some known facts on the sharpness of the above bound. If a zero-dimensional scheme $S \subset \mathbf{P}_K^N$ lies on a rational normal curve, then we have an equality, $i(S) = \lceil(\deg(S) - 1)/N\rceil$. On the other hand, assume that a zero-dimensional scheme $S \subset \mathbf{P}_K^N$ is in uniform position and $\deg(S)$ is large enough. If the equality $i(S) = \lceil(\deg(S) - 1)/N\rceil$ holds, then S lies on a rational normal curve, see, e.g., [9, 17].

Now we consider a generic hyperplane section $S \subset \mathbf{P}_K^N$ of a nondegenerate projective curve over an algebraically closed field K such that S does not have the uniform position property. So you may assume $\text{char}(K) > 0$. Under the condition that $N \geq 3$ and $\deg(S)$ is large enough, if S does not have the uniform position property, then $i(S) \leq \lceil(\deg(S) - 1)/N\rceil - 1$, which is proved in classical Castelnuovo's method. Let $S \subset \mathbf{P}_K^N$ be a generic hyperplane section of a nondegenerate projective curve with $\deg(S)$ large enough. Without assuming S is in uniform position, if the equality $i(S) = \lceil(\deg(S) - 1)/N\rceil$ holds, then S lies on a rational normal curve.

A regularity bound

$$\text{reg}(X) \leq \lceil(\deg(X) - 1)/\text{codim}(X)\rceil + \max\{k(X)\dim(X), 1\}$$

is known for a nondegenerate projective variety X, see [10, 13], where $k(X)$ is the leasr integer such that X is k-Buchsbaum. Conversely, under the assumption that a nondegenerate projective variety X is ACM, that is, the coordinate ring of X is Cohen-Macaulay, if $\text{reg}(X) \leq \lceil(\deg(X) - 1)/\text{codim}(X)\rceil + 1$ and $\deg(X)$ is large enough, then X is a variety of minimal degree, see [11, 15]. Moreover, there gives a classification of nondegenerate projective non-ACM varieties X attaining a regularity bound $\text{reg}(X) = \lceil(\deg(X) - 1)/\text{codim}(X)\rceil + k(X)\dim(X)$. In [9], under the assumption that $\deg(X)$ is large enough and $\text{char}(K) = 0$, it is shown that a projective non-ACM variety having the equality must be a curve on a rational ruled surface, that is, on a Hirzeburch surface.

Now we state our main results.

Theorem 1([2]). Let $S \subset \mathbf{P}_K^N$ be a generic hyperplane section of a nondegenerate projective curve $C \subset \mathbf{P}_K^{N+1}$ with $d = \deg(C)$ over an algebraically closed field K. If $d \geq \max\{N^2 + 2N + 2, 25\}$ and $i(S) = \lceil (d-1)/N \rceil$, then S lies on a rational normal curve.

Theorem 2([2]). Let X be a nondegenerate projective variety in $\mathbf{P}_K^N$ over an algebraically closed field K. Assume that $k(X) > 0$, $\deg(X) \geq \max\{2\,\mathrm{codim}(X)^2 + \mathrm{codim}(X) + 2, 25\}$ and $\mathrm{reg}(X) = \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + k(X)\dim(X)$. Then X is a curve on a 2-dimensional rational normal scroll.

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AN INTRODUCTION TO TIGHT CLOSURE

KAREN E. SMITH

Lecture One: Main Properties of Tight Closure

In these two talks, I wish to give a brief introduction to the subject of tight closure, aimed at commutative algebraists who have not before studied this topic. The first talk will focus mainly on the definition and basic properties, and the second talk on some applications. Before beginning, I want to thank the organizers, Professors Hashimoto and Yoshida, for kindly offering me this honor of speaking, and for their great generosity during my visit to Japan. I also wish to thank Professor Kei-ichi Watanabe for his generosity in hosting me in Tokyo, and especially my friend and collaborator Nobuo Hara, for his generosity and patience in hosting me, and in helping me find my way around Tokyo and to Kashikojima.

Tight closure was introduced by Mel Hochster and Craig Huneke about twelve years ago [HH1]. Today it is still a subject of very active research, with an ever increasing list of applications. Applications include areas like the study of Cohen-Macaulayness. For example, the famous Hochster-Roberts theorem on the Cohen-Macaulayness of rings of invariants has a simple tight closure proof [HH1]. Also, the existence of big Cohen-Macaulay algebras was proved with ideas from tight closure [HH3], and the existence of "arithmetic Macaulayfications" in some cases was discovered with tight closure [AHS], [Ku]. Tight closure has provided us with greater insight into integral closure, and into the homological theorems that grew out of Serre's work on multiplicities. For example, it gives us simple proofs of the Briançon-Skoda Theorem, the Syzygy Theorem of Evans and Griffith and of the monomial conjecture (in mixed characteristic) [HH1]. Tight closure provided the inspiration for results on the simplicity of rings of differential operators on certain rings of invariants [SV], and it has produced "uniform" Artin-Rees theorems [Hu1]. There are also numerous applications to and connections with algebraic geometry, such as in the study of singularities [W], [S1], [Ha1], of vanishing theorems [Ha2], [HS], [Ha4], and of adjoint linear series [S4], [S7], where the work of Japanese mathematicians has had a particularly strong impact. In the second lecture, I will summarize some of these applications to algebraic geometry, although of course, there will not be enough time to do any of them any justice.

Let us begin with our task in the first lecture: to introduce the definition and main properties of tight closure.

Tight closure is a closure operation performed on ideals in a commutative, Noetherian ring containing a field (that is, of "equi-characteristic"). The tight closure

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of an ideal I is an ideal I^* containing I. The definition is based on reduction to characteristic p, where the Frobenius (or $p-th$ power map) is then used. To keep things as simple as possible, we treat only the characteristic p case here.

Definition. *Let R be a Noetherian domain of prime characteristic p, and let I be an ideal with generators $(y_1, \ldots, y_r)$. An element z is defined to be in the tight closure I^* if there exists a non-zero element c of R such that*

$$(*) \qquad cz^{p^e} \in (y_1^{p^e}, \ldots, y_r^{p^e})$$

for all $e \gg 0$.

Loosely speaking, the tight closure consists of all elements that are "almost" in I as far as the Frobenius map is concerned. Indeed, if we take the $p^e - th$ root of $(*)$ above, we see that

$$c^{1/p^e} z \in IR^{1/p^e}$$

for $e \gg 0$. As e goes to infinity, $1/p^e$ goes to zero, so in some sense c^{1/p^e} goes to 1 (this idea can be made precise with valuations). So z is "almost in" I, at least after applying the Frobenius map.

It is not important to restrict to the case where R is a domain; we can define tight closure in an arbitrary Noetherian ring of characteristic p by requiring that c is not in any minimal prime. However, because most theorems about tight closure reduce to the domain case, we treat only the domain case in these lectures.

Example. Let R be the hypersurface ring

$$\frac{k[x,y,z]}{(x^3+y^3-z^3)},$$

where k is any field whose characteristic is not 3. Then

$$(x,y)^* = (x,y,z^2).$$

Indeed, if k has characteristic p, we can write

$$(z^2)^q = (x^3+y^3)^{\frac{2q-r}{3}} z^r$$

where $r = 1$ or 2 and $q = p^e$. Expanding this expression as

$$z^r \sum \binom{2q-r}{i} x^{3i} y^{2q-r-3i},$$

it is easy to see that each monomial $x^m y^n$ appearing in the sum has either $m \geq q$ or $n \geq q$ unless both m and n equal $q-1$ (which only happens in the case where $p = 1 \bmod 3$). So we can take $c = x$ (or y), and conclude that $cz^q \in (x^q, y^q)$ for all $q = p^e$. Thus $z^2 \in (x,y)^*$. A similar argument can be used to show that z is not in $(x,y)^*$. Because this works for all p (except $p = 3$), we declare that z^2, but not z, is in the tight closure of (x,y) also in characteristic zero. So $(x,y)^* = (x,y,z^2)$ in every characteristic $p \geq 0$ except $p = 3$.

The definition of tight closure takes some getting used to. Fortunately, one can understand many applications of tight closure if one simply accepts the following properties of tight closure as axioms:

Main Properties of Tight Closure.

(1) *If R is regular, then all ideals of R are tightly closed.*
(2) *If $R \hookrightarrow S$ is an integral extension, then $IS \cap R \subset I^*$ for all ideals I of R.*
(3) *If R is local, with system of parameters $x_1, \ldots, x_d$, then $(x_1, \ldots, x_i) : x_{i+1} \subset (x_1, \ldots, x_i)^*$ ("Colon Capturing").*
(4) *If μ denotes the minimal number of generators of I, then $\overline{I^\mu} \subset I^* \subset \bar{I}$, where for any ideal J, $\overline{J}$ denotes the integral closure of J.*
(5) *If $R \longrightarrow S$ is any ring map, $I^*S \subset (IS)^*$ ("Persistence").*

For the remainder of this lecture, we will discuss these five main properties, their proofs and main consequences. Some of the five require some mild hypotheses; precise statements will be given. All of them are true in any equicharacteristic ring (although Property 2 is not interesting in characteristic zero). All of them are quite elementary to prove, at least in the main settings, with the exception of Property 5 which requires a new idea. We will stick to the prime characteristic case, and simply remark that "by reduction to characteristic p", one can prove the characteristic zero case without essential difficulty.

Note that one important property is omitted from the list. Any decent closure operation ought to commute with localization, but amazingly, we still do not know that tight closure does.

Open Problem. *If U is a multiplicative system in a ring R, is*

$$I^*R[U^{-1}] = (IR[U^{-1}])^*?$$

It is easy to see that one direction holds, namely, $I^*R[U^{-1}] \subset (IR[U^{-1}])^*$. Indeed, if $z \in I^*$, then we have the equations $cz^{p^e} = a_{1e}y_1^{p^e} + \cdots + a_{re}y_r^{p^e}$ in R. Expanding to $R[U^{-1}]$, the same equations show that $\frac{z}{1}$ is in the tight closure of $IR[U^{-1}]$. This is a very special case of Property 5 above. On the other hand, the other direction is not known in any non-trivial case (see, however, [AHH], [S6]).

The localization problem is probably the biggest open problem in tight closure theory. It is remarkable that the theory is so powerful while such a basic question remains unsolved. The power is derived from the five main properties above, which we now discuss.

Property One: All ideals are tightly closed in a regular ring.

It is easy to see why all ideals are tightly closed in a regular ring. For example, consider the special case where (R, m) is local domain and the Frobenius map is finite. This is not a very restrictive assumption from our point of view, because we are usually interested in the local case anyway; also the Frobenius map is finite in a large class of interesting rings— for example, for any algebra essentially of a finite type over a perfect field or for any complete local ring with a perfect residue field.

We have a descending chain of subrings of R

$$R \supset R^p \supset R^{p^2} \supset R^{p^3} \supset \ldots$$

Because R is regular, the ring R is a free module considered over each one of the subrings R^{p^e}. Indeed, the Frobenius map is flat for any regular ring; but because we have assumed that R is local and the Frobenius map is finite, we actually get that R is free over R^{p^e}. This means that, for any non-zero c, we can find an R^{p^e}-linear splitting

$$R \xrightarrow{\phi} R^{p^e}$$
$$c \mapsto 1$$

so long as e is large enough that c is not in the expansion of the maximal ideal of R^{p^e} to R (that is, $c \notin m^{[p^e]}$, where $m^{[p^e]}$ denotes the ideal of R generated by the $p^e - th$ powers of the generators of m).

Now if we have an ideal $I = (y_1, \ldots, y_r)$ of R and an element $z \in I^*$, then we can find equations

$$cz^{p^e} = a_1 y_1^{p^e} + \cdots + a_r y_r^{p^e}$$

for all large e. Applying the R^{p^e}-linear map ϕ above, we see that

$$z^{p^e} = \phi(a_1) y^{p^e} + \cdots + \phi(a_r) y_r^{p^e}$$

where each coefficient $\phi(a_i)$ is some element of R^{p^e}. By taking the $p^e - th$ root of this equation, we see that z is an R-linear combination of $y_1, \ldots, y_r$. Thus $z \in I$, and $I^* = I$ for all ideals of R.

This completes the proof that all ideals are tightly closed in a regular ring, at least in the special case we considered. The general case (of prime characteristic) is not much harder. The point is that flatness of Frobenius in a regular ring. See [HH1].

Property Two: Elements mapped to I after integral extension are in I^*.

We now prove Property 2: if $R \hookrightarrow S$ is an integral extension of domains of prime characteristic, and I is an ideal of R, then $IS \cap R \subset I^*$.

The following lemma will be useful also in the proof of Property 3.

Key Lemma. *If $R \hookrightarrow S$ is a module finite extension of domains, and d is any fixed non–zero element of S, then there is an R-linear map, $S \xrightarrow{\phi} R$ sending d to a non-zero element of R.*

The point in the proof of the Lemma is that after tensoring with the fraction field, K, of R, we have an inclusion $K \hookrightarrow K \otimes_R S$, where the latter is simply a finite dimensional vector space over K. So of course there is a K-linear splitting $K \otimes S \xrightarrow{\psi} K$ sending $1 \otimes d$ to 1. Thinking of S as a subset of $K \otimes S$, we look at where ψ sends each of a set of R-module generators $\{s_1, \ldots, s_d\}$ for S, say $\psi(s_i) = \frac{r_i}{t_i} \in K$. Now we can define ϕ to be the map $t\psi$ where t is the product of the t_i. This map is R-linear, and sends each s_i to an element of R. The lemma is proved.

To prove Property 2, let $z \in R$ be any element in $IS \cap R$. This means we can write

$$z = a_1 y_1 + \cdots + a_r y_r$$

where $a_i \in S$ and the y_i's generate I. Because this expression involves only finitely many elements from S there is no loss of generality in assuming S is module finite over R. Now, raising this equation to the $p^e - th$ power, we have

$$z^{p^e} = a_1^{p^e} y_1^{p^e} + \cdots + a_r^{p^e} y_r^{p^e}.$$

Using the lemma, we find an R-linear map $S \xrightarrow{\phi} R$ sending 1 to some non-zero element $c \in R$. Applying ϕ to this equation, we have

$$cz^{p^e} = \phi(a_1^{p^e}) y_1^{p^e} + \cdots + \phi(a_r^{p^e}) y_r^{p^e}.$$

This is an equation now in R, showing that $z \in I^*$. Property 2 is proved.

Essentially the same argument shows the stronger property: if $R \hookrightarrow S$ is an integral extension of prime characteristic domains and I is an ideal of R, then $(IS)^* \cap R \subset I^*$.

Property 2, unlike the other four properties, is interesting only in prime characteristic. For example, if R is any normal domain containing $\mathbb{Q}$, then R splits off of every finite integral extension S (using the trace map). In this case, $IS \cap R = I$ for every ideal of R and every integral extension S.

Property 2 can be phrased in terms of the absolute integral closure. For any domain R, the absolute integral closure R^+ of R is the integral closure of R in an algebraic closure of its fraction field. In other words, R^+ is the direct limit of all finite integral extensions of R. Property 2 can be stated: $IR^+ \cap R \subset I^*$ for all ideals I of R. This leads to the following interesting problem.

Open Problem. *Let R be a domain of prime characteristic. Is $IR^+ \cap R = I^*$ for all ideals R?*

In addition to providing a very nice characterization of tight closure, an affirmative answer to this question would immediately solve the localization problem. Indeed, it is easy to check that the closure operation defined by expansion to the absolute integral closure and contraction back to R commutes with localization.

There is no non-trivial class of rings in which this open problem has been solved. However, we do have the following result.

Theorem [S1]. *Let R be a locally excellent[1] domain of prime characteristic. Then $I^* = IR^+ \cap R$ for all parameter ideals I of R.*

A "parameter ideal" is any ideal I generated by n-elements where n is the height of I; if R is local, an ideal is a parameter ideal if and only if it is generated by part of a system of parameters.

[1]Virtually all rings the commutative algebraist on the street is likely to run across are locally excellent, but see [Mats] for a definition.

As we see from the theorem, tight closure commutes with localization for parameter ideals. However, this does not follow from the theorem because this fact is used in its proof. See instead [AHH].

The proof of this theorem is somewhat involved, so we do not sketch it here; see [S1]. The result has been generalized to a larger class of ideals, including ideals generated monomials in the parameters, by Aberbach [A].

Property Three: Colon Capturing.

Property 3, the colon capturing property of tight closure, is particularly instrumental in applications of tight closure to problems about Cohen-Macaulayness. Of course, if R is a Cohen-Macaulay local ring with system of parameters $x_1, \dots, x_d$, then by definition,

$$(x_1, \dots, x_i) : x_{i+1} \subset (x_1, \dots, x_i)$$

for each $i = 1, 2, \dots, d-1$. Colon capturing says that, even for rings that are not Cohen-Macaulay, the colon ideal $(x_1, \dots, x_i) : x_{i+1}$ is at least contained in $(x_1, \dots, x_i)^*$. Loosely speaking, tight closure captures the failure of a ring to be Cohen-Macaulay.

We now prove the colon capturing property of tight closure: if R is a local domain (satisfying some mild hypothesis to be made soon precise) and $x_1, \dots, x_d$ is a system of parameters for R, then

$$(x_1, \dots, x_i) : x_{i+1} \subset (x_1, \dots, x_i)^*$$

for each $i = 1, \dots, d-1$.

Let us first assume that R is complete. In this case, we can express R as a module finite extension of the power series subring $k[[x_1, \dots, x_d]]$, where k is a field isomorphic to the residue field of R.

Suppose that $z \in (x_1, \dots, x_i) :_R x_{i+1}$. Consider the ring A contained in R obtained by adjoining the element z to the power series ring $k[[x_1, \dots, x_d]]$. Observe that the ring A is Cohen-Macaulay; in fact, A is a hypersurface ring because its dimension is d and its embedding dimension is $d+1$ (or d, if z happens to be in power series ring already).

Now, because $z \in (x_1, \dots, x_i) :_R x_{i+1}$, we can write

$$z x_{i+1} = a_1 x_1 + \cdots + a_i x_i$$

for some elements a_i in R. Raising this equation to the $p^e - th$ power, we have

$$z^{p^e} x_{i+1}^{p^e} = a_1^{p^e} x_1^{p^e} + \cdots + a_i^{p^e} x_i^{p^e}$$

Because the inclusion $A \hookrightarrow R$ is a module finite finite extension, we can use the Key Lemma to find an A–linear map $R \xrightarrow{\phi} A$ sending 1 to some non-zero element $c \in A$. This yields equations

$$c z^{p^e} x_{i+1}^{p^e} = \phi(a_1^{p^e}) x_1^{p^e} + \cdots + \phi(a_i^{p^e}) x_i^{p^e}$$

where the $\phi(a_j^{p^e})$ are just some elements of A.

In other words,

$$cz^{p^e} \in (x_1^{p^e}, \dots, x_i^{p^e}) :_A x_{i+1}^{p^e}$$

in the ring A. But A is Cohen-Macaulay, and $x_1^{p^e}, \dots, x_d^{p^e}$ is a system of parameters for A, so we see

$$cz^{p^e} \in (x_1^{p^e}, \dots, x_i^{p^e})$$

for all e. This shows that $z \in (x_1, \dots, x_i)^*$ in R (also in A, but it is R we care about). Thus $(x_1, \dots, x_i) :_R x_{i+1} \subset (x_1, \dots, x_i)^*$, and the proof of the colon capturing property is complete— at least for complete local domains.

Inspecting the proof, we see that we have not used the completeness of R in a crucial way: what we need is that R the domain is a finite extension of a regular subring. So this proof also works for algebras essentially of finite type over a field (the required regular subring is supplied by Noether normalization) and in many other settings. In fact, colon capturing holds for any ring module finite and torsion free over a regular ring. See [HH1] and [Hu2] for different proofs and more general statements.

The philosophy of colon capturing holds for other ideals involving parameters. For example, if I and J are any ideals generated by monomials in a system of parameters $\{x_0, \dots, x_d\}$, one can compute $I : J$ formally as if the x_i's are the indeterminates of a polynomial ring. Then the actual colon $I : J$ is contained in the tight closure of the 'formal' colon ideal. Furthermore, even more is true: we have $I^* : J$ is contained in the tight closure of the formal colon ideal. Essentially the same proof gives these stronger results with very small effort. For an explicit example, let $x_0, \dots, x_d$ be a system of parameters in a domain R. Then

$$(x_0^t, \dots, x_d^t) : (x_0 x_1 \dots x_d) \subset (x_0^{t-1}, \dots, x_d^{t-1})^*$$

and even

$$(x_0^t, \dots, x_d^t)^* : (x_0 x_1 \dots x_d) \subset (x_0^{t-1}, \dots, x_d^{t-1})^*.$$

One reason for tight closure's effectiveness is that these sorts of manipulations can often help us prove a general statement about parameters if we already have an argument for a regular sequence.

Some Consequences of the First Three Properties.

It follows immediately from the colon capturing property that if R is a local ring in which all ideals are tightly closed, then R must be Cohen-Macaulay. Indeed, if all parameter ideals are tightly closed, then colon capturing implies that R is Cohen–Macaulay. This leads us to define two important new classes of rings.

Definition. *A ring R is weakly F-regular if all ideals are tightly closed. A ring R is F-rational if all parameter ideals are tightly closed.*

So far we have seen the following implications: Regular $\Longrightarrow$ weakly F-regular $\Longrightarrow$ F-rational $\Longrightarrow$ Cohen-Macaulay. The first implication is Property 1, while the last implication is Property 3.

The reason the adjective "weakly" modifies "F–regular" above goes back to the localization problem. Unfortunately, we do know whether the property that all ideals are tightly closed is preserved under localization. The term "F–regular" is reserved for rings R in which all ideals are tightly closed not just in R, but also in every localization of R. That is, we have the following special case of the localization problem:

Open Problem. *If R is weakly F–regular, and $U \subset R$ is any multiplicative system, is the localization $R[U^{-1}]$ also weakly F–regular?*

This problem is much easier than the localization problem itself. Indeed, it has been shown in a number of cases. For example, Hochster and Huneke showed the answer is yes when R is Gorenstein [HH2], [HH4]. This was later generalized to the $\mathbb{Q}$–Gorenstein case, and even to the case where there are only isolated non $\mathbb{Q}$–Gorenstein points, by MacCrimmon [M]. Using this, it is possible to see that weakly F-regular is equivalent to F-regular in dimensions three and less. (These statements require some mild assumption on R, such as excellence). Recently, an affirmative answer was given also for finitely generated $\mathbb{N}$-graded algebras over a field [LyS]. By contrast, the full localization problem has not been solved in any of these cases.

The problem is reminiscent of an analogous problem in commutative algebra that looked quite difficult in the mid-century: is the localization of a regular ring still regular? With Serre's introduction of homological algebra to commutative algebra, the problem suddenly became quite easy. Perhaps a similar revelation is necessary in tight closure theory.

Returning to the applications of the first three properties, we now prove the following easy, but important, theorem.

Theorem [HH1]. *Let $R \subset S$ be an inclusion of rings that splits as a map of R-modules. If S is (weakly) F-regular, then R is (weakly) F-regular.*

The proof is simple. Suppose that I is an ideal of R and that $z \in I^*$. This means that for some non-zero c, $cz^{p^e} \in I^{[p^e]}$ where $I^{[p^e]}$ denotes the ideal generated by the p^e-th powers of the generators of I. Expanding to S, we have $cz^{p^e} \in (IS)^{[p^e]}$, so that $z \in (IS)^*$. But all ideals of S are tightly closed, and so $z \in IS$. Now applying the splitting $S \longrightarrow R$ (which sends 1 to 1 R-linearly), we see that $z \in I$ in R as well. This completes the proof.

The importance of this Theorem lies in the following corollaries.

Corollary. *Any ring (containing a field) which is a direct summand of a regular ring is Cohen-Macaulay.*

The proof is obvious: a regular ring is F-regular by Property 1, so any direct summand is also F-regular. By Property 3, this summand is Cohen-Macaulay.

Corollary (The Hochster-Roberts Theorem). *The ring of invariants of a linearly reductive group acting linearly on a regular ring is Cohen-Macaulay.*

This is essentially a special case of the previous corollary because the so-called Reynold's operator gives us a splitting of R^G from R.

We emphasize that both the Theorem and its corollaries make sense and are true in characteristic zero. Thus even though there are very fewer linearly reductive groups in prime characteristic, the Hochster-Roberts Theorem for reductive groups over the complex numbers has been proved here by reduction to characteristic p. To be fair, we have not proved Properties 1 and 3 in characteristic zero (nor even given a precise definition of tight closure in characteristic zero). However, if one accepts the existence of a closure operation in characteristic zero satisfying Properties 1 and 3, then we have proved that the Hochster-Roberts Theorem follows.

We now mention one of the crown jewels of tight closure theory.

Theorem [HH3]. *Let R be an excellent local domain of prime characteristic. Then the absolute integral closure R^+ of R is a Cohen–Macaulay R-module.*

We can see that this must be true as follows. Let $x_1, \ldots, x_d$ be a system of parameters. Suppose $z \in (x_1, \ldots, x_i) : x_{i+1}$. By the colon capturing property, $z \in (x_1, \ldots, x_i)^*$. But for parameter ideals, tight closure is the same as the contraction of the expansion to R^+ (see the discussion of Property 2). Thus $z \in (x_1, \ldots, x_i)R^+ \cap R$. This holds for all i, so $x_1, \ldots, x_d$ is a regular sequence on R^+, and the Theorem is "proved". Unfortunately, this is not an honest proof because the proof that $I^* = IR^+ \cap R$ for parameter ideals I uses the Cohen-Macaulayness of R^+.

Property 4: Relationship to integral closure.

Property 4 is really two statements. First, the tight closure is contained in the integral closure for any ideal I. Second, the integral closure of I^μ (where μ is the least number of generators of I) is contained in the tight closure I^*.

The point in proving both statements is the following alternative definition of the integral closure $\bar{J}$ of an ideal J in a domain R: an element $z \in \bar{J}$ if and only if there exists a non-zero c in R such that $cz^n \in J^n$ for all (equivalently, for infinitely many) $n \gg 0$. (This can be easily proved equivalent to the more standard definition of integral closure by recalling another characterization of integral closure: $\bar{J}$ consists of all elements z such that $z \in JV$ for all discrete valuation rings V lying between R and its fraction field.) Note that in particular, $I^* = \overline{I}$ for all principle ideals I.

Now, with this definition of the integral closure, it is immediately clear that the tight closure of any ideal is contained in the integral closure. Indeed, since the

$p^e - th$ power of the generators of I are contained in the $p^e - th$ power of I, we have

$$cz^{p^e} \in I^{[p^e]} \subset I^{p^e}$$

for all e. So any z in I^* is in $\bar{I}$.

For the second statement, suppose that $z \in \overline{I^\mu}$. This means that there exists a non–zero c such that for all n, $cz^n \in I^{\mu n}$. If $y_1, \ldots, y_\mu$ generate I, then $I^{\mu n}$ is generated by monomials of degree μn in the y_i. But if $y_1^{a_1} y_2^{a_2} \ldots y_\mu^{a_\mu}$ is such a monomial, at least one a_i must be greater than or equal to n. So

$$cz^n \in I^{\mu n} \subset (y_1^n, \ldots, y_\mu^n)$$

for all n. In particular, this holds for $n = p^e$, for all e, and we conclude that $z \in I^*$. The proof that $\overline{I^\mu} \subset I^*$ is complete.

The statement that $\overline{I^\mu} \subset I^*$ is sometimes called the Briançon-Skoda Theorem. The original Briançon-Skoda Theorem stated that for any ideal I in a ring of convergent complex power series, the integral closure of the μ-th power of I is contained in I, where μ is the minimal number of generators of I [BS]. This statement was later generalized by Lipman and Sathaye to more general regular local rings and then by Lipman and Tessier to the so-called 'pseudo-rational' local rings (for a ring essentially of finite type over a field of characteristic zero, pseudo-rational is equivalent to rational singularities) [LS], [LT]. Tight closure gives an extremely simple proof of the Briançon-Skoda theorem for any regular ring containing a field: $\overline{I^\mu} \subset I^* \subset I$, where the first inclusion follows from Property 4 and the second by Property 1. But better still, tight closure explains what happens in a non-regular ring as well.

The original motivating problem for the Briançon-Skoda theorem is said to be due to J. Mather: if f is a germ of an analytic function vanishing at the origin in $\mathbb{C}^n$, find a uniform k (depending only on n) such that f^k is in the ideal generated by the partial derivatives of f. The Briançon-Skoda theorem tell us that we can take $k = n$. Indeed, it is easy to check that $f \in \overline{J_f} = \overline{(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})}$. So $f^n \in \overline{J_f}^n \subset \overline{J_f^n} \subset J$. It is remarkable how easy the tight closure proof is for this problem that once seemed very difficult.

Before moving on to Property 5, we consider one more comparison of tight and integral closure. Let I be an m-primary ideal in a local domain of dimension d. Recall the Hilbert-Samuel function defined by

$$HS(n) = \text{ length } R/I^n.$$

This function is eventually a polynomial in n, and its normalized leading coefficient

$$\lim_{n \to \infty} \frac{d!}{n^d} HS(n)$$

is called the Hilbert-Samuel multiplicity of I. Analogously, when R is of characteristic p, we can define the Hilbert–Kunz function

$$HK(e) = \text{ length } R/I^{[p^e]}.$$

This function has polynomial growth in p^e, and its leading coefficient

$$\lim_{e\to\infty} \frac{1}{(p^e)^d} HK(e)$$

is called the Hilbert-Kunz multiplicity of I.

As is well known, the integral closure of I is the largest ideal containing I having the same Hilbert-Samuel multiplicity (assuming the completion of R is equidimensional). What is also fairly straightforward to prove is that the tight closure of I is the largest ideal containing I having the same Hilbert-Kunz multiplicity (assuming the completion of R is reduced and equidimensional) [HH1]. In this sense, tight closure is a natural analog of integral closure.

Hilbert–Kunz functions are interesting and mysterious, with important connections to tight closure theory and surprising interactions with number theory. Much has been proved about them by Paul Monsky, among others; see, for example, [Mo]. Please refer to the lecture of Professor Kei-ichi Watanabe for the latest ideas about them, and to the bibliography of [Hu2] for more references on this topic.

Property Five: Persistence of Tight Closure.

The persistence property states: whenever $R \longrightarrow S$ is a map of rings containing a field, $I^*S \subset (IS)^*$. In other words, any element in the tight closure of an ideal I of R will "persist" in being in the tight closure of I after expansion to any R–algebra.

Before discussing the precise hypothesis necessary, let us consider what is involved in proving such a statement. Suppose $z \in I^*$ where I is an ideal in domain R. Thus there exists a non-zero c such that

$$cz^{[p^e]} \in I^{[p^e]}$$

for all large e. Expanding to S, of course, the same relationship holds in S (using the same letters to denote the images of c, z, and I in S). This would seem to say that the image of z is in $(IS)^*$, which is what we need to show. The problem is that c may be in the kernel of the map $R \longrightarrow S$. Thus we need to find a c that "witnesses" $z \in I^*$ but is not in this kernel.

Unlike the first four properties, Property 5 does not follow immediately from the definition. The new idea we need is the idea of a *test element.*

Definition. *An element c in a prime characteristic ring R is said to be in the test ideal of R if, for all ideals I and all elements $z \in I^*$, we have $cz^{p^e} \in I^{[p^e]}$ for all e. An element c is a test element if it is in the test ideal but not in any minimal prime of R.*

Note that the definition of the test ideal requires that $cz^{p^e} \in I^{[p^e]}$ for all e, not just for all sufficiently large e. We could also define the *asymptotic test ideal* as above but require only that $cz^{p^e} \in I^{[p^e]}$ for $e \gg 0$. An interesting fact is that the aymptotic test ideal is a D-module—that is, it is a submodule of the module R under the action of the ring of all $\mathbb{Z}$-linear differential operators on R. See [S2].

For more on the general theory of D-modules in prime characteristic, please refer to Professor Kaneda's lectures.

It is not at all obvious that there exists a non-zero test ideal for a ring R. Fortunately, however, it is not very difficult to prove the following.

Theorem [HH2]. *Let R be a ring of prime characteristic, and assume that the Frobenius map of R is finite. If c is an element of R such that the localization R_c is regular, then c has a power which is a test element. That is, the test ideal contains an ideal defining the non–regular locus of Spec R.*

In a later paper, Hochster and Huneke prove this without the assumption that the Frobenius map is finite, imposing the weaker and more technical hypothesis of being finitely generated over an excellent local ring. Although the theorem stated above for rings in which Frobenius is finite is quite easy to prove, the proof in the more general setting is difficult and technical; see [HH4].

Note that in any ring R, the element 1 is a test element if and only if R is weakly F-regular. We expect that much more is true:

Conjecture. *The test ideal defines precisely the non–F–regular locus in Spec R.*

The conjecture is proved in some cases, such as for (excellent local) Gorenstein rings [HH4] and for rings $\mathbb{N}$-graded over a field [LyS].

Having introduced the idea of a test element, we resume our discussion of persistence. First of all, we should say that Property 5 is not known to hold in the generality we've stated; some mild hypothesis on R is needed. The problem is in finding test elements for R.

Let us now sketch the proof of persistence. Let $R \xrightarrow{\phi} S$ be a map of domains.[2] As we remarked above, persistence is trivially true when ϕ is injective, so factoring ϕ as a surjection followed by an injection, we might as well assume ϕ is surjective. Now factor ϕ as a sequence of surjections

$$R \longrightarrow R/P_1 \longrightarrow R/P_2 \longrightarrow \ldots \longrightarrow R/(ker\phi) = S,$$

where $P_1 \subset P_2 \subset \cdots \subset (\ker\phi)$ is a saturated chain of prime ideals contained in the kernel of ϕ. By considering each map separately, we see that we might as well assume that the kernel of the map $R \xrightarrow{\phi} S$ has height one.

Now if R is normal, then the non-regular locus of R is defined by an ideal J of height two or more. But as we mentioned above, this means that the test ideal has height two or more,[3] so we can find a c which is a test element but not in the kernel of ϕ. The proof is complete in the case R is normal.

[2] Like most proofs in tight closure theory, the proof reduces immediately to the case where both R and S are domains.

[3] This requires some hypothesis on R, such as finite generation over an excellent local ring, so that R satifies the conclusion of Hochster and Huneke's theorem about test elements above. In practice, all rings we run across will satisfy this hypothesis.

Finally, it is not difficult to reduce the problem to the case where R is normal, using Property 2. What happens is the normalization $\tilde{R}$ of R maps to an integral extension $\tilde{S}$ of S, namely the domain $\tilde{S}$ obtained by killing a prime of $\tilde{R}$ lying over the kernel of ϕ. The map $\tilde{R} \xrightarrow{\tilde{\phi}} \tilde{S}$ restricts to the map $R \xrightarrow{\phi} S$. Now if $z \in I^*$ in R, then $z \in (I\tilde{R})^*$, and so $\tilde{\phi}(z) \in (I\tilde{S})^*$ because we know persistence holds when the source ring is normal. By Property 2 (or really, by the same proof used to prove Property 2), we see that $\phi(z) \in (I\tilde{S})^* \cap S \subset (IS)^*$. This completes the proof of persistence.

We have completed the proofs and discussion of the five main properties of tight closure. It is natural to ask whether the five main properties characterize tight closure. They do not, or at least, not obviously. For example, in characteristic p, the 'plus closure' $IR^+ \cap R$ satisfies Properties 1,2, 3, and 5, and in all cases where it can be checked, it satisfies Property 4 as well. On the other hand, since we expect $I^* = IR^+ \cap R$, this is perhaps not very convincing.

A more interesting question is whether we can define a closure operation for rings that do not contain a field (that is, in 'mixed characteristic') which satisfies Properties 1 through 5. If so, many theorems that can now be proved only for rings containing a field, such as the homological conjectures that grew out of Serre's work on multiplicities, would suddenly admit "tight closure" proofs. The only serious attempts at defining such a closure operation in mixed characteristic are due to Mel Hochster, but so far none has proved successful; see, for example, [Ho2].

I hope it is clear from this lecture that the main ideas in tight closure theory are remarkably simple and elegant, and also that they have have far-reaching consequences. In the second lecture, we will look more closely at applications of tight closure to algebraic geometry.

Lecture Two: Three Applications of Tight Closure

At the beginning of the first lecture, we mentioned that tight closure is applicable to a wide range of problems in commutative algebra and related fields. In this lecture, we will discuss in greater detail how tight closure has increased our insight in three areas of algebraic geometry: adjoint linear systems (Fujita's Freeness Conjecture), vanishing theorems for cohomology (Kodaira Vanishing), and singularities. We will mainly discuss the first of these, giving a tight closure proof of Fujita's freeness conjecture for globally generated line bundles, but we point out connections with the other two topics as they arise.

In all three areas, characteristic p methods are used to prove characteristic zero theorems. The unifying theme for the tight closure approach to these three problems is the action of the Frobenius operator on local cohomology.

Reduction to Characteristic p.

Reduction to characteristic p is easiest to understand by example. Say we want to study the affine scheme associated to the ring

$$\frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}.$$

We instead consider the "fibration"

$$\operatorname{Spec} \frac{\mathbb{Z}[x,y,z]}{(x^3+y^3+z^3)} \longrightarrow \operatorname{Spec} \mathbb{Z}.$$

The fiber over a closed point $(p) \in \operatorname{Spec} \mathbb{Z}$ is the characteristic p scheme

$$\operatorname{Spec} \frac{\mathbb{Z}/(p)[x,y,z]}{(x^3+y^3+z^3)},$$

whereas the fiber over the generic point $(0) \in \operatorname{Spec} \mathbb{Z}$ is the original scheme

$$\operatorname{Spec} \frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}.$$

For the sorts of questions we are interested in here (which are ultimately cohomological) the following philosophy holds: what is true for the generic fiber is true for a Zariski dense set of closed fibers, and conversely, what is true for a Zariski dense set of closed fibers is true for the generic fiber. So in order to study the ring $\frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}$, we consider the ring $\frac{\mathbb{Z}/(p)[x,y,z]}{(x^3+y^3+z^3)}$ for a "generic" p.

The same approach works even if we take $\mathbb{C}$ as the ground field. Indeed,

$$\operatorname{Spec} \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)},$$

is obtained from $\operatorname{Spec} \frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}$ by the flat base change $\mathbb{Q} \to \mathbb{C}$. Again, from the point of view of the types of questions we will consider, we might as well study $\frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3)}$, and hence $\frac{\mathbb{Z}/(p)[x,y,z]}{(x^3+y^3+z^3)}$ for a "generic" p.

The philosophy holds for any scheme of finite type over a field of characteristic zero. For example, if we are interested in the ring

$$R = \frac{\mathbb{C}[x,y,z]}{(\pi x^3 + \sqrt{17} y^3 + z^3)},$$

we set $A = \mathbb{Z}[\pi, \sqrt{17}]$ and consider the fibration

$$\operatorname{Spec} \frac{A[x,y,z]}{(\pi x^3 + \sqrt{17} y^3 + z^3)} \longrightarrow \operatorname{Spec} A.$$

Again, we might as well study the prime characteristic ring $\frac{A/\mu[x,y,z]}{(\pi x^3 + \sqrt{17}y^3 + z^3)}$, where μ is a generic maximal ideal in A. Each A/μ is a finite field, so these closed fibers are all "characteristic p models", for varying p, of the original ring R.

In general, if

$$R = \frac{k[x_1, \dots, x_n]}{(F_1, \dots, F_r)},$$

where k is a field of characteristic zero, we let $A = \mathbb{Z}[\text{coefficients of the } F_i\text{'s}] \subset k$ and set

$$R_A = \frac{A[x_1, \dots, x_n]}{(F_1, \dots, F_r)}.$$

Then the map

$$\text{Spec } R_A \longrightarrow \text{ Spec } A$$

(or the map $A \to R_A$) will be called a *family of models* for Spec R (or R). The generic fiber is the original scheme Spec R (after extending the field if necessary) and a generic (or typical) closed fiber is a *characteristic p model* of Spec R. We will prove theorems about R by establishing the same statement for a generic characteristic p model of R, that is, "for all large p."

The idea of a family of models can be used to define concepts in characteristic zero which seemingly only make sense in prime characteristic. For example, we can define F-regularity and F-rationality for finitely generated algebras over a field in this way.

Definition. *Let R be a finitely generated algebra over a field of characteristic zero. Then R is said to have F-regular type if R admits a family of models $A \to R_A$ in which a Zariski dense set of closed fibers are F-regular. (This does not depend on the choice of the the family of models.)*

Similarly, we can define weakly F-regular type, F-rational type, or F-split type for any finitely generated algebra over a field of characteristic zero. (In characteristic p, F-split means that the Frobenius map splits, that is, $R^p \subset R$ splits as a R^p-module map.)

There is a subtlety in the meaning of F–regularity for algebras of characteristic zero. As we've said in the first lecture, the operation of tight closure can be defined for any ring containing a field, so it makes sense to define a finitely generated algebra over a field of characteristic zero to be weakly F-regular if all ideals are tightly closed. This is a priori different from the condition of weakly F-regular type. We expect that these notions are equivalent, but this remains unsolved. See [Ho3].

The notions of F-rational type and F–regular type turn out to be intimately connected with the singularities that come up in the minimal model program. The first theorem in this direction explains the name "F-rational".

Theorem [S3], [Ha1]. *A finitely generated algebra over a field of characteristic zero has F-rational type if and only if it has rational singularities.*

The concept of rational singularities is very important in birational algebraic geometry. Recall that by definition, a ring R has rational singularities if and only if it is normal and it admits a desingularization X for which $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.)

Because Nobuo Hara has lectured on this theorem before in Japan, we will not dwell on it, although we will later mention some ideas in the proof. Now move on the application of tight closure to Fujita's freeness conjecture, where many related ideas appear.

Application of Tight Closure to Adjoint Linear Series.

Let X be a smooth projective variety of dimension d, and let $\mathcal{L}$ be an ample invertible sheaf on X. We are interested in the adjoint line bundles $\omega_X \otimes \mathcal{L}^n$, for $n > 0$. Because $\mathcal{L}$ is ample, we know that for large n, this adjoint bundle is globally generated. Fujita's freeness conjecture provides an effective version of this statement.

Fujita's Freeness Conjecture. *With X and $\mathcal{L}$ as above, the sheaf $\omega_X \otimes \mathcal{L}^{d+1}$ is globally generated.*

The conjecture is known in characteristic zero in dimension up to four [R], [EL], [Ka]. See [Ko] for a survey. In arbitrary characteristic, the best that is known is given by the following theorem.

Theorem [S4]. *If X is a smooth projective variety of dimension d and $\mathcal{L}$ is a globally generated ample line bundle on X, then $\omega_X \otimes \mathcal{L}^{d+1}$ is globally generated.*

See [S7] for a recent improvement of this result.

Our next task is to prove this theorem, that is, to establish Fujita's Freeness Conjecture for globally generated line bundles. This will give a good overview of some of the methods that can be used in applying tight closure to algebro-geometric questions.

If X has characteristic zero, the first step is to reduce to the characteristic p case using the standard technique we described. So it is enough to prove the theorem in the case that X has prime characteristic.

A good way to study an ample line bundle on a projective variety X is to build the section ring

$$S = \oplus_{n \geq 0} H^0(X, \mathcal{L}^n).$$

This is a finitely generated, $\mathbb{N}$-graded ring whose associated projective scheme recovers X. Its dimension is $d+1$. Assuming that X is irreducible, every section ring S will be a domain. If X is smooth, then S has (at worst) an isolated singularity at the unique homogeneous maximal ideal m. The invertible sheaf $\mathcal{L}^n$ corresponds to the graded S-module $S(n)$, the S-module S with degrees shifted by n.

Fujita's Freeness Conjecture is equivalent to the following more commutative algebraic statement.

Fujita's Freeness Conjecture in terms of local cohomology. *If (S, m) is a section ring with an isolated non-smooth point, then $H_m^{d+1}(S)$ has the following property: there exists an integer N such that for all $\eta \in H_m^{d+1}(S)$ of degree less than N, η has a non-zero S-multiple of degree $-d-1$.*

The proof of the equivalence of this statement with Fujita's Conjecture is not difficult. This is essentially the dual statement (using Matlis duality for S or Serre duality for X). Details can be found in [S4].

To prove Fujita's Conjecture, we will tackle this local cohomological conjecture. First we describe a convenient way to think about elements in the local cohomology module $H_m^{d+1}(S)$.

Let $x_0, x_1, \ldots, x_d$ be a system of parameters for S of degree one. Such a system of parameters exists by our assumption that $\mathcal{L}$ is globally generated (after enlarging the ground field if necessary). The local cohomology module $H_m^{d+1}(S)$ can be computed as the cokernel of the following map

$$S_{x/x_0} \oplus S_{x/x_1} \cdots \oplus S_x \longrightarrow S_x$$

$$\left(\frac{s_0 x_0^t}{x^t}, \frac{s_1 x_1^t}{x^t}, \ldots, \frac{s_d x_d^t}{x^t}\right) \mapsto \frac{\sum_{i=0}^{d}(-1)^i s_i x_i^t}{x^t}$$

where x denotes the product $x_0 x_1 \ldots x_d$ of the x_i's. This is the last map in the Cech complex for computing the cohomology of the sheaf of $\mathcal{O}_X$-algebras $\bigoplus_{i=0}^{\infty} \mathcal{O}_X(nL)$ with respect to the affine cover of X given by the $d+1$ open sets U_i where x_i does not vanish. More generally, the local cohomology modules $H_m^i(S)$ can be computed as the cohomology of this Cech complex, so that $H_m^i(S) = \bigoplus_{n \in \mathbf{Z}} H^{i-1}(X, \mathcal{L}^n)$, for all $i > 1$.

We represent elements of $H_m^{d+1}(S)$ by fractions $[\frac{z}{x^t}]$, with the square bracket reminding us of the equivalence relation on fractions. If the degree of η is $-n$, we see that $-n = \deg z - t(d+1)$.

It is easy to see that if $z \in (x_0^t, x_1^t, \ldots, x_d^t)$, then $\eta = [\frac{z}{x^t}]$ must be zero, by thinking about the image of the map above. Unfortunately, the converse is false. However, we have the following interesting observation.

Lemma. *If $\eta = [\frac{z}{x^t}] = 0$, then $z \in (x_0^t, \ldots, x_d^t)^*$.*

The Lemma is easily proved: if $\eta = [\frac{z}{x^t}] = 0$, then this means that for some integer s, we have

$$[\frac{z}{x^t}] = [\frac{x^s z}{x^{t+s}}] = 0$$

where now $x^s z \in (x_0^{t+s}, \ldots, x_d^{t+s})$. Thus

$$z \in (x_0^{t+s}, \ldots, x_d^{t+s}) : x^s$$

so by colon capturing, $z \in (x_0^t, \ldots, x_d^t)^*$. The Lemma is proved.

The Frobenius Action on Local Cohomology.

The Frobenius action on local cohomology is the main idea in the proof of Fujita's Freeness Conjecture for globally generated line bundles, and in the proof of the equivalence of rational singularities with F-rationality. It is also the central point in the relationship between tight closure and the Kodaira Vanishing theorem. The idea of using the Frobenius action on local cohomology to study tight closure first appears in the work of Richard Fedder and Kei–ichi Watanabe [FW].

The Frobenius action

$$H_m^{d+1}(S) \xrightarrow{F} H_m^{d+1}(S)$$

is easy to understand. Indeed, Frobenius acts in a natural way on each module $S_{x_{i_1}\dots x_{i_r}}$ in the Cech complex defining the local cohomology modules; it simply raises fractions to their $p-th$ powers. This action obviously commutes with the boundary maps, so that it induces a natural action on the local cohomology modules. In particular, the Frobenius action on $H_m^{d+1}(S)$ is given by

$$\eta = [\frac{z}{x^t}] \mapsto \eta^p = [\frac{z^p}{(x^t)^p}].$$

Using this, it makes sense to define tight closure for submodules of $H_m^{d+1}(S)$ by mimicking the definition for ideals. For example, we can define the tight closure of the zero submodule in $H_m^{d+1}(S)$:

$$0^* = \{\eta \in H_m^{d+1}(S) \mid \text{ there exists } c \neq 0 \text{ with } c\eta^{p^e} = 0 \text{ for all } e \gg 0\}.$$

The tight closure of zero in $H_m^{d+1}(S)$ is an important gadget. One can show that it is the unique maximal proper submodule of $H_m^{d+1}(S)$ stable under the action of Frobenius [S3].

Returning to the proof of Fujita's Freeness Conjecture, we observe the following two facts.

(1) $\eta = [\frac{z}{x^t}] \in 0^*$ if and only if $z \in (x_1^t, \dots, x_d^t)^*$.
(2) Any test element c kills 0^*.

These two facts are straightforward to prove using nearly the same argument as in the proof of the Lemma above.

Now the proof can be summarized in two main steps. First we show that if $\eta \in H_m^{d+1}(S)$ does not have a multiple of degree $-d-1$, then η is in 0^*. Next we show that 0^* vanishes in all sufficiently small degrees. Obviously, upon completion of these two steps, the proof is complete.

Step One: if $\eta \in H_m^{d+1}(S)$ does not have a multiple of degree $-d-1$, then η is in 0^*.

The main point is colon capturing. Assume on the contrary, that an element $\eta = [\frac{z}{x^t}]$ of degree $-n$ has no non-zero multiple of degree $-d-1$. This means that every element of S of degree $n-d-1$ must kill η. In particular,

$$(x_0, \ldots, x_d)^{n-d-1}\eta = 0.$$

By the Lemma, this means that

$$(x_0, \ldots, x_d)^{n-d-1} z \in (x_0^t, \ldots, x_d^t)^*,$$

or in other words,

$$z \in (x_0^t, \ldots, x_d^t)^* : (x_0, \ldots, x_d)^{n-d-1}.$$

Now we use colon capturing. We manipulate the parameters $x_0, \ldots, x_d$ formally as if they are the indeterminants of a polynomial ring, in which case the colon ideal (ignoring the $*$) would be easily computed to be

$$(x_0^t, \ldots, x_d^t) + (x_0, \ldots, x_d)^{(d+1)(t-1)-(n-d-1)+1}.$$

Colon capturing says that the actual colon ideal is contained in the tight closure of this "formal" colon ideal, that is,

$$z \in [(x_0^t, \ldots, x_d^t) + (x_0, \ldots, x_d)^{(d+1)(t)-(n)+1}]^*.$$

But note that the degree z is $(d+1)t - n$ (because $\eta = [\frac{z}{x^t}]$ has degree $-n = \deg z - (d+1)t$). Thus

$$z \in [(x_0^t, \ldots, x_d^t) + (x_0, \ldots, x_d)^{\deg z+1}]^*.$$

A moment's thought reveals that this forces

$$z \in (x_0^t, \ldots, x_d^t)^*.$$

Indeed, if

$$cz^q \in (x_0^t, \ldots, x_d^t)^{[q]} + [(x_0, \ldots, x_d)^{\deg z+1}]^{[q]},$$

we see immediately that because the degree of c is fixed, the degrees of the generators of $[(x_0, \ldots, x_d)^{\deg z+1}]^{[q]}$ are much larger than the degree of cz^q, so that cz^q must in fact be in the ideal $(x_0^t, \ldots, x_d^t)^{[q]}$ for large $q = p^e$. But by Fact (1) above, then we see that

$$\eta = [\frac{z}{x^t}] \in 0^*,$$

and the proof of step one is complete.

Step two: 0^* vanishes in sufficiently small degrees.

The point is to consider the test elements of S. Because X is smooth, the section ring S has an isolated singularity. This means that the defining ideal of the non-regular locus of S is m-primary. As we mentioned in Lecture 1, this implies that the test ideal of S (of all elements that "witness" all tight closure relations) contains an m-primary ideal. But according to Fact 2 above, the test ideal of S annihilates 0^*, so that 0^* is killed by an m-primary ideal. This says that 0^* has finite length, so of course, it must vanish eventually in all degrees sufficiently small. This completes the proof of step two, and thus the proof of Fujita's Freeness Conjecture for globally generated line bundles.

Experts will notice that the argument above does not really require that X be smooth. We used smoothness only in Step 2, to conclude that 0^* is finite length. But 0^* is of finite length more generally, and is in fact equivalent to the variety X being F-rational (or F-rational type in characteristic zero). Thus Fujita's Freeness Conjecture holds for any globally generated ample line bundle on a projective F-rational (type) variety.

We should remark that Fujita's Freeness Conjecture for globally generated line bundles can also be proved, in characteristic zero, using the Kodaira vanishing theorem. As far as I know, however, tight closure provides the only proof in prime characteristic. Interestingly, the Frobenius action on local cohomology seems to act as a substitute for Kodaira Vanishing. There is a good reason for this: it turns out that Kodaira vanishing theorem is equivalent to a statement about the action on Frobenius on local cohomology modules.

Tight Closure and Kodaira Vanishing.

Recall the classical Kodaira Vanishing Theorem:

Kodaira Vanishing. *If X is a smooth projective variety of characteristic zero, and $\mathcal{L}$ is any ample invertible sheaf on X, then $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$.*

The Kodaira Vanishing Theorem is false in characteristic p, although it can be proved by reduction to characteristic p [DI]. See also [EV].

Let $S = \oplus_{n \geq 0} H^0(X, \mathcal{L}^n)$ be the section ring of the pair $(X, \mathcal{L})$. Unwinding definitions using the point of view that local cohomology can be computed from the Cech complex of the $\mathcal{O}_X$–algebra $\oplus \mathcal{L}^n$, Kodaira Vanishing is seen to be equivalent to

$$H^i_m(S) \text{ vanishes in negative degree for all } i \text{ with } 1 < i < \dim S.$$

Because S has (at worst) an isolated non-Cohen-Macaulay point at m, we know that each $H^i_m(S)$ is supported at m, and hence must vanish in degrees sufficiently small. So we could state the Kodaira Vanishing Theorem as follows: *the Frobenius action on a dense set of characteristic p models for S is injective in negative degrees on $H^i_m(S)$, for $1 < i < \dim S$.*

Although it may sound a bit silly, this way of stating the vanishing of local cohomology in negative degree has the advantage of making sense also for the top local cohomology module $H^{\dim S}_m(S)$. In fact, the injective action of Frobenius on $H^{d+1}_m(S)$ in negative degrees is a new and important phenomenon, a natural generalization of the Kodaira Vanishing Theorem, which is not at all apparent otherwise. This extension to the top local cohomology module was conjectured to be true and called "Strong Kodaira Vanishing" in [HS]. The conjecture was proved in a beautiful paper of Nobuo Hara [Ha1], and in fact, turns out to be the main point in his proof that a rationally singular variety (of characteristic zero) must be of F–rational type. (See also [MS].)

The injective action of Frobenius on the negative degree part of local cohomolgy can be re-interpreted in terms of tight closure of parameter ideals. Using ideas similar to the ideas we used in the proof of Fujita's Conjecture to translate statements about the Frobenius action on $\eta = [\frac{z}{x^t}]$ into statements about the tight closure of $(x_0^t, x_1^t, \dots, x_d^t)$, we get a tight closure version of Kodaira Vanishing.

Kodaira Vanishing in terms of Tight Closure [HS]. *Let S be a section ring of a pair $(X, \mathcal{L})$ where X is a smooth variety of characteristic zero and $\mathcal{L}$ is an ample invertible sheaf of $\mathcal{O}_X$-modules. Then for any proper subset $x_0, \dots, x_k$ of a system of (homogeneous) parameters for S, where $\deg x_i \gg 0$,*

$$(x_0, \dots, x_k)^* \subset \sum_{i=0}^{k} (x_0, \dots, \hat{x}_i, \dots, x_k)^* + S_{\geq D}$$

where D is the sum of the degrees of the x_i's.

This theorem is equivalent to the Kodaira Vanishing Theorem. Just as Kodaira Vanishing can fail in prime characteristic, so can this tight closure statement. However, the statement holds when S is a generic characteristic p model for a section ring of characteristic zero, that is, "for large p."

By allowing the possibility that we have a full system of parameters in the above version of the Kodaira Vanishing Theorem, we get the strong Kodaira Vanishing Theorem. In fact, if $x_0, x_1, \dots, x_d$ is a full system of parameters for a section S as above, we get a more precise statement.

Strong Kodaira Vanishing [HS] [H]. *Let S be an $\mathbb{N}$-graded ring over a field of characteristic zero, and let $x_0, x_1, \dots, x_d$ be a full system of (homogeneous) parameters for S, with $\deg x_i \gg 0$. Then*

$$(x_0, \dots, x_d)^* = \sum_{i=0}^{d} (x_0, \dots, \hat{x}_i, \dots, x_d)^* + S_{\geq D}$$

where D is the sum of the degrees of the x_i's.

The reason we get equality here is that $S_{\geq D}$ is contained in $(x_0, \dots, x_d)^*$, as can be verified with the Briançon-Skoda theorem (Property 4).

It is possible to say precisely how large the degrees of the x_i's must be in the statements of Kodaira and strong Kodaira vanishing in terms of tight closure. In both theorems, each x_i should have degree larger than a, where a is the a-invariant of S. By definition (due to Goto and Watanabe), the a-invariant is the largest integer n such that $H_m^{\dim S}(S)$ is non-zero in degree n.

The strong form of Kodaira Vanishing is conjectured in [HS], where the idea of the "monomial property of a d^+ sequence" due to Goto and Yamagishi is used. It is proved in [HS] for rings of dimension two, from which it is shown that the Kodaira Vanishing Theorem follows for any normal surface of dimension two. In full generality, however, the statement was not known until Nobuo Hara proved the injectivity of the Frobenius action on the negatively graded part of local cohomology [Ha1]. Hara has since greatly generalized his work; see [Ha3].

Tight Closure and Singularities.

Finally, we summarize some more connections between tight closure and singularities in algebraic geometry.

Let X be a normal variety of characteristic zero. Assume that X is $\mathbb{Q}$-Gorenstein, that is, that the reflexive sheaf ω_X represents a torsion element K_X in the (local) class group of X. In other words, the Weil divisor class K_X is assumed to have a multiple which is locally principle.

Consider a desingularization $\tilde{X} \xrightarrow{\pi} X$ of X, where the exceptional divisor is a simple normal crossings divisor with components $E_1, \ldots, E_n$. Write

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i=1}^{n} a_i E_i$$

for some unique rational numbers a_i. To understand this expression, suppose that rK_X is locally principle, so that it makes sense to pull it back; then compare to $rK_{\tilde{X}}$. The difference is some divisor supported on the exceptional set, hence of the form $\sum_{i=1}^{n} m_i E_i$. Dividing by r, we arrive at the above expression, where 'equality' means numerical equivalence of $\mathbb{Q}$-divisors. See [KMM].

In general, the a_i's can be any rational number, although if X is smooth, we can easily see that each a_i will be a positive integer. This leads us to the following restricted class of singularities.

Definition. *The variety X has log-terminal singularities if all $a_i > -1$, and has log-canonical singularities if all $a_i \geq 1$. (This is independent of the choice of desingularization.)*

The relationship to tight closure is is evidenced by the following theorem.

Theorem. *Let X be a normal $\mathbb{Q}$-Gorenstein variety of characteristic zero. X has F-regular type if and only if X has log-terminal singularities.*

This theorem follows immediately from the equivalence of rational singularities and F-rational type discussed earlier, using the "canonical cover trick". Indeed, assuming X is local, set

$$Y = \text{ Spec } \{\mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \mathcal{O}_X(2K_X) \oplus \ldots \mathcal{O}_X((r-1)K_X)\}$$

where r is such that $\mathcal{O}_X(rK_X)$ is isomorphic to $\mathcal{O}_X$ via a fixed isomorphism (so that we can define a ring structure on $\mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \mathcal{O}_X(2K_X) \oplus \ldots \mathcal{O}_X((r-1)K_X))$.

The natural map $Y \longrightarrow X$ is called the canonical cover of X. It is easy to check that when X is Cohen-Macaulay, the canonical cover Y is Gorenstein, and that the map is étale in codimension one. With these properties, it is not hard to show the following two facts:

(1) (Kawamata) Y has rational singularities if and only if X has log-terminal singularities.
(2) (K.-i. Watanabe) Y has F-rational type if and only if X has F-regular type.

Thus the equivalence of F-regular type with log-terminal singularities follows from the equivalence of F-rational type with rational singularities.

There are some subtleties involved in the argument using the canonical cover. Watanabe's argument shows F-rationality for Y is equivalent to *strong F-regularity* for X. Strong F-regularity is a technical condition conjectured to be equivalent to weak F-regularity (when both are defined), introduced because it, unlike weak F-regularity, is easily shown to pass to localizations [HH2]. However, in the case of $\mathbb{Q}$–Gorenstein rings, weak and strong F-regularity turn out to be equivalent [M].

The first proof that F-regular type $\mathbb{Q}$-Gorenstein singularities are log-terminal is due to Kei-ichi Watanabe and uses a different argument [W]. This different argument also produces the following nice result.

Theorem [W]. *Let X be a variety satisfying the conditions above. If X is of F–split type, then X has log–terminal singularities. (Recall, a local ring of characteristic p is F-split if the inclusion $R^p \hookrightarrow R$ splits as a map of R^p modules.)*

A very interesting open problem that has deep connections with number theory is the following.

Open Problem. *If X has log-canonical singularities, does X have F–split type?*

Further Reading on Tight Closure.

The original tight closure paper of Hochster and Huneke [HH1] is still an excellent introduction to the subject. There are also a number of expository articles on tight closure. Craig Huneke's book *Tight Closure and its Applications* [Hu2] is an good place for a beginning commutative algebra student to learn the subject; it contains several applications more or less disjoint from the ones discussed in detail here. It also contains an appendix by Mel Hochster [Ho3] discussing tight closure in characteristic zero. Another nice survey is [Ho1], which contains a list of open problems; although the article is now seven years old, many of these problems remain open. A more recent view is provided by the expository article [B]. The article [S5] is a survey written for algebraic geometers. Huneke's "Tight Closure and Geometry" is another nice read for algebraists [Hu3]. All these sources, but especially [Hu2], contain long bibliographies to direct the reader to numerous research articles on tight closure.

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Dutta multiplicity in characteristic 0

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1 Introduction

Multiplicities defined by limits of Euler characteristics over powers of the Frobenius map in positive characteristic were introduced by Dutta [1], who showed that in many respects they have better properties than ordinary Euler characteristics. In particular, if A is a local ring of dimension d, the Dutta multiplicity of a nonexact complex of free modules of length d with homology of finite length is always positive; this fact was proven and used in an essential way to prove the Intersection Theorem for rings of mixed characteristic in Roberts [13].

More recently, the concept of Dutta multiplicity has been generalized to rings of arbitrary characteristic using localized Chern characters by Kurano [7]. While this definition agrees with the original one for rings of positive characteristic, the lack of a simple construction in terms of limits over the Frobenius map makes it more difficult to prove many of its properties. In this paper we show that the positivity result mentioned above can be generalized to homomorphic images of regular local rings of arbitrary characteristic containing a field.

The basic idea is to reduce to the case of positive characteristic using the "Metatheorem" of Hochster [6], which states that if a set of equations has a solution in any ring containing a field, then it has a solution in some ring of positive characteristic. To apply this theorem, we use an alternate description of Dutta multiplicity in terms of Adams operations. The theory of Adams operations can be considered to be a generalization of the Frobenius map to arbitrary characteristic, but it is defined on the Grothendieck group of complexes rather than induced by a map of rings, so the theory, like that of Dutta multiplicities, is somewhat more difficult than in the case of positive characteristic.

In the first section we give a precise statement of the theorem and introduce notation. In the next few sections we prove the necessary statements on Adams operations, after which we prove the positivity result. In the last section we prove a generalization of a result of Dutta on the positivity of intersection multiplicities.

1 Notation and results

In this section we state the main result on the positivity of Dutta multiplicity. We first recall the original definition of Dutta multiplicity in positive characteristic.

Let A be a local ring, and let $\mathbb{F}.$ be a bounded complex of A-modules. If $\mathbb{F}.$ has homology of finite length, we define the Euler characteristic $\chi(\mathbb{F}.)$ to be

$$\chi(\mathbb{F}.) = \sum_t (-1)^t \ell(H_t(\mathbb{F}.)),$$

where $H_t(\mathbb{F}.)$ is the t-th homology module of $\mathbb{F}.$ and $\ell(H_t(\mathbb{F}.))$ denotes its length.

Let A be a complete local ring of positive characteristic p with perfect residue field, and let $\mathbb{F}.$ be a bounded complex of finitely generated free A-modules with homology of finite length. Let d be the dimension of A. If we denote $\mathbb{F}.^e$ the tensor product of $\mathbb{F}.$ with the e-th power of the Frobenius map, then the *Dutta multiplicity* is defined to be

$$\chi_\infty(\mathbb{F}.) = \lim_{e \to \infty} \frac{\chi(\mathbb{F}.^e)}{p^{de}}. \qquad (1)$$

Let X be a scheme of finite type over a regular scheme. A bounded complex of locally free $\mathcal{O}_X$-modules of finite rank is called a *perfect* complex. For a perfect $\mathcal{O}_X$-complex $\mathbb{F}.$, we define the *support* of $\mathbb{F}.$ by

$$\mathrm{Supp}(\mathbb{F}.) = \bigcup_t \mathrm{Supp}(H_t(\mathbb{F}.)).$$

The support of $\mathbb{F}.$ is a closed set of X consisting of those points at which $\mathbb{F}.$ is not exact. Let $\mathbb{F}.$ be a perfect $\mathcal{O}_X$-complex with support in Y. We denote the *localized Chern character* with respect to the perfect complex $\mathbb{F}.$ by

$$\mathrm{ch}_Y^X(\mathbb{F}.) = \bigoplus_{i \geq 0} \mathrm{ch}_i(\mathbb{F}.)$$

If no confusion is possible, we denote it simply by $\mathrm{ch}(\mathbb{F}.)$. We refer the reader to Fulton [3] or Roberts [14] for the definition and basic properties of localized Chern characters. We recall that localized Chern characters are defined as operators on the Chow group, and that if η is a cycle of dimension j in $\mathrm{A}_j(X)_{\mathbb{Q}}$, then $\mathrm{ch}_i(\mathbb{F}.)(\eta)$ is an element of $\mathrm{A}_{j-i}(Y)_{\mathbb{Q}}$.

Definition 1.1 Let (A, m) be a homomorphic image of a regular local ring. Put $d = \dim A$. Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Then, we define the *Dutta multiplicity* of a perfect A-complex $\mathbb{F}.$ by letting

$$\chi_\infty(\mathbb{F}.) = \mathrm{ch}_d(\mathbb{F}.) \cap [\mathrm{Spec}\, A].$$

By definition, $\chi_\infty(\mathbb{F}.) \in \mathrm{A}_0(\mathrm{Spec}\, A/m)_{\mathbb{Q}} = \mathbb{Q} \cdot [\mathrm{Spec}\, A/m]$. Hence, we may regard the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ as a rational number.

Remark 1.2 Let (A, m) be a complete equi-characteristic Noetherian local ring such that the residue class field is perfect of positive characteristic. Then the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ as above coincides with the original one defined by Dutta [1] using the Frobenius endomorphism (Szpiro [16], Roberts [13], [14]).

We also make one comment on the relation of Dutta multiplicity to the usual Euler characteristic. Let $X = \mathrm{Spec}\, A$, where A has dimension d. Let $[\mathrm{Spec}\, A]$ be the element of the Chow group of A in dimension d defined by taking the sum of $\ell_{A_\mathfrak{p}}(A_\mathfrak{p})[\mathrm{Spec}\, A/\mathfrak{p}]$, where the sum is taken over all prime ideals of A with $\dim A/\mathfrak{p} = d$. By the local Riemann-Roch theorem (Fulton [3], Example 18.3.12), there is an element of $\mathrm{A}_*(X)_{\mathbb{Q}}$ denoted

$$\tau_A(A) = \tau_d(A) + \tau_{d-1}(A) + \cdots + \tau_0(A),$$

where $\tau_i(A) \in \mathrm{A}_i(X)_{\mathbb{Q}}$ for each i, $\tau_d(A) = [\mathrm{Spec}\, A]$, and

$$\chi(\mathbb{F}.) = \sum_{i=0}^{d} \mathrm{ch}_i(\mathbb{F}.)(\tau_i(A)).$$

We refer to τ_A as the Riemann-Roch map. Comparing this expression for the Euler characteristic to Dutta multiplicity, we see that the Dutta multiplicity of $\mathbb{F}.$ is equal to the term in this sum for which $i = d$. If $\tau_i(A) = 0$ for $i < d$ (a situation which sometimes occurs; see the discussion at the end of section 5), the Euler characteristic and the Dutta multiplicity of $\mathbb{F}.$ are equal.

The main aim in the present paper is to prove the following theorem:

Theorem 1.3 *Let (A, m) be a homomorphic image of a regular local ring. Assume that A contains a field. Put $d = \dim A$. Let*

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$*. (In particular, the support is not empty, and $\mathbb{F}.$ is not exact.) Then we have $\chi_\infty(\mathbb{F}.) > 0$*

The above theorem was proved by Roberts [13] if (A, m) is a complete equi-characteristic Noetherian local ring such that the residue class field is perfect of positive characteristic. It played an essential role in his proof of the New Intersection Theorem in the mixed characteristic case. By Theorem 1.3, we know that this proof of the New Intersection Theorem is valid for all Noetherian local rings.

It is open whether Theorem 1.3 is true or not without the assumption that A contains a field. If A has a test module in the sense of [10], then we do not need the assumption that A have equal characteristic. We refer the reader to [10] for definition and basic properties of test modules.

We prove Theorem 1.3 in Section 4.

Describing Theorem 1.3 in geometric terms, we obtain the following corollary.

Corollary 1.4 *Let X be a scheme of finite type over a regular scheme and Y a closed subset of X. Let d be a positive integer. Suppose that $\mathbb{F}.$ is a perfect $\mathcal{O}_X$-complex with* $\mathrm{Supp}(\mathbb{F}.) = Y$*. Let Z be a closed integral subscheme of X and W an irreducible component of $Z \cap Y$ such that* $\mathrm{codim}_Z W = d$*. Assume that $\mathbb{F}. \otimes_{\mathcal{O}_X} \mathcal{O}_{Z,W}$ is quasi-isomorphic to a perfect $\mathcal{O}_{Z,W}$-complex of length d. (A perfect complex $\mathbb{G}.$ is said to be of length d if $G_i = 0$ for $i \notin [0, d]$, $G_0 \neq 0$ and $G_d \neq 0$.)*

If the local ring $\mathcal{O}_{Z,W}$ contains a field, then the coefficient of $[W]$ in $\mathrm{ch}_d(\mathbb{F}.) \cap [Z] \in \mathrm{A}_*(Y \cap Z)_{\mathbb{Q}}$ *is positive.*

Proof. This corollary is a translation of Theorem 1.3 into geometric language; we leave the details to the reader.

q.e.d.

In Section 5, we give an application of Theorem 1.3 to intersection multiplicities.

2 Adams Operations

In this section we collect the facts about Adams operations which will be necessary for the proof of Theorem 1.3. We refer the reader to Gillet-Soulé [4] or Soulé [15] for details on the definition and basic properties of Adams operations.

For a closed subset Y of X, we denote by $\mathrm{K}_0^Y(X)$ the *Grothendieck group of perfect $\mathcal{O}_X$-complexes with supports in Y*, i.e., $\mathrm{K}_0^Y(X) = Z/R$, where Z is the free abelian group generated by the set of perfect $\mathcal{O}_X$-complexes whose supports are contained in Y and R is the subgroup of Z generated by the following two types of relations:

- $[\mathbb{F}.] - [\mathbb{G}.]$, if there exists a quasi-isomorphism from $\mathbb{F}.$ to $\mathbb{G}.$;
- $[\mathbb{F}.] + [\mathbb{H}.] - [\mathbb{G}.]$, if there is an exact sequence of complexes

$$0 \to \mathbb{F}. \to \mathbb{G}. \to \mathbb{H}. \to 0.$$

Adams operations are defined on $\mathrm{K}_0^Y(X)$ using the λ-ring structure on $\mathrm{K}_0^Y(X)$, which is in turn defined by defining operators λ^r on $\mathrm{K}_0^Y(X)$ using exterior powers. It has been shown by M. Hashimoto (unpublished) that if the underlying ring contains a field of characteristic zero, the λ-operations can be defined directly using exterior powers of complexes. In general, however, a more complicated definition using simplicial objects is required. We refer to Gillet-Soulé [4] for details of this construction. The properties of λ^r we need are summarized in the following proposition.

Proposition 2.1 *1. If $\mathbb{F}.$ is a perfect complex with $F_t = 0$ for $t \neq 0$, then $\lambda^r(\mathbb{F}.)$ is the r-th exterior power of F_0 (in degree zero).*

2. *If* $\mathbb{F}.$ *is a complex of the form*

$$\cdots \to 0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to 0 \to \cdots,$$

where $\mathcal{O}_X(-D)$ *and* $\mathcal{O}_X$ *have degrees 1 and 0 respectively, then*

$$[\lambda^r(\mathbb{F}.)] = (-1)^{r-1}[\mathbb{F}. \otimes \mathcal{O}_X(-(r-1)D)]$$

in $\mathrm{K}_0^Y(X)$.

3. *Let* $\mathbb{F}.$ *be a perfect complex of free modules over a local ring, and let the rank of* F_t *be* r_t *for each* t. *Then* $\lambda^r(\mathbb{F}.)$ *can be represented by a complex* $\mathbb{G}.$, *where the rank of* G_i *for each* i *depends only on the values of* r_t, *and where the boundary maps are defined by matrices with entries which are polynomials in the entries of the matrices of* $\mathbb{F}.$ *with coefficients in* $\mathbb{Z}$. *These polynomials also depend only on the numbers* r_t.

Proof. We refer to Gillet-Soulé [4] for verification of these facts. The second statement is a global version of their Lemma 4.12.

q.e.d.

For a closed subset Y of X and an integer $k \geq 1$, we next define a map $\psi^k : \mathrm{K}_0^Y(X) \to \mathrm{K}_0^Y(X)$, which is called the k-th *Adams operation.* Adams operations are defined in terms of the λ-operations by the following formula, where $\cup$ denotes the operation induced by the tensor product of complexes.

$$\psi^k - \psi^{k-1} \cup \lambda^1 + \cdots + (-1)^{k-1}\psi^1 \cup \lambda^{k-1} + (-1)^k k\lambda^k = 0. \qquad (2)$$

Adams operations are functorial with respect to pullback of perfect complexes and are multiplicative with respect to the tensor product. We note also the localized Chern characters are additive with respect to short exact sequences, multiplicative with respect to the tensor product, and functorial with respect to flat pullback and proper push-forward (see Fulton [3], Chapter 18).

3 The relation between Adams operations and localized Chern characters

In this section we prove the following theorem.

Theorem 3.1 *Let X be a scheme of finite type over a regular scheme. Let Y be a closed subset of X and let $\mathbb{F}.$ be a perfect $\mathcal{O}_X$-complex with support in Y. Then, for any $k \geq 1$ and $i \geq 0$, we have*

$$\mathrm{ch}_i(\psi^k(\mathbb{F}.)) = k^i \,\mathrm{ch}_i(\mathbb{F}.).$$

The proof of Theorem 3.1 uses the splitting principle for complexes and an explicit computation of Adams operations in the case of an elementary complex. We recall that an *elementary complex* is a complex of one of the two forms:

1. A locally free sheaf $\mathcal{L}$ of rank one in degree zero.

2. A complex of the form

$$0 \to \mathcal{L} \otimes \mathcal{O}_X(-D) \to \mathcal{L} \to 0,$$

   where D is an effective Cartier divisor and $\mathcal{L}$ is a locally free sheaf of rank one.

By the splitting principle for complexes (see Fulton [3], Example 18.3.12 or Roberts [14], section 12.2), there exists a projective map from Y to X such that the pullback of $\mathbb{F}.$ to Y has a filtration with quotients which are elementary complexes with shifts of degrees. Thus, using the functorial properties of localized Chern characters and of Adams operations, together with their additivity, it suffices to prove the result in the case of an elementary complex (see either of the references cited above for the details of this procedure).

We first compute the Adams operations for elementary complexes.

Lemma 3.2 *Let X be a scheme.*

(a) Let $\mathcal{L}$ be a line bundle (locally free sheaf of rank 1). Then, for any $k \geq 1$, we have $\psi^k(\mathcal{L}) = \mathcal{L}^{\otimes k}$, where we regard $\mathcal{L}$ as a complex concentrated in degree 0.

(b) Let D be an effective Cartier divisor on X. For a positive integer n, we define a perfect complex $\mathbb{E}.^{(n)}$ with support D as

$$E_t^{(n)} = \begin{cases} \mathcal{O}_X & \text{if } t = 0, \\ \mathcal{O}_X(-nD) & \text{if } t = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{O}_X(-nD) \to \mathcal{O}_X$ *is the natural inclusion. We put* $\mathbb{E}. = \mathbb{E}^{(1)}_.$. *Then,*

$$[\mathbb{E}^{(n)}_.] = [\mathbb{E}.] + [\mathbb{E}. \otimes \mathcal{O}_X(-D)] + \cdots + [\mathbb{E}. \otimes \mathcal{O}_X(-(n-1)D)]$$

in $\mathrm{K}_0^D(X)$.

(c) *With notation as above, for any* $k \geq 1$, $\psi^k(\mathbb{E}.) = [\mathbb{E}^{(k)}_.]$.

Proof. By the first part of Proposition 2.1, we have

$$\lambda^k(\mathcal{L}) = \begin{cases} \mathcal{L} & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Therefore, by equation (2) of section 2, we obtain $\psi^k(\mathcal{L}) = \mathcal{L}^{\otimes k}$ for any $k \geq 1$.

To prove (b), we take the total complex of

$$\begin{array}{ccc} \mathcal{O}_X(-nD) & = & \mathcal{O}_X(-nD) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(-(n-1)D) & \to & \mathcal{O}_X. \end{array}$$

Taking the filtration of this double complex by rows, and using that the top row is exact, we see that the total complex is quasi-isomorphic to $\mathbb{E}^{(n-1)}_.$. Taking filtration by columns , we conclude that it is also equal to

$$[\mathbb{E}^{(n)}_.] - [\mathbb{E}. \otimes \mathcal{O}_X(-(n-1)D)]$$

in $\mathrm{K}_0^D(X)$. Therefore, (b) follows by induction on n.

We now prove (c) by induction on k. By Proposition 2.1, we have

$$\lambda^k(\mathbb{E}.) = (-1)^{k-1}[\mathbb{E}. \otimes \mathcal{O}_X(-(k-1)D)]$$

for $k \geq 1$. In particular, we have $\psi^1(\mathbb{E}.) = \mathbb{E}. = \lambda^1(\mathbb{E}.)$. Suppose $k \geq 2$. Assertion (c) follows easily from (b), equation (2) of section 2, and the fact, proven in the same way as (b), that $[\mathbb{E}. \otimes \mathbb{E}.] = [\mathbb{E}.] - [\mathbb{E}. \otimes \mathcal{O}_X(-D)]$. **q.e.d.**

We now prove Theorem 3.1 for elementary complexes. We may assume that X is an integral scheme, and we have only to prove that

$$\mathrm{ch}_i(\psi^k(\mathbb{F}.)) \cap [X] = k^i \, \mathrm{ch}_i(\mathbb{F}.) \cap [X]$$

for $k \geq 1$ and $i \geq 0$.

First, assume that F_0 is a line bundle and $F_t = 0$ for $t \neq 0$. In this case the localized Chern character of $\mathbb{F}.$ is the Chern character of F_0 as a sheaf, and (by Proposition 18.1 (a) in [3] or [14], section 11.4), we have

$$\mathrm{ch}_i(\mathbb{F}.) = \mathrm{ch}_i(F_0) = \frac{1}{i!} c_1(F_0)^i,$$

where $c_1(F_0)$ is the first Chern class of the line bundle F_0. Therefore, by Lemma 3.2 (a), we have

$$\mathrm{ch}_i(\psi^k(\mathbb{F}.)) = \mathrm{ch}_i(F_0^{\otimes k}) = \frac{1}{i!}\{k \cdot c_1(F_0)\}^i = k^i \, \mathrm{ch}_i(\mathbb{F}.).$$

Next, we suppose that $Y \neq X$ and that $\mathbb{F}.$ is an elementary complex concentrated in degrees 0 and 1. Then, there exist a line bundle $\mathcal{L}$ and an elementary complex $\mathbb{E}.$ defined by an effective Cartier divisor D as in Lemma 3.2 (b) such that $\mathbb{F}. = \mathbb{E}. \otimes \mathcal{L}$. Note that Y contains D by definition. By the multiplicativity of localized Chern characters (Example 18.1.5 in Fulton [3]), we have

$$\mathrm{ch}(\mathbb{F}.) = \mathrm{ch}(\mathcal{L}) \cdot \mathrm{ch}(\mathbb{E}.).$$

On the other hand, by the multiplicative property of Adams operations (4.11 A2) in Gillet-Soulé [4]) and Lemma 3.2 (c), we have

$$\mathrm{ch}(\psi^k(\mathbb{F}.)) = \mathrm{ch}(\psi^k(\mathcal{L}) \otimes \psi^k(\mathbb{E}.)) = \mathrm{ch}(\psi^k(\mathcal{L})) \cdot \mathrm{ch}(\mathbb{E}.^{(k)}).$$

By Corollary 18.1.2 in Fulton [3] or Theorem 11.4.5 in Roberts [14], we have

$$\mathrm{ch}(\mathbb{E}.) = 1 - e^{-D}.$$

Therefore, we have

$$\begin{aligned} \mathrm{ch}(\mathbb{F}.) &= \mathrm{ch}(\mathcal{L}) \cdot \mathrm{ch}(\mathbb{E}.) = e^c \cdot (1 - e^{-D}) \\ \mathrm{ch}(\psi^k(\mathbb{F}.)) &= \mathrm{ch}(\psi^k(\mathcal{L})) \cdot \mathrm{ch}(\mathbb{E}.^{(k)}) = e^{kc} \cdot (1 - e^{-kD}), \end{aligned}$$

where $c = c_1(\mathcal{L})$. Since $\mathrm{ch}_i(\mathbb{F}.)$ (resp. $\mathrm{ch}_i(\psi^k(\mathbb{F}.))$) is the homogeneous component of $\mathrm{ch}(\mathbb{E}.)$ (resp. $\mathrm{ch}(\psi^k(\mathbb{F}.))$) in degree i with respect to the total degree in variables c and D, we obtain $\mathrm{ch}_i(\psi^k(\mathbb{F}.)) = k^i \, \mathrm{ch}_i(\mathbb{F}.)$. This completes the proof of Theorem 3.1.

q.e.d.

4 Proof of the Main Theorem

In this section we prove Theorem 1.3.

If A is a complete equi-characteristic local ring, and if the residue class field of A is perfect of characteristic $p > 0$, then $\chi_\infty(\mathbb{F}.)$ coincides with the original Dutta multiplicity (Remark 1.2). Using this fact, the theorem was proven in this case by Roberts [13], [14]:

Now let A be an arbitrary homomorphic image of a regular local ring containing a field of positive characteristic. By a standard construction (see [5], section 0.6.8), there exists a flat map $f : A \to B$ of relative dimension 0 such that B is also a homomorphic image of a regular local ring and such that the residue field of B is perfect. We recall (see Fulton [3], Appendix B.2.5) that to say that f has relative dimension 0 means that for all prime ideals $\mathfrak{p}$ of A, and for all primes ideals $\mathfrak{q}$ of B minimal over $f(\mathfrak{p})B$, we have $\dim A/\mathfrak{p} = \dim B/\mathfrak{q}$. We can then complete the local ring B; the map from B to its completion $\hat{B}$ is flat of relative dimension zero since a homomorphic image of a regular local ring is formally equidimensional (or quasi-unmixed; see Matsumura [11], Section 31). Then, using the compatibility of localized Chern classes with flat pullback (see Fulton [3] or Roberts [14]; although Fulton assumes that maps are of finite type, the proof is valid also in the case we are considering), letting m denote the maximal ideal of A, we have

$$\ell_{\hat{B}}(\hat{B}/m\hat{B}) \cdot \chi_\infty(\mathbb{F}.) = \chi_\infty(\mathbb{F}. \otimes_A \hat{B}).$$

Hence, since $\chi_\infty(\mathbb{F}. \otimes_A \hat{B}) > 0$, we have $\chi_\infty(\mathbb{F}) > 0$.

We now present the main part of the proof, which is the reduction of the characteristic zero case to the case of positive characteristic. The main theorem we use, as mentioned in the introduction, is the Metatheorem of Hochster [6], which states that if a system of equations has a solution over a ring of characteristic zero satisfying certain properties, then it has a solution over a ring of positive characteristic. The precise statement of this lemma in the form we need is as follows (Kurano [8]). In this lemma, a solution of an ideal $\mathfrak{a}$ of a polynomial ring means a sequence of elements which gives zero when evaluated at any polynomial in $\mathfrak{a}$.

Lemma 4.1 *Let $\mathfrak{a}$ be an ideal of the polynomial ring*

$$\mathbb{Z}[Y_1, \ldots, Y_n, X_1, \ldots, X_d, G_1, \ldots, G_l, W_1, \ldots, W_k]$$

over $\mathbb{Z}$. Suppose that a regular local ring R containing $\mathbb{Q}$ has a solution

$$y_1, \dots, y_n, x_1, \dots, x_d, g_1, \dots, g_l, w_1, \dots, w_k$$

of $\mathfrak{a}$ such that $y_1, \dots, y_n$ forms a regular system of parameters for R and $\overline{x_1}, \dots, \overline{x_d}$ forms a system of parameters for $R/(g_1, \dots, g_l)$, where $\overline{x_i}$ stands for the homomorphic image of x_i for each i. Then there exists a regular local ring R' which satisfies the following two conditions; (1) R' is essentially of finite type over a field of positive characteristic, (2) R' has a solution

$$y'_1, \dots, y'_n, x'_1, \dots, x'_d, g'_1, \dots, g'_l, w'_1, \dots, w'_k$$

of $\mathfrak{a}$ such that $y'_1, \dots, y'_n$ forms a regular system of parameters for R' and $\overline{x'_1}, \dots, \overline{x'_d}$ forms a system of parameters for $R'/(g'_1, \dots, g'_l)$.

Thus what we need to prove Theorem 1.3 is to express the positivity of Dutta multiplicity in terms of equations of the type described in Lemma 4.1.

We assume that we have a counterexample to the positivity of Dutta multiplicity over a ring A which is a homomorphic image of a regular local ring and which contains a field of characteristic zero. Let the dimension of A be d. Then there exists a complex

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

such that the Dutta multiplicity is not positive. We can represent the maps of free modules by matrices, and the fact that $\mathbb{F}.$ is a complex is expressed by the fact that the entries in these matrices form the solution to a set of polynomials. In addition, Lemma 3.11 of Kurano [8] states that we can add more variables and new polynomials to the system so that every solution to the new system defines a complex $\mathbb{G}.$ such that

$$\ell(H_t(\mathbb{G}.)) = \ell(H_t(\mathbb{F}.))$$

for all t, so that in particular we have $\chi(\mathbb{G}.) = \chi(\mathbb{F}.)$.

In addition, the third part of Proposition 2.1 states that the λ-operations can be defined by a finite set of polynomials. Combining this with the inductive definition of Adams operations in equation (2) of section 2, we may conclude that for any integer e, there exists a set of polynomials with coefficients in $\mathbb{Z}$ such that $\psi^e(\mathbb{F}.)$ can be represented in the Grothendieck group by

$[\mathbb{G}.] - [\mathbb{H}.]$ where the entries in the matrices of $\mathbb{G}.$ and $\mathbb{H}.$ are given by these polynomials. We would like to emphasize that these polynomials depend only on e and the ranks of the free modules F_i.

Combining this fact with Lemma 3.11 of Kurano [8] quoted above, it suffices to show that we can express the Dutta multiplicity in terms of a finite number of Euler characteristics $\chi(\psi^e(\mathbb{F}.))$ for a finite number of values of e. We do this in the following lemma.

Lemma 4.2 *Let (A, m) be a homomorphic image of a regular local ring and put $\dim A = d$. Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$.*

1. *For any $k \geq 2$, we have*

$$\chi_\infty(\mathbb{F}.) = \lim_{e \to \infty} \frac{\chi(\psi^e(\mathbb{F}.))}{e^d} = \lim_{e \to \infty} \frac{\chi((\psi^k)^e(\mathbb{F}.))}{k^{de}}. \tag{3}$$

2. *There exist rational numbers $a_1, \ldots, a_{d+1}$ (depending only on d) such that*

$$\chi_\infty(\mathbb{F}.) = \sum_{e=1}^{d+1} a_e \cdot \chi(\psi^e(\mathbb{F}.)).$$

Proof. Let $\tau_A : \mathrm{K}_0 \operatorname{Spec} A_\mathbb{Q} \to \mathrm{A}_* \operatorname{Spec} A_\mathbb{Q}$ be the Riemann-Roch map for A. Put $\tau_A([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0$, where $\tau_i \in \mathrm{A}_i \operatorname{Spec} A_\mathbb{Q}$ for each i. Note that $\tau_d = [\operatorname{Spec} A]$ is satisfied by the top term property (Theorem 18.3 (5) in Fulton [3]). Then, by Example 18.3.12 in [3] and Theorem 3.1, we have

$$\begin{aligned}
\chi(\psi^e(\mathbb{F}.)) &= \mathrm{ch}(\psi^e(\mathbb{F}.)) \cap \tau_A([A]) \\
&= \sum_{i=0}^{d} \mathrm{ch}_i(\psi^e(\mathbb{F}.)) \cap \tau_i \\
&= \sum_{i=0}^{d} e^i \,\mathrm{ch}_i(\mathbb{F}.) \cap \tau_i \\
\chi((\psi^k)^e(\mathbb{F}.)) &= \mathrm{ch}((\psi^k)^e(\mathbb{F}.)) \cap \tau_A([A]) \\
&= \sum_{i=0}^{d} \mathrm{ch}_i((\psi^k)^e(\mathbb{F}.)) \cap \tau_i \\
&= \sum_{i=0}^{d} k^{ei} \,\mathrm{ch}_i(\mathbb{F}.) \cap \tau_i.
\end{aligned}$$

The first statement immediately follows from the equations above.

Put $\alpha_e = \chi(\psi^e(\mathbb{F}.))$ and $\beta_i = \mathrm{ch}_i(\mathbb{F}.) \cap \tau_i$. Then we have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d+1 & \cdots & (d+1)^d \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_d \end{pmatrix}.$$

We denote by L the $(d+1)$ by $(d+1)$ matrix in the equation as above. Obviously L is invertible. Let $a_1, \ldots, a_{d+1}$ be the last column of L^{-1}. Then, we have

$$\chi_\infty(\mathbb{F}.) = \beta_d = \sum_{e=1}^{d+1} a_e \cdot \chi(\psi^e(\mathbb{F}.)).$$

q.e.d.

Now, we combine the above results to prove Theorem 1.3 in the case A contains a field of characteristic 0.

Let (A, m) be a homomorphic image of regular local ring that contains a field of characteristic 0. Put $d = \dim A$. Let

$$\mathbb{F}. \ : \ 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with $\mathrm{Supp}(\mathbb{F}.) = \{m\}$.

Let $\mathbb{G}^1_., \ldots, \mathbb{G}^{d+1}_., \mathbb{H}^1_., \ldots, \mathbb{H}^{d+1}_.$ be perfect A-complexes with supports in $\{m\}$ such that $\psi^e(\mathbb{F}.) = [\mathbb{G}^e_.] - [\mathbb{H}^e_.]$ as in the discussion preceding Lemma 4.2. Then, by Lemma 4.1 and Lemma 3.11 of Kurano [8], we can find a d-dimensional Noetherian local ring (B, m') and perfect B-complexes

$$\mathbb{E}. \ : \ 0 \to E_d \to \cdots \to E_0 \to 0$$
$$\mathbb{R}^1_., \ \ldots, \ \mathbb{R}^{d+1}_., \ \mathbb{S}^1_., \ \ldots, \ \mathbb{S}^{d+1}_.$$

that satisfy following four conditions:

(i) B is essentially of finite type over a field of positive characteristic.

(ii) The supports of perfect B-complexes $\mathbb{E}.$, $\mathbb{R}^1_., \ldots, \mathbb{R}^{d+1}_., \mathbb{S}^1_., \ldots, \mathbb{S}^{d+1}_.$ are contained in $\{m'\}$.

(iii) $\psi^e(\mathbb{E}.) = [\mathbb{R}^e_.] - [\mathbb{S}^e_.]$ for $e = 1, \ldots, d+1$.

(iv) We have equalities $\ell_A(H_t(\mathbb{F}.)) = \ell_B(H_t(\mathbb{E}.))$, $\ell_A(H_t(\mathbb{G}^e_.)) = \ell_B(H_t(\mathbb{R}^e_.))$, and $\ell_A(H_t(\mathbb{H}^e_.)) = \ell_B(H_t(\mathbb{S}^e_.))$ for any t and $e = 1, \ldots, d+1$.

Then, by the conditions (iii) and (iv), we have

$$\chi(\psi^e(\mathbb{F}.)) = \chi(\mathbb{G}^e_.) - \chi(\mathbb{H}^e_.) = \chi(\mathbb{R}^e_.) - \chi(\mathbb{S}^e_.) = \chi(\psi^e(\mathbb{E}.))$$

for $e = 1, \ldots, d+1$. Hence, since $\mathbb{E}.$ is defined over a ring of positive characteristic, Theorem 1.3 is true for $\mathbb{E}.$, so we have

$$\chi_\infty(\mathbb{F}.) = \chi_\infty(\mathbb{E}.) > 0.$$

Thus the Dutta multiplicity of $\mathbb{F}.$ is positive. **q.e.d.**

Remark 4.3 Compare the equation (1) in Section 1 and (3) in Lemma 4.2.

For a Noetherian local ring A containing a field of characteristic $p > 0$ and a perfect A-complex $\mathbb{F}.$, $F^e(\mathbb{F}.)$ denotes a perfect A-complex defined by matrices whose entries are the p^e-th powers of those of the complex $\mathbb{F}.$.

Then, by a remark following Proposition 4.13 in Gillet-Soulé [4], we have $(\psi^p)^e(\mathbb{F}.) = F^e(\mathbb{F}.)$ for $e > 0$.

Furthermore, if A is equi-characteristic complete local ring with perfect residue field of characteristic p, then we have $\chi(\mathbb{F}.^e) = \chi(F^e(\mathbb{F}.))$. Therefore we obtain $\chi((\psi^p)^e(\mathbb{F}.)) = \chi(\mathbb{F}.^e)$ in the case.

5 An application to intersection multiplicities

In this section we prove a generalization of a result of Dutta [2] on the positivity of intersection multiplicities.

Corollary 5.1 *Let (A, m) be a homomorphic image of a regular local ring with $d = \dim A$. Assume that $\tau_A([A]) \in \mathrm{A}_d \mathrm{Spec}\, A_{\mathbb{Q}}$. Let M and N be finitely generated A-modules such that*

- $\mathrm{pd}_A M < \infty$, $\mathrm{pd}_A N < \infty$,
- $0 < \ell_A(M \otimes_A N) < \infty$,

- $\dim M + \dim N = \dim A$,
- $\operatorname{depth} M = \dim M$,
- *for any prime ideal Q in the support of N with* $\dim(A/Q) = \dim(N)$, *A/Q contains a field,*

where $\operatorname{pd}_A M$ *(resp.* $\operatorname{pd}_A N$*) denotes the projective dimension of M (resp. N). Then*

$$\sum_i (-1)^i \ell_A(\operatorname{Tor}_i^A(M, N)) > 0.$$

Put $X = \operatorname{Spec} A$, $Y = \operatorname{Supp} M$ and $Z = \operatorname{Supp} N$. Then, we have $Y \cap Z = \operatorname{Spec} A/m$ (as a set) and $\dim Y + \dim Z = d$ by our assumption. Put $s = \dim Y$ and $t = \dim Z$.

Let $\mathbb{F}.$ and $\mathbb{G}.$ be minimal A-free resolutions of M and N respectively. Then $\mathbb{F}.$ and $\mathbb{G}.$ are perfect A-complexes with supports Y and Z. Then, by the local Riemann-Roch formula (Fulton [3], Example 18.3.12), we have

$$\begin{aligned} & \sum_{i \geq 0} (-1)^i \ell_A(\operatorname{Tor}_i^A(M, N)) \cdot [\operatorname{Spec} A/m] \\ = \; & \operatorname{ch}(\mathbb{F}. \otimes \mathbb{G}.) \cap \tau_A([A]) \\ = \; & \operatorname{ch}_d(\mathbb{F}. \otimes \mathbb{G}.) \cap \tau_A([A]) \\ = \; & \sum_{p+q=d} \operatorname{ch}_p(\mathbb{F}.) \operatorname{ch}_q(\mathbb{G}.) \cap \tau_A([A]), \end{aligned}$$

where the second equality uses the assumption that $\tau_A([A]) \in \mathrm{A}_d X_{\mathbb{Q}}$. Let p and q be nonnegative integers such that

1. $p + q = d$
2. $\operatorname{ch}_p(\mathbb{F}.) \operatorname{ch}_q(\mathbb{G}.) \cap \tau_A([A]) \neq 0$,

Since $0 \neq \operatorname{ch}_q(\mathbb{G}.) \cap \tau_A([A]) \in \mathrm{A}_{d-q} Z_{\mathbb{Q}}$ and $\dim Z = t$, we have $0 \leq d - q \leq t$. On the other hand, by the commutativity of localized Chern characters [12], we have

$$\operatorname{ch}_p(\mathbb{F}.) \operatorname{ch}_q(\mathbb{G}.) \cap \tau_A([A]) = \operatorname{ch}_q(\mathbb{G}.) \operatorname{ch}_p(\mathbb{F}.) \cap \tau_A([A]).$$

Therefore, reversing the roles of p and q, we have $\mathrm{ch}_p(\mathbb{F}.) \cap \tau_A([A]) \neq 0$, so $0 \leq d - p \leq s$. Since $d - q \leq t$ and $d - p \leq s$, we have $p = t$ and $q = s$. Hence, we get

$$\sum_{i \geq 0} (-1)^i \ell_A(\mathrm{Tor}_i^A(M, N)) \cdot [\mathrm{Spec}\, A/m] = \mathrm{ch}_t(\mathbb{F}.)\, \mathrm{ch}_s(\mathbb{G}.) \cap \tau_A([A]).$$

Let I be the annihilator of N, that is, $I = \{x \in A \mid xN = 0\}$. Then, by the local Riemann-Roch formula, we have

$$\tau_{A/I}([N]) = \mathrm{ch}(\mathbb{G}.) \cap \tau_A([A]) \in \mathrm{A}_*(\mathrm{Spec}\, A/I)_{\mathbb{Q}}.$$

Therefore $\mathrm{ch}_s(\mathbb{G}.) \cap \tau_A([A])$ coincides with the top term of $\tau_{A/I}([N])$, that is,

$$\mathrm{ch}_s(\mathbb{G}.) \cap \tau_A([A]) = \sum_Q \ell_{A_Q}(N_Q) \cdot [\mathrm{Spec}\, A/Q] \in \mathrm{A}_t(\mathrm{Spec}\, A/I)_{\mathbb{Q}}$$

by Theorem 18.3 (5) in Fulton [3], where the sum is taken over all $Q \in \mathrm{Supp}(N)$ with $\dim(A/Q) = \dim(N)$.

Thus, we have

$$\begin{aligned} \mathrm{ch}_t(\mathbb{F}.)\, \mathrm{ch}_s(\mathbb{G}.) \cap \tau_A([A]) &= \sum_Q \ell_{A_Q}(N_Q) \cdot \mathrm{ch}_t(\mathbb{F}.) \cap [\mathrm{Spec}\, A/Q] \\ &= \sum_Q \ell_{A_Q}(N_Q) \cdot \mathrm{ch}_t(\mathbb{F}. \otimes A/Q) \cap [\mathrm{Spec}\, A/Q]. \end{aligned}$$

On the other hand, by the Auslander-Buchsbaum formula, we have

$$\mathrm{pd}_A M = \mathrm{depth}\, A - \mathrm{depth}\, M \leq d - s = t,$$

since $\dim M = \mathrm{depth}\, M$ by our assumption. Therefore the length of the complex $\mathbb{F}. \otimes A/Q$ is less than or equal to $t = \dim N = \dim A/Q$. Note that each homology module of $\mathbb{F}. \otimes A/Q$ has finite length and $\mathbb{F}. \otimes A/Q$ is not exact. (By the New Intersection Theorem [13], A must be a Cohen-Macaulay ring.) Since A/Q is equi-characteristic, we have $\mathrm{ch}_t(\mathbb{F}. \otimes A/Q) \cap [\mathrm{Spec}\, A/Q] > 0$ for each Q as above by Theorem 1.3. Thus the intersection multiplicity of M and N is positive. **q.e.d.**

Remark 5.2 We note several cases in which the assumption that $\tau_A([A]) \in \mathrm{A}_d\mathrm{Spec}\, A_{\mathbb{Q}}$ is satisfied. Assume that A is a homomorphic image of a regular local ring such that $\dim A = d$.

1. If A is a complete intersection, then $\tau_A([A]) \in \mathrm{A}_d\mathrm{Spec}\, A_{\mathbb{Q}}$ by Corollary 18.1.2 in Fulton [3]. On the other hand, there exists a Gorenstein ring A such that $\tau_A([A]) \notin \mathrm{A}_d\mathrm{Spec}\, A_{\mathbb{Q}}$ (see [9]).

2. Let S be a d-dimensional regular local ring such that a finite group G acts S. Assume that $A = S^G$ is the invariant subring. Since $\mathrm{A}_*\mathrm{Spec}\, S_{\mathbb{Q}} \simeq \mathrm{K}_0\,\mathrm{Spec}\, S_{\mathbb{Q}} \simeq \mathbb{Q}$ by the singular Riemann-Roch theorem, we have
$$\mathrm{A}_i\mathrm{Spec}\, S_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & (i = d) \\ 0 & (i \neq d) \end{cases}.$$
On the other hand, by Example 1.7.6 in Fulton [3], we have $\mathrm{A}_*\mathrm{Spec}\, A_{\mathbb{Q}} = (\mathrm{A}_*\mathrm{Spec}\, S_{\mathbb{Q}})^G$. Therefore, we have $\mathrm{A}_*\mathrm{Spec}\, A_{\mathbb{Q}} = \mathrm{A}_d\mathrm{Spec}\, A_{\mathbb{Q}}$. In particular, $\tau_A([A]) \in \mathrm{A}_d\mathrm{Spec}\, A_{\mathbb{Q}}$ is satisfied.

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HILBERT-KUNZ MULTIPLICITY
– MANY QUESTIONS AND VERY FEW ANSWERS

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0. Introduction

The notion of the Hilbert-Kunz multiplicity began with [Ku1] and [M1]. Although it is not a very new notion, it got a new meaning with the theory of tight closures of ideals to characterize the tight closure of $\mathfrak{m}$-primary ideals.

Also, it is very desirable that this multiplicity has some "geometric" meaning and measures some kind of "goodness" among the family of singularities with the same multiplicity, because Hilbert-Kunz multiplicity is finer than usual one.

In this article, we list some questions about Hilbert-Kunz multiplicity and give partial answers to some of them.

This is a report on joint works [WY1, WY2] with Ken-ichi Yoshida (Nagoya University).

Let $(A, \mathfrak{m})$ be a local ring of characteristic $p > 0$ of $\dim A = d$. We always use the letter q as a power $q = p^e$ of p. Also, for an ideal I of A, we write $I^{[q]} = (a^q | a \in I)$.

Definition (0.1). (Hilbert-Kunz multiplicity) Let I be an $\mathfrak{m}$-primary ideal and M an A-module. Then we define Hilbert-Kunz multiplicity of M with respect to I (resp. I, resp.A) by the following formula.

$$e_{HK}(I, M) = \lim_{e \to \infty} \frac{l_A(M/I^{[p^e]}M)}{p^{de}}$$
$$e_{HK}(I) = e_{HK}(I, A)$$
$$e_{HK}(A) = e_{HK}(\mathfrak{m}, A).$$

Fundamental Properties (0.2).

(1) $e(I) \geq e_{HK}(I) \geq e(I)/d!$, *where $e(I)$ is the usual multiplicity.*
(2) *If J is a parameter ideal, then $e_{HK}(J) = e(J)$.*
(3) ([HH]) If $I \subset I' \subset I^*$, then $e_{HK}(I) = e_{HK}(I')$. The concerse is true if $\widehat{A}$ is reduced and equi-dimensional. (Where I^* is the tight closure of I.)
(4) If A is regular, then for every I, $e_{HK}(I) = l_A(A/I)$.
(5) ([BCP],[WY]) If $A \subset B$, where B is a local domain finite over A of rank r. Then $e_{HK}(I) = \dfrac{e_{HK}(IB)}{r}$. In particular, if B is regular, then $e_{HK}(I) = \dfrac{l_B(B/IB)}{r}$.

In [WY1], we showed the following.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem (0.3).

(1) *If A is unmixed* $(\mathrm{Ass}(\widehat{A}) = \mathrm{Min}(\widehat{A}))$ *and if* $e_{HK}(A) = 1$*, then A is regular.*

(2) ([WY]) If A is unmixed and if $\mathfrak{q}$ is a parameter ideal of A which satisfies the condition # (i) and # (ii) below, then $e(\mathfrak{q}) \geq l_A(A/\mathfrak{q}^*)$.

(i) a_i is $A_{i-1}/\mathrm{H}^0_{\mathfrak{m}}(A_{i-1})$-regular, where $A_{i-1} = A/(a_1, \dots, a_{i-1})A$ for each $i = 1, \dots, d-1$.

(ii) $0 :_{A_{d-1}} a_d = 0 :_{A_{d-1}} a_d^2$.

Let us recall here some terminology related to the theory of tight closure of ideals.

Definition (0.4).

(1) *A is weakly F-regular if $I^* = I$ for every $\mathfrak{m}$-primary (and hence for every) ideal I.*

(2) *A is strongly F-regular if A is reduced, $A^{1/p}$ is a finite A module and for every $c \in A$, which is not contained in any minimal prime ideal of A, there exists $q = p^e$, a homomorphism $\phi : A^{1/q} \to A$ of A modules such that $\phi(c^{1/q}) = 1$.*

(3) *A is F-rational if there exists a (or, equivalently, every) parameter ideal* $\mathfrak{q}$, $\mathfrak{q}^* = \mathfrak{q}$.

Remark. If A is Gorenstein, the three notions above are equivalent. Also, weakly and strongly F-regular are supposed to be equivalent if $A^{1/p}$ is a finite A module and proved for dimension ≤ 3. If A is a quotient of a Cohen-Macaulay local ring and if A is F-rational, then A is Cohen-Macaulay and normal.

In what follows, A is an unmixed Noetherian local of characteristic $p > 0$. We always assume $d = \dim A > 1$, since in dimension 1 $e_{HK}(I) = e(I)$ for every I. Also, we always treat $\mathfrak{m}$-primary ideals.

1. Fundamental Questions.

Question (1.1). *Are $e_{HK}(I)$ or $e_{HK}(I, M)$ rational numbers ?*

Seibert [Se] proved that $e_{HK}(I) \in \mathbb{Q}$ if A is Cohen-Macaulay and has finite Cohen-Macaulay type (that is, A has only finite number of non-isomorphic indecomposable maximal Cohen-Macaulay modules).

His method is to consider the decomposition of $A^{1/q}$ as a Cohen-Macaulay A module for $q \gg 1$.

If A is a normal semigroup ring and I is an ideal generated by "monomials", we can see that $e_{HK}(I) \in \mathbb{Q}$ by the same method and $e_{HK}(I)$ can be calculated by summing-up the "volume" of areas defined by linear ineqaulities with coefficients in $\mathbb{Q}$. (Also N. Hara pointed out $e_{HK}(I) \in \mathbb{Q}$ can be seen by direct computation in this case.)

The easiest case with $e_{HK}(I) \in \mathbb{Q}$ is the case where A has a regular ring B as a finite overring. ((0.1), (6)). In this case, $r.e_{HK}(I)\mathbb{Z}$, where r is the rank of B as A module.

What is the meaning of denominator of $e_{HK}(I)$ in general? Do $e_{HK}(I)$ have a common denominator when I runs through all $\mathfrak{m}$-primary ideals of A?

Question (1.2). *What is the minimal value of $e_{HK}(A) > 1$ in dimension d?*

If A is unmixed, $e_{HK}(A) = 1$ if and only if A is regular. So it is most probable that next smallest $e_{HK}(A)$ is given by A defined by the quadratic form of maximal rank ($A = k[[X_0, \dots, X_d]]/(X_0^2 + \dots + X_d^2)$ for $p > 2$).

For $d = 2, 3$, $e_{HK}(A) = 3/2, 4/3$ respectively and for $d = 2$, it is shown that this gives the smallest $e_{HK}(A)$ after 1.

For $d = 4$, Monsky claims that $e_{HK}(A) = (29p^2 + 15)/(24p^2 + 12)$ [Mo2]. He also claims that $e_{HK}(A)$ is a certain rational function f_d of p in dimension d and the value of f_d at infinity is 1+the co-efficient of x^d in the power series expansion of $\sec x + \tan x$!

Also, we are interested in the values taken as $e_{HK}(A)$ for some A (of fixed dimension d). When $d = 2$, we have shown in [WY1] that $e(A) = 2 - 1/n$ for some integer n if $e_{HK}(A) < 2$. Also, we have the example ([WY1], (5.6));

Example (1.2.1). *Let* $R = k\left[T, xT^a, x^{-1}T^b, \frac{1}{x+1}T^c\right]$ *and* A *be the localization of* R *at the unique graded maximal ideal. Then*

$$e_{HK}(A) = 3 - \frac{a+b+c}{ab+bc+ca}.$$

If we fix a, b and let c tend to infinity, then the limit will be $3 - 1/(a+b)$. But up to now, we have only accumulation of $e_{HK}(A)$ only from below and not from above.

Question (1.3). *What is the relation between* $e_{HK}(A)$ *and* $e(A)$ *or* $e_{HK}(I)$ *and* $e(I)$ *?*

We know that $e_{HK}(I) = e(I)$ if I is generated by a system of parameters. But if we take high power of any I, the ratio $e_{HK}(I^n)/e(I^n)$ tends to the other extreme.

Theorem(1.4). *For any* $\mathfrak{m}$*-primary ideal* I *of* A *and for any positive integer* n*, we have the following inequalities:*

$$\frac{e(I^n)}{d!} \le e_{HK}(I^n) \le \frac{\binom{n+d-1}{d}}{n^d} e(I^n).$$

In particular,

$$\lim_{n\to\infty} \frac{e_{HK}(I^n)}{e(I^n)} = \frac{1}{d!}.$$

To prove this, we may replace I with its minimal reduction J and then taking a regular local ring over which A is finite, we can reduce to the case of a regular local ring.

This shows that the ratio $e_{HK}(I)/e(I)$ tells us how far is I from parameter ideals.

Also, we have the following thorem in dimension 2.

Theorem (1.5). *Let* A *be a Cohen-Macaulay local ring of characteristic* $p > 0$ *with* $\dim A = 2$*. Assume that* $k = A/\mathfrak{m}$ *is algebraically closed. Then* $e_{HK}(A) \ge (e(A)+1)/2$ *and equality holds if and only if* $G := G_{\mathfrak{m}}(A) \cong (k[X,Y])^{(e(A))}$*, where* G *is the associated graded ring with respect to the maximal ideal and* $(k[X,Y])^{(e(A))}$ *is the subring of* $k[X,Y]$ *generated by all forms of degree* $e(A)$.

The proof is contained in §2.

We have some conjectures concerning the multiplicities and colength of ideals.

Conjecture (1.6). *Let I be an $\mathfrak{m}$-primary ideal of A.*

1. $e_{HK}(I) \geq l_A(A/I^*)$.
2. *When A is Cohen-Macaulay,* $e_{HK}(I) \geq l_A(A/I)$.
3. *When* $pd_A A/I < \infty$, $e_{HK}(I) = l_A(A/I)$.
4. *If* $e(\mathfrak{q}) = l_A(A/\mathfrak{q}^*)$ *for some parameter ideal, then A is F-rational (and hence Cohen-Macaulay).*

There are several related questions.

Question (1.7). *Fix A and take $I' \supset I$ with $l_A(I'/I) = 1$. Then what is the possible values of $e_{HK}(I) - e_{HK}(I')$?*

Examples.

1. *If A is regular,* $e_{HK}(I) - e_{HK}(I') = 1$ *for every* $I' \supset I$ *with* $l_A(I'/I) = 1$.
2. *If* $A = k[[x, y, z]]/(xz - y^2)$, *the possible values of* $e_{HK}(I) - e_{HK}(I')$ *are* $1/2, 1$ *and* $3/2$.

Is the number of possible such values finite for a given A ? It is easy to see that the maximum of this value is taken in the case $A \supset \mathfrak{m}$. We then ask for the minimum of this value. Of course, if A is not F-regular, there are ideals with $I^* \neq I$ and by (0.2) (3), $e_{HK}(I^*) = e_{HK}(I)$; the difference is 0. So we consider the case A is F-regular.

Proposition (1.8).

1. *Assume A is Gorenstein and F-regular, let $\mathfrak{q}$ be a parameter ideal and put* $\mathfrak{q}' := \mathfrak{q} : \mathfrak{m}$. *Then for every $I' \supset I$ with* $l_A(I'/I) = 1$, $e_{HK}(I) - e_{HK}(I') \geq e_{HK}(\mathfrak{q}) - e_{HK}(\mathfrak{q}')$.
2. *If A is F-regular, then for every $I' \supset I$ with* $l_A(I'/I) = 1$, $e_{HK}(I) - e_{HK}(I') \geq e_{HK}(0, S; E)$, *where* $E = E_A(A/\mathfrak{m})$ *be the injective envelope of the residue field,* $S = [0 :_E \mathfrak{m}]$ *and* $e_{HK}(0, S; E)$ *is the "relative HK multiplicity" defined below.*

Definition (1.9).

1. *(relative HK multiplicity) Let $N \subset N'$ be finitely generated submodules of M with* $Supp(N'/N) \subset \{\mathfrak{m}\}$. *Then we put*

$$e_{HK}(N, N'; M) = \lim l_A((N')^{[q]}/N^{[q]})/q^d.$$

2. *(minimal HK multiplicity) If A is strongly F-regular, we put*

$$m_{HK}(A) = e_{HK}(0, S; E)$$

and call it "minimal HK multiplicity of A.

The condition $m_{HK}(A) > 0$ is equivalent to A is strongly F-regular.

It is clear that $e_{HK}(A) \geq e_{HK}(I) - e_{HK}(I') \geq m_{HK}(A)$ for every $I' \supset I$ with $l_A(I'/I) = 1$. If A is a F-regular hypersurface of multiplicity 2, then $m_{HK}(A) = 2 - e_{HK}(A)$. If $\dim A$ becomes very big, then there are rings A for which $e_{HK}(A)$ is very near to 1. This means $e_{HK}(I)$ is very near to $l_A(A/I)$ for every I.

Proof of (1.8). It is easy to see that (1) is a special case of (2). Let a, z be a generator of I'/I and S, respectively. Then it suffices to show that for $b \in A$, if $ba^q \in I^{[q]}$, then $bz^q = 0$, where $z^q = z \otimes 1$ (the image of z under the map $F^e : E \to E \otimes F^e(A)$).

By Matlis duality, z can be written as $z = ay$, where $y \in [0 : I]_E$. Then $bz^q = b(ay)^q = (ba^q)y^q = 0$ since $ba^q \in I^{[q]}$ and $y \in [0 : I]_E$.

Remark. If A is F-regular and not Gorenstein, it is not clear that if there exists (or how to take) ideals $I' \supset I$ with $l_A(I'/I) = 1$ and $e_{HK}(I) - e_{HK}(I') = m_{HK}(A)$.

If A is not weakly F-regular, then there exists ideals $I' \supset I$ with $l_A(I'/I) = 1$ and $e_{HK}(I) = e_{HK}(I')$. But what occurs to positive values of this difference ?

Question (1.10). *If A is not strongly F-regular, have the set of possible values*

$$\{e_{HK}(I) - e_{HK}(I') \mid I' \supset I \text{ with } l_A(I'/I) = 1\}$$

has 0 as accumulation point ?

Actually, the most difficult part of the theory of Hilbert-Kunz multiplicity is to compute $e_{HK}(I)$ or $e_{HK}(A)$ except for very few fortunate examples as in (0.2)(5).

Problem. *Compute $e_{HK}(A)$ ($e_{HK}(I)$) for some class of rings (ideals).*

As far as the author knows, the existing examples of $e_{HK}(A)$ are only the followings. (I apology in advance for my lack of knowledge.)

Examples.

(1) *Some classes of hypersurfaces computed in* [HM] *and other papers. (There are several papers coputing HK multiplicities of some classes of hypersurffaces.)*
(2) *The case where A has regular ring as finite overring.*
(3) [BCP] *Cone of "Segre embeddings" of $\mathbb{P}^n \times \mathbb{P}^m$.*
(4) *Example* (1.2.1).

2. Rings of Dimension 2 with "minimal" HK multiplicity

In this section, we will prove Theorem (1.5).

First, let us recall the proof of the inequality $e_{HK}(A) \geq (e(A)+1)/2$. First observation was the following Lemma.

Lemma (2.1). ([WY1, (5.5)], also [W, (3.3)]]) Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. If $\mathfrak{m}/\mathfrak{q}^*$ minimally generated by r elements (note that $r \leq e(A) - 1$), then $e_{HK}(A) \geq \dfrac{r+2}{2(r+1)} e(A)$.

Since $\dfrac{r+2}{2(r+1)}$ is a decreasing function of r, the minimal value is attained when $r = e(A) - 1$ and we get $e_{HK}(A) \geq (e(A) + 1)/2$.

Assume $G_\mathfrak{m}(A) \cong (k[X,Y])^{(e(A))}$. Then since $e_{HK}((k[X,Y])^{(e(A))} = (e(A) + 1)/2$ (this is easy from (0.2)(5) and $e_{HK}(A) \leq e_{HK}(G(A))$ in general (cf. [WY1,(2.15)]), we get $e_{HK}(A) = (e(A) + 1)/2$.

Next, assume $e_{HK}(A) = (e(A) + 1)/2$. Then by the argument after (2.1), we should have $\mathfrak{m}^2 = \mathfrak{m}\mathfrak{q}$. By [Sa], we know that $G_\mathfrak{m}(A)$ is Cohen-Macaulay with $a(G) < 0$. we use the following Theorem, which gives a characterization of "curves of minimal degree".

Theorem (2.2). *Let R be a Cohen-Macaulay graded ring of dimension 2 over a field $k = R_0$, generated by elements of degree 1. Assume R is Cohen-Macaulay with $a(R) < 0$ and multiplicity e. Then we have;*

(1) *Let I be a graded ideal of R of pure height 0 with* $\operatorname{depth} R/I > 0$. *Then I is a Cohen-Macaulay R module generated by elements of degree 1 and R/I is Cohen-Macaulay with $a(R/I) < 0$.*

(2) *If R is not reduced, there is $x \neq 0 \in R_1$ with $x^2 = 0$. Also, R_{red} is Cohen-Macaulay with $a(R_{red}) < 0$.*

(3) *If R is reduced and take I, J to be intersections of some minimal prime ideals with $I \cap J = (0)$ and no common minimal primes, then we can take a system of minimal genarators $\{x, y_1, \dots, y_s, z_1, \dots, z_t\}$ of R_1 so that $I = (y_1, \dots, y_s)$ and $J = (z_1, \dots, z_t)$.*

In this case, $R/(I + J) = k[x]$, $\deg(R/I) = t$ *and* $\deg(R/J) = s$.

First, we will show Theorem (1.5) from (2.2). We will show that if G is not a domain, then this term becomes strictly less after divided by q^2 and then the Hilbert-Kunz multiplicity becomes strictly bigger.

Now, if G is *not* reduced and let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be such that $in(x)^2 = 0$ as in (2.2)(2). Since $x^2 \in \mathfrak{m}^3$, we have $x^q \in \mathfrak{m}^{(3q)/2} \subset \mathfrak{m}^{(r+2)q/(r+1)}$ and we have

$$e_{HK}(A) \geq \frac{e(A)+1}{2} + \frac{1}{2e(A)}.$$

Next, if G is reduced and *not* an integral domain. Then take a minimal generator system $\{x, y_1, \dots, y_s, z_1, \dots, z_t\}$ of $\mathfrak{m}$ corresponding to those basis of G_1 as in (2.2)(3). Then if we put $I_1 = (y_1, \dots, y_s)$ and $I_2 = (z_1, \dots, z_t)$, we have $I_1.I_2 \subset \mathfrak{m}^3$ and $y_i^q I_2^{q/2(r+1)} \subset \mathfrak{m}^{(r+2)q/(r+1)}$ and by (2.2)(3), we have strict inequality in (2.0.1). Thus we have shown that if $e_{HK}(A) = (e(A)+1)/2$, $G_{\mathfrak{m}}(A)$ is an integral domain. Then it is well known that 2-dimensional graded domain with $a(R) = -1$ is isomorphic to a Veronese subring.

Now, we will prove (2.2). In this part, a "graded ring" always mean a graded ring generated by degree 1 with $R_0 = k$ a field.

Also, in the following proof, we write $f(t) \geq 0$ if $f(t)$ is a polynomial whose coefficients are non-negative integers.

(1) First, recall that if R is an unmixed graded ring, then there is a finite over ring $\overline{R}$ with $\operatorname{depth} \overline{R} \geq 2$ and $\overline{R}/R$ has finite length. We always use $\overline{R}$ in this sense.

Now let I be a graded ideal of pure height 0 with $\operatorname{depth} R/I > 0$. From the exact sequence,

$$(*) \qquad 0 \to I \to R \to R/I \to 0$$

we easily see that I is a Cohen-Macaulay R-module. Also, since $H^2_{\mathfrak{m}}(R)$ surjects to $H^2_{\mathfrak{m}}(R/I) = H^2_{\mathfrak{m}}(\overline{R/I})$, we also have $a(\overline{R/I}) < 0$.

Now, let us compute the Poincaré series of each terms. We will write

$$P(M,t) = \sum_{n \in \mathbf{Z}} \dim_k M_n t^n.$$

Also, since $R, \overline{R/I}$ are Cohen-Macaulay with $a(R), a(\overline{R/I}) < 0$,

$$P(R,t) = \frac{1+(e-1)t}{(1-t)^2} \quad \text{and} \quad P(\overline{R/I},t) = \frac{1+at}{(1-t)^2}.$$

Also, since I is also Cohen-Macaulay, $P(I,t) = \dfrac{g(t)}{(1-t)^2}$, where $g(t) = P(I/(x,y)I,t) \geq 0$.

Now, from (*), we have

$$P(R,t) = P(I,t) + P(R/I,t) = P(I,t) + P(\overline{R/I},t) - P((\overline{R/I})/(R/I),t).$$

Hence

$$\frac{1+(e-1)t}{(1-t)^2} = \frac{g(t)}{(1-t)^2} + \frac{1+at}{(1-t)^2} - f(t),$$

where $f(t) \geq 0$. Now it is easy to see that the above equality is possible only when $f(t) = 0$ and $g(t) = (e-1-a)t$. This shows that R/I is Cohen-Macaulay and I is generated by elements of degree 1.

If R is not reduced, let I be the nilradical of R and J_r be the component of pure height 0 of I^r for $r \geq 1$. Since $\operatorname{depth} R/J_r > 0$, (1) shows that J_r is generated by degree 1 for every r.

Now, since $I^r = 0$ for some r, we can take r so that $J_r \neq 0$ and $J_{r+1} = 0$. If x is a generator of J_r, then $x^2 \in J_{r+1} = 0$.

Now let I, J be as in (3). Then we have the exact sequence

$$(**) \qquad 0 \to R \to R/I \oplus R/J \to R/(I+J) \to 0.$$

As is in the proof of (1), we have the equality of Poincaré series (we can easily see that $R/(I+J)$ is Cohen-Macaulay of dimension 1 from (**)).

$$\frac{1+(e-1)t}{(1-t)^2} = \frac{1+at}{(1-t)^2} + \frac{1+bt}{(1-t)^2} - f_1(t) - f_2(t) - \frac{g(t)}{1-t}.$$

Where $P(R/(I+J),t) = \dfrac{g(t)}{1-t}$ and $f_1(t), f_2(t), g(t) \geq 0$. Since $g(0) = 1$ (being a Poincaré series of a graded ring), we can easily see that $f_1(t) = f_2(t) = 0$, $g(t) = 1$ and $a+b = e-2$. Hence I, J are generated respectively by $b+1, a+1$ elements of degree 1. Since $\mathfrak{m}$ is generated by $e+1$ elements, we have the desired conclusion.

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Introduction to $\mathcal{D}$-modules in positive characteristic

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This is an expositin of some of the recent developments in the theory of D-modules in positive characteristic, especially of B. Haastert's [H87, 88] and of R. Bøgvad's holonomicity [Bø95], [Bø]. Due partly to the local nature of Bøgvad's definition of holonomicity we will describe the theory ring theoretically, not in terms of sheaves on varieties.

The author's interest has resulted in the selection of the topics presented here; he learned the subject from Haastert's dissertation [H86] in his hope of application to the representation thory of algebraic groups in positive characteristic after the great success in characteristic 0, the solution of the Kazhdan-Lusztig conjecture by J.L. Brylinski and Kashiwara M. [BK] and by A. Beilinson and J. Bernstein [BB]. In this direction one will need a G-equivariant theory, G an algebraic group (cf. [Bø95], [K]). For works before Haastert refer, e.g., to S.P. Smith [S86, 87].

The present note is based on a graduate seminar at OCU. The author is grateful to the participants for their patience. Special thanks are due to R. Bøgvad for making available prepublication access to [Bø]. The author is also grateful to the audience of the symposium the questions from whom have contributed to a little expansion of the manuscript. Compared to the enormous successes in characteristic 0 we are still at a very primitive stage in positive characteristic. We hope this may be of a little help in recruiting new researchers to the subject.

Notations. Throughout the exposition $\mathfrak{k}$ will denote an algebraically closed field.

$\otimes = \otimes_{\mathfrak{k}}$ unless otherwise specified

$\mathbf{Alg}_A$ the category of commutative A-algebras, A a commutative ring

$\mathbf{dom}_A$ the full subcategory of $\mathbf{Alg}_A$ consisting of algebraic A-domains, i.e., A-domains of finite type

$A\mathbf{Alg}$ the category of A-algebras

$R\mathbf{Rng}$ the category of R-rings, R a ring, consisting of the pairs (S, f) of a ring S and a ring homomorphism $f : R \to S$

$R\mathbf{Mod}$ the category of left R-modules

$R\mathbf{mod}$ the full subcategory of $R\mathbf{Mod}$ consisting of R-modules of finite type

$\mathbf{Mod}R$ the category of right R-modules

$\mathbf{Mod}_A$ the category of A-modules when there is no need to distiguish left and right

1° Generalities [HS], [EGAIV]

Let $A \in \mathbf{Alg}_{\mathfrak{k}}$.

(1.1) We make $\mathbf{Mod}_{\mathfrak{k}}(A, A)$ into an $A \otimes A$-module or an (A, A)-bimodule by setting

$$(a \otimes b) \cdot f = af(b?) = a \cdot f \cdot b, \quad f \in \mathbf{Mod}_{\mathfrak{k}}(A, A), \quad a, b \in A.$$

Define $[\ ,\ ] : \mathbf{Mod}_{\mathfrak{k}}(A, A) \times \mathbf{Mod}_{\mathfrak{k}}(A, A) \to \mathbf{Mod}_{\mathfrak{k}}(A, A)$ by $[f, g] = fg - gf$. Set $\mathit{Diff}_{-1}(A) = 0$ and define inductively on $n \in \mathbb{N}$

$$\mathit{Diff}_n(A) = \{f \in \mathbf{Mod}_{\mathfrak{k}}(A, A) | [a, f] \in \mathit{Diff}_{n-1}(A) \quad \forall a \in A\}.$$

Then $\forall n \in \mathbb{N}$,

$$\mathit{Diff}_n(A) \leq \mathbf{Mod}_{\mathfrak{k}}(A, A) \quad \text{in } A \otimes A\mathbf{Mod},$$

called the module of differential operators on A of order $\leq n$. One has

$$\mathit{Diff}_n(A) \leq \mathit{Diff}_{n+1}(A)$$

and under the composition

$$\mathit{Diff}_n(A)\mathit{Diff}_m(A) \subseteq \mathit{Diff}_{n+m}(A),$$

hence if we let $D(A) = \cup_{n \in \mathbb{N}} \mathit{Diff}_n(A)$, then

$$D(A) \leq \mathbf{Mod}_{\mathfrak{k}}(A, A) \quad \text{in } \mathfrak{k}\mathbf{Alg},$$

called the ring of differential operastors on A. There is a $\mathfrak{k}$-algebra homomorphism from A to $D(A)$ sending each $a \in A$ to the multiplication by a. Note, however, that the image of A is not central in $D(A)$ in general, hence $D(A)$ is an A-ring. The left (resp. right) A-module structure on $D(A)$ induced by the left (resp. right) regular action of $D(A)$ coincides with the one induced by j_1 (resp. j_2) on the $A \otimes A$-module structure, where $j_i \in \mathbf{Alg}_{\mathfrak{k}}(A, A \otimes A)$, $i \in \{1, 2\}$, with $j_1 : a \mapsto a \otimes 1$ and $j_2 : a \mapsto 1 \otimes a$. We will often suppress A in what follows when there can be no confusion.

(1.2) To compute $D = D(A)$, let $\mu \in \mathbf{Alg}_{\mathfrak{k}}(A \otimes A, A)$ via $a \otimes b \mapsto ab$ and set $I_A = I_{A/\mathfrak{k}} = \ker \mu$. Then

$$I_A = \sum_{a \in A} A\mathrm{d}_A a = \sum_{a \in A} (\mathrm{d}_A a)A \quad \text{with} \quad \mathrm{d}_A a = \mathrm{d}_{A/\mathfrak{k}} a = 1 \otimes a - a \otimes 1.$$

For each $n \in \mathbb{N}$ one has [HS, 2.2.3]

(1)
$$\mathit{Diff}_n = \mathit{Diff}_n(A) = \{f \in \mathbf{Mod}_{\mathfrak{k}}(A, A) | I_A^{n+1} \cdot f = 0\}.$$

Hence if we regard $A \otimes A$ as a left A-module via j_1 via $a \mapsto a \otimes 1$, then

$$(2) \qquad \begin{array}{ccc} \mathbf{Mod}_{\mathfrak{k}}(A, A) & \xrightarrow{\sim} & A\mathbf{Mod}(A \otimes A, A) \\ \uparrow & \circlearrowleft & \uparrow \\ \mathit{Diff}_n(A) & \overset{\sim}{\dashrightarrow} & A\mathbf{Mod}(A \otimes A/I_A^{n+1}, A). \end{array}$$

Set $P_A^n = P_{A/\mathfrak{k}}^n = A \otimes A/I_A^{n+1}$, called the ring of the n-th normal invariants of the diagonal imbedding, that is naturally an $A \otimes A$-algebra and also an A-algebra via j_1. Then (2) reads as

$$(3) \qquad \mathit{Diff}_n(A) \simeq (P_A^n)^\vee = A\mathbf{Mod}(P_A^n, A),$$

under which the $A \otimes A$-module structure is transferred onto $(P_A^n)^\vee$ such that

$$(a \otimes b) \cdot f = f((a \otimes b)?).$$

If $A = \mathfrak{k}[a_1, \ldots, a_r]$, then $I_A = (\mathrm{d}_A a_i | 1 \leq i \leq r)$, from which one finds that

(4) if A is algebraic over $\mathfrak{k}$, then each $\mathit{Diff}_n(A), n \in \mathbb{N}$, is of finite type in both $A\mathbf{Mod}$ and $\mathbf{Mod}A$.

(1.3) Using the standard filtration $(\mathit{Diff}_n)_{n\in\mathbb{N}}$ on $D = D(A)$ we form the associated graded $\mathfrak{k}$-algebra

$$\mathrm{gr}D = \coprod_{n\in\mathbb{N}} \mathit{Diff}_n/\mathit{Diff}_{n-1}.$$

By induction on n and m one checks, using the Jacobi identity, that $[\mathit{Diff}_n, \mathit{Diff}_m] \subseteq \mathit{Diff}_{n+m-1}$, hence

(1) $\quad \mathrm{gr}D$ is naturally a commutative A-algebra.

Let also $\mathrm{gr}(P_A) = \coprod_{n\in\mathbb{N}} I_A^n/I_A^{n+1}$. If $\Omega_A = \Omega_{A/\mathfrak{k}} = I_A/I_A^2$, there is a natural surjection of graded A-algebras

$$(2) \qquad \mathrm{S}_A^{\cdot}(\Omega_A) \to \mathrm{gr}(P_A).$$

2° The Weyl algebra and holonomicity in characteristic 0 [E]

Let $A = \mathfrak{k}[x] = \mathfrak{k}[x_1, \ldots, x_n]$ be the polynomial $\mathfrak{k}$-algebra in indeterminates $x_i, 1 \leq i \leq n$. We call $D = D(\mathfrak{k}[x])$ the Weyl algebra in n indeterminates.

(2.1) To compute D, let $A[y] = A[y_1, \ldots, y_n]$ be the polynomial A-algebra in the y_i. If $y^\alpha = \prod_{i=1}^n y_i^{\alpha_i}$, $\alpha \in \mathbb{N}^n$, there is an isomorphism in $A\mathbf{Alg}$

$$(1) \qquad P_A^m \to A[y]/(y^\alpha | |\alpha| = m+1) \quad \text{via} \quad 1 \otimes x_i \mapsto x_i + y_i \quad \forall i,$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$. If $\lambda_\alpha \in A[y]^\vee$ with $\lambda_\alpha(y^\beta) = \delta_{\alpha\beta}$, then

$$(A[y]/(y^\alpha \,|\, |\alpha| = m+1)^\vee = \coprod_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|\le m}} A\lambda_\alpha.$$

Let $\partial^\alpha \in \mathit{Diff}_m$ corresponding to λ_α under the bijection $\mathit{Diff}_m \simeq (P_A^m)^\vee \simeq (A[y]/(y^\alpha \,|\, |\alpha| = m+1)^\vee$ induced by (1). Then for each $\beta \in \mathbb{N}^n$

$$\partial^\alpha(x^\beta) = \lambda_\alpha((x+y)^\beta) = \lambda_\alpha(\prod_{i=1}^n (x_i+y_i)^{\beta_i}) = \prod_{i=1}^n \binom{\beta_i}{\alpha_i} x_i^{\beta_i-\alpha_i} = \binom{\beta}{\alpha} x^{\beta-\alpha},$$

hence symbolically we may write

$$\partial^\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Then

$$D = D(A) = \coprod_{\alpha\in\mathbb{N}^n} A\partial^\alpha = \coprod_{\alpha\in\mathbb{N}^n} \partial^\alpha A \tag{2}$$

with

$$\partial^\alpha\partial^\beta = \binom{\alpha+\beta}{\alpha}\partial^{\alpha+\beta} = \partial^\beta\partial^\alpha. \tag{3}$$

Moreover, if $A[z]$ is the polynomial algebra in z_α, $\alpha \in \mathbb{N}^n \setminus 0$, then we have an isomorphism in $\mathbf{Alg}_A$

$$A[z]/(z_\alpha z_\beta - \binom{\alpha+\beta}{\alpha} z_{\alpha+\beta} \,|\, \alpha,\beta \in \mathbb{N}^n) \to \mathrm{gr}D \quad \text{via} \quad z_\alpha \mapsto \partial^\alpha \quad \forall\alpha. \tag{4}$$

If $\zeta^\alpha = \prod_{i=1}^n \zeta_i^{\alpha_i}$ with $\zeta_i = \mathrm{d}_A x_i$ in P_A^m, $m \ge |\alpha|$, then by (1)

$$P_A^m = \coprod_{|\alpha|\le m} A\zeta^\alpha, \tag{5}$$

hence $I_A^m/I_A^{m+1} = \coprod_{|\alpha|=m} A\zeta^\alpha$. It follows that the natural map

$$\mathrm{S}^{\cdot}_A(\Omega_A) \to \mathrm{gr}(P_A) \quad \text{is bijective.} \tag{6}$$

Then in $\mathbf{Mod}_A$

$$\begin{aligned} \mathrm{gr}D &= \coprod_m (\mathit{Diff}_m/\mathit{Diff}_{m-1}) \simeq \coprod_m (P_A^m)^\vee/(P_A^{m+1})^\vee \\ &\simeq \coprod_m (I_A^m/I_A^{m+1})^\vee = \coprod_m \mathrm{gr}_m(P_A)^\vee \\ &\simeq \coprod_m \mathrm{S}_A^m(\Omega_A)^\vee = \mathrm{S}^{\cdot}_A(\Omega_A)^{\vee,\mathrm{gr}} \quad \text{the graded dual of } \mathrm{S}^{\cdot}_A(\Omega_A). \end{aligned} \tag{7}$$

Moreover, if $\Delta : \mathrm{S}^{\cdot}_A(\Omega_A) \to \mathrm{S}^{\cdot}_A(\Omega_A) \otimes_A \mathrm{S}^{\cdot}_A(\Omega_A)$ via $\mathrm{d}_A a \mapsto 1 \otimes \mathrm{d}_A a + \mathrm{d}_A a \otimes 1$ $\forall a \in A$, then $\mathrm{S}^{\cdot}_A(\Omega_A)^{\vee,\mathrm{gr}}$ comes equipped with a structure of $\mathbf{Alg}_A$ such that

$$f_i f_j = (f_i \otimes f_j) \circ \Delta, \quad f_i \in \mathrm{S}^i_A(\Omega_A)^\vee, f_i \in \mathrm{S}^j_A(\Omega_A)^\vee.$$

One checks that (7) is an isomorphism of $\mathbf{Alg}_A$:

(8) $$\mathrm{gr}D \simeq \mathrm{S}^{\cdot}_A(\Omega_A)^{\vee,\mathrm{gr}} \quad \text{in } \mathbf{Alg}_A.$$

(2.2) Throughout the rest of §2 we assume $\mathrm{ch}\mathfrak{k} = 0$.

Set $\partial_i = \partial^{\epsilon_i}$ with $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i-th place, $1 \leq i \leq n$. Then $\partial^\alpha = \alpha! \prod_{i=1}^n \partial_i^{\alpha_i}$ $\forall \alpha \in \mathbb{N}$, hence from (2.1.2)

(1) $$D = \mathfrak{k}[x_i, \partial_i | 1 \leq i \leq n].$$

If $\partial_\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$, then we also get

(2) $$D = \coprod_{\alpha,\beta\in\mathbb{N}^n} \mathfrak{k} x^\alpha \partial_\beta,$$

hence D admits a presentation as $\mathfrak{k}$-algebra

(3) $$D \simeq \mathrm{T}_{\mathfrak{k}}(x_i, \partial_i | 1 \leq i \leq n)/([x_i, x_j], [\partial_i, \partial_j], [\partial_i, x_j] - \delta_{i,j} | 1 \leq i \leq n),$$

where $\mathrm{T}_{\mathfrak{k}}(?)$ denotes the tensor algebra over $\mathfrak{k}$ in ?. Using the presentation, one can define the formal Fourier transforms

(4) $\mathrm{Four} \in \mathfrak{k}\mathbf{Alg}(D, D)$ (resp. $\mathrm{Four}' \in \mathfrak{k}\mathbf{Alg}(D, D^{\mathrm{op}})$)
via $x_i \mapsto \partial_i, \partial_i \mapsto -x_i$ (resp. $x_i \mapsto \partial_i, \partial_i \mapsto x_i$) $\forall i$.

From (2.1.4) one has in $\mathbf{Alg}_A$

(5) $\mathrm{gr}D \simeq A[z_1, \ldots, z_n]$ the polynomial A-algebra in z_i, $1 \leq i \leq n$.

(2.3) There is another filtration on D, called the Bernstein filtration, defined by

$$\mathfrak{B}_j = \sum_{\substack{\alpha,\beta\in\mathbb{N}^n \\ |\alpha|+|\beta|=\leq j}} \mathfrak{k} x^\alpha \partial_\beta, \quad j \in \mathbb{N}.$$

Note that each $\mathfrak{B}_j$ is finite dimensional and that $\mathfrak{B}_i\mathfrak{B}_j = \mathfrak{B}_{i+j}$ $\forall i, j$. Let $\mathrm{gr}^{\mathfrak{B}} D = \coprod_{j\geq 0} \mathfrak{B}_j/\mathfrak{B}_{j-1}$ with $\mathfrak{B}_{-1} = 0$. From (2.2.2) one has in $\mathbf{Alg}_{\mathfrak{k}}$

(1) $\mathrm{gr}^{\mathfrak{B}} D \simeq \mathfrak{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial $\mathfrak{k}$-algebra in the x_i and y_j
via $x_i \mapsto x_i, \partial_i \mapsto y_i$,

hence from (2.2.5)

(2) $$\mathrm{gr}^{\mathfrak{B}} D \simeq \mathrm{gr}D \quad \text{in } \mathbf{Alg}_{\mathfrak{k}}.$$

(2.4) By (2.3.1) we can study D-modules via $\mathrm{gr}^{\mathfrak{B}} D$. Let $M \in D\mathbf{Mod}$. A filtration Γ on M is an ascending chain $0 = \Gamma_{-1} \leq \Gamma_0 \leq \Gamma_1 \leq \cdots \leq M = \cup_j \Gamma_j$ of finite dimensional $\mathfrak{k}$-subspaces of M such that $\mathfrak{B}_i \Gamma_j \subseteq \Gamma_{i+j}$ $\forall i,j$. Then the associated graded module of M is $\mathrm{gr}^{\Gamma} M = \coprod_{j \geq 0} (\Gamma_j / \Gamma_{j-1})$, that carries a structure of $\mathrm{gr}^{\mathfrak{B}} D$-module such that

$$\pi_i(a)\pi_j^{\Gamma}(m) = \pi_{i+j}^{\Gamma}(am), \quad a \in \mathfrak{B}_i, m \in M_j,$$

where $\pi_i : \mathfrak{B}_i \to \mathfrak{B}_i / \mathfrak{B}_{i-1}$ and $\pi_j^{\Gamma} : \Gamma_j \to \Gamma_j / \Gamma_{j-1}$ are the quotients. We say the filtration on M is good iff $\mathrm{gr}^{\Gamma} M$ is a $\mathrm{gr}^{\mathfrak{B}} D$-module of finite type.

(2.5) **Proposition.** *Let $M \in D\mathbf{Mod}$ with a filtration Γ.*

(i) *If Γ is good, then M is of finite type over D.*

(ii) *Γ is good iff there is j_0 such that for each $i \in \mathbb{N}$ and $j \geq j_0$ it holds that $\mathfrak{B}_i \Gamma_j = \Gamma_{i+j}$.*

(iii) *If M is of finite type over D, then M admits a good filtration.*

(iv) *Let Γ' be another filtration on M. If Γ is good, there is $j_1 \in \mathbb{N}$ such that $\forall j \in \mathbb{N}, \Gamma_j \subseteq \Gamma'_{j_1+j}$. If Γ' is also good, there is $j_2 \in \mathbb{N}$ such that $\forall j \in \mathbb{N}, \Gamma'_{j-j_2} \subseteq \Gamma_j \subseteq \Gamma'_{j+j_2}$.*

(2.6) **Corollary.** *If $M \in D\mathbf{Mod}$ is of finite type, then M is noetherian. In particular, D is left and right noetherian.*

Proof. M admits by (2.5.iii) a good filtration, say Γ. If $N \leq M$ in $D\mathbf{Mod}$, define a filtration Γ' on N by $\Gamma'_. = \Gamma_. \cap N$. Then $\mathrm{gr}^{\Gamma'} N \leq \mathrm{gr}^{\Gamma} M$ in $\mathbf{Mod}_{\mathrm{gr}^{\mathfrak{B}} D}$. As Γ is good and as $\mathrm{gr}^{\mathfrak{B}} D$ is noetherian by (2.3.1), $\mathrm{gr}^{\Gamma'} N$ remains finite over $\mathrm{gr}^{\mathfrak{B}} D$. Then N is finite over D by (2.5.i).

(2.7) Let $M \in D\mathbf{Mod}$ of finite type. Then M admits a good filtration by (2.5.iii), say Γ. Let χ_M^{Γ} be the Hilbert polynomial of $\mathrm{gr}^{\Gamma} M = \coprod_{i \geq 0} \Gamma_i / \Gamma_{i-1}$: $\chi_M^{\Gamma} \in \mathbb{Q}[y]$ a polynomial in y such that

$$\chi_M^{\Gamma}(r) = \sum_{i=0}^{r} \dim_{\mathfrak{k}} \Gamma_i / \Gamma_{i-1} = \dim_{\mathfrak{k}} \Gamma_r \quad \forall r >> 0.$$

If $\chi_M^{\Gamma} = a_d y^d + \cdots + a_1 y + a_0$ with $a_i \in \mathbb{Q}$ and $a_d \neq 0$, we call $d = d(M)$ the dimension of M and $e = e(M) = d! a_d \in \mathbb{Z}$ the multipicity of M. If Γ' is another good filtration on M, by (2.5.iv) there is $r_2 \in \mathbb{N}$ such that for all $r >> 0$

$$\chi_M^{\Gamma'}(r - r_2) = \dim \Gamma'_{r-r_2} \leq \dim \Gamma_r = \chi_M^{\Gamma}(r) \leq \dim \Gamma'_{r+r_2} = \chi_M^{\Gamma'}(r + r_2),$$

hence the dimension and the multiplicity of M are independent of the choice of the good filtrations on M.

(2.8) By definition the Bernstein filtration is a good filtration on D with

$$\dim \mathfrak{B}_j = \sum_{i=0}^{j} \mathfrak{k}[x_1,\dots,x_n,y_1,\dots,y_n]_i \quad \text{by (2.3.1)}$$
$$= \binom{2n+j}{j} = \frac{1}{(2n)!} j^{2n} + \text{lower degree terms},$$

hence

(1) $$d(D) = 2n \quad \text{and} \quad e(D) = 1.$$

If $M \in D$**Mod** of finite type, as $\mathrm{gr}^\Gamma M$ is of finite type over $\mathrm{gr}^{\mathfrak{B}} D$ for a good filtration Γ on M,

(2) $$d(M) \leq d(D) = 2n.$$

The basic theorem here is

(2.9) **Bernstein's inequality.** *If $M \in D$**Mod** is nonzero of finite type, then $d(M) \geq n$.*

(2.10) One then defines $M \in D$**Mod** to be holonomic iff either (i) $M = 0$ or (ii) $M \neq 0$ and $d(M) = n$. For example, $A = \mathfrak{k}[x]$ is a holonomic $D(A)$-module.

∵ Let $\Gamma_j = \coprod_{|\alpha|\leq j} \mathfrak{k}x^\alpha$. As $\mathfrak{B}_i \ni x^\alpha$ for each $\alpha \in \mathbb{N}^n$ with $|\alpha| = i$, $\mathfrak{B}_i\Gamma_j = \Gamma_{i+j}$, hence Γ is good on A by (2.5.ii) with

$$\dim \Gamma_j = \binom{n+j}{j} = \frac{1}{n!} j^n + \text{lower degree terms}.$$

It follows that $d(A) = n$ and $e(A) = 1$.

From the definition the holonomicity is closed under taking submodules, quotients, and extensions.

3° p-filtrations [H86, 87]

Throughout the rest of the exposition assume $\mathrm{ch}\mathfrak{k} = p > 0$.

(3.1) Let $\mathfrak{k}[x] = \mathfrak{k}[x_1,\dots,x_n]$ the polynomial algebra in the x_i, $1 \leq i \leq n$. If $\mathrm{ev}_1 : D(\mathfrak{k}[x]) \to \mathfrak{k}[x]$ via $\delta \mapsto \delta(1)$ and if $D^+(\mathfrak{k}[x]) = \ker(\mathrm{ev}_1) = \coprod_{\alpha\in\mathbb{N}^n\setminus 0} \mathfrak{k}[x]\partial^\alpha$, [H87, 1.3.5] shows

(1) $$D^+(\mathfrak{k}[x]) \text{ is NOT of finite type over } D(\mathfrak{k}[x]).$$

Then $\mathrm{gr}(D^+(\mathfrak{k}[x])) \simeq (z_\alpha | \alpha \in \mathbb{N}^n \setminus 0)$ is not of finite type over $\mathrm{gr}D(\mathfrak{k}[x])$ in (2.1.4), hence

(2) $$\mathrm{gr}D(\mathfrak{k}[x]) \text{ is NOT noetherian},$$

and the definition of holonomicity in §2 breaks down in positive characteristic.

Fortunately, there is another filtration in characteristic p, called the p-filtration, to replace the standard filtration on $D(A)$, $A \in \mathbf{Alg}_{\mathfrak{k}}$ of finite type. For each $r \in \mathbb{N}$ let $A^{(r)} = \{a^{p^r} | a \in A\}$ and set

$$D_r(A) = \mathbf{Mod}_{A^{(r)}}(A, A).$$

Recall the $A \otimes A$-module structure on $\mathbf{Mod}_{\mathfrak{k}}(A, A)$: $(a \otimes b) \cdot f = af(b?)$. For each $\phi \in \mathbf{Mod}_{\mathfrak{k}}(A, A)$

(3) $$\phi \in D_r(A) \quad \text{iff} \quad \mathrm{d}_A(a^{p^r}) \cdot \phi = 0 \;\; \forall a \in A.$$

If $A = \mathfrak{k}[a_1, \ldots, a_s]$, as $I_A = \sum_{i=1}^{s}(A \otimes A)\mathrm{d}_A a_i$,

$$I_A{}^{sp^r} \subseteq \sum_{u \in I_A}(A \otimes A)u^{p^r} \subseteq I_A{}^{p^r},$$

hence by (3) and (1.2.1)

$$\mathit{Diff}_{p^r-1}(A) \subseteq D_r(A) \subseteq \mathit{Diff}_{sp^r-1}(A).$$

It follows that $D(A) = \cup_r D_r(A)$. We call $(D_r(A))_r$ the p-filtration on $D(A)$. In fact, [MN, 1.2.2](cf. [K, 1.2])

(4) $$D_r(A) = A[\mathit{Diff}_{p^r-1}(A)].$$

Note that $A^{(r)}$ is central in $D_r(A)$, making $D_r(A)$ into an $A^{(r)}$-algebra.

(3.2) From now on we will restrict our attentions to algebraic $\mathfrak{k}$-domains. Let $\mathbf{dom}_{\mathfrak{k}}$ be the category of algebraic $\mathfrak{k}$-domains, and let $A \in \mathbf{dom}_{\mathfrak{k}}$ and $B \in \mathbf{dom}_A$. Recall that B is called nette over A iff $\Omega_{B/A} = 0$; étale over A iff B is nette and flat over A; and smooth over A iff there is a partition $B = \sum_{i=1}^{r} Bb_i$ and $n \in \mathbb{N}$ such that for each i the localization B_{b_i} is étale over the polynomial A-algebra $A[x_1, \ldots, x_n]$ in the x_j.

Throughout the rest of §3 we let $A \in \mathbf{dom}_{\mathfrak{k}}$ étale over the polynomial $\mathfrak{k}$-algebra $\mathfrak{k}[x] = \mathfrak{k}[x_1, \ldots, x_n]$ in the x_i with structure homomorphism $\theta : \mathfrak{k}[x] \to A$. From the first fundamental exact sequence [M, 25.1] we have $\Omega_{A/\mathfrak{k}} \simeq A \otimes_{\mathfrak{k}[x]} \Omega_{\mathfrak{k}[x]/\mathfrak{k}}$ in $A\mathbf{Mod}$, hence

(1) $$\Omega_{A/\mathfrak{k}} = \coprod_{i=1}^{n} A\mathrm{d}_{A/\mathfrak{k}}\theta(x_i).$$

We call $(\theta(x_i))_{1 \le i \le n}$ a regular system of parametres on A. Put $z_i = \theta(x_i) \; \forall i$. As A is smooth over $\mathfrak{k}$, by [EGAIV, 16.10] the natural map

(2) $$\mathrm{S}^{\cdot}_A(\Omega_{A/\mathfrak{k}}) \to \mathrm{gr}(P_{A/\mathfrak{k}}) \quad \text{is bijective.}$$

Then for each $m \in \mathbb{N}$

(3) $$P_A^m = P_{A/\mathfrak{k}}^m = \coprod_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le m}} A\zeta_A^\alpha \quad \text{with} \quad \zeta_A^\alpha = \prod_{i=1}^{n} \zeta_i^{\alpha_i}, \zeta_i^{\alpha_i} = (\mathrm{d}_A z_i)^{\alpha_i},$$

taking the A-dual of which yields

$$(4) \qquad Diff_m(A) = \coprod_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le m}} A\partial_A^\alpha \quad \text{with} \quad \partial_A^\alpha(z^\beta) = \binom{\beta}{\alpha} z^{\beta-\alpha},$$

hence

$$(5) \qquad D(A) = \coprod_{\alpha\in\mathbb{N}} A\partial_A^\alpha \simeq A\otimes_{\mathfrak{k}[x]} D(\mathfrak{k}[x]) \quad \text{sending } 1\otimes\partial^\alpha \text{ to } \partial_A^\alpha\,.$$

As before we will often suppress A in the following.

(3.3) **Lemma [EGA0, 21.1.7].** *Let $C \in \mathbf{Alg}_{\mathfrak{k}}$ be a $C^{(1)}$-algebra of finite type. If $\Omega_{C/\mathfrak{k}} = \sum_{\lambda\in\Lambda} C\mathrm{d}_{C/\mathfrak{k}}c_\lambda$, then $C = C^{(1)}[c_\lambda|\lambda\in\Lambda]$.*

(3.4) **Proposition.** *For each $r\in\mathbb{N}$*

$$A = \coprod_{\alpha\in[0,p^r-1]^n} A^{(r)}z^\alpha \quad \textit{in } \mathbf{Mod}_{A^{(r)}} \textit{ with } \quad z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}.$$

Proof. By (3.3) we have $A = \mathfrak{k}[A^{(1)}, z]$, hence $A = \sum_{\alpha\in[0,p-1]^n} A^{(1)}z^\alpha$. If $0 = \sum_\alpha a_\alpha z^\alpha$, $a_\alpha \in A^{(1)}$, then in $\Omega_{A/\mathfrak{k}}$

$$\begin{aligned} 0 = \mathrm{d}_A(\sum_\alpha a_\alpha z^\alpha) &= \sum_\alpha \mathrm{d}_A(a_\alpha z^\alpha) = \sum_\alpha\{(\mathrm{d}_A a_\alpha)z^\alpha + a_\alpha \mathrm{d}_A(z^\alpha)\} \\ &= \sum_\alpha a_\alpha \sum_i \frac{\partial z^\alpha}{\partial z_i}\mathrm{d}_A(z_i) \quad \text{as } a_\alpha\in A^{(1)} \\ &= \sum_\alpha a_\alpha \sum_i \alpha_i z^{\alpha-\epsilon_i}\mathrm{d}_A(z_i), \end{aligned}$$

hence $0 = \sum_\alpha a_\alpha\alpha_i z^{\alpha-\epsilon_i} = \sum_{\alpha\ge\epsilon_i} a_\alpha\alpha_i z^{\alpha-\epsilon_i}$ $\forall i$. By induction on α one obtains $a_\alpha = 0$ $\forall\alpha\in[0,p-1]^n$. It follows that $A = \coprod_\alpha A^{(1)}z^\alpha$. Then, as A is a domain, $A^{(1)} = \coprod_{\alpha\in[0,p-1]^n} A^{(2)}z^{p\alpha}$, hence

$$A = \coprod_{\alpha\in[0,p-1]^n}(\coprod_{\alpha\in[0,p-1]^n} A^{(2)}z^{p\alpha})z^\alpha = \coprod_{\alpha\in[0,p^2-1]^n} A^{(2)}z^\alpha.$$

Iterate to get the assertion.

(3.5) **Corollary(Cartier-Chase equivalence).** *For each $r\in\mathbb{N}$ we have an isomorphism of $A^{(r)}$-algebras*

$$D_r \simeq \mathrm{M}_{p^{rn}}(A^{(r)}) \quad \textit{the } p^{rn}\times p^{rn}\textit{-matrix algebra over } A^{(r)},$$

hence D_r is Morita equivalent to $A^{(r)}$. Explicitly, if we equip $\mathbf{Mod}_{A^{(r)}}(A, A^{(r)})$ with a structure of $\mathbf{Mod}D_r$ such that $\phi\cdot\delta = \phi\circ\delta$, $\phi\in\mathbf{Mod}_{A^{(r)}}(A,A^{(r)})$, $\delta\in D_r$, then we have isomorphisms

$$A\otimes_{A^{(r)}} \mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \to D_r \quad \textit{in } D_r\mathbf{Mod}D_r \textit{ via } \quad a\otimes\phi\mapsto a\phi$$

and

$$\mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \otimes_{D_r} A \to A^{(r)} \quad \textit{in } \mathbf{Mod}_{A^{(r)}} \textit{ via } \quad \phi \otimes a \mapsto \phi(a),$$

hence $\mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \otimes_{D_r} ? : D_r\mathbf{Mod} \to \mathbf{Mod}_{A^{(r)}}$ *is an equivalence of categories with quasi-inverse* $A \otimes_{A^{(r)}} ?$.

(3.6) An A^∞-module is a projective system $(M_r, \pi_r)_{r \in \mathbb{N}}$ of $M_r \in \mathbf{Mod}_{A^{(r)}}$ and $\pi_r \in \mathbf{Mod}_{A^{(r)}}(M_r, M_{r-1})$ such that $A^{(r-1)} \otimes_{A^{(r)}} \pi_r : A^{(r-1)} \otimes_{A^{(r)}} M_r \to M_{r-1}$ is an isomorphism of $A^{(r-1)}$-modules. E.g., $(A^{(r)}, A^{(r)} \hookrightarrow A^{(r-1)})_r$ forms an A^∞-module.

If $(M_r, \pi_r)_r$ is an A^∞-module, then for each $r \in \mathbb{N}$ one will have isomorphisms in $A\mathbf{Mod}$

$$\begin{aligned} A \otimes_{A^{(r)}} M_r &\simeq A \otimes_{A^{(r-1)}} A^{(r-1)} \otimes_{A^{(r)}} M_r \simeq A \otimes_{A^{(r-1)}} M_{r-1} \\ &\simeq \cdots \simeq A \otimes_{A^{(0)}} M_0 \simeq M_0. \end{aligned}$$

If we make $A \otimes_{A^{(r)}} M_r$ into a left D_r-module by setting $\delta \cdot (a \otimes m) = \delta(a) \otimes m \ \forall r \in \mathbb{N}$, then we get a commutative diagram in $D_r\mathbf{Mod}$

$$\begin{array}{ccc} A \otimes_{A^{(r)}} M_r & \dashrightarrow & A \otimes_{A^{(r+1)}} M_{r+1} \\ & \nwarrow^{\sim}_{A \otimes_{A^{(r)}} A^{(r)} \otimes_{A^{(r+1)}} \pi_{r+1}} & \Big| \wr \\ & & A \otimes_{A^{(r)}} A^{(r)} \otimes_{A^{(r+1)}} M_{r+1}, \end{array}$$

hence $M_0 \simeq \varinjlim_r (A \otimes_{A^{(r)}} M_r)$ comes equipped with a structure of $D\mathbf{Mod}$. If $\mathcal{C}(A^\infty)$ denotes the category of A^∞-modules,

Theorem [H87, 2.2.4]. *There is a categorical equivalence*

$$D\mathbf{Mod} \to \mathcal{C}(A^\infty) \quad \textit{via} \quad M \mapsto (\mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \otimes_{D_r} M, \pi_r)_r$$

with quasi-inverse $(M_r)_r \mapsto \varinjlim_r A \otimes_{A^{(r)}} M_r$, *where* π_r *is defined by the commutative diagram*

$$\begin{array}{ccccc} \phi \otimes m & \mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \otimes_{D_r} M & \overset{\pi_r}{\dashrightarrow} & \mathbf{Mod}_{A^{(r-1)}}(A, A^{(r-1)}) \otimes_{D_{r-1}} M & \psi \otimes m \\ & \searrow & \searrow & \downarrow \wr & \downarrow \\ & \phi \otimes m & & \mathbf{Mod}_{A^{(r)}}(A, A^{(r-1)}) \otimes_{D_r} M & \psi \otimes m. \end{array}$$

Proof. To define π_r, the right vertical map is bijective as the map becomes so upon tensoring with A over $A^{(r-1)}$ and as A is faithfully flat over $A^{(r-1)}$.

(3.7) Let $M \in D\mathbf{Mod}$. To realize $\mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \otimes_{D_r} M$ inside M, let $\sigma = \sigma_{A,r} \in \mathbf{Mod}_{A^{(r)}}(A, A^{(r)})$ be a retraction of the inclusion $A^{(r)} \to A$, called an r-th Frobenius splitting, e.g., the projection along the decomposition $A = \coprod_{\alpha \in [0, p^r - 1]^n} A^{(r)} z^\alpha$. We will often regard σ as an idempotent $\mathrm{inc} \circ \sigma$ of D_r. Then

(1)
$$\begin{array}{ccc} \mathbf{Mod}_{A^{(r)}}(A, A^{(r)}) \otimes_{D_r} M & \overset{\sim}{\dashrightarrow} & \sigma M \\ {\scriptstyle \mathbf{Mod}_{A^{(r)}}(A, \mathrm{inc}) \otimes_{D_r} M} \big\downarrow & \circlearrowleft & \big\downarrow \\ D_r \otimes_{D_r} M & \underset{\sim}{\longrightarrow} & M, \end{array}$$

hence σM is independent of the choice of the Frobenius splittings, which we will denote by $M^{(r)}$. The notation is consistent in case $M = A$. As σ is an idempotent of D_r,

(2) $$M^{(r)} = \sigma M = \{m \in M \mid (1-\sigma)m = 0\} \text{ is a direct summand of } M \text{ in } \mathbf{Mod}_{A^{(r)}}.$$

Recall the evaluation $\mathrm{ev}_1 \in \mathbf{Mod}_{D_r}(D_r, A)$ at 1. Another intrinsic definition of $M^{(r)}$ is given by

(3) $$M^{(r)} = \{m \in M \mid \ker(\mathrm{ev}_1)m = 0\}.$$

∵ $\sigma M \subseteq \mathrm{RHS} \subseteq \mathrm{Ann}_M(1-\sigma)$.

(3.8) Let $\eta \in \mathbf{Alg}_{\mathfrak{k}}(A, B)$ be an étale extension of $\mathfrak{k}$-domains. Then $\Omega_{B/\mathfrak{k}} \simeq B \otimes_A \Omega_{A/\mathfrak{k}}$, hence $\Omega_{B/\mathfrak{k}} = \coprod_{i=1}^n B d_{B/\mathfrak{k}}\eta(z_i)$, and $(\eta(z_i))_i$ forms a regular system of parameteres on B. Put $w_i = \eta(z_i)\ \forall i$. As in (3.2.5)

(1) $$D(B) \simeq B \otimes_A D(A) \quad \text{in } B\mathbf{Mod} \text{ sending } 1 \otimes \partial_A^\alpha \text{ to } \partial_B^\alpha,$$

where $\partial_B^\alpha(w^\beta) = \binom{\beta}{\alpha} w^{\beta-\alpha}$ with $w^\beta = \prod_{i=1}^n w_i^{\beta_i}$.

On the other hand, we have in $\mathfrak{k}\mathbf{Alg}$

(2) $$\begin{array}{ccc} D_r(A) & \xrightarrow{D_r(\eta)} & D_r(B) \\ \sim \Big| & \circlearrowleft & \Big| \sim \\ \mathrm{M}_{p^{rn}}(A^{(r)}) & \xrightarrow[\mathrm{M}_{p^{rn}}(\eta^{(r)})]{} & \mathrm{M}_{p^{rn}}(B^{(r)}). \end{array}$$

Given $\lambda \in \mathbb{N}^n$, take $r >> 0$ such that $\partial_A^\lambda \in D_r(A)$. If $e_{\alpha\beta}^A \in D_r(A)$ with $e_{\alpha\beta}^A(z^\gamma) = \delta_{\beta\gamma} z^\alpha\ \forall \alpha, \beta, \gamma \in [0, p^r - 1]^n$, then

(3) $$\partial_A^\lambda = \sum_\alpha \binom{\alpha}{\lambda} e_{\alpha-\lambda,\alpha}^A,$$

hence $D_r(\eta)(\partial_A^\lambda) = \partial_B^\lambda$. It follows that the bijection (1) is compatible with $B \otimes_A D_r(\eta)$:

(4) $$\begin{array}{ccc} B \otimes_A D(A) & \xrightarrow[\sim]{(1)} & D(B) \\ \uparrow & \circlearrowleft & \uparrow \\ B \otimes_A D_r(A) & \xrightarrow[B \otimes D_r(\eta)]{\sim} & D_r(B). \end{array}$$

In particular,

$$D(A) \to D(B) \quad \text{via} \quad \sum_\alpha a_\alpha \partial_A^\alpha \mapsto \sum_\alpha \eta(a_\alpha)\partial_B^\alpha, \quad a_\alpha \in A$$

is a $\mathfrak{k}$-algebra homomorphism, which we will denote by $D(\eta)$. As η is injective, $D(\eta)$ is injective.

∵ We may assume A is local. Then by Auslander-Buchsbaum A is a UFD. There are

$h \in A$ and $g \in B$ such that B_g is a standard étale extension of A_h [R, Th. V.1.1], i.e., there is a monic irreducible $u \in A_h[t]$, t an indeterminate, with $\frac{du}{dt} \neq 0$ and $v \in A_h[t]$ such that

$$\begin{array}{ccccc} A & \hookrightarrow & A_h & \hookrightarrow & (A_h[t]/(u))_v \\ {\scriptstyle\eta}\downarrow & & \circlearrowleft & & \downarrow\wr \\ B & & \hookrightarrow & & B_g. \end{array}$$

(3.9) Let us complement some ring theoretic properties of $D = D(A)$ for A a smooth $\mathfrak{k}$-domain. Although D is not noetherian if $\mathrm{Krull\,dim}A \neq 0$, [H87, 1.3.3] shows

(1) $$D \text{ is a left coherent ring,}$$

i.e., every left ideal of D of finite type is finitely presented, and [S87, 3.7] finds

(2) $$\mathrm{gl\,dim}D = \mathrm{Krull\,dim}A.$$

In case $A = \mathfrak{k}[x_1, \ldots, x_n]$, [S86, 2.9]/[H86, 1.8.8] shows that

(3) $$D \text{ is a simple ring.}$$

4° Bøgvad's holonomicity [Bø]

Let $A = \mathfrak{k}[x] = \mathfrak{k}[x_1, \ldots, x_n]$ be the polynomial $\mathfrak{k}$-algebra in the x_i, $D = D(A)$, $D_r = D_r(A)$, $r \in \mathbb{N}$, and $M \in D\mathbf{Mod}$.

(4.1) Let $\sigma = \sigma_r$ be an r-th Frobenius splitting of A and let $\Delta_r = \mathfrak{k}[\partial^\alpha | \alpha \in [0, p^r - 1]^n]$ in D. Recall from (3.1.4)

(1) $$D_r = A[\Delta_r].$$

Note also that

(2) Δ_r admits a structure of commutative and cocommutative Hopf algebra over $\mathfrak{k}$ with comultiplication $\partial^\alpha \mapsto \sum_{\beta+\gamma=\alpha} \partial^\beta \otimes \partial^\gamma$, counit $\varepsilon_r = \mathrm{ev}_1|_{\Delta_r}$, and the antipode $\partial^\alpha \mapsto (-1)^{|\alpha|}\partial^\alpha$.

Let $V_r = \coprod_{\alpha \in [0, p^r-1]^n} \mathfrak{k}x^\alpha$. As each ∂^α, $\alpha \in [0, p^r - 1]^n \setminus 0$, annihilates $A^{(r)}$, Δ_r acts on $A^{(r)}$ via ε_r. Then

(3) $$A \simeq V_r \otimes A^{(r)} \quad \text{in } \mathbf{Mod}_{\Delta_r}$$

with Δ_r acting on the RHS by the comultiplication. More generally, we have in $\mathbf{Mod}_{\Delta_r}$

(4) $$\begin{aligned} M &\simeq A \otimes_{A^{(r)}} M^{(r)} \quad \text{by the Cartier-Chase equivalence} \\ &\simeq V_r \otimes M^{(r)}. \end{aligned}$$

(4.2) If P is a $\mathfrak{k}$-subspace of M, we will write V_rP for the subspace $\{\sum_i v_i m_i | v_i \in V_r, m_i \in P\}$. It is easy to check

Lemma [Bø, Lem. 2.2.1]. *Let P and Q be two $\mathfrak{k}$-linear subspaces of $M^{(r)}$.*

(i) *There is a commutative diagram of $\mathfrak{k}$-linear spaces*

$$\begin{array}{ccc} V_r \otimes M^{(r)} & \xrightarrow{(4.1.4)} & M \\ \uparrow & & \uparrow \\ V_r \otimes P & \overset{\sim}{\dashrightarrow} & V_r P. \end{array}$$

(ii) $V_r P \cap V_r Q = V_r(P \cap Q)$.

(iii) $V_r P + V_r Q = V_r(P + Q)$.

(iv) $V_r P \subseteq V_r Q$ *iff* $P \subseteq Q$. *In particular,* $V_r P = V_r Q$ *iff* $P = Q$.

(4.3) If $S \subseteq A$, we will write $S^{[r]}$ for $\{s^{p^r} | s \in S\}$.

Lemma [Bø, Lem. 2.2.2]. *For each* $r, s \in \mathbb{N}$ *with* $s \leq r$

$$V_r = V_s V_{r-s}{}^{[s]}.$$

(4.4) **Lemma.** *Let* $r, s, j \in \mathbb{N}$.

(i) $M^{(r+j)} \subseteq M^{(r)}$. *If* $M \neq 0$ *and* $j \neq 0$, $M^{(r+j)} \subset M^{(r)}$. *In particular, M is infinite dimensional if not* 0.

(ii) $V_s^{[r]} M^{(r+j)} \subseteq M^{(r)}$.

Proof. (i) As $\sigma_r \sigma_{r+j} = \sigma_r$ in D, $M^{(r+j)} = \sigma_{r+j} M = \sigma_r \sigma_{r+j} M \subseteq \sigma_r M = M^{(r)}$. Moreover, $V_r \otimes M^{(r)} \simeq M \simeq V_{r+j} \otimes M^{(r+j)}$ by (4.1.4). Hence if $M \neq 0$ and if $j \neq 0$, we must have $M^{(r+j)} \neq M^{(r)}$.

(ii) $\forall a \in V_s$ and $m \in M^{(r+j)}$, $\sigma_r(a^{p^r} m) = a^{p^r} \sigma_r m = a^{p^r} m$ as $m \in M^{(r)}$ by (i), hence $a^{p^r} m \in \sigma_r M = M^{(r)}$.

(4.5) Let Q be a finite dimensional subspace of M. By (4.2.iii) for each $r \in \mathbb{N}$ there is a unique maximal $\mathfrak{k}$-subspace $\Phi^r(Q)$ of $M^{(r)}$ such that $V_r \Phi^r(Q) \subseteq Q$. For each $j \in \mathbb{N}$ one has $V_j^{[r]} \Phi^{r+j}(Q) \subseteq M^{(r)}$ from (4.4.ii). As $Q \supseteq V_{r+j} \Phi^{r+j}(Q) = V_r V_j^{[r]} \Phi^{r+j}(Q)$ by (4.3), the maximality of $\Phi^r(Q)$ implies

(1)
$$V_j^{[r]} \Phi^{r+j}(Q) \subseteq \Phi^r(Q).$$

Set $\tau^r(Q) = V_r \Phi^r(Q)$. Then

$$\begin{aligned} \tau^{r+1}(Q) &= V_{r+1} \Phi^{r+1}(Q) \\ &= V_r V_1^{[r]} \Phi^{r+1}(Q) \quad \text{by (4.3)} \\ &\subseteq V_r \Phi^r(Q) \quad \text{by (1)} \\ &= \tau^r(Q), \end{aligned}$$

hence we have obtained a filtration $Q = \tau^0(Q) \geq \tau^1(Q) \geq \ldots$, on Q in $\mathbf{Mod}_{\mathfrak{k}}$ that Bøgvad calls the canonical filtration of Q.

As $\dim \tau^r(Q) = \dim V_r \Phi^r(Q) = \dim V_r \dim \Phi^r(Q) = p^{nr} \dim \Phi^r(Q)$,

$$\dim \Phi^r(Q) = p^{-nr} \dim \tau^r(Q), \tag{2}$$

hence $\tau^r(Q) = 0$ if $p^{nr} > \dim Q$. That enables us to define

$$t(Q) = \sum_{i \geq 0} p^{-ni} \dim(\tau^i(Q)/\tau^{i+1}(Q)).$$

If $Q \neq 0$, then

$$0 < t(Q) = \dim Q - (1 - p^{-n}) \sum_{i \geq 1} p^{(1-i)n} \dim \tau^i(Q) \leq \dim Q. \tag{3}$$

As $\tau^{r+1}(Q) = V_{r+1}\Phi^{r+1}(Q) = V_r V_1^{[r]} \Phi^{r+1}(Q)$ with $V_1^{[r]}\Phi^{r+1}(Q) \subseteq M^{(r)}$,

$$\dim V_1^{[r]}\Phi^{r+1}(Q) = \frac{1}{\dim V_r} \dim \tau^{r+1}(Q) = p^{-nr} \dim \tau^{r+1}(Q),$$

hence

$$\begin{aligned} p^{-nr} \dim(\tau^r(Q)/\tau^{r+1}(Q)) &= p^{-nr}(\dim \tau^r(Q) - \dim \tau^{r+1}(Q)) \\ &= \dim \Phi^r(Q) - \dim V_1^{[r]}\Phi^{r+1}(Q) \quad \text{by (2)} \\ &= \dim(\Phi^r(Q)/V_1^{[r]}\Phi^{r+1}(Q)) \quad \text{by (1) again.} \end{aligned}$$

It follows that

$$t(Q) = \sum_{i \geq 0} \dim(\Phi^i(Q)/V_1^{[i]}\Phi^{i+1}(Q)). \tag{4}$$

(4.6) **Another characterization of t [Bø, Prop. 2.3.2].** *Let Q be a finite dimensional $\mathfrak{k}$-subspace of M and let $\kappa \in \mathbb{N}$. The following are equivalent:*

(i) *For each $i \in \mathbb{N}$ there is a $\mathfrak{k}$-subspace Q_i of $M^{(i)}$ such that $Q = \sum_{i \geq 0} V_i Q_i$ and that $\sum_{i \geq 0} \dim Q_i \leq \kappa$.*

(ii) *$t(Q) \leq \kappa$.*

In particular, if $Q = V_r Q_r$ with $Q_r \leq M^{(r)}$, then $t(Q) \leq \dim Q_r$, but the equality need not hold.

(4.7) **Bøgvad's definition of holonomicity.** We say $M \in D(\mathfrak{k}[x])\mathbf{Mod}$ is (filtration) holonomic iff there is $\kappa \in \mathbb{N}$ and a family $(M_r)_{r \in \mathbb{N}}$ of finite dimensional $\mathfrak{k}$-subspaces of M, called a generating sequence of M, such that

(i) any $m \in M$ is contained in all but finitely many M_r's

(ii) $t(M_r) \leq \kappa$ for each $r \in \mathbb{N}$.

The minimal value of such $\kappa((M_r)_{r\in\mathbb{N}})$ is called the multiplicity of M, denoted $e(M)$.

(4.8) Let $r, s \in \mathbb{N}$. For each $\delta \in D_s(A^{(r)}) = \mathbf{Mod}_{(A^{(r)})^{(s)}}(A^{(r)}, A^{(r)})$ one can regard $\delta \circ \sigma_r \in D_{r+s}$, hence obtains a $\mathfrak{k}$-algebra homomorphism

$$(1) \qquad D(A^{(r)}) \to \sigma_r D \sigma_r \quad \text{via} \quad \delta \mapsto \sigma_r \circ (\delta \circ \sigma_r) \circ \sigma_r = \delta \circ \sigma_r.$$

Then $D(A^{(r)})$ can act on M through (1):

$$\delta \cdot m = (\delta \circ \sigma_r)m, m \in M, \delta \in D(A^{(r)}).$$

As $\sigma_r(\delta \cdot \sigma_r m) = \sigma_r(\delta \circ \sigma_r)\sigma_r m = (\delta \circ \sigma_r)\sigma_r m = \delta \cdot \sigma_r m$ for each $m \in M$, $D(A^{(r)})$ stabilizes $\sigma_r M = M^{(r)}$, hence

$$(2) \qquad M^{(r)} \leq M \quad \text{in } D(A^{(r)})\mathbf{Mod}.$$

As $\mathfrak{k}[x]^{(r)} = \mathfrak{k}[x^{p^r}] \simeq \mathfrak{k}[x]$ in $\mathbf{Alg}_{\mathfrak{k}}$ via $x_i^{p^r} \mapsto x_i$ for each i, one can define $(M^{(r)})^{(s)}$ with respect to the $D(A^{(r)})$-module structure on $M^{(r)}$. If $\sigma_s^{(r)}$ is an s-th Frobenius splitting of $A^{(r)}$, then

$$(3) \qquad (M^{(r)})^{(s)} = \sigma_s^{(r)} M^{(r)} = \sigma_s^{(r)} \sigma_r M = \sigma_{s+r} M = M^{(r+s)}.$$

Note also that for each $j \in \mathbb{N}$

$$V_j^{[r]} = \coprod_{\alpha\in[0,p^j-1]^n} \mathfrak{k}(x^\alpha)^{p^r} = \coprod_{\alpha\in[0,p^j-1]^n} \mathfrak{k}(x^{p^r})^\alpha.$$

If Q is a finite dimensional $\mathfrak{k}$-subspace of $M^{(r)}$, one can define the canonical filtration on Q with respect to $D(A^{(r)})$, denoted $\tau_r^{\cdot}(Q)$, by

$$\tau_r^j(Q) = V_j^{[r]} \Phi_r^j(Q)$$

with $\Phi_r^j(Q)$ maximal $\mathfrak{k}$-subspace of $(M^{(r)})^{(j)} = M^{(r+j)}$ such that $V_j^{[r]}\Phi_r^j(Q) \subseteq Q$. Then set

$$t_r(Q) = \sum_{i\geq 0} p^{-ni} \dim\{\tau_r^i(Q)/\tau_r^{i+1}(Q)\}.$$

(4.9) **Lemma [Bø, Lem. 2.3.2].** *Let Q be a finite dimensional $\mathfrak{k}$-subspace of $M^{(r)}$.*

(i) $\forall j \in \mathbb{N}$, $\Phi_r^j(Q) = \Phi^{r+j}(V_r Q)$.

(ii) $\forall j \in \mathbb{N}$

$$\tau^j(V_r Q) = \begin{cases} V_r \tau_r^{j-r}(Q) & \text{if } j \geq r \\ V_r Q & \text{otherwise.} \end{cases}$$

(iii) $t(V_r Q) = t_r(Q)$.

Proof. As $V_rQ \supseteq V_r\tau_r^j(Q) = V_rV_j^{[r]}\Phi_r^j(Q) = V_{r+j}\Phi_r^j(Q)$ by (4.3) with $\Phi_r^j(Q) \subseteq (M^{(r)})^{(j)} = M^{(r+j)}$,

$$\text{(1)} \qquad \Phi_r^j(Q) \subseteq \Phi^{r+j}(V_rQ)$$

by the maximality of the latter. In particular, $V_rQ = V_r\Phi_r^0(Q) \subseteq V_r\Phi^r(V_rQ) = \tau^r(V_rQ)$, hence $\tau^j(V_rQ) = V_rQ \ \forall j \leq r$. Also (1) implies that

$$V_r\tau_r^j(Q) = V_{r+j}\Phi_r^j(Q) \subseteq V_{r+j}\Phi^{r+j}(V_rQ) = \tau^{r+j}(V_rQ).$$

On the other hand,

$$V_rV_j^{[r]}\Phi^{r+j}(V_rQ) = V_{r+j}\Phi^{r+j}(V_rQ) = \tau^{r+j}(V_rQ) \subseteq V_rQ \quad \text{with } V_j^{[r]}\Phi^{r+j}(V_rQ) \subseteq M^{(r)}.$$

It follows from (4.2.iv) that $V_j^{[r]}\Phi^{r+j}(V_rQ) \subseteq Q$. As $\Phi^{r+j}(V_rQ) \subseteq M^{(r+j)} = (M^{(r)})^{(j)}$, $\Phi^{r+j}(V_rQ) \subseteq \Phi_r^j(Q)$ by definition. Hence together with (1) one obtains that $\Phi^{r+j}(V_rQ) = \Phi_r^j(Q)$ and that

$$\text{(2)} \qquad \tau^{r+j}(V_rQ) = V_{r+j}\Phi^{r+j}(V_rQ) = V_{r+j}\Phi_r^j(Q) = V_rV_j^{[r]}\Phi_r^j(Q) = V_r\tau_r^j(Q),$$

verifying (i) and (ii). Finally,

$$\begin{aligned}
t(V_rQ) &= \sum_{i\geq 0} p^{-ni}\dim(\tau^i(V_rQ)/\tau^{i+1}(V_rQ)) = \sum_{i\geq 0} p^{-ni}\{\dim\tau^i(V_rQ) - \dim\tau^{i+1}(V_rQ)\}\\
&= \sum_{i\geq r} p^{-ni}\{\dim V_r\tau_r^{i-r}(Q) - \dim V_r\tau_r^{i-r+1}(Q)\} \quad \text{by (ii)}\\
&= \sum_{i\geq r} p^{-ni}\dim V_r\{\dim\tau_r^{i-r}(Q) - \dim\tau_r^{i-r+1}(Q)\} \quad \text{by (4.2.i) as } Q \subseteq M^{(r)}\\
&= \sum_{i\geq r} p^{-n(i-r)}\dim(\tau_r^{i-r}(Q)/\tau_r^{i-r+1}(Q))\\
&= \sum_{j\geq 0} p^{-nj}\dim(\tau_r^j(Q)/\tau_r^{j+1}(Q)) = t_r(Q).
\end{aligned}$$

(4.10) **Examples [Bø, 4.1].** (i) A is holonomic with $e(A) = 1$.

⊙ We have $A = \cup_r V_r$. $\forall r$ and $j \in \mathbb{N}$, by (4.3)

$$\tau^j(V_r) = V_j\Phi^j(V_r) = \begin{cases} V_jV_{r-j}{}^{[j]} = V_r & \text{if } j \leq r\\ 0 & \text{otherwise,}\end{cases}$$

hence $t(V_r) = p^{-nr}\dim\tau^r(V_r) = p^{-nr}\dim V_r = p^{-nr}p^{nr} = 1$.

Alternatively, as $V_r = V_r\mathfrak{k}$, one could take $\kappa = 1$ in (4.6). Or by (4.9) one has $t(V_r) = t_r(\mathfrak{k}) = 1$.

(ii) If $f \in A \setminus 0$, A_f is holonomic with $e(A_f) \leq \prod_{j=1}^n(1 + \deg_{x_j} f)$.

(iii) Let $\mathfrak{a} \trianglelefteq A$. If $P \in \mathbf{Mod}_A$, let

$$\Gamma_{\mathfrak{a}}(P) = \{m \in P | \mathrm{supp}(m) \subseteq V(\mathfrak{a}) \text{ closed subset of Spec}A \text{ defined by } \mathfrak{a}\}.$$

Let $A^{\cdot}$ be an injective resolution of A in $D\mathbf{Mod}$ and set $\mathrm{H}^{\cdot}_{\mathfrak{a}}(A) = \mathrm{H}^{\cdot}(\Gamma_{\mathfrak{a}}(A^{\cdot}))$. Then each $\mathrm{H}^i_{\mathfrak{a}}(A)$, $i \in \mathbb{N}$, is holonomic.

(4.11) **Theorem [Bø, Th. 3.2.1].** *All submodules, quotients and extensions of holonomic D-modules remain holonomic.*

(4.12) **Theorem [Bø, Th. 3.5.3].** *Let N be a D-submodule of M. If M is holonomic, then $e(M) = e(N) + e(M/N)$. In particular, any holonomic modules has finite length bounded by its multiplicity.*

(4.13) In the absense of noetherianity of D the holonomicity is a strong finiteness condition. Eg., D^+ is not of finite type in $D\mathbf{Mod}$. Moreover, according to J.T. Stafford's theorem [Bj, 1.8.18] if R is a simple ring of infinite length as a left R-module, then any left R-module of finite length must be finite, hence from (3.8.3) one obtains

Corollary [Bø, Prop. 3.2.4]. *Any holonomic D-module is cyclic.*

(4.14) **Remark.** Mebkhout and Narvaez-Macarro [MN, 1.2.4] proposes after (3.9.2) to call $M \in D\mathbf{Mod}$ of finite presentation type holonomic iff $\mathrm{Ext}^i_D(M, D) = 0 \ \forall i \neq \mathrm{Krull\,dim}A$. In [MN], however, the category of their holonomic modules is yet to be verified to form an abelian category.

Assume $\mathrm{Krull\,dim}A \geq 1$. Then $D^+ = \ker(\mathrm{ev}_1)$ is not of finite type [H87, 1.3.5], hence

(1) $\qquad A$ is not of finite presentation type in $D(A)\mathbf{Mod}$.

It follows that A is not holonomic in the sense of [MN]. Moreover, [H86, 1.8.5]/[S86, 2.20] shows that if A admits a regular system of parameters, then

(2) $$\mathrm{Ext}^i_{D\mathbf{Mod}}(A, D) = 0 \quad \forall i \neq 1 \quad \text{while} \quad \mathrm{Ext}^1_{D\mathbf{Mod}}(A, D) \neq 0.$$

5° The direct and inverse images [H88], [Bø]

In this section we fix $\theta \in \mathbf{Alg}_{\mathfrak{k}}(A, B)$ with A and B both smooth over $\mathfrak{k}$. We will define two functors ${}^{\theta}? : D(A)\mathbf{Mod} \to D(B)\mathbf{Mod}$, called the inverse image functor, and $\int^0_{\theta} : D(B)\mathbf{Mod} \to D(A)\mathbf{Mod}$, called the direct image functor. The terms "inverse" and "direct" refer to the associated morphism $\mathfrak{Sp}_{\mathfrak{k}}(\theta) : \mathfrak{Sp}_{\mathfrak{k}}(B) \to \mathfrak{Sp}_{\mathfrak{k}}(A)$ of the affine spectra. The holonomicity of $M \in D(B)\mathbf{Mod}$ is defined by taking a finite affine open cover $(\mathfrak{Sp}_{\mathfrak{k}}(B_{b_i}))_i$ of $\mathfrak{Sp}_{\mathfrak{k}}(B)$ with each B_{b_i} an étale extension of the polynomial algebra $\mathfrak{k}[x] = \mathfrak{k}[x_1, \ldots, x_n]$, $n = \mathrm{Krull\,dim}B$. We say M is holonomic iff $\int^0_{\theta_i}(M \otimes_B B_{b_i})$ is holonomic for each i, where $\theta_i : \mathfrak{k}[x] \to B_{b_i}$ is the structure homomorphism.

(5.1) Geometrically, the following should be clear:

Lemma. *Let $a \in A$.*

(i) $A_a \otimes_A D_r(A) \simeq D_r(A_a) = \mathbf{Mod}_{(A_a)^{(r)}}(A_a, A_a)$ *in* $A_a\mathbf{Rng}$.

(ii) $(A_a)^{(r)} \otimes_{A^{(r)}} D_r(A) \simeq D_r(A_a)$ *in* $(A_a)^{(r)}\mathbf{Alg}$.

(iii) $A_a \otimes_A D(A) \simeq D(A_a)$ *in* $A_a\mathbf{Rng}$.

Proof. As A is locally free of finite rank over $A^{(r)}$ and as $(A_a)^{(r)} = (A^{(r)})_{a^{p^r}}$, (ii) is standard. (iii) will follow from (i) by taking the direct limit.

For (i) let $a_1 + \cdots + a_s = 1$ be a partition of 1 in A with each A_{a_i} admitting a regular system of parametres. Then $a_1^{p^r} + \cdots + a_s^{p^r} = 1$. If $b = a_i$,

$$\begin{aligned}
&(A_a \otimes_A D_r(A)) \otimes_{A^{(r)}} (A_b)^{(r)} \\
&\quad \simeq \mathbf{Mod}_{A^{(r)}}(A, A_a) \otimes_{A^{(r)}} (A_b)^{(r)} \quad \text{as } A \text{ is of finite presentation type over } A^{(r)} \\
&\quad \simeq \mathbf{Mod}_{(A_b)^{(r)}}(A \otimes_{A^{(r)}} (A_b)^{(r)}, A_a \otimes_{A^{(r)}} (A_b)^{(r)}) \simeq \mathbf{Mod}_{(A_b)^{(r)}}(A_b, A_{ab}) \\
&\quad \simeq \mathbf{Mod}_{(A_{ab})^{(r)}}((A_{ab})^{(r)} \otimes_{(A_b)^{(r)}} A_b, A_{ab}) \simeq \mathbf{Mod}_{(A_{ab})^{(r)}}(A_{ab}, A_{ab}) \\
&\quad \simeq \mathbf{Mod}_{(A_{ab})^{(r)}}(A_a \otimes_{(A_a)^{(r)}} (A_{ab})^{(r)}, A_a \otimes_{(A_a)^{(r)}} (A_{ab})^{(r)}) \\
&\quad \simeq \mathbf{Mod}_{(A_a)^{(r)}}(A_a, A_a) \otimes_{(A_a)^{(r)}} (A_{ab})^{(r)} \simeq D_r(A_a) \otimes_{A^{(r)}} (A_b)^{(r)},
\end{aligned}$$

hence the assertion.

(5.2) Let, heuristically, $D_{\theta\leftarrow} = B \otimes_A D(A)$ and $D_{\theta,r\leftarrow} = B \otimes_A D_r(A)$, $r \in \mathbb{N}$. We make $D_{\theta\leftarrow}$ into a $(D(B), D(A))$-bimodule as follows. As A is locally free of finite rank over $A^{(r)}$, arguing as in (5.1) yields an isomorphism of $A^{(r)}$-modules

$$(1) \qquad B \otimes_A D_r(A) \simeq \mathbf{Mod}_{A^{(r)}}(A, B),$$

the RHS of which carries a structure of $D_r(B)\mathbf{Mod}D_r(A)$ such that

$$(2) \qquad \delta_B \cdot \phi \cdot \delta_A = \delta_B \circ \phi \circ \delta_A, \quad \delta_B \in D_r(B), \delta_A \in D_r(A), \phi \in \mathbf{Mod}_{A^{(r)}}(A, B).$$

As the structure is compatible with respect to r, $D_{\theta\leftarrow} \simeq \varinjlim_r D_{\theta,r\leftarrow}$ comes equipped with a structure of $D(B)\mathbf{Mod}D(A)$.

We then define the inverse image functor ${}^{\theta}? : D(A)\mathbf{Mod} \to D(B)\mathbf{Mod}$ via $M \mapsto D_{\theta\leftarrow} \otimes_{D(A)} M$ with ${}^{\theta}M$ called the inverse image of M. Note that

$$(3) \qquad {}^{\theta}M = B \otimes_A D(A) \otimes_{D(A)} M \simeq B \otimes_A M \quad \text{in } B\mathbf{Mod}$$

under which the $D_r(B)$-module structure is transferred onto $B \otimes_A M$ such that $\delta \cdot (b \otimes m)$, $\delta \in D_r(B)$, corresponds to $(\delta \circ b\theta) \otimes m$ in $\mathbf{Mod}_{A^{(r)}}(A, B) \otimes_{D_r(A)} M$.

(5.3) **Lemma [H86, 1.8.1].** *For each $r \in \mathbb{N}$ one has A projective in $D_r(A)\mathbf{Mod}$.*

Proof. As A is locally free over $A^{(r)}$ of rank p^{rn}, $n = \operatorname{Krull\,dim} A$, $A/A^{(r)}$ is projective of rank $p^{rn} - 1$ in $\mathbf{Mod}_{A^{(r)}}$, hence

$$(1) \qquad A^{(r)} \hookrightarrow A \text{ admits a retraction } \sigma \text{ in } \mathbf{Mod}_{A^{(r)}}.$$

Then $\mathrm{ev}_1 : D_r(A) \to A$ splits in $D_r(A)\mathbf{Mod}$ via $a \mapsto a\sigma$, $a \in A$.

(5.4) **Proposition [H88, 1.2].** *Each $D_{\theta,r\leftarrow}$, $r \in \mathbb{N}$, is projective of finite type in $D_r(B)\mathbf{Mod}$, hence $D_{\theta\leftarrow}$ is flat in $D(B)\mathbf{Mod}$.*

Proof. As A is projective of finite type over $A^{(r)}$, A is a direct summand of $A^{(r)\oplus m}$ in $\mathbf{Mod}_{A^{(r)}}$ for some $m \in \mathbb{N}$. Then $D_{\theta,r\leftarrow} \simeq \mathbf{Mod}_{A^{(r)}}(A, B)$ is in $D_r(B)\mathbf{Mod}$ a direct summand of

$$\mathbf{Mod}_{A^{(r)}}(A^{(r)\oplus m}, B) \simeq \mathbf{Mod}_{A^{(r)}}(A^{(r)}, B)^{\oplus m} \simeq B^{\oplus m},$$

with $D_r(B)$ acting on B in the standard way. As B is projective in $D_r(B)\mathbf{Mod}$ by (5.3), so is $D_{\theta,r\leftarrow}$. Then $D_{\theta,r\leftarrow}$ is flat in $D_r(B)\mathbf{Mod}$ by taking the direct limit.

(5.5) If $\eta \in \mathbf{Alg}_{\mathfrak{k}}(B, C)$ with C smooth over $\mathfrak{k}$, for each $r \in \mathbb{N}$ one has in $D_r(C)\mathbf{Mod}D_r(A)$

$$D_{\eta,r\leftarrow} \otimes_{D_r(B)} D_{\theta,r\leftarrow} = C \otimes_B D_r(B) \otimes_{D_r(B)} B \otimes_A D_r(A) \simeq C \otimes_A D_r(A) = D_{\eta\circ\theta,\leftarrow},$$

hence taking the direct limit yields,

Proposition. ${}^{\eta}? \circ {}^{\theta}? \simeq {}^{(\eta\circ\theta)}? : D(A)\mathbf{Mod} \to D(C)\mathbf{Mod}$.

(5.6) The definition of the direct image functor is much harder [H88, 7.1]. Let $\omega_A = \wedge^n \Omega_{A/\mathfrak{k}}$, $n = \mathrm{rk}_A \Omega_{A/\mathfrak{k}}$, and $\omega_B = \wedge^\ell \Omega_{B/\mathfrak{k}}$, $\ell = \mathrm{rk}_B \Omega_{B/\mathfrak{k}}$. Set, again heuristically, $D_{\theta\rightarrow} = \omega_B \otimes_B \{B \otimes_A (D(A) \otimes_A \omega_A^\vee)\}$ and $D_{\theta,r\rightarrow} = \omega_B \otimes_B \{B \otimes_A (D_r(A) \otimes_A \omega_A^\vee)\}$, $r \in \mathbb{N}$, where $\omega_A^\vee = \mathbf{Mod}_A(\omega_A, A)$. We will equip $D_{\theta,r\rightarrow}$ with a structure of $D_r(A)\mathbf{Mod}D_r(B)$, compatibly with respect to r, to make $D_{\theta\rightarrow} \simeq \varinjlim_r D_{\theta,r\rightarrow}$ into a $(D(A), D(B))$-bimodule.

If A and B both admit regular systems of parameters, however, the structure is easy to describe. Thus let $(z_i)_{1\le i\le n}$ (resp. $(w_j)_{1\le j\le \ell}$) be a regular system of parametres on A (resp. B). Define $^* \in \mathbf{Mod}_{\mathfrak{k}}(D(A), D(A))$ by

$$\sum_\alpha a_\alpha \partial_A^\alpha \mapsto \sum_\alpha (-1)^{|\alpha|} \partial_A^\alpha a_\alpha, \quad a_\alpha \in A.$$

On the Weyl algebra of characteristic 0 the map * is just the composite of Fourier transforms Four $\circ$ Four$'$ from (2.2.4).

Lemma. *Assume A admits a regular system of parametres $(z_i)_i$. Then the $\mathfrak{k}$-linear map * is an antiinvolution in $\mathfrak{k}\mathbf{Alg}$; for each $\delta_1, \delta_2 \in D(A)$*

$$(\delta_1\delta_2)^* = \delta_2^*\delta_1^*.$$

Proof. One quickly reduces the problem to showing $(\partial_A^\alpha b)^* = (-1)^{|\alpha|} b \partial_A^\alpha$ $\forall \alpha$ and $b \in A$. Take $r >> 0$ so that $\partial_A^\alpha \in D_r(A)$. We may assume $b = b' z^\gamma$ for some $b' \in A^{(r)}$ and $\gamma \in [0, p^r - 1]^n$. Write $\partial_A^\alpha z^\gamma = \sum_\nu a_\nu \partial_A^\nu$, $a_\nu \in A$. Then

$$\begin{aligned}(\partial_A^\alpha b)^* &= (b' \partial_A^\alpha z^\gamma)^* \quad \text{as } b' \in A^{(r)} \\ &= (\sum_\nu b' a_\nu \partial_A^\nu)^* = (\sum_\nu (-1)^{|\nu|} \partial_A^\nu a_\nu) b' = (\sum_\nu a_\nu \partial_A^\nu)^* b' = (\partial_A^\alpha z^\gamma)^* b'\end{aligned}$$

while $(-1)^{|\alpha|}b\partial_A^\alpha = (-1)^{|\alpha|}b'z^\gamma\partial_A^\alpha = (-1)^{|\alpha|}z^\gamma\partial_A^\alpha b'$. Hence we have only to show $(\partial_A^\alpha z^\gamma)^* = (-1)^{|\alpha|}z^\gamma\partial_A^\alpha$.

Put $C = \mathbb{Z}[z] = \mathbb{Z}[z_1, \ldots, z_n]$ the polynomial $\mathbb{Z}$-algebra in the z_i's. Write $\partial_C^\alpha z^\gamma = \sum_\nu c_\nu\partial_C^\nu$ in $D(C)$ with $c_\nu \in C$ and $\partial_C^\nu \in D(C)$ such that $\partial_C^\nu(z^\mu) = \binom{\mu}{\nu}z^{\mu-\nu}\ \forall\mu$. As $\partial_A^\alpha z^\gamma$ is completely determined by the evaluation at z^μ, $\mu \in \mathbb{N}$, we must have $\partial_A^\alpha z^\gamma = \sum_\nu \overline{c_\nu}\partial_A^\nu$ with $\overline{c_\nu}$ the image of c_μ in A. Then in $D(A)$

$$(1) \qquad (\partial_A^\alpha z^\gamma)^* = (\sum_\nu \overline{c_\nu}\partial_A^\nu)^* = \sum_\nu (-1)^{|\nu|}\partial_A^\nu\overline{c_\nu}.$$

But $D(C)$ naturally sits inside $D(\mathbb{Q}[z])$, and on $D(\mathbb{Q}[z])$ our * is an antiinvolution coinciding with Four $\circ$ Four$'$. It follows on $D(C)$ that $(-1)^{|\alpha|}z^\gamma\partial_C^\alpha = (\partial_C^\alpha z^\gamma)^* = (\sum_\nu c_\nu\partial_C^\nu)^* = \sum_\nu(-1)^{|\nu|}\partial_C^\nu c_\nu$. Then

$$\begin{aligned}(-1)^{|\alpha|}z^\gamma\partial_A^\alpha &= \sum_\nu(-1)^{|\nu|}\partial_A^\nu\overline{c_\nu} \quad \text{as both sides are determined by the evaluation at the } z^\mu\text{'s} \\ &= (\partial_A^\alpha z^\gamma)^* \quad \text{by (1)},\end{aligned}$$

as desired.

(5.7) Assume still that A and B both have regular systems of parametres. By (3.1.4) the antiinvolution * stabilizes each $D_r(A)$ and $D_r(B)$. As $D_{\theta,r\to} \simeq B \otimes_A D_r(A) = D_{\theta,r\leftarrow}$ in $\mathbf{Mod}_\mathfrak{k}$, transfer the structure of $D_r(B)\mathbf{Mod}D_r(A)$ on $D_{\theta,r\leftarrow}$ onto $D_{\theta,r\to}$ twisted by * to make it a $(D_r(A), D_r(B))$-bimodule: $\forall\delta_A \in D_r(A), \delta_B \in D_r(B)$ and $\phi \in D_{\theta,r\to}$, set

$$\delta_A \cdot \phi \cdot \delta_B = \delta_B^*\phi\delta_A^* = \delta_B^* \circ \phi \circ \delta_A^*$$

regarding $\phi \in \mathbf{Mod}_{A^{(r)}}(A, B)$. This is compatible with respect to r, so that $D_{\theta\to} \simeq \varinjlim_r D_{\theta,r\to}$ comes equipped with a structure of $D(A)\mathbf{Mod}D(B)$.

(5.8) Back to the general smooth $\mathfrak{k}$-algebras A and B, we need the Cartier operators $C_r \in \mathbf{Mod}_{A^{(r+1)}}(\omega_{A^{(r)}}, \omega_{A^{(r+1)}}), r \in \mathbb{N}$, to check that the local $D(A)\mathbf{Mod}D(B)$-structures defined in (5.7) are compatible with each other to define the structure on the whole of $D_{\theta\to}$.

Locally, if A admits a regular system of parametres $(z_i)_{1\le i\le n}$, C_r is described as follows [H88, 5.3]. Put $z_i^{(r)} = z_i^{p^r}\ \forall i$. As A is étale over $\mathfrak{k}[x]$, $A^{(r)}$ is étale over $\mathfrak{k}[x]^{(r)}$, hence $(z_i^{(r)})_i$ forms a regular system of parametres on $A^{(r)}$.

$\because$ We have $\Omega_{A^{(r)}/\mathfrak{k}} \simeq A^{(r)} \otimes_{\mathfrak{k}[x]^{(r)}} \Omega_{\mathfrak{k}[x]^{(r)}/\mathfrak{k}}$.

If $\mathrm{d}_{A^{(r)}}z^{(r)} = \mathrm{d}_{A^{(r)}/\mathfrak{k}}z^{(r)} = \mathrm{d}_{A^{(r)}/\mathfrak{k}}z_1^{(r)} \wedge \cdots \wedge \mathrm{d}_{A^{(r)}/\mathfrak{k}}z_n^{(r)}$,

$$\omega_{A^{(r)}} = A^{(r)}\mathrm{d}_{A^{(r)}}z^{(r)} = \coprod_{\alpha\in[0,p-1]^n} A^{(r+1)}(z^{(r)})^\alpha\mathrm{d}_{A^{(r)}}z^{(r)}.$$

Then C_r is given by

$$(1) \qquad (z^{(r)})^\alpha\mathrm{d}_{A^{(r)}}z^{(r)} \mapsto \begin{cases}\mathrm{d}_{A^{(r+1)}}z^{(r+1)} & \text{if } \alpha = (p-1, \ldots, p-1) \\ 0 & \text{otherwise.}\end{cases}$$

Like an A^∞-module in (3.6) Haastert defines an R-A^∞-module to be a direct system $(M^r, \pi^r)_{r\in\mathbb{N}}$ of $M^r \in \mathbf{Mod}_{A^{(r)}}$ and $\pi^r \in \mathbf{Mod}_{A^{(r+1)}}(M^r, M^{r+1})$ such that $M^r \to \mathbf{Mod}_{A^{(r+1)}}(A^{(r)}, M^{r+1})$ via $m \mapsto \pi_r(?m)$ is an isomorphism of $A^{(r)}$-modules. Then

$$
\begin{aligned}
(2) \qquad \mathbf{Mod}_{A^{(r)}}(A, M^r) &\simeq \mathbf{Mod}_{A^{(r)}}(A, \mathbf{Mod}_{A^{(r+1)}}(A^{(r)}, M^{r+1})) \\
&\simeq \mathbf{Mod}_{A^{(r+1)}}(A^{(r)} \otimes_{A^{(r)}} A, M^{r+1}) \quad \text{by adjunction} \\
&\simeq \mathbf{Mod}_{A^{(r+1)}}(A, M^{r+1}).
\end{aligned}
$$

The resulting bijection $\mathbf{Mod}_{A^{(r)}}(A, M^r) \simeq \mathbf{Mod}_{A^{(r+1)}}(A, M^{r+1})$ belongs to $\mathbf{Mod}D_r(A)$ with $D_r(A)$ acting on A. By iteration $\mathbf{Mod}_{A^{(r)}}(A, M^r) \simeq \mathbf{Mod}_{A^{(0)}}(A, M^0) \simeq M^0$ in $\mathbf{Mod}D_0(A) = \mathbf{Mod}A$, and hence M^0 comes equipped with a structure of $\mathbf{Mod}D(A)$.

If $\mathcal{C}_R(A^\infty)$ denotes the category of R-A^∞-modules,

(5.9) **Proposition [H88, 5.2].** *There is a categorical equivalence*

$$\mathbf{Mod}D(A) \to \mathcal{C}_R(A^\infty) \quad \textit{via} \quad M \mapsto (M \otimes_{D_r(A)} A, \pi^r)_r$$

with quasi-inverse $(M^r)_r \mapsto \varinjlim_r \mathbf{Mod}_{A^{(r)}}(A, M^r)$, *where* $\pi^r : M \otimes_{D_r(A)} A \to M \otimes_{D_{r+1}(A)} A$ *is given by* $m \otimes a \mapsto m \otimes a$.

(5.10) The Cartier operators induce for each $r \in \mathbb{N}$ an isomorphism of $A^{(r)}$-modules

$$(1) \qquad \omega_{A^{(r)}} \to \mathbf{Mod}_{A^{(r+1)}}(A^{(r)}, \omega_{A^{(r+1)}}) \quad \text{via} \quad w \mapsto C_r(?w).$$

Locally, in the notation of (5.8), the isomorphism is given by

$$(2) \qquad (z^{(r)})^\alpha \mathrm{d}_{A^{(r)}} z^{(r)} \mapsto \xi_{(p-1,\dots,p-1)-\alpha}, \quad \alpha \in [0, p-1]^n,$$

where $\xi_\beta \in \mathbf{Mod}_{A^{(r+1)}}(A^{(r)}, \omega_{A^{(r+1)}})$ is given by $\xi_\beta((z^{(r)})^\alpha) = \delta_{\alpha\beta}\mathrm{d}_{A^{(r+1)}} z^{(r+1)}$ $\forall \alpha, \beta \in [0, p-1]^n$.

It now follows that $(\omega_{A^{(r)}}, C_r) \in \mathcal{C}_R(A^\infty)$, and hence by (5.9)

$$(3) \qquad \omega_A \text{ carries a structure of } \mathbf{Mod}D(A).$$

Locally again the structure is described as follows [H88, 5.5]. Let $\Phi_r \in \mathbf{Mod}A(\omega_A, \mathbf{Mod}_{A^{(r)}}(A, \omega_{A^{(r)}}))^\times$ induced by $C_0, \dots, C_{r-1}$ as in (5.8.2) to equip ω_A with the structure of $\mathbf{Mod}D_r(A)$. If $\mathrm{d}_A z = \mathrm{d}_A z_1 \wedge \cdots \wedge \mathrm{d}_A z_n$, then for each $\alpha \in [0, p^r - 1]^n$

$$(4) \qquad \Phi_r(\mathrm{d}_A z)(z^\alpha) = \delta_{\alpha,(p^r-1,\dots,p^r-1)}\mathrm{d}_{A^{(r)}} z^{(r)} = \partial_A^{(p^r-1,\dots,p^r-1)}(z^\alpha)\mathrm{d}_{A^{(r)}} z^{(r)}.$$

But $\partial_A^{(p^r-1,\dots,p^r-1)} = e^A_{0,(p^r-1,\dots,p^r-1)}$ in the notation of (3.8.3). In particular, $\partial_A^{(p^r-1,\dots,p^r-1)}$ is $A^{(r)}$-linear. It follows that

$$\Phi_r(\mathrm{d}_A z)(a) = \partial_A^{(p^r-1,\dots,p^r-1)}(a)\mathrm{d}_{A^{(r)}} z^{(r)} \quad \forall a \in A.$$

More generally, for each $a, b \in A$

$$
\begin{aligned}
(5) \qquad \Phi_r(b\mathrm{d}_A z)(a) &= \Phi_r(\mathrm{d}_A z)(ba) \quad \text{as } \Phi_r \text{ is } A\text{-linear} \\
&= \partial_A^{(p^r-1,\dots,p^r-1)}(ba)\mathrm{d}_{A^{(r)}} z^{(r)}
\end{aligned}
$$

Then for each $\alpha \in [0, p^r - 1]^n \setminus 0$

$$\begin{aligned}\Phi_r((b\mathrm{d}_A z) \cdot \partial_A^\alpha)(a) &= \Phi_r(b\mathrm{d}_A z)(\partial_A^\alpha a) \quad \text{by definition} \\ &= \partial_A^{(p^r-1,\ldots,p^r-1)}(b\partial_A^\alpha(a))\mathrm{d}_{A^{(r)}} z^{(r)} \\ &= (\partial_A^{(p^r-1,\ldots,p^r-1)} \circ [b, \partial_A^\alpha])(a)\mathrm{d}_{A^{(r)}} z^{(r)} \quad \text{as } \partial_A^{(p^r-1,\ldots,p^r-1)}\partial_A^\alpha = 0 \\ &= \Phi_r((\mathrm{d}_A z) \cdot [b, \partial_A^\alpha])(a),\end{aligned}$$

hence $(b\mathrm{d}_A z) \cdot \partial_A^\alpha = (\mathrm{d}_A z) \cdot [b, \partial_A^\alpha]$. Then

$$(b\mathrm{d}_A z) \cdot (a\partial_A^\alpha) = ((\mathrm{d}_A z) \cdot b) \cdot (a\partial_A^\alpha) = (\mathrm{d}_A z) \cdot (ba\partial_A^\alpha) = (\mathrm{d}_A z) \cdot [ba, \partial_A^\alpha].$$

It follows that the right $D_r(A)$-action on ω_A is given for each $a, b \in A$ and for $\alpha \in [0, p^r-1]^n$ by the inductive formula

(6) $$(a\mathrm{d}_A z) \cdot (b\partial_A^\alpha) = \begin{cases} ab\mathrm{d}_A z & \text{if } \alpha = 0 \\ (\mathrm{d}_A z) \cdot [ab, \partial_A^\alpha] & \text{otherwise.}\end{cases}$$

(5.11) Let $M \in D(A)\mathbf{Mod}$. If $(M_r, \pi_r) \in \mathcal{C}(A^\infty)$ corresponding to M in (3.6), then $(\omega_{A^{(r)}} \otimes_{A^{(r)}} M_r, \pi^r) \in \mathcal{C}_R(A^\infty)$ with π^r defined by the commutative diagram

$$\begin{array}{ccc} \omega_{A^{(r)}} \otimes_{A^{(r)}} M_r & \overset{\pi_r}{\dashrightarrow} & \omega_{A^{(r+1)}} \otimes_{A^{(r+1)}} M_{r+1} \\ \uparrow{\scriptstyle \omega_{A^{(r)}} \otimes_{A^{(r+1)}} \pi_{r+1}}\ \wr & & \uparrow{\scriptstyle C_r \otimes_{A^{(r+1)}} M_{r+1}} \\ \omega_{A^{(r)}} \otimes_{A^{(r)}} (A^{(r)} \otimes_{A^{(r+1)}} M_{r+1}) & \underset{\sim}{\longrightarrow} & \omega_{A^{(r)}} \otimes_{A^{(r+1)}} M_{r+1}. \end{array}$$

Hence $\omega_A \otimes_A M = \omega_{A^{(0)}} \otimes_{A^{(0)}} M_0$ comes equipped with a structure of $\mathbf{Mod}D(A)$. Locally, in the notation of (5.8), the structure is given by

(1) $$(\mathrm{d}_A z \otimes m) \cdot \delta = \mathrm{d}_A z \otimes \delta^* m, \quad m \in M, \delta \in D(A).$$

On the other hand, if $N \in \mathbf{Mod}D(A)$, let $(N^r, '\pi^r) \in \mathcal{C}_R(A^\infty)$ corresponding to N in (5.9). Then $(\mathbf{Mod}_{A^{(r)}}(\omega_{A^{(r)}}, N^r), \pi'_r) \in \mathcal{C}(A^\infty)$ with π'_r defined by the commutative diagram

$$\begin{array}{ccc} \mathbf{Mod}_{A^{(r)}}(\omega_{A^{(r)}}, N^r) & \overset{\pi'_r}{\dashrightarrow} & \mathbf{Mod}_{A^{(r-1)}}(\omega_{A^{(r-1)}}, N^{(r-1)}) \\ \downarrow{\scriptstyle \mathbf{Mod}_{A^{(r)}}(C_{r-1}, N^r)} & & \\ \mathbf{Mod}_{A^{(r)}}(\omega_{A^{(r-1)}}, N^r) & & \wr \downarrow{\scriptstyle \mathbf{Mod}_{A^{(r-1)}}(\omega_{A^{(r-1)}}, '\pi^{r-1})} \\ \wr\Big| & & \\ \mathbf{Mod}_{A^{(r)}}(A^{(r-1)} \otimes_{A^{(r-1)}} \omega_{A^{(r-1)}}, N^r) & \underset{\text{adjunction}}{\overset{\sim}{\longrightarrow}} & \mathbf{Mod}_{A^{(r-1)}}(\omega_{A^{(r-1)}}, \mathbf{Mod}_{A^{(r)}}(A^{(r-1)}, N^r)). \end{array}$$

Hence $\mathbf{Mod}A(\omega_A, N) = \mathbf{Mod}_{A^{(0)}}(\omega_{A^{(0)}}, N^0)$ comes equipped with a structure of $D(A)\mathbf{Mod}$ such that locally

(2) $$(\delta \cdot \xi)(\mathrm{d}_A z) = \xi(\mathrm{d}_A z)\delta^*, \quad \delta \in D(A), \xi \in \mathbf{Mod}A(\omega_A, N).$$

Putting these together one obtains

(5.12) **Theorem [H88, 6.1].** *There is a categorical equivalence*

$$D(A)\mathbf{Mod} \to \mathbf{Mod}D(A) \quad \textit{via} \quad M \mapsto \omega_A \otimes_A M$$

$$\begin{array}{ccc} D(A)\mathbf{Mod} & \dashrightarrow & \mathbf{Mod}D(A) \\ \sim\downarrow & \circlearrowleft & \downarrow\sim \\ \mathcal{C}(A^\infty) & \xrightarrow[\sim]{} & \mathcal{C}_R(A^\infty) \end{array}$$

with quasi-inverse $N \mapsto \mathbf{Mod}A(\omega_A, N)$.

(5.13) We are now ready to describe the structure of $D(A)\mathbf{Mod}D(B)$ on $D_{\theta\to}$. Note first that $D(A) \otimes_A \omega_A^\vee \simeq \mathbf{Mod}A(\omega_A, D(A))$ carries two compatible structures of $D(A)\mathbf{Mod}$, one given by the right regular action on $D(A)$ transferred by (5.12) and the other induced by the left regular action on $D(A)$: $\delta \cdot \xi = \delta \circ \xi$, $\delta \in D(A)$, $\xi \in \mathbf{Mod}A(\omega_A, D(A))$. Use the second left $D(A)$-module structure on $D(A) \otimes_A \omega_A^\vee$ to form the tensor product

$$D_{\theta\leftarrow} \otimes_{D(A)} (D(A) \otimes_A \omega_A^\vee) = B \otimes_A D(A) \otimes_{D(A)} (D(A) \otimes_A \omega_A^\vee) \simeq B \otimes_A (D(A) \otimes_A \omega_A^\vee),$$

that admits a structure of $(D(B) \otimes D(A))\mathbf{Mod}$ inheriting from $D_{\theta\leftarrow} \in D(B)\mathbf{Mod}D(A)$ and from the first left $D(A)$-module structure on $D(A) \otimes_A \omega_A^\vee$. Finally, the left $D(B)$-module structure is transferred to the right by (5.12) on $D_{\theta\to} = \omega_B \otimes_B \{B \otimes_A (D(A) \otimes_A \omega_A^\vee)\}$ to make $D_{\theta\to} \in D(A)\mathbf{Mod}D(B)$.

Likewise $D_{\theta,r\to}$ is equipped with a structure of $D_r(A)\mathbf{Mod}D_r(B)$ such that

(1) $$D_{\theta\to} \simeq \varinjlim_r D_{\theta,r\to} \quad \text{in } D(A)\mathbf{Mod}D(B).$$

One then defines the direct image functor

$$\int_\theta^0 : D(B)\mathbf{Mod} \to D(A)\mathbf{Mod} \quad \text{via} \quad M \mapsto D_{\theta\to} \otimes_{D(B)} M$$

and for each $r \in \mathbb{N}$

$$\int_{\theta,r}^0 : D_r(B)\mathbf{Mod} \to D_r(A)\mathbf{Mod} \quad \text{via} \quad M \mapsto D_{\theta,r\to} \otimes_{D_r(B)} M.$$

By (1)

(2) $$\int_\theta^0 \simeq \varinjlim_r \int_{\theta,r}^0 .$$

(5.14) If $\eta \in \mathbf{Alg}_{\mathfrak{k}}(B, C)$ with C smooth over $\mathfrak{k}$, one has in $D_r(A)\mathbf{Mod}D_r(C)$ for each $r \in \mathbb{N}$

$$\begin{aligned} D_{\theta,r\to} &\otimes_{D_r(B)} D_{\eta,r\to} \\ &= \{\omega_B \otimes_B (B \otimes_A (D_r(A) \otimes_A \omega_A^\vee))\} \otimes_{D_r(B)} \{\omega_C \otimes_C (C \otimes_B (D_r(B) \otimes_B \omega_B^\vee))\} \\ &\simeq \omega_C \otimes_C (C \otimes_A (D_r(A) \otimes_A \omega_A^\vee)) = D_{\eta\circ\theta,r\to}. \end{aligned}$$

$\because$ If $w_i \in \omega_B$ and $\xi_i \in \omega_B^\vee$ such that $\sum_i w_i \otimes \xi_i \mapsto \mathrm{id}_{\omega_B}$ under the natural bijection $\omega_B \otimes_B \omega_B^\vee \simeq \mathbf{Mod}_B(\omega_B, \omega_B)$, the map

$$w \otimes 1 \otimes \delta \otimes \xi \mapsto \sum_i w_i \otimes 1 \otimes \delta \otimes \xi \otimes w \otimes 1 \otimes 1 \otimes \xi_i$$

is locally a bijection from $\omega_C \otimes_C (C \otimes_A (D_r(A) \otimes_A \omega_A^\vee))$ to $\{\omega_B \otimes_B (B \otimes_A (D_r(A) \otimes_A \omega_A^\vee))\} \otimes_{D_r(B)} \{\omega_C \otimes_C (C \otimes_B (D_r(B) \otimes_B \omega_B^\vee))\}$.

Hence taking the direct limit yields

Proposition. (i) *For each* $r \in \mathbb{N}$

$$\int_{\theta,r}^0 \circ \int_{\eta,r}^0 \simeq \int_{\eta\circ\theta,r}^0 .$$

(ii) $\int_\theta^0 \circ \int_\eta^0 \simeq \int_{\eta\circ\theta}^0$.

(5.15) **Proposition [H88, 7.2].** (i) $D_{\theta,r\to}$ *is locally projective of finite type in* $\mathbf{Mod}D_r(B)$ *for each* $r \in \mathbb{N}$.

(ii) $D_{\theta\to}$ *is flat in* $\mathbf{Mod}D(B)$.

Proof. (ii) will follow from (i) by (5.13.1).

(i) If B admits a regular system of parametres, it is enough to show by (5.4) and (5.7) that $D_r(B)$ is projective in $\mathbf{Mod}D_r(B)$ under the right action $\delta_1 \cdot \delta_2 = \delta_2^* \delta_1$. But that right action coincides with the right regular action under the bijection $^* : D_r(B) \to D_r(B)$.

Then $D_{\theta,r\to}$ is flat in $\mathbf{Mod}D_r(B)$, in general, as $(B_b)^{(r)} \otimes_{B^{(r)}} D_{\theta,r\to} \otimes_{D_r(B)} ? \simeq \{(B_b)^{(r)} \otimes_{B^{(r)}} D_{\theta,r\to}\} \otimes_{D_r(B_b)} \{(B_b)^{(r)} \otimes_{B^{(r)}} ?\}$ is exact by above if B_b admits a regular system of parametres.

(5.16) Unlike in characteristic 0

Corollary. *The direct image functors* $\int_{\theta,r}^0$ *and* $\int_\theta^0$ *are exact.*

(5.17) **Lemma.** *Assume that* θ *is étale and that* A *and* B *both admit regular systems of parameters. Let* $M \in D(B)\mathbf{Mod}$.

(i) *If we regard* M *also as a left* $D_r(A)$*-module via* $D_r(\theta)$ *of (3.8),*

$$\int_{\theta,r}^0 M \simeq M \quad \text{in } D_r(A)\mathbf{Mod}.$$

(ii) *If we regard* M *also as in* $D(A)\mathbf{Mod}$ *via* $D(\theta)$, *then*

$$\int_\theta^0 M \simeq M \quad \text{in } D(A)\mathbf{Mod}.$$

*In particular, $M^{(r)}$ is the same whether M is regarded as $D(B)$-module or as $D(A)$-module. Hence also $\int_\theta^0 M \simeq A \otimes_{A^{(r)}} M^{(r)}$ in $D_r(A)$***Mod** *with $D_r(A)$ acting on the RHS by the formula $\delta \cdot (a \otimes m) = \delta(a) \otimes m$, $a \in A$, $m \in M^{(r)}$, $\delta \in D_r(A)$.*

Proof. (i) By the hypothesis one has bijections in $\mathbf{Mod}_{\mathfrak{k}}$

$$D_{\theta,r\to} \simeq B \otimes_A D_r(A) \simeq D_r(B)$$

under which the $D_r(A)\mathbf{Mod}D_r(B)$-structure is transferred onto $D_r(B)$ such that

(1) $$\delta_A \cdot \delta \cdot \delta_B = \delta_B^* \circ \delta \circ (D_r(\theta)(\delta_A))^*, \quad \delta_A \in D_r(A), \delta, \delta_B \in D_r(B).$$

For if $\delta' \in \mathbf{Mod}_{A^{(r)}}(A, B)$ corresponds to $\delta \in D_r(B)$ under the bijection

$$\mathbf{Mod}_{A^{(r)}}(A, B) \simeq B \otimes_A D_r(A) \xrightarrow{B \otimes_A D_r(\theta)} D_r(B),$$

then

$$\begin{aligned} a\partial_A^\alpha \cdot \delta \cdot \delta_B &= \delta_B^* \circ \delta' \circ (a\partial_A^\alpha)^* \quad \text{in } \mathbf{Mod}_{A^{(r)}}(A, B) \\ &= \delta_B^* \circ \delta' \circ (-1)^{|\alpha|}\partial_A^\alpha a \\ &= \delta_B^* \circ \delta \circ (-1)^{|\alpha|}\partial_B^\alpha \theta(a) \quad \text{in } D_r(B) \\ &= \delta_B^* \circ \delta \circ (\theta(a)\partial_B^\alpha)^* = \delta_B^* \circ \delta \circ (D_r(\theta)(a\partial_A^\alpha))^*. \end{aligned}$$

Hence one obtains bijections in $\mathbf{Mod}_{\mathfrak{k}}$

$$\int_{\theta,r}^0 M = D_{\theta,r\to} \otimes_{D_r(B)} M \simeq D_r(B) \otimes_{D_r(B)} M \simeq M$$

such that $D_r(B) \otimes_{D_r(B)} M \ni \delta_B^* \otimes m = (1 \cdot \delta_B) \otimes m = 1 \otimes \delta_B m$ is mapped to $\delta_B m \in M$, that tansferres the structure of $D_r(A)\mathbf{Mod}$ on $\int_{\theta,r}^0$ onto M such that $\delta_A \cdot m = (D_r(\theta)(\delta_A))m$.

$$\begin{array}{ccccc} M & & m & \dashrightarrow & (D_r(\theta)(\delta_A))m \\ \sim\downarrow & & \downarrow & & \uparrow \\ D_r(B) \otimes_{D_r(B)} M & & 1 \otimes m & \xrightarrow[\delta_A]{} & (D_r(\theta)(\delta_A))^* \otimes m. \end{array}$$

The first assertion of (ii) follows from (i) by taking the direct limit. If σ_A is an r-th Frobenius splitting on A, $D_r(\theta)(\sigma_A) = 1 \otimes \sigma_A$ is an r-th Frobenius splitting on B. Hence

$$\begin{aligned} M^{(r)} &= D_r(\theta)(\sigma_A)M \quad \text{if } M \text{ is regarded as } D(A)\text{-module} \\ &= M^{(r)} \quad \text{if } M \text{ is regarded as } D(B)\text{-module}. \end{aligned}$$

Then by Cartier-Chase $\int_{\theta\to} M \simeq A \otimes_{A^{(r)}} M^{(r)}$ in $D_r(A)\mathbf{Mod}$ with $D_r(A)$ acting on the RHS by $\delta \cdot (a \otimes m) = \delta(a) \otimes m$.

(5.18) Let A be a smooth $\mathfrak{k}$-algebra with a finite open cover $(A_{a_i})_i$ such that each A_{a_i} is étale over $\mathfrak{k}[x]$ under structure homomorphism $\theta_i : \mathfrak{k}[x] \to A_{a_i}$. If $\mathrm{i}_j : A \hookrightarrow A_{a_j}$, then

$$\int_{\mathrm{i}_j}^0 M \simeq A_{a_j} \otimes_A M \quad \text{in } D(A_{a_j})\mathbf{Mod}$$

with $D(A_{a_j}) \simeq A_{a_j} \otimes_A D(A)$ acting naturally on the RHS. Bøgvad defines in [Bø, 2.5.1] that M be holonomic on A iff each $\int^0_{i_j \circ \theta_j} M$ is holonomic in $D(\mathfrak{k}[x])$**Mod**, and checks that the definition is independent of the choice of the étale coverings of A.

Moreover, he shows

(5.19) **Theorem [Bø, Prop. 6.1 and Prop. 6.2].** *The direct and the inverse image functors $\int^0_\theta$ and $^\theta?$ preserve holonomicity.*

(5.20) **Remark.** Assume for simplicity that A and B both admit regular systems of parametres. Then $\int^0_\theta D(B) \simeq \{B \otimes_A D(A)\} \otimes_{D(B)} D(B) \simeq B \otimes_A D(A)$ is NOT of finite type in $D(A)$**Mod** if B is not of finite type over A. This also shows that the holonomicity is a strong finiteness condition.

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Errata. We take this opportunity to correct some innacuracies in [K] :

p. 103 line -7 There was no need of introducing "left" and "right" A-rings. An A ring is just a ring C together with a ring homomorphism $A \to C$. Accordingly, "both $\mathfrak{X}$**Rng** and **Rng**$\mathfrak{X}$" in (1.2) should be read as simply "in $\mathfrak{X}$**Rng**".

(1.1.1) $A \otimes \Omega^1_{\mathfrak{k}[\mathbf{A}^N]/\mathfrak{k}}$ should be read as $A \otimes_{\mathfrak{k}[\mathbf{A}^N]/\mathfrak{k}} \Omega^1_{\mathfrak{k}[\mathbf{A}^N]/\mathfrak{k}}$

p. 109 line 1 $\mathcal{D}_{\mathfrak{Y},r}$ should be read as $\mathcal{D}_{\mathfrak{X},r}$

p. 110 line -3 "finite open" should be read as "finite affine open".

(2.7) Proposition. $B = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$ should be read as $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$. Assume in the proposition that C admits a regular system of parameters.

(2.7.2) $$\partial^{\mathbf{t}}_C = \sum_{\mathbf{n} \in [0, p^r - 1]^N} \binom{\mathbf{n}}{\mathbf{t}} e^C_{\mathbf{n}-\mathbf{t},\mathbf{n}}$$

(2.7.3) The middle term should be replaced by $\delta^*_A \circ \delta \circ (-1)^{|\mathbf{n}|} \partial^{\mathbf{n}}_A \mathfrak{f}^o(c)$

(2.9) Proposition. "$\mathfrak{f}^+$ is right adjoint to $\int^0_{\mathfrak{f}} |_{\mathcal{D}^{\mathfrak{X}}_{\mathfrak{Y}}\mathrm{qc}}$" should be replaced by "$\mathfrak{f}^+|_{\mathcal{D}^{\mathfrak{X}}_{\mathfrak{Y}}\mathrm{qc}}$ is right adjoint to $\int^0_{\mathfrak{f}}$".

(2.9)–(2.13) (2.7 should be read as (2.8.

(3.1) (5) and (6) should be read as (1) and (2), respectively.

On the Sequences Relevant of the Relation Type

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Introduction.

Let A be a commutative ring and $a_1,\dots,a_n$ a sequence of elements of A. We denote the ideal $(a_1,\dots,a_n)$ by I.

Consider the following conditions on the sequences $a_1,\dots,a_n$ (p denotes a positive integer).

Definition 1.([1])

(★) $[(a_1,\dots,a_{i-1})I^{p-1} : a_i] \cap I^p = (a_1,\dots,a_{i-1})I^{p-1}$ for $1 \leqq \forall i \leqq n$.

(★★) $[(a_1,\dots,a_{i-1})I^{p} : a_i] \cap I^p = (a_1,\dots,a_{i-1})I^{p-1}$ for $1 \leqq \forall i \leqq n$.

We note the condition (★) is equivalent to the following condition essentially based on Kühl[3].

$[(a_1,\dots,a_{i-1}) : a_i] \cap I^p = (a_1,\dots,a_{i-1})I^{p-1}$ for $1 \leqq \forall i \leqq n$.

To explain some known results, we will need the following definitions

Definition 2.([2]) Elements $a_1,\dots,a_n$ are said to form a d-sequence in A if for each integer i=1,...,n and each integer k=i,...,n.

$$(a_1,\dots,a_{i-1}) : a_i a_k = (a_1,\dots,a_{i-1})$$

Definition 3.([1]) Let R(I) be the Rees algebra of I. Then, there exists a canonical surjection $\phi : B = A[X_1,\dots,X_n] \longrightarrow R(I)$ with $\phi(X_i) = a_i t$. We put $K = \ker\ \phi$. Let K_j denote the ideal of B generated by $f \in K$ of degree at most j. Let r be the least positive integer $K_r = K$.

The invariant r is called the relation type of the ideal I. An ideal of relation type 1 is said to be of linear type.

Costa [1] Proved the following;

(1) The condition (★) holds for i=1,...,n and for p=1 if and only if $a_1,\dots,a_n$ is a d-sequence.

(2) If (★) holds for all $p \geqq 1$, then I is of linear type.

(3) If (★★) holds for all $p \geqq 1$, then I is of linear type.

Raghavan [4] generalized the above results to the following;

(1)' If (★) holds for p=m, then it holds for all $p \geqq m$.

(2)' If (★) holds for p=m, then the relation type of $I \leqq m$.

(3)' If (★★) holds for p=m, then every relation of degree (m+1) on $a_1,\dots,a_n$ belongs to the ideal K_m.

In this paper we will consider the following question.

Question .

① What is the difference between the condition (★) and (★★) ?

② What can one say concerning the relationship between these sequences and d-sequences?

Concerning to the question ①, Raghavan gives the following example;

Example([4]). Let A be the subring $k[x,xy,y^3,y^4]$ of the polynomial ring k[x,y] over a field k. Consider the sequence y^6,x in A. Then we can see

(1) x,y^6 is a d-sequence in A,

(2) y^6,x is not a d-sequence in A,

(3) (★★) holds for y^6,x and for all $p \geqq 1$,

and

(4) (★) does not hold for y^6,x and for any $p \geqq 1$.

(We remark that the sequence x,y^4 in the example of Raghavan[4] does not satisfy the above situation and that the method of Raghavan's proof is incorrect.)

This example leads us to consider the following

Problem. Suppose that a,b a d-sequence. If (★) holds for b,a and for some $p \geqq 1$, is the sequence b,a a d-sequence? More generally, suppose that $a_1,\ldots,a_n$ is a d-sequence. Let $b_1,\ldots,b_n$ be a permutation of the sequence $a_1,\ldots,a_n$. If (★) holds for this sequence $b_1,\ldots,b_n$ and for some $p \geqq 1$, is $b_1,\ldots,b_n$ a d-sequence?

1. An answer to the problem.

In this section, we will give an answer to the above problem concerning d-sequence. Our result is

Theorem 1.1. Let $a_1,\ldots,a_n$ be a d-sequence. Suppose that $a_{\sigma(1)},\ldots,a_{\sigma(n)}$ satisfies (★) for all permutation σ and for some $p \geqq 1$, then $a_{\sigma(1)},\ldots,a_{\sigma(n)}$ is a d-sequence.

(We note that we can take the above p by virtue of the result (1)' of Raghavan)

Proof. To prove the assertion, we may assume $\sigma=(i,i+1)$. Moreover, we may assume that i = 1. Hence, we have only to show that $a_2,a_1,a_3,\ldots,a_n$ satisfies (★) for some p, then $a_2,a_1,a_3,\ldots,a_n$ is a d-sequence. For this purpose, we prove

Claim. ① $(0):a_2 \cap I = (0)$

② $(a_2):a_1 \cap I = (a_2)$.

①: Let x be an element of $(0):a_2 \cap I$. Then we have $xa^{p-1} \in (0):a_2 \cap I^p = (0)$ from (★). This implies $x \in (0):a_1 \cap I = (0)$ as $a_1,\ldots,a_n$ is a d-sequence.

②: Let y be an element of $(a_2):a_1 \cap I$. Then , we have $ya^{p-1} \in (a_2)I^{p-1}:a_1 \cap I^p = (a_2)I^{p-1}$ from (★). thus we can express $ya^{p-1} = za_2$ for some $z \in I^{p-1}$. As $a_1,\ldots,a_n$ is a d-sequence, we have $z \in (a_1) \cap$

$I^{p-1}=(a_1)I^{p-2}$. Therefore, $z = a_1z_1$ for some $z_1 \in I^{p-2}$, which implies $ya^{p-2} = a_2z_1$. Repeating this method, we see $y \in (a_2)$.

Corollary1.2. Suppose that a,b is a d-sequence and that (★) holds for b,a and for some $p \geqq 1$, then b,a is a d-sequence.

Now, assume that $a_1,\ldots,a_n$ is a d-sequence and that $a_{\sigma(1)},\ldots,a_{\sigma(n)}$ satisfies (★) for some permutation σ and for some $p \geqq 1$. Then we have the followings;

Lemma 1.3 $(a_{\sigma(1)},\ldots,a_{\sigma(n-1)}) : a_{\sigma(n)} \cap I = (a_{\sigma(1)},\ldots,a_{\sigma(n-1)})$.

Proposition 1.4. If $n \leqq 4$, $a_{\sigma(1)},a_{\sigma(2)}, a_{\sigma(3)},a_{\sigma(4)}$ is a d-sequence.

Proof. Put $b_i = a_{\sigma(i)}$ for $i=1,\ldots,4$. Using the same method as the one in the claim of Theorem 1.1and from Lemma 1.3, we have only to prove the followings.

Claim. $(b_1,b_2)\, I^{k-1} : b_3 \cap I^k = (b_1,b_2)I^{k-1}$ for $1 \leqq k \leqq p$.

Proof. We prove this assertion by the descending induction on k. However, the proof of this assertion is not easy. Thus, we omit the details.

We close this section with the following problem.

Problem. Suppose that $a_1,\ldots,a_n$ is a d-sequence. Let $b_1,\ldots,b_r$ $(r \leqq n)$ be a sequence of elements contained in I. Suppose that (★) holds for this sequence and for some $p \geqq 1$. Then, is $b_1,\ldots,b_r$ a d-sequence?

2. On the filter-regular sequence in G(I).

In this section, we will explain the relationship between the conditions (★),(★★) and the filter-regular sequences.

We begin with

Definition2.1. Let $S=\bigoplus_{i \geqq 0} S_i$ be a standard graded ring over a commutative ring S_0. For any graded ideal L in S, we denote by L_i the homogeneous part of degree i of L. Let $z_1,\dots,z_n$ be a sequence of homogeneous elements of S. We call $z_1,\dots,z_n$ a filter-regular sequence if $[(z_1,\dots,z_j):z_{j+1}]_i = (z_1,\dots,z_j)_i$ for $j=1,\dots,n-1$ and for all large i.

The following two results follows from the definition.

Lemma 2.2. Let $f_1=a_1t,\dots,f_n=a_nt$ in R(I). T.F.A.E.

(1) $f_1,\dots,f_n$ is a filter-regular sequence.

(2) $a_1,\dots,a_n$ satisfies (★★) for all large p.

Lemma 2.3. Let $g_1=\overline{a_1t},\dots,g_n=\overline{a_nt}$ in G(I). T.F.A.E.

(1) $g_1,\dots,g_n$ is a filter-regular sequence.

(2) $\{[(a_1,\dots,a_{i-1})I^p+I^{p+2}]:a_i\}\cap I^p = (a_1,\dots,a_{i-1})I^{p-1}+I^p$ for $i=1,\dots,n$ and for all large p.

Proposition 2.4. If $a_1,\dots,a_n$ satisfies (★) for some p, then $g_1,\dots,g_n$ is a filter-regular sequence in G(I).

Proof. Let x be an element of $\{[(a_1,\dots,a_{i-1})I^p+I^{p+2}]:a_i\}\cap I^p$ and express $xa_i=\sum_{j=1}^{i-1}x_ja_j+y$ for some $x_j\in I^p$ and $y\in I^{p+2}$. Set $y=z+z_ia_i+\dots+z_na_n$ for some $z\in(a_1,\dots,a_{i-1})I^{p+1}$ and $z_j\in I^{p+1}$. Then we have $z_n\in (a_1,\dots,a_{n-1})I^p:a_n\cap I^{p+1}=(a_1,\dots,a_{n-1})I^p$,since the condition (★) for i=n. Hence, we can express $y=z'+z'_ia_i+\dots+z'_{n-1}a_{n-1}$ for some $z'\in (a_1,\dots,a_{i-1})I^p$ and $z'_j\in I^{p+1}$. Repeating this method, we have $x\in (a_1,\dots,a_{i-1})I^{p-1}+I^p$.

In general, the converse of Proposition 2.4 dose not hold. However, we have following:

Proposition 2.5. For a fixed integer m, the following conditions are equivalent.

(1) $a_1,\dots,a_n$ satisfies (★) for p = m.

(2) $a_1,\dots,a_n$ satisfies (★★) for all $p\geqq m$ and $g_1,\dots,g_n$ is a filter-regular sequence in G(I).

The following proposition gives a characterization of the condition (★★).

Proposition 2.6. For a fixed integer m, the following conditions are equivalent.

(1) $a_1,\dots,a_n$ satisfies (★★) for all $p \geqq m$.

(2) $f_1=a_1t,\dots,f_n=a_nt$ satisfies (★) for $p = m$ in R(I).

Combining these results, we have

Theorem 2.7. For a fixed integer m, the following conditions are equivalent.

(1) $a_1,\dots,a_n$ satisfies (★) for $p = m$.

(2) $f_1=a_1t,\dots,f_n=a_nt$ satisfies (★) for $p = m$ in R(I) and $g_1=\overline{a_1t},\dots,g_n=\overline{a_nt}$ is a filter-regular sequence in G(I).

Finally, we have the following corollary without the noetherian assumption.

Corollary 2.8.([5.Cor.5.7]) Let A be a commutative ring and $a_1,\dots,a_n$ a sequence of elements of A. Then, the following conditions are equivalent.

(1) $a_1,\dots,a_n$ is a d-sequence.

(2) $f_1=a_1t,\dots,f_n=a_nt$ is a d-sequence and $g_1=\overline{a_1t},\dots,g_n=\overline{a_nt}$ is a filter-regular sequence.

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On the Gorensteinness in graded rings associated to ideals

Shiro Goto and Shin-ichiro Iai

1 Introduction

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Assume that A is a homomorphic image of a Gorenstein ring and the field $A/\mathfrak{m}$ is infinite. For an ideal I in A we define

$$\mathcal{R}'(I) = \textstyle\sum_{i \in \mathbb{Z}} I^i t^i \quad \text{and}$$
$$\mathcal{G}(I) = \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I)$$

which we call the extended Rees algebra and the associated graded ring, respectively. The purpose of our paper is to study the conditions for certain class of ideals under which associated graded rings are Gorenstein rings. To state our result precisely, we set up the following notation. Let $I(\neq A)$ be an ideals in A of height s and let J be a minimal reduction of I, hence $J \subseteq I$ and $I^{n+1} = JI^n$ for some $n \geq 0$. We put $r_J(I) = min\{n \geq 0 | I^{n+1} = JI^n\}$ and call it the reduction number of I with respect to J. Let $\lambda(I)$ denote the analytic spread of I, that is, $\lambda(I) := \dim A/\mathfrak{m} \otimes \mathcal{G}(I)$. We put $l = \lambda(I)$. Then as is well-known, J is minimally generated by l-elements ([NR]). Let $J = (a_1, \ldots, a_l)$ and $K = (a_1, \ldots, a_s)$. Let $\bigcap_{Q \in Ass_A A/I} I(Q)$ denote a primary decomposition of I and let $U = U(I) := \bigcap_{Q \in Assh_A A/I} I(Q)$ where $Assh_A A/I := \{Q \in \mathrm{V}(I) | \dim A/I = \dim A/Q\}$. We put $R' = \mathcal{R}'(I)$ and $G = \mathcal{G}(I)$. Let K_A, $K_{R'}$, and K_G denote respectively the canonical module of A, R', and G, and let $a = a(G)$ denote the a-invariant of G (see [GW](3.1.4) for definition). Our purpose is to prove the following.

Theorem 1.1. *Assume that $l = s + 1$ and $I_Q = K_Q$ for all $Q \in Assh_A A/I$. Then the following two conditions are equivalent.*

(1) G is a Gorenstein ring.

(2) (i) A is a Gorenstein ring.

(ii) depth $A/I \geq d - s - 1$.

(iii) $r_J(I) \leq 1$.

(iv) $I = (JU + K) :_U I$.

If I is unmixed, then the condition (iv) in (2) is naturally satisfied. Hence we get

Corollary 1.2. *Let A be a Gorenstein ring. Suppose that I is an unmixed ideal with $l = s+1$ and $I_Q = K_Q$ for all $Q \in Assh_A A/I$. Then G is a Gorenstein ring if and only if depth $A/I \geq d - s - 1$ and $r_J(I) \leq 1$.*

In [HH] the analytic deviation of I was defined to be $ad(I) = l - s$. Ideals having analytic deviation 0 are called equimultiple. For equimultiple ideals Gorenstein properties of G have been closely studied. Ideals satisfying the condition in this theorem are located in one of classes to be explored next to the equimultiple ideals. If $d = 1$, above theorem is directly covered by a theorem given by [GNa] (1.1). Our result is generalization of this theorem. And in the section 2 we will give an another proof of the case of $d = 1$, which is based on the structure of the I-filtration $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ of K_A with $\mathcal{R}'(\omega) = \sum_{i\in\mathbb{Z}} \omega_i t^i \cong K_{\mathcal{R}'}$ ([HSV]). We shall prove Theorem 1.1 in section 3.

2 Preliminaries

The goal in this section is to prove our Theorem 1.1 in the case of $d = 1$. We begin with the following structure theorem of the canonical module of R' due to [HSV].

Proposition 2.1. *Suppose that G is a Cohen-Macaulay ring. Then there exists a unique I-filtration $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ of K_A such that*
(1) $\omega_i = K_A$ for $i \ll 0$ and
(2) There is an isomorphism $K_{R'} \cong \sum_{i\in\mathbb{Z}} \omega_i t^i$ of graded R'-modules.
When this is the case, we have $K_G \cong \bigoplus_{i\geq -a} \omega_{i-1}/\omega_i$ as graded G-modules.

Proof. From the canonical isomorphism $[K_{R'}]_{t^{-1}} \cong K_A[t, t^{-1}]$, we get the injective homomorphism

$$\sigma : K_{R'} \to K_A[t, t^{-1}]$$

of graded R'-modules. We set $\omega_i = \{x \in K_A | xt^i \in \sigma([K_{R'}]_i)\}$, then it is easy to check that $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ satisfies the condition (1), (2), and the last assertion. We shall prove uniqueness. From the condition (1), the homomorphism $\rho : K_{R'} \to K_A[t, t^{-1}]$ induced by the condition (2) is localization of $K_{R'}$ with respect to t^{-1}, and σ is so. Then we have a commutative diagram

$$\begin{array}{ccc} K_A & \xrightarrow{\sigma} & K_A[t, t^{-1}] \\ \| & & \downarrow \wr \\ K_A & \xrightarrow{\rho} & K_A[t, t^{-1}]. \end{array}$$

Moreover a graded autmorphism of $K_A[t, t^{-1}]$ is a multiplication of an unit in A. Hence we get $Im\,\sigma = Im\,\rho$. This proves uniqueness. □

The above I-filtration ω is called the canonical I-filtration of K_A. In what follows we assume that $l = s + 1$ and $I_Q = K_Q$ for all $Q \in Assh_A A/I$. Then we note the following results due to [GNN].

Lemma 2.2 ([GNN] (2.8),(2.9)). *Suppose G is a Gorenstein ring. Then $r_J(I) \leq 1$ and $a = -s$.*

To prove our theorem we receive suggestions from the structure of the canonical I-filtration $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ of K_A described as follows.

Proposition 2.3. *Suppose A is a Gorenstein ring of $d = 1$. Let $s = 0$ and $l = 1$. Assume that $a = 0$ and $r_J(I) \leq 1$. Then*

(1) $\omega_0 = \begin{cases} JU :_U I, & if \quad r_J(I) = 1 \\ U, & if \quad r_J(I) = 0 \end{cases}$;

(2) $\omega_i = I w_{i-1}$ *for all* $i \geq 1$.

Proof. We put $\mathfrak{a} = 0 :_A I$, $\bar{A} = A/\mathfrak{a}$, and $T = \mathcal{G}(I\bar{A})$. Recall that G and T are Cohen-Macaulay ([GNa](2.5)). Let $\varphi : G \to T$ denote the canonical epimorphism of the associated graded rings. Then by [GNa](2.3) we have an exact sequence $0 \to \mathfrak{a} \to G \xrightarrow{\varphi} T \to 0$ of graded G-modules. Taking the K_G-dual of this, we get an exact sequence

$$0 \to K_T \to K_G \to A/U \to 0$$

of graded G-modules. Hence by $a = 0$ we have $a(T) = -1$ or 0. Let $\bar{T} = T/a_1tT$. Recall that a_1t is T-regular [[GNa] (3.2)]. And it is easy to check $a(\bar{T}) = 0$ if and only if $r_J(I) = 0$. Therefore we have

$$a(\bar{T}) = \begin{cases} 0 & (r_J(I) = 0) \\ 1 & (r_J(I) = 1) \end{cases}$$

as $a(\bar{T}) = a(T) + 1$. Let $b = a(\bar{T})$. We may assume that $K_{\bar{A}} = U$ and $K_{\bar{A}/a_1\bar{A}} = U/a_1U$. Let $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$, $\eta = \{\eta_i\}_{i\in\mathbb{Z}}$, and $\theta = \{\theta_i\}_{i\in\mathbb{Z}}$ denote the canonical I, $I\bar{A}$, and $I\bar{A}/a_1\bar{A}$-filtration of A, U, and U/a_1U respectively. Then because $I\bar{A}/a_1\bar{A}$ is an equimultiple ideal, we can compute the canonical filtration $\theta = \{\theta_i\}_{i\in\mathbb{Z}}$ as follows (see [I]).

$$\begin{array}{ll} \vdots & \\ \theta_{-b-2} & = U/a_1U \\ \theta_{-b-1} & = U/a_1U \\ \theta_{-b} & = (0) :_{U/a_1U} (I\bar{A}/a_1\bar{A})^b = \begin{cases} JU :_U I/a_1U & (r_J(I) = 1) \\ (0) & (r_J(I) = 0) \end{cases} \\ \theta_{-b+1} & = (0) :_{U/a_1U} (I\bar{A}/a_1\bar{A})^{b-1} = (0) \\ \vdots & \end{array}$$

On the other hand, notice that a_1t is also $\mathcal{R}'(I\bar{A})$-regular, and we have an isomorphism $\mathcal{R}'(I\bar{A}/a_1\bar{A}) \cong \mathcal{R}'(I\bar{A})/a_1t\mathcal{R}'(I\bar{A})$. We put $S = \mathcal{R}'(I\bar{A})$ and $\bar{S} = \mathcal{R}'(I\bar{A}/a_1\bar{A})$. Then we have an exact sequence

$$0 \to S(-1) \xrightarrow{a_1t} S \to \bar{S} \to 0$$

of graded S-modules. Take the S-dual of this sequence. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_S & \xrightarrow{a_1t} & K_S(1) & \xrightarrow{f} & K_{\bar{S}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle inclusion} & & \downarrow{\scriptstyle inclusion} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & U[t,t^{-1}] & \xrightarrow{a_1t} & U[t,t^{-1}](1) & \xrightarrow{canonical} & \frac{U[t,t^{-1}]}{a_1tU[t,t^{-1}]}(1) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \wr\ t^{-1}} & & \downarrow{\scriptstyle \wr\ \psi} & & \\ 0 & \longrightarrow & U[t,t^{-1}] & \xrightarrow{a_1} & U[t,t^{-1}] & \xrightarrow{canonical} & \frac{U}{a_1U}[t,t^{-1}] & \longrightarrow & 0 \end{array}$$

, then the composite $\psi \circ \phi$ is the localization of $K_{\bar{S}}$ with respect to t^{-1}. Therefore we get

$$\theta_i = \frac{\eta_{i+1} + a_1 U}{a_1 U} \quad \text{for all } i \in \mathbb{Z}.$$

Hence putting $c = a(T)$ we have

$$\begin{aligned}
&\vdots \\
\eta_{-c-2} &= U \\
\eta_{-c-1} &= U \\
\eta_{-c} &= \begin{cases} JU :_U I/a_1 U & (r_J(I) = 1) \\ IU & (r_J(I) = 0) \end{cases}
\end{aligned}$$

We furthermore have

Claim 1. $\eta_i = I\eta_{i-1}$ for all $i \geq -c+1$.

Proof. Let $\mathfrak{M} = \mathfrak{m}T + T_+$, the unique graded maximal ideal in T. and apply the local cohomology functors $H^i_{\mathfrak{M}}(*)(i \in \mathbb{Z})$ to an exact sequence

$$0 \to T(-1) \xrightarrow{a_1 t} T \to \bar{T} \to 0$$

of graded T-modules. Then we have

$$0 \to H^0_{\mathfrak{M}}(\bar{T}) \to H^1_{\mathfrak{M}}(T) \xrightarrow{a_1 t} H^1_{\mathfrak{M}}(T).$$

And as $a_1 tT \subseteq \mathfrak{M}$, applying the functor $Hom_T(T/\mathfrak{M}, *)$ to the above sequence, we get the isomorphism $Soc\, H^0_{\mathfrak{M}}(\bar{T}) \cong [Soc\, H^1_{\mathfrak{M}}(T)](-1)$. As $I\bar{A}/a_1\bar{A}$ is an equimultiple ideal and as $r_J(I) \leq 1$, then $Soc\, H^0_{\mathfrak{M}}(\bar{T})$ is concentrated in degree b or degree b and degree $b-1$ (see, e.g., [I]). So $Soc\, H^1_{\mathfrak{M}}(T)$ is concentrated in degree c or degree c and degree $c-1$. Thus K_T is generated by elements of degree $-c$ or degree $-c$ and degree $-c+1$. This completes the proof of the claim 1.

Claim 2. $\eta_i \subseteq \omega_i$ for all $i \in \mathbb{Z}$

Proof. We consider an exact sequence

$$0 \to \mathfrak{a}^* \to R' \to S \to 0$$

of graded R'-modules where $\mathfrak{a}^* = \mathfrak{a}A[t, t^{-1}] \cap R'$. Taking the K_G-dual of this short exact sequence, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_S & \longrightarrow & K_{R'} & \longrightarrow & Hom_{R'}(\mathfrak{a}^*. K_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U[t, t^{-1}] & \xrightarrow{f} & A[t, t^{-1}] & \xrightarrow{g} & \frac{A}{U}[t, t^{-1}] & \longrightarrow & 0
\end{array}$$

where file maps are the localizations with respect to t^{-1}, and the bottom row is what the top row induces. Then f is a multiplication of element in A which is an unit in $A/\mathfrak{a}$. Hence the assertion follows.

We will show that $\eta_i = \omega_i$ for all $i \geq 0$. To do this, look at the homogeneous components

$$\begin{array}{l} 0 \to U/\eta_0 \to A/\omega_0 \to A/U \to 0 \\ 0 \to \eta_0/\eta_1 \to \omega_0/\omega_1 \to 0 \\ 0 \to \eta_1/\eta_2 \to \omega_1/\omega_2 \to 0 \\ \vdots \end{array}$$

in the exact sequence $0 \to K_T \to K_G \to A/U \to 0$. So it is easy to check $\omega_0 \subseteq U$. and by induction i we know that $\eta_i = \omega_i$ for all $i \geq 0$. This complete the proof of Prop 2.3 □

As consequence of Prop 2.3, we have

Theorem 2.4 ([GNa] (1.1)). *Let $d = 1$, $l = 1$, and $s = 0$. Assume that $I_Q = (0)$ for all $Q \in Assh_A A/I$. Then the following two conditions are equivalent.*

(1) G is a Gorenstein ring.

(2) (i) A is a Gorenstein ring.

(ii) $r_J(I) \leq 1$.

(iii) $I = JU :_U I$.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. To do this. we need the following theorem due to [I].

Theorem 3.1. *Assume that A is a Gorenstein local ring. Let $\mathcal{A} = \{\mathfrak{p} \in \mathrm{V}(I) \mid ht_A \mathfrak{p} = \dim G_\mathfrak{p}/\mathfrak{p}G_\mathfrak{p}\}$. Then $\mathcal{A}$ is a nonempty finite set and the following three conditions are equivalent.*

(a) G is a Gorenstein ring.

(b) There is an injective homomorphism $G(a) \hookrightarrow K_G$ of graded G-modules.

(c) $G_\mathfrak{p}$ is a Gorenstein ring with $a = a(G_\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{A}$.

Moreover our proof of Theorem 1.1 is based on the reduction to the case where the base ring A have dimension l. For that, the following corollary of the Theorem 2.4 is one of our key.

Corollary 3.2. *Let $d = l = s + 1$. Assume that $I_Q = (0)$ for all $Q \in Assh_A A/I$. Then the following two conditions are equivalent.*

(1) G is a Gorenstein ring.

(2) (i) A is a Gorenstein ring.

(ii) $r_J(I) \leq 1$.

(iii) $I = (JU + K) :_U I$.

We are now ready to prove Theorem 1.1.

Proof. To begin with we note that if the condition (ii), (iii), $l = s+1$, and $I_Q = K_Q$ for all $Q \in Assh_A A/I$, then G is a Cohen-Macaulay ring (see, e.g., [GNN]). Hence we may assume the condition (i), (iii), and G is a Cohen-Macaulay ring with $a = -s$ (see [B] and [GH]). And then it is enough to show (1) $\iff$ *(iv)*. First we assume the condition (1). Then we have

Claim 3. $r_{J_Q}(I_Q) = 1$ *for all* $Q \in \mathcal{A}$ *with* $I_Q \neq U_Q$.

Proof. We take any $Q \in \mathcal{A}$ with $I_Q \neq U_Q$, so that $ht_A Q = s+1$ and $\lambda(I_Q) = s+1$. Therefore J_Q is a minimal reduction of I_Q. If $ht_{A_Q} I_Q = s$, then by $U_Q = U(I_Q)$ the assertion is true. If $ht_{A_Q} I_Q = s+1$, then by I_Q is $\mathfrak{m}$-primary ideal we get $-s = a(G_Q) = r_{J_Q}(I_Q) - \lambda(I_Q)$. Hence $r_{J_Q}(I_Q) = 1$. □

Let us take any $Q \in Ass_A A/I$. It is enough to show $I_Q = (J_Q U_Q + K_Q) :_{U_Q} I_Q$ since the inclusion $I \subseteq (JU+K) :_U I$ is obvious. If $ht_A Q = s$, then by $I_Q = U_Q$ it is trivial. Assume $ht_A Q = s+1$. Then $\lambda(I_Q) = s+1$ and J_Q is a minimal reduction of I_Q. If $ht_{A_Q} I_Q = s$, then by Corollary 3.2 we have $I_Q = (J_Q U(I_Q) + K_Q) :_{U(I_Q)} I_Q$. Since $U_Q = U(I_Q)$, then the equality is true. If $ht_{A_Q} I_Q = s+1$, then by claim 3 $r_{J_Q}(I_Q) = 1$. And I_Q is a $\mathfrak{m}$-primary ideal, so that by [O] $I_Q = J_Q :_{A_Q} I_Q = (J_Q U_Q + K_Q) :_{U_Q} I_Q$, as $U_Q = A_Q$. On the other hand assume that the condition (iv). Then it is easy to check the following claim.

Claim 4. $r_{J_Q}(I_Q) = 1$ *for all* $Q \in \mathcal{A}$ *with* $I_Q \neq U_Q$.

By Theorem 3.1 it is enough to show G_Q is a Gorenstein ring with $a(G_Q) = -s$ for all $Q \in \mathcal{A}$. If $ht_A Q = s$, then by $I_Q = K_Q$ the assertion is obvious. Assume $ht_A Q = s+1$. Then $\lambda(I_Q) = s+1$ and J_Q is a minimal reduction of I_Q. If $ht_{A_Q} I_Q = s$, then $(I_Q)_\mathfrak{p} = (K_Q)_\mathfrak{p}$ for all $\mathfrak{p} \in Assh_{A_Q} A_Q/I_Q$. Hence by [GH] we have $a(G_Q) = -s$. Furthermore as $U_Q = U(I_Q)$ we get $I_Q = (J_Q U(I_Q) + K_Q) :_{U(I_Q)} I_Q$. Thus by Corollary 3.2 G_Q is a Gorenstein ring. If $ht_{A_Q} I_Q = s+1$, then by claim 4 $r_{J_Q}(I_Q) = 1$. And I_Q is a $\mathfrak{m}$-primary ideal, so that $a(G_Q) = r_{J_Q}(I_Q) - \lambda(I_Q) = -s$. Moreover we have $I_Q = (J_Q U_Q + K_Q) :_{U_Q} I_Q = J_Q :_{A_Q} I_Q$, as $U_Q = A_Q$. Then by [O] G_Q is a Gorenstein ring. we have completed the proof of Theorem 1.1. □

We would like to close this paper with the following corollary.

Corollary 3.3. *Let* $P \in Spec\, A$ *with* $\dim A/P = 2$ *and* $\lambda(P) = d-1$. *Assume that* A *is a Gorenstein ring and* A_P *is a regular ring. Then the following conditions are equivalent.*

(1) $\mathcal{G}(P)$ *is a Gorenstein ring.*

(2) There is an minimal reduction L *of* P *with* $r_L(P) \leq 1$.

When this is the case, if $d = 4$, *then the Rees algebra* $\mathcal{R}(P)$ *is a Gorenstein ring.*

ON THE $\mathbb{I}$-INVARIANT OF THE ASSOCIATED GRADED RINGS OF POWERS OF $\mathfrak{m}$-PRIMARY IDEALS

Kikumichi Yamagishi

§1 Introduction

In this note we shall discuss some asymptotic behaviours on the $\mathbb{I}$-invariants of the associated graded rings of powers of $\mathfrak{m}$-primary ideals.

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. We denote by $\mathbb{I}(A)$ the following invariant of A, namely

$$\mathbb{I}(A) := \sum_{i=0}^{d-1} \binom{d-1}{i} l_A(\mathrm{H}^i_{\mathfrak{m}}(A)).$$

Then $0 \leq \mathbb{I}(A) \leq \infty$ clearly, and A is Cohen-Macaulay if and only if $\mathbb{I}(A) = 0$. This is just the "Buchsbaum invariant" of A, if A is a Buchsbaum ring, cf., [SV].

Let I be an $\mathfrak{m}$-primary ideal I of A. We denote by $\mathrm{G}(I)$ the associated graded ring of I, namely

$$\mathrm{G}(I) := \bigoplus_{n \geq 0} I^n / I^{n+1}.$$

Moreover M denotes its unique graded maximal ideal, i.e., $M = \mathfrak{m}\mathrm{G}(I) + \mathrm{G}(I)_+$. We simply denote by $\mathbb{I}(\mathrm{G}(I))$ in stead of $\mathbb{I}(\mathrm{G}(I)_M)$.

Since I is $\mathfrak{m}$-primary, $\mathbb{I}(\mathrm{G}(I)) \geq \mathbb{I}(A)$ holds clearly, cf. [T]. We mention that most Buchsbaum associated graded rings $\mathrm{G}(I)$ satisfy the equality $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$, cf. [G1], [G2] and [N] etc., and it was shown by the author that the associated graded ring $\mathrm{G}(I)$ must be Buchsbaum if the equality $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ holds (and if A is so), see [Y]. Moreover we also know that the study of behaviours of associated graded rings such that $\mathbb{I}(\mathrm{G}(I)) > \mathbb{I}(A)$ has started quite newly, maybe Goto's work on $\mathfrak{m}$-primary ideals of minimal multiplicity [G3] is the firstest one.

On the other hand, In 1976, G. Valla studied the associated graded rings of powers of ideals generated by A-regular sequences, where A is Cohen-Macaulay, [V], cf. also [B]. Moreover, as described as in Corollary 2 below, it is also very interested in that the equality $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ guarantees the equality $\mathbb{I}(\mathrm{G}(I^n)) = \mathbb{I}(A)$ for all $n \geq 2$ too.

Motivated these works before, we shall discuss the following question:

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Problem. *How does $\mathbb{I}(\mathrm{G}(I^n))$ behave when n increases? More precisely, does it become $\mathbb{I}(A)$ or the other constant when n increases?*

Our answer to the question above is stated as follows.

Theorem 1. *Let I be an $\mathfrak{m}$-primary ideal. Then $\mathbb{I}(\mathrm{G}(I)) \geq \mathbb{I}(\mathrm{G}(I^n)) \geq \mathbb{I}(A)$ holds for all $n \geq 2$. Moreover, if $\mathbb{I}(\mathrm{G}(I)) < \infty$, then $\mathbb{I}(\mathrm{G}(I^n))$ becomes a constant for all n large enough.*

As consequences of our theorem we also get the following.

Corollary 2. *One has the following statements.*

(1) *If $\mathrm{G}(I)$ is Cohen-Macaulay, then $\mathrm{G}(I^n)$ is so for all $n \geq 2$: cf.* [HR].

(2) *If the equality $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ holds, then the equality $\mathbb{I}(\mathrm{G}(I^n)) = \mathbb{I}(A)$ holds for all $n \geq 2$: cf.* [Y].

§2 Outline of the proof

Let $(A, \mathfrak{m})$ still be a Noetherian local ring of dimension $d > 0$ and I an $\mathfrak{m}$-primary ideal of A. Untill the end of this note we shall keep the following notations:

$\mathrm{G}(I)$, the associated graded ring of I;
$M := \mathfrak{m}\mathrm{G} + \mathrm{G}_+$, the unique homogeneous maximal ideal of G ;
$\mathrm{R}(I) := \bigoplus_{n\geq 0} I^n$, the Rees algebra of I;
$N := \mathfrak{m}\mathrm{R} + \mathrm{R}_+$, the unique homogeneous maximal ideal of R.
$\mathrm{R}'(I) := \bigoplus_{n\in\mathbf{Z}} I^n$, the extended Rees algebra of I.

When we set $I = (a_1, a_2, \ldots, a_w)$, the Rees algebra $\mathrm{R}(I)$ and the extended Rees algebra $\mathrm{R}'(I)$ are respectively regared as A-subalgebras of $A[t]$ and $A[t, t^{-1}]$, where t is an indeterminate over A, as follows,

$$\mathrm{R}(I) = A[a_1t, a_2t, \ldots, a_wt]\ , \quad \mathrm{R}'(I) = A[a_1t, a_2t, \ldots, a_wt, t^{-1}].$$

Moreover, for any integer $n > 0$ we define the graded $\mathrm{R}(I)$-algebra, writing $\mathrm{G}'_n(I)$, as follows:

$$\mathrm{G}'_n(I) := \mathrm{R}'(I)/t^{-n} \cdot \mathrm{R}'(I) = \bigoplus_{\alpha > -n} I^\alpha / I^{\alpha+n}.$$

It is clear that $\mathrm{G}'_n(I)$ is a finitely generated graded $\mathrm{R}(I)$-module too. We sometimes denote by G, R, R$'$ and G'_n omitting the letter I for simplicity.

Let $W = \oplus_{\alpha\in\mathbf{Z}} W_\alpha$ be a graded module. For an integer $n > 0$ we denote by $(W)^{(n)}$ the Veronese submodule of W of order n, i.e., $(W)^{(n)} := \oplus_{\alpha\in\mathbf{Z}} W_{n\alpha}$, and for any integer m we denote by $W(m)$ the shifted module of order m, i.e., $[W(m)]_\alpha := W_{m+\alpha}$ for each $\alpha \in \mathbb{Z}$.

Then with these notation we have

Lemma 3. *The following statements are true.*

(1) *$(\mathrm{G}'_n)^{(n)} \cong \mathrm{G}(I^n)$, as graded algebras.*

(2) *There exists an exact sequence of graded* R*-modules as follows:*

$$0 \longrightarrow \mathrm{G}'_{n-1}(1) \longrightarrow \mathrm{G}'_n \longrightarrow \mathrm{G} \longrightarrow 0.$$

Here we introduce one more useful notation. Let i, j be integers. We denote by $[i, j]$ the set of integers n such that $i \leq n \leq j$. Of course, $[i, j] = \emptyset$ if $i > j$.

Lemma 4. *Suppose that* $[\mathrm{H}_N^p(\mathrm{R})]_\alpha = (0)$ *for all* $0 \leq p \leq d$ *and* $\alpha \neq 0$. *Then, for any integers* $n \geq 1$ *and* $0 \leq i < d$, *the following statements are hold.*

(1) $[\mathrm{H}_N^i(\mathrm{G}'_n)]_\alpha = (0)$ *for* $\alpha \notin [-n, 0]$.
(2) $[\mathrm{H}_N^i(\mathrm{G}'_n)]_0 \cong [\mathrm{H}_N^i(\mathrm{G})]_0$ *and* $[\mathrm{H}_N^i(\mathrm{G}'_n)]_{-n} \cong [\mathrm{H}_N^i(\mathrm{G})]_{-1}$.
(3) $a(\mathrm{G}'_n) = a(\mathrm{G}) \leq 0$.

Moreover we also have $\mathbb{I}(\mathrm{G}(I^n)) = \mathbb{I}(\mathrm{G})$ *and* $a(\mathrm{G}(I^n)) \leq 0$.

Proof of Theorem 1. To (1): Let $n \geq 2$, $0 \leq i < d$ and $\alpha \in \mathbb{Z}$. By the eaxct sequence of (2) in Lemma 3, we easily see that

$$\begin{aligned} l_A([\mathrm{H}_N^i(\mathrm{G}'_n)]_{n\alpha}) &\leq l_A([\mathrm{H}_N^i(\mathrm{G}'_{n-1})]_{n\alpha+1}) + l_A([\mathrm{H}_N^i(\mathrm{G})]_{n\alpha}) \\ &\leq \dots \leq \sum_{j=0}^{n-1} l_A([\mathrm{H}_N^i(\mathrm{G})]_{n\alpha+j}) \end{aligned}$$

By (1) of lemma 3, we have $l_A([\mathrm{H}_P^i(\mathrm{G}(I^n))]_\alpha) = l_A([\mathrm{H}_N^i(\mathrm{G}'_n)]_{n\alpha})$, for each $\alpha \in \mathbb{Z}$, where $P := \mathfrak{m} \cdot \mathrm{R}(I^n) + \mathrm{R}(I^n)_+$, and hence

$$l_A(\mathrm{H}_P^i(\mathrm{G}(I^n))) \leq l_A(\mathrm{H}_N^i(\mathrm{G})).$$

To (2): Assume that $\mathbb{I}(\mathrm{G}) \leq \infty$. Then $\mathbb{I}(\mathrm{R}) \leq \infty$ too. Hence we can find an integer $t > 0$ such that $[\mathrm{H}_P^i(\mathrm{R}(I^n))]_\alpha = (0)$ for all $0 \leq i \leq d$, $\alpha \neq 0$ and $n \geq t$. By Lemma 4 we conclude that

$$\mathbb{I}(\mathrm{G}(I^n)) = \mathbb{I}(\mathrm{G}(I^{nt})) = \mathbb{I}(\mathrm{G}(I^t))$$

for all $n \geq t$. This completes the proof of Theorem 1.

§3 Examples

Example 5 [G3, Example (3.2)]. Let $k[[X, Y]]$ be a formal power series ring over an infinite field k and put

$$A := k[[X,Y]]^{(5)}\ , \quad I := (X^5, X^4Y, XY^4, Y^5)A\ , \quad \mathfrak{q} := (X^5, Y^5)A.$$

Then we know the following.

(1) A is a 2-dimensional Cohen-Macaulay ring and I is an $\mathfrak{m}$-primary ideal of A. Moreover $I^3 = \mathfrak{m}^3$, $\mathfrak{m}I = \mathfrak{m}\mathfrak{q}$, $I^4 = \mathfrak{q}I^3$ and $I^3 \neq \mathfrak{q}I^2$ hold, and thus I possesses minimal multiplicity and $\mathfrak{q}$ is a minimal reduction of I.
(2) $\mathrm{G}(I)$ is not a Buchsbaum ring and $\mathbb{I}(\mathrm{G}(I)) = 8$.
(3) $\mathrm{G}(I^2)$ is a Buchsbaum ring and $\mathbb{I}(\mathrm{G}(I^2)) = 4$.
(4) $\mathrm{G}(I^n)$ is a Cohen-Macaulay ring, and hence $\mathbb{I}(\mathrm{G}(I^n)) = 0$, for $n \geq 3$.

Example 6 [G3, Example (9.5)]. Let $k[X,Y,U,V,W]$ be a polynomial ring over an infinite field k and let us consider

$$\mathfrak{a} := (X,Y)(X,Y,W) + (W^2 - XU - YV)$$
$$S := k[X,Y,U,V,W]/\mathfrak{a} \cong k[x,y,u,v,w]\ , \quad \mathfrak{M} := S_+$$
$$A := S_{\mathfrak{M}}\ , \quad I := (u,v,w)A\ , \quad \mathfrak{q} := (u,v)A.$$

Then we have the following.

(1) A is a 2-dimensional Cohen-Macaulay ring with maximal embedding dimension, i.e., $\mathfrak{m}^2 = \mathfrak{q}\mathfrak{m}$, and $e(A) = 4$. Moreover $I^3 = \mathfrak{q}I^2$, $\mathfrak{m}I = \mathfrak{m}\mathfrak{q}$ and $I^2 \neq \mathfrak{q}I$ hold, hence I possesses minimal multiplicity and $\mathfrak{q}$ is a minimal reduction of I.

(2) $\operatorname{depth} \mathrm{R}(I) = 2$ and $\mathrm{H}^2_N(\mathrm{R}(I)) = \underline{k}$.

(3) $\operatorname{depth} \mathrm{G}(I^n) = 1$ and $\mathrm{H}^1_P(\mathrm{G}(I^n)) = \underline{k}(1)$, where $P := \mathfrak{m}\cdot \mathrm{R}(I^n) + \mathrm{R}(I^n)_+$, and thus $\mathbb{I}(\mathrm{G}(I^n)) = 1 > 0 = \mathbb{I}(A)$ holds for all $n > 0$.

§4 Two theorems on associated graded rings of $\mathfrak{m}$-primary ideals whose reduction numbers are at most one

In the rest of this note, we shall discuss the Buchsbaumness of the associated graded rings, especially some supplementary result to the works by Y. Nakamura (Meiji University) [N].

Now we assume that A is a Buchsbaum ring of dimension $d > 0$ and the residue field $A/\mathfrak{m}$ of A is infinite. As is well-known, if the equality $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ holds, then $\mathrm{G}(I)$ (and hence $\mathrm{G}(I^n)$ for all $n \geq 2$) is Buchsbaum too, see [Y]. Moreover, even though $\mathrm{G}(I)$ is Buchsbaum, the equality $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ does not necessarily hold. In [G3] we can find such Buchsbaum rings $\mathrm{G}(I)$, namely it occurs $\mathbb{I}(\mathrm{G}(I)) > \mathbb{I}(A) = 0$.

To describe our result, we need one more notation. For a parameter ideal $\mathfrak{q}$ of A, we define the ideal $\Sigma(\mathfrak{q})$ as follows:

$$\Sigma(\mathfrak{q}) := \sum_{i=1}^{d} [(a_1, \dots, \widehat{a_i}, \dots, a_d) : a_i] + \mathfrak{q},$$

where we are setting $\mathfrak{q} = (a_1, a_2, \dots, a_d)$. Since A is Buchsbaum, this definition of the ideal $\Sigma(\mathfrak{q})$ is independent on the particular choice of a system $a_1, a_2, \dots, a_d$ which generates the given parameter ideal $\mathfrak{q}$.

Then we have the following results.

Theorem 7. *The following two statements are equivalent.*

(1) $\mathrm{G}(I)$ *is a Buchsbaum ring such that* $[\mathrm{H}^i_M(\mathrm{G}(I))]_\alpha = (0)$ *for* $\alpha \neq -i$ $(0 \leq i < d)$ *and* $a(\mathrm{G}(I)) \leq 1 - d$.

(2) *Some (resp. every) minimal reduction* $\mathfrak{q}$ *of* I *satisfies* $I^2 = \mathfrak{q}I$ *and* $I \cap \Sigma(\mathfrak{q}) = \mathfrak{q}$.

When this is the case, it holds that $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ *and* $[\mathrm{H}^d_M(\mathrm{G}(I))]_{1-d} \cong I/\mathfrak{q}$.

With this notations we have the following.

Theorem 8. *The following two statements are equivalent.*

(1) $\mathrm{G}(I)$ *is a Buchsbaum ring such that* $[\mathrm{H}^i_M(\mathrm{G}(I))]_\alpha = (0)$ *for* $\alpha \neq 1 - i$ $(0 \leq i < d)$ *and* $a(\mathrm{G}(I)) \leq 1 - d$.

(2) *Some (resp. every) minimal reduction* $\mathfrak{q}$ *of* I *satisfies* $I^2 = \mathfrak{q}I$ *and* $I \supseteq \Sigma(\mathfrak{q})$.

When this is the case, it holds that $\mathbb{I}(\mathrm{G}(I)) = \mathbb{I}(A)$ *and* $[\mathrm{H}^d_M(\mathrm{G}(I))]_{1-d} \cong I/\Sigma(\mathfrak{q})$.

Example 9. Let us further assume $d \geq 2$, and let $x_1, x_2, \ldots, x_d$ be a system of parameters for A. Put

$$I := (x_1^2, x_2^2, \ldots, x_d^2, \prod_{i=1}^{d} x_i)\ , \quad \mathfrak{q} := (x_1^2, x_2^2, \ldots, x_d^2).$$

Then we have the following.

(1) $\mathfrak{q}$ is a minimal reduction of I such that $I^2 = \mathfrak{q}I$.

(2) $I \cap \Sigma(\mathfrak{q}) = \mathfrak{q}$.

(3) $\mathrm{G}(I^n)$ is a Buchsbaum ring by Theorem 7 and the equality $\mathbb{I}(\mathrm{G}(I^n)) = \mathbb{I}(A)$ holds for all $n > 0$.

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Prime divisors of 2-dimensional regular local rings and their valuation ideals

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1 Backgrounds and definitions

In this section we review backgrounds and definitions concerning prime divisors of two-dimensional regular local rings and their valuation ideals. For detailed explanations we refer to [3], [5]–[7], [16]–[17], and most frequently to [5].

Let (R, m, k) be a 2–dimensional regular local ring with infinite residue field k. Let K denote the quotient field of R. A valuation v of K is said to dominate R if v has nonnegative values on R and strictly positive values on m. Let $(V, m(v), k(v))$ denote the corresponding valuation ring of v. Then, the residual transcendence degree $tr.deg_k(k(v))$ of v over R is either 0 or 1 [1, Theorem 1]. If the residual transcendence degree is 0 (1, respectively), then v is called 0–dimensional (1–dimensional, respectively).

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We refer to [2] and [16] for theories of 0–dimensional valuation ideals in two-dimensional regular locals and to [7] and [15] for more recent development. In this paper, we study properties of 1–dimensional valuations of K. A prime divisor of R is a 1–dimensional valuation of K, and such a v is necessarily a discrete rank one valuation [1, Theorem 1].

Let v be a prime divisor of R. Then, the semigroup $v(R) = \{v(r) | r \in R\backslash\{0\}\}$ is a sub-semigroup of the set of nonnegative integers. We call $v(R)$ the value semigroup of v on R. It is known that this semigroup $v(R)$ is symmetric, i.e., there exists some integer z such that $a \in v(R)$ if and only if $z - a \notin v(R)$ for all integer a [6, Corollary (4.4)], [8, Theorem 1].

An ideal I of R is a v–ideal if it is a contracted ideal of some ideal of V, i.e., if $IV \cap R = I$. As v dominates R, $m(v) \cap R = m$ and hence m is the largest proper v–ideal in R. For an ideal I, define $v(I) = \min\{v(r) | r \in I\}$. If I is a v–ideal then the set $I' = \{r \in R | v(r) > v(I)\}$ is also a v–ideal and called the v–successor of I. If this is the case, then I is also called the v–predecessor of I'. Since v is rank one and discrete, there is an infinite descending sequence of v–ideals in R:

$$R \supset m = I_0 \supset I_1 \supset I_2 \supset \cdots\cdots$$

In this paper, we study the prime divisors of R with respect to their valuation ideals in R and their corresponding values. One simple example of a prime divisor of R is the m–adic order valuation ord_m of K which will be often denoted by o:

Example 1.1. Let $m = (x, y)$ and k be algebraically closed. Let $v = ord_m$ be the m–adic order valuation of K, i.e., $ord_m(a) = r$ if $a \in m^r \backslash m^{r+1}$ for $a \in R$. Then, the corresponding valuation ring is

$$Ord_R = V = R[\frac{m}{x}]_{(x)}$$

whose maximal ideal $m(v) = (x)$, and the residue field $k(v)$ of v is purely trascendental over k which is generated by the image of $\frac{y}{x}$ over k [17]. The valuation ideals of ord_m in R are the set of ideals $m(v)^n \cap R = m^n$ for all $n \geq 1$ and therefore

$$m \supset m^2 \supset m^3 \supset m^4 \supset \cdots\cdots$$

is the sequence of all the v–ideals in R. Furthermore the value semigroup of ord_m on R is $\mathbf{N}$, the set of all nonnegative integers.

Throughout these notes v will denote a prime divisor of R and we assume that the residue field k of R is algebraically closed. By an ideal we mean an m–primary ideal. The theory of prime divisors can be found in Zariski's foundation of the theory of integrally closed ideals in two-dimensional regular local rings [17, Appendix 5]. An ideal I of R is integrally closed (or complete) if it is an intersection of valuation ideals of prime divisors of R, i.e., $I = \overline{I}$, where

$$\overline{I} = \bigcap_{K \supset V \supset R} (IV \cap R).$$

An ideal I is said to be simple if whenever $I = JL$, then either $J = R$ or $L = R$. Three main structure theorems of Zariski in [17] are the followings:

Theorem 1.2. ([17, Theorem 3, p.386]) *Every m–primary integally closed ideal I is uniquely factored into products of finitely many simple integrally closed ideals as follows:*

$$I = I_1^{k_1} I_2^{k_2} \cdots I_n^{k_n}$$

Theorem 1.3. ([17, Theorem 2', p.385]) *A product of integrally closed ideals is integrally closed.*

Theorem 1.4. ([17, (E), p.391]) *There is a one-one correspondence:*

$$\{\, I \,|\, \text{simple } m\text{–primary integrally closed ideal} \,\}$$
$$\updownarrow$$
$$\{\, v \,|\, \text{prime divisor of R} \,\}$$

To describe the correspondence in the last theorm, we need to discuss quadratic transformations of R first. The ring S is called a first quadratic transformation of R if $S = R[m/x]_N$ for some height 2 maximal ideal N of $R[m/x]$ containing $mR[m/x]$ and for some $x \in m \backslash m^2$. If I is an m–primary ideal such that $o(I) = r$, then $IS = x^r I'$, for some ideal I' of S which is called the ideal transform of I in S. Conversely, if J is an $m(S)$–primary ideal of S then the inverse transform of J in R is the ideal $\tilde{J} = x^r J \cap R$ with the least such r. If S is a 2–dimensioanl regular local ring containing R in K, there exists a finite quadratic sequence

$$R = R_0 \subset R_1 \subset \cdots\cdots \subset R_n = S$$

where each R_{i+1} is a first quadratic transformation of R_i for $0 \leq i \leq n$ [1, Theorem 3]. Such S is called a point infinitely near to R [6, (1.1), p.225].

The above one-one correspondence was given as follows [17, (E), p.391]. For any simple m–primary integrally closed ideal I, there exists a quadratic sequence

$$R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_t,$$

where the transform of I in R_t is the maximal ideal m_t and v is the m_t–adic order valuation of K. Conversely, given prime divisor v of R, one can find a quadratic sequence

$$R = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_\ell,$$

where v is the m_ℓ–adic order valuation and the inverse transform of m_ℓ in R is the simple m–primary integrally closed ideal of R. The above two quadratic sequences coincide and the number $t = \ell$ is defined to be the rank of I and denoted by $rank(I) = t$.

Let I be a simple integrally closed ideal of rank t which is associated to the prime divisor v. Zariski gave the description of v–ideals as follows.

Theorem 1.5. ([17, (F), p.392]) *Let I be a simple integrally closed ideal of R of rank t and associated to the prime divisor v. Let P_i be the simple integrally closed ideal in R whose transform in R_i is the maximal ideal of R_i. Then*

- $m = P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_t = I$
- *each P_i is a v–ideal and every v–ideal in R is a power product of these $t+1$ simple v–ideals.*

As in the above theorem, we see that there are only finitely many simple v–ideals

$$m = P_0 \supset P_1 \supset P_2 \supset \cdots\cdots \supset P_t = I$$

in the sequence of all the v–ideals

$$m = I_0 \supset I_1 \supset I_2 \supset \cdots\cdots \supset I_s = I \supset I_{s+1} \supset \cdots\cdots$$

In [9, Theorem A.2, Appendix], Lipman further showed that the above sequence of v–ideals is saturated from m to I, i.e., $\lambda(I_{i-1}/I_i) = 1$ for $1 \leq i \leq s$. This is no longer true for the v–ideals smaller than I. In [9, Theorem 3.1], the length between any two consecutive v–ideals $J \supset J'$ smaller than I is measured to be the least integer ν such that I^ν does not divide J. With these two theorems we can compute the length between any two consecutive v–ideals.

Therefore, now we turn our concern to describing simple v–ideals with their minimal generators and then to finding factorizations of composite v–ideals into products of simple v–ideals.

Finding generators of all the v–ideals was solved in the case of a power series ring over an algebraically closed field of characteristic 0 in [16]. In [15], Spivakovsky proved a structure theorem for prime divisors when k is algebraically closed. By using combinatorics and dual graphs, Spivakovsky described a generating sequence of v which then completely determines the given prime divisor v.

A factorization of a composite v–ideal in terms of a product of simple v–ideals was shown in [7, Theorem (4.11)]. In [10, Theorem 3.1], we proved that there exists a unique integrally closed ideal adjacent to I from above, i.e., the v–predecessor I_{s-1} of I is the unique integrally closed ideal containing I such that $\lambda(I_{s-1}/I) = 1$. Lipman proved this uniqueness in more general context, and further obtained the factorization of I_{s-1} by using proximity [7, Theorem (4.11)]. Note that for two 2–dimensional regular local rings $S \supset T$ in K, S is said to be proximate to T, denoted by $S \succ T$, if $m(T)$–adic order valuation ring Ord_T contains S. If S is a first quadratic transformation of R, then $S \succ R$ since $S = R[m/x]_N$ and $Ord_R = R[m/x]_{(x)}$ and $N \supset (x)$. In the case of k being algebraically closed, the above result of Lipman asserts that the v–predecessor I_{s-1} of I is either simple P_{t-1} or product $P_{t-1}P_i$ of two simple v–ideals for some $0 \leq i \leq t-2$. In the former, we say that I is free and proximate to P_{t-1}. In the later, we say that I is satellite and proximate to P_{t-1} and P_i as Hoskin used with 0–dimensional valuation ideals in [2].

In this paper, we further find minimal generating sets of simple v–ideals and the unique factorizations of composite v–ideals especially when I has small orders.

Since I is integrally closed, if $o(I) = r$ then $\mu(I) = r + 1$ due to Hilbert-Burch, Lipman, and Rees. For a proof, we refer [5, Corollary (3.2)], [3, Proposition 2.3], or [4, Theorem 2.1]. Therefore, we first need to find a set of minimal generators (a, b) or (a, b, c) of the simple integrally closed ideal I when $o(I) = 1$ or $o(I) = 2$, then find sets of minimal genetators of all simple v–ideals. Further, we describe the factorizations of composite v–ideals into products of simple ones by using the invariants $|v(x) - v(y)| = b_v$ and $rank(I)$, where $m = (x, y)$. The organization of the paper is as follows.

In Section 2, we obtain minimal generating sets of simple v–ideals and describe factorizations of other composite v–ideals in terms of products of simple v–ideals for prime divisors associated to simple m–primary integrally closed ideals of order 1. We also find the symmetric value semigroup $v(R)$ for such v and compute the length between any two consecutive v–ideals explicitly.

In Section 3, we obtain minimal generating sets of simple simple v–ideals and describe factorizations of composite v–ideals in terms of products of simple v–ideals for prime divisors associated to simple m–primary integrally closed ideal of order 2. We also find the value semigroup $v(R)$ of v on R and compute the length between any two consecutive v–ideals explicitly.

The most of the results in Section 2 and Section 3 with more detailed proofs can be found in [13] and [14].

2 Integrally closed ideals of order one

Let us assume that $m = (x, y)$ and denote $ord_m(I) = o(I)$ if there is no confusion. Let N denote the set of all nonnegative integers. If an integrally closed ideal I is of order 1, then it is simple and hence is associated to a unique prime divisor v of R. Assume that $rank(I) = t$. If $t = 0$, then $I = m = (x, y)$ and we know that the value semigroup of v on R and the sequence of v–ideals are as follows:

- $v(R) = \mathrm{N}$
- $m \supset m^2 \supset m^3 \supset \cdots\cdots$
- $\lambda(m^n/m^{n+1}) = n + 1$

for all $n \geq 0$. Now we assume that $t \geq 1$.

Ideals of rank 1 are called the first-neighborhood ideals and they are of the form either (x, y^2) or $(y - \alpha x, x^2)$ for some $\alpha \in k$ [12, Theorem 2.1]. In [11] it was shown the followoing one-one correspondences:

$$\{\, I \mid \text{first-neighborhood ideal of R} \,\}$$
$$\updownarrow$$
$$\{\, S \mid \text{first quadratic transformation of R} \,\}$$

If v is the prime divisor associated to a first-neighborhood ideal I, then the value semigroup of v on R and the sequence of v–ideals are as follows:

- $v(R) = \mathbf{N}$
- $m \supset I \supset mI \supset I^2 \supset mI^2 \supset I^3 \supset \cdots\cdots$
- $\lambda(I^n/mI^n) = \mu(I^n) = n+1$
- $\lambda(mI^n/I^{n+1}) = n+1$

for all $n \geq 0$ [11], [13, Theorem 3.4].

What we used in the above length computation are Lipman's intersection multiplicity formula and reciprocity formula whose more general statements can be found in [5]–[7]. The point basis of an m–primary ideal I is defined to be the finite set of nonnegative integers:

$$B(I) = \{ord_S(I_S) | K \supset S \supset R\},$$

where ord_S is the $m(S)$–adic order valuation of K and I_S is the ideal transform of I in an infinitely near point S of R [6, (1.2), p.225]. The intersection multiplicity(number) of two m–primary ideals I and J is the positive integer defined by:

$$(I \cdot J) = \sum_{K \supset S \supset R} ord_S(I_S) ord_S(J_S)$$

[6, (1.6), p.228]. This intersection number can be obtained by measuring the lengths in R as follows when k is algebraically closed.

Theorem 2.1. (Intersection multiplicity formula, [5, Corollary (3.7)]) *Let I and J be m–primary integrally closed ideals of R. Then, $(I \cdot J) = \lambda(R/IJ) - \lambda(R/I) - \lambda(R/J)$, where $(I \cdot J)$ denotes the intersection multiplicity of I and J.*

Theorem 2.2. (Reciprocity formula, [7, Corollary (4.8)]) *Let v and w be prime divisors associated to I and J, respectively. Then, $v(J) = w(I)$.*

This intersection multiplicity together with reciprocity formula enable us to measure the lengths between ideals $IL \supset JL$ if $I \supset J$ and L is another integrally closed ideal as follows.

Lemma 2.3. (Length formula, [9, Remark 2.2]) *Let $I \supset J$ be two integrally closed ideals and L be another integrally closed ideal. Then, $\lambda(IL/JL) = \lambda(I/J) + (J \cdot L) - (I \cdot L)$.*

The above length formula is useful to find out the length between any two successive v–ideals. We first find a necessary and sufficient condition for an m–primary ideal I of order one to be integrally closed, i.e., $I = \overline{I}$ [13, Theorem 2.1].

Theorem 2.4. *Let $m = (x, y)$ and I be an m-primary integrally closed ideal of order 1. Then,*

$$I = \overline{I} \text{ with } rank(I) = t \text{ if and only if } I = (x - \alpha y^s, y^{t+1})$$

for $1 \leq s \leq t + 1$ and $\alpha \in k$.

Proof. ($\Longleftarrow$) Assume $t \geq 2$. If $\alpha = 0$, then $I = (x, y^{t+1})$ and this is integrally closed [11, Lemma 3.5]. If $\alpha \neq 0$ and $s = 1$, then by replacing $x - \alpha y$ by x' we get $I = (x', y^{t+1})$ for $m = (x', y)$. This is integrally closed by the case of $\alpha = 0$. Assume $\alpha \neq 0$ and $s > 1$. Use induction on t in this case. Let $t = 2$ and $I = (x - \alpha^s, y^3)$. Since $\mu(I) = o(I) + 1$, I is contracted. Since $y^* \not| c(I) = x^*$, I is contracted from $R' = R[\frac{m}{y}]_N$, where $N = (\frac{x}{y}, y)$ is the maximal ideal of $R[\frac{m}{y}]$ containing the ideal transform I' of I. The ideal transform $I' = (\frac{x}{y} - \alpha y^{s-1}, y^2)$ is a first-neighborhood ideal of R' in both cases $s = 2$ and $s = 3$. Since the transform is integrally closed, so is I [17]. Assume $t > 2$. The ideal transform I' of I in R' is of the form $(\frac{x}{y} - \alpha y^{s-1}, y^t)$ in the two-dimensional regular local ring R' with the maximal ideal $m' = (\frac{x}{y}, y)R'$. By induction hypothesis, the transform I' is integrally closed and $rank(I') = t - 1$. Since $IR' \cap R = yI' \cap R = I$, I is also integrally closed and $rank(I) = t$.

($\Longrightarrow$) Assume that $t \geq 2$. Let v be the prime divisor associated to I. Since $v(m) = o(I) = 1$ by reciprocity formula, we may assume that $v(y) = 1$ for $m = (x, y)$. Let

$$m = P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_t = I$$

be the sequence of all simple v-ideals. However, $o(P_i) = 1$ for all i since $o(I) = 1$. Since $v(m) = 1$, the value semigroup of v in R is $v(R) = \mathbf{N}$, where $\mathbf{N}$ is the set of nonnegative integers. Since $m = P_0 \neq P_1$, $v(P_1) = 2$ and hence $y^2 \in P_1$. Since P_1 is a first-neighborhood ideal containing y^2, we have $P_1 = (x - \alpha y, y^2)$ for some $\alpha \in k$. By replacing $x - \alpha y$ by x, we may assume that $P_1 = (x, y^2)$. Since $rank(I) = t$, there is a unique sequence of quadratic transformations

$$R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_t,$$

where the transform of P_i in R_i becomes the maximal ideal for each $0 \leq i \leq t$ [17, Theorem (F), Appendix 5]. Let m_i be the maximal ideal of R_i, o_i the m_i–adic order valuation, and I_i the transform of I in R_i for each i. By reciprocity, $o_i(I_i) = v(m_i)$ for all i. Hence $o_i(I_i) \leq o_{i-1}(I_{i-1})$ since $m_{i-1} = m_i \cap R_{i-1}$ for all $i \geq 1$. Since $o(I) = 1$, we have $o_i(I_i) = 1$ for all $0 \leq i \leq t$. Therefore, $v(I) = e(I) = (I \cdot I) = \sum_{i=0}^{t} 1^2 = t+1$. Since there are $t+1$ distinct simple v–ideals $P_i's$, $v(P_i) = i+1$ for $0 \leq i \leq t$. Since $v(y^i) = i$ for each i, $y^i \in P_{i-1}$ for $1 \leq i \leq t+1$. In particular, $y^{t+1} \in I = P_t$ and y^{t+1} is part of a minimal reduction of I since $v(y^{t+1}) = v(I) = t+1$ [9, Lemma 1.9]. Since $o(I) = 1$, we may assume that

$$I = (ax + by + f, y^{t+1})$$

for either a or b is a unit and $o(f) = r \geq 2$. Let $f = \sum_{i=0}^{r} a_i x^{r-i} y^i$.

(1) Assume that a and b are both units.

Then,

$$\begin{aligned} I &= (ax + by + \sum_{i=0}^{r} a_i x^{r-i} y^i, y^{t+1}) \\ &= (x(a + a_0 x^{r-1}) + y(b + \sum_{i=1}^{r} a_i x^{r-i} y^{i-1}), y^{t+1}). \end{aligned}$$

Since $r \geq 2$ and a, b are units, $a + a_0 x^{r-1}$ and $b + \sum_{i=1}^{r} a_i x^{r-i} y^{i-1}$ are also units, therefore we may assume that $I = (x - \alpha y, y^{t+1})$ for some $\alpha \neq 0$ in k. We are done in this case.

(2) Assume that b is a unit, but a is not a unit.

Let $a = a'x + a''y$ for some $a', a'' \in R$. Then,

$$\begin{aligned} I &= (ax + by + \sum_{i=0}^{r} a_i x^{r-i} y^i, y^{t+1}) \\ &= (x(a' + a_0 x^{r-1}) + y(b + a''x + \sum_{i=1}^{r} a_i x^{r-i} y^{i-1}), y^{t+1}). \end{aligned}$$

Since $b + a''x + \sum_{i=1}^{r} a_i x^{r-i} y^{i-1}$ is a unit, we may assume that

$$I = (y + x(c + dx^{r-1}), y^{t+1})$$

for some $c, d \in R$. We claim that c is a unit. Suppose c is not a unit. Then, by continuously replacing c by $c'x + c''y$, we eventually may assume that $I = (y + ex^s + gx^\ell, y^{t+1})$ for some $s, \ell \geq 2$ and units e, g. Since $c(I) = y^*$, I is contracted from $R' = R[\frac{y}{x}]_{(\frac{y}{x}, x)}$ which is dominated by v. Therefore, $v(\frac{y}{x}) > 0$ and $v(y) > v(x)$ imply that $v(x) = 0$ since $v(y) = 1$. This is a contraction since v dominates R. Therefore, c should be a unit. Since c is a unit and $r \geq 2$, then we may assume that $I = (y - \beta x, y^{t+1})$ for some $\beta \neq 0 \in k$. Therefore, $I = (x - \alpha y, y^{t+1})$ for a unit $\alpha = \beta^{-1} \neq 0$ as desired.

(3) Assume that a is a unit, but b is not a unit.

In this case,

$$I = (ax + by + f, y^{t+1}) = (x + f', y^{t+1})$$

for some $f' \in R$ of order at least 2. Say $f' = \sum_{i=0}^{r} a_i' x^{r-i} y^i$ for $r \geq 2$. Then,

$$\begin{aligned} I &= (x + \sum_{i=0}^{r} a_i' x^{r-i} y^i, y^{t+1}) \\ &= (x(1 + \sum_{i=0}^{r-1} a_i' x^{r-i-1} y^i) + a_r' y^r, y^{t+1}), \end{aligned}$$

hence we may assume that $I = (x + a_r'' y^r, y^{t+1})$. If a_r'' is not a unit, keep replacing a_r'' by $cx + dy$ and get finally $I = (x - \alpha y^s, y^{t+1})$ for some $\alpha \neq 0 \in k$ and for $s \geq 2$ since $s \geq r \geq 2$. If $s \geq t + 1$ then $I = (x, y^{t+1})$ and if $s < t + 1$ then $I = (x - \alpha y^s, y^{t+1})$.

¿From (1)–(3), we obtain I in the form $(x - \alpha y^s, y^{t+1})$ for some $\alpha \in k$ and $1 \leq s \leq t + 1$. □

The above theorem classifies all integrally closed ideals of order one, i.e., the simple valuation ideals of order one in terms of their minimal generating sets and of their ranks. By comparing the v–values, we also can find minimal generating sets of all the simple v–ideals.

Corollary 2.5. *Let I and v be as in Theorem 2.4. Assume $1 \leq s < t + 1$. Then,*

$$\begin{aligned} m = (x, y) &\supset (x, y^2) \supset \cdots \supset (x, y^s) \supset (x - \alpha y^s, y^{s+1}) \\ &\supset (x - \alpha y^s, y^{s+2}) \supset \cdots\cdots \supset (x - \alpha y^s, y^{t+1}) = I \end{aligned}$$

is the set of all simple v–ideals.

Proof. It is easy to see that $v(y) = 1$, $v(x) = s$, and $v(x - \alpha y^s) = t + 1$. Therefore, $v(x, y^i) = i$ for $i \leq s$ and $v(x - \alpha y^s, y^i) = i$ for $i \geq s + 1$, hence they are contained in $t + 1$ different v–ideals. However, the sequence of all the v–ideals from I to m is saturated [9, Lipman, Theorem A.2],

$$\begin{aligned} m = (x, y) \;&\supset\; (x, y^2) \supset \cdots \supset (x, y^s) \supset (x - \alpha y^s, y^{s+1}) \\ &\supset\; (x - \alpha y^s, y^{s+2}) \supset \cdots\cdots \supset (x - \alpha y^s, y^{t+1}) = I \end{aligned}$$

is indeed the set of all the v–ideals containing I with the v–values $1, 2, \ldots, t + 1$.

The rest of the sequence of v–ideals is as follows.

Theorem 2.6. *Let I be a simple integrally closed ideal of order 1 with rank $t \geq 2$ associated to the prime divisor v. Let $\ell = (t + 1)q + i$ for $0 \leq i \leq t$. Then,*

- $v(R) = \mathbf{N}$
- $I_\ell = P_i I^q$
- $v(I_\ell) = \ell + 1$, *and*
- $\lambda(I_\ell / I_{\ell+1}) = q + 1$.

Proof. We have the sequence of all simple v–ideals as follows:

$$m = P_0 \supset P_1 \supset \cdots \supset P_{t-1} \supset P_t = I.$$

Since $v(m) = 1$ and $o(I) = 1$, it is readily shown that $v(R) = \mathbf{N}$, where $\mathbf{N}$ is the set of nonnegative integers. We also proved that the above sequence is the set of all v–ideals containing I, i.e., $I_\ell = P_\ell$ if $0 \leq \ell \leq t$. Let $\ell = (t+1)q + i \geq t + 1$ for $0 \leq i \leq t$. We also know that $P_i I^q$ is a v–ideal for any $0 \leq i \leq t$ and for any $q \geq 1$ by [9, Theorem 3.1]. Then, $v(P_i I^q) = (i + 1) + (t + 1)q = \ell + 1 = v(I_\ell)$ implies that $I_\ell = P_i I^q$. By intersection multiplicity formula, we obtain

$$\begin{aligned} \lambda(I_\ell / I_{\ell+1}) &= \lambda(P_i I^q / P_{i+1} I^q) \\ &= \lambda(P_i / P_{i+1}) + q(v(P_{i+1}) - v(P_i)) \\ &= q + 1 \end{aligned}$$

for all $\ell \geq 0$.

3 Simple integrally closed ideals of order two

In Section 2, we found minimal generators of simple v–ideals when v is associated to a simple integrally closed ideal I of order one. In particular, it was clear to see that the v–predecessor I_{s-1} is also of order one and simple, i.e., such I is free.

If I is simple integrally closed of order two, then the v–predecessor need not be simple. We need to first find out whether a simple integrally closed ideal has simple(or composite) v–predecessor or not. Let us denote $I_{s-1} = J$. In [7, Theorem (4.11)], Lipman showed that J is the product of simple v–ideals $P_i's$ associated to $R_i's$ to which R_t is proximate. However, there are at most two such $R_i's$, one of them is R_{t-1} since R_t is a first quadratic transformation of R_{t-1} [7, (1.3), p.294]. Hence we have either $J = P_{t-1}$ or $J = P_{t-1}P_i$ for some $0 \leq i \leq t-2$ since k is algebraically closed. Let us consider the sequence of simple v–ideals

$$m = P_0 \supset P_1 \supset \cdots \supset P_t = I$$

where I is a simple integrally closed ideal associated to v such that $o(I) = 2$ and $rank(I) = t$.

Let us assume that $m = (x, y)$ and I is contracted from $R_1 = R[\frac{m}{y}]_N$ and $m(v) \cap R[\frac{m}{y}] = N$, hence we may assume that $v(x) > v(y)$. By reciprocity formula, $v(m) = v(y) = o(I) = 2$. Let $v(x) = 2 + b$ for $b \geq 1$. Then integer b, denoted by b_v, plays an important role in studying the simple v–ideals other than I. First, we use the value difference $b_v = b$ to determine which powers m^j of m is a v–ideal, how many simple v–ideals have order 1, and how many do have order 2 among t nonmaximal simple v–ideals [9, Theorem 1.2], [14].

Proposition 3.1. *Let $b \geq 1$ and $rank(I) = t$. Then,*

(i) *m^j is a v–ideal iff $1 \leq j \leq \lceil \frac{2}{b} \rceil$*

(ii) *the number of non-maximal order 1 simple v–ideals is $\lceil \frac{b+1}{2} \rceil$*

(iii) *the number of order 2 simple v–ideals is $t - \lceil \frac{b+1}{2} \rceil$.*

Proof. If $b = 1$, then m^2 is a v–ideal [9, Theorem 1.2]. Hence $P_1 = (x, y^2)$ is the only one non-maximal simple v–ideal. Therefore, assume $b \geq 2$. Then, $\lceil \frac{2}{b} \rceil = 1$ and hence m is the only v–ideal among all the powers of m by [9, Theorem 1.2]. For the rest of the proof, we consider two cases when b is odd and when b is even independently. We refer [14] for details.

The conductor ideal (or adjoint ideal) of v is the v–ideal C such that for any successive v–ideals $C \supset J \supset J' \supset I$, $\lambda(J/J') = 1$ and $v(J') = v(J) + 1$ [6, Definition (2.1)] and it is known that $C = L : m$ for the largest v–ideal L of order $o(I)$ [6, Theorem (2.2)]. Using this and Proposition 3.1, we easily obtain the conductor ideal of v in our case, i.e., $P_{\lceil \frac{b-1}{2} \rceil}$ is the conductor ideal of v. A proof can be found in [14] in which two cases when b is odd and when b is even are considered separately.

In either case $b = 2k$ or $b = 2k+1$, $\lceil \frac{b-1}{2} \rceil = k$, $\lceil \frac{b+1}{2} \rceil = k+1$, and $\lceil \frac{b+3}{2} \rceil = k+2$. Therefore, P_k is the conductor ideal and P_{k+2} is the only satellite simple v–ideal in either case.

Among simple v–ideals of v,

$$m \supset P_1 \supset \cdots \supset P_{\lceil \frac{b+1}{2} \rceil} = P_{k+1} \supset P_{k+2} \supset \cdots \supset P_t = I,$$

each P_i is associated to a unique prime divisor v_i which is also associated to R_i for $0 \leq i \leq t$. For a simple v–ideal P_i for $i \geq k+2$,

$$m \supset P_1 \supset \cdots \supset P_{\lceil \frac{b+1}{2} \rceil} = P_{k+1} \supset P_{k+2} \supset \cdots \supset P_i$$

is still the sequence of simple v_i–ideals. Furthermore, the sequence of all v_i–ideals from m to P_i coincides with the sequence of v–ideals from m to P_i by [9, Lipman, Theorem A.2]. Since $v_i(m) = o(P_i) = 2$ and v_i dominates $R_1 = R[\frac{m}{y}]_N$, we know that $v_i(m) = v_i(y) = 2 < v_i(x)$. We show now that $v_i(x) = v(x)$, i.e., $v_i(x) - v_i(y) = b$, and that $v_i(R) = v(R)$ for such v_i.

It has been shown that if $J \supset I$ are adjacent simple integrally closed ideal, then $o(J) = o(I)$ and $w(R) = v(R)$ if w, v are the prime divisors associated to J and I, respectively [8, Theorem 2]. Let us denote $|v_i(x) - v_i(y)| = b_{v_i}$ for each v_i. Let us assume that $v(y) = 2 < v(x) = 2 + b$ always.

Proposition 3.2. *Let $b = b_v \geq 1$ and $rank(I) = t$. Let v_i be the prime divisor associated to the simple v–ideal P_i for $k+2 \leq i \leq t-1$. Then,*

$$v_i(y) = v(y), \; b_{v_i} = b_v, \; v_i(R) = v(R).$$

Proof. Refer [14].

Now we use the above results to find minimal generating sets of simple v–ideals for the prime divisor v associated to a simple integrally closed ideal I of order 2. First consider the sequence of simple v–ideals:

$$m = P_0 \supset P_1 \supset \cdots \supset P_{\lceil \frac{b-1}{2} \rceil} \supset P_{\lceil \frac{b+1}{2} \rceil} \supset P_{\lceil \frac{b+3}{2} \rceil} \supset \cdots \supset P_t = I.$$

By Proposition 3.1, we know that $o(P_i) = 1$ for $0 \le i \le \lceil \frac{b+1}{2} \rceil$ and $o(P_i) = 2$ for $\lceil \frac{b+3}{2} \rceil \le i \le t$. Each P_i is a v–ideal, but is also associated to the unique prime divisor v_i for $\lceil \frac{b+3}{2} \rceil \le i \le t$. ¿From Proposition 3.2, we know that $v_i(x) - v_i(y) = v(x) - v(y)$ for all $\lceil \frac{b+3}{2} \rceil \le i \le t$, so that we use induction on $t = rank(I)$ in various places.

If $b = v(x) - v(y) = 1$ and $rank(I) = 2$, then we easily can show that $I = (x^2, xy^2, y^3)$ and adjacent to $m(x, y^2)$, i.e., I is satellite and

$$m \supset (x, y^2) \supset m^2 \supset m(x, y^2) \supset (x^2, xy^2, y^3) = I$$

is the sequence of v–ideals from m to I. If $b = v(x) - v(y) = 1$ and $rank(I) \ge 3$, then the following is shown in [14]:

Theorem 3.3. *Let $b = 1$ and $rank(I) = t$ for $t \ge 3$. Then,*

(i) *I is free*

(ii) *$I = (x^2 - \alpha y^3 - \beta xy^2, xy^{\lceil \frac{t+1}{2} \rceil}, y^{\lceil \frac{t+4}{2} \rceil})$ for some $\alpha \ne 0$ and $\beta \in k$*

(iii) *$v(x^2 - \alpha y^3 - \beta xy^2) = e(I) = t + 4$*

(iv) *$v(R) = \mathbb{N} \backslash \{1\}$.*

Proof. Refer [14]. □

Now we consider the case of $b \ge 2$ and $t \ge 4$. If $b \ge 2$, then $rank(I) \ge 3$. As always, we let $v(y) = 2 = v(m)$ and $v(x) = 2 + b$ for $m = (x, y)$. Since $b \ge 2$, m is the only v–ideal among the powers of maximal ideal, i.e., m^2 is not a v–ideal. Recall that P_{k+2} is the only satellite simple v–ideal and all the other simple v–ideals are free for both $b = 2k$ and $b = 2k + 1$ cases. This satellite simple v–ideal plays an important role in the following theorems. We will find the simple v–ideal other than P_{k+1} to which P_{k+2} is proximate in the following theorems.

We can find minimal generating sets of the simple integrally closed ideal I in two cases. The first case is when b is odd and $b \ge 3$ and the other is when b is even and $b \ge 2$. Both of the proofs can be found in [14].

Theorem 3.4. *Let $m = (x, y)$, $b = 2k + 1$ for $k \geq 1$, and $rank(I) = t \geq 4$. Then,*

(i) *P_{k+2} is proximate to P_{k+1} and P_k*

(ii) $v(R) = \mathbf{N} \backslash \{1, 3, 5, ..., b\}$

(iii) *for some $\alpha \neq 0$ and $\beta \in k$, and $2 \leq i \leq \lceil \frac{t-k-1}{2} \rceil$,*

$$I = (x^2 - \alpha y^{2k+3} - \beta x y^{k+i}, x y^{\lceil \frac{t+k+1}{2} \rceil}, y^{\lceil \frac{t+3k+4}{2} \rceil})$$

(iv) $v(x^2 - \alpha y^{2k+3} - \beta x y^{k+i}) = e(I) = t + 3k + 4.$

Theorem 3.5. *Let $m = (x, y)$, $b = 2k$ for $k \geq 1$, and $rank(I) = t \geq 3$. Then,*

(i) *P_{k+2} is proximate to P_{k+1} and P_k*

(ii) $v(R) = \mathbf{N} \backslash \{1, 3, 5, ..., b + 1\}$

(iii) *for some $\alpha \neq 0$ and $\beta \in k$,*

$$I = ((x - \alpha y^{k+1})^2 - \beta y^{2k+3}, (x - \alpha y^{k+1}) y^{\lceil \frac{t+k+1}{2} \rceil}, y^{\lceil \frac{t+3k+4}{2} \rceil})$$

(iv) $v((x - \alpha y^{k+1})^2 - \beta y^{2k+3}) = e(I) = t + 3k + 4.$

We found minimal generating sets of all simple v–ideals and described the sequence of v–ideals from m to I in the proof of the above theorems. Now we can describe the rest of v–ideals and find their factorizations in terms of power products of simple v–ideals P_i's to obatin the infinite descending sequence of v–ideals completely.

Theorem 3.6. *Let I, v, b, t, $C = P_k$ be as in Theorem 3.4 and Theorem 3.5. Then, the sequence of v–ideals is as follows:*

(i) *v–ideals from m to I are*

$$\begin{aligned} m = P_0 &\supset P_1 \supset \cdots\cdots \supset P_k \supset P_{k+1} \\ &\supset mP_k \supset mP_{k+1} \supset P_1P_k \supset P_1P_{k+1} \supset \cdots \supset P_kP_k \supset P_kP_{k+1} \\ &\supset P_{k+2} \supset P_{k+3} \supset \cdots \supset P_{t-1} \supset P_t = I \end{aligned}$$

(ii) $\forall n \geq 1$,

$$I^n \supset mP_{t-1}I^{n-1} \supset mI^n$$

(iii) $\forall n \geq 1$,

$$\begin{aligned} mI^n &\supset P_1P_{t-1}I^{n-1} \supset P_1I^n \\ &\supset P_2P_{t-1}I^{n-1} \supset P_2I^n \\ &\supset \cdots\cdots\cdots \supset P_{k-1}I^n \\ &\supset P_kP_{t-1}I^{n-1} \supset P_kI^n = CI^n \end{aligned}$$

(iv) $\forall n \geq 1$,

$$\begin{aligned} CI^n = P_kI^n &\supset P_{k+1}I^n \\ &\supset mP_kI^n \supset mP_{k+1}I^n \\ &\supset P_1P_kI^n \supset P_1P_{k+1}I^n \\ &\supset \cdots\cdots \supset P_{k-1}P_{k+1}I^n \\ &\supset P_kP_kI^n \supset P_kP_{k+1}I^n \\ &\supset P_{k+2}I^n \supset P_{k+3}I^n \supset \cdots \supset P_{t-1}I^n \supset P_tI^n = I^{n+1}. \end{aligned}$$

Proof. Use intersection multiplicity formula, recipricity formula, length formula, and [9, Theorem 3.1]. We refer [14] for detailed proofs. □

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Projective plane curves whose complements have $\overline{\kappa} = 1$

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1 Introduction

All algebraic varieties considered in this paper are defined over the field of complex numbers $\mathbf{C}$. Let C be an irreducible curve on a projective plane $\mathbf{P}^2$. It is important to study its complement $X := \mathbf{P}^2 - C$ with respect to its logarithmic Kodaira dimension $\overline{\kappa}(X)$ (cf. Iitaka [2] for the definition and the relevant results of logarithmic Kodaira dimension), which takes one of $-\infty, 0, 1$ or 2. With respect to $\overline{\kappa}$, the following theorems are important (cf. Miyanishi [5] and Kawamata [3]):

Theorem 1.1 *Let S be a non-singular affine surface with $\overline{\kappa}(S) = -\infty$. Then S has an $\mathbf{A}^1$-fibration:*

$$\varphi : S \to B$$

where B is a smooth curve.

Theorem 1.2 *Let S be a non-singular affine surface with $\overline{\kappa}(S) = 1$. Then S has a $\mathbf{C}^*$-fibration:*

$$\varphi : S \to B$$

where $\mathbf{C}^*$ means an affine line with one point deleted off and B is a smooth curve.

In consideration of Theorem 1.1, Miyanishi and Sugie researched an irreducible curve C on $\mathbf{P}^2$ such that $\overline{\kappa}(\mathbf{P}^2 - C) = -\infty$, and obtain the following theorem (cf. Miyanishi-Sugie [6]).

Theorem 1.3 *Let C be an irreducible curve on $\mathbf{P}^2$ such that $\overline{\kappa}(\mathbf{P}^2 - C) = -\infty$. Then the following assertions hold true:*

There exists an irreducible linear pencil Λ on $\mathbf{P}^2$ such that some multiple mC ($m \geq 1$) of C is a member of Λ, where Λ satisfies the following properties:

1: Λ has only one base point, say p_0, and for a general member C' of Λ, $C' - \{p_0\}$ is isomorphic to an affine line $\mathbf{A}^1$; the point p_0 is, therefore, a one-place point of C'.

2: All members of Λ are irreducible, and Λ has at most two multiple members.

3: Let d be the degree of a general member C' of Λ; if $d = 1$ then Λ is the pencil of lines through p_0; if $d > 1$, let l_0 be the tangent line of a general member of Λ, i.e., a line of maximal contact with a general member of Λ at p_0; if $dl_0 \in \Lambda$ then dl_0 is the unique multiple member of Λ; if $dl_0 \notin \Lambda$ then Λ has two multiple members, say aF and bF, where F and G are irreducible and a, b are positive integers ≥ 2 such that $a = \deg(G), b = \deg(F), d = ab$ and $\text{g.c.d.}(a, b) = 1$; C is said to be of the first kind (resp. of the second kind) if $dl_0 \in \Lambda$ (resp. $dl_0 \notin \Lambda$).

On the other hand, with respect to $\overline{\kappa}(\mathbf{P}^2 - C)$ for an irreducible curve C on $\mathbf{P}^2$, the following assertions hold true (cf. Tsunoda [9] or Wakabayashi [10]):

Theorem 1.4 *Let C be an irreducible curve on $\mathbf{P}^2$. Then,*

(1) If C is irrational and $\deg(C) \geq 4$ then $\overline{\kappa}(\mathbf{P}^2 - C) = 2$.

(2) If C is rational and $\sharp\text{Sing}(C) \geq 3$, then $\overline{\kappa}(\mathbf{P}^2 - C) = 2$.

(3) If C is rational, $\sharp\text{Sing}(C) = 2$ and at least one of singularities is not a cuspidal point, then $\overline{\kappa}(\mathbf{P}^2 - C) = 2$.

(4) If C is rational, $\sharp\text{Sing}(C) = 2$ and both singularities are cuspidal points, then $\overline{\kappa}(\mathbf{P}^2 - C) \geq 1$.

Analogous to Theorem 1.3, and in consideration of Theorem 1.2 and Theorem 1.4, our problem is the following:

PROBLEM 1.5 *Let C be an irreducible curve on $\mathbf{P}^2$ with two cuspidal points and with $\overline{\kappa}(\mathbf{P}^2 - C) = 1$. Then we want to investigate the structure of its complement $X := \mathbf{P}^2 - C$ and ask for the construction of such a curve C. Furthermore, if possible, we want to ask for the homogeneous polynomial defining C (up to* PGL(2; **C**)*).*

Note that if C is a rational curve on $\mathbf{P}^2$ with two cuspidal points, then its complement is a **Q**-homology plane (cf. Section 3 for the definition of **Q**-homology planes). Problem 1.5 above can be interpreted as the classification of **Q**-homology planes with logarithmic Kodaira dimension 1, each of which is obtained by excluding an irreducible curve from $\mathbf{P}^2$.

As seen in Section 4, we can roughly classify such curves as in Problem 1.5 into two types, say *of the first type* and *of the second type* (cf. Theorem 2.1, (3) for the definition). In the first type, we can ask for the homogeneous polynomials by which such curves are defined. But, in the second type, it is hard to ask for the homogeneous polynomials by which such curves are defined. I think this difficulty is mainly due to the fact that the configuration of $\bar{F_1}$ (cf. Section 4) is not necessarily a linear chain. In this present article, we shall give an answer to Problem 1.5, for the case of the first type, only.

2 Main Theorem

In this section, we shall give some partial answers for Problem 1.5. Namely:

Theorem 2.1 *Let C be an irreducible curve on $\mathbf{P}^2$ with two cuspidal points and with $\overline{\kappa}(\mathbf{P}^2 - C) = 1$. Then the following assertions hold true:*

(1) There exists an irreducible linear pencil Λ on $\mathbf{P}^2$ such that the restriction of Φ_Λ (:= the rational mapping defined by Λ) on $X := \mathbf{P}^2 - C$ gives an untwisted $\mathbf{C}^*$-fibration onto $\mathbf{A}^1$:

$$\varphi = \Phi_\Lambda|_X : X \to \mathbf{A}^1.$$

(2) Λ has a unique reducible member, say $\bar{F_1}$, which consists of two components, say $\bar{F_{11}}$ and $\bar{F_{12}}$, and has a unique irreducible, multiple member, say $\bar{F_2}$. Furthermore, C itself is a member of Λ.

(3) The configuration of the unique reducible fibre of φ, $F_1 := \bar{F_1} \cap X = m_{11}F_{11} + m_{12}F_{12}$, where $F_{1j} := \bar{F_{1j}} \cap X$ for $j = 1, 2$, is one of the following two cases:

1: $F_{11} \cong F_{12} \cong \mathbf{A}^1$ *and* $F_{11} \cap F_{12} \neq \emptyset$.

2: $F_{11} \cong \mathbf{A}^1, F_{12} \cong \mathbf{C}^*$ *and* $F_{11} \cap F_{12} = \emptyset$.

If C is in the case 1 (resp. case 2), we say this curve C *of the first type* (resp. *of the second type*). We can solve the problem of Section 1 for curves of the first type. Namely,

Theorem 2.2 *There exists a bijective correspondence between the set of pairs of positive integers (d_0, d_1) such that $d_1 \geq 2, d_0 > 2d_1$ and* g.c.d.$(d_0, d_1) =$ *1, and the set of irreducible curves C on $\mathbf{P}^2$ such that $\overline{\kappa}(\mathbf{P}^2 - C) = 1$ and of the first type (up to* PGL(2; C)*). This correspondence is given by:*

$$(d_0, d_1) \mapsto C(d_0, d_1) := \{X^{d_1}Y^{d_0-d_1} + Z^{d_0} = 0\}.$$

3 Terminologies and Preliminary Results

In the subsequent section, we shall use the following terminologies and notations.

Linear pencil means linear system of dimension 1. The *base point* of the linear pencil Λ on a smooth surface V is a point on V which all members of Λ pass through. The set of all base points of the linear pencil Λ is denoted by Bs Λ. A *quasi-section* (resp. *cross-section*) of the linear pencil Λ on a smooth surface V is a curve H on V such that $H \cap \mathrm{Bs}\,\Lambda = \emptyset$ and $(H.C) > 0$ (resp. $(H.C) = 1$) for a general member C of Λ. A $\mathbf{P}^1$-fibration (resp. $\mathbf{A}^1$-fibration or $\mathbf{C}^*$-fibration) means a surjective morphism whose general fibres are irreducible and reduced and isomorphic to $\mathbf{P}^1$ (resp. $\mathbf{A}^1$ or $\mathbf{C}^*$). A $(-n)$-*curve* is a non-singular rational curve such that its self-intersection number is $(-n)$.

The surjective morphism $\varphi : S \to B$ from a smooth surface onto a smooth curve is called to be an *untwisted* $\mathbf{C}^{(n*)}$*-fibration* , if S has a smooth compactification $S \subset \bar{S}$ such that the $\mathbf{P}^1$- fibration from $\bar{S}$, which is an extension of φ, has exactly $(n+1)$-cross sections contained in the boundary $\bar{S} - S$.

For any topological space T, $e(T)$ denotes its Euler number, i.e., $e(T) := \Sigma_{i\geq 0}\dim_{\mathbf{Q}}(-1)^i H_i(T;\mathbf{Q})$.

A smooth affine surface S is called to be a $\mathbf{Q}$- *homology plane* , if $H_i(S;\mathbf{Q}) = 0$ for all $i > 0$. Note that if S is a $\mathbf{Q}$-homology plane, then $e(S) = 1$.

We shall define an Euclidean transformation associated with some datum and an EM-transformation (see below for the definitions), which play very important roles in this article.

Let V_0 be a smooth projective surface, let p_0 be a point on V_0 and let l_0 be an irreducible curve on V_0 such that p_0 is a simple point of l_0. Let d_0 and d_1 be positive integers such that $d_1 < d_0$. With respect to $d_1 < d_0$, we shall perform an Euclidean algorithm in the following way and find positive integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$:

$$\begin{cases} d_0 &= q_1 d_1 + d_2 & d_2 < d_1 \\ d_1 &= q_2 d_2 + d_3 & d_3 < d_2 \\ \cdots & \cdots\ \cdots & \cdots \\ d_{\alpha-2} &= q_{\alpha-1}d_{\alpha-1} + d_\alpha & d_\alpha < d_{\alpha-1} \\ d_{\alpha-1} &= q_\alpha d_\alpha & q_\alpha > 1 \end{cases}$$

Let $N := \Sigma_{s=1}^{\alpha} q_s$. Define the infinitely near points p_i's of p_0 (for $1 \leq i < N$) and the blow-up $\sigma_i : V_i \to V_{i-1}$ with a center at p_{i-1} (for $1 \leq i \leq N$) inductively in the following way:

(i) p_i is an infinitely near point of order one of p_{i-1} for $1 \leq i < N$.

(ii) Let $E_i := \sigma_i^{-1}(p_{i-1})$ for $1 \leq i \leq N$ and let $E(s,t) := E_i$ if $i = q_1 + \ldots + q_{s-1} + t$ with $1 \leq s \leq \alpha$ and $1 \leq t \leq q_s$, where we set $q_0 := 0$. The point p_i is an intersection point of the proper transform of $E(s-1, q_{s-1})$ on V_i and the exceptional curve $E(s,t)$ if $i = q_1 + \ldots + q_{s-1} + t$ with $1 \leq s \leq \alpha$ and $1 \leq t \leq q_s$ ($1 \leq t < q_\alpha$ if $s = \alpha$).

Then the composite $\sigma := \sigma_1 \ldots . \sigma_N$ is called to be an *Euclidean transformation associated with the datum* $\{p_0, l_0, d_0, d_1\}$ (cf. Miyanishi [4] p.95). The weighted dual graph of Supp$(\sigma^{-1}(l_0))$ is the following (cf. Figure 1), where $E_0 := \sigma'(l_0)$ and we denote the proper transform of $E(s,t)$ on V_N by the same notation $E(s,t)$ for $1 \leq s \leq \alpha$ and $1 \leq t \leq q_s$.

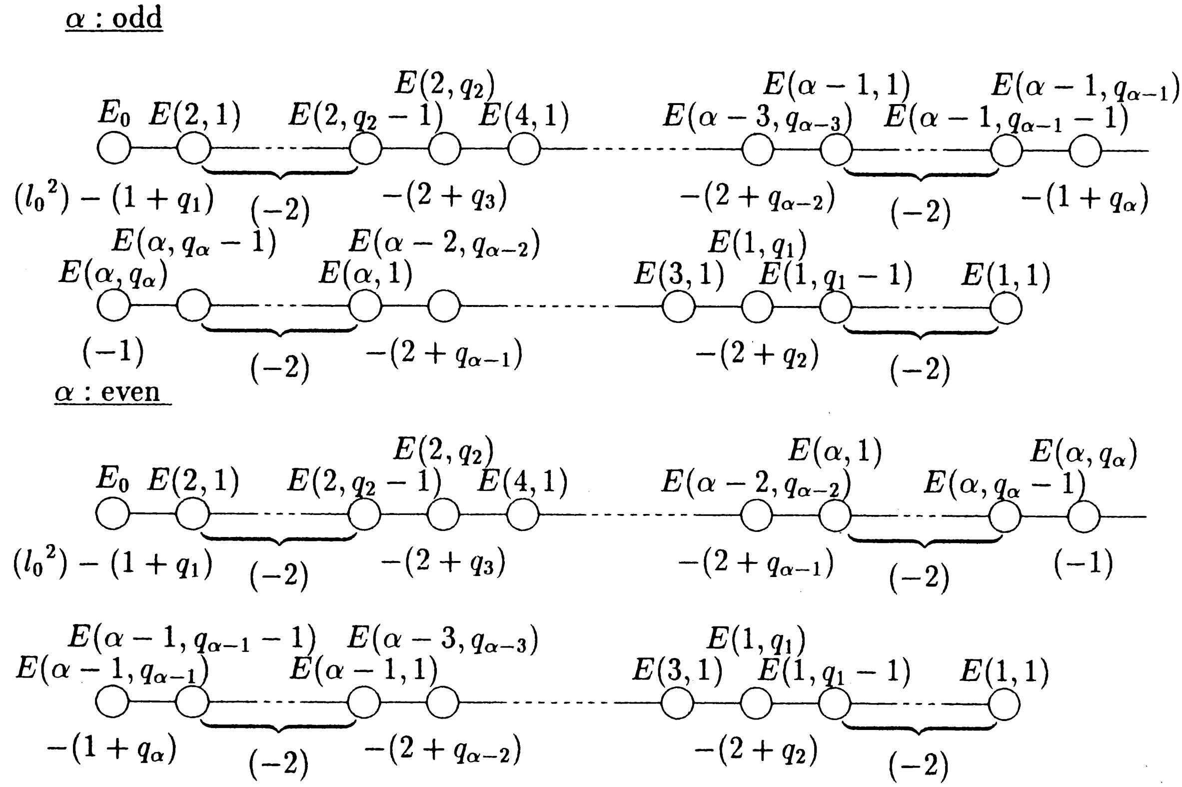

Figure 1:

If C_0 is an irreducible curve on V_0 such that p_0 is a one-place point of C_0 with $d_0 = i(C_0.l_0; p_0)$(:=the local intersection number of C_0 and l_0 at p_0) $> d_1 = \text{mult}_{p_0}(C_0)$(:= multiplicity of C_0 at p_0), then the proper transform $C_i := (\sigma_1.\cdots.\sigma_i)'(C_0)$ passes through p_i so that $(C_i.E(s,t)) = d_s$ and the intersection number of C_i with the proper transform of $E(s-1, q_{s-1})$ on V_i is $d_{s-1} - td_s$, where $i = q_1 + \ldots + q_{s-1} + t$ with $1 \leq s \leq \alpha$ and $1 \leq t \leq q_s$. The smaller one of d_s and $d_{s-1} - td_s$ is the multiplicity of C_i at p_i, since C_0 has a point p_0 as a one-place point. Note that the proper transform $\sigma'(C_0)$ on V_N meets only with the last exceptional curve $E(\alpha, q_\alpha)$ with order d_α, and does not meet with the other exceptional curves obtained in the process of σ and $E_0 := \sigma'(l_0)$.

Let V_0, p_0 and l_0 be as above. Let $r > 0$ be a positive integer. An *equi-multiplicity transformation of length* r with center at p_0 (or an *EM-transformation* , for short) is the composite $\tau = \tau_1.\cdots.\tau_r$ of blow-ups defined as follows:

For $1 \le i \le r$, $\tau_i : V_i \to V_{i-1}$ is the blow-up with a center at p_{i-1} and p_i is a point on $\tau_i^{-1}(p_{i-1})$ other than the point $\tau_i'(\tau_{i-1}^{-1}(p_{i-2})) \cap \tau_i^{-1}(p_{i-1})(\tau_1'(l_0) \cap \tau_1^{-1}(p_0)$ if $i = 1)$.

A related notion is the (e, i)-transformation defined in Miyanishi [4] p.100.

If C_0 is an irreducible curve on V_0 such that p_0 is a one-place point of C_0 with $d_0 := i(C_0.l_0; p_0)$ equal to $d_1 := \mathrm{mult}_{p_0}(C_0)$. Then $p_1 := \tau_1'(C_0) \cap \tau_1^{-1}(p_0)$ differs from $\tau_1'(l_0) \cap \tau_1^{-1}(p_0)$. If $d_1^{(1)} := \mathrm{mult}_{p_1}(\tau_1'(C_0))$ equals to $d_0 = d_1 = i(\tau_1'(C_0).\tau_1^{-1}(p_0); p_1)$, then $p_2 := (\tau_1.\tau_2)'(C_0) \cap \tau_2^{-1}(p_1)$ differs from the point $\tau_2'(\tau_1^{-1}(p_0)) \cap \tau_2^{-1}(p_1)$. Thus this process can be repeated as long as the intersection number of the proper transform of C with the last exceptional curve equals the multiplicity of the proper transform of C at the intersection point.

We use the following elementary but important results with respect to the singular fibre of the $\mathbf{P}^1$-fibration (cf. Miyanishi [4] p.115).

Lemma 3.1 *Let $f : V \to B$ be a surjective morphism from a smooth projective surface V onto a smooth complete curve B such that almost all fibres of f are isomorphic to $\mathbf{P}^1$, i.e., f is a $\mathbf{P}^1$-fibration. Let $F := n_1C_1 + \ldots + n_rC_r$ be a reducible singular fibre of f, where C_i is an irreducible curve, $C_i \ne C_j$ if $i \ne j$, $n_i > 0$ for $1 \le i \le r$ and $r \ge 2$.*

Then we have the following:

(1) The greatest common divisor g.c.d.$(n_1, \ldots, n_r)$ *is* 1, *and* $\mathrm{Supp}(F) = \bigcup_{i=1}^{r} C_i$ *is connected.*

(2) For $1 \le i \le r$, C_i is isomorphic to $\mathbf{P}^1$, and $(C_i^2) < 0$.

(3) For $i \ne j$, $(C_i.C_j) = 0$ or 1.

(4) For distinct three indices i, j and l, $C_i \cap C_j \cap C_l = \emptyset$.

(5) At least one of C_i's, say C_1, is an exceptional curve of the first kind, i.e., (-1)-curve.

(6) If one of n_i's, say n_1, equals 1, then there exists an exceptional curve of the first kind among C_i's with $2 \le i \le r$.

A generalization of Lemma 3.1 is the following (cf. Miyanishi [*ibid*]):

Lemma 3.2 *Let V be a smooth projective surface and let Λ be an irreducible linear pencil on V such that general member of Λ are rational curves. Let $F := n_1C_1 + \ldots + n_rC_r$ be a reducible member of Λ such that $r \geq 2$, where C_i is an irreducible curve, $C_i \neq C_j$ if $i \neq j$, and $n_i > 0$ for $1 \leq i \leq r$.*

Then we have the following:

(1) If $C_i \cap \mathrm{Bs}\Lambda = \emptyset$, then C_i is isomorphic to $\mathbf{P}^1$ and $({C_i}^2) < 0$.

(2) If $C_i \cap C_j \neq \emptyset$ for $i \neq j$ and $C_i \cap C_j \cap \mathrm{Bs}\Lambda = \emptyset$, then $C_i \cap C_j$ consists of one point where C_i and C_j intersect each other transversally.

(3) For distinct three indices i, j and l, if $C_i \cap C_j \cap C_l \cap \mathrm{Bs}\Lambda = \emptyset$, then $C_i \cap C_j \cap C_l = \emptyset$.

(4) Assume that $({C_i}^2) < 0$ whenever $C_i \cap \mathrm{Bs}\Lambda \neq \emptyset$. Then the set $\mathcal{S} := \{C_i; C_i$ is a component of F such that $C_i \cap \mathrm{Bs}\Lambda = \emptyset\}$ is non-empty and there exists an exceptional curve of the first kind in $\mathcal{S}$ other than C_1.

(5) With the same assumption as in (4) above, if a component in the set $\mathcal{S}$, say C_1, has multiplicity $n_1 = 1$ in F, then there exists an exceptional curve of the first kind in $\mathcal{S}$ other than C_1.

With respect to the Euler number, we shall use the following result (cf. Suzuki [8] and Zaidenberg [12]).

Lemma 3.3 *Let S be a smooth affine surface and let $\varphi : S \to B$ be a morphism onto a smooth curve B. Then we have the following equality:*

$$e(S) = e(B)e(F) + \Sigma_i(e(F_i) - e(F))$$

where F is a general fibre of φ and the summation is over all the singular fibres of φ. Furthermore, $e(F_i) \geq e(F)$ for all i and the equality occurs if and only if either $F \cong \mathbf{C}$ or $F \cong \mathbf{C}^*$ and $(F_i)_{\mathrm{red}} \cong F$.

The following lemma is a straightforward computation. So we shall omit the proof.

Lemma 3.4 *Let d_0 and d_1 be positive integers such that $d_0 > d_1$ and* g.c.d.$(d_0, d_1) =$ *1. We shall perform an Euclidean algorithm with respect to $d_1 < d_0$ and obtain positive integers $d_2, \ldots, d_\alpha = 1$ and $q_1, \ldots, q_\alpha$ as in the definition of an*

Euclidean transformation. Let $q_s' := q_{\alpha+1-s}$ *for* $1 \leq s \leq \alpha$. *Define positive integers* $b(s,t)$ *for* $1 \leq s \leq \alpha$ *and* $1 \leq t \leq q_s'$ *as follows:*

$$\begin{array}{lcll} b(1,t) & := & 1+t & 1 \leq t \leq q_1' \\ b(2,t) & := & b(1,q_1') + tb(1,q_1'-1) & 1 \leq t \leq q_2' \\ b(s,t) & := & b(s-1,q_{s-1}') + tb(s-1,q_{s-1}'-1) & 2 \leq s \leq \alpha, 1 \leq t \leq q_s' \end{array}$$

Then it follows that $b(\alpha - i, q_{\alpha-i}' - 1) = d_i$ *for* $0 \leq i \leq \alpha - 1$.

4 Proof of Theorem 2.1

Let C be an irreducible curve on $\mathbf{P}^2$ with two cuspidal points and with $X := \mathbf{P}^2 - C$ has logarithmic Kodaira dimension $\overline{\kappa}(X) = 1$. Then by Theorem 1.2, there exists a non-empty Zariski open subset $U \subseteq X$ and an irreducible linear pencil Λ on $\mathbf{P}^2$ such that $\varphi := \Phi_\Lambda|_U$ gives a $\mathbf{C}^*$-fibration. For distinct general members C_1, C_2 of Λ, $C_1 \cap U \cong C_2 \cap U \cong \mathbf{C}^*$ and $(C_1 \cap C_2) \cap U = \emptyset$. Thus C_1 has one or two points on the curve C and $\mathrm{Bs}\Lambda$ consists of one or two points. If a point $p \in \mathrm{Bs}\Lambda$ is contained in X, we shall consider a succession of blow-ups with centers at p (including its infinitely near points), say $\sigma : V \to \mathbf{P}^2$, such that $E \cap \mathrm{Bs}(\sigma'(\Lambda)) = \emptyset$, where E is the last exceptional curve obtained in the process of σ. Then $\sigma^{-1}(X)$ has a structure of an $\mathbf{A}^1$-fibration, so it follows that $\overline{\kappa}(\sigma^{-1}(X)) = -\infty$. On the other hand, since $\sigma^{-1}(X)$ is obtained from X via several blow-ups, we have $\overline{\kappa}(X) = \overline{\kappa}(\sigma^{-1}(X))$. This is a contradiction to our condition $\overline{\kappa}(X) = 1$. Thus it follows that $\mathrm{Bs}\Lambda \cap X = \emptyset$ and φ can be extended to a $\mathbf{C}^*$-fibration:

$$\varphi : X \to B \subseteq \mathbf{P}^1.$$

Thus the assertion (1) of Theorem 2.1 is proved.

If the general member C' of Λ meets with C at only one point, say p_1, then $\mathrm{Bs}\Lambda = \{p_1\}$ and C is contained in a member of Λ, say Δ. Let $\tilde{\sigma}$ be the elimination of base points of Λ and let $\tilde{\Delta}$ be the member of $\tilde{\Lambda} := \tilde{\sigma}'(\Lambda)$, corresponding to Δ. Let $\tilde{C} := \tilde{\sigma}'(C)$ be the proper transform by $\tilde{\sigma}$ of C. Note that the linear pencil $\tilde{\Lambda}$ defines a $\mathbf{P}^1$-fibration and $\tilde{\Delta}$ is the fibre of this

$\mathbf{P}^1$-fibration. Since C has two cuspidal points, $\tilde{C}$ has at least one singular point. This is a contradiction to Lemma 3.1 (esp. (2)). Hence, we know that the general members of Λ meet with exactly two points. Then $\mathrm{Bs}\Lambda$ consists of one or two points.

We shall prove, in fact, the following:

Lemma 4.1 *The case where* $\mathrm{Bs}\Lambda = \{$ one point $\}$ *does not occur.*

Proof. Indeed, assume, on the contrary, that $\mathrm{Bs}\Lambda$ consists of only one point, say $\mathrm{Bs}\Lambda = \{p_0\}$. Then for a general member C_1 of Λ, $C_1 - \{p_0\} \cong \mathbf{A}^1$ and has a point p_0 as a one-place point. Moreover for distinct general members C_1, C_2 of Λ, it follows that $C_1 \cap C_2 = \{p_0\}$, and $C_1 - (C_1 \cap X) - \{p_0\}$ and $C_2 - (C_2 \cap X) - \{p_0\}$ are distinct points on C. Then no members of Λ contains C as a component since C intersects with C_1 and C_2 at distinct points which are other than p_0. Therefore $\varphi = \Phi_\Lambda|_X$ is a $\mathbf{C}^*$-fibration onto $\mathbf{P}^1$.

Let $\tilde{\sigma} : \tilde{V} \to \mathbf{P}^2$ be the shortest succession of blow-ups with centers at p_0 (including its infinitely near points) such that $\tilde{\Lambda} := \tilde{\sigma}'(\Lambda)$ has no base points. Note that this process $\tilde{\sigma}$ is the composite of Euclidean transformations and EM-transformations (cf. Section 3, for the definitions), which are uniquely determined by general members of Λ since general members of Λ have a point p_0 as a one-place point.

Among the boundary curves contained in $D := \tilde{V} - X$ (where we identify $\tilde{\sigma}^{-1}(X)$ with X), the last exceptional curve obtained in the process of $\tilde{\sigma}$, say $\tilde{H}$, and $\tilde{C} := \tilde{\sigma}'(C)$ are cross-sections of $\tilde{\Lambda}$ and all other components are contained in some members of $\tilde{\Lambda}$. Note that $\tilde{\Lambda}$ defines a $\mathbf{P}^1$-fibration $\Phi_{\tilde{\Lambda}} : \tilde{V} \to \mathbf{P}^1$ such that its restriction onto X coincides with φ. Since D contains two cross-sections of $\tilde{\Lambda}$, it follows that φ is an untwisted $\mathbf{C}^*$-fibration. By looking at the configuration of $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_0))$, we know that $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_0))$ consists of at most two connected components except for $\tilde{H}$ (cf. Section 3, Figure 1). Therefore we know that Λ has at most two multiple members by Lemma 3.1 (esp.(1)).

Let $F_1, \ldots, F_r$ exhaust all singular fibres of $\varphi = \Phi_\Lambda|_X = \Phi_{\tilde{\Lambda}}|_X : X \to \mathbf{P}^1$. Let $\tilde{F}_i$ be a member of $\tilde{\Lambda}$, corresponding to F_i for $1 \le i \le r$. We write $\tilde{F}_i$ as:

$$\tilde{F}_i = \Sigma_{j=1}^{s_i} \mu_{ij} C_{ij} + \Sigma_{j=s_i+1}^{t_i} \nu_{ij} E_{ij}$$

where $C_{ij} \cap X \neq \emptyset$ and $E_{ij} \cap X = \emptyset$.

On the other hand, since $\tilde{H}$ and $\tilde{C}$ are cross-sections of $\tilde{\Lambda}$, we can write:

$$\tilde{H} - \tilde{C} \sim \Sigma_{i,j}\alpha_{ij}C_{ij} + (\text{a linear combination of } E_{ij}{}'s).$$

Then associated with the fibration $\varphi : X \to \mathbf{P}^1$, we know that Pic$(X)$ is an abelian group defined by the following generators and the relations:

$$\mathrm{Pic}(X) = \left\langle \begin{array}{c} [C_{ij}] \\ i = 1, \ldots, r \\ j = 1, \ldots, s_i \end{array} \middle| \begin{array}{c} \Sigma_{j=1}^{s_1}\mu_{1j}[C_{1j}] = \ldots = \Sigma_{j=1}^{s_r}\mu_{rj}[C_{rj}] \\ \\ \Sigma_{i,j}\alpha_{i,j}[C_{ij}] = 0 \end{array} \right\rangle.$$

Therefore we can obtain:

$$\text{rank Pic}(X) \geq \Sigma_{i=1}^{r} s_i - (r-1) - 1.$$

On the other hand, since $X = \mathbf{P}^2 - C$, it is clear that Pic(X) is a finite group. Hence rank Pic$(X) = 0$. This implies that $s_1 = \ldots = s_r = 1$, i.e., all fibres of φ are irreducible (but at most two fibres of φ might be multiple).

Let Δ_1 and Δ_2 be members of Λ, corresponding to multiple fibres of φ if they exist at all, and let $Q_1 := \varphi(\Delta_1)$ and $Q_2 := \varphi(\Delta_2)$. All members of Λ other than Δ_1 and Δ_2 are irreducible and reduced. Thus X contains a Zariski open subset $U := \varphi^{-1}(\mathbf{P}^1 - \{Q_1, Q_2\}) \cong \mathbf{C}^* \times \mathbf{C}^*$. But then $\overline{\kappa}(X) \leq \overline{\kappa}(U) = \overline{\kappa}(\mathbf{C}^* \times \mathbf{C}^*) = 0$, which is a contradiction to our condition $\overline{\kappa}(X) = 1$.

Therefore the case where Bs$\Lambda = \{$ one point $\}$ does not occur.

Q.E.D.

Hence in the subsequent argument we have only to consider the case where Bs$\Lambda = \{$ two points $\} = \{p_1, p_2\}$. For a general member C' of Λ, it follows that $C' - \{p_1, p_2\} \cong \mathbf{C}^*$. So it is clear that C' has points p_1 and p_2 as one-place points. Furthermore, since C' meets with C at only Bs$\Lambda = \{p_1, p_2\}$, C is a component of a member of Λ, say $\Delta \in \Lambda$, and C is a rational curve with two cuspidal points p_1 and p_2.

The shortest succession of blow-ups with centers at Bs$\Lambda = \{p_1, p_2\}$ (including their infinitely near points), say $\tilde{\sigma} : \tilde{V} \to \mathbf{P}^2$, such that the proper transform $\tilde{\Lambda} := \tilde{\sigma}'(\Lambda)$ has no base points is constructed in the following way:

Construction of $\tilde{\sigma}$: For a general member C' of Λ, let l_1 (resp. l_2) be a tangent line of C' at p_1 (resp. p_2). Set $d_{1,0} := i(C'.l_1; p_1), d_{1,1} := \mathrm{mult}_{p_1}(C'), d_{2,0} := i(C'.l_2; p_2)$ and $d_{2,1} := \mathrm{mult}_{p_2}(C')$. Note that it follows that $d_{1,0} > d_{1,1}$. Indeed, otherwise, then C' is a line. Hence Λ is a linear pencil composed of lines. So C is also a line. But then $\overline{\kappa}(X) = -\infty$, which is a contradiction. Similarly we have $d_{2,0} > d_{2,1}$. Let σ_1 be the shortest composite of Euclidean transformations and EM-transformations which starts with an Euclidean transformation associated with the datum $\{p_1, l_1, d_{1,0}, d_{1,1}\}$ (cf. Section 3), such that $\mathrm{Bs}(\sigma_1{}'(\Lambda)) \cap \bar{H}_1 = \emptyset$, where $\bar{H}_1$ is the last exceptional curve obtained in the process of σ_1.

Next let $\bar{p_2} := \sigma_1{}^{-1}(p_2), \bar{C'} := \sigma_1{}'(C')$ and $\bar{l_2} := \sigma_1{}'(l_2)$. Note that the process of σ_1 is concentrated on a point p_1, so it follows that $i(C'.l_2; p_2) = i(\bar{C'}.\bar{l_2}; \bar{p_2})$ and $\mathrm{mult}_{p_2}(C') = \mathrm{mult}_{\bar{p_2}}(\bar{C'})$.

Let $\sigma_2 : \tilde{V} \to \bar{V}$ be the shortest composite of Euclidean transformations and EM-transformations which starts with an Euclidean transformation associated with the datum $\{\bar{p_2}, \bar{l_2}, d_{2,0}, d_{2,1}\}$ such that $\mathrm{Bs}((\sigma_1.\sigma_2)'(\Lambda)) \cap \tilde{H}_2 = \emptyset$, where $\tilde{H}_2$ is the last exceptional curve obtained in the process of σ_2.

Then the composite $\tilde{\sigma} := \sigma_1.\sigma_2 : \tilde{V} \to \mathbf{P}^2$ is the one as required.

Note that among the boundary curves contained in $D := \tilde{V} - X$, $\tilde{H}_1 := \sigma_2{}'(\bar{H}_1)$ and $\tilde{H}_2$ are cross-sections of $\tilde{\Lambda} := \tilde{\sigma}'(\Lambda)$, and all other components are contained in some members of $\tilde{\Lambda}$. Furthermore, $\tilde{\Lambda}$ defines a $\mathbf{P}^1$-fibration $\Phi_{\tilde{\Lambda}} : \tilde{V} \to \mathbf{P}^1$ such that its restriction onto X coincides with $\varphi = \Phi_\Lambda|_X$. So it is clear that φ is an untwisted $\mathbf{C}^*$-fibration. By looking at the configuration of $\mathrm{Supp}(\tilde{\sigma}^{-1}(\{p_1, p_2\}))$, we know that Λ has at most two multiple members by Lemma 3.1 (esp.(1)).

Let Δ be the member of Λ containing C as a component. Then the follwing two cases can occur:

$$\begin{cases} \text{Case I.} & \mathrm{Supp}(\Delta) \supset C \\ \text{Case II.} & \mathrm{Supp}(\Delta) = C \end{cases}$$

Note that since C is a rational curve with two cuspidal points, $X = \mathbf{P}^2 - C$ is a $\mathbf{Q}$-homology plane.

At first, we shall consider Case I, above. We prove, in fact, the following:

Lemma 4.2 *Case I above does not occur.*

Proof. Assume, on the contrary, that Case I does occur. Then $\varphi = \Phi_\Lambda|_X$ is an untwisted $\mathbf{C}^*$-fibration onto $\mathbf{P}^1$. We write Δ as:

$$\Delta = mC + \Sigma_{k=1}^{l} m_k C_k$$

where C_k is an irreducible component of Δ and $m_k > 0$ $(1 \leq k \leq l)$. Let $\tilde{\Delta}$ be a member of $\tilde{\Lambda}$, corresponding to Δ. We write $\tilde{\Delta}$ as:

$$\tilde{\Delta} = m\tilde{C} + \Sigma_{k=1}^{l} m_k \tilde{C}_k + \Sigma_j \nu_j E_j$$

where $\tilde{C} := \tilde{\sigma}'(C), \tilde{C}_k := \tilde{\sigma}'(C_k)$ $(1 \leq k \leq l)$ and $E_j \cap X = \emptyset (\forall j)$.

Let $F_i (\neq \Delta \cap X)$ for $1 \leq i \leq r$ exhaust all singular fibres of φ other than $\Delta \cap X$ and let $\tilde{F}_i$ be a member of $\tilde{\Lambda}$, corresponding to F_i. We write $\tilde{F}_i$ as:

$$\tilde{F}_i = \Sigma_{j=1}^{s_i} \mu_{ij} C_{ij} + \Sigma_{j=s_i+1}^{t_i} \nu_{ij} E_{ij}$$

where $C_{ij} \cap X \neq \emptyset$ and $E_{ij} \cap X = \emptyset$ for $1 \leq i \leq r$.

On the other hand, since $\tilde{H}_1$ and $\tilde{H}_2$ are cross-sections of $\tilde{\Lambda}$, we can write:

$$\tilde{H}_1 - \tilde{H}_2 \sim \beta\tilde{C} + \Sigma_{k=1}^{l} \beta_k \tilde{C}_k + \Sigma_{i,j} \alpha_{ij} C_{ij} + (\text{a linear combination of } E_{ij}{}' s).$$

Then associated with the fibration $\varphi : X \to \mathbf{P}^1$, we know that Pic$(X)$ is an abelian group defined by the following generators and the relations:

$$\mathrm{Pic}(X) = \left\langle \begin{array}{c} [\tilde{C}_1], \cdots, [\tilde{C}_l] \\ C_{ij} \\ i = 1, \cdots, r \;\; j = 1, \cdots, s_i \end{array} \middle| \begin{array}{c} \Sigma_{k=1}^{l} m_k [\tilde{C}_k] = \\ \Sigma_{j=1}^{s_1} \mu_{1j} [C_{1j}] = \ldots = \Sigma_{j=1}^{s_r} \mu_{rj} [C_{rj}] \\ \Sigma_{k=1}^{l} \beta_k [\tilde{C}_k] + \Sigma_{i,j} \alpha_{ij} [C_{ij}] = 0 \end{array} \right\rangle.$$

Hence we have:

$$\text{rank Pic}(X) \geq l + \Sigma_{i=1}^{r} s_i - r - 1.$$

But since Pic(X) is a finite group, so rank Pic$(X) = 0$. Hence it follows that $l = 1$ and $s_1 = \ldots = s_r = 1$. Namely, Δ contains one component other than C, say C_1, and all fibres of $\varphi : X \to \mathbf{P}^1$ are irreducible. We write Δ as $\Delta = mC + m_1 C_1$, where m and m_1 are positive integers. If g.c.d.$(m, m_1) > 1$, then some of exceptional curve obtained in the process of $\tilde{\sigma}$ must be contained in the member $\tilde{\Delta}$ of $\tilde{\Lambda}$, corresponding to Δ by Lemma 3.1 (esp.(1)). Furthermore, at most one member of Λ other than Δ, say Δ_1 if it exists at all, is multiple, and all members of Λ other than Δ and Δ_1 are irreducible and reduced. Let $Q := \varphi(\Delta)$ and $Q_1 := \varphi(\Delta_1)$. Then X contains a Zariski open subset $U := \varphi^{-1}(\mathbf{P}^1 - \{Q, Q_1\})$, which is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. Hence $\overline{\kappa}(X) \leq \overline{\kappa}(U) = 0$, which is a contradiction to our condition $\overline{\kappa}(X) = 1$. So it follows that g.c.d.$(m, m_1) = 1$. Then Λ contains exactly two irreducible, multiple members, say Δ_1 and Δ_2, which are not Δ. Indeed, otherwise, by the same argument as above, X contains a Zariski open subset U, which is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. This is a contradiction. Let $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ be members of $\tilde{\Lambda}$, corresponding to Δ_1 and Δ_2, respectively. Then each exceptional curve obtained in the process of $\tilde{\sigma}$ other than $\tilde{H}_1$ and $\tilde{H}_2$, is contained in $\tilde{\Delta}_1$ or $\tilde{\Delta}_2$ by Lemma 3.1 (esp.(1)).

On the other hand, for a general member C' of Λ, C_1 must intersect with C' at p_1 or p_2. Indeed, otherwise, C_1 intersects with C' at some point which is not in BsΛ by Bezout's theorem. Then C_1 is not contained in a member of Λ, which is a contradiction. Via the process of $\tilde{\sigma}$, $\tilde{C} := \tilde{\sigma}'(C)$ and $\tilde{C}_1 := \tilde{\sigma}'(C_1)$ intersect each other at a point which lies on $\tilde{H}_1$ or $\tilde{H}_2$, since all exceptional curves obtained in the process of $\tilde{\sigma}$ other than $\tilde{H}_1$ and $\tilde{H}_2$ are contained in $\tilde{\Delta}_1$ or $\tilde{\Delta}_2$. This is a contradiction. Thus Case I does not occur. Q.E.D.

Thus in the subsequent arguments, we have only to consider Case II, i.e., Supp$(\Delta) = C$. Then φ is an untwisted $\mathbf{C}^*$- fibration from X onto an affine line $\mathbf{A}^1$. Let F_i for $1 \leq i \leq r$ exhaust all singular fibres of φ and let $\tilde{F}_i$ be a member of $\tilde{\Lambda}$, corresponding to F_i for $1 \leq i \leq r$. For each $1 \leq i \leq r$, we write $\tilde{F}_i$ as:

$$\tilde{F}_i = \Sigma_{j=1}^{s_i} \mu_{ij} C_{ij} + \Sigma_{j=s_i+1}^{t_i} \nu_{ij} E_{ij}$$

where $C_{ij} \cap X \neq \emptyset$ and $E_{ij} \cap X = \emptyset$.

On the other hand, since $\tilde{H}_1$ and $\tilde{H}_2$ are cross-sections of $\tilde{\Lambda}$, we can write:

$\tilde{H}_1 - \tilde{H}_2 \sim \Sigma_{i,j} \alpha_{ij} C_{ij} +$ (a linear combination of $E_{ij}'s$ and components of $\tilde{\Delta}$)

where $\tilde{\Delta}$ is a member of $\tilde{\Lambda}$, corresponding to Δ.

Then Pic(X) is an abelian group defined by the following generators and the relations:

$$\mathrm{Pic}(X) = \left\langle \begin{array}{c} [C_{ij}] \\ i = 1, \cdots, r \\ j = 1, \cdots, s_i \end{array} \middle| \begin{array}{c} \Sigma_{j=1}^{s_1} \mu_{1j}[C_{1j}] = \ldots = \Sigma_{j=1}^{s_r} \mu_{rj}[C_{rj}] = 0 \\ \\ \Sigma_{i,j} \alpha_{ij}[C_{ij}] = 0 \end{array} \right\rangle .$$

Thus we have:

$$\text{rank Pic}(X) \geq \Sigma_{i=1}^{r} s_i - r - 1.$$

But since Pic(X) is a finite group, it follows that $s_1 \leq 2$ and $s_2 = \ldots = s_r = 1$, by changing indices if necessary. However, if $s_1 = s_2 = \ldots = s_r = 1$, we write $\tilde{F}_i$ as:

$$\tilde{F}_i = m_i C_i + \Sigma_j \nu_{ij} E_{ij}$$

for $1 \leq i \leq r$. Let $N := \prod_{i=1}^{r} m_i$. Then it is easy to see that:

$$N(\tilde{H}_1 - \tilde{H}_2) \sim \Sigma_{i,j}\gamma_{ij}E_{ij} + (\text{a linear combination of components } \tilde{\Delta}'s).$$

Thus we can obtain a non-trivial relation with respect to the images in $\mathrm{Pic}(\tilde{V})$ of the boundary components contained in $D := \tilde{V} - X$. But since X is a **Q**-homology plane as remarked before Lemma 4.2, it follows that $\Gamma(\mathcal{O}_X)^* = \mathbf{C}^*$ (cf. Miyanishi and Sugie [7], Lemma 1.1). Thus we get a contradiction. Hence we can deduce that $s_1 = 2, s_2 = \ldots = s_r = 1$, i.e., Λ has exactly one reducible fibre, say $\bar{F}_1$, which consists of two components, say $\bar{F}_{11}$ and $\bar{F}_{12}$.

If $\bar{F}_{11}$ and $\bar{F}_{12}$ intersect each other at a point other than p_1 and p_2, then $\bar{F}_{1j}$ passes through p_j for $j = 1, 2$ and they meet in one point transversally by Lemma 3.2 (esp.(2)). Then $F_{11} \cong F_{12} \cong \mathbf{A}^1$, where $F_{1j} := \bar{F}_{1j} \cap X$ for $j = 1, 2$. Note that in this case $\bar{F}_{1j}$ are lines by Bezout's theorem. We call C in this case to be *of the first type* .

On the other hand, if $\bar{F}_{11}$ and $\bar{F}_{12}$ intersect each other at p_1 or p_2, then $F_{11} \cong \mathbf{A}^1$ and $F_{12} \cong \mathbf{C}^*$ and $F_{11} \cap F_{12} = \emptyset$, where $F_{1j} := \bar{F}_{1j} \cap X$ for $j = 1, 2$. We call C in this case to be *of the second type* . Note that it is impossible that both $\bar{F}_{11}$ and $\bar{F}_{12}$ pass through the points p_1 and p_2. Indeed, otherwise, then φ has a reducible fibre F_1 such that $(F_1)_{\mathrm{red}}$ is a disjoint union of two $\mathbf{C}^{*\prime}$s. This is a contradiction by noting that $e(X) = 1$ and by Lemma 3.3.

For C in the first type, if we write $\bar{F}_1 = m_{11}\bar{F}_{11} + m_{12}\bar{F}_{12}$, then at least one of m_{11} and m_{12} is larger than 1. Indeed, assume, on the contrary, that $m_{11} = m_{12} = 1$, then Λ is a linear pencil of curves of degree 2. But since C is a member of Λ, it follows that $\bar{\kappa}(X) = -\infty$, which is a contradiction. Thus we know that some of exceptional curves obtained in the process of $\tilde{\sigma}$ are contained in a member of $\tilde{\Lambda}$, corresponding to $\bar{F}_1$, say $\tilde{F}_1$. By looking at the configuration of $\mathrm{Supp}(\tilde{\sigma}^{-1}(\{p_1, p_2\})$, we know that Λ has at most one irreducible, multiple member. But if all members of Λ other than $\bar{F}_1$ are irreducible and reduced, then X contains a Zariski open subset which is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. This is a contradiction. Hence Λ has exactly one irreducible, multiple member, say $\bar{F}_2$, and $\bar{F}_2 \neq \Delta$.

For C of the second type, let $\tilde{F}_1$ be a member of $\tilde{\Lambda}$, corresponding to $\bar{F}_1$. Then $\tilde{F}_1$ contains some exceptional curves obtained in the process of $\tilde{\sigma}$. Indeed, otherwise, the proper transforms of $\bar{F}_{11}$ and $\bar{F}_{12}$ by $\tilde{\sigma}$ meet at a point which lies on $\tilde{H}_1$ or $\tilde{H}_2$. This is a contradiction. By the same argument a for the first type, we know that Λ has exactly one irreducible, multiple member, say $\bar{F}_2$, and $\bar{F}_2 \neq \Delta$.

In both cases of the first type and of the second type, all members of Λ other than $\bar{F}_1$ and $\bar{F}_2$ are irreducible and reduced. Hence it follows that $\Delta = C$.

Thus we have proved the assertions of Theorem 2.1.

5 Proof of Theorem 2.2

In this section, we shall consider irreducible curves on $\mathbf{P}^2$ of the first type. Let C be an irreducible curve on $\mathbf{P}^2$ such that $\overline{\kappa}(X) = 1$, where $X := \mathbf{P}^2 - C$ and of the first type, i.e., with the notations as in the proof of Theorem 2.1, the unique reducible fibre of $\varphi : X \to \mathbf{A}^1$, say $F_1 = m_{11}F_{11} + m_{12}F_{12}$ is of the form $F_{11} \cong F_{12} \cong \mathbf{A}^1$ and $F_{11} \cap F_{12} \neq \emptyset$. Let F_2 be the unique irreducible, multiple fibre of φ. Let $\bar{F}_1 = m_{11}\bar{F}_{11} + m_{12}\bar{F}_{12}$ (resp. $\bar{F}_2$) be the member of Λ, corresponding to F_1 (resp. F_2), where $\bar{F}_{1j}$ is the component, corresponding to F_{1j} for $j = 1, 2$. Let $\tilde{F}_1$ (resp. $\tilde{F}_2$) be the member of $\tilde{\Lambda}$, corresponding to $\bar{F}_1$ (resp. $\bar{F}_2$). Let C_{11}, C_{12} and C_2 be components in the fibres $\tilde{F}_1$ and $\tilde{F}_2$, corresponding to $\bar{F}_{11}, \bar{F}_{12}$ and $\bar{F}_2$, respectively. Then we note the following:

Lemma 5.1 *With the notations and the assumptions as above. Then the dual graphs of $\tilde{F}_1$ and $\tilde{F}_2$ are linear chains, i.e., the dual graph of $\tilde{F}_i$ is a tree and there are no vertices sprouting to three or more other vertices in $\tilde{F}_i$ $(i = 1, 2)$.*

Proof. Note that, by the construction of $\tilde{\sigma}$ (cf. Section 4), all exceptional curves obtained in the process of $\tilde{\sigma}$ other than $\tilde{H}_1$ and $\tilde{H}_2$ have self-intersection numbers ≤ -2. If the (weighted) dual graph of $\tilde{F}_1$ is not a linear chain, then the configuration of $\tilde{F}_1 \cup \tilde{H}_1 \cup \tilde{H}_2$ is the following (cf. Figure 2).

Note that exactly one of C_{11} and C_{12} must be a (-1)-curve by Lemma 3.1. Starting with the contraction of C_{11} or C_{12}, after successive contractions of (-1)-curves, we can obtain a smooth fibre of a $\mathbf{P}^1$-fibration which is an image of the component of $\tilde{F}_1$ intersecting with $\tilde{H}_1$. However, in the process of this successive contractions, we are compled to contract the (-1)-curve E in Figure 3. This is a contradiction by Lemma 3.1.

Similarly, we know that $\tilde{F}_2$ is also a linear chain.

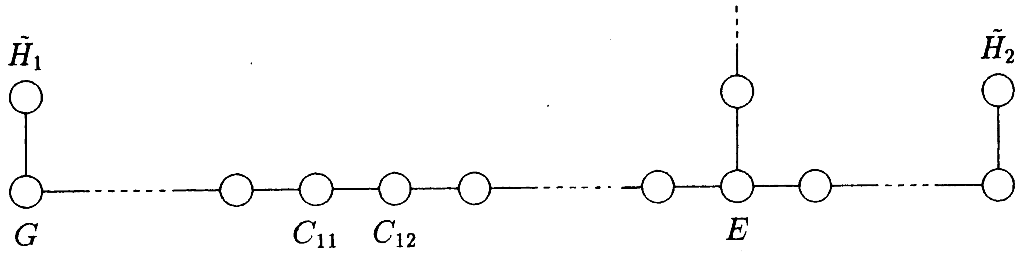

Figure 2:

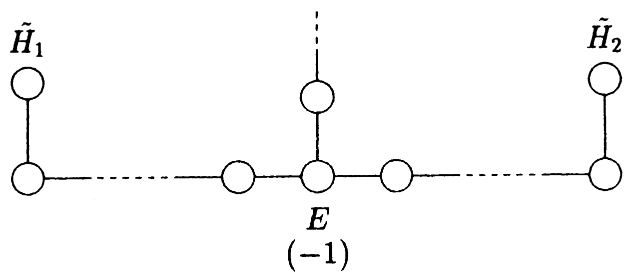

Figure 3:

Q.E.D.

By Lemma 5.1, it follows that both configurations of $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_1))$ and $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_2))$ are rational linear chains. In order that $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_i))$ is a linear chain, σ_i (cf. Section 4 for the notations σ_1 and σ_2) must be one of the following for $i = 1, 2$:

1. σ_i is a unique Euclidean transformation associated with a datum $\{p_i, l_i, d_{i,0}, d_{i,1}\}$ (cf. Section 4) such that g.c.d.$(d_{i,0}, d_{i,1}) = 1$.

2. σ_i is a composite of Euclidean transformations and EM-transformations:

$$\sigma_i = \sigma_i^{(1)}.\tau_i^{(1)}.\cdots.\sigma_i^{(n_i-1)}.\tau_i^{(n_i-1)}.\sigma_i^{(n_i)}.\tau_i^{(n_i)} \quad \text{with } n_i \geq 2$$

where $\sigma_i^{(j)}$ is the j-th Euclidean transformation in σ_i associated with a datum $\mathcal{D}_i^{(j)} := \{p_i^{(j)}, l_i^{(j)}, d_{i,0}^{(j)}, d_{i,1}^{(j)}\}$ (we set $p_i^{(1)} := p_i, l_i^{(1)} := l_i, d_{i,0}^{(1)} := d_{i,0}$ and $d_{i,1}^{(1)} := d_{i,1}$) and $\tau_i^{(j)}$ is the j-th EM-transformation in σ_i. $\tau_i^{(j)}$ might be an identity morphism. But in order that $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_i))$ is a linear chain, with respect to the j-th datum $\mathcal{D}_i^{(j)}$ of $\sigma_i^{(j)}$, $d_{i,1}^{(j)}$ must divide $d_{i,0}^{(j)}$ for $1 \leq j < n_i$ $(i = 1, 2)$.

We shall prove, in fact, the following:

Lemma 5.2 *At least one of σ_1 and σ_2 is in the case 1.*

Proof. At first, note that both $\bar{F}_{11}$ and $\bar{F}_{12}$ are lines as remarked in Section 4. Furthermore, since $\bar{F}_{11}$ (resp. $\bar{F}_{12}$) meets with C at only p_1 (resp. p_2), $\bar{F}_{11}$ (resp. $\bar{F}_{12}$) is a tangent line of a general member of Λ at p_1 (resp. p_2). Now assume, on the contrary, that both σ_1 and σ_2 are in the case **2.**, above. After the first Euclidean transformation $\sigma_1^{(1)}$ associated with a datum $\mathcal{D}_1^{(1)}$, the configuration of $\sigma_1^{(1)'}(\bar{F}_{11}) \cup \mathrm{Supp}(\sigma_1^{(1)^{-1}}(p_1))$ is the following (cf. Figure 4) ,where the component A is the last exceptional curve obtained

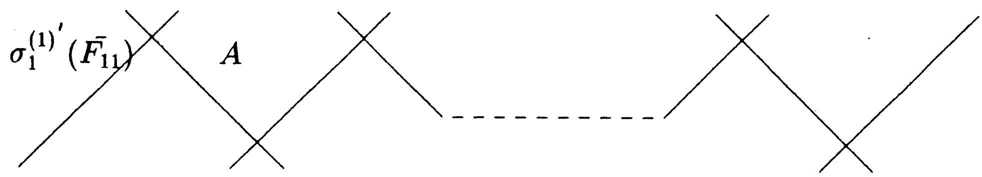

Figure 4:

in the process of $\sigma_1^{(1)}$. In order to complete σ_1, we must perform saveral blow-ups with center at a point (including its infinitely near point) which is an intersection point of the proper tramsform by $\sigma_1^{(1)}$ of a general member of Λ and the component A in Figure 4, say Q_1. Note that this point Q_1 differs from a point $\sigma_1^{(1)'}(\bar{F}_{11}) \cap A$. But in order that the fibres $\tilde{F}_1$ and $\tilde{F}_2$ are linear chains, it follows that $\sigma_1^{(1)'}(\bar{F}_2) \cap A = Q_1, n_1 = 2$, $\tau_1^{(1)}$ and $\tau_1^{(2)}$ are identity morphism and $\sigma_1^{(2)}$ is an Euclidean transformation associated with the datum $\mathcal{D}_1^{(2)} = \{Q_1, A, d_{1,0}^{(2)} = d_{1,1}^{(1)}, 1\}$. By the same argument, σ_2 holds similar conditions. Then $\tilde{\sigma}$ is expressed as $\tilde{\sigma} = \sigma_1^{(1)}.\sigma_1^{(2)}.\sigma_2^{(1)}.\sigma_2^{(2)}$ and the weighted dual graph of $\tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{H}_1 \cup \tilde{H}_2$ is the following (cf. Figure 5):

Therefore it follows that $(C^2) = d^2 = d_{1,0}d_{1,1} + d_{2,0}d_{2,1} + d_{1,1} + d_{2,1}$, where $d := \deg(C)$. By noting that $d = d_{1,0} = d_{2,0}$,we obtain an equation:

$$d^2 - (d_{1,1} + d_{2,1})d - (d_{1,1} + d_{2,1}) = 0.$$

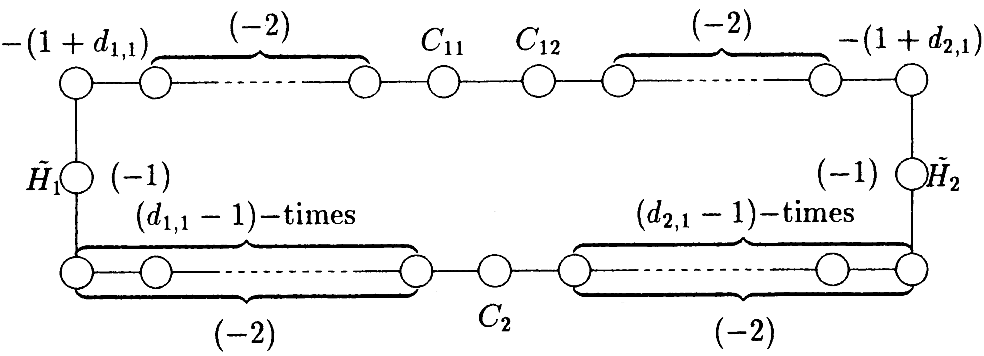

Figure 5:

We solve the above equation with respect to d and obtain:

$$d = \frac{-a \pm \sqrt{a(a+4)}}{2}$$

where $a := d_{1,1} + d_{2,1} \geq 2$. For any $a \geq 2$, d can not be an integer. This is a contradiction.

Q.E.D.

By Lemma 5.2, we may and shall assume that σ_1 is a unique Euclidean transformation associated with a datum $\mathcal{D}_1^{(1)} := \{p_1, l_1 = \bar{F_{11}}, d_{1,0}, d_{1,1}\}$ such that g.c.d$(d_{1,0}, d_{1,1}) = 1$. Let $d_0 := d_{1,0}$ and $d_1 := d_{1,1}$. As in Section 3, we shall perform an Euclidean algorithm associated with $d_0 > d_1$, and obtain positive integers $d_2, \cdots, d_\alpha = 1$ and $q_1, \cdots, q_\alpha$. The weighted dual graph of Supp$(\tilde{\sigma}^{-1}(p_1))$ is like as in Figure 1. Note that the case where $\alpha = 1$ does not occur. Indeed, otherwise, then C_{11} intersects with the cross-section $\tilde{H}_1$. Hence the multiplicity of C_{11} in the fibre $\tilde{F}_1$ is 1. But then by Lemma 5.3, (3-1) below, it follows that $\overline{\kappa}(X) = -\infty$. This is a contradiction. Thus we know that $\alpha \geq 2$.

Supp$(\tilde{\sigma}^{-1}(p_1))$ has two connected components except for the last exceptional curve $E(\alpha, q_\alpha) = \tilde{H}_1$. Among these two connected components, one which consists of all $E(s,t)'$s with s:even possibly except for $E(\alpha, q_\alpha)$ is contained in the fibre $\tilde{F}_1$, and another one which consists of all $E(s,t)'$s with s:odd possibly except for $E(\alpha, q_\alpha)$ is contained in the fibre $\tilde{F}_2$. Note that in the fibre $\tilde{F}_2$, C_2 must intersect with $E(1,1)$. Indeed, otherwise, the dual graph of $\tilde{F}_2$ is not a linear chain. Furthermore, note that in $\tilde{F}_1$, exactly one of C_{11} and C_{12} is a unique (-1)-curve, and in $\tilde{F}_2$, C_2 is a unique (-1)-curve.

In consideration of Lemma 3.1 and Lemma 5.1, and by noting that the part $\mathrm{Supp}(\tilde{\sigma}^{-1}(\{p_1,p_2\}))$ is contracted to smooth points p_1 and p_2 by successive contractions of (-1)-curves, the weighted dual graph of $\tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{H}_1 \cup \tilde{H}_2$ must be the following configuration (cf. Figure 6), where the part $A + E(\alpha, q_\alpha) + B$ is like as in Figure 1. In the part C and D in Figure 6, the left terminal components intersect with C_{12} and C_2, respectively. In particular, we note that $q_1 \geq 2$. One of C_{11} and C_{12} is a unique (-1)-curve in the fibre $\tilde{F}_1$. But if C_{11} is a (-1)-curve, it follows that $-1 = ({C_{11}}^2) = ({\tilde{F}_{11}}^2) - (1 + q_1) = -q_1$, i.e., $q_1 = 1$. This is a contradiction. Hence we know that C_{12} is a (-1)-curve.

Conversely, if the weighted dual graph of $\tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{H}_1 \cup \tilde{H}_2$ is as in Figure 6 with $({C_{11}}^2) = -q_1$ and $({C_{12}}^2) = -1$, then we can contract all components of $\tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{H}_1 \cup \tilde{H}_2$ other than C_{11}, C_{12} and C_2, starting with contractions of (-1)-curves $\tilde{H}_1$ and $\tilde{H}_2$.

In consideration of these observations, for given positive integers d_0 and d_1 such that $d_1 \geq 2, d_0 > 2d_1$ and g.c.d.$(d_0, d_1) = 1$, we can construct an irreducible curve $C(d_0, d_1)$ (see below for this notation) on $\mathbf{P}^2$ such that $\bar{\kappa}(\mathbf{P}^2 - C(d_0, d_1)) = 1$ and of the first type in the following way:

Construction of $C(d_0, d_1) \subset \mathbf{P}^2$:

Let d_0 and d_1 be positive integers such that $d_1 \geq 2, d_0 > 2d_1$ and g.c.d.$(d_0, d_1) = 1$. We shall perform an Euclidean algorithm with respect to $d_0 > d_1$ as in Section 3, and obtain positive integers $d_2, \cdots, d_\alpha = 1$ and $q_1, \cdots, q_\alpha$. Let l, l_1 and l_2 be distinct three fibres of a $\mathbf{P}^1$-bundle $\Sigma_1 \to \mathbf{P}^1$, where Σ_1 is a Hirzebruch surface of degree 1. Let M_1 be a minimal section on Σ_1 and M_2 be a cross-section of $\Sigma_1 \to \mathbf{P}^1$ such that $M_1 \cap M_2 = \emptyset$. Then $M_2 \sim M_1 + l$.

At first, we shall perform blow-ups with centers at $Q_1 := l_1 \cap M_2$ and $Q_2 := l_2 \cap M_2$. We denote this process by θ. We denote an intersection point of the proper transform of l_1 (resp. l_2) and an obtained exceptional curve by the same notation Q_1 (resp. Q_2). We shall perform blow-ups q_α-times with centers at Q_1 and its infinitely near points lying over the proper transforms of an exceptional curve obtained in the process of θ, and perform blow-ups q_α-times with centers at Q_2 and its infinitely near points lying over the proper

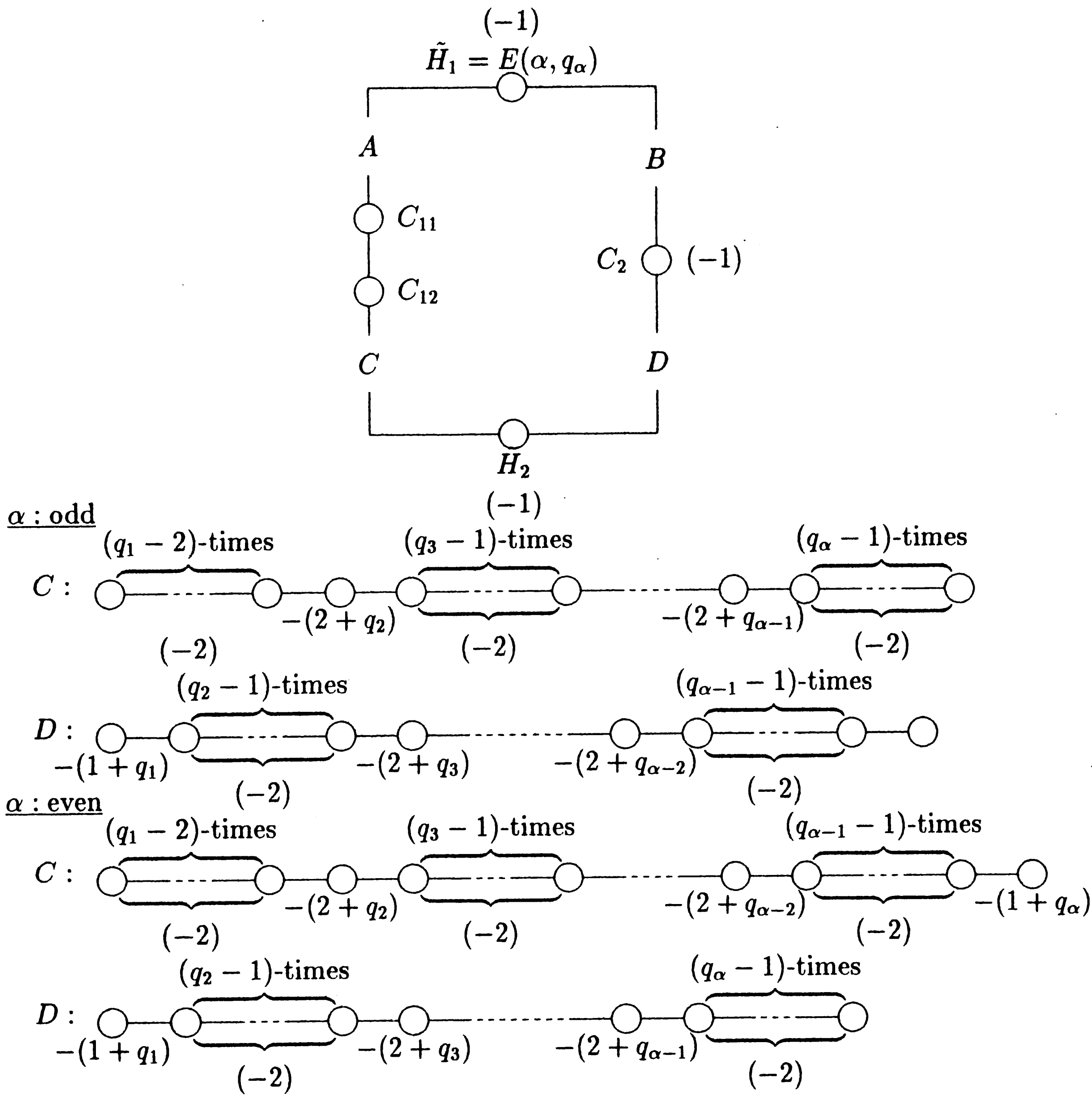
(−1)
H̃1 = E(α, qα)
A
B
C11
C12
C2
(−1)
C
D
H2
(−1)
α : odd
(q1 − 2)-times
(q3 − 1)-times
(qα − 1)-times
C :
−(2 + q2)
−(2 + qα−1)
(−2)
(−2)
(−2)
(q2 − 1)-times
(qα−1 − 1)-times
D :
−(1 + q1)
−(2 + q3)
−(2 + qα−2)
(−2)
(−2)
α : even
(q1 − 2)-times
(q3 − 1)-times
(qα−1 − 1)-times
C :
−(2 + q2)
−(2 + qα−2)
−(1 + qα)
(−2)
(−2)
(−2)
(q2 − 1)-times
(qα − 1)-times
D :
−(1 + q1)
−(2 + q3)
−(2 + qα−1)
(−2)
(−2)

Figure 6:

transforms of l_2. We denote this process by θ_α. We denote an intersection point on the fibre $(\theta.\theta_\alpha)^*(l_1)$ (resp. $(\theta.\theta_\alpha)^*(l_2)$) of the last and the last but one exceptional curves obtained in the peocess of θ_α by the same notation Q_1 (resp. Q_2). We shall perform blow-ups $q_{\alpha-1}$-times with centers at Q_1 and Q_2 and their infinitely near points lying over the proper transforms of the last but one exceptional curves obtained in the process of θ_α. We denote this process by $\theta_{\alpha-1}$.

Similarly, we can construct $\theta_\alpha, \theta_{\alpha-1}, \cdots, \theta_2$ associated with $q_2, \cdots, q_\alpha$. Finally, we denote an intersection point of the last and the last but one exceptional curves in the peocess of $\theta.\theta_\alpha.\dots.\theta_2$ on the fibre $(\theta.\theta_\alpha.\dots.\theta_2)^*(l_1)$ (resp. $(\theta.\theta_\alpha.\dots.\theta_2)^*(l_2)$) by the same notation Q_1 (resp. Q_2). We shall perform blow-ups (q_1-2)-times (resp. (q_1-1)-times) with the centers at Q_1 (resp. Q_2) including its infinitely near points lying over the proper transforms of the last but one exceptional curve in the process of $\theta.\theta_\alpha.\dots.\theta_2$. We denote this process by θ_1. Thus we obtained a birational morphism $\varrho := \theta.\theta_\alpha.\dots.\theta_1 : \tilde{V} \to \Sigma_1$. Let $\tilde{F_1} := \varrho^*(l_1), \tilde{F_2} := \varrho^*(l_2), \tilde{l} := \varrho^*(l), \tilde{H_1} := \varrho^*(M_1)$ and $\tilde{H_2} := \varrho^*(M_2)$. Let C_{11} be the component with self-intersection number $= -q_1$ and let C_{12} be the unique (-1)-curve in the fibre $\tilde{F_1}$. Let C_2 be the unique (-1)-curve in the fibre $\tilde{F_2}$. Then the weighted dual graph of $\tilde{F_1} \cup \tilde{F_2} \cup \tilde{H_1} \cup \tilde{F_2}$ is exactly like as in Figure 6. Note that the multiplicity of C_{11} (resp. C_{12}) in the fibre $\tilde{F_1}$ is d_1 (resp. $d_0 - d_1$) by Lemma 3.4. Similarly, note that the multiplicity of C_2 in the fibre $\tilde{F_2}$ is d_0.

We can contract all components in $\tilde{F_1} \cup \tilde{F_2} \cup \tilde{H_1} \cup \tilde{H_2}$ except for C_{11}, C_{12} and C_2, starting with contractions of $\tilde{H_1}$ and $\tilde{H_2}$, to smooth points on $\mathbf{P}^2$, say p_1 and p_2. We shall denote this contraction process by $\tilde{\sigma} : \tilde{V} \to \mathbf{P}^2$. Let $C(d_0, d_1)$ be an image of $\tilde{l}$ by $\tilde{\sigma}$. It is easy to see that the curves $\bar{F_{11}} := \tilde{\sigma}(C_{11}), \bar{F_{12}} := \tilde{\sigma}(C_{12})$ and $\bar{F_2} := \tilde{\sigma}(C_2)$ are lines. Furthermore, it is clear that $\bar{F_{11}}, \bar{F_{12}}$ and $\bar{F_2}$ don't have a common point. Thus we may assume that $\bar{F_{11}}, \bar{F_{12}}$ and $\bar{F_2}$ are defined by $X = 0, Y = 0$ and $Z = 0$, respectively with respect to suitable homogeneous coordinates X, Y and Z on $\mathbf{P}^2$. Note that $C(d_0, d_1)$ is a member of a linear pencil spanned by $d_1\bar{F_{11}} + (d_0 - d_1)\bar{F_{12}}$ and $d_0\bar{F_2}$, say Λ. Therefore we can deduce that $C(d_0, d_1)$ is defined by $X^{d_1}Y^{d_0-d_1} + Z^{d_0} = 0$.

On the other hand, we can use Lemma 5.3 (3-1) to calculate the value of $\overline{\kappa}(X)$, where $X := \mathbf{P}^2 - C(d_0, d_1)$. In effect, by the construction, it follows that X is a $\mathbf{Q}$-homology plane with an untwisted $\mathbf{C}^*$-fibration, $\varphi := \Phi_\Lambda|_X : X \to \mathbf{A}^1$ onto an affine line. Furthermore, φ has a unique reducible fibre

$F_1 := d_1F_{11} + (d_0 - d_1)F_{12}$, where $F_{1j} := \bar{F_{1j}} \cap X$ for $j = 1, 2$, which is of the form $F_{11} \cong F_{12} \cong \mathbf{A}^1$ and $F_{11} \cap F_{12} \neq \emptyset$. Hence, by Lemma 5.3 (3-1), it follows that $\overline{\kappa}(X) = 1$ if and only if $1 - \frac{1}{d_1} - \frac{1}{d_0} > 0$. Since we assume that $d_1 \geq 2$, consequently we have $\overline{\kappa}(X) = 1$.

Conversely, if a plane curve C on $\mathbf{P}^2$ is defined by $X^{d_1}Y^{d_0-d_1} + Z^{d_0} = 0$ with $d_1 \geq 2, d_0 > 2d_1$ and g.c.d.$(d_0, d_1) = 1$, then C has two cuspidal points $p_1 = (0 : 1 : 0)$ and $p_2 = (1 : 0 : 0)$, and the configurations of $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_1))$ and $\mathrm{Supp}(\tilde{\sigma}^{-1}(p_2))$ are exactly like $A + E(\alpha, q_\alpha) + B$ and $C + \tilde{H}_2 + D$ in Figure 6, respectively, where $\tilde{\sigma}$ is the shortest succession of blow-ups with centers at p_1 and p_2 (including their infinitely near points) such that all components of $\mathrm{Supp}(\tilde{\sigma}^{-1}(C))$ are non-singular and they intersect each other transversally if they meet at all, i.e., $\mathrm{Supp}(\tilde{\sigma}^{-1}(C))$ is S.N.C. Furthermore, $\overline{\kappa}(\mathbf{P}^2 - C) = 1$ and C is of the first type.

With the notations as above, for given two pairs of positive intagers, say (d_0, d_1) and (e_0, e_1) such that $d_1 \geq 2, e_1 \geq 2, d_0 > 2d_1, e_0 > 2e_1$, g.c.d.$(d_0, d_1) = 1$ and g.c.d.$(e_0, e_1) = 1$, $C(d_0, d_1) = C(e_0, e_1)$ (up to projective equivalence PGL(2; $\mathbf{C}$)) if and only if $d_0 = e_0$ and $d_1 = e_1$. Thus we obtain an injection from the set of pairs of positive integers (d_0, d_1) such that $d_1 \geq 2, d_0 > 2d_1$ and g.c.d.$(d_0, d_1) = 1$ to the set of irreducible curves on $\mathbf{P}^2$ such that whose complements have logarithmic Kodaira dimension 1 and of the first type up to PGL(2; $\mathbf{C}$). This correspondence is given by $(d_0, d_1) \mapsto C(d_0, d_1) = \{X^{d_1}Y^{d_0-d_1} + Z^{d_0} = 0\}$. The surjectivity of this correspondence is easily seen from the argument before the construction of $C(d_0, d_1)$. Therefore we can obtain the required bijection of Theorem 2.2.

we used the following lemma in the above arguments. For this result, we shall refer to Miyanishi and Sugie [7] Lemma 2.15 and Lemma 2.16 with a few improvements.

Lemma 5.3 *Let S be a $\mathbf{Q}$-homology plane with an untwisted $\mathbf{C}^*$-fibration onto an affine line $\phi : S \to \mathbf{A}^1$. Then the following assertions hold true:*

(1) ϕ has a unique reducible fibre, say G_0, which consists of two components, say $G_{0,1}$ and $G_{0,2}$. Let $m_{0,1}$ and $m_{0,2}$ be multiplicities of $G_{0,1}$ and $G_{0,2}$ in G_0, respectively. All singular fibres of ϕ other than G_0 are irreducible, multiple fibres (if they exist at all). Let G_i exhaust all irreducible, multiple fibres of ϕ (if they exist at all) for $1 \leq i \leq r$. Then G_i is of the form $G_i = m_i\mathbf{C}^*$ with $m_i > 1$ for $1 \leq i \leq r$.

(2) $\mathrm{Supp}(G_0) = G_{0,1} \cup G_{0,2}$ *has one of the following two configurations:*
1. $G_{0,1} \cong G_{0,2} \cong \mathbf{A}^1$ *and* $G_{0,1} \cap G_{0,2} \neq \emptyset$.
2. $G_{0,1} \cong \mathbf{A}^*$, $G_{0,2} \cong \mathbf{C}^*$ *and* $G_{0,1} \cap G_{0,2} = \emptyset$.
(3-1) If the case is in 1, then $\overline{\kappa}(S) = 1, 0$ *or* $-\infty$ *if and only if*

$$r - \frac{1}{\min(m_{0,1}, m_{0,2})} - \Sigma_{i=1}^{r} \frac{1}{m_i} > 0, \ = 0 \text{ or } \ < 0, \text{ respectively.}$$

(3-2) If the case is in 2, then $\overline{\kappa}(S) = 1, 0$ *or* $-\infty$ *if and only if*

$$r - \frac{1}{m_{0,2}} - \Sigma_{i=1}^{r} \frac{1}{m_i} > 0, \ = 0 \text{ or } \ < 0, \text{ respectively.}$$

As a easy corollary of Theorem 2.2, we have the following:

Corollary 5.4 *Let* $n \geq 2$. *Then there exists an irreducible curve* C *on* $\mathbf{P}^2$ *of degree* $= 2n+1$ *such that* $\overline{\kappa}(\mathbf{P}^2 - C) = 1$ *and of the first type.*

Proof. Indeed, it is enough to take $C(2n+1, n) = \{X^n Y^{n+1} + Z^{2n+1} = 0\} \subset \mathbf{P}^2$ as C.

Q.E.D.

REMARK **5.5** *Note that, even if* C *and* C' *are irreducible curves on* $\mathbf{P}^2$ *such that with same degree,* $\overline{\kappa}(\mathbf{P}^2 - C) = \overline{\kappa}(\mathbf{P}^2 - C') = 1$, *and of the first type, it is not necessarily* $C = C'$ *(up to* $\mathrm{PGL}(2;\mathbf{C})$*). For example,* $C(17,2) = \{X^2Y^{15} + Z^{17} = 0\}, C(17,3) = \{X^3Y^{14} + Z^{17} = 0\}, C(17,4) = \{X^4Y^{13} + Z^{17} = 0\}, C(17,5) = \{X^5Y^{12} + Z^{17} = 0\}, C(17,6) = \{X^6Y^{11} + Z^{17} = 0\}, C(17,7) = \{X^7Y^{10} + Z^{17} = 0\}$ *and* $C(17,8) = \{X^8Y^9 + Z^{17} = 0\}$ *are irreducible curves on* $\mathbf{P}^2$ *of the first type such that whose complements have logarithmic Kodaira dimension* 1. *But they are not projectively equivalent each other.*

REMARK **5.6** *Among all curves which are of the first type and their complements have logarithmic Kodaira dimension* 1, *the lowest degree curve is* $C(5,2) = \{X^2Y^3 + Z^5 = 0\}$. *This curve is, in effect, listed up in Yoshihara [11].*

6 Appendix

We have'nt solve Problem 1.5 for the case of the second type yet, completely. The difficulty in this type is mainly due to the fact that, with the notations as in Section 4, the configuration of $\tilde{F}_1$ is not a linear chain. We can construct, even in this type, such an irreducible curve in Problem 1.5, geometrically and concretely with some additional assumption. However it is hard to ask for the homogeneous polynomial defining such a curve, concretely.

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ON AUTOMORPHISM GROUP OF $A^{[1]}$

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1. Introduction

Let D be an integral domain. We say that a D-algebra A is *D-invariant*, if for any D-algebra B such that $A^{[1]} \cong_D B^{[1]}$ we have always $A \cong_D B$, where $A^{[1]}$ and $B^{[1]}$ are polynomial rings in one variable over A and B respectively.

If D is a field and A is an integral domain such that $D \subset A$ with the transcendence degree $\operatorname{tr.deg}_D A = 1$, then A is always D-invariant as is shown in [1]. A generalization of this theorem to the case where D is a Dedekind domain or a discrete valuation ring does not necessarily hold (cf. [3]). Until now, no normal non-D-invariant D-algebras A with $\operatorname{tr.deg}_D A = 1$ are known.

Problem 1.1. Let D be a Dedekind domain, say $D = k^{[1]}$ for a field k, and let A be an integral domain finitely generated over D with $\operatorname{tr.deg}_D A = 1$. If A is normal or more strongly regular, then is A D-invariant?

There is a close connection between Problem 1.1 and the following cancellation problem for two-dimensional regular affine domains over a field k.

Problem 1.2. Is every two-dimensional regular affine k-domain k-invariant?

In case $k = \mathbf{C}$ counterexamples to Problem 1.2 are given by Danielewski [5]. (See also [9] for the cancellation problem.)

In this article for any field k we will give counterexamples to Problem 1.1 and show that some of those examples are also counterexamples to Problem 1.2. Moreover using these examples we will show regular affine k-domains A and B which satisfy the following two conditions:
(1) $A^{[1]} \cong_k B^{[1]}$.
(2) $\operatorname{Aut}_k A \not\cong_k \operatorname{Aut}_k B$.

2. Preliminaries

Throughout the paper we assume all rings to be commutative with 1. In particular k is always assumed to be a field. We set up the notations for later use.

For a commutative ring R, $R^{[n]}$ means a polynomial ring in n variables over R. It should be noticed that n elements $F_1, \ldots, F_n \in R^{[n]}$ which generate $R^{[n]}$ over R are always algebraically independent over R (see the proof of (1.1) in [1]). We denote by R^* the unit group of R. The notations $x_1, \ldots, x_n, y_1, \ldots, y_m$ are always used as indeterminates over k.

For a polynomial $f(x)$ in one variable x, $\deg f(x)$ denotes the x-degree of $f(x)$ as usual.

For an ideal I of a ring R, $(I, y_1, ..., y_m)$ means the ideal of the polynomial ring $R[y_1, ..., y_m]$ generated by $I, y_1, ..., y_m$. For convenience sake we often write $I[y_1, ..., y_m]$ for the ideal of $R[y_1, ..., y_m]$ generated by I.

For a k-algebra A, $\mathrm{Aut}_k A$ is the groups of k-automorphisms of A.

For a k-algebra S and k-homomorphism $\alpha \colon A \to B$ of k-algebras A and B, the notation $\alpha \otimes S$ denotes S-homomorphism $A \otimes_k S \to B \otimes_k S$ canonically extended from α.

3. Equivalent ideals

Definition 3.1. Let k be a field and let I be an ideal of a k-algebra A. We say that I is *equivalent* to an ideal J of a k-algebra B, if there exists a k-isomorphism $\sigma \colon A \to B$ such that the image $\sigma(I)$ is equal to J. The equivalence relation will be written $I \sim J$. If I is equivalent to the ideal $(x_{m+1}, ..., x_n)$ of $k[x_1, ..., x_n]$ for some $m \leq n$, then I is said to be *rectifiable.* (When $m = n$, $(x_{m+1}, ..., x_n)$ means the zero ideal.) Let C and D be k-algebras. A k-homomorphism $\alpha \colon A \to C$ is also said to be *equivalent* to a k-homomorphism $\beta \colon B \to D$, if there exist k-isomorphisms $\sigma \colon A \to B$ and $\theta \colon C \to D$ such that $\theta\alpha = \beta\sigma$. This equivalence relation is also written $\alpha \sim \beta$. In particular if α is equivalent to the k-homomorphism

$$\beta \colon k[x_1, ..., x_n] \to k[x_1, ..., x_m]$$

defined by $\beta(x_i) = x_i$ $(i = 1, ..., m)$ and $\beta(x_j) = 0$ $(j = m+1, ..., n)$, then α is said to be *rectifiable.*

Note that if $\alpha \sim \beta$, then $\ker\alpha \sim \ker\beta$.

Lemma 3.2 (see [6]). *Let C be a k-algebra and let*

$$\alpha, \gamma \colon k[x_1, ..., x_n, y_1, ..., y_m] \to C$$

be two surjective k-homomorphisms. If

$$\alpha(y_j) = 0 \ (j = 1, ..., m), \ \ \gamma(x_i) = 0 \ (i = 1, ..., n),$$

then there exist

$$f_j \in k[x_1, ..., x_n] \ (j = 1, ..., m)$$

and

$$g_i \in k[y_1, ..., y_m] \ (i = 1, ..., n)$$

such that

$$\alpha(x_i) = \gamma(g_i), \ \ \gamma(y_j) = \alpha(f_j)$$

for $i = 1, ..., n$ and $j = 1, ..., m$. If we set $\tau = \tau_1\tau_2$ where

$$\tau_i \in \mathrm{Aut}_k \, k[x_1, ..., x_n, y_1, ..., y_m]$$

are defined by

$$\tau_1 \colon (x_1, ..., x_n, y_1, ..., y_m) \to (x_1 + g_1, ..., x_n + g_n, y_1, ..., y_m)$$

and

$$\tau_2 \colon (x_1, ..., x_n, y_1, ..., y_m) \to (x_1, ..., x_n, y_1 - f_1, ..., y_m - f_m),$$

then we have $\alpha = \gamma\tau$. In particular, α is equivalent to γ.

Proof. The existence of the elements f_j and g_i easily follows from the fact

$$B = \alpha(k[x_1, ..., x_n]) = \gamma(k[y_1, ..., y_m]).$$

Since

$$\gamma\tau_1 \colon (x_1, ..., x_n, y_1, ..., y_m) \to (\alpha(x_1), ..., \alpha(x_n), \gamma(y_1), ..., \gamma(y_m)),$$

we have

$$\gamma\tau_1\tau_2(x_i) = \gamma\tau_1(x_i) = \alpha(x_i) \ (i = 1, ..., n)$$

and

$$\gamma\tau_1\tau_2(y_j) = \gamma(y_j) - \alpha(f_j) = 0 \ (j = 1, ..., m)$$

which shows $\gamma\tau = \gamma\tau_1\tau_2 = \alpha$ as required.

Corollary 3.3. *Let C be a k-algebra generated by m elements over k and let*

$$\alpha, \beta \colon k[x_1, ..., x_n, y_1, ..., y_m] \to C$$

be surjective k-homomorphisms with

$$\alpha(y_j) = \beta(y_j) = 0 \ (j = 1, ..., m).$$

Then we have $\alpha \sim \beta$. In particular, if $C = k^{[m]}$, then α and β are both rectifiable.

Proof. The k-algebra C is of the form $C = k[c_1, ..., c_m]$ for some $c_j \in B$ $(j = 1, ..., m)$. So the k-homomorphism

$$\gamma \colon k[x_1, ..., x_n, y_1, ..., y_m] \to C$$

defined by

$$\gamma(x_i) = 0 \ (i = 1, ..., n), \ \gamma(y_i) = c_j \ (j = 1, ..., m)$$

is surjective. Thus by Lemma 3.2 we have $\alpha \sim \gamma$ and $\beta \sim \gamma$, which shows $\alpha \sim \beta$, as desired.

Corollary 3.4. *Let I and J be ideals of $R = k[x_1, ..., x_n]$ such that*

$$R/I \cong_k R/J.$$

If R/I is generated by m elements over k as a k-algebra, then we have

$$(I, y_1, ..., y_m) \sim (J, y_1, ..., y_m)$$

as ideals of $R[y_1, ..., y_m]$. In particular, if $R/I \cong_k k^{[m]}$, then $(J, y_1, ..., y_m)$ and $(I, y_1, ..., y_m)$ are both rectifiable.

Proof. This is an immediate consequence of Corollary 3.3.

4. Rees algebras

Let k be a field and let $(M, +)$ be a commutative monoid. Recall that a k-algebra A is called an M-graded k-algebra, if A is the direct sum

$$A = \bigoplus_{\lambda \in M} \Gamma_\lambda$$

of k-modules Γ_λ satisfying the condition $\Gamma_\lambda \Gamma_\nu \subset \Gamma_{\lambda+\nu}$. The map

$$\Gamma \colon \lambda \to \{\Gamma_\lambda \mid \lambda \in M\}$$

is called the M-grading of A. In particular if Γ is injective, i.e. $\lambda = \nu$ whenever $\Gamma_\lambda = \Gamma_\nu$, then the M-grading Γ is said to be faithful.

Let B be an M-graded k-algebra by Δ. For a k-subalgebra L of both A and B, an L-isomorphism $\sigma\colon A \to B$ such that $\sigma(\Gamma_\lambda) = \Delta_\lambda$ for any $\lambda \in M$ is called an M-graded L-isomorphism. If there exists such an M-graded L-isomorphism, then we say that Γ is L-isomorphic to Δ and write $\Gamma \cong_L \Delta$.

If I is an ideal of a k-algebra R, then we write $\mathcal{R}_R(I)$ for the Rees ring $R[t, t^{-1}I]$ as a subring of Laurent polynomial ring $R[t, t^{-1}]$ with an indeterminate t (cf. [8, p.120]). By assigning the degree 1 to t we may consider $\mathcal{R}_R(I)$ as a $\mathbb{Z}$-graded k-algebra. So the $\mathbb{Z}$-grading Γ of $\mathcal{R}_R(I)$ is given by

$$\Gamma_\lambda = \begin{cases} t^\lambda I^{-\lambda} & (\lambda < 0) \\ t^\lambda R & (\lambda \geq 0). \end{cases}$$

Such a grading is called the *Rees* $\mathbb{Z}$*-grading.* Note that $\mathcal{R}_R(I)$ may be also considered as a $k[t]$-algebra.

Proposition 4.1. *Let Γ and Δ be the Rees $\mathbb{Z}$-gradings of $\mathcal{R}_R(I)$ and $\mathcal{R}_S(J)$ respectively. Then the following three conditions are equivalent:*
(1) $\Gamma \cong_k \Delta$.
(2) $\Gamma \cong_{k[t]} \Delta$.
(3) $I \sim J$.

Proof. (1)⇒(3): Suppose $\Gamma \cong_k \Delta$. Then we can find a k-isomorphism

$$\sigma : \mathcal{R}_R(I) \to \mathcal{R}_S(J)$$

such that $\sigma(\Gamma_\lambda) = \Delta_\lambda$. In particular we have

$$\sigma(t^i R) = \sigma(\Gamma_i) = \Delta_i = t^i S$$

for $i = 0, 1$ and

$$\sigma(t^{-1}I) = \sigma(\Gamma_{-1}) = \Delta_{-1} = t^{-1}J.$$

Therefore the restriction $\overline{\sigma} = \sigma \mid_R$ is a k-isomorphism from R to S satisfying the condition

$$\overline{\sigma}(I) = \sigma(tRt^{-1}I) = tSt^{-1}J = J,$$

which proves (1) ⇒ (3).
(3)⇒(2): Suppose $I \sim J$. Then we can find a k-isomorphism $\overline{\sigma}\colon R \to S$ such that $\overline{\sigma}(I) = J$. Note that $\overline{\sigma}$ can be extended uniquely to the $k[t]$-isomorphism $R[t, t^{-1}] \to S[t, t^{-1}]$ which will be written σ'. Then it is easy to verify that the restriction

$$\sigma = \sigma' \mid_{\mathcal{R}_R(I)}\colon \mathcal{R}_R(I) \to \mathcal{R}_S(J)$$

is a required $\mathbb{Z}$-graded k-isomorphism and the proof of (3)⇒(2) follows.
(2)⇒(1): This follows immediately from the definition and we are through.

Corollary 4.2. *Let I and J be ideals of $R = k[x_1, ..., x_n]$ $(= k^{[n]})$ such that $R/I \cong R/J$. If R/I is generated by m elements over k, then*

$$\mathcal{R}_R(I)^{[m]} \cong_{k[t]} \mathcal{R}_R(J)^{[m]}.$$

In particular if $R/I \cong_k k^{[m]}$, then

$$\mathcal{R}_R(I)^{[m]} \cong_{k[t]} k[t]^{[n+m]}.$$

Proof. Let us set $B = R[y_1, ..., y_m]$. If we put $I' = (I, y_1, ..., y_m)$ and $J' = (J, y_1, ..., y_m)$ as ideals of B, then by Corollary 3.4, we have $J' \sim I'$. So from Proposition 4.1 it follows that

$$\mathcal{R}_B(I') \cong_{k[t]} \mathcal{R}_B(J').$$

On the other hand we have

$$\mathcal{R}_B(I') = \mathcal{R}_R(I)[t^{-1}y_1, ..., t^{-1}y_m],$$

and

$$\mathcal{R}_B(J') = \mathcal{R}_R(J)[t^{-1}y_1, ..., t^{-1}y_m].$$

Therefore

$$\mathcal{R}_R(I)^{[m]} \cong_{k[t]} \mathcal{R}_R(J)^{[m]}, \tag{4.1}$$

because $t^{-1}y_1, ..., t^{-1}y_m$ are algebraically independent over both $\mathcal{R}_R(I)$ and $\mathcal{R}_R(J)$. In particular if $R/I \cong_k k^{[m]}$, then we can choose J so that $J = (x_{m+1}, ..., x_n)$. Since

$$\mathcal{R}_R(J)^{[m]} \cong_{k[t]} k[t]^{[n+m]},$$

we have

$$\mathcal{R}_R(I)^{[m]} \cong_{k[t]} k[t]^{[n+m]} \tag{4.2}$$

as a special case of (4.1), which completes the proof.

5. Non-cancelable affine surfaces

Lemma 5.1. *Let $D[x]$ be a polynomial ring in one variable x over an integral domain D. Let $A = D[x, w^{-1}f(x)]$ and $B = D[x, w^{-1}g(x)]$ be D-subalgebras of $D[w^{-1}, x]$ for a nonzero element $w \in D$ and monic polynomials $f(x), g(x) \in D[x]$ with* $\deg f(x) > 1$ *and* $\deg g(x) > 1$. *If there exists an isomorphism $\sigma\colon A \to B$ such that $\sigma(D) = D$, then σ satisfies the following three conditions:*
(1) $\sigma(D[x]) = D[x]$.
(2) $\sigma(wD) = wD$.
(3) $\sigma(I) = J$ *for ideals* $I = (f(x), w)$ *and* $J = (g(x), w)$ *of* $D[x]$.

Proof. If $w \in D^*$, then the lemma is obvious. Thus we may assume $w \notin D^*$ from the first. Notice that, since $f(x)$ is monic, the polynomial $wy - f(x)$ in two variables x and y is a prime element of $D[x, y]$ $(= D^{[2]})$. So

$$A \cong_{D[x]} D[x, y]/(wy - f(x)).$$

Similarly we have

$$B \cong_{D[x]} D[x, y]/(wy - g(x)),$$

and hence

$$B/wB \cong_{D[x]} D[x, y]/JD[x, y] \cong_{D[x]} (D[x]/J)^{[1]}, \tag{5.1}$$

which implies that

$$wB \cap D[x] = J.$$

On the other hand by assumption we have $\sigma(x) \in D[w^{-1}, x]$ and $K[x] = K[\sigma(x)]$ for the quotient field K of D. In other words $\sigma(x)$ is of the form $\sigma(x) = ax + b$ for some

$a, b \in D[w^{-1}]$ with $a \neq 0$. Thus we can find an integer $s > 0$ so that $w^s(ax+b) \in D[x]$. Since $\sigma(x) \in B$, the element $w^s(ax+b)$ is also contained in wB, and therefore

$$w^s(ax+b) \in D[x] \cap wB = J.$$

So we have

$$w^s(ax+b) \equiv g(x)r(x) \pmod{wD[x]}$$

for some $r(x) \in D[x]$. From the hypothesis that $g(x)$ is a monic polynomial with $\deg g(x) > 1$ it easily follows that

$$r(x) \equiv 0 \pmod{wD[x]}.$$

Therefore

$$w^s(ax+b) \equiv 0 \pmod{wD[x]},$$

which means

$$w^{(s-1)}(ax+b) \in D[x].$$

As a result we obtain

$$w^{(s-i)}(ax+b) \in D[x],$$

for any $i = 0, ..., s$ by induction on i. In particular we have $ax+b \in D[x]$, that is $D[\sigma(x)] \subset D[x]$. In a similar way we can also verify that $D[\sigma^{-1}(x)] \subset D[x]$. Summarizing these two inclusions we get the condition (1).

Now for the proof of (2) and (3), we put $F = \sigma(w^{-1}f(x))$ for convenience sake. Then by (1) we have

$$B = \sigma(A) = D[x, F].$$

So given a subset S of B if we write $\overline{S}$ for the image of S under the canonical surjection $B \to B/wB$, then

$$\overline{B} = \overline{\sigma(A)} = \overline{D}[\overline{x}, \overline{F}].$$

Moreover we have also $\overline{B} = \overline{D}[\overline{x}]^{[1]}$ by the isomorphism (5.1), the image $\overline{F}$ must be algebraically independent over $\overline{D}[\overline{x}]$ (see the proof of (1.1) in [1]). On the other hand by the definition of F we have an equation

$$\overline{\sigma(w)}\ \overline{F} - \overline{\sigma(f(x))} = 0,$$

which implies

$$\overline{\sigma(w)} = \overline{\sigma(f(x))} = 0,$$

because $\sigma(w) \in D$ and $\sigma(f(x)) \in D[x]$. This shows that

$$\sigma(w) \in J \cap D = wD$$

and

$$\sigma(f(x)) \in J.$$

Applying the same argument to σ^{-1}, we also get $\sigma^{-1}(w) \in wD$ and $\sigma^{-1}(g(x)) \in I$. Sum up these results to see that σ also satisfies both conditions (2) and (3) in the lemma, which completes the proof.

Theorem 5.2. *Let D be an integral domain over a field k and let x be an indeterminate over D. Given a nonzero element $w \in D$ and monic polynomials $f(x), g(x) \in k[x]$ with $\deg f(x) > 1$ and $g(x) > 1$, let us set $A = D[x, w^{-1}f(x)]$ and $B = D[x, w^{-1}g(x)]$ as D-subalgebras of $D[x, w^{-1}]$. Then we have the following:*
(1) If $k[x]/(f(x)) \cong_k k[x]/(g(x))$, then $A^{[1]} \cong_D B^{[1]}$.
(2) There exists a k-isomorphism $\sigma\colon A \to B$ such that $\sigma(D) = D$, if and only if we can find $a \in D^*$ and $b \in D$ so that

$$f(ax+b) \equiv a^n g(x) \pmod{wD[x]} \tag{5.2}$$

for $n = \deg f(x)$.

Proof. If $w \in D^*$, then the theorem is obvious and we may assume $w \notin D^*$ from the first.
(1): Suppose $k[x]/(f(x)) \cong_k k[x]/(g(x))$. Since $w \neq 0$ and $w \notin D^*$, the element w is algebraically independent over k, and therefore by Corollary 4.2 we have

$$k[x, w, w^{-1}f(x)]^{[1]} \cong_{k[w]} k[x, w, w^{-1}g(x)]^{[1]}.$$

On the other hand, as is noticed in the proof of Lemma 5.1, we have

$$A \cong_{D[x]} D[x, y]/(wy - f(x)),$$

so that

$$A \cong_D k[w, x, y]/(wy - f(x)) \otimes_{k[w]} D \cong_D k[x, w, w^{-1}f(x)] \otimes_{k[w]} D.$$

Similarly we have

$$B \cong_D k[x, w, w^{-1}g(x)] \otimes_{k[w]} D.$$

Summing up these isomorphisms, we see

$$A^{[1]} \cong_D k[x, w, w^{-1}f(x)]^{[1]} \otimes_{k[w]} D \cong_D k[x, w, w^{-1}g(x)]^{[1]} \otimes_{k[w]} D \cong_D B^{[1]}$$

as required.
(2): First, for the proof of the 'only if' part, suppose that there exists a k-isomorphism $\sigma\colon A \to B$ such that $\sigma(D) = D$. If we set $I = (f(x), w)$ and $J = (g(x), w)$ as ideals of $D[x]$, then by Lemma 5.1 (1) the image $\sigma(x)$ is of the form $\sigma(x) = ax + b$ for $a \in D^*$ and $b \in D$. Moreover by Lemma 5.1 (2) we can easily verify that $\sigma(I) = (\sigma(f(x)), w)$ as the ideal of $D[x]$. So by Lemma 5.1 (3) we have

$$J = \sigma(I) = (\sigma(f(x)), w) = (f(ax+b), w) \ (\subset D[x]).$$

This is equivalent to the condition (5.2) in (2), because $f(x)$ and $g(x)$ are monic polynomials in $k[x]$ with $\deg f(x) = n > 1$. The 'only if' part follows.

Next in order to prove the 'if' part we suppose that the condition (5.2) in (2) holds. Consider $\sigma' \in \mathrm{Aut}_K K[x]$ defined by $\sigma'(x) = ax + b$, then the image $\sigma'(f(x))$ is of the form

$$\sigma'(f(x)) = a^n g(x) + wr(x)$$

for some $r(x) \in D[x]$. Thus

$$\sigma'(A) = D[\sigma'(x), w^{-1}f(\sigma'(x))] = D[x, w^{-1}(a^n g(x) + wr(x))] = B$$

and the restriction $\sigma = \sigma'|_A$ gives rise to a D-isomorphism $A \to B$, which completes the proof of the 'if' part, and we are through.

Corollary 5.3. *Let u and x be two indeterminates over a field k and let $w(u)$ and $q(u)$ be nonzero polynomials of $k[u]$ such that $w(\zeta) = 0$ but $q(\zeta) \neq 0$ for some $\zeta \in k$. Let us set $A = D[x, w(u)^{-1}f(x)]$ and $B = D[x, w(u)^{-1}g(x)]$ for $D = k[u, q(u)^{-1}]$ and $f(x), g(x) \in k[x]$ with $\deg f(x) > 1$ and $\deg g(x) > 1$. Then A and B are both two-dimensional affine k-domains satisfying the following:*
(1) If $k[x]/(f(x)) \cong_k k[x]/(g(x))$, then $A^{[1]} \cong_D B^{[1]}$.
(2) If $(f(x)) \not\sim (g(x))$, then $A \not\cong_D B$.
(3) If $(f(x)) \not\sim (g(x))$ and $q(u) \notin k$, then $A \not\cong_k B$.
(4) If $f(x)$ (resp. $g(x)$) is separable, then A (resp. B) is regular.

Proof. (1): This is an immediate consequence of Theorem 5.2 (1).
(2): If $A \cong_D B$ on the contrary, then by Theorem 5.2 (2) there exist $a \in D^*$ and $b \in D$ such that

$$f(ax + b) \equiv a^n g(x) \pmod{w(u)D[x]}. \tag{5.3}$$

By hypothesis we can choose $\bar{a}, \bar{b} \in k$ so that

$$\bar{a} - a,\ \bar{b} - b \in (u - \zeta)D,$$

and hence substituting ζ for u into the equations on the both sides of the congruence (5.3) we have

$$f(\bar{a}x + \bar{b}) = \bar{a}^n g(x),$$

because $w(\zeta) = 0$ by assumption. This shows that $(f(x)) \sim (g(x))$, contrary to the hypothesis. The part (2) follows.
(3): Assume, for a contradiction, that there exists a k-isomorphism $\sigma\colon A \to B$. Then

$$\sigma(A^*) = \sigma(D^*) = D^* = \sigma(B^*),$$

and in particular

$$\sigma(q(u)) \in D^* \subset D.$$

Furthermore since $q(u) \notin k$ by hypothesis, the field extension $k(\sigma(u))/k(\sigma(q(u)))$ is algebraic, and hence every element of $\sigma(D)$ is algebraic over $k[\sigma(u)](\subset D)$, which shows that $\sigma(D) \subset D$, because D is algebraically closed in B. The reverse inclusion $D \subset \sigma(D)$ can be also obtained in a similar way. Thus $\sigma(D) = D$ and we can apply the same argument as in the proof of the part (2) to get the contradiction, as required.
(4): Suppose that $f(x)$ is separable. For the proof of the part (4) it is sufficient to show that the localization A_M is regular for any maximal ideal M of A. First assume $w(u) \notin M$. Then $A_M = D[x]_P$ for $P = M \cap D[x]$, because

$$A \subset D[x, w(u)^{-1}] \subset D[x]_P.$$

Clearly $D[x]_P$ is regular, so is A_M. Next assume $w(u) \in M$. Then there exists an element $v(u) \in D$ such that

$$w(u) \in v(u)D = D \cap M.$$

If I denotes the ideal $(f(x), v(u))$ of $D[x]$, then by hypothesis $D[x]/I$ is the direct product of a finite number of fields. Furthermore from the congruence (5.1) it follows that

$$A/v(u)A \cong D[x, y]/I[y].$$

Thus $A/v(u)A$ may be considered as a polynomial ring in one variable over $D[x]/I$. So every ideal of $A/v(u)A$ is principal and in particular $M/v(u)A$ is a principal ideal of

$A/v(u)A$. Therefore we can find an element $h \in A$ so that $M = (v(u), h)$. Note also that M is a height two prime ideal of A. Summing up these facts, we see that A_M is a two-dimensional regular local ring, as desired.

Remark 5.4. For A and B in Corollary 5.3, due to Makar-Limanov [7], we have also $A \not\cong_k B$ in the case where $k = \mathbf{C}$, $w(u) = u^e$ $(e > 1)$, $q(u) \in \mathbf{C}^*$.

Corollary 5.5. *Let k be an algebraically closed field and let ξ be a primitive n-th root of 1 for $n > 2$ with $\operatorname{ch} k \nmid n$. In Corollary 5.3 let us set*

$$\begin{aligned} f(x) &= x^n - 1, \\ g(x) &= x^n - \alpha x - 1, \\ q(u) &= u, \\ w(u) &= (u-1)(u-\xi)(u-\zeta), \end{aligned}$$

where α and ζ are nonzero elements of k such that

$$\zeta \notin \{\xi^{-1}, \xi, \xi^2\},\ \zeta^2 \notin \{1, \xi\}, \tag{5.4}$$

and

$$\alpha^n \neq n^n(1-n)^{1-n}. \tag{5.5}$$

Then A and B are two-dimensional regular affine k-domains such that $A^{[1]} \cong_k B^{[1]}$ but $\operatorname{Aut}_k A \not\cong \operatorname{Aut}_k B$.

Proof. Clearly $f(x)$ is separable. In order to prove that $g(x)$ is separable, it is enough to show $\gcd(g(x), g'(x)) = 1$ for the formal derivative $g'(x) = nx^{n-1} - \alpha$ of $g(x)$. Since

$$g(x) = n^{-1}xg'(x) + \alpha n^{-1}(1-n)x - 1,$$

we see $\gcd(g(x), g'(x)) = 1$ if and only if $g(\alpha^{-1}n(1-n)^{-1}) \neq 0$ which is equivalent to the condition (5.5). Therefore $g(x)$ is also separable. Thus by Corollary 5.3 (4) A and B are two-dimensional regular affine k-domains. Moreover both $k[x]/(f(x))$ and $k[x]/(g(x))$ are k-isomorphic to the direct product k^n of n copies of k and hence by Corollary 5.3 (1) we have $A^{[1]} \cong_k B^{[1]}$.

For the proof of the assertion $\operatorname{Aut}_k A \not\cong \operatorname{Aut}_k B$, we consider the structures of the groups $\operatorname{Aut}_k A$ and $\operatorname{Aut}_k B$. Let τ be an element of $\operatorname{Aut}_k A$. First we claim that $\tau(u) = u$. Since

$$D^* = A^* = \tau(A^*) = \tau(D^*),$$

the image $\tau(u)$ is of the form $\tau(u) = cu^e$ for some $c \in k^*$ and $e = \pm 1$. On the other hand from Lemma 5.1 (2) it follows that $\tau(w(u)D) = w(u)D$. Therefore

$$w(cu^e) = bu^d w(u) \tag{5.6}$$

for some $b \in k^*$ and $d \in \mathbf{Z}$. Substituting $1, \xi, \zeta$ for u into the both sides of the equation (5.6), we see that

$$w(c) = w(c\xi^e) = w(c\zeta^e) = 0.$$

Namely $c, c\xi^e, c\zeta^e$ are the distinct roots of the equation $w(u) = 0$, and therefore as subsets of k we obtain

$$\{c, c\xi^e, c\zeta^e\} = \{1, \xi, \zeta\}.$$

By the assumption (5.4) we can easily verify that $c = e = 1$, that is $\tau(u) = u$. The claim follows. On the other hand by Lemma 5.1 the image $\tau(x)$ is of the form

$$\tau(x) = au^s x + r(u) \ (a \in k^*, \ s \in \mathbf{Z}, \ r(u) \in D)$$

such that

$$f(\tau(x)) \equiv a^n u^{sn} f(x) \pmod{w(u)D[x]}, \tag{5.7}$$

namely

$$f(\tau(x)) = a^n u^{sn} f(x) + w(u)h(u,x)$$

for some $h(u,x) \in D[x]$. Substituting $1, \ \xi, \ \zeta$ for u into the equation just above we can easily verify that the congruence (5.7) holds if and only if $r(u) \in w(u)D$ and $a^n = \zeta^{sn} = 1$. Note that the condition $a^n = \zeta^{sn} = 1$ is equivalent to the condition

$$a = \xi^i \ (0 \leq i < n), \ sn \equiv 0 \pmod{l},$$

where l denotes the least common multiple of n and m if ζ is a primitive m-th root of 1 for some $m > 1$; otherwise $l = 0$. As a consequence τ is a k-automorphism of A if and only if τ is a k-homomorphism $A \to A$ defined by

$$\tau(u) = u, \ \tau(x) = \xi^i u^{lj} x + w(u)v(u) \ (0 \leq i < n, \ j \in \mathbf{Z}, \ v(u) \in D). \tag{5.8}$$

Now consider the subgroup

$$G = \left\{ \begin{pmatrix} \xi^i u^{lj} & w(u)v(u) \\ 0 & 1 \end{pmatrix} \middle| \ 0 \leq i < n, \ j \in \mathbf{Z}, \ v(u) \in D \right\}$$

of the general linear group $GL_2(D)$. If we set

$$M = \left\{ \begin{pmatrix} 1 & w(u)v(u) \\ 0 & 1 \end{pmatrix} \middle| \ v(u) \in D \right\}, \ N = \left\{ \begin{pmatrix} \xi^i u^{lj} & 0 \\ 0 & 1 \end{pmatrix} \middle| \ 0 \leq i < n, \ j \in \mathbf{Z} \right\},$$

then it is easy to see that M and N are subgroups of G such that G is the semidirect product $M \rtimes N$ of M and N. Note that

$$M \cong D^+, \ N \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & (l = 0) \\ \mathbf{Z}/n\mathbf{Z} \oplus \mathbf{Z} & (l > 0), \end{cases}$$

where D^+ denotes the additive group of D. On the other hand the map $\mathrm{Aut}_k A \to G$ defined by

$$\tau \to \begin{pmatrix} \xi^i u^{lj} & w(u)v(u) \\ 0 & 1 \end{pmatrix}$$

is clearly a group isomorphism. So we have

$$\mathrm{Aut}_k A \cong \begin{cases} D^+ \rtimes \mathbf{Z}/n & (l = 0) \\ D^+ \rtimes (\mathbf{Z} \oplus \mathbf{Z}/n) & (l > 0). \end{cases} \tag{5.9}$$

By similar argument we can also prove

$$\mathrm{Aut}_k B \cong \begin{cases} D^+ & (l = 0) \\ D^+ \rtimes \mathbf{Z} & (l > 0). \end{cases} \tag{5.10}$$

In other words, if $\tau \in \operatorname{Aut}_k B$, then in the same way as the case of $\tau \in \operatorname{Aut}_k A$ we see that τ is given as the element of $\operatorname{Aut}_D B$ so that $\tau(x)$ is of the form

$$\tau(x) = au^s x + r(u) \ (a \in k^*, s \in \mathbf{Z}, r(u) \in D).$$

But in the present case τ ($\in \operatorname{Aut}_k B$) satisfies the condition

$$g(\tau(x)) \equiv a^n u^{sn} g(x) \pmod{w(u)D[x]}$$

instead of (5.7). So if we write N_1 for the subgroup of N generated by the element

$$\begin{pmatrix} u^l & 0 \\ 0 & 1 \end{pmatrix} \ (\in N),$$

where l is as in (5.8), then using a similar argument to the proof of (5.9) we can verify that $a = 1$, $s = lj$ ($j \in \mathbf{Z}$) and $\operatorname{Aut}_k B \cong M \rtimes N_1$. The isomorphism (5.10) follows. Now it is easy to see that $\operatorname{Aut}_k A \not\cong \operatorname{Aut}_k B$, which completes the proof of the corollary.

Remark 5.6. Let us set

$$A = k[t, x, y, z]/(tz + y^{p^q} + x + x^{qp})$$

for a field k of ch $= p > 0$ and a prime integer q with $q \neq p$, where t, x, y, z are indeterminates over k. Note that A can be regarded as a $k[t]$-algebra in an obvious way. Then we have

(1) $A^{[1]} \cong_{k[t]} k[t]^{[3]}$,

(2) $A \not\cong_{k[t]} k[t]^{[2]}$

as is shown in [2]. In particular $k[t]^{[2]}$ is not $k[t]$-invariant. The fact (1) now easily follows from Corollary 4.2, because

$$\begin{aligned} k[x,y]/(y^{p^q} + x + x^{qp}) &\cong_k k[-t^{p^q}, t + t^{qp}] \\ &= k[t] \\ &= k^{[1]}. \end{aligned}$$

The fact (2) is obtained as an immediate consequence of the following:

(3) If $\sigma \in \operatorname{Aut}_k A$ such that $\sigma(k[t]) = k[t]$, then we have $\sigma(t) = at$ for some $a \in k^*$.

The proof of the assertion (3) is similar to one of Lemma 5.1. This fact leads us to the following conjecture.

Conjecture 5.7. *For a k-algebra A in Remark 5.6 we have*

$$\operatorname{Aut}_k A \not\cong \operatorname{Aut}_k k^{[3]}.$$

The affirmative solution of this conjecture shows that A is a counterexsample to the cancellation problem for polynomial rings in three variables over a field k, which asks if $R = k^{[3]}$ for any k-algebra R such that $R^{[1]} \cong_k k^{[4]}$.

The content of present article except Remark 5.6 and Conjecture 5.7 is a part of [4].

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Local cohomology modules with respect to monomial ideals

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Introduction

A *simplicial complex* Δ on the *vertex set* $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that (*) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define a *Stanley-Reisner ideal* I_Δ of Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

In this article we always assume $\Delta \ne \emptyset$ i.e., $I_\Delta \ne A$. But we include the case $\Delta = \{\emptyset\}$ i.e., $I_\Delta = \boldsymbol{m} := (x_1, x_2, \ldots, x_n)$.

Hochster proved the following theorem:

THEOREM (Hochster's formula, cf. [St_2, Theorem 4.1]).

$$F(H^r_{\boldsymbol{m}}(k[\Delta]),\ t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{r-|F|-1}(\mathrm{link}_\Delta F; k) \prod_{i \in F} \frac{t_i^{-1}}{1 - t_i^{-1}}.$$

where $H^r_{\boldsymbol{m}}(k[\Delta])$ denote the r-th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $\boldsymbol{m}$, and $F(\cdot, t)$ is a $\mathbf{Z}^n$-graded Hilbert series.

The next theorem is inspired by the above formula:

MAIN THEOREM.

$$F(H^r_{I_\Delta}(A),\ t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{n-r-|F|-1}(\mathrm{link}_\Delta F; k) \prod_{i \in [n] \setminus F} \frac{t_i^{-1}}{1 - t_i^{-1}} \prod_{j \in F} \frac{1}{1 - t_j}.$$

where $H^r_{I_\Delta}(A)$ denote the r-th local cohomology module of A with respect to I_Δ.

The proof is Main Theorem, which is presented in §2, is similar to the computation of Hilbert series of local cohomology groups of semigroup rings in Stanley[St$_1$]. We identify a graded piece of Čech complex with a cochain complex of some simplicial complex and calculate its cohomology.

As applications, we consider the following problems according as Huneke [Hu].

Problem 1. When are $H^r_{I_\Delta}(A) = 0$?

Problem 2. When are $H^r_{I_\Delta}(A)$ finitely generated?

Problem 3. When are $H^r_{I_\Delta}(A)$ Artinian?

We show some criteria using local cohomology modules of $k[\Delta]$ with respect to the graded maximal ideal $\boldsymbol{m}$.

As a by-product we show a new (but not useful) criterion on Buchsbaum property of Stanley-Reisner rings $k[\Delta]$:

COROLLARY. *The following conditions are equivalent:*
(1)$H^r_{I_\Delta}(A)$ is an Artinian $\mathbf{Z}^n$-graded A-module for every $r \neq \mathrm{ht} I_\Delta$.
(2)$k[\Delta]$ is Buchsbaum.

§1. Preliminaries

We first fix notation. Let $\mathbf{N}$(resp. $\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). Let $|S|$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings. We refer the reader to, e.g., [Br–He], [Hi$_1$], [Ho] and [St$_2$] for the detailed information about combinatorial and algebraic background.

A *simplicial complex* Δ on the *vertex set* $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ such that (*) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $|F| = i+1$. We set $d = \max\{|F| \mid F \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d-1$.

We say that a simplicial complex Δ is *spanned* by $\{\sigma_1, \cdots, \sigma_s\}$ if $\Delta = \mathbf{2}^{\sigma_1} \cup \cdots \cup \mathbf{2}^{\sigma_s}$, where $\mathbf{2}^\sigma$ is the family of all subsets of σ.

For a simplicial complex Δ on the vertex set $[n]$, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

Then Δ^* is also a simplicial complex on the vertex set $[n]$.

If F is a face of Δ, then we define a subcomplex $\mathrm{link}_\Delta F$ as follows:

$$\mathrm{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\},$$

which we call *link* of F in Δ.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Next we define Hilbert series. Let k be a field, and A a polynomial ring $k[x_1, x_2, \ldots, x_n]$ of n-variables over k.

We consider A as the $\mathbf{Z}^n$- graded algebra $A = \bigoplus_{\alpha \in \mathbf{N}^n} A_\alpha$ with $\deg x_j = e_j$ where e_j is a j-th unit vector.

Let M be a $\mathbf{Z}^n$-graded A-module with $\dim_k M_\alpha < \infty$ for all $\alpha \in \mathbf{Z}^n$, where $\dim_k M_\alpha$ denotes the dimension of M_α as a k-vector space. Put $t = (t_1, t_2, \ldots, t_n)$.

The *Hilbert series* of M is defined by

$$F(M, t) = \sum_{\alpha \in \mathbf{Z}^n} (\dim_k M_\alpha) t^\alpha.$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n}$.

We consider $k[\Delta]$ as the $\mathbf{Z}^n$- graded A-module $k[\Delta] = \bigoplus_{i \geq 0} k[\Delta]_i$.

§2. Proof of the main theorem

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. We fix a simplicial complex Δ. Put $I_\Delta = (m_1, m_2, \ldots, m_\mu)$, where $\{m_1, m_2, \ldots, m_\mu\}$ is a minimal generating set of monomials for I_Δ.

Consider the following Čech complex;

$$C^\bullet = \bigotimes_{i=1}^{\mu}(0 \longrightarrow A \longrightarrow A_{m_i} \longrightarrow 0)$$
$$= 0 \longrightarrow A \xrightarrow{\delta^1} \bigoplus_{1\le i\le\mu} A_{m_i} \xrightarrow{\delta^2} \bigoplus_{1\le i<j\le\mu} A_{m_i m_j} \xrightarrow{\delta^3} \cdots \xrightarrow{\delta^\mu} A_{m_1 m_2 \cdots m_\mu} \longrightarrow 0.$$

We describe δ^{r+1} as follows; Put $R := A_{m_{i_1} m_{i_2} \cdots m_{i_r}}$ and $\{j_1, j_2, \ldots, j_s\} = \{1, 2, \cdots, \mu\} \setminus \{i_1, i_2, \ldots, i_r\}$, where $j_1 < j_2 < \ldots < \ldots j_s$ and $r + s = \mu$.
Let $\psi_{j_p} : R \longrightarrow R_{m_{j_p}}$ be a natural map. For $u \in R$, we have

$$\delta^{r+1}(u) = \sum_{p=1}^{s}(-1)^{|\{q; i_q < j_p\}|}\psi_{j_p}(u) = \sum_{p=1}^{s}(-1)^{j_p - p}\psi_{j_p}(u).$$

For $F \in \mathbf{2}^{[n]}$, we define $x^F := x^{\sum_{i\in F} e_i}$.
We define a simplicial complex $\Delta(F)$ on the vertex set $[\mu]$ by

$$\Delta(F) = \{\{i_1, i_2, \ldots, i_r\};\ \ x^F \mid \prod_{j\in\{1,2,\cdots\mu\}\setminus\{i_1,i_2,\ldots,i_r\}} m_j\}.$$

For $\alpha \in \mathbf{Z}^n$, there is a unique decomposition $\alpha = \alpha_+ - \alpha_-$ such that $\alpha_+, \alpha_- \in \mathbf{N}^n$ and supp $\alpha_+ \cap$supp $\alpha_- = \emptyset$. Then we have supp $\alpha_- = \{i; \alpha_i < 0\}$.

LEMMA 2.1.(cf. [St$_1$, Lemma 2.5]). *For $\alpha \in \mathbf{Z}^n$ give an orientation for $\Delta(\text{supp }\alpha_-)$ by $1 < 2 < \cdots < \mu$. Then we have the following isomorphism of complexes: $C^\bullet_\alpha \cong \tilde{C}_\bullet(\Delta(\text{supp }\alpha_-))$ such that $C^r_\alpha \cong \tilde{C}_{\mu-r-1}(\Delta(\text{supp }\alpha_-))$.*

For $F \in \mathbf{2}^{[n]}$, we define a covering C of $(\Delta^*)_F$ as follows: Let $F_1, F_2, \ldots, F_\mu$ be the facets of Δ^*. Let Δ_i be a simplicial complex spaned by $F_i \cap F$. Then Δ_i is a subcomplex of $(\Delta^*)_F$ and $(\Delta^*)_F = \cup_{i=1}^{\mu}\Delta_i$. We define $C := \{\Delta_i \mid i = 1, 2 \ldots, \mu\}$.

LEMMA 2.2 . We have

$$\Delta(F)^* = \text{Nerve}(C).$$

Proof.

$$\Delta(F)^* = \{\{i_1, i_2, \ldots, i_r\} \mid \{1, 2, \cdots \mu\} \setminus \{i_1, i_2, \ldots, i_r\} \notin \Delta(F)\}$$

$$
\begin{aligned}
&= \{\{i_1, i_2, \ldots, i_r\} \mid F \not\subset \bigcup_{j\in\{i_1,i_2,\ldots,i_r\}} ([n] \setminus F_j)\} \\
&= \{\{i_1, i_2, \ldots, i_r\} \mid p \notin \bigcup_{j\in\{i_1,i_2,\ldots,i_r\}} ([n] \setminus F_j) \text{ for some } p \in F\} \\
&= \{\{i_1, i_2, \ldots, i_r\} \mid F \cap \bigcap_{j\in\{i_1,i_2,\ldots,i_r\}} F_j \neq \emptyset\} \\
&= \{\{i_1, i_2, \ldots, i_r\} \mid \bigcap_{j\in\{i_1,i_2,\ldots,i_r\}} (F \cap F_j) \neq \emptyset\} \\
&= \{\{i_1, i_2, \ldots, i_r\} \mid \bigcap_{j\in\{i_1,i_2,\ldots,i_r\}} \Delta_j \neq \{\emptyset\}\} \\
&= \text{Nerve}(C).
\end{aligned}
$$

Q.E.D.

Proof of the main theorem. We have the following isomorphisms:

$$
\begin{aligned}
H^r(C^\bullet)_\alpha &\cong \tilde{H}_{\mu-r-1}(\Delta(\text{supp } \alpha_-); k) \\
&\cong \tilde{H}^{r-2}(\Delta(\text{supp } \alpha_-)^*; k) \\
&\cong \tilde{H}^{r-2}(\text{Nerve } (C); k) \\
&\cong \tilde{H}^{r-2}((\Delta^*)_{\text{supp } \alpha_-}; k) \quad \text{(by Nerve Lemma)} \\
&\cong \tilde{H}_{|\text{supp } \alpha_-|-r-1}(\text{link}_\Delta([n] \setminus \text{supp } \alpha_-); k)
\end{aligned}
$$

$$
\begin{aligned}
&F(H^r_{I_\Delta}(A),\ t) \\
&= \sum_{\substack{\alpha\in \mathbf{Z}^n \\ [n]\setminus \text{supp } \alpha_-\in\Delta}} \dim_k \tilde{H}_{|\text{supp } \alpha_-|-r-1}(\text{link}([n] \setminus \text{supp } \alpha_-); k) t^\alpha \\
&= \sum_{F\in\Delta} \dim_k \tilde{H}_{n-r-|F|-1}(\text{link}_\Delta F; k) \prod_{i\in V\setminus F} \frac{t_i^{-1}}{1-t_i^{-1}} \prod_{j\in F} \frac{1}{1-t_j}.
\end{aligned}
$$

Q.E.D.

§3. Some applications

First we consider the vanishing and Noetherian property on $H^\bullet_{I_\Delta}(A)$.

COROLLARY 3.1. *Assume $I_\Delta \neq (0)$. Put $d = \dim k[\Delta]$ and $h = \text{ht} I_\Delta$. For a fixed r, the following conditions are equivalent:*
(1)$H^{h+r}_{I_\Delta}(A)$ is a finitly generated $\mathbf{Z}^{\,n}$-graded A-module.

(2)$H^{h+r}_{I_\Delta}(A) = 0$.
(3)$H^{d-r}_{\boldsymbol{m}}(k[\Delta]) = 0$.

COROLLARY 3.2(Lyubeznik). *Put* $\mathrm{cd}(A, I_\Delta) := \max\{r \mid H^r_{I_\Delta}(A) \neq 0\}$. *Then we have*

$$\mathrm{cd}(A, I_\Delta) = \mathrm{pd}_A k[\Delta].$$

In particular, the following conditions are equivalent:
(1)$H^r_{I_\Delta}(A) = 0$ *for every* $r \neq \mathrm{ht} I_\Delta$.
(2)$k[\Delta]$ *is Cohen-Macaulay.*

For the Artinian property, we study the module structure of $H^\bullet_{I_\Delta}(A)$.

For $\beta \in \mathbf{N}^n$, we have supp $\alpha_- \supset$ supp $(\alpha+\beta)_-$. Then we have $\Delta(\mathrm{supp}\ \alpha_-) \subset \Delta(\mathrm{supp}\ (\alpha+\beta)_-)$. There is a natural map $\theta_\beta : \tilde{H}_\bullet(\Delta(\mathrm{supp}\ \alpha_-); k) \longrightarrow \tilde{H}_\bullet(\Delta(\mathrm{supp}\ (\alpha+\beta)_-); k)$.

PROPOSITON 3.3.

$$H^r_{I_\Delta}(A) \cong \bigoplus_{\alpha \in \mathbf{Z}^n} \tilde{H}_{\mu-r-1}(\Delta(\mathrm{supp}\ \alpha_-); k),$$

as $\mathbf{Z}^n$*-graded A-modules, where the A-module structure of the right-hand side is defined by* $x^\beta u = \theta_\beta(u)$ *for* $u \in \tilde{H}_{\mu-r-1}(\Delta(\mathrm{supp}\ \alpha_-); k)$ *and* $\beta \in \mathbf{N}^n$.

Proof. We have to check that the following diagram is commute:

$$\begin{array}{ccc} H^r_{I_\Delta}(A)_\alpha & \xrightarrow{\cong} & \tilde{H}_{\mu-r-1}(\Delta(\mathrm{supp}\ \alpha_-); k) \\ {\scriptstyle x^\beta}\downarrow & & \downarrow{\scriptstyle \theta_\beta} \\ H^r_{I_\Delta}(A)_{\alpha+\beta} & \xrightarrow{\cong} & \tilde{H}_{\mu-r-1}(\Delta(\mathrm{supp}\ (\alpha+\beta)_-); k) \end{array}$$

Note that $(H^r_{I_\Delta}(A))_\alpha$ is a subquotient of $\bigoplus(A_{m_{i_1}m_{i_2}\cdots m_{i_r}})_\alpha$. Suppose $(A_{m_{i_1}m_{i_2}\cdots m_{i_r}})_\alpha \neq 0$. Then we can take $(0, \ldots, x^\alpha, \ldots 0) \in \bigoplus(A_{m_{i_1}m_{i_2}\cdots m_{i_r}})_\alpha$, where $x^\alpha \in (A_{m_{i_1}m_{i_2}\cdots m_{i_r}})_\alpha$. This element corresponds to $\pm\{[\mu]\setminus\{i_1, i_2, \ldots, i_r\}\}$. We have $(A_{m_{i_1}m_{i_2}\cdots m_{i_r}})_{\alpha+\beta} \neq 0$. And $(0, \ldots, x^{\alpha+\beta}, \ldots 0) \in \bigoplus(A_{m_{i_1}m_{i_2}\cdots m_{i_r}})_{\alpha+\beta}$ coresponds $\pm\{[\mu]\setminus\{i_1, i_2, \ldots, i_r\}\}$. This indeuces the desired diagram commutative. Q.E.D.

COROLLARY 3.4. *For* $\beta \in \mathbf{N}^n$ *such that* supp $\alpha_- =$ supp $(\alpha+\beta)_-$, *we have an isomorphism*

$$H^r_{I_\Delta}(A)_\alpha \xrightarrow{x^\beta} H^r_{I_\Delta}(A)_{\alpha+\beta}.$$

COROLLARY 3.5. *We have*

$$\mathrm{Soc}\ H^r_{I_\Delta}(A) := \mathrm{Ann}\ _{H^r_{I_\Delta}(A)}\boldsymbol{m} \subset H^r_{I_\Delta}(A)_{(-1,-1,\ldots,-1)}$$

And

$$\dim_k \mathrm{Soc}\ H^r_{I_\Delta}(A) \leq \dim_k H^{n-r}_{\boldsymbol{m}}(k[\Delta])_{(0,0,\ldots,0)} = \dim_k \tilde{H}_{n-r-1}(\Delta; k).$$

In particular, $\mathrm{Soc}\ H^r_{I_\Delta}(A)$ *is finitely generated.*

LEMMA 3.6. *For a* $\mathbf{Z}^n$*-graded A-module M such that the Hilbert series* $F(M,t)$ *belongs to* $\mathbf{Z}[t_1^{-1}, t_2^{-1}, \ldots t_n^{-1}]$, *we have* $\mathrm{Supp}M \subset \{\boldsymbol{m}\}$.

Since $H^r_{I_\Delta}(A)$ is an Artinian if and only if $\mathrm{Soc}\ H^r_{I_\Delta}(A)$ is finitely generated and $\mathrm{Supp}H^r_{I_\Delta}(A) \subset \{\boldsymbol{m}\}$, we have:

COROLLARY 3.7. *The following conditions are equivalent for a fixed r:*
(1)$H^r_{I_\Delta}(A)$ *is an Artinian* $\mathbf{Z}^n$*-graded A-module.*
(2)$H^{n-r}_{\boldsymbol{m}}(k[\Delta])$ *is a finite-dimensional vector space over k.*

COROLLARY 3.8. *The following conditions are equivalent:*
(1)$H^r_{I_\Delta}(A)$ *is an Artinian* $\mathbf{Z}^n$*-graded A-module for every* $r \neq \mathrm{ht}I_\Delta$.
(2)$k[\Delta]$ *is Buchsbaum.*

COROLLARY 3.9. *Take r such that* $r \in \{\mathrm{ht}P \mid P \in \mathrm{Ass}A/I_\Delta\}$. *Then:*
(1)$H^r_{I_\Delta}(A)$ *is not a finitely generated* $\mathbf{Z}^n$*-graded A-module for* $I_\Delta \neq (0)$.
(2)$H^r_{I_\Delta}(A)$ *is not an Artinian* $\mathbf{Z}^n$*-graded A-module for* $I_\Delta \neq (0), \boldsymbol{m}$.

Remark. Dr. T. Kawasaki suggested to consider the natural map

$$\mathrm{Ext}^{h+r}(A/I_\Delta, A) \longrightarrow H^{h+r}_{I_\Delta}(A).$$

and taught the author that this map is injective. It is also shown in Mustaţă [Mu]. There is, in fact, a more general treatment there. The author found [Mu] after proving all assertions in this article, but before finishing typing this manuscript.

And it is easy to show that the above map is an essential extension by Proposition 3.3. Some of corollaries (e.g., Corollaries 3.1, 3.2, 3.7) also follow from this fact.

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Modules with certain homology

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1 Introduction

Let R be a Gorenstein local ring with dimension d. We consider the following subcategories of the category mod R consisting of finitely generated R-modules:

$\mathcal{CM}(R)$: the category of maximal Cohen-Macaulay modules.
$\mathcal{F}(R)$: the category of modules with finite projective dimensions.

Let us remind the definition of Cohen-Macaulay approximation.

Theorem 1.1 (Auslander-Buchweitz [3], Kato [7]) *An arbitrary $M \in$ mod R has the following exact sequences.*

$$\begin{array}{rl} 0 \to Y_M \to X_M \to M \to 0 & \textit{(Cohen-Macaulay approximation)}, \\ 0 \to M \to Y^M \to X^M \to 0 & \textit{(finite projective hull)}, \\ 0 \to X(M) \to M \oplus P \to Y(M) \to 0 & \textit{(origin extension)} \end{array}$$

where $Y_M, Y^M, Y(M) \in \mathcal{F}(R)$, $X_M, X^M, X(M) \in \mathcal{CM}(R)$, and P is a free module. And $Y(M) \overset{st}{\cong} Y^M$, $Y_M \overset{st}{\cong} \Omega^1_R(Y(M))$ and $X(M) \overset{st}{\cong} X_M \overset{st}{\cong} \Omega^1_R(X^M)$.

Two modules M and M' are called stablly equivalent and denoted as $M \overset{st}{\cong} M'$ if projective modules P and P' exist such that $M \oplus P \cong M' \oplus P'$. A stable module refers to a module without non-zero free summand. The projective stabilization $\underline{mod\ R}$ is defined as follows.

- Each object of $\underline{mod\ R}$ is an object of $mod\ R$.
- For $A, B \in mod\ R$, a set of morphisms from A to B is $\underline{\mathrm{Hom}_R}(A, B) := \mathrm{Hom}_R(A, B)/\mathcal{P}(A, B)$ where $\mathcal{P}(A, B) := \{f \in \mathrm{Hom}_R(A, B) \mid f \text{ factors through some projective module}\}$. Each element is denoted as $\underline{f} = f \ mod\ \mathcal{P}(A, B)$.

Lemma 1.2 *For $M \in mod\ R$, the minimal origin extension remains exact when dualized by R. Putting $T(M) := tr\ X(tr\ M)$ and $N(M) := tr\ Y(tr\ M)$, we have an exact sequence*

$$0 \to N(M) \to M \oplus P' \to T(M) \to 0$$

with some free module P'.

We observe that

$$\mathrm{Ext}^i_R(M, R) \cong \mathrm{Ext}^i_R(Y(M), R) \quad (1 \leq i), \qquad \Omega^d_R(M) \overset{st}{\cong} \Omega^d_R(X(M)).$$

In other words, $Y(M)$ is homological extract of M and $X(M)$ syzygical. For given modules $E_1, \ldots, E_s$ and L, our aim is to find the modules M with the property that $\mathrm{Ext}^i_R(M, R) \cong E_i \quad (1 \leq i \leq s), \qquad \Omega^s_R(M) \cong L$.

Of course every module cannot be an extension module dualized by R. Here are conditions for a module to be an extension module, or a syzygy module.

Remark 1.3 *For $M \in mod\ R$ and an integer $r > 0$, put $E := Ext^r_R(M, R)$. Then $Ext^i_R(E, R) = 0 \quad (i < r)$.*

Lemma 1.4 (Auslander-Bridger [2]) *Let $r > 0$ be an integer. The following are equivalent for $M \in mod\ R$.*

1) There exists $M' \in mod\ R$ such that $M \overset{st}{\cong} \Omega^r_R(M')$.

2) $Ext^i_R(tr\ M, R) = 0 \quad (1 \leq i \leq r)$.

3) $M \overset{st}{\cong} \Omega^r_R tr \Omega^r_R tr M$.

In the case $L = 0$, we already have one solution.

Lemma 1.5 (Kawasaki [8]) *Let $L = 0$ and $E_1, \ldots, E_s$ be R-modules satisfying* $\dim E_i \leq d - i \quad (1 \leq i \leq s)$. *Put $M := \oplus_{i=1}^{s} tr\Omega_R^{i-1} E_i$. Then M has the property that $Ext_R^i(M, R) \cong E_i \quad (1 \leq i \leq s), \qquad \Omega_R^s(M) = 0$.*

We generalize the above to construct disired modules. The similar method is also given by Amasaki [1].

Definition 1.6 *For R-modules $E_1, \ldots E_s$, and L, $E_s \lfloor E_{s-1} \lfloor \ldots \lfloor E_1; L$ denotes the set of the sequences of modules $(C_{s+1}, C_s, \ldots, C_1)$ such that*

$$C_{s+1} = L, \quad 0 \to E_i \to C_i \to \Omega_R^1(C_{i+1}) \to 0.$$

Theorem 1.7 *Let $E_1, \ldots, E_s$ and L be R-modules satisfying* $\dim E_i \leq d - i \quad (1 \leq i \leq s)$ *and $Ext_R^j(tr\ C, R) = 0 \quad (1 \leq j \leq s)$. Every module M with*

$$\Omega_R^s(M) \cong L, \qquad Ext_R^i(M, R) \cong E_i \quad (1 \leq i \leq s)$$

corresponds to an element of $E_s \lfloor E_{s-1} \lfloor \ldots \lfloor E_1; tr\ L$.

proof Correspondence to $E_s \lfloor E_{s-1} \lfloor \ldots \lfloor E_1; tr\ L$. If an R-module M satisfies

$$\mathrm{Ext}_R^j(M, R) \cong E_j \quad (1 \leq j \leq s), \qquad \Omega_R^s(M) \overset{st}{\cong} L,$$

put $C_i := tr\ \Omega_R^{i-1}(M) \quad (1 \leq i \leq s+1)$. Consider the free resolution of M:

$$\cdots \to F_{M i+1} \overset{d_{F_{M i+1}}}{\to} F_{M i} \overset{d_{F_{M i}}}{\to} F_{M i-1} \to \cdots \to F_{M0} \to M \to 0$$

Take the R-dual complex:

$$\cdots \leftarrow (F_{M i+1})^* \overset{d_{F_{M i+1}}^*}{\leftarrow} (F_{M i})^* \overset{(d_{F_{M i}})^*}{\leftarrow} (F_{M i-1})^* \leftarrow \cdots$$

Then we have the exact sequences

$$0 \to \mathrm{Ext}_R^i(M, R) \to Coker\, d_{F_{M i}}^* \to Coim\ d_{F_{M i+1}}^* \to 0$$

where $Coker\, d_{F_{M i}}^* \cong C_i$ and $Coim\ d_{F_{M i+1}}^* \cong \Omega_R^1(C_{i+1})$.

Correspondence from $E_s \lfloor E_{s-1} \lfloor \ldots \lfloor E_1; tr\ L$. Let $(C_{s+1}, C_s, \ldots, C_1) \in E_s \lfloor E_{s-1} \lfloor \ldots \lfloor E_1; tr\ L$. That is, $C_{s+1} = tr\ L$, and

$$\theta_i : 0 \to E_i \to C_i \overset{r_i}{\to} \Omega_R^1(C_{i+1}) \to 0 \quad (1 \leq i \leq s)$$

are exact. We claim that $M := trC_1$ satisfies $\mathrm{Ext}^j_R(M,R) \cong E_j \quad (1 \leq j \leq s)$, and $\Omega^s_R(M) \overset{st}{\cong} L$. To show the claim, first take the projective presentation of C_i $Q^i \overset{d^i}{\to} P^i \overset{\rho}{\to} C_i$ as $P^i = Q^{i+1}$; such a presentation is obtained as follows. The homomorphism $r_i : C_i \to \Omega^1_{C_{i+1}}(\cong) Im\, d^{i+1}$ on θ_i induces $\tilde{r_i} : P^i \to Q^{i+1}$. Since $Im\,(d^{i+1} \circ \tilde{r_i}) = Im\,(r_i \circ \rho) = Im\, d^{i+1}$, we may put $d^{i+1} := d^{i+1} \circ \tilde{r_i}$. We get the complex

$$P^\bullet : P^0 \overset{d^1}{\to} P^1 \to \cdots \overset{d^i}{\to} P^i \overset{d^{i+1}}{\to} P^{i+1} \to \cdots \to P^s \overset{d^{s+1}}{\to} P^{s+1}$$

which induces exact sequences

$$\begin{array}{ccccccc} 0 \to H^i(P^\bullet) \to & P_i/Im\, d^i & \to & P^i/Ker\, d^{i+1} & \to 0. \\ & \| & & \| & \\ & C_i & \overset{r_i}{\to} & \Omega^1_R(C_{i+1}) & \end{array}$$

Thus $H^i(P^\bullet) \cong E_i \quad (1 \leq i \leq s)$.

Secondly, we shall prove that the R-dual complex $P^{\bullet *}$ is exact. The map $r_i^* : (\Omega^1_R(C_{i+1}))^* \to C_i^* = Ker\,(d^{i*})$ is an isomorphism since $E_i^* = 0$. On the other hand, the natural homomorphism $Im\,(d^{i+1*}) \to (Im\, d^{i+1})^*$ is nothing but the natural homomorphism $\Omega^1_R tr\ (C_{i+1}) \to \Omega^2_R\ tr\Omega^1_R(C_{i+1})$. If $\mathrm{Ext}^1_R C_{i+1} = 0$, Lemma 1.4 tells us $tr\ (C_{i+1}) \to\ tr\Omega^1_R(C_{i+1})$ is an isomorphism. Therefore we have only to check $\mathrm{Ext}^1_R(C_{i+1}, R) = 0$. In general, $\mathrm{Ext}^j_R(C_i, R) \cong \mathrm{Ext}^{j+1}_R(C_{i+1}, R) \cong \cdots \cong \mathrm{Ext}^{s+1-i+j}_R(C_{s+1}, R) = 0$ for $0 < j < i \leq s$ because $\mathrm{Ext}^j_R(E_i, R) = 0 \quad (j < i)$ and $\mathrm{Ext}^l_R(tr\ L, R) = 0 \quad (1 \leq l \leq s)$.

Finally, we get a free complex $(F_\bullet, d_{F_\bullet})$ as the combination of $P^{\bullet *}$ and the free resolution of $Ker\, d^{s+1*} = \Omega^2_R(L)$:

- $\cdots \to F_{s+3} \to F_{s+2} \to \Omega^2_R(L)$ is exact.
- $F_i = P^{i*}$, $d_{Fi} = d^i$ for $1 \leq i \leq s+1$. $F_0 = P^{0*}$.
- d_{Fs+2} is the composite of F_{s+2} ? $\Omega^2_R(L)$ and $\Omega^2_R(L) \hookrightarrow P^{s+1*}$.

Then $F_\bullet$ is a free resolution of $M := tr\ C_1$ hence $\mathrm{Ext}^j_R(M,R) = H^j(P^\bullet) \cong E_j \quad (1 \leq j \leq s)$ and $\Omega^s_R(M) \overset{st}{\cong} L$. (q.e.d.)

2 Diversity

From now on, we assume that R is complete. (We use completeness only for the proof of Proposition 2.1.) For the simplicity, let $s = d$. Our process of finding an element of $E_d \lfloor E_{d-1} \lfloor \dots \lfloor E_1; tr\ L$ is actually by two steps: First, find a module $Y \in \mathcal{F}(R)$ such that $\mathrm{Ext}^i_R(M, R) \cong E_i \quad (1 \le l \le d)$. This is obtained as an element of $E_d \lfloor E_{d-1} \lfloor \dots \lfloor E_1; 0$. Second, find a module $M \in \mathrm{mod}\ R$ such that $Y(M) \overset{st}{\cong} Y$ and $X(M) \overset{st}{\cong} \Omega_R^{-d}(L)$. Indeed, the condition $\mathrm{Ext}^l_R(tr\ L, R) = 0 \quad (1 \le l \le d)$ implies that L is a maximal Cohen-Macaulay module. Hence the condition $\Omega^d_R(M) \overset{st}{\cong} L$ means $X(M) \overset{st}{\cong} \Omega_R^{-d}(L)$. Since $Y^M \overset{st}{\cong} Y(M)$ and $X^M \overset{st}{\cong} \Omega_R^{-1}(X(M))$, the second step is equivalent to find a homomorphism $Y \to \Omega_R^{-d-1}(L)$.

In the previous section, we have already given an explicit method for the first step. Alternatively, in this section we shall further discuss the second step. Here we shall show only the rough sketch of the proofs. For the detail, please ask for [5].

Proposition 2.1 ([5], [6]) *Let $Y \in \mathcal{F}(R)$ and $X \in \mathcal{CM}(R)$. Every homomorphism factors through TY via φ_Y. Moreover φ_Y induces an isomorphism* $\underline{Hom}_R(Y, X)/\underline{Aut}_R(Y) \times \underline{Aut}_R(X) \cong \underline{Hom}_R(TY, X)/\underline{Aut}_R(TY) \times \underline{Aut}_R(X)$.

From this proposition above, to get a module with the property that $Y(M) \overset{st}{\cong} Y$ and $X^M \overset{st}{\cong} X$, we have only to calculate $\underline{Hom}_R(TY, X)$ instead of $\underline{Hom}_R(Y, X)$. In other words, if $TY \overset{st}{\cong} TY'$, each module M with the property that $Y(M) \overset{st}{\cong} Y$ one to one corresponds to a module M' with $Y(M') \overset{st}{\cong} Y'$. Hence for the sake of classification of *mod* R via $\mathcal{F}(R)$, it is necessary to know $T\mathcal{F}(R)$.

The following is the most extreme example with respect to the diversity of $TY \quad (Y \in \mathcal{F}(R))$.

Example 2.2 *Let R be a ring $k[[x, y]]/(xy)$. For every indecomposable module Y with finite projective dimension, $TY \cong R/xR \oplus R/yR$.*

In general, the whole image $T\mathcal{F}(R)$ is obtained by a part of $\mathcal{F}(R)$:

Proposition 2.3

$$\begin{aligned} \{TY \mid Y \in \mathcal{F}(R)\} &= \{TY \mid pdY = 1\} \\ &= \{X^M \mid M : \begin{array}{l} \textit{Cohen-Macaulay module with codimension one;} \\ \dim M = depthM = d-1. \end{array} \} \end{aligned}$$

Lemma 2.4 *The folllowing are equivalent for $M \in$ mod R.*

1) *There exits an integer $s \geq 0$ such that $M_{\mathfrak{p}} \cong (R_{\mathfrak{p}})^s$ for every minimal prime ideal $\mathfrak{p}$.*

2) *There exists an exact sequence $0 \to R^s \to M \to M' \to 0$ with the property that $M'^* = 0$.*

We say that *fc0-condition holds for M* if an R-module M satisfies either (hence both) of above two cnditions.

We easily observe the following.

Corollary 2.5 *If R is a domain, then fc0-condition holds for every $M \in$ mod R.*

Proposition 2.6 *The following are equivalent.*

1) *fc0-condition holds for every $M \in$ mod R.*

2) *fc0-condition holds for every $X \in CM(R)$.*

proof. The nontrivial implication 2) $\Rightarrow$ 1) comes from the following Lemma 2.7 together with Cohen-Macaulay approximation.

Lemma 2.7 *If M or tr M is of finite projective dimension, then fc0-condition holds for M.*

The first equation of Proposition 2.3 is also obtained from Lemma 2.7.

Theorem 2.8 *The following are equivalent.*

1) $\underline{\{TY \mid Y \in \mathcal{F}(R)\}} = \underline{\mathcal{C}M(R)}$.

2) For every $X \in \mathcal{C}M(R)$*, there exists a Cohen-Macaulay module* X' *with codimension one such that* $X \overset{st}{\cong} X(X')$.

3) For every $M \in$ *mod* R*, fc0-condition holds.*

4) R *is an integral domain.*

proof. The equivalence between 1) and 2) is straightforward from Proposition 2.3. The condition 2) is nothing but the condition 2) in Proposition 2.6 which is equivalent to 3). Corollary 2.5 says that 4) implies 3).

Now it remains to show the implication from 3) to 4). If R has two distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, $R/\mathfrak{p}$ should have different ranks when localized by $\mathfrak{p}$ and $\mathfrak{q}$. On the other hand, suppose $\mathfrak{p}$ is the only minimal prime ideal in R. From the assumption 3), we get $\mathfrak{p}R_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is a set of the non-zero-divisors in R, $\mathfrak{p} = 0$. (q.e.d.)

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On heights and grades of determinantal ideals

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1 Introduction

The present note is just a report of our recent work. Thus many of the proofs are omitted, and we shall advise the reader to be referred to our paper [MY].

In the book "Syzygies" [EG], Evans and Griffith stated and used a theorem which is, under mild conditions, equivalent to the following

Theorem 1.1. *Let R be a commutative noetherian ring with identity and A an $m \times n$ matrix with entries in R. Suppose that $\mathrm{ht}(I_t(A)) \geq l$ and $n \geq t+l-1$. Then there is a lower triangular unipotent matrix V such that*

$$\mathrm{ht}(I_t(A'_{\leq n-j})) \geq l-j \qquad \text{for } j = 1, \ldots, l-1,$$

where $A' = AV$.

(See below for the notation.)

In this note we establish the following more general and stronger result.

Theorem 1.2. *With the same R and A as above, suppose that $\mathrm{ht}(I_t(A)) \geq l$ (or $\mathrm{grade}(I_t(A)) \geq l$ resp.). Then there is a lower triangular unipotent matrix V such that*

$$\mathrm{ht}(I_t(A'_{\leq t+j-1})) \geq j \qquad \text{for } j = 1, \ldots, l$$

(or $\mathrm{grade}(I_t(A'_{\leq t+j-1})) \geq j$ for $j = 1, \ldots, l$ resp.), where $A' = AV$.

Using this result (and a more general one), we establish generalizations of Serre's theorem and Bourbaki's theorem, and some kind of results of this direction. The main result of this note also has a version which is described in terms of Fitting invariants. We will use this version to establish a result of Forster-Swan type.

Notation and Convention

- R will always denote a commutative noetherian ring with identity.
- $\mathbf{N}$ denote the set of all non-negative integers.
- If A is an $m \times n$ matrix with entries in R, we denote by $I_t(A)$ the ideal generated by all t-minors of A. If $t \leq 0$, we define $I_t(A)$ to be the unit ideal R and if $t > \max(m, n)$, then we define $I_t(A)$ to be the zero ideal (0).
- With A as above and j an integer with $1 \leq j \leq n$, we denote by $A_{\leq j}$ the $m \times j$ matrix consisting of the first j columns of A.
- When denoting a free R-module by R^n, elements of R^n are described by row vectors, and if
$$R^m \xrightarrow{A} R^n$$
is an R-linear map, A is an $m \times n$ matrix with entries in R and the map is obtained by multiplying A from the right.
- If M is an R-module and $x_1, \ldots, x_n \in M$, the R-submodule generated by $x_1, \ldots, x_n$ is denoted by $(x_1, \ldots, x_n)$ or $(x_1, \ldots, x_n)R$.
- If M is a finitely generated R-module and $\mathfrak{p}$ is a prime ideal of R, we denote the number of elements in a minimal system of generators of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ by $\mu(M_\mathfrak{p})$.

2 Values of prime ideals with finiteness conditions

Before stating our main theorem, we argue and state several examples on the values of prime ideals which satisfy a condition, which we call the "finiteness condition".

Although in the major applications of our main theorem we use height or depth as the values of prime ideals, it is also applicable to any values of prime ideals with finiteness conditions. And the variety of values of prime ideals with finiteness conditions increases the applicability of our theorem.

Let $|\cdot|' \colon \operatorname{Spec} R \to \mathbf{N} \cup \{\infty\}$ be a function defined on $\operatorname{Spec} R$. We call such a function a value of prime ideals. For any value of prime ideals $|\cdot|'$, we define for any ideal I of R
$$|I| := \inf(\{|\mathfrak{p}|' \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \supseteq I\} \cup \{\infty\}).$$
Note that $|I| \leq |I'|$ if $I \subseteq I'$ and $|R| = \infty$.

If the condition

(FC) If I is an ideal of R such that $|I| < \infty$, then the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I,\ |\mathfrak{p}|' = |I|\}$ is a finite set.

is satisfied, then we say that the value $|\cdot|'$ satisfies the finiteness condition ((FC) for short).

We list several examples of values of prime ideals which satisfy (FC). Note that $|\mathfrak{p}| \neq |\mathfrak{p}|'$ in general as we will see in (ii) of the following example.

Example 2.1. (i) If we put $|\mathfrak{p}|' = \operatorname{ht} \mathfrak{p}$ for any prime ideal $\mathfrak{p}$, then $|\cdot|'$ satisfies (FC) and $|I| = \operatorname{ht} I$ for any ideal I of R.

(ii) If we put $|\mathfrak{p}|' = \operatorname{depth} R_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$, then $|\cdot|'$ satisfies (FC) and $|I| = \operatorname{grade} I$ for any ideal I of R.

(iii) Suppose that $\dim R$ is finite. If we put $|\mathfrak{p}|' = \dim R - \dim R/\mathfrak{p}$ for any prime ideal $\mathfrak{p}$, then $|\cdot|'$ satisfies (FC) and $|I| = \dim R - \dim R/I$.

Before stating the next examples, we recall several definitions from [Swa]. For any ideal I of R, we set

$$\text{j-rad}\, I := \bigcap_{\mathfrak{m} \in \operatorname{Max} R, \mathfrak{m} \supseteq I} \mathfrak{m}$$

and call it the j-radical of I, where $\operatorname{Max} R$ is the set of all maximal ideals of R. We also set

$$\text{j-Spec}\, R := \{\mathfrak{p} \in \operatorname{Spec} R \mid \text{j-rad}\, \mathfrak{p} = \mathfrak{p}\}.$$

It is easily verified that if I is an ideal of R and if

$$\text{j-rad}\, I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$$

is the minimal primary decomposition of j-rad I, then $\mathfrak{p}_i \in \text{j-Spec}\, R$ for any i. We also define

$$\text{j-dim}\, R := \sup\{t \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t,\ \mathfrak{p}_0, \ldots, \mathfrak{p}_t \in \text{j-Spec}\, R\},$$

and for any $\mathfrak{p} \in \text{j-Spec}\, R$ and for any ideal I of R, we set

$$\text{j-ht}\, \mathfrak{p} := \max\{t \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t = \mathfrak{p},\ \mathfrak{p}_0, \ldots, \mathfrak{p}_t \in \text{j-Spec}\, R\},$$

and

$$\text{j-ht}\, I := \inf\{\text{j-ht}\, \mathfrak{p} \mid \mathfrak{p} \in \text{j-Spec}\, R,\ \mathfrak{p} \supseteq I\}.$$

Now we state the following

Example 2.2. (i) If we set

$$|\mathfrak{p}|' := \begin{cases} \text{j-ht}\,\mathfrak{p} & \text{if } \mathfrak{p} \in \text{j-Spec}\, R, \\ \infty & \text{otherwise,} \end{cases}$$

then $|\cdot|'$ satisfies (FC), since

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I,\ |\mathfrak{p}|' = |I|\} \subseteq \operatorname{Min}(R/\text{j-rad}\, I),$$

and $|I| = \text{j-ht}\, I$.

(ii) Suppose $\text{j-dim}\, R$ is finite. Then if we set

$$|\mathfrak{p}|' := \begin{cases} \text{j-dim}\, R - \text{j-dim}\, R/\mathfrak{p} & \text{if } \mathfrak{p} \in \text{j-Spec}\, R, \\ \infty & \text{otherwise,} \end{cases}$$

$|\cdot|'$ satisfies (FC), since

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I,\ |\mathfrak{p}|' = |I|\} \subseteq \operatorname{Min}(R/\text{j-rad}\, I),$$

and $|I| = \text{j-dim}\, R - \text{j-dim}\, R/I$ in this case.

Next we state examples of values of prime ideals with (FC) defined in relation to a finitely generated R-module M.

Example 2.3. (i) If we set

$$|\mathfrak{p}|' := \begin{cases} \dim M_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Supp} M, \\ \infty & \text{otherwise,} \end{cases}$$

then $|\cdot|'$ satisfies (FC) and $|I| = \operatorname{ht}((I + \operatorname{Ann} M)/\operatorname{Ann} M)$.

(ii) If we set $|\mathfrak{p}|' := \operatorname{depth} M_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$, then $|\cdot|'$ satisfies (FC) and $|I| = \operatorname{grade}(I, M)$, where the depth of the zero module and the $\operatorname{grade}(I, M)$ of the module M with $IM = M$ is defined to be ∞.

Once we have a value of prime ideals, we can define new values related to a set of prime ideals. In fact, let U be a subset of $\operatorname{Spec} R$ and $|\cdot|'$ a value of prime ideals. We set

$$|\mathfrak{p}|'_U := \begin{cases} |\mathfrak{p}|' & \text{if } \mathfrak{p} \in U, \\ \infty & \text{otherwise.} \end{cases}$$

Example 2.4. (i) Let $|\cdot|'$ be a value of prime ideals with (FC) and $U = V(J)$, the set of all prime ideals containing J, where J is an ideal of R. Then $|\cdot|'_U$ satisfies (FC) and $|I|_U = |I + J|$ for any ideal I of R.

(ii) If U is a subset of $\operatorname{Spec} R$ which is closed under generalization (i.e., $\mathfrak{p} \in U$ and $\mathfrak{p} \supseteq \mathfrak{q} \in \operatorname{Spec} R$ imply $\mathfrak{q} \in U$), and $|\mathfrak{p}|' = \operatorname{ht} \mathfrak{p}$ (or $|\mathfrak{p}|' = \operatorname{depth} R_{\mathfrak{p}}$ resp.) for any prime ideal $\mathfrak{p}$, then $|\cdot|'_U$ satisfies (FC).

(iii) Combining (i) and (ii), we see that if U is a constructible set and $|\mathfrak{p}|' = \operatorname{ht}\mathfrak{p}$ (or $|\mathfrak{p}|' = \operatorname{depth} R_{\mathfrak{p}}$ resp.), then $|\cdot|'_U$ satisfies (FC).

(iv) Let $|\cdot|'$ be a value of prime ideals with (FC) and $k \in \mathbf{N}$. Set $U := \{\mathfrak{p} \in \operatorname{Spec} R \mid |\mathfrak{p}|' \leq k\}$. Then for any ideal I with $|I|_U < \infty$, $|I|_U = |I| \leq k$ and $|\cdot|'_U$ satisfies (FC).

3 Theorems of determinantal ideals

In this section we state our main result. First we recall the definition of Fitting invariants of a finitely generated R-module M.

Definition 3.1. Let

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

be a presentation of M. Then the ideal $I_{n-k}(A)$ is called the k-th Fitting invariant of M and denoted by $\mathcal{F}_k(M)$.

It is well known and easily verified that the Fitting invariants are independent of the presentation. It is also verified that

$$\mathcal{F}_0(M) = R \Longleftrightarrow M = 0,$$

and

$$\mu(M_{\mathfrak{p}}) \leq k \Longleftrightarrow \mathcal{F}_k(M) \not\subseteq \mathfrak{p}$$

for any $\mathfrak{p} \in \operatorname{Spec} R$ and any $k \in \mathbf{N}$.

Now we state

Theorem 3.2. *Let M be a finitely generated R-module, $|\cdot|'$ a value of prime ideals with (FC), m, n, t, $l \in \mathbf{N}$ and $x_1, \ldots, x_n \in M$. Suppose*

$$|\mathcal{F}_{m+i}(M)| > i \qquad \text{for } 0 \leq i \leq l,$$

and

$$|\mathcal{F}_{m-t}(M/(x_1, \ldots, x_n))| > l.$$

Then there are $x'_1, \ldots, x'_n \in M$ such that

$$x'_i = x_i + (\text{a linear combination of } x_{i+1}, \ldots, x_n)$$

for $i = 1, \ldots, n$ and

$$|\mathcal{F}_{m-k}(M/(x'_1, \ldots, x'_s))| > \min(s-k, l) \qquad \text{for } 0 \leq s \leq n \text{ and } -l \leq k \leq t.$$

Proof. See [MY].

As a corollary we obtain the following

Theorem 3.3. *Let $|\cdot|'$ be a value of prime ideals with (FC), A an $m \times n$ matrix with entries in R, t an integer with $0 \leq t \leq \min(m,n)$ and $l \in \mathbf{N} \cup \{\infty\}$. Suppose*

$$|I_t(A)| = l.$$

Then there is a lower triangular unipotent matrix V such that

$$|I_t(A'_{\leq t+i-1})| \geq i \qquad \text{for } 1 \leq i \leq l,$$

where $A' = AV$.

Proof. We set $M := R^m$ and, violating the convention written in introduction, we denote elements of R^m by column vectors. Let us denote the i-th column vector of A by x_i. Then the result follows by applying Theorem 3.2. See [MY] for the details. ∎

4 Application I. Serre's theorem and Bourbaki's theorem

In this section we apply Theorem 3.3 to show some results that generalize Serre's theorem and Bourbaki's theorem. First we state it in the most general form.

Theorem 4.1. *Let $|\cdot|'$ be a value of prime ideals with (FC), U a subset of* $\operatorname{Spec} R$, *M a finitely generated R-module and k, r non-negative integers. Suppose*

(i) $\mathfrak{p} \in U \Longrightarrow |\mathfrak{p}| \leq k$.

(ii) $\mathfrak{p} \notin U \Longrightarrow |\mathfrak{p}|' > k$.

(iii) $\operatorname{Ass} R \subseteq U$.

(iv) $M_\mathfrak{p} \simeq R_\mathfrak{p}^r$ *for any* $\mathfrak{p} \in U$.

(v) $r \geq k$.

Then there exists a short exact sequence

$$0 \longrightarrow R^{r-k} \longrightarrow M \longrightarrow C \longrightarrow 0$$

such that

$$C_\mathfrak{p} \simeq R_\mathfrak{p}^k \qquad \text{for any } \mathfrak{p} \in U.$$

Proof. Take a presentation

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

of M. Then we see by (iv) that

$$I_{n-r}(A) \not\subseteq \mathfrak{p}, \quad I_{n-r+1}(A)_{\mathfrak{p}} = (0)$$

for any $\mathfrak{p} \in U$. So it follows form (ii) that

$$|I_{n-r}(A)| > k.$$

Therefore, by changing the basis of R^n if necessary, we may assume that

$$|I_{n-r}(A_{\leq n-r+k})| > k$$

by Theorem 3.3.

Define R-modules C and K by the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & & K & & \\
 & & \downarrow & & \downarrow & & \\
 & & R^{r-k} & = & R^{r-k} & & \\
 & & \downarrow & & \downarrow & & \\
R^m & \xrightarrow{A} & R^n & \longrightarrow & M & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
R^m & \xrightarrow{A_{\leq n-r+k}} & R^{n-r+k} & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where the horizontal and the vertical lines are exact.

Then it is verified that $K = 0$ and $C_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^k$ for any $\mathfrak{p} \in U$. See [MY] for the details. ∎

As a corollary, we state a result that generalize Serre's theorem.

Corollary 4.2. *Let M be a finitely generated projective R-module of constant rank r. If there exists a value $|\cdot|'$ of prime ideals such that*

$$d(R) := \sup\{|\mathfrak{m}| \mid \mathfrak{m} \in \mathrm{Max}(R)\} < r,$$

then M has a free direct summand.

Remark 4.3. (i) If we put $|\mathfrak{p}|' = \mathrm{ht}\,\mathfrak{p}$, then $d(R) = \dim R$ and Corollary 4.2 becomes Serre's theorem [Ser].

(ii) If we put

$$|\mathfrak{p}|' := \begin{cases} \text{j-ht}\,\mathfrak{p} & \text{if } \mathfrak{p} \in \text{j-Spec}\, R, \\ \infty & \text{otherwise,} \end{cases}$$

then $d(R) = \text{j-dim}\, R$ and Corollary 4.2 becomes Eisenbud-Evans theorem [EE].

(iii) If we put $|\mathfrak{p}|' = \operatorname{depth} R_{\mathfrak{p}}$, we get another result of this type.

Proof of Corollary 4.2. Set $U = \operatorname{Spec} R$ and $k = d(R)$. Then all the assumptions of Theorem 4.1 are satisfied. So we have a short exact sequence

$$0 \longrightarrow R^{r-k} \longrightarrow M \longrightarrow C \longrightarrow 0$$

such that

$$C_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^{k} \qquad \text{for any } \mathfrak{p} \in \operatorname{Spec} R.$$

Then C is projective since it is locally free, and therefore R^{r-k} is a direct summand of M. Note $r > k$. ∎

We also deduce a result which is a generalization of Bourbaki's theorem [Bou].

Corollary 4.4. *Let M be a finitely generated R-module and r a positive integer. Suppose*

(i) $\operatorname{depth} R_{\mathfrak{p}} \geq 2 \Longrightarrow \operatorname{depth} M_{\mathfrak{p}} \geq 1$.

(ii) $\operatorname{depth} R_{\mathfrak{p}} \leq 1 \Longrightarrow M_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^{r}$.

Then there exists a short exact sequence

$$0 \longrightarrow R^{r-1} \longrightarrow M \longrightarrow I \longrightarrow 0$$

such that I is an ideal of R and $I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ if $\operatorname{depth} R_{\mathfrak{p}} \leq 1$.

Remark 4.5. If the assumptions of Corollary 4.4 are satisfied for $r = 0$, then $M = 0$. In fact, if $M \neq 0$, take $\mathfrak{p} \in \operatorname{Ass} M$. Then by (i), $\operatorname{depth} R_{\mathfrak{p}} \leq 1$ since $\operatorname{depth} M_{\mathfrak{p}} = 0$. Therefore $M_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^{r} = 0$. Contradiction.

Proof of Corollary 4.4. We set

$$\begin{aligned} &|\mathfrak{p}|' := \operatorname{depth} R_{\mathfrak{p}} \qquad \text{for any } \mathfrak{p} \in \operatorname{Spec} R, \\ &U := \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{depth} R_{\mathfrak{p}} \leq 1\} \text{ and} \\ &k := 1. \end{aligned}$$

Then all the assumptions of Theorem 4.1 are satisfied and we see that there exists a short exact sequence

$$0 \longrightarrow R^{r-1} \longrightarrow M \longrightarrow C \longrightarrow 0$$

such that

$$C_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \qquad \text{if } \operatorname{depth} R_{\mathfrak{p}} \leq 1.$$

It is verified that C is isomorphic to an ideal of R. See [MY] for the details. ∎

We deduce Bourbaki's theorem as a corollary to Corollary 4.4.

Corollary 4.6 (Bourbaki's theorem [Bou]). *Let R be a normal domain and M a finitely generated torsion-free R-module. Then there is a free submodule F of M such that M/F is isomorphic to an ideal of R.*

Proof. We may assume that $M \neq 0$. Set $r = \operatorname{rank} M$. Since M is torsion-free, we see that $\operatorname{Ass} M = \{(0)\}$. So the condition (i) of Corollary 4.4 is satisfied. And if $\operatorname{depth} R_{\mathfrak{p}} \leq 1$, then $M_{\mathfrak{p}}$ is a torsion-free module of rank r over a PID $R_{\mathfrak{p}}$, we see that $M_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^r$. The result follows from Corollary 4.4. ∎

We also state another corollary of Corollary 4.4.

Corollary 4.7. *Let M be a finitely generated R-module of finite projective dimension. Suppose that M is a first syzygy and* $\operatorname{Spec} R$ *is connected. Then there is a free submodule F of M such that M/F is isomorphic to an ideal of R and $(M/F)_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Spec} R$ with* $\operatorname{depth} R_{\mathfrak{p}} \leq 1$.

Proof. Let

$$0 \longrightarrow P_h \longrightarrow P_{h-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{4.8}$$

be a finite projective resolution of M. Then, since $\operatorname{Spec} R$ is connected, $\operatorname{rank}(P_i)_{\mathfrak{p}}$ is independent of $\mathfrak{p} \in \operatorname{Spec} R$.

Setting

$$r = \sum_{i=0}^{h} (-1)^i \operatorname{rank}(P_i)_{\mathfrak{p}},$$

we shall show that the assumptions of Corollary 4.4 are satisfied.

First, since M is a first syzygy, $\operatorname{Ass} M \subseteq \operatorname{Ass} R$ and therefore the condition (i) of Corollary 4.4 is satisfied. And if $\operatorname{depth} R_{\mathfrak{p}} = 1$, then $\mathfrak{p} \notin \operatorname{Ass} M$ and $\operatorname{pd} M_{\mathfrak{p}} < \infty$. It follows from Auslander-Buchsbaum formula [AB] that $M_{\mathfrak{p}}$ is free and we see that $\operatorname{rank} M_{\mathfrak{p}} = r$ by the exact sequence (4.8). We also have that $M_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^r$ in case $\operatorname{depth} R_{\mathfrak{p}} = 0$ by the same way. Therefore the assumptions of Corollary 4.4 are satisfied and we are done. ∎

5 Application II. Forster-Swan theorem

In this section we state a result of Forster-Swan type as a corollary to Theorem 3.2. First we state it in a general form.

Theorem 5.1. *Let M be a finitely generated R-module and $|\cdot|'$ a value of prime ideals with (FC) such that*

$$d(R) = \sup\{|\mathfrak{m}| \mid \mathfrak{m} \in \operatorname{Max}(R)\} < \infty.$$

If we put

$$t := \sup\{\mu(M_{\mathfrak{p}}) + d(R) - |\mathfrak{p}|' \mid \mathfrak{p} \in \operatorname{Supp} M\},$$

M can be generated by t elements.

By setting

$$|\mathfrak{p}|' := \begin{cases} \text{j-dim}\, R - \text{j-dim}\, R/\mathfrak{p} & \text{if } \mathfrak{p} \in \text{j-Spec}\, R, \\ \infty & \text{otherwise}, \end{cases}$$

we have the following

Corollary 5.2 (Forster-Swan theorem [For] and [Swa]). *Let R be a noetherian ring with* j-dim $R < \infty$ *and M a finitely generated R-module. Then M can be generated by*

$$t = \sup\{\mu(M_\mathfrak{p}) + \text{j-dim}\, R/\mathfrak{p} \mid \mathfrak{p} \in (\text{j-Spec}\, R \cap \operatorname{Supp} M)\}$$

elements.

Proof of Theorem 5.1. If $|\cdot|'$ is a value of prime ideals with (FC), then it is easily verified that the restriction of $|\cdot|'$ to $\operatorname{Spec} R/I$ also satisfy (FC) for any ideal I of R. So by considering $R/\operatorname{Ann} M$ instead of R, we may assume that $\operatorname{Supp} M = \operatorname{Spec} R$.

Take a system of generators $x_1, \ldots, x_n$ of M and put $l = d(R)$ and $m = t - l$. Then for any non-negative integer i and any prime ideal $\mathfrak{p}$ with $|\mathfrak{p}|' \leq i$, we have

$$\mu(M_\mathfrak{p}) \leq t - d(R) + |\mathfrak{p}|' \leq m + i.$$

Therefore $\mathcal{F}_{m+i}(M) \not\subseteq \mathfrak{p}$. Thus it follows that

$$|\mathcal{F}_{m+i}(M)| > i.$$

On the other hand, since $\mathcal{F}_0(M/(x_1, \ldots, x_n)) = R$, we have

$$|\mathcal{F}_{m-m}(M/(x_1, \ldots, x_n))| > l.$$

Therefore, by Theorem 3.2, there are $x'_1, \ldots, x'_n \in M$ such that

$$|\mathcal{F}_{m-k}(M/(x'_1, \ldots, x'_s))| > \min(s - k, l) \qquad \text{for } 0 \leq s \leq n \text{ and } -l \leq k \leq m.$$

In particular,

$$|\mathcal{F}_0(M/(x'_1, \ldots, x'_{m+l}))| > l.$$

(If $n \leq t = m + l$, there is nothing to prove. So we are assuming that $n > m + l$.) Since $l = d(R)$, it follows that

$$\mathcal{F}_0(M/(x'_1, \ldots, x'_t)) = R,$$

that is, $(x'_1, \ldots, x'_t) = M$. ∎

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