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Preface

This is the Proceedings of the 21-th Symposium on Commutative Algebra in Japan (held on November 23-26, 1999), which was financially supported by Professor Masanori Ishida of Tohoku University (the Grant-in-Aid for Scientific Researches in Japan). This time we had many guest speakers from abroad, including Professor Steven Dale Cutkosky and Professor Eero Hyry. I would like to express my hearty thanks for their excellent lectures. We also had many participants from Korea, our neighbors, and I wish this conference has provided a good opportunity to grow up a further friendship between Korea and Japan.

January 22, 2000

Shiro Goto

序

これは第21回可換環論シンポジウムの報告集です。このシンポジウムは，東北大学の石田正典教授の科研費（基盤研究(A)(1)）のサポートの下に，1999年11月23日から26日にかけて東京都多摩センターで行われました。今回は，Steven Dale Cutkosky 教授 (University of Missouri, USA) や Eero Hyry 教授 (National Defence College, Finland) の他に，韓国からも多数の方が参加されました。彼らの熱意ある優れた講演に謝意を表します。また，このシンポジウムが韓国と日本の間の更なる友情を育む上で，幾許かの寄与があれば誠に幸いに思います。

2000年1月22日

明治大学　後藤四郎

Program

Tuesday, Nov. 23

16:00 ∼ 16:05	opening
16:05 ∼ 16:55	Shiro Goto (Meiji Univ.) Good ideals in Gorenstein local rings
17:05 ∼ 17:35	Mee-Kyoung Kim (Sungkyunkwan Univ.) Equimultiple good ideals
19:00 ∼ 20:00	Eero Hyry (National Defence College) Cohen-Macaulay multi-Rees algebras
20:15 ∼ 20:45	Satoshi Haraikawa (Meiji Univ.) Good ideals in idealizations

Wednesday Nov. 24

9:00 ∼ 9:40	Kikumichi Yamagishi (Himeji Dokkyo Univ.) Buchsbaumness of the Rees algebras of $\mathfrak{m}$-primary ideals whose reduction numbers are at most one
9:55 ∼ 10:45	Koji Nishida (Chiba Univ.) On filtrations having small analytic deviation
11:00 ∼ 12:00	Steven Dale Cutkosky (Univ. of Missouri) Monomialization of morphisms
13:20 ∼ 14:00	Masanori Ishida (Tohoku Univ.) Toric varieties and schemes based on semigroups I
14:15 ∼ 15:05	Kei-ichi Watanabe (Nihon Univ.) · Ken-ichi Yoshida (Nagoya Univ.) Hilbert-Kunz multiplicity, McKay correspondence and good ideals in 2-dimensional rational singularities
15:20 ∼ 16:00	Kisuk Lee (Sookmyung Women's Univ.) A note on column-invariants of local rings
16:15 ∼ 17:05	Takesi Kawasaki (Tokyo Metropolitan Univ.) Arithmetic Macaulayfication of local rings
17:20 ∼ 17:50	Shin-ichiro Iai (Meiji Univ.) Three theorems on Gorensteinness in associated graded rings

19:00 ∼ 20:00	Eero Hyry (National Defence College) The coefficient ideal and the canonical module of a Rees algebra
20:15 ∼ 20:45	Hidefumi Ohsugi (Osaka Univ.) Compressed polytopes

Thursday Nov. 25

9:00 ∼ 10:40	Mutsumi Amasaki (Hiroshima Univ.) Existence of homogeneous prime ideals fitting into long Bourbaki sequences
9:55 ∼ 10:45	Kazuhiko Kurano (Tokyo Metropolitan Univ.) Dutta multiplicities and Test modules
11:00 ∼ 12:00	Steven Dale Cutkosky (Univ. of Missouri) Asymptotic regularity
13:20 ∼ 14:00	Masanori Ishida (Tohoku Univ.) Toric varieties and schemes based on semigroups II
14:15 ∼ 15:05	Mitsuyasu Hashimoto (Nagoya Univ.) Equivariant twisted inverse pseudo-functors without equivariant compactification
15:20 ∼ 16:00	Guangfeng Jiang (Tokyo Metropolitan Univ.) Conormal modules and primitive ideals
16:15 ∼ 17:05	Kohji Yanagawa (Osaka Univ.) Sheaves on posets and squarefree modules over normal semigroup rings
17:20 ∼ 17:50	Tokuji Araya (Kyoto Univ.) Mutation of exceptional sequence on 1-dimensional Gorenstein ring
18:30 ∼ 20:30	Banquet

Friday Nov. 26

9:30 ∼ 10:10	Tadahito Harima (Shikoku Univ.) A note on graded Betti numbers of Gorenstein rings
10:25 ∼ 11:05	Yong-su Shin (Sungshin Women's Univ.) Extremal Minimal Resolutions and Decompositions of a Finite Set of Points in $\mathbb{P}^n$
11:20 ∼ 12:00	Mitsuhiro Miyazaki (Kyoto Univ. of education) Properties of the discrete counterpart of an algebra with straightening laws

Contents

GOOD IDEALS IN GORENSTEIN LOCAL RINGS

SHIRO GOTO

1. INTRODUCTION.

This is a joint work [GIW] with Shin-ichiro Iai and Kei-ichi Watanabe. And what I want to do in my talk is to study certain $\mathfrak{m}$-primary ideals in Gorenstein local rings. So, in what follows, let A denote a Gorenstein local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let I be an $\mathfrak{m}$-primary ideal in A. Then

Definition (1.1). We say that I is a good ideal in A, if I contains a parameter ideal Q in A as a reduction and the associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is a Gorenstein ring with $\mathrm{a}(G(I)) = 1 - d$. Here $\mathrm{a}(G(I))$ denotes the a-invariant of $G(I)$.

In this definition the latter condition is rather strong. Actually, good ideals in our sense are good ones next to the parameter ideals in A and they are characterized in the following way.

Proposition (1.2). *Let I be an $\mathfrak{m}$-primary ideal in A and assume that I contains a parameter ideal Q in A as a reduction. Then the following conditions are equivalent.*

(1) *I is a good ideal in A.*
(2) *$I^2 = QI$ and $I = Q : I$.*
(3) *$I^2 = QI$ and $\ell_A(A/I) = \frac{1}{2}\ell_A(A/Q)$.*
(4) *$I^3 \subseteq Q^2$ and $I = Q : I$.*
(5) *The extended Rees algebra $R' = \mathrm{R}'(I)$ of I is a Gorenstein ring with $\mathrm{K}_{R'} \cong R'(2-d)$.*

If $d \geq 1$, you may add the following.

(6) *$I^n = Q^n : I$ for all $n \in \mathbb{Z}$.*

When this is the case, $\mathrm{K}_{A/I} \cong I/Q$, $\mathrm{r}(A/I) = \mu_A(I) - d \geq 1$, and $\mathrm{e}_I^0(A) = 2\ell_A(A/I)$. Here $\ell_A(*)$ and $\mu_A(*)$ denote respectively the length and the number of generators.

More or less, the conditions in Proposition (1.2) directly follow from Definition (1.1) and it is not very difficult to check their equivalence. Here the point is as follows : as far

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as I know, the characterization (1.2) is almost all we know about general good ideals. So, in my talk I am very eager to develop further the theory of good ideals and my purpose is to answer the following questions.

Problem (1.3). (1) Determine all the good ideals in a given Gorenstein local ring.

(2) Characterize those Gorenstein local rings A for which the sets

$$\mathcal{X}_A = \{I \mid I \text{ is a good ideal in } A\}$$

of good ideals are finite.

(3) When $\mathcal{X}_A = \emptyset$?

On the other hand the notion of good ideal is directly generalized to that of equimultiple ideals. And this generalization contains somewhat unexpected results, which Mee-kyoung Kim will talk about in her lecture.

2. The case where $\dim A = 0$.

By (1.2), in the case where $\dim A = 0$, our ideal I is good in A if and only if $I = (0) : I$. We also have that if $\mathcal{X}_A \neq \emptyset$, then the length $\ell_A(A)$ of A must be even. Therefore it seems quite natural to guess the converse is also true. However this is not the case and we have the following.

Theorem (2.1). *Let k be a field. Then the following conditions are equivalent.*

1. *The field k contains a square root $\sqrt{\alpha}$ for any element $\alpha \in k$.*
2. *Let $A = \sum_{n \geq 0} A_n$ be a finite-dimensional Gorenstein graded k-algebra with $k = A_0$. Then $\mathcal{X}_A \neq \emptyset$ if and only if $2 \mid \dim_k A$.*

This theorem says that if the base field k is large enough, then $\mathcal{X}_A \neq \emptyset$ if and only if $2 \mid \ell_A(A)$ at least for finite-dimensional Gorenstein graded k-algebras $A = \sum_{n \geq 0} A_n$ with $k = A_0$. Here let me note the following, which is an immediate consequence of my proof of Theorem (2.1).

Corollary (2.2). *For each $\alpha \in k$ with no square root in k there exists at least one finite-dimensional Gorenstein graded k-algebra $A = A(\alpha)$ with $k = A_0$ such that $2 \mid \dim_k A$ but $\mathcal{X}_A = \emptyset$.*

Let me give an example to illustrate the theorem.

Example (2.3). Let $k[X, Y]$ be the polynomial ring in two variables over a field k and let $A = k[X, Y]/(X^n - Y^n, XY)$ $(n \geq 2)$. Then $\sharp \mathcal{X}_A \leq 2$. More explicitly we have

1. $\mathcal{X}_A = \emptyset$ if and only if n is even and $\alpha^2 \neq -1$ for any $\alpha \in k$.
2. $\sharp \mathcal{X}_A = 1$ if and only if n is odd, or n is even and $\mathrm{ch}\, k = 2$.

If $k = \mathbb{R}$ (the field of real numbers), the situation is very bad. In fact, we have

Example (2.4). Let $n \geq 1$ be an integer and let $R = k[X_i, Y_i \mid 1 \leq i \leq n]$ be the polynomial ring in $2n$ variables over a field k. Let $\mathfrak{a} = (X_i^2 - Y_i^2, X_iY_i \mid 1 \leq i \leq n)$ in R and put $A = R/\mathfrak{a}$. Then we have

(1) $\mathrm{a}(A) = 2n$ and $\dim_k A = 4^n$.

(2) $\mathcal{X}_A = \emptyset$ if $k = \mathbb{R}$.

One of the simplest way to construct Artinian Gorenstein local rings is the idealization. Let $(R, \mathfrak{n})$ be an Artinian local ring and let $E = \mathrm{E}_R(R/\mathfrak{n})$ be the injective envelope of $R/\mathfrak{n}$. Let $A = R \ltimes E$ denote the idealization. Hence $A = R \oplus E$ as R-modules and the multiplication in A is given by $(a, x) \cdot (b, y) = (ab, ay + bx)$ and the ring A is an Artinian Gorenstein local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times E$ ([R]). In this situation we have a certain structure theorem of good ideals in $A = R \ltimes E$, which Satoshi Haraikawa will talk about in his lecture.

3. The case where $\dim A = 1$.

If $\dim A = 1$, there is a beautiful correspondence theorem between the set $\mathcal{X}_A$ and the set $\mathcal{Y}_A$ of certain overrings of A. To explicitly state the result, let $K = \mathrm{Q}(A)$ be the total quotient ring of A. We denote by $\mathcal{Y}_A$ the set of Gorenstein A-subalgebras C of K such that $C \supsetneq A$ but the A-module C is finitely generated.

Let me begin with the following.

Lemma (3.1) (cf. [L1], Lemma 1.11). *Suppose that $I^2 = aI$ with $a \in I$ and let $C = \{\frac{i}{a} \mid i \in I\}$ in K. Then*

(1) *C is an A-subalgebra of K and $I = aC$. Hence C is a finitely generated A-module.*

(2) *$C = A$ if and only if $I = aA$.*

(3) *$A :_K C = aA : I$.*

(4) *C is a Gorenstein ring if $I = aA : I$.*

(5) *$C = I :_K I$.*

Now let me choose $I \in \mathcal{X}_A$. Then since $I^2 = aI$ and $I = aA : I$ for some $a \in I$, by (3.1) we get the Gorenstein A-subalgebra $C = \{\frac{i}{a} \mid i \in I\}$ of K. As $aA \neq I$, $C \neq A$ whence $C \in \mathcal{Y}_A$. Because $C = I :_K I$ by (3.1) (5), the map $\varphi : \mathcal{X}_A \to \mathcal{Y}_A$, $I \mapsto C$ is well-defined, that is independent of the choice of the element $a \in I$ and it is an injection, since $A :_K C = I$ ((3.1)(3)). We furthermore have the following.

Theorem (3.2). *There is a one-to-one correspondence between $\mathcal{X}_A$ and $\mathcal{Y}_A$, which sends each $I \in \mathcal{X}_A$ to $\mathrm{End}_A I = I :_K I$ and takes back each $C \in \mathcal{Y}_A$ to $\mathrm{Hom}_A(C, A) = A :_K C$. The correspondence reverses the inclusion and one has the equality*

$$\ell_A(C/A) = \ell_A(A/A :_K C)$$

for all $C \in \mathcal{Y}_A$.

It is often much easier to compute the set $\mathcal{Y}_A$ than the set $\mathcal{X}_A$. Let me give one example.

Theorem (3.3). *Let* A *be a one-dimensional reduced complete local ring with* $\mathrm{e}(A) = 2$ *and let* $B = \overline{A}$ *denote the normalization of* A*. Then*

(1) *Every intermediate ring* $A \subseteq C \subseteq B$ *is a Gorenstein ring.*
(2) $\sharp\mathcal{X}_A = \ell_A(B/A)$.
(3) *The set* $\mathcal{X}_A$ *is totally ordered with respect to inclusion.*

Hence there is a unique chain $A = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_n = B$ *of rings between* A *and* B *where* $n = \ell_A(B/A)$.

Typical examples of one-dimensional reduced complete local rings A with $\mathrm{e}(A) = 2$ are as follows.

Example (3.4). Let k be a field and $n \geq 1$ an integer. For a local ring R let $\kappa(R)$ denote the residue class field of R.

(I) ($\sharp\mathrm{Ass}A = 2$) Let $A = k[[X,Y]]/(X) \cap (X + Y^n)$, where $k[[X,Y]]$ denotes the formal power series ring in two variables X, Y over k.
(II) (A is an integral domain with $[\kappa(B) : \kappa(A)] = 1$) Let $A = k[[t^2, t^{2n+1}]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal power series ring in one variable over k. Then $B = k[[t]]$.
(III) (A is an integral domain with $[\kappa(B) : \kappa(A)] = 2$) Let K/k be an extension of fields with $[K : k] = 2$ and choose $\theta \in K$ so that $K = k + k\theta$. Let $A = k[[t, \theta t^n]] \subseteq K[[t]]$, where $K[[t]]$ denotes the formal power series ring in one variable over K. Then $B = K[[t]]$.

For these rings A we always have $\sharp\mathcal{X}_A = n$.

Unless $\widehat{A}$ is reduced, the set $\mathcal{X}_A$ is no more finite, even though A has multiplicity 2.

Example (3.5). Let k be a field and $A = k[[X,Y]]/(Y^2)$. Let x, y denote respectively the reduction of $X, Y \bmod (Y^2)$. Then

(1) $\mathcal{X}_A = \{(x^i, y) \mid i \geq 1\}$. Hence the set $\mathcal{X}_A$ is infinite, which is a totally ordered set with respect to inclusion.

(2) Every module-finite extension of A in K is a Gorenstein ring.

Hence between A and B there is a unique chain

$$A = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_i = R\left[\frac{y}{x^i}\right] \subsetneq \cdots \subsetneq B$$

of rings, consisting of module-finite extensions of A, where $R = k[[x]]$ in A.

One of the direct consequences of Theorem (3.2) shows that good ideals I and J in A must coincide, once $I \cong J$ as A-modules. Hence the set $\mathcal{X}_A$ is finite, if A has finite CM-representation type. However, in general the number of good ideals depends on the base field. In fact, let k be a field and $B = k[[t]]$ the formal power series ring over k. Then we have the following.

Example (3.6). Let $A_1 = k[[t^3, t^4]]$ and $A_2 = k[[t^4, t^5, t^6]]$. Then

(1) $\mathcal{X}_{A_1} = \{(t^6, t^7, t^8), (t^4, t^6)\}$ and $\mathcal{Y}_{A_1} = \{B, k[[t^2, t^3]]\}$.

(2) $\mathcal{X}_{A_2} = \{(t^8, t^9, t^{10}, t^{11}), (t^6, t^8, t^9)\} \cup \{(t^4 - \lambda t^5, t^6 - \lambda^2 t^8) \mid \lambda \in k\}$ and $\mathcal{Y}_{A_2} = \{B, k[[t^2, t^3]]\} \cup \{A_2[t^2 + \lambda t^3] \mid \lambda \in k\}$. Hence $\sharp \mathcal{X}_{A_2} = \sharp k + 2$ and so the set $\mathcal{X}_{A_2}$ is infinite if so is k.

This example (3.6) also shows the inequality $\ell_A(\overline{A}/A) > \sharp \mathcal{X}_A$ may occur if $\mathrm{e}(A) \geq 3$.

Now you may wonder when $\mathcal{X}_A = \emptyset$. As for the question I have partial answers only. Let me state them.

Proposition (3.7). *Suppose that $A/\mathfrak{m}$ is infinite. Then the following conditions are equivalent.*

(1) $\mathcal{X}_A = \emptyset$.

(2) *Every $\mathfrak{m}$-primary ideal I in A with the Gorenstein associated graded ring $\mathrm{G}(I)$ is principal.*

(3) *Every $\mathfrak{m}$-primary ideal I in A for which Proj $\mathrm{R}(I)$ is a Gorenstein scheme is principal.*

Proposition (3.8). *The following conditions are equivalent.*

(1) *A is a discrete valuation ring.*

(2) *The completion $\widehat{A}$ of A is reduced and $\mathcal{X}_A = \emptyset$.*

I suspect that A is a discrete valuation ring if $\mathcal{X}_A = \emptyset$.

4. The case where $\dim A = 2$.

If $\dim A = 2$, we have the following characterization of good ideals in A.

Theorem (4.1). *Suppose that* $\dim A = 2$*. Let I be an $\mathfrak{m}$-primary ideal in A and assume that I contains a parameter ideal Q as a reduction. Then the following nine conditions are equivalent to each other.*

(1) $I \in \mathcal{X}_A$.

(2) $I^\ell \in \mathcal{X}_A$ *for all* $\ell \geq 1$.

(3) $I^2 = QI$ *and* $I^\ell \in \mathcal{X}_A$ *for some* $\ell \geq 1$.

(4) *The Hilbert function $\ell_A(A/I^{n+1})$ is a polynomial in n for all $n \geq 0$ and $I^\ell \in \mathcal{X}_A$ for some $\ell \geq 1$.*

(5) $\mathrm{G}(I)$ *is a Cohen-Macaulay ring and* $I^\ell \in \mathcal{X}_A$ *for some* $\ell \geq 1$.
(6) $I^n = Q^n : I$ *for all* $n \in \mathbb{Z}$.
(7) $\ell_A(A/I^n) = n^2 \ell_A(A/I)$ *for all* $n \geq 1$.
(8) *The Rees algebra* $R = \mathrm{R}(I)$ *of* I *is a Cohen-Macaulay ring and* $\mathrm{K}_R \cong R_+$ *as graded* R*-modules.*
(9) *The extended Rees algebra* $R' = \mathrm{R}'(I)$ *of* I *is a Gorenstein ring and* $\mathrm{K}_{R'} \cong R'$ *as graded* R'*-modules.*

When this is the case, the equality

$$\mu_A(I^n) = n\mu_A(I) - n + 1$$

holds true for all $n \geq 1$.

Here let me note that conditions (1) and (2) in Theorem (4.1) are not equivalent to each other, unless $\dim A = 2$. We actually have in the case where $\dim A = 1$ that $\ell = 1$ if $\ell \geq 1$ and $I^\ell \in \mathcal{X}_A$. And even though $I^\ell \in \mathcal{X}_A$ for all $\ell \gg 0$, the ideal I is not necessarily a good ideal in A. Let me give one example.

Example (4.2). Let $k[[X,Y,Z]]$ be the formal power series ring over a field k and let $A = k[[X,Y,Z]]/(Z^2 - XY)$. Let x, y, and z denote respectively the reduction of X, Y and Z mod $(Z^2 - XY)$. We put $I = (x^2, y^2, xz, yz)$. Then $I^\ell \in \mathcal{X}_A$ for all $\ell \geq 2$ but $I \notin \mathcal{X}_A$.

However, if $I^\ell \in \mathcal{X}_A$ for some $\ell \geq 1$, the powers I^n of I are good ideals in A for all $n \geq N+1$, where N denotes the least integer $N \geq 0$ such that the Hilbert function $\ell_A(A/I^{n+1})$ of I is a polynomial in n for all $n \geq N$. Namely

Theorem (4.3). *Let I be an $\mathfrak{m}$-primary ideal in A which contains a parameter ideal Q in A as a reduction. Let $N \geq 0$ be an integer and suppose that the equality*

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+2}{2} - \mathrm{e}_I^1(A)\binom{n+1}{1} + \mathrm{e}_I^2(A)$$

holds true for all integers $n \geq N$. Assume that $I^\ell \in \mathcal{X}_A$ for some $\ell \geq 1$. Then the following assertions hold true.

(1) $\mathrm{e}_I^2(A) = 0$ *and* $\mathrm{e}_I^1(A) = \dfrac{1}{2}\mathrm{e}_I^0(A)$.
(2) $\ell_A\left(A/(I^k)^{n+1}\right) = (n+1)^2 \ell_A(A/I^k)$ *for all integers* $k \geq N+1$ *and* $n \geq 0$.
(3) $I^k \in \mathcal{X}_A$ *if* $k \geq N+1$.
(4) $I \in \mathcal{X}_A$ *if and only if* $\mathrm{H}_{\mathfrak{M}}^1(G) = (0)$.

Here $G = \mathrm{G}(I)$ *and* $\mathfrak{M} = \mathfrak{m}G + G_+$.

Recall that a two-dimensional Noetherian local ring R is rational if R is normal and there exists a desingularization $X \to \operatorname{Spec} R$ with $\mathrm{H}^1(X, \mathcal{O}_X) = (0)$ ([L2]). In the case where our Gorenstein local ring A is rational, the theory of good ideals are closely related to that of adjoints $\widetilde{I}$ of ideals I in the sense of J. Lipman ([L3]). He proved that the equality $\widetilde{I} = Q : I$ holds true for any $\mathfrak{m}$-primary integrally closed ideal I in A and for any minimal reduction Q of I, if A is a two-dimensional Gorenstein rational local ring. Therefore for an $\mathfrak{m}$-primary ideal I in A we have that $I \in \mathcal{X}_A$ if and only if $I = \widetilde{I}$. Added to it, thanks to Kato's Rieman-Roch theorem [K], we have the following characterization of good ideals. Let $f : X \to \operatorname{Spec} A$ be a desingularization and let I be an $\mathfrak{m}$-primary ideal in A. Then we say that I is represented on X if $I\mathcal{O}_X$ is invertible. With this notation my characterization is stated as follows.

Theorem (4.4). *Let A be a two-dimensional Gorenstein excellent rational local ring. Let I be an $\mathfrak{m}$-primary ideal in A which contains a parameter ideal Q in A as a reduction. Then the following conditions are equivalent.*

(1) $I \in \mathcal{X}_A$.
(2) *I is integrally closed, which is represented on the minimal resolution of* $\operatorname{Spec} A$.
(3) $I = \widetilde{I}$.

Here let me explore the simplest case. In general it is possible by the same manner to explicitly describe the good ideals for any two-dimensional Gorenstein rational excellent local rings, which Kei-ichi Watanabe will also talk about in his lecture.

Example (4.5). Let $k[[X, Y, Z]]$ be the formal power series ring over a field k. Let $A = k[[X,Y,Z]]/(Z^2 - XY)$. Then the minimal resolution of $\operatorname{Spec} A$ is given by $\operatorname{Proj} \mathrm{R}(\mathfrak{m})$ and we have $\mathcal{X}_A = \{\mathfrak{m}^\ell \mid \ell \geq 1\}$.

I close this section with the following.

Remark (4.6). (1) According to an argument of Huneke and Swanson [HS], in the case where $\dim A = 2$ we have $\operatorname{pd}_A A/I = \infty$ for all $I \in \mathcal{X}_A$. Hence $\mathcal{X}_A = \emptyset$ if A is a regular local ring with $\dim A \leq 2$. I don't know whether the converse is also true.

(2) If A is a rational local ring, the product of two good ideals is again a good ideal in A. This is no more true, unless A is a rational singularity. For example, let $A = k[[X,Y,Z]]/(Z^2 - X^nY^n)$ $(n \geq 2)$. Then we have the following two ideals $\mathfrak{m}$, $I = (x^n, y^n, z)$ are good in A but $\mathfrak{m}I \notin \mathcal{X}_A$. So, the condition that the product of any two good ideals is again a good ideal might characterize rational singularities.

5. The case where $\dim A \geq 3$.

If $d = \dim A \geq 3$, we have the following.

Theorem (5.1). *The set $\mathcal{X}_A$ is necessarily infinite if* $\dim A \geq 3$.

The proof is quite easy. We just look at the ideal $I = (X_1, X_2, X_3)^2 + (X_4, \cdots, X_d)$, where $X_1, X_2, \cdots, X_d$ is a system of parameters of A. Then the ideal I is always a good ideal in A, so that we have $\sharp\mathcal{X}_A = \infty$. However, if you are interested in the question whether the set

$$\mathcal{X}_Q = \{I \in \mathcal{X}_A \mid I \text{ contains } Q \text{ as a reduction}\}$$

is empty or not for a given parameter ideal Q in A, the problem turns more complicated. To conclude my talk let me add a few results about the question. Let $R = k[X_1, X_2, X_3]$ be the polynomial ring in three variables over a field k. Let $a_1, a_2, a_3 \geq 1$ be integers and $Q = (X_1^{a_1}, X_2^{a_2}, X_3^{a_3})$. We denote by $\mathcal{X}_Q$, or simply by $\mathcal{X}_{(a_1,a_2,a_3)}$ the set of ideals J in R which are generated by monomials in X_1, X_2, X_3 and such that $J \supseteq Q$, $J^2 = QJ$, and $J = Q : J$. Then clearly $JR_{\mathfrak{M}} \in \mathcal{X}_{R_{\mathfrak{M}}}$ for all $J \in \mathcal{X}_{(a_1,a_2,a_3)}$, where $\mathfrak{M} = (X_1, X_2, X_3)$. Added to it, because $a_i = b_i$ for all $i = 1, 2, 3$ if $\mathcal{X}_{(a_1,a_2,a_3)} \cap \mathcal{X}_{(a_1,a_2,a_3)} \neq \emptyset$, we immediately see that $\mathcal{X}_{R_{\mathfrak{M}}}$ is infinite if $\mathcal{X}_{(a_1,a_2,a_3)} \neq \emptyset$ for infinitely many vectors (a_1, a_2, a_3) with $a_i's \geq 1$. From this viewpoint the next result might have its own interest.

Theorem (5.2). *Let $a_1, a_2, a_3 \geq 1$ be integers. Then $\mathcal{X}_{(a_1,a_2,a_3)} = \emptyset$ if and only if one of the following conditions is satisfied.*

(1) $\{a_1, a_2, a_3\} \ni 1$.

(2) $2 \nmid a_1a_2a_3$.

(3) $(a_1, a_2, a_3) = (2, 2, odd)$, $(2, odd, 2)$, *or* $(odd, 2, 2)$.

The most striking consequence of Theorem (5.2) is the following.

Corollary (5.3). *Suppose that* $\min\{a_1, a_2, a_3\} \geq 3$. *Then* $\mathcal{X}_{(a_1,a_2,a_3)} \neq \emptyset$ *if and only if* $2 \mid a_1a_2a_3$.

This gives an alternative proof of Theorem (5.1) and we get

Corollary (5.4). $\sharp\mathcal{X}_{k[[X_1,X_2,X_3]]} = \infty$.

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EQUIMULTIPLE GOOD IDEALS

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ABSTRACT. Let $I(\neq A)$ be an ideal with $ht_A(I) = s$ in a Gorenstein local ring (A,m) of $dim(A) = d \geq 1$ and assume that I contains an ideal $Q = (a_1, a_2, \cdots, a_s)$ in A as a reduction. Then we say that I is the $s^{\underline{th}}$ good ideal in A if the following conditions are satisfied : (1) $I^2 = QI$, (2) $I = Q : I$, and (3) A/I is a Cohen-Macaulay local ring. Let $\mathcal{X}_A^1$ be the set of the first good ideals in A. In this paper we will give a correspondence theorem between the set $\mathcal{X}_A^1$ and the set $\mathcal{Y}_A$ of certain overrings of A.

1. INTRODUCTION

This is a joint work with Professor Shiro Goto.

Let A be a Gorenstein local ring of dimension $d \geq 1$ and m denote the maximal ideal in A. Let $K = Q(A)$ be the total quotient ring of A. Let $I(\neq A)$ be an ideal in A and $s = ht_A(I)$. Assume that I contains an ideal $Q = (a_1, \cdots, a_s)$ in A as a reduction; hence $Q \subseteq I$ and $I^{n+1} = QI^n$ for some $n \geq 0$. We put $r_Q(I) = min\{n \geq 0 \mid I^{n+1} = QI^n\}$ and call it the reduction number of I with respect to Q. Let $H_m^i(*)$ $(i \in \mathbb{Z})$ stand for the $i^{\underline{th}}$ local cohomology functor of A with respcet to m. We denote by $\mu_A(M)$ the number of elements in a minimal system of generators for M as an A-module and $\ell_A(M)$ the lenght of M as an A-module. We define

$$R(I) = A[It] \subseteq A[t],$$

$$R'(I) = A[It,t^{-1}] \subseteq A[t,t^{-1}],$$

and

$$G(I) = R'(I)/t^{-1}R'(I)$$

(here t denotes an indeterminate over A), which we call the Rees algebra, the extended Rees algebra, and the associated graded ring of I, respectively.

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The concept of good ideal was first introduced by Goto, Iai, and Watanabe ([GIW]). In [GIW] they developed the theory of m-primary good ideals in a Gorenstein local ring. Let A be a Gorenstein local ring with the maximal ideal m and $d = dim(A)$. Let I denote an m-primary ideal in A and assume that I contains a parameter ideal $Q = (a_1, \cdots, a_d)$ of A as a reduction. We put $G = G(I)$ and $\mathfrak{M} = mG + G_+$. Let $[H^d_{\mathfrak{M}}(G)]_n$ denote the homogeneous component of degree n in the $d^{\underline{th}}$ local cohomology module $H^d_{\mathfrak{M}}(G)$ with respect to $\mathfrak{M}$. Let $a(G) = max\{n \in \mathbb{Z} \mid [H^d_{\mathfrak{M}}(G)]_n \neq (0)\}$ ([GW, (3.1.4)]). Then they defined that the m-primary ideal I is good if G is a Gorenstein ring and $a(G) = 1 - d$ and they explored that this condition is equivalent to saying that $I^2 = QI$ and $I = Q : I$ ([GIW]).

Let $I(\neq A)$ be an ideal in a Gorestein local ring A and $s = ht_A(I)$. Assume that I contains an ideal $Q = (a_1, \cdots, a_s)$ in A as a reduction. Then we say that I is good ideal in A of height s if the associated graded ring $G(I)$ is a Gorenstein ring with $a(G(I)) = 1-s$. Let $\mathcal{X}^s_A$ denote the set of good ideals in A of height s. We denote by $\mathcal{Y}_A$ the set of Gorenstein A-subalgebras C of K such that $C \supsetneq A$ but the A-module C is finitely generated. The purpose of this paper is to give the following correspondence theorem in the case where $ht_A(I) = 1$.

THEOREM 1.1. Let $I(\neq A)$ be an ideal in a Gorenstein local ring A and $1 = ht_A(I)$. Assume that I contains an element a in A such that $I^{n+1} = aI^n$ for some $n \geq 0$. Then we have :

(1) There is an one-to-one correspondence between the sets $\mathcal{X}^1_A$ and $\mathcal{Y}_A$, which sends each $I \in \mathcal{X}^1_A$ to $End_A I = (I :_A I)$ and takes back each $C \in \mathcal{Y}_A$ to $Hom_A(C, A) = A :_K C$.

(2) The correspondence reverses the inclusion.

As consequence of the theorem, we have

COROLLARY 1.2. Let $I(\neq A)$ be an ideal in A and $1 = ht_A(I)$ and assume that $I^2 = aI$ for some $a \in I$. Suppose $\mu_A(I) \geq 2$ and let $C = \{\frac{x}{a} \mid x \in I\}$. Then C is a Gorenstein ring if and only if $aA : I \in \mathcal{X}^1_A$ and A/I is a Cohen-Macaulay local ring.

The proofs of Theorem 1.1 and Corollary 1.2 will be given in Section 2. Section 3 is to discuss the question when $\mathcal{X}^1_A = \emptyset$.

Throughout this paper, A denotes a Gorenstein local ring with maximal ideal m and $dim(A) = d \geq 1$ and $K = Q(A)$ denotes the total quotient ring of A.

2. Proof of Theorem 1.1.

Let $I(\neq A)$ be an ideal in A and $1 = ht_A(I)$. Assume that I contains an element a in A such that $I^{n+1} = aI^n$ for some $n \geq 0$. Notice that the element a ia a regular element in A. To begin with we note the following. This is known by [GIW] but let us give a brief proof for completeness.

Lemma 2.1. ([GIW]) Suppose $I^2 = aI$ and let $C = \{\frac{x}{a} \mid x \in I\}$ in K. Then we have :
(1) C is an A-subalgebra of K and $I = aC$. Hence C is a finitely generated A-module.
(2) $C = A$ if and only if $I = aA$.
(3) $A :_K C = aA : I$.
(4) $C = I :_K I$.

Proof. (1) Since $I^2 = aI$ and $a \in I$, C is a subring of K containing A. We have $C \cong I$ as A-module, because $I = aC$ and the element a is a regular element in A. Hence C is a module-finite extension of A.
(2) The assertion (2) is clear, because $I = aC$ and the element a is a regular element in A.
(3) Since the element a is a unit in K, we have

$$A :_K C = aA :_K aC.$$

Moreover we have $aA :_K aC = aA : I$, because $I = aC$ and C contains the identity element.
(4) Since $I = aC$ and the element a is a unit in K, we see

$$I :_K I = aC :_K aC = C :_K C.$$

Furthermore we have $C :_K C = C$, because C contains the identity element. This completes the proof of all our assertions.

Lemma 2.2. Suppose $I^2 = aI$, $I = aA : I$ and A/I is a Cohen-Macaulay local ring. Let $C = \{\frac{x}{a} \mid x \in I\}$. Then C is a Gorenstein ring.

Proof. Suppose $I = aA : I$. Then we have an isomorphism $A :_K C \cong C$ of C-modules, by $I = aC$ and the Lemma 2.1.(3). So we have

$$C \cong A :_K C = A : C \cong Hom_A(C,A).$$

Let apply the functor $H^i_m(*)$ to the exact sequence $0 \to C \to A \to A/I \to 0$ of A-modules. And we get

$$H^i_m(C) = 0 \qquad \text{for} \quad i \neq d,$$

so that C is a Cohen-Macaulay ring. By [BH, Theorem 3.3.7], we have $C \cong K_C$ as C-modules, where K_C is a canonical module of C. Hence C is a Gorenstein ring.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. (1) Let $I \in \mathcal{X}^1_A$. Then since A/I is a Cohen-Macaulay local ring, $I^2 = aI$, and $I = aA : I$ for some $a \in I$, by Lemma 2.1 and 2.2, we have the Gorenstein A-subalgebra $C = \{\frac{i}{a} \mid i \in I\}$ of K. Since $aA \neq I$, we have $C \neq A$ whence $C \in \mathcal{Y}_A$. Because $C = I :_K I$ by Lemma 2.1.(4), the map $\varphi : \mathcal{X}^1_A \longrightarrow \mathcal{Y}_A$ given by $\varphi(I) = C$ for all $I \in \mathcal{X}^1_A$, is well-defined, that is independent of the choice of the elements $a \in I$. Suppose that $\varphi(I) = \varphi(J)$, where $I, J \in \mathcal{X}^1_A$. Then since $I = aA : I$ for some $a \in I$ and $J = bA : J$ for some $b \in J$, by Lemma 2.1.(3). We have

$$I = aA : I = A :_K \varphi(I) = A :_K \varphi(J) = bA : J = J,$$

so that the map φ is injective. Let $C \in \mathcal{Y}_A$ and put $I = A :_K C$. Then $I \neq A$ as $C \neq A$. Since C is a Gorenstein A-subalgebra of K, we see

$$\begin{aligned} C \cong K_C &\cong Ext_A^{dim(A)-dim(C)}(C,K_A) \\ &\cong Hom_A(C,A) \\ &\cong A :_K C, \end{aligned}$$

whence I is a projective C-module of rank 1. Thus $I = aC$ for some $a \in I$, whence we have

$$I^2 = (aC)(aC) = a^2C = aaC = aI.$$

Therefore I contains aA as a reduction, so that $ht_A(I) = ht_A(aA) = 1$, because a is a regular element in A. Hence $C = I/a$ and so by Lemma 2.1.(3), $I = aA : I$. Applying the functor $H^i_m(*)$ to the exact sequence $0 \to C \to A \to A/I \to 0$ of A-modules, we get

$$H^i_m(A/I) = 0 \qquad \text{for} \quad i \neq d-1,$$

since A and C are d-dimensional Gorenstein rings, whence A/I is a Cohen-Macaulay local ring. Thus $I \in \mathcal{X}^1_A$ and $\varphi(I) = I :_K I = C$ by Lemma 2.1.(4). Hence the map

$\varphi : \mathcal{X}_A^1 \longrightarrow \mathcal{Y}_A$ is surjective.

(2) Let $C, D \in \mathcal{Y}_A$.

CLAIM : $C \supseteq D$ if and only if $A :_K C \subseteq A :_K D$.

Proof of Claim. ($\Rightarrow$) This is trivial. ($\Leftarrow$) By the bijective map φ, we have

$$I :_K I = C \qquad \text{and} \qquad J :_K J = D.$$

for some $I, J \in \mathcal{X}_A^1$. Let $x \in D$. Then since $A :_K C \subseteq A :_K D$, we have $xI \subseteq xJ \subseteq J$. By the definition of $\mathcal{X}_A^1$, there exists an element $a \in I$ such that $aA : I = I$. Write $xa = b$ for some $b \in J$. Then

$$bI = xaI = axI \subseteq aJ \subseteq aA,$$

so that $b \in I$. Hence we see

$$a(xI) = (ax)I = bI \subseteq I^2 = aI.$$

Since a is a non-zero-divisor in A, we have $xI \subseteq I$. Thus $x \in I :_K I$. This completes the proof of Theorem 1.1.

COROLLARY 2.3. Let I, $J \in \mathcal{X}_A^1$. Assume that $I \cong J$ as A-modules. Then $I = J$.

Proof. Since $I \cong J$ as A-modules, there exists an element $\alpha \in A$ which is a unit in K such that the map $\rho : I \longrightarrow J$ given by $\rho(x) = \alpha x$ for $x \in I$ is the isomophism. Then since $J = \alpha I$, we have

$$J :_K J = \alpha I :_K \alpha I = I :_K I \quad \text{in} \quad \mathcal{Y}_A.$$

Hence $I = J$ by Theorem 1.1.

COROLLARY 2.4. Let $I(\neq A)$ be an ideal in A and $1 = ht_A(I)$ and assume that $I^2 = aI$ for some $a \in I$. Let $\ell \geq 1$ be an integer. Then $\ell = 1$ if $I^\ell \in \mathcal{X}_A^1$.

Proof. Suppose $\ell \geq 2$. Since $(I^\ell)^2 = a^\ell I^\ell$, I^ℓ contains $a^\ell A$ as a reduction, whence $C = \{\frac{x}{a^\ell} \mid x \in I^\ell\} \in \mathcal{Y}_A$ by Theorem 1.1. Hence $C = \{\frac{i}{a} \mid i \in I\}$ as $I^\ell = a^{\ell-1}I$. Therefore $I \subseteq A :_K C$ as $I = aC$, which is impossible because $A :_K C = a^\ell A : I^\ell = I^\ell$ by Lemma 2.1.(3).

Proof of Corollary 1.2. We have $I = aC$. Then since $I^2 = aI$, by Lemma 2.1.(3) we have

(a) $$A :_K C = aA : I.$$

Hence $C \neq A$ because $\mu_A(I) \geq 2$. Therefore if C is a Gorenstein ring, then $C \in \mathcal{Y}_A$ by Lemma 2.1.(1), whence $A :_K C \in \mathcal{X}_A^1$ by Theorem 1.1. Thus we have $aA : I \in \mathcal{X}_A^1$ by (a) and A/I is a Cohen-Macaulay local ring, because we apply the functor $H_m^i(*)$ to the exact sequence $0 \to C \to A \to A/I \to 0$. Conversely suppose $aA : I \in \mathcal{X}_A^1$ and A/I is a Cohen-Macaulay local ring. By Theorem 1.1, there exists $D \in \mathcal{Y}_A$ such that $A :_K D = aA : I$, whence $A :_K D = A :_K C$ by (a). Applying the functor $H_m^i(*)$ to the exact sequence $0 \to C \to A \to A/I \to 0$, we get $H_m^i(C) = 0$ for $i \neq d$, so that C is a Cohen-Macaulay ring. Notice that $C \cong A :_K (A :_K C)$ and $D \cong A :_K (A :_K D)$ ([BH, Theorem 3.3.10]), because A is a Gorestein local ring and C, D are maximal Cohen-Macaulay A-modules. Thus we have

$$D \cong A :_K (A :_K D) = A :_K (A :_K C) \cong C,$$

so that C is a Gorenstein ring. This completes the proof of Corollary 1.2.

EXAMPLE 2.5. Let $A = k[[s^4,st,t^2]]$ be the formal power series ring over a field k. Then $\mathcal{X}_A^1 = \{(t^2,s^2t^2)A\}$.

Proof Let $\overline{A}$ denote the normalization of A. Then $\overline{A} = k[[s^2,st,t^2]]$. Hence $\overline{A} \in \mathcal{Y}_A$ since $\overline{A} = A[s^2]$. By Theorem 1.1, we see $A : \overline{A} \in \mathcal{X}_A^1$. Since $\overline{A} = A[s^2] = A + As^2$, we have

$$A : \overline{A} = A : s^2 = (t^2,s^2t^2)A,$$

whence $(t^2,s^2t^2)A \in \mathcal{X}_A^1$. Let $C \in \mathcal{Y}_A$. Then $A \subsetneq C \subseteq \overline{A} \subseteq K = Q(A)$. Hence $A : C \supseteq A : \overline{A} = (t^2,s^2t^2)A$. So we take an arbitary ideal $J (\neq A)$ in A such that $J \supseteq (t^2,s^2t^2)A = t^2\overline{A}$. Consider the following exat sequence

$$0 \longrightarrow Ker\psi \longrightarrow A/t^2\overline{A} \xrightarrow{\psi} A/J \longrightarrow 0.$$

Notice that $A/t^2\overline{A} \cong k[[X,Y,Z]]/(Y^4 - XZ^2,Y^2,Z) \cong k[[X,Y]]/(Y^2)$, where $X \longmapsto s^4$, $Y \longmapsto st$, and $Z \longmapsto t^2$. Then $Ker\psi = (Y)$, whence $J = (st,t^2)A$. Thus we see $\mathcal{X}_A^1 \subseteq \{(t^2,s^2t^2)A,\ (st,t^2)A\}$.

CLAIM : $J = (st,t^2)A$ is not the $1^{\underline{th}}$ good ideal in A.

Proof of Claim. Suppose J is the $1^{\underline{th}}$ good ideal in A. Then $J^2 = fJ$ for some $f \in J$. Hence $f \mid st$ and $f \mid t^2$ in $B = k[[s,t]]$. Since f is a reduction of J, we see $\overline{fB} = fB$, whence $f = ut$ for some unit u in B. Write $u = c+as+bt+g$, where $c \neq 0$ in k, $a ,b \in k$, and $g \in B$. Then $f = ut = ct + ast + bt^2 + g$, whence $c = 0$ because $t \notin A$, which is absurd. This proves Claim.
Therefore we have $\mathcal{X}_A^1 = \{(t^2,s^2t^2)A\}$ and $\mathcal{Y}_A = \{\overline{A}\}$.

EXAMPLE 2.6. Let (B,n) be a regular local ring with $dim(B) > 1$ and let $A = B \ltimes B = (B[X]/(X^2))$ denote the idealization of B over B (here X denotes an indeterminate over B). Let x denote the reduction of X mod (X^2). Then $\mathcal{X}_A^1 = \{(\alpha,x)A \mid 0 \neq \alpha \in n\}$.

Proof. Let $I = (\alpha,x)$, where $0 \neq \alpha \in n$. Then $I^2 = (\alpha^2,\alpha x)A = \alpha I$ and $ht_A(I) = 1$ since αA is a reduction of I. A/I is 1-dimensional Gorestein local ring, because $A/I \cong B/\alpha B$. Since $I^2 = \alpha I$, we see $I \subseteq \alpha A : I$. Let $f \in \alpha A : I$. Write $f = b_0 + b_1 x$, where $b_0,b_1 \in B$. Then $f\alpha = b_0\alpha + b_1 x\alpha \in \alpha A$, and $fx = b_0 x + b_1 x^2 \in \alpha A \Longleftrightarrow b_0 \in \alpha A$ since $x^2 = 0$. Hence $I = \alpha A : I$. Thus $I \in \mathcal{X}_A^1$. Let $C \in \mathcal{Y}_A$. Then $A \subsetneq C \subseteq K = Q(A)$, where $Q(A)$ is the total quotient field of A. Since $\overline{A} = B + Fx$, where $F = Q(B)$, we see $C = B \oplus L$ with $Bx \subsetneq L \subseteq Fx$. Since $B \cong L$, we have $B \subsetneq Bg \subseteq F$ for some $0 \neq g \in B$. Write $g = \frac{b}{a}$, where a,b are non-zero coprime in B. Since $B \subsetneq Bg$, we have $1_B = c\,\frac{b}{a}$ for some $c \in B$, whence $b = 1$, because a and b are coprime. Hence $g = \frac{1}{a}$, where $0 \neq a \in n$. Thus $C = B + B\,\frac{x}{a}$, where $0 \neq a \in n$.

CLAIM : $A : C = aB + Bx$.

Proof of Claim. Let $\omega \in (A : C)$. Write $\omega = b_0 + b_1 x$, where $b_0,b_1 \in B$. Then

$$\begin{aligned} \omega C \subseteq A &\Longleftrightarrow \omega(B + B\,\frac{x}{a}) \subseteq A = B + Bx \\ &\Longleftrightarrow \omega\,\frac{x}{a} \in A \\ &\Longleftrightarrow \frac{b_0}{a} \in B \\ &\Longleftrightarrow b_0 \in aB \\ &\Longleftrightarrow aB + Bx. \end{aligned}$$

By Theorem 1.1, we have $A : C \in \{(\alpha,x)A \mid 0 \neq \alpha \in n\}$. Therefore $\mathcal{X}_A^1 = \{(\alpha,x)A \mid 0 \neq \alpha \in n\}$.

3. THE ESTIMATION OF THE SET $\mathcal{X}_A^1$.

Let $S = \oplus_{n \geq 0} S_n$ be a Noetherian graded ring and assume that S contains a unique graded maximal ideal $\mathfrak{M}$. We denote by $H^i_{\mathfrak{M}}(*)$ $(i \in \mathbb{Z})$ the $i^{\underline{th}}$ local cohomology functor of S with respect to $\mathfrak{M}$. For each graded S-module E and $n \in \mathbb{Z}$, let $[H^i_{\mathfrak{M}}(E)]_n$ denote the homogeneous component of the graded S-module $H^i_{\mathfrak{M}}(E)$ of degree n. If $S_n = (0)$ for all $n < 0$ and E is a finitely generated graded S-module, we have $[H^i_{\mathfrak{M}}(E)]_n = (0)$ for all $n \gg 0$ and $i \in \mathbb{Z}$. We put

$$a(E) = max\{n \in \mathbb{Z} \mid [H^e_{\mathfrak{M}}(E)]_n \neq (0)\}$$

with $e = dim_S E$ and call it the a-invariant of E ([GW, (3.1.4)]). For each $p \in \mathbb{Z}$, let $E(p)$ stand for the graded S-module, whose underlying S-module coincides with that of E and whose graduation is given by $[E(p)]_n = E_{p+n}$ for all $n \in \mathbb{Z}$. We denote by K_S the graded canonical module of S, if it exists.

PROPOSITION 3.1. Suppose $I \in \mathcal{X}_A^1$ and let $R = R(I)$. Then $K_R \cong R_+$ as graded R-modules.

Proof. Let $a \in I$ with$I^2 = aI$. We put $P = R(aA)$ and $at \in R_1$. Then at is transcendental over A and P is the polynomial ring over A, so that P is a Gorenstein ring with $K_P \cong P(-1)$. By [BH, 3.6.12], we have the following isomorphisms

$$K_R \cong Hom_P(R,K_P) \cong Hom_P(R,P(-1)) \cong [Hom_P(R,P)](-1) \cong (P :_P R)(-1).$$

CLAIM : $(P :_P R) = IR$.

Proof of Claim. Since $I \in \mathcal{X}_A^1$, we have $I^{n+1} = a^n I$ for all $n \geq 0$. Hence $IR \cdot R = IR = IP \subset P$. Therefore we get $IR \subset (P :_P R)$. Let $a^n t^n \in (P :_P R)$, where $n \geq 0$. Then $a^n x = a^{n+1}$ for all $x \in I$, so that $a^n \in (a^{n+1} : I) = I^{n+1}$. Thus we get $a^n t^n \in I^{n+1} t^n = [IR]_n$.

Then since $IR \cong R_+(1)$ and by Claim, we have $K_R(1) \cong R_+(1)$. Thus $K_R \cong R_+$. This completes the proof of Proposition 3.1.

We are now discussing the question when $\mathcal{X}_A^1 = \emptyset$. To do this we need the following two lemmas.

LEMMA 3.2. Let $I(\neq A)$ be an ideal with $ht_A(I) = 1$. Let $a \in I$ such that $I^{n+1} = aI^n$ for some $n \geq 0$. Assume that $\mu_A(I) \geq 2$. Let $r = r_{aA}(I)$. Then $I^r \in \mathcal{X}_A^1$ if $G(I)$ is a Gorenstein ring.

Proof. Since $\mu_A(I) \geq 2$, we have $r \geq 1$. Hence $(I^r)^2 = a^r I^r$ and $ht_A(I^r) = 1$, because $a^r A$ is a reduction of I^r. Thus $a(G(I)) = r_{a^r A}(I^r) - ht_A(I^r) = 1 - 1 = 0$. Since $a(G(I)) \equiv -1(\mathrm{mod} r)$, $G(I^r)$ is a Gorenstein ring, by [Hy, Theorem 2.4].Thus $I^r \in \mathcal{X}_A^1$ by Definition.

LEMMA 3.3. Let$I(\neq A)$ be an ideal with $ht_A(I) = 1$. Let $a \in I$ such that $I^{n+1} = aI^n$ for some $n \geq 0$. Assume that $\mu_A(I) \geq 2$. Let $r = r_{aA}(I)$. Then $a^r A : I^r \in \mathcal{X}_A^1$ if $Proj R(I)$ is a Gorenstein scheme.

Proof. Let $R = R(I)$. Then since $\sqrt{R_+} = \sqrt{(at)R}$, we have $Proj R(I) = Spec(C)$, where $C = I^r/a^r = A[I/a]$. Hence C is a Gorenstein ring. We have $I^r = a^r C$ and so $(I^r)^2 = (a^r C)(a^r C) = a^r a^r C = a^r I^r$. We claim $\mu_A(I^r) \geq 2$. In fact, suppose that $\mu_A(I^r) = 1$. Then $I^r = a^r A$ by [S, Chap.2, Theorem 1.7]. Let $x \in I$. Then $a^{r-1}x \in I^r$, we get $a^{r-1}x \in a^r A$, whence $a \in aA$. Thus $I = aA$. This is impossible, because $\mu_A(I) \geq 2$ by our assumption. Thus $a^r A : I^r \in \mathcal{X}_A^1$ by Corollary 1.2.

THEOREM 3.4. Suppose that A/m is infinite.Then the following conditions are equivalent.

(1) $\mathcal{X}_A^1 = \emptyset$.

(2) Every equimultiple ideal I with $ht_A(I) = 1$ in A for which $G(I)$ is Gorenstein ring is principal.

(3) Every equimultiple ideal I with $ht_A(I) = 1$ in A for which $Proj R(I)$ is Gorenstein scheme is principal.

Proof. Because the field A/m is infinite, every equimultiple ideal I with $ht_A(I) = 1$ in A contains an element $a \in I$ such that $I^{n+1} = aI^n$ for some $n \geq 0$ (cf.[NR]).

(1)⇒(2). Suppose there exists an equimultiple ideal I with $ht_A(I) = 1$ in A such that $G(I)$ is a Gorenstein ring and $\mu_A(I) \geq 2$. By Lemma 3.2,we have $I^r \in \mathcal{X}_A^1$, where $r = r_a A(I)$. This is impossible, because $\mathcal{X}_A^1 = \emptyset$ by our assumption.

(2)⇒(1). Suppose $\mathcal{X}_A^1 \neq \emptyset$ and let $I \in \mathcal{X}_A^1$. Choose $a \in I$ so that $I^2 = aI$, $G(I)$ is a Gorenstein ring, and $ht_A(I) = 1$. By (2), I is a principal ideal, i.e., $I = aA$. This is absurd.

(1)$\Rightarrow$(3). Suppose there exists an equimultiple ideal I with $ht_A(I) = 1$ in A such that $ProjR(I)$ is a Gorenstein scheme and $\mu_A(I) \geq 2$. By Lemma 3.3, we have $a^r A : I^r \in \mathcal{X}_A^1$, where $r = r_{aA}(I)$. This is impossible, because $\mathcal{X}_A^1 = \emptyset$ by our assumption.

(3)$\Rightarrow$(1) Suppose $\mathcal{X}_A^1 \neq \emptyset$ and let $I \in \mathcal{X}_A^1$. Choose $a \in I$ so that $I^2 = aI$ and $I = aA : I$. Then $C = I/a$ is a Gorenstein ring by Theorem 1.1. Hence $ProjR(I) = Spec(C)$ is a Gorenstein scheme, so that the ideal I has to be principal, i.e., $I = aA$. This is absurd.

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COHEN-MACAULAY MULTI-REES ALGEBRAS

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1. Introduction

Let $(A, \mathfrak{m})$ be a local ring, and let $I_1, \ldots, I_r \subset A$ be ideals of positive height. The multi-Rees algebra $R_A(I_1, \ldots, I_r) = A[I_1 t_1, \ldots, I_r t_r] \subset A[t_1, \ldots, t_r]$ where $t_1, \ldots, t_r$ are variables. Multi-Rees algebras are connected to multiple blowing up. The purpose of this lecture is to report on some recent work on their Cohen-Macaulay and Gorenstein properties ([5]). In investigating homological properties of a Rees algebra of a single ideal the corresponding form ring plays a central role. Unfortunately, in the case of several ideals this method seems not to work anymore. Our approach is to compare the properties of the multi-Rees algebra $R_A(I_1, \ldots, I_r)$ to those of the corresponding diagonal subring which is the usual Rees algebra $R_A(I_1 \cdots I_r)$ of the product $I_1 \cdots I_r$. It has already been proven in [7, Corollary 2.10] that the Cohen-Macaulayness of $R_A(I_1, \ldots, I_r)$ implies that of $R_A(I_1 \cdots I_r)$. However, easy examples show that the converse is not true. We therefore seek additional conditions which would make $R_A(I_1, \cdots, I_r)$ Cohen-Macaulay in this case. We start by characterizing in Theorem 3.4 the Cohen-Macaulay property of $R_A(I_1, \ldots, I_r)$ in terms of the sheaf cohomology of the corresponding multi-projective scheme $\operatorname{Proj} R_A(I_1, \ldots I_r)$. For usual Rees algebras this was done by J. Lipman in [11, Theorem 4.1]. The case $r = 2$ was treated in [7, Theorem 2.5]. However, in the general case we have to follow a quite different line of argument. We then utilize the fact that $\operatorname{Proj} R_A(I_1, \ldots, I_r)$ is isomorphic to the usual blowup $\operatorname{Proj} R_A(I_1 \cdots I_r)$. If the dimension of the closed fiber is small, then most of the sheaf cohomology is known to vanish. This helps us to state the conditions for the Cohen-Macaulayness of $R_A(I_1, \ldots, I_r)$ when the product $I_1 \cdots I_r$ has analytic spread less than three (s. Theorem 3.8 and Theorem 3.10). We apply our results to joint reductions in Theorem 4.1. Finally, we investigate the Gorenstein property of $R_A(I_1, \ldots, I_r)$. If $R_A(I_1, \ldots, I_r)$ is Cohen-Macaulay, it turns out in Theorem 5.2 that the Gorensteiness of the diagonal $R_A(I_1 \cdots I_r)$ is equivalent to that of $R_A(I_1, \ldots, I_r)$.

2. Preliminaries

In this section we fix some notation and recall some general facts about multi-graded rings. We always assume that all rings and schemes are Noetherian. We also assume that all schemes and morphisms are separated. If $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^r$ and $n_j < n'_j$ for $j = 1, \ldots, r$, we write $\mathbf{n} < \mathbf{n}'$. Let $\mathbf{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ $(j = 1, \ldots, r)$ be the canonical base elements of $\mathbb{Z}^r$. Moreover, we set $\mathbf{1} = (1, \ldots, 1)$.

Let $S = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} S_{\mathbf{n}}$ be an r-graded ring finitely generated over $S_{\mathbf{0}}$ by elements in degrees $\mathbf{1}_1, \ldots, \mathbf{1}_r$. The diagonal subring of S is the graded ring $S^{\Delta} = \bigoplus_{n \in \mathbf{N}} S_{n, \ldots, n}$.

The irrelevant ideal of S is $S^+ = \bigoplus_{\mathbf{n}>0} S_{\mathbf{n}}$. The multiprojective scheme $\operatorname{Proj} S$ is defined in the usual way by using the multihomogeneous prime ideals $P \subset S$ which do not contain S^+. The quasi-coherent sheaf corresponding to $S(\mathbf{n})$ is invertible for every $\mathbf{n} \in \mathbb{Z}^r$. We denote it by $\mathcal{O}_Z(\mathbf{n})$. Note that multiprojective schemes are projective: if $Z^\Delta = \operatorname{Proj} S^\Delta$, the inclusion $S^\Delta \longrightarrow S$ induces an isomorphism $f\colon Z \longrightarrow Z^\Delta$ such that $f^*(\mathcal{O}_{Z^\Delta}(n)) = \mathcal{O}_Z(n, \dots, n)$ for all $n \in \mathbb{Z}$. In the following we take $S = R_A(I_1, \dots, I_r)$ where $I_1, \dots, I_r \subset A$ are ideals of positive height. However, many of our results could be formulated for more general multigraded rings, too (cp. [7]). We set $Z = \operatorname{Proj} S$. Note that $\dim S = d + r$ and $\dim Z = d$.

3. The Cohen-Macaulay property of a multi-Rees algebra

We are now going to characterize the Cohen-Macaulay property of S in terms of sheaf cohomology. We consider the Rees algebra $T = R_S(S^+)$. Note that

$$T = \bigoplus_{k\geq 0} \Big(\bigoplus_{\mathbf{n}\geq 0} S_{n_1+k,\dots,n_r+k} t^k \Big).$$

Set $W = \operatorname{Proj} T$. In the following lemma which is crucial for the proof of our first main Theorem 3.4, we observe that W can in fact be considered as a vector bundle over Z.

Lemma 3.1. $W = \mathbb{V}(\mathcal{O}_Z(\mathbf{1}_1) \oplus \cdots \oplus \mathcal{O}_Z(\mathbf{1}_r))$.

Outline of the proof. Write $T = S[S^+t]$ where t is a variable. Cover W with open affine sets $D_+(st) = \operatorname{Spec} T_{(st)}$ where $s \in S_{\mathbf{1}}$. Since $T_k = \bigoplus_{\mathbf{n}\geq 0} S_{n_1+k,\dots,n_r+k} t^k$, we observe that $T_{(st)} = \bigoplus_{\mathbf{n}\geq 0} (S(\mathbf{n}))_{(s)}$. But then $W = \mathcal{S}pec \bigoplus_{\mathbf{n}\geq 0} \mathcal{O}_Z(\mathbf{n}) = \mathcal{S}pec \operatorname{Sym}(\mathcal{O}_Z(\mathbf{1}_1) \oplus \cdots \oplus \mathcal{O}_Z(\mathbf{1}_r))$. □

Next we want to calculate the sheaf cohomology modules $H^i_F(W, \mathcal{O}_W)$ with supports in $F = W \times_S S/\mathfrak{M}$ where $\mathfrak{M}$ denotes the homogeneous maximal ideal of S. Note that these are r-graded S-modules. In fact, a look at the Sancho de Salas sequence ([11, p. 150]) shows that $H^i_F(W, \mathcal{O}_W) = [H_{\mathfrak{N}}(T^+)]_0$ where $\mathfrak{N}$ is the homogeneous maximal ideal and $T^+ = \bigoplus_{k>0} T_k$ the usual irrelevant ideal of T. Set $E = Z \times_A A/\mathfrak{m}$.

Proposition 3.2. *As a graded S-module*

$$H^i_F(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n}<0} H^{i-r}_E(Z, \mathcal{O}_Z(\mathbf{n}))$$

for all $i \geq 0$.

Outline of the proof. We utilize the local cohomology sheaves $\mathcal{H}^i_D(\mathcal{O}_W)$ with supports in $D = W \times_S S/\mathfrak{A}$ where $\mathfrak{A} = \bigoplus_{\mathbf{n}\neq 0} S_{\mathbf{n}}$ (for basic facts about local cohomology sheaves we refer to [1, §1.]). Let $\pi\colon W \longrightarrow Z$ be the canonical morphism. As $\mathfrak{M} = \mathfrak{m} \oplus \mathfrak{A}$, the functor $\Gamma_F(W, \,\cdot\,)$ equals to the composite $\Gamma_{\pi^{-1}(E)}(W, \mathcal{H}^0_D(\,\cdot\,))$. We thus have a spectral sequence

$$E_2^{p,q} = H^p_{\pi^{-1}(E)}(W, \mathcal{H}^q_D(\mathcal{O}_W)) \Rightarrow H_F(W, \mathcal{O}_W).$$

We can now cover Z with open affine sets $U = \operatorname{Spec} B$ such that $\pi^{-1}(U) = \operatorname{Spec} B[t_1, \dots, t_r]$ where $t_1, \dots, t_r$ are variables. Also $\pi^{-1}(U) \cap D = V(t_1, \dots, t_r)$.

It is well-known from [2, Proposition 2.1.12] that

$$H^i_{(t_1,\dots,t_r)}(B[t_1,\dots,t_r]) = \begin{cases} 0 & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{0}} Bt_1^{n_1}\cdots t_r^{n_r} & \text{if } i = r. \end{cases}$$

This leads to the observation that

$$\pi_*(\mathcal{H}^i_D(\mathcal{O}_W)) = \begin{cases} 0 & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{0}} \mathcal{O}_Z(\mathbf{n}) & \text{if } i = r. \end{cases}$$

But then the spectral sequence considered above degenerates. The claim follows, because by the affiness of π $H^p_{\pi^{-1}(E)}(W, \mathcal{H}^q_D(\mathcal{O}_W)) = H^p_E(Z, \pi_*(\mathcal{H}^q_D(\mathcal{O}_W)))$. □

A second ingredient in the proof of Theorem 3.4 is the following proposition ([8, Lemma 1.1]) which originates from [11, Lemma 4.2]. It relates vanishing of local cohomology, sheaf cohomology and sheaf cohomology with supports to each other.

Proposition 3.3. *Let T be a standard graded ring defined over a local ring $(B, \mathfrak{n})$. Let $\mathfrak{N}$ denote the homogeneous maximal ideal of T. Set $W = \operatorname{Proj} T$ and $F = W \times_B B/\mathfrak{n}$. Let N be a graded T-module. Let $k \in \mathbb{Z}$. Then the following conditions are equivalent:*
1) $[H^i_{\mathfrak{N}}(N)]_k = 0$ for all $i \geq 0$;
2) The canonical homomorphism $H^i_{\mathfrak{n}}(N_k) \longrightarrow H^i_F(W, \widetilde{N}(k))$ is an isomorphism for all $i \geq 0$;
3) The canonical homomorphism $N_k \longrightarrow \Gamma(W, \widetilde{N}(k))$ is an isomorphism and one has $H^i(W, \widetilde{N}(k)) = 0$ for $i > 0$;
4) $[H^i_{T^+}(N)]_k = 0$ for all $i \geq 0$.

We are now ready to prove

Theorem 3.4. *Let A be a local ring of dimension d, and let $I_1, \dots, I_r \subset A$ be ideals of positive height. Then $R_A(I_1, \dots, I_r)$ is Cohen-Macaulay if and only if the following conditions are satisfied*
1) $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = I_1^{n_1} \cdots I_r^{n_r}$ for all $\mathbf{n} \geq \mathbf{0}$;
2) $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geq \mathbf{0}$;
3) $H^i_E(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i < d$ and $\mathbf{n} < \mathbf{0}$.

Outline of the proof. Consider T as an $(r+1)$-graded ring. Let Q denote the corresponding irrelevant ideal. As a first step we will show that $[H^i_{\mathfrak{N}}(T)]_{\mathbf{n},k} = 0$ for all $i \geq 0$ and $\mathbf{n} < \mathbf{0}$, $k \geq 0$. Because $T = \bigoplus_{\mathbf{n}\geq \mathbf{0}, k\geq 0} S_{n_1+k,\dots,n_r+k}$, there is an obvious isomorphism $Q \longrightarrow T^+$ which maps an element in $T_{\mathbf{n},k}$ to the corresponding element of $T_{\mathbf{n}-\mathbf{1},k+1}$. By using the long exact sequences of cohomology corresponding to the exact sequences

$$0 \longrightarrow T^+ \longrightarrow T \longrightarrow T/T^+ \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Q \longrightarrow T \longrightarrow T/Q \longrightarrow 0$$

one now checks that for any $k \geq 0$ and $\mathbf{n} < \mathbf{0}$ there are isomorphisms

$$[H^i_{\mathfrak{N}}(T)]_{\mathbf{n},k} = [H^i_{\mathfrak{N}}(Q)]_{\mathbf{n},k} = [H^i_{\mathfrak{N}}(T^+)]_{\mathbf{n}-\mathbf{1},k+1} = [H^i_{\mathfrak{N}}(T)]_{\mathbf{n}-\mathbf{1},k+1}.$$

As $[H^i_{\mathfrak{N}}(T)]_{\mathbf{n},k} = 0$ for $k \gg 0$, the above claim follows.

The Sancho de Salas sequence (s. [11, p. 150]) now gives an r-graded sequence

$$\dots \longrightarrow [H^i_{\mathfrak{N}}(T)]_0 \longrightarrow H^i_{\mathfrak{M}}(S) \longrightarrow H^i_F(W, \mathcal{O}_W) \longrightarrow \dots \quad .$$

As $[H^i_{\mathfrak{N}}(T)]_{\mathbf{n};0} = 0$ for $\mathbf{n} < \mathbf{0}$, this implies by Proposition 3.2 that there is an isomorphism $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = H^{i-r}_E(Z, \mathcal{O}_Z(\mathbf{n}))$ for $\mathbf{n} < \mathbf{0}$. We thus see that $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = 0$ for $i < d+r$ and $\mathbf{n} < \mathbf{0}$ if and only if 3) holds. On the other hand, the Sancho de Salas sequence also implies that

$$[H^i_{\mathfrak{N}}(T)]_0 = \bigoplus_{\text{some } n_j \geq 0} [H^i_{\mathfrak{M}}(S)]_{\mathbf{n}}.$$

Recall from [3, Lemma 2.1] that $[H^{d+r}_{\mathfrak{M}}(S)]_{\mathbf{n}} = 0$ if some $n_j \geq 0$. Therefore $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = 0$ for all $i < d+r$ and $\mathbf{n} \in \mathbb{Z}^r$ such that $n_j \geq 0$ for some j if and only if $[H^i_{\mathfrak{N}}(T)]_0 = 0$ for all $i \geq 0$. But according to Proposition 3.3 this is equivalent to having $\Gamma(W, \mathcal{O}_W) = S$ and $H^i(W, \mathcal{O}_W) = 0$ for $i > 0$. The claim follows, because $\Gamma(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n} \geq 0} \Gamma(Z, \mathcal{O}_Z(\mathbf{n}))$ and $H^i(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n} \geq 0} H^i(Z, \mathcal{O}_Z(\mathbf{n}))$. □

Remark 3.1. Note the formula $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = H^{i-r}_E(Z, \mathcal{O}_Z(\mathbf{n}))$ for $i \geq 0$ and $\mathbf{n} < \mathbf{0}$.

Because $\operatorname{Proj} R_A(I_1 \cdots I_r) \cong Z$, it follows in particular that the Cohen-Macaulay property of $R_A(I_1, \ldots, I_r)$ implies that of $R_A(I_1 \cdots I_r)$. This recovers [7, Corollary 2.10]. The converse implication does not hold in general (s. [7, Example 2.11]). In comparing the Cohen-Macaulay properties of $R_A(I_1, \ldots, I_r)$ and $R_A(I_1 \cdots I_r)$ the main point is now to understand how the vanishing of the cohomology of the sheaves $\mathcal{O}_Z(n, \ldots, n)$ ($n \in \mathbb{Z}$) affects the vanishing of the cohomology of $\mathcal{O}_Z(\mathbf{n})$ ($\mathbf{n} \in \mathbb{Z}^r$). The following lemma, which can be proven by means of a Castelnuovo-Mumford type lemma (s. [5]), is therefore very essential for our arguments:

Lemma 3.5. *Let $m \in \mathbb{Z}$ and $p \in \mathbb{N}$.*
a) Suppose that $H^i(Z, \mathcal{O}_Z(m-i, \ldots, m-i)) = 0$ for all $i > p$. Then

$$H^i(Z, \mathcal{O}_Z(n_1 - i, \ldots, n_r - i)) = 0$$

for all $i > p$ and $\mathbf{n} \geq (m, \ldots, m)$. Moreover, in the case $p = 0$ we also have

$$\Gamma(Z, \mathcal{O}_Z(\mathbf{n} + \mathbf{1}_j)) = I_j \Gamma(Z, \mathcal{O}_Z(\mathbf{n}))$$

for $\mathbf{n} \geq (m-1, \ldots, m-1, m, m-1, \ldots, m-1)$ and $j = 1, \ldots, r$.
b) Suppose that $H^i_E(Z, \mathcal{O}_Z(m-i, \ldots, m-i)) = 0$ for all $i < p$. Then

$$H^i_E(Z, \mathcal{O}_Z(n_1 - i, \ldots, n_r - i)) = 0$$

for all $i < p$ and $\mathbf{n} \leq (m, \ldots, m)$.

Let $I \subset A$ be an ideal of positive height. Set $X = \operatorname{Proj} R_A(I)$. As the closed fiber of the canonical projection $X \longrightarrow \operatorname{Spec} A$ has dimension $\leq \ell(I) - 1$, it is well-known from [2, Corollaire (4.2.2)] that $H^i(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X if $i \geq \ell(I)$. When the ideal has small analytic spread, we thus see that most of the sheaf cohomology vanishes. The next lemma shows that the same also holds for the sheaf cohomology with supports in the closed fiber $E = X \times_A A/\mathfrak{m}$.

Lemma 3.6. *If X is Cohen-Macaulay, then $H^i_E(X, \mathcal{L}) = 0$ for any invertible sheaf $\mathcal{L}$ on X if $i \leq d - \ell(I)$.*

Outline of the proof. We may assume that A is complete. By the above the claim is then a consequence of the local–global duality of Lipman ([11, p. 188]) according to which $H^i_E(X, \mathcal{L}) = \operatorname{Hom}_A(H^{d-i}(X, \omega_X \otimes \mathcal{L}^{-1}), E_A(k))$. □

The values of m which we can use in Lemma 3.5, can now be found using

Lemma 3.7. *Set $\ell = \ell(I)$ and $a = a(G_A(I))$. If $R_A(I)$ is Cohen-Macaulay, then*
a) $H^i(X, \mathcal{O}_X(\ell - 1 - i)) = 0$ for all $i > 0$;
b) $H^i_E(X, \mathcal{O}_X(d - \ell - i)) = 0$ for all $i < d$.

Suppose, moreover that A is Cohen-Macaulay. Then
a') $H^i(X, \mathcal{O}_X(\ell + a - i)) = 0$ for all $i > 0$;
b') $H^i_E(X, \mathcal{O}_X(d - \ell + 1 - i)) = 0$ for all $i < d$.

Outline of the proof. By taking $r = 1$ in Theorem 3.4 we immediately see that a) holds. Taking into account Lemma 3.6 also b) follows. To prove a') we first observe that the Cohen-Macaulayness of $R_A(I)$ now implies that of the form ring $G_A(I)$. Let $\mathfrak{M}$ denote the maximal homogeneous ideal of $R_A(I)$. Then $[H^i_{\mathfrak{M}}(G_A(I))]_n = 0$ for all $i \geq 0$ and $n > a$. Set $Y = \operatorname{Proj} G_A(I)$. By Proposition 3.3 we get $H^i(Y, \mathcal{O}_Y(n)) = 0$ for all $i > 0$ and $n > a$. By standard arguments this further gives $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i > 0$ and $n > a$. Thus a') follows. Finally, a look at the Sancho de Salas sequence

$$\dots \longrightarrow H^i_{\mathfrak{m}}(A) \longrightarrow H^i_E(X, \mathcal{O}_X) \longrightarrow [H^{i+1}_{\mathfrak{M}}(R_A(I))]_0 \longrightarrow \dots$$

([11, p. 150]) shows that $H^i_E(X, \mathcal{O}_X) = 0$ for $i < d$. This implies b'). □

For ideals whose product has analytic spread two, we have the following result:

Theorem 3.8. *Let A be a local ring, and let $I_1, \dots, I_r \subset A$ be ideals of positive grade such that $\ell(I_1 \cdots I_r) \leq 2$. Then $R_A(I_1, \dots, I_r)$ is Cohen-Macaulay if and only if $R_A(I_1 \cdots I_r)$ is Cohen-Macaulay and $(I_{j_1} \cdots I_{j_k}) : I_{j_l} = I_{j_1} \cdots I_{j_{l-1}} I_{j_{l+1}} \cdots I_{j_k}$ for all $1 \leq j_1 < \dots < j_k \leq r$ and $1 \leq l \leq k$.*

Outline of the proof. It is a general fact that if $I_1, \dots, I_r \subset A$ are ideals of positive grade, then

$$(\dagger) \qquad \Gamma(Z, \mathcal{O}_Z(\mathbf{n} - \mathbf{m})) = \Gamma(Z, \mathcal{O}(\mathbf{n})) :_{\Gamma(Z, \mathcal{O}_Z)} (I_1^{m_1} \cdots I_r^{m_r})$$

for all $\mathbf{n} \geq \mathbf{m} \geq \mathbf{0}$. This fact together with Theorem 3.4 now easily implies that the conditions of the theorem are necessary.

Let us prove that they are also sufficient. Since $R_A(I_1 \cdots I_r)$ is Cohen-Macaulay and $\operatorname{Proj} R_A(I_1 \cdots I_r) \cong Z$, Lemma 3.5 together with Lemma 3.7 implies that $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geq \mathbf{0}$. Moreover, since $\Gamma(Z, \mathcal{O}_Z(\mathbf{1})) = I_1 \cdots I_r$, we get $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = I_1^{n_1} \cdots I_r^{n_r}$ for all $\mathbf{n} \geq \mathbf{1}$. By (†) one now checks that this in fact holds for all $\mathbf{n} \geq \mathbf{0}$. Finally, set $E = Z \times_A A/\mathfrak{m}$. By applying again Lemma 3.7 and Lemma 3.5 we see that $H^i_E(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i < d$ and $\mathbf{n} < \mathbf{0}$. The claim then follows from Theorem 3.4. □

Example 3.1. In general $R_A(I, J)$ need not to be Cohen-Macaulay even if it is normal and $R_A(IJ)$ is Cohen-Macaulay. By [6, Example 3.6] it is possible to find a three-dimensional regular local ring A and an ideal $I_0 \subset A$ such that $R_A(I_0)$ does not have rational singularities although it is normal and Cohen-Macaulay. We can also find an ideal an ideal $J_0 \subset A$ such that $Y = \operatorname{Proj} R_A(J_0)$ is regular and that $I_0\mathcal{O}_Y$ is invertible. Then $\operatorname{Proj} R_A(I_0, J_0) \cong \operatorname{Proj} R_A(I_0 J_0) \cong Y$ is normal. It is not difficult to see that this implies $I_0^p J_0^q = \overline{I_0^p J_0^q}$ for $p, q \gg 0$. Take $I = I_0^N$ and $J = J_0^N$ where $N \gg 0$. Then $R_A(I, J)$ is normal. Moreover, it still holds that $R_A(I)$ does not have rational singularities (use e.g. [8, Proposition 2.1]). Now $R_A(I, J)$ is not Cohen-Macaulay. To see this, consider the blowup $Z = \operatorname{Proj} R_{R_A(I)}(JR_A(I))$. By looking at the affine open sets which cover Z, one easily checks that $Z = \mathbb{V}(I\mathcal{O}_Y)$.

In particular, Z is regular. The Cohen-Macaulayness of $R_A(I,J)$ would then by [11, Theorem 4.1] imply that $R_A(I)$ has rational singularities. Finally note that since A has rational singularities, by using [11, Theorem 4.1] again we may choose $N \gg 0$ in such a way that $R_A(IJ)$ is Cohen-Macaulay.

J. Verma showed in [12, Theorem 3.4] that in a two-dimensional regular local ring integrally closed ideals primary to the maximal ideal always have Cohen-Macaulay multi-Rees algebras. This can be generalized as follows:

Corollary 3.9. *Let A be a Cohen-Macaulay local ring which satisfies the condition (R_2). Let $I_1,\ldots,I_r \subset A$ be integrally closed equimultiple ideals of height two such that also $I_1\cdots I_r$ is equimultiple of height two. Then $R_A(I_1,\ldots,I_r)$ is Cohen-Macaulay if and only if $A/I_1\cdots I_r$ is Cohen-Macaulay*

With the preceeding techniques one can handle the case of analytic spread three, too. For details we refer again to [5].

Theorem 3.10. *Let A be a Cohen-Macaulay local ring of dimension d, and let $I,J \subset A$ be ideals of positive height such that $\ell(IJ)=3$. Let $\mathfrak{M}$ and $\mathfrak{N}$ denote the homogeneous maximal ideals of $R_A(I)$ and $R_A(J)$ respectively. Then $R_A(I,J)$ is Cohen-Macaulay if and only if the following conditions are satisfied*
1) $R_A(IJ)$ is Cohen-Macaulay;
2) $I^2J^2 : I = IJ^2$ and $I^2J^2 : J = I^2J$;
3) $[H^i_{\mathfrak{M}}(JR_A(I))]_0 = 0$ and $[H^i_{\mathfrak{N}}(IR_A(J))]_0 = 0$ for all $i < d+1$.

We also have the following result concerning rational singularities of $R_A(I,J)$:

Theorem 3.11. *Let A be an excellent local ring of equicharacteristic zero. Let $I,J \subset A$ be ideals of positive height such that $\ell(IJ)=3$. If $R_A(I)$, $R_A(J)$ and $R_A(IJ)$ have rational singularities, then so does $R_A(I,J)$.*

Outline of the proof. Since $Z \cong \operatorname{Proj} R_A(IJ)$ has rational singularities, it is enough to show that $H^i(Z,\mathcal{O}_Z(p,q)) = 0$ for all $i>0$ and $p,q \geq 0$. Indeed, then also $W = \operatorname{Proj} R_S(S^+) = \mathbb{V}(\mathcal{O}_Z(1,0)\oplus\mathcal{O}_Z(0,1))$ has rational singularities. But $H^i(W,\mathcal{O}_W) = \bigoplus_{p,q\geq 0} H^i(Z,\mathcal{O}_Z(p,q)))$. Since $R_A(IJ)$ has rational singularities, we have $H^i(Z,\mathcal{O}_Z(n,n)) = 0$ for all $i>0$ and $n \geq 0$ (s. [8, Proposition 2.3]). By Lemma 3.5 we now get $H^i(Z,\mathcal{O}_Z(p,q)) = 0$ for all $i>0$ and $p,q \geq 2-i$. It remains to prove that $H^1(Z,\mathcal{O}_Z(p,0)) = 0$ and $H^1(Z,\mathcal{O}_Z(0,q)) = 0$ for all $p,q \geq 0$. But this is clear, since $R_A(I)$ and $R_A(J)$ have rational singularities. □

When the product IJ has a small reduction number, we need not to assume anything about the analytic spread $\ell(IJ)$:

Theorem 3.12. *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of positive dimension, and let $I,J \subset A$ be $\mathfrak{m}$-primary ideals such that $r(IJ) \leq 1$. Then $R_A(I,J)$ is Cohen-Macaulay if and only if $IJ : I = J$ and $IJ : J = I$.*

Example 3.2. Let k be a field. Let $A = k[x,y,z]_{(x,y,z)}$ where x,y,z are variables. Take $I = (x^2,y,z)A$ and $J = \mathfrak{m}$, where $\mathfrak{m} = (x,y,z)$ is the maximal ideal of A. Then I and J are normal ideals. Moreover, $IJ = (x^3,y^2,z^2,xy,xz,yz)$ has a minimal reduction $(x^3+y^2+z^2, xy, xz+yz)$, and one checks that $r(IJ)=1$. Therefore $R_A(I,J)$ is Cohen-Macaulay. One easily sees here that $R_A(J)/IR_A(J) = k[U,V,W]/(x^2,xV,xW)$ where U,V and W be variables. It is then not difficult to check that if $\mathfrak{N}$ denotes the homogeneous maximal ideal of $R_A(J)$, then $[H^1_{\mathfrak{M}}(R_A(J)/IR_A(J))]_n \neq 0$ for all $n<0$.

4. An appication to joint reductions

Joint reductions were introduced by D. Rees. Let us first recall the definition from [10, p. 218]. Let $\mathbf{q} \in \mathbb{N}^r$. A set $\{a_{i,j} \in I_i \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ is called a joint reduction of $I_1, \ldots, I_r$ of type $\mathbf{q}$ if

$$I_1^{n_1} \cdots I_r^{n_r} = \sum_{i=1}^{r} (a_{i,1}, \ldots, a_{i,q_i}) I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_r^{n_r}$$

for all $\mathbf{n} \gg \mathbf{0}$. It is known that if A has an infinite residue field, then joint reductions always exist when $|\mathbf{q}| \geq \ell(I_1 \cdots I_r)$

Theorem 4.1. *Let A be a local ring, and let $I_1, \ldots, I_r$ be ideals of positive height. Let $\{a_{i,j} \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ be a joint reduction of $I_1, \ldots, I_r$ of type $\mathbf{q}$. If $R_A(I_1, \ldots, I_r)$ is Cohen-Macaulay, then*

$$I_1^{n_1} \cdots I_r^{n_r} = \sum_{i=1}^{r} (a_{i,1}, \ldots, a_{i,q_i}) I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_r^{n_r}$$

for all $\mathbf{n} \geq \mathbf{q}$.

Outline of the proof. Since

$$S_{\mathbf{n}} = (a_{1,1}t_1, \ldots, a_{1,q_1}t_1) S_{\mathbf{n}-\mathbf{1}_1} + \cdots + (a_{r,1}t_r, \ldots, a_{r,q_r}t_r) S_{\mathbf{n}-\mathbf{1}_r}$$

for $\mathbf{n} \gg \mathbf{0}$, we observe that the elements $a_{i,j}t_i$ induce an epimorphism $\sigma \colon \mathcal{F} \longrightarrow \mathcal{O}_Z$ where $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}_Z(-\mathbf{1}_i)^{\oplus q_i}$. It follows that the corresponding Koszul complex

$$K(\mathcal{F}, \sigma) \colon \quad \wedge^{|\mathbf{q}|} \mathcal{F} \longrightarrow \ldots \longrightarrow \wedge^j \mathcal{F} \longrightarrow \wedge^{j-1} \mathcal{F} \longrightarrow \ldots \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z$$

is exact. By Theorem 3.4 $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and and $\mathbf{n} \geq \mathbf{0}$. Moreover, we have $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = S_{\mathbf{n}}$ for all $\mathbf{n} \geq \mathbf{0}$. Chasing the complex $K(\mathcal{F}, \sigma) \otimes \mathcal{O}_Z(\mathbf{n})$ now yields an epimorphism $\Gamma(Z, \mathcal{F}(\mathbf{n})) \longrightarrow \Gamma(Z, \mathcal{O}(\mathbf{n}))$, which implies the claim. □

Note that in the case $r = 1$ this recovers the well-known result of Johnston and Katz in [9] saying that the Cohen-Macaulayness of a Rees algebra $R_A(I)$ implies that the reduction number $r_J(I) \leq \ell(I) - 1$ for any $\ell(I)$-generated reduction $J \subset I$.

5. The Gorenstein property of a multi-Rees algebra

Proposition 5.1. *Suppose that A is a homomorphic image of a Gorenstein local ring. Then the canonical module*

$$\omega_S = \bigoplus_{\mathbf{n} \geq \mathbf{1}} \Gamma(Z, \omega_Z(\mathbf{n})).$$

Outline of the proof. Recall first from [3, Lemma 2.1] that $[H_{\mathfrak{M}}^{d+r}(S)]_{\mathbf{n}} = 0$ if some $n_j \geq 0$. By Remark 3.1 we then get $H_{\mathfrak{M}}^{d+r}(S) = \bigoplus_{\mathbf{n} < \mathbf{0}} H_E^d(Z, \mathcal{O}_Z(\mathbf{n}))$. The claim now follows by taking Matlis-duals, because by the local–global duality of Lipman ([11, p. 188]) $H_E^d(Z, \mathcal{O}_Z(\mathbf{n})) = \operatorname{Hom}_A(\Gamma(Z, \omega_Z(-\mathbf{n})), E_A(k))$. □

Theorem 5.2. *Let A be a local ring, and let $I_1, \ldots, I_r \subset A$ be ideals of positive height. If $R_A(I_1, \ldots, I_r)$ is Cohen-Macaulay, then $R_A(I_1, \ldots, I_r)$ is Gorenstein if and only if $R_A(I_1 \cdots I_r)$ is Gorenstein.*

Outline of the proof. We may assume that A is complete. Set $R = R_A(I_1 \cdots I_r)$. Note that by Proposition 5.1 $\omega_R = (\omega_S)^{\Delta}$. In particular, when $\omega_S = S(-\mathbf{1})$, this means that $\omega_R = R(-1)$. The Gorensteiness of S therefore implies that of R. Conversely, suppose that $\omega_R = R(-1)$. Then, for any $\mathbf{n} \geq \mathbf{1}$,

$$\begin{aligned}\Gamma(Z, \omega_Z(\mathbf{n})) &= \operatorname{Hom}_A(I_1^{n-n_1} \cdots I_r^{n-n_r}, \Gamma(Z, \omega_Z(n, \dots, n))) \text{ (cp. [4, Theorem 2.2.])}\\ &= \operatorname{Hom}_A(I_1^{n-n_1} \cdots I_r^{n-n_r}, \Gamma(Z, \mathcal{O}_Z(n-1, \dots, n-1)))\\ &= \Gamma(Z, \mathcal{O}_Z(\mathbf{n}-\mathbf{1}))\end{aligned}$$

where $n = \max\{n_1, \dots, n_r\}$. Because $\Gamma(Z, \mathcal{O}_Z(\mathbf{n}-\mathbf{1})) = S_{\mathbf{n}-\mathbf{1}}$ by Theorem 3.4, the claim follows.

□

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THE COEFFICIENT IDEAL AND THE CANONICAL MODULE OF A REES ALGEBRA

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1. Introduction

Let $(A, \mathfrak{m})$ be a local ring of dimension d having an infinite residue field, and let $I \subset A$ be an ideal of positive height h and analytic spread ℓ. Let $J \subset I$ an ideal. The coefficient ideal of I relative to J, denoted by $\mathfrak{a}(I, J)$, is by definition the largest ideal $\mathfrak{a} \subset A$ such that $I\mathfrak{a} = J\mathfrak{a}$. It is easy to see that the coefficient ideal always exists. Also note that if $\mathfrak{a}(I, J)$ contains a non zero-divisor, then an application of the 'determinant trick' shows that necessarily $\overline{I} = \overline{J}$ i.e. J is a reduction of I. This notion was introduced by Aberbach and Huneke in their article [1] where they proved a new Briançon-Skoda type theorem. Their theorem, called a Briançon-Skoda theorem with coefficients, states that if $(A, \mathfrak{m})$ is a regular local ring containing a field and $I \subset A$ is an $\mathfrak{m}$-primary ideal, then $\overline{I^{n+d-1}} \subset \mathfrak{a}(I, J)J^n$ ([1, Theorem 2.7]) for all minimal reductions $J \subset I$ and all integers $n \geq 0$. In their article Aberbach and Huneke raise the question whether the coefficient ideal $\mathfrak{a}(I, J)$ might be independent of the minimal reduction $J \subset I$. It is shown in [5, Theorem 3.4] that the answer is positive if the Rees algebra $R_A(I)$ is Cohen-Macaulay. It also comes out that the coefficient ideal is closely related to the canonical module $\omega_{R_A(I)}$. The purpose of the present lecture is to outline the proof of this result and discuss some of its corollaries (see Theorem 3.1 and Corollaries 3.2 and 3.3 below). For more details, we refer to [5].

Our approach has been influenced by the article [11] of Lipman. Assuming that A is a regular local ring essentially of finite type over a field of characteristic zero, he showed that $\overline{I^{n+\ell-1}} \subset \operatorname{adj}(I^{\ell-1})J^n$ where $\operatorname{adj}(I^{\ell-1})$ denotes the adjoint ideal of $I^{\ell-1}$ (see [11, p. 745 and p. 747]). The adjoint $\operatorname{adj}(I^{\ell-1})$ can be defined as $\Gamma(Y, I^{\ell-1}\omega_Y)$ where ω_Y denotes the canonical sheaf of Y and $Y \longrightarrow \operatorname{Spec} A$ is any desingularization such that $I\mathcal{O}_Y$ is invertible. The A-module $\Gamma(Y, I^{\ell-1}\omega_Y)$ is considered here as an ideal of A by means of the trace homomorphism $\Gamma(Y, \omega_Y) \longrightarrow \omega_A = A$. One has $\operatorname{adj}(I^{\ell-1}) \subset \mathfrak{a}(I, J)$. Moreover, when $d = 2$ and I is an integrally closed $\mathfrak{m}$-primary ideal, Lipman proved in [11, Proposition 3.3] that $\mathfrak{a}(I, J) = \operatorname{adj}(I) = J : I$. Note in particular that in this case the Rees algebra $R_A(I)$ is always known to have rational singularities (use [9, Proposition 1.2] and [6, Proposition 2.1]).

In the following all rings and schemes are assumed to be Noetherian. We set $R = R_A(I)$ and $X = \operatorname{Proj} R_A(I)$. The maximal homogeneous ideal of R is denoted by $\mathfrak{M}$.

2. Preliminaries on the canonical module of a Rees algebra.

The purpose of this section is to recall some known properties of the canonical module of a Rees algebra needed in the sequel. This topic has been investigated by several authors (see e.g. [4], [2],[7] and [14]). Here we want to emphasize in particular the connection of the canonical module to the canonical sheaf of the corresponding projective scheme.

Suppose from now on that A is a homomorphic image of a Gorenstein local ring. By definition the canonical sheaf $\omega_X = H^{-d}(\mathcal{R}^\bullet_X)$ where $\mathcal{R}^\bullet_X$ is the dualizing complex of X. We have $\mathcal{R}^\bullet_X = f^!(\widetilde{R^\bullet_A})$ where $f\colon X \longrightarrow \operatorname{Spec} A$ is the canonical projection and $R^\bullet_A$ is a normalized dualizing complex of A. The canonical sheaf ω_X is defined up to an isomorphism. By the general theory of duality we get a trace morphism $\mathbf{R}\Gamma(X, \mathcal{R}^\bullet_X) \longrightarrow R^\bullet_A$ (see [3, Chapter VII, Corollary 3.4]). Taking cohomology gives a trace homomorphism $\Gamma(X, \omega_X) \longrightarrow \omega_A$. This is injective. In the following we shall always consider $\Gamma(X, \omega_X)$ as a submodule of ω_A by means of this homomorphism. The trace homomorphism corresponds to the canonical homomorphism $H^d_{\mathfrak{m}}(A) \longrightarrow H^d_E(X, \mathcal{O}_X)$ where $E = X \times_A A/\mathfrak{m}$, via the isomorphisms

$$\operatorname{Hom}_A(\Gamma(X, \omega_X), E_A(k)) = H^d_E(X, \mathcal{O}_X) \quad \text{and} \quad \operatorname{Hom}_A(\omega_A, E_A(k)) = H^d_{\mathfrak{m}}(A)$$

(see [12, the proof of Lemma 4.2] or [13, p. 110]). The canonical sheaf equals to the associated sheaf of the canonical module ω_R. However, even the following is true:

Proposition 2.1. *As a graded R-module* $\omega_R = \bigoplus_{n \geq 1} \Gamma(X, I^n \omega_X)$.

Outline of the proof. Consider the Sancho de Salas sequence

$$\ldots \longrightarrow H^d_{\mathfrak{m}}(R_n) \longrightarrow H^d_E(X, \mathcal{O}_X(n)) \longrightarrow [H^{d+1}_{\mathfrak{M}}(R)]_n \longrightarrow 0.$$

This implies that $[H^{d+1}_{\mathfrak{M}}(R)]_n = H^d_E(X, \mathcal{O}_X(n))$ for all $n < 0$. Taking the Matlis-duals and using the isomorphism $H^d_E(X, \mathcal{O}_X(n)) = \operatorname{Hom}_A(\Gamma(X, \omega_X(-n)), E_A(k))$, now gives $[\omega_R]_n = \Gamma(X, \omega_X(n)) = \Gamma(X, I^n \omega_X))$ for all $n > 0$. Because $[\omega_R]_n = 0$ for $n \leq 0$ by [2, Part I, 6.3], the claim follows. □

Remark 2.1. We list here some facts which will be used later (for the proofs, see [5]):

1) $\Gamma(X, \omega_X) \supset \Gamma(X, I\omega_X) \supset \Gamma(X, I^2\omega_X) \supset \ldots$;

2) $\Gamma(X, I^n\omega_X) = \operatorname{Hom}_A(I, \Gamma(X, I^{n+1}\omega_X))$ for all $n \geq 0$;

3) Set $G = G_A(I)$. Recall that the a-invariant $a(G) = \sup\{n \in \mathbb{Z} \mid [H^d_{\mathfrak{M}}(S)]_n \neq 0\}$. When A is Cohen-Macaulay, there exists a monomorphism

$$0 \longrightarrow \bigoplus_{n \geq 1} \Gamma(X, I^{n-1}\omega_X)/\Gamma(X, I^n\omega_X) \longrightarrow \omega_G$$

which is an isomorphism if R is Cohen-Macaulay. In the case $a(G) < 0$, this gives

$$\omega_A = \Gamma(X, \omega_X) = \Gamma(X, I\omega_X) = \cdots = \Gamma(X, I^{-a(G)-1}\omega_X)$$

so that $I^{n+a(G)+1}\omega_A \subset \Gamma(X, I^n\omega_X)$ for $n \geq 0$.

Let $J \subset I$ be any reduction, and let $r \geq 0$ be any integer such that $I^{r+1} = JI^r$. Set $Y = \operatorname{Proj} R_A(J)$. We now have a finite morphism $\pi\colon X \longrightarrow Y$, which can be utilized to express ω_X in terms of ω_Y. Indeed, given a canonical sheaf ω_X, we know that there exists an isomorphism $\pi_*\omega_X \cong \mathcal{H}om_Y(\pi_*\mathcal{O}_X, \omega_Y)$. Note that $\pi_*\mathcal{O}_X = I^r\mathcal{O}_Y(-r)$. Taking into account the commutative diagram

$$\begin{array}{ccc} \Gamma(X,\omega_X) & \xrightarrow{\text{trace}} & \omega_A \\ \wr\downarrow & & \uparrow\text{trace} \\ \mathrm{Hom}_Y(\pi_*\mathcal{O}_X,\omega_Y) & \longrightarrow & \mathrm{Hom}_Y(\mathcal{O}_Y,\omega_Y) = \Gamma(Y,\omega_Y) \end{array}$$

(see [3, Chapter III, Theorem 10.5]), one can now check that the following holds:

Proposition 2.2. *Let ω_X be a fixed canonical sheaf of X. Then*

$$\Gamma(X, I^n\omega_X) = \Gamma(Y, J^{n+r}\omega_Y) :_{\omega_A} I^r$$

for all $n \geq 0$ as submodules of ω_A.

Finally, we need information about the generating degrees of ω_R. It is then important to observe that from a certain degree on elements of ω_R can be expressed as linear combinations of the generators in such a way that the coefficients lie in the subring $R_A(J) \subset R$. This is based on a version of the Castelnuovo-Mumford lemma. Its proof is analogous to that of [13, Lemma 5.1].

Lemma 2.3. *Let Y be a scheme, and let $\mathcal{L}$ be an invertible sheaf on Y generated by finitely many global sections $s_1, \ldots, s_\ell \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a coherent sheaf on Y. Let $m \in \mathbb{Z}$. If $H^i(Y, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$ for all $i > 0$, then the induced homomorphism $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)^{\oplus \ell} \longrightarrow \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{n+1})$ is surjective for $n \geq m$.*

Proposition 2.4. *Suppose that R is Cohen-Macaulay. If* $\operatorname{depth} A/I^n \geq d - m - n$ *for $n = 1, \ldots, \ell - m - 1$, then $\Gamma(X, I^{n+1}\omega_X) = J\Gamma(X, I^n\omega_X)$ for all reductions $J \subset I$ and all integers $n \geq m$.*

Outline of the proof. Observe first that we have $J\mathcal{O}_X = I\mathcal{O}_X$. Generators of J therefore determine global sections which generate $I\mathcal{O}_Y$. By Lemma 2.3 we then only need to show that $H^i(X, \omega_X(m-i)) = 0$ for $i > 0$. Recall that by the local-global duality of Lipman ([10, p. 188])

$$\mathrm{Hom}_A(H^i(X, \omega_X(m-i)), E_A(k)) = H_E^{d-i}(X, \mathcal{O}_X(i-m))$$

where $E = X \times_A A/\mathfrak{m}$. The proof can now be completed using the the Sancho de Salas sequence

$$\ldots \longrightarrow [H_{\mathfrak{M}}^{d-i}(R)]_{i-m} \longrightarrow H_{\mathfrak{m}}^{d-i}(R_{i-m}) \longrightarrow H_E^{d-i}(X, \mathcal{O}_X(i-m)) \longrightarrow \ldots$$

(see [12, p. 150]). □

3. The main result

Let A be a local ring. Recall that an ideal $I \subset A$ is said to satisfy the condition G_k if $\mu(I_\mathfrak{p}) \leq \operatorname{ht}\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ with $\operatorname{ht}\mathfrak{p} \leq k-1$.

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d. Let $I \subset A$ be an ideal with positive height h and analytic spread ℓ. Suppose that I satisfies G_ℓ and that* $\operatorname{depth} A/I^n \geq d - h - n + 1$ *for $n = 1, \ldots, \ell - h$. If R is Cohen-Macaulay, then $\mathfrak{a}(I, J) = \Gamma(X, I^{h-1}\omega_X)$ as ideals of A for all minimal reductions $J \subset I$. In particular, $\mathfrak{a}(I, J)$ is independent of the minimal reduction J. Moreover, $\mathfrak{a}(I, J) = J^r : I^r$ where $r = r_J(I)$. In the case $r \leq \ell - h + 1$, we even have $\mathfrak{a}(I, J) = J : I = \cdots = JI^{r-1} : I^r$.*

Outline of the proof. We sketch here the proof of the first part of the theorem. It follows from Proposition 2.4 that $\Gamma(X, I^h\omega_X) = J\Gamma(X, I^{h-1}\omega_X) = I\Gamma(X, I^{h-1}\omega_X)$. Thus $\Gamma(X, I^{h-1}\omega_X) \subset \mathfrak{a}(I, J)$. It remains to prove that $\mathfrak{a}(I, J) \subset \Gamma(X, I^{h-1}\omega_X)$. Since $I\mathfrak{a}(I, J) = J\mathfrak{a}(I, J)$, we also have $I^r\mathfrak{a}(I, J) = J^r\mathfrak{a}(I, J)$. So $\mathfrak{a}(I, J) \subset J^r : I^r$. Because of [8, Theorem 2.2 and Remark 2.7] we may apply [8, Lemma 2.3 b)] to find a generating sequence $a_1, \ldots, a_\ell$ of J which is a d-sequence. According to [15, p. 33] this implies that $a(G_A(J)) \leq -h$. Set $Y = \operatorname{Proj} R_A(J)$. We now obtain from Remark 2.1 3) that $J^r \subset \Gamma(Y, J^{h-1+r}\omega_Y)$. But then Proposition 2.2 implies that $\mathfrak{a}(I, J) \subset J^r : I^r \subset \Gamma(Y, J^{h-1+r}\omega_Y) : I^r = \Gamma(X, I^{h-1}\omega_X)$. □

We now show that the canonical module is completely determined by the coefficient ideal.

Corollary 3.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d. Let $I \subset A$ be an ideal with positive height h and analytic spread ℓ. Suppose that I satisfies G_ℓ and that* $\operatorname{depth} A/I^n \geq d - h - n + 1$ *for $n = 1, \ldots, \ell - h$. If R is Cohen-Macaulay, then*

$$\omega_R = \bigoplus_{n=1}^{h-2} (\mathfrak{a}(I, J) : I^{h-1-n}) \oplus \bigoplus_{n \geq \max\{h-1, 1\}} I^{n-h+1}\mathfrak{a}(I, J).$$

Outline of the proof. By Theorem 3.1 we have $\Gamma(X, I^{h-1}\omega_X) = \mathfrak{a}(I, J)$. Therefore Proposition 2.4 implies that $\Gamma(X, I^n\omega_X) = I^{n-h+1}\mathfrak{a}(I, J)$ for all $n \geq h-1$. On the other hand, according to Remark 2.1 2) we have $\Gamma(X, I^n\omega_X) = \mathfrak{a}(I, J) : I^{h-1-n}$ for $n < h - 1$. The claim then follows from Proposition 2.1.

□

Let A be a Gorenstein local domain essentially of finite type over a field of characteristic zero. Given an ideal $I \subset A$, the adjoint ideal of I, denoted by $\operatorname{adj}(I)$, is defined as $\operatorname{adj}(I) = \Gamma(Y, I\omega_Y)$ where $Y \longrightarrow \operatorname{Spec} A$ is any proper birational morphism such that Y has rational singularities and that $I\mathcal{O}_Y$ is invertible. When A is regular, this definition coincides with the one given in [11, p. 140] (see [11, Proposition 1.3.1]). The result of Lipman mentioned in the introduction ([11, Proposition 3.3]) can now be generalized as follows:

Corollary 3.3. *Let $(A, \mathfrak{m})$ be a Gorenstein local domain essentially of finite type over a field of characteristic zero and let $I \subset A$ be an equimultiple ideal of height h. If R is normal and Cohen-Macaulay, then R has rational singularities if and only if $\mathfrak{a}(I, J) = \operatorname{adj}(I^{h-1})$ for some (and then also for all) minimal reductions $J \subset I$.*

Outline of the proof. Since R is normal and Cohen-Macaulay, it has rational singularities if and only if X has rational singularities (see e.g. [6, Proposition 2.1]). Let $f : Y \longrightarrow X$ be a desingularization. By definition X has rational singularities if and only if $f_*\omega_Y = \omega_X$. Since $\Gamma(X, f_*\omega_Y(n)) = \Gamma(Y, I^n\omega_Y)$ for all $n \geq 0$, this is the case if and only if $\operatorname{adj}(I^n) = \Gamma(X, I^n\omega_X)$ for $n \gg 0$. Observe here that the trace homomorphism $f_*\omega_Y \longrightarrow \omega_X$ induces an inclusion $\operatorname{adj}(I^n) \subset \Gamma(X, I^n\omega_X)$. By Remark 2.1 2) we have $\operatorname{Hom}_A(I, \Gamma(X, I^n\omega_X)) = \Gamma(X, I^{n-1}\omega_X)$. Similarly, $\operatorname{Hom}_A(I, \Gamma(Y, I^n\omega_Y)) = \Gamma(Y, I^{n-1}\omega_Y)$. It follows that if R has rational singularities, then necessarily $\operatorname{adj}(I^n) = \Gamma(X, I^n\omega_X)$ for all $n \geq 0$. By Theorem 3.1 we then get $\mathfrak{a}(I, J) = \operatorname{adj}(I^{h-1})$ for any minimal reduction $J \subset I$. Conversely, suppose that $\mathfrak{a}(I, J) = \operatorname{adj}(I^{h-1})$ for some minimal reduction $J \subset I$. By [11, p. 745 and p. 747] we know that $\operatorname{adj}(I^n) = I^{n-h+1}\operatorname{adj}(I^{h-1})$ for $n \geq h-1$. On the other hand, using Proposition 2.4 we have $\Gamma(X, I^n\omega_X) = I^{n-h+1}\mathfrak{a}(I,J)$ for $n \geq h-1$. Hence $\Gamma(X, I^n\omega_X) = \operatorname{adj}(I^n)$ for $n \geq h-1$ which means that X and so also R have rational singularities. □

In this context we also want to recall the following result (see [6, Theorem 3.2])

Theorem 3.4. *Let A be a regular local ring of dimension d essentially of finite type over a field of characteristic zero. Let $I \subset A$ be a normal $\mathfrak{m}$-primary ideal such that $r(I) \leq 1$. Then $R_A(I)$ has rational singularities if and only if* $\operatorname{adj} I^{d-2} = A$.

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Good ideals in idealizations

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この研究は後藤四郎先生の指導の下に行われたものである. Gorenstein 局所環 $(A, \mathfrak{m})$ 内の ideal I は, ある非負整数 n に対して等式 $I^{n+1} = QI^n$ を満たすような parameter ideal Q を含む $\mathfrak{m}$-primary ideal であると仮定する. Ideal I の随伴次数環 $\oplus_{n\geq 0} I^n/I^{n+1}$ が Gorentein 環であってその a-不変量が $1 - \dim A$ であるという条件をみたす時, この ideal I のことを A 内の good ideal であると呼ぶ. 目下論文 [GIW] に於て, good ideals の基礎理論が展開されつつあるが, その一部と, [GIW] 内に報告出来なかった具体例の計算の詳細はここに報告して, 記録に留めたいと思う. さて, 環 A の次元が 0 であることを仮定すれば, A の ideal I が good であるということは, 次の条件で置き換えられることが一般に知られている.

Proposition. $\dim A = 0$ であると仮定せよ. 次の 3 条件は互いに同値である.

(1) $I \in X_A$.

(2) $I = (0) \underset{A}{:} I$.

(3) $I^2 = (0)$ であって, かつ $\ell_A(I) = \frac{1}{2}\ell_A(A)$ が成立する.

ここで私は $(R, \mathfrak{n})$ を Artin 局所環とし, $E = \mathrm{E}_R(R/\mathfrak{n})$ を R の剰余体 $R/\mathfrak{n}$ の入射内包とした時, E の R 上でのイデアル化として得られる 0 次元 Gorenstein 局所環 $A = R \ltimes E$ に対し, 環 A 内の good ideals の構造解析を実行してみたいと思う. 例えば ideal $(0) \times E \subset A$ を見るに, 上の補題により, この ideal $(0) \times E$ が A 内の good ideal であることが確かめられる. より一般に, ideal $I \subset A$ について, I が等式 $I \cap R = (0)$ をみたす A 内の good ideal であることと, ある R-代数としての A の自己同型射 ξ を取り $I = \xi((0) \times E)$ と表される

ことが同値である, ということが知られている. そこで, 私は A 内の good ideals について, 次の様な予想を立て, この研究を開始した. その予想とは, A 内の good ideals の集合 X_A について等式

$$X_A = \left\{ \xi(\mathfrak{a} \times L) \,\middle|\, \mathfrak{a} \subset R:\ ideal,\ \mathfrak{a}^2 = (0),\ L = (0) \underset{E}{:} \mathfrak{a},\ \xi \in \mathrm{Aut}_{R-alg}\, A \right\}$$

が成立しているのではないか, ということである.

しかし, 実際にはこの様に表すことが出来ない good ideals も数多く存在することが確かめられた. その議論に入る前に, まず, 一般的な A 内の good ideal の構造定理を述べておく.

Theorem. I を環 A の *ideal* とすれば, I が A 内の *good ideal* であることと, 以下に述べる 4 条件を満たす, 等式 $\mathfrak{a}^2 = (0)$ が成立する R のイデアル $\mathfrak{a}$ と $R/\mathfrak{a}$-線形写像 $h : (0) \underset{E}{:} \mathfrak{a} \to R/\mathfrak{a}$ の組 $(\mathfrak{a}, h)$ が存在することは, 互いに同値である. その条件とは

(i) 任意の $x, y \in L$ に対し等式 $h(x)h(y) = 0, h(x)y + h(y)x = 0$ が成り立つ.

(ii) $a, b \in R$ が $\overline{a}, \overline{b} \in h(L)$ をみたすならば, $ab = 0$ である.

(iii) $a \in R$ が $\overline{a} \in h(L)$ をみたすなら, 任意の $x \in E$ に対して $ax \in L$ であって更に $h(ax) = 0$ が成り立つ.

(iv) $I = \left\{ (a, x) \,\middle|\, a \in R,\ x \in L\ s/t\ \overline{a} = h(x) \right\}$.

である. ここで, $a \in R$ に対して $\overline{a}$ は mod $\mathfrak{a}$ での *reduction* (a mod $\mathfrak{a}$) を表す. なおこのとき, このような組 $(\mathfrak{a}, h)$ はイデアル I に対して唯一通りに定まり, 特に $\mathfrak{a} = I \cap R$ となる.

今, Artin 局所環 R 内で $\mathfrak{a}^2 = (0)$ である様な ideal $\mathfrak{a}$ を取り, 写像 h として零写像を取れば, これらの組 $(\mathfrak{a}, h)$ は定理の条件 (i),(ii),(iii) をみたし, 条件 (iv) の如く環 A 内の good ideal I を求めると, 実は $I = \mathfrak{a} \times L$ となるので, ideal $\mathfrak{a} \times L$ は A の good ideal であり, 従って, 任意の $\xi \in \mathrm{Aut}_{R-alg}\, A$ に対して $\xi(\mathfrak{a} \times L)$ は A 内の good ideal である.

この定理を用いて, 以下に定める Artin 局所環 $(R, \mathfrak{n})$ に対して入射内包 $E = \mathrm{E}_R(R/\mathfrak{n})$ を求め, 次元 0 の Gorenstetin 局所環 $A = R \ltimes E$ に対して, 環 A 内の good ideals を全て決定しようと思う. 即ち, k を体とし, 整数 $n \geq 3$ として

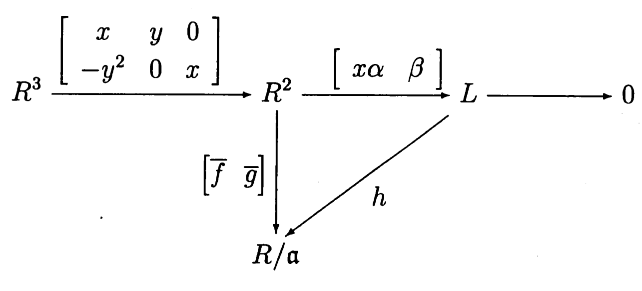
$$R = \frac{k[X, Y]}{(X^n, Y^n, XY)}$$

とする. $x = X \mod (X^n, Y^n, XY)$, $y = Y \mod (X^n, Y^n, XY)$ とおく. $\alpha = \widehat{x^{n-1}}, \beta = \widehat{y^{n-1}}$ を環 R の基底 $\{1, x^i, y^j | 1 \leq i, j \leq n-1\}$ の双対基底とすれば $E = R\alpha + R\beta$ であることがわかる.

定理によれば, 環 A 内の good ideal を探すには, まず, 等式 $\mathfrak{a}^2 = (0)$ をみたす R の ideal $\mathfrak{a}$ を探すことが必要となる. その様な R の ideal $\mathfrak{a}$ と, その ideal $\mathfrak{a}$ に対して $L := (0) \underset{E}{:} \mathfrak{a}$ を全て求めると以下の如く書き出すことが出来る.

$\mathfrak{a}$	(x^{n-r})	(y^{n-s})	(x^{n-r}, y^{n-s})	$(x^{n-r} + \lambda y^{n-s})$
L	$Rx^r\alpha + R\beta$	$R\alpha + Ry^s\beta$	$Rx^r\alpha + Ry^s\beta$	$R\cdot(y^{s-1}\beta - \lambda x^{r-1}\alpha)$

但し, $0 \leq r, s \leq \frac{n}{2}$, $0 \neq \lambda \in k$ とする. この時, 対応する $R/\mathfrak{a}$-線型写像 $h : L \to R/\mathfrak{a}$ を求めるには次の様なことを行えばよい. 例えば, 簡単な場合で $n = 3$, $\mathfrak{a} = (x^2)$ とする. すると $L = Rx\alpha + R\beta$ であって, この L は次の表現を持つ.



故に, 写像 $h : L \to R/\mathfrak{a}$ を求めることと, 等式 $\begin{bmatrix}\overline{f} & \overline{g}\end{bmatrix} \begin{bmatrix} x & y & 0 \\ -y^2 & 0 & x \end{bmatrix} = [0\ 0\ 0]$ をみたす写像 $\begin{bmatrix}\overline{f} & \overline{g}\end{bmatrix} : R^2 \to R/\mathfrak{a}$ $(f, g \in R)$ を求めることは同値であり, 写像の合成が $[0\ 0\ 0]$ であるという条件を f と g によって容易に表すことができる. 更に, 写像 h が定理の残りの条件を満たす条件を f と g の言葉で書き出すことができる. この例において, 実際に写像 h を求めると, 元 $h \in \mathrm{Hom}_{R/\mathfrak{a}}(L, R/\mathfrak{a})$ は char $k \neq 2$ ならば, $h = 0$ のみであって, もし char $k = 2$ ならば $c \in k$ を取り $h(x\alpha) = 0$, $h(\beta) = \overline{cy^2}$ となる.

この様にして, 環 R の ideal $\mathfrak{a}$ に対して, 写像 h を求め, 環 $A = R \ltimes E$ 内の good ideals を一覧表にすれば, 下記の結果が得られる. まず char $k \neq 2$ のときは

$\mathfrak{a}$	$I \in X_A$ で $\mathfrak{a} = I \cap R$ を満たす
(x^{n-r}) $n \geq 2r+2$	$R{\cdot}(ay^{n-1}, x^r\alpha) + R{\cdot}(-ax^{n-r-1}, \beta) + \mathfrak{a}\times(0)$ $(a \in k)$
(x^{n-r}) $n < 2r+2$	$(x^{n-r}) \times (Rx^r\alpha + R\beta)$
(y^{n-s}) $n \geq 2s+2$	$R{\cdot}(-ay^{n-s-1}, \alpha) + R{\cdot}(ax^{n-1}, y^s\beta) + \mathfrak{a}\times(0)$ $(a \in k)$
(y^{n-s}) $n < 2s+2$	$(y^{n-s}) \times (R\alpha + Ry^s\beta)$
(x^{n-r}, y^{n-s}) $n \geq 2r+2$ *and* $n \geq 2s+2$	$R{\cdot}(ay^{n-s-1}, x^r\alpha) + R{\cdot}(-ax^{n-r-1}, y^s\beta) + \mathfrak{a}\times(0)$ $(a \in k)$
(x^{n-r}, y^{n-s}) $n < 2r+2$ *or* $n < 2s+2$	$(x^{n-r}, y^{n-s}) \times (Rx^r\alpha + Ry^s\beta)$
$(x^{n-r} + \lambda y^{n-s})$ $0 \neq \lambda \in k$	$(x^{n-r} + \lambda y^{n-s}) \times R{\cdot}(y^{s-1}\beta - \lambda x^{r-1}\alpha)$

であって, もし char $k=2$ であるならば

$\mathfrak{a}$	$I \in X_A$ で $\mathfrak{a} = I \cap R$ を満たす
(x^{n-r}) $n \geq 2r+2$	$R\cdot\left(ay^{n-1} + \sum_{\frac{n}{2}\leq i\leq n-r-1} b_i x^i, x^r\alpha\right)$ $+R\cdot\left(-ax^{n-r-1} + \sum_{\frac{n}{2}\leq i\leq n-1} c_i y^i, \beta\right) + \mathfrak{a}\times(0)$ $(a, b_i, c_i \in k)$
(x^{n-r}) $n < 2r+2$	$R\cdot(0, x^r\alpha) + R\cdot\left(\sum_{\frac{n}{2}\leq i\leq n-1} c_i y^i, \beta\right) + \mathfrak{a}\times(0)$ $(c_i \in k)$
(y^{n-s}) $n \geq 2r+2$	$R\cdot\left(ay^{n-s-1} + \sum_{\frac{n}{2}\leq i\leq n-1} b_i x^i, \alpha\right)$ $+R\cdot\left(-ax^{n-1} + \sum_{\frac{n}{2}\leq i\leq n-s-1} c_i y^i, y^s\beta\right) + \mathfrak{a}\times(0)$ $(a, b_i, c_i \in k)$
(y^{n-s}) $n < 2s+2$	$R\cdot\left(\sum_{\frac{n}{2}\leq i\leq n-1} b_i x^i, \alpha\right) + R\cdot(0, y^s\beta) + \mathfrak{a}\times(0)$ $(b_i \in k)$
(x^{n-r}, y^{n-s}) $n \geq 2r+2$ *and* $n \geq 2s+2$	$R\cdot\left(ay^{n-s-1} + \sum_{\frac{n}{2}\leq i\leq n-r-1} b_i x^i, x^r\alpha\right)$ $+R\cdot\left(-ax^{n-r-1} + \sum_{\frac{n}{2}\leq i\leq n-s-1} c_i y^i, y^s\beta\right) + \mathfrak{a}\times(0)$ $(a, b_i, c_i \in k)$
(x^{n-r}, y^{n-s}) *otherwise*	$R\cdot\left(\sum_{\frac{n}{2}\leq i\leq n-r-1} b_i x^i, x^r\alpha\right) + R\cdot\left(\sum_{\frac{n}{2}\leq i\leq n-s-1} c_i y^i, y^s\beta\right) + \mathfrak{a}\times(0)$ $(b_i, c_i \in k)$
$(x^{n-r} + \lambda y^{n-s})$ $0 \neq \lambda \in k$	$R\cdot\left(\sum_{\frac{n}{2}\leq i\leq n-r} a_i x^i + \sum_{\frac{n}{2}\leq i\leq n-s} b_i y^i, y^{s-1}\beta - \lambda x^{r-1}\alpha\right) + \mathfrak{a}\times(0)$ $(a_i, b_i \in k)$

となる. 特に $n=3$, char $k \neq 2$ とすれば, 下の様な表になる.

$\mathfrak{a}$	$I \in X_A$ で $\mathfrak{a} = I \cap R$ を満たす
(0)	$R\cdot(ay^2, \alpha) + R\cdot(-ax^2, \beta)\ \ (a \in k)$
(x^2)	$(x^2) \times (Rx\alpha + R\beta)$
(y^2)	$(y^2) \times (R\alpha + Ry\beta)$
(x^2, y^2)	$(x^2, y^2) \times (Rx\alpha + Ry\beta)$
$(x^2 + \lambda y^2)$ $0 \neq \lambda \in k$	$(x^2 + \lambda y^2) \times R\cdot(\beta - \lambda\alpha)$

この仮定の下では, 対応する R の ideal $\mathfrak{a}$ が (0) でないならば, 環 A 内の good ideal は全て $\mathfrak{a} \times L$ という型をし, 一方で $\mathfrak{a} = (0)$ ならば, 環 A 内の good ideal $I = R\cdot(ay^2, \alpha) + R\cdot(-ax^2, \beta)$ $(a \in k)$ は, $\xi \in \mathrm{Aut}_{R-alg}\, A$ を

$$\xi(a, r_1\alpha + r_2\beta) := \left(a + r_1 ay^2 + r_2(-ax^2), r_1\alpha + r_2\beta\right)$$

と定めれば, 等式 $I = \xi\left((0) \times E\right)$ が成立する. 次に $n=3$, char $k = 2$ とすれば

$\mathfrak{a}$	$I \in X_A$ で $\mathfrak{a} = I \cap R$ を満たす	$\theta \in \mathcal{H}_E$
(0)	$R\cdot(ay^2 + bx^2, \alpha) + R\cdot(-ax^2 + cy^2, \beta)$ $(a, b, c \in k)$	$\theta(\alpha) = ay^2 + bx^2$ $\theta(\beta) = -ax^2 + cy^2$
(x^2)	$R\cdot(0, x\alpha) + R\cdot(cy^2, \beta) + (x^2)\times(0)$ $(c \in k)$	$\theta(\alpha) = 0$ $\theta(\beta) = cy^2$
(y^2)	$R\cdot(bx^2, \alpha) + R\cdot(0, y\beta) + (y^2)\times(0)$ $(b \in k)$	$\theta(\alpha) = bx^2$ $\theta(\beta) = 0$
(x^2, y^2)	$(x^2, y^2) \times (Rx\alpha + Ry\beta)$	
$(x^2 + \lambda y^2)$ $0 \neq \lambda \in k$	$R\cdot(x^2 + by^2, \beta - \lambda\alpha) + (x^2 + \lambda y^2)\times(0)$ $(b \in k)$	$\theta(\alpha) = (-\lambda)^{-1}x^2$ $\theta(\beta) = by^2$

この表で第3列目の集合 $\mathcal{H}_E$ は $\mathfrak{a} = (0)$ とした時の定理の条件 (i) を満たす $\mathrm{Hom}_R(E, R)$ の R-部分加群を表し, 任意の元 $\theta \in \mathcal{H}_E$ に対して $\xi_\theta \in \mathrm{Aut}_{R-alg}\, A$ を $\xi_\theta(a, z) := (a + \theta(z), z)$ と定めれば, 等式 $I = \xi_\theta(\mathfrak{a} \times L)$ が成り立つ. 例えば $\mathfrak{a} = (x^2)$ に注目してみよう. すると, 等式 $I \cap R = (x^2)$ を満たすような環 A 内の good ideal I は, 元 $c \in k$ を取って $I = R\cdot(0, x\alpha) + R\cdot(cy^2, \beta) + (x^2) \times (0)$ の形に表すことができる. ここで, 写像 θ を $\theta(\alpha) = 0$, $\theta(\beta) = cy^2$ によって定まる $\mathcal{H}_E$ の元とし, そして $\xi_\theta \in \mathrm{Aut}_{R-alg}\, A$ を取ると, 任意の元 $(a, z) \in A$ に対して $z = r_1\alpha + r_2\beta$ とおく時 $\xi_\theta(a, z) = (a + r_2(cy^2), z)$ であって, 更に

等式 $I = \xi_\theta(\mathfrak{a} \times L)$ が成立する.

従って $n = 3$ であれば, 体 k の標数に関係なく, 始めの予想が正しいことがわかる.

しかしながら $n = 4$, char $k = 2$ の仮定において $\mathfrak{a} = (x^3)$ に対応する A 内の good ideal I を

$$I = R{\cdot}(y^3 + x^2, x\alpha) + R{\cdot}(-x^2, \beta) + (x^3) \times (0)$$

と定めれば, 如何なる $\xi \in \mathrm{Aut}_{R\text{-}alg}\, A$ に対しても等式 $I = \xi(\mathfrak{a} \times L)$ が成立することは有り得ない. つまり, この good ideal I は私の当初の予想の反例の 1 つである.

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BUCHSBAUMNESS OF THE REES ALGEBRAS OF $\mathfrak{m}$-PRIMARY IDEALS WHOSE REDUCTION NUMBERS ARE AT MOST ONE

KIKUMICHI YAMAGISHI

1. INTRODUCTION

The aim of this note is to investigate the conditions for blowing-up rings, especially the Rees algebras $\mathrm{R}(\mathfrak{a}) := \bigoplus_{n\geq 0} \mathfrak{a}^n$ of $\mathfrak{m}$-primary ideals $\mathfrak{a}$, to be Buchsbaum. In particular, we shall mainly discuss the case where the reduction numbers of such $\mathfrak{m}$-primary ideals are at most one.

Throughout this note, let $(A, \mathfrak{m})$ be a Buchsbaum ring of dimension $d > 0$ and $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal of A. We always assume that the residue field $A/\mathfrak{m}$ is infinite.

In 1998, Y. Nakamura [N] gave us the equivalent condition for the associated graded rings of $\mathfrak{m}$-primary ideals, whose reduction numbers are at most one, to be Buchsbaum. Recently the author proved the same argument without any hypotheses about the reduction numbers of $\mathfrak{m}$-primary ideals [Ya].

Concerning the Buchsbaumness of Rees algebras, however, we have already known a quite few works. In 1985, J. Stückrad [St] proved that for any parameter ideal $\mathfrak{q}$ of A the Rees algebra $\mathrm{R}(\mathfrak{q})$ is Buchsbaum too. Moreover, in 1999, S. Goto [G2] brought us an epoch-making work. Namely, he introcduced a new notion, called an $\mathfrak{m}$-primary ideal of *minimal multiplicity*, and studied the structures of Rees algebras associated to such ideals of minimal multiplicity.

On the other hand, in January 1997, Nakamura (see Corollary 2 below) tackled with this theme and proved a partial generalization of the Stückrad's theorem in the case where the dimension of given Buchsbaum rings are very small.

Now we shall state our main result as follows.

Theorem 1. *Suppose that, for some minimal reduction $\mathfrak{q}$ of $\mathfrak{a}$, the following three conditions are satisfied:*

(i). $\mathfrak{a}^2 = \mathfrak{q}\mathfrak{a}$ *holds;*

(ii) *writing* $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$ *the equality*

$$(a_1^2, a_2^2, \ldots, a_d^2) \cap \mathfrak{a}^n = (a_1^2, a_2^2, \ldots, a_d^2)\mathfrak{a}^{n-2}$$

holds for $3 \leq n \leq d+1$; *and*

(iii) $\mathfrak{a}$ *has minimal multiplicity, i.e.,* $\mathfrak{m}\mathfrak{a} \subseteq \mathfrak{q}$ *holds.*

Then $\mathrm{R}(\mathfrak{a})_+ := \bigoplus_{n>0} \mathfrak{a}^n$ *is a Buchsbaum* $\mathrm{R}(\mathfrak{a})$*-module. Moreover, the Rees algebra* $\mathrm{R}(\mathfrak{a})$ *is a Buchsbaum ring if* $d \geq 2$.

We mention that our reslut is a generalization of the Stückrad's theorem, hence the Nakamura's result below.

Corollary 2 (Nakamura). *Suppose that, for some minimal reduction* $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$ *of* $\mathfrak{a}$*, the following four conditions are satisfied:*

(i) $\mathfrak{a}^2 = \mathfrak{q}\mathfrak{a}$ *holds;*
(ii) $\mathfrak{a} \supseteq \sum_{i=1}^{d} [(a_1, \ldots, \widehat{a_i}, \ldots, a_d) : a_i]$ *holds;*
(iii) $\mathfrak{a}$ *has minimal multiplicity; and*
(iv) $2 \leq d \leq 4$.

Then the Rees algebra $\mathrm{R}(\mathfrak{a})$ *is a Buchsbaum ring too.*

Further consequences of our Theorem 1 are stated as follows:

Corollary 3. *Let* A *be a Buchsbaum ring of maximal embedding dimension. Then the Rees algebra* $\mathrm{R}(\mathfrak{m})$ *is a Buchsbaum ring. Moreover* $\mathrm{R}(\mathfrak{m})_+$ *and* $\mathrm{R}(\mathfrak{m})_{\geq 2} := \bigoplus_{n \geq 2} \mathfrak{m}^n$ *are Buchsbaum modules over* $\mathrm{R}(\mathfrak{m})$ *too.*

Applying the similar arguments on Buchsbaum modules we also have the following corollary concerning *linear maximal* Buchsbaum modules, which were introduced by K. Yoshida [Yo]. Namely

Corollary 4. *Let* E *be a linear maximal Buchsbaum* A*-module. Then both* $\mathrm{R}_{\mathfrak{m}}(E)$ *and* $\mathrm{R}_{\mathfrak{m}}(E)_+$ *are Buchsbaum modules over* $\mathrm{R}(\mathfrak{m})$.

Remark 5. In the case where $\dim A = 1$, unfortunately, the last part of Theorem 1 is not necessarily true in general. Namely, there exist counterexamples such that the Rees algebras $\mathrm{R}(\mathfrak{a})$ are not Buchsbaum rings, even though $\mathrm{R}(\mathfrak{a})_+$ are Buchsbaum modules.

2. Quasi-Buchsbaumness of $\mathrm{R}(\mathfrak{a})_+$

Throughout this note, let us keep the notation as follows:

$\mathrm{R} := \bigoplus_{n \geq 0} \mathfrak{a}^n$, the Rees algebra of $\mathfrak{a}$;

$\mathrm{R}_+ := \bigoplus_{n > 0} \mathfrak{a}^n$, the homogeneous ideal of positive degree of R;

$\mathrm{G} := \bigoplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$, the associated graded ring of $\mathfrak{a}$;

$\mathfrak{M} := \mathfrak{m}\mathrm{R} + \mathrm{R}_+$, the unique homogeneous maximal ideal of R.

When we are setting $\mathfrak{a} = (a_1, a_2, \ldots, a_u)$ the Rees algebra R is usually regarded as the A-subalgebra of the polynomial ring $A[t]$, where t is an indeterminate over A;

$$\mathrm{R} = A[a_1 t, a_2 t, \ldots, a_u t].$$

¿From now on, let us assume that, for some minimal reduction $\mathfrak{q} := (a_1, a_2, \ldots, a_d)$ of $\mathfrak{a}$, the following three condition are satisfied:

(i) $\mathfrak{a}^2 = \mathfrak{q}\mathfrak{a}$ holds;
(ii) the equality

$$(a_1^2, a_2^2, \ldots, a_d^2) \cap \mathfrak{a}^n = (a_1^2, a_2^2, \ldots, a_d^2)\mathfrak{a}^{n-2}$$

holds for $3 \leq n \leq d+1$,

(iii) $\mathfrak{a}$ has minimal multiplicity, i.e., $\mathfrak{m}\mathfrak{a} \subseteq \mathfrak{q}$ holds.

Then we begin with the following.

Lemma 6. *One has the following statements.*

(1)

$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}_+)]_n = \begin{cases} \mathfrak{a} \cap \mathrm{H}^0_{\mathfrak{m}}(A) & (n=1) \\ (0) & (else); \end{cases}$$

and $\mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}_+) = (0)$. *Therefore* R_+ *is a Buchsbaum* R*-module for* $d=1$.

(2)

$$[\mathrm{H}^2_{\mathfrak{N}}(\mathrm{R}_+)]_n = \begin{cases} [\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G})]_{-1} & (n=0) \\ (0) & (else); \end{cases}$$

(3) *For each* $3 \le i \le d$,

$$[\mathrm{H}^i_{\mathfrak{N}}(\mathrm{R}_+)]_n = \begin{cases} \mathrm{H}^{i-1}_{\mathfrak{m}}(A) & (n \in [3-i, 0]) \\ [\mathrm{H}^{i-1}_{\mathfrak{N}}(\mathrm{G})]_{1-i} & (n = 2-i) \\ (0) & (else). \end{cases}$$

Lemma 7. *One has the following statements.*

(1) *There are exact sequences of graded* R*-modules as follows:*

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}_+) \longrightarrow \mathrm{R}_+ \longrightarrow \mathrm{R}(\mathfrak{a} + U/U)_+ \longrightarrow 0,$$

where we put $U = \mathrm{H}^0_{\mathfrak{m}}(A)$.

(2) *Suppose that* $\operatorname{depth} A > 0$, *and put* $a := a_1$. *Then there are short exact sequences of graded* R*-modules as follows:*

$$0 \longrightarrow \mathrm{R}_+ \xrightarrow{a} \mathrm{R}_+ \longrightarrow \mathrm{R}_+/a\mathrm{R}_+ \longrightarrow 0;$$

$$0 \longrightarrow \mathrm{G}(-1) \longrightarrow \mathrm{R}_+/a\mathrm{R}_+ \longrightarrow \mathrm{R}(\mathfrak{a}/(a))_+ \longrightarrow 0.$$

$$0 \longrightarrow \mathrm{R}_+(-1) \xrightarrow{at} \mathrm{R}_+ \longrightarrow \mathrm{R}_+/at \cdot \mathrm{R}_+ \longrightarrow 0;$$

$$0 \longrightarrow A(-1) \xrightarrow{a} \mathrm{R}_+/at \cdot \mathrm{R}_+ \longrightarrow \mathrm{R}(\mathfrak{a}/(a))_+ \longrightarrow 0.$$

Combining these facts we get the following, which we shall need later in order to use induction on $d := \dim A$. We refer [Su] for the details on quasi-Buchsbaumness of rings/modules.

Proposition 8. R_+ *is a quasi-Buchsbaum* R*-module. In particular one has*

$$at \cdot \mathrm{H}^i_{\mathfrak{N}}(\mathrm{R}_+) = (0) \quad for \quad 0 \le i \le d.$$

3. Reduction step to the case where depth $A > 0$

Let us write $u := \mu_A(\mathfrak{a})$ and $v := \mu_A(\mathfrak{m})$, where $\mu_A(*)$ denotes the minimal number of generators of an A-module. Moreover, for a set S we denote by $|S|$ the number of all elements in S. Recall that, for integers i, j, we denote by $[i, j]$ the set of integers n such that $i \leq n \leq j$. Of course, $[i, j] = \emptyset$ if $i > j$. Then we have the following.

Lemma 9. *There exist systems of elements in A, say $a_1, a_2, \dots, a_u$ and $x_1, x_2, \dots, x_v$, which satisfy the following conditions:*

(1) *$a_1, a_2, \dots, a_u$ is a minimal system of generators of $\mathfrak{a}$;*
(2) *any d-elements of $a_1t, a_2t, \dots, a_ut$ in R form a system of parameters for G, i.e., this is equivalent to say that there is an integer $r \geq 0$ such that $\mathfrak{a}^{r+1} = (a_i \mid i \in I)\mathfrak{a}^r$ for all $I \subseteq [1, u]$ with $|I| = d$.*
(3) *$x_1, x_2, \dots, x_v$ is a minimal system of generators of $\mathfrak{m}$;*
(4) *any d-elements of $a_1, \dots, a_u, x_1, \dots, x_v$ form a system of parameters for A.*

Let $a_1, a_2, \dots, a_u$ and $x_1, x_2, \dots, x_v$ be a systems of elements in A satisfying four conditions in Lemma 9 above. We put $\underline{at} := a_1t, a_2t, \dots, a_ut$ and $\underline{x} := x_1, x_2, \dots, x_v$. Let $\mathrm{K}^{\cdot}(\underline{at}, \underline{x}; \mathrm{R}_+)$ be the Koszul (co-)complex generated by the system $\underline{at}, \underline{x}$ over the graded R-module R_+ and we denote it by $\mathrm{K}^{\cdot}(\mathfrak{N}; \mathrm{R}_+)$ simply. Also $\mathrm{H}^{\cdot}(\mathfrak{N}; \mathrm{R}_+)$ denotes the cohomology modules of the Koszul complex $\mathrm{K}^{\cdot}(\mathfrak{N}; \mathrm{R}_+)$.

In order to lead the Buchsbaumness of R_+, we enoughly show that the canonical map

$$\phi^i_{\mathrm{R}_+} : \mathrm{H}^i(\mathfrak{N}; \mathrm{R}_+) \longrightarrow \mathrm{H}^i_{\mathfrak{N}}(\mathrm{R}_+)$$

is surjective for all $0 \leq i \leq d$, see [SV, Theorem (2.15) in Chap. I]. The next section shall be devoted to proving this statement by induction on $d := \dim A$. Before doing it, we have to prepare the reduction step, which allows to reduce our problem to the case where depth $A > 0$. We begin with the following.

Lemma 10. *Let $\tau : \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}) \longrightarrow \mathrm{R}$ and $\tau_+ : \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}_+) \longrightarrow \mathrm{R}_+$ be canonical inclusions. Then one has the following statements.*

(1) *Suppose that $d \geq 2$. Then the canonical map induced by τ*

$$\tau^i : \mathrm{H}^i(\mathfrak{N}; \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R})) \longrightarrow \mathrm{H}^i(\mathfrak{N}; \mathrm{R})$$

is injective for all $0 \leq i \leq d$, and moreover

$$[\tau^{d+1}]_n : [\mathrm{H}^{d+1}(\mathfrak{N}; \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}))]_n \longrightarrow [\mathrm{H}^{d+1}(\mathfrak{N}; \mathrm{R})]_n$$

is injective for $1 - d \leq n < 0$.

(2) *The canonical map induced by τ_+*

$$\tau^i_+ : \mathrm{H}^i(\mathfrak{N}; \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}_+)) \longrightarrow \mathrm{H}^i(\mathfrak{N}; \mathrm{R}_+)$$

is injective for all $0 \leq i \leq d + 1$.

Proposition 11. *Suppose that $d \geq 2$. Then one has the following statements.*

(1) R *is a Buchsbaum ring if and only if* $\mathrm{R}(\mathfrak{a}+U/U)$ *is so, where we put* $U := \mathrm{H}^0_{\mathfrak{m}}(A)$.

(2) R_+ *is a Buchsbaum* R*-module if and only if* $\mathrm{R}(\mathfrak{a}+U/U)_+$ *is so.*

Corollary 12. *The following statements are true.*

(1) R_+ *is a Buchsbaum* R*-module for* $d = 2$.

(2) *Suppose that $d \geq 2$. Then* $\phi^i_{\mathrm{R}} : \mathrm{H}^i(\mathfrak{N}; \mathrm{R}) \longrightarrow \mathrm{H}^i_{\mathfrak{N}}(\mathrm{R})$ *is surjective for $i \leq 2$. Hence* R *is a Buchsbaum ring for* $d = 2$.

(3) *Suppose that $d \geq 3$. Then* R *is a Buchsbaum ring if* R_+ *is a Buchsbaum* R*-module.*

4. Proof of main results

Before giving the proof of our results, we need two more facts.

Lemma 13. *The following statements hold.*

(1) $\mathfrak{a} \cap U(a)$ *is a Buchsbaum A-module of dimension d such that*

$$\mathrm{H}^0_{\mathfrak{m}}(\mathfrak{a} \cap U(a)) = (0) \quad \textit{and} \quad \mathrm{H}^1_{\mathfrak{m}}(\mathfrak{a} \cap U(a)) = U(a)/\mathfrak{a} \cap U(a).$$

(2) $A/\mathfrak{a} \cap U(a)$ *is a Buchsbaum A-module of dimension $d-1$ such that*

$$\mathrm{H}^0_{\mathfrak{m}}(A/\mathfrak{a} \cap U(a)) = U(a)/\mathfrak{a} \cap U(a).$$

(3) *There is an exact sequence of graded* R*-modules:*

$$0 \longrightarrow (\mathfrak{a} \cap U(a))(-1) \stackrel{\rho}{\longrightarrow} \mathrm{R}_+/at \cdot \mathrm{R}_+ \longrightarrow \mathrm{R}(\mathfrak{a} + U(a)/U(a))_+ \longrightarrow 0.$$

(4) *The following sequence of local cohomology modules*

$$0 \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(\mathfrak{a} \cap U(a))(-1) \longrightarrow \mathrm{H}^i_{\mathfrak{N}}(\mathrm{R}_+/at \cdot \mathrm{R}_+) \longrightarrow \mathrm{H}^i_{\mathfrak{N}}(\mathrm{R}(\mathfrak{a} + U(a)/U(a))_+) \longrightarrow 0$$

is exact for $0 \leq i < d$.

Proposition 14. *The canonical map*

$$\rho^i : \mathrm{H}^i(\mathfrak{N}; (\mathfrak{a} \cap U(a))(-1)) \longrightarrow \mathrm{H}^i(\mathfrak{N}; \mathrm{R}_+/at \cdot \mathrm{R}_+)$$

is injective for all $0 \leq i \leq d$.

Now we are ready to prove our main results, Theorem 1 and Corollaries 2–4.

Outline of the proof of main results. For the case where $d = 2$ we have already shown our statement by Corollary 12. Now let $d \geq 3$ and assume that our statement is true for $d-1$. By Proposition 11, we may further assume that $\operatorname{depth} A > 0$. Now put $a := a_1$ and $U(a) := (a) : \mathfrak{m}$. Look at the following short exact sequence:

$$0 \longrightarrow (\mathfrak{a} \cap U(a))(-1) \stackrel{\rho}{\longrightarrow} \mathrm{R}_+/at \cdot \mathrm{R}_+ \longrightarrow \mathrm{R}(\mathfrak{a} + U(a)/U(a))_+ \longrightarrow 0.$$

By the hypothesis of induction on d, the graded module $\mathrm{R}(\mathfrak{a}+U(a)/U(a))_+$ is Buchsbaum. Acoording to Lemma 15 below, we consequently get the Buchsbaumness of $\mathrm{R}_+/at \cdot \mathrm{R}_+$ from Lemma13, Proposition 14 and the observation above. Finally consider the next short exact sequence of graded R-modules:

$$0 \longrightarrow \mathrm{R}_+(-1) \stackrel{at}{\longrightarrow} \mathrm{R}_+ \longrightarrow \mathrm{R}_+/at \cdot \mathrm{R}_+ \longrightarrow 0.$$

Since $\mathrm{R}_+/at \cdot \mathrm{R}_+$ is Buchsbaum, we naturally get the Buchsbaumness of R_+ itself by Proposition 8. This completes the proof of Theorem 1. Further results also follow at once.

Lemma 15. *Let* $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ *be an exact sequence of finitely generated* A*-modules. Suppose that the following three conditions are satisfied:*

(i) $\dim E = \dim E' = \dim E'' > 0$;
(ii) E' *and* E'' *are Buchsbaum* A*-modules;*
(iii) *the following sequence of local cohomology modules*

$$0 \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(E') \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(E) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(E'') \longrightarrow 0$$

is exact for $0 \leq i < s$, *where* $s := \dim E$.

Then the following statements are equivalent.

(1) E *is a Buchsbaum* A*-module.*
(2) *The sequence of Koszul cohomology modules*

$$0 \longrightarrow \mathrm{H}^i(\mathfrak{m}; E') \longrightarrow \mathrm{H}^i(\mathfrak{m}; E) \longrightarrow \mathrm{H}^i(\mathfrak{m}; E'') \longrightarrow 0$$

is exact for $0 \leq i < s$.
(3) *The canonical map induced from the monomorphism* $E' \longrightarrow E$ *above*

$$\mathrm{H}^i(\mathfrak{m}; E') \longrightarrow \mathrm{H}^i(\mathfrak{m}; E)$$

is injective for $0 \leq i \leq s$.

5. One dimensional case

In the case where $\dim A = 1$ we have the following characterization for the Rees algebras to be Buchsbaum.

Theorem 16. *Let* A *be a Noetherian local ring of dimension* $d = 1$ *and* $\mathfrak{a}$ *an* $\mathfrak{m}$*-primary ideal of* A. *Then the Rees algebra* R *is a Buchsbaum ring if and only if the following four conditions are satisfied:*

(i) A *is a Buchsbaum ring;*
(ii) $\mathfrak{a}^2 = a\mathfrak{a}$ *for some (resp. every) minimal reduction* (a) *of* $\mathfrak{a}$;
(iii) $\mathfrak{a}$ *has minimal multiplicity, i.e.,* $\mathfrak{m}\mathfrak{a} \subseteq (a)$ *for some (resp. every) minimal reduction* (a) *of* $\mathfrak{a}$;
(iv) *for some (resp. every)* A*-basis of* $\mathfrak{m}$, *say* $\{x_1, x_2, \ldots, x_v\}$, *it holds that*

$$x_j\mathfrak{a} : \mathfrak{a} \subseteq (x_j) : \mathfrak{m}$$

for all $1 \leq j \leq v$.

When this is the case, the associated graded ring G *is a Buchsbaum ring too.*

Unfortunately, without the condition (iv) in Theorem 16, the Rees algebra R is not necessarily a Buchsbaum ring, though it is quasi-Buchsbaum.

Example 17. Let $k[[X,Y,Z,W]]$ be a formal power series ring over an infinite field k. Now consider

$$A := k[[X,Y,Z,W]]/(XZ, YZ, XW, YW, Y^2+W^2) = k[[x,y,z,w]],$$

and four $\mathfrak{m}$-primary ideals of A as follows:

$$\mathfrak{a}_0 := (a^2, y^2),\ \mathfrak{a}_1 := (x^2, y^2, z^2),\ \mathfrak{a}_2 := (x^2, y^2, z^2, xy) \text{ and } \mathfrak{a}_3 := \mathfrak{m}^2,$$

where we put $a := x + z$ and $\mathfrak{m}$ is the maximal ideal of A. Then

(1) A is a Buchsbaum ring of dimension 1 and $\mathfrak{m}^3 = a\mathfrak{m}^2$.
(2) $\mathrm{H}^0_\mathfrak{m}(A) = y^2A = w^2A \cong k$, and hence $\mathrm{H}^0_\mathfrak{m}(A) \subset \mathfrak{a}_i \subseteq \mathfrak{m}^2$ for each $i = 1, 2, 3$.
(3) $\mathfrak{a}_i^2 = a^2\mathfrak{a}_i$, $\mathfrak{m}\mathfrak{a}_i \subseteq a^2A$ and $l_A(\mathfrak{a}_i/\mathfrak{a}_0) = i$ for each $0 \leq i \leq 3$.
(4) $\mathrm{R}(\mathfrak{a}_0)$ and $\mathrm{R}(\mathfrak{a}_1)$ are Buchsbaum rings.
(5) $\mathrm{R}(\mathfrak{a}_2)$ and $\mathrm{R}(\mathfrak{a}_3)$ are not Buchsbaum rings, but they are quasi-Buchsbaum.

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On filtrations having small analytic deviation

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1 Introduction

Let A be a d-dimensional Noetherian local ring with the maximal ideal $\mathfrak{m}$ and let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a family of ideals in A such that (i) $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{Z}$, (ii) $F_0 = A$ and $F_1 \neq A$, and (iii) $F_mF_n \subseteq F_{m+n}$ for all $m, n \in \mathbb{Z}$. In this report we simply call such $\mathcal{F}$ a filtration. When a filtration $\mathcal{F}$ is given, we can consider the following algebras:

$$\begin{aligned} \mathrm{R}(\mathcal{F}) &= \sum_{n\geq 0} F_nT^n \subseteq A[T] \ \ (T \text{ is an indeterminate}),\\ \mathrm{R}'(\mathcal{F}) &= \sum_{n\in\mathbb{Z}} F_nT^n \subseteq A[T, T^{-1}], \ \text{ and}\\ \mathrm{G}(\mathcal{F}) &= \mathrm{R}'(\mathcal{F})/T^{-1}\mathrm{R}'(\mathcal{F}) = \bigoplus_{n\geq 0} F_n/F_{n+1}\,. \end{aligned}$$

Those algebras are respectively called the Rees algebra, the extended Rees algebra, and the form ring associated to $\mathcal{F}$. We always assume that $\mathrm{R}(\mathcal{F})$ is Noetherian and $\dim \mathrm{R}(\mathcal{F}) = d + 1$. Typical examples of filtration are constructed from an ideal I. For example, setting $F_n = I^n$, we get the I-adic filtration. Symbolic filtration of I is defined by setting $F_n = I^{(n)}$, where $I^{(n)} = \bigcap_{\mathfrak{p}\in \mathrm{Min}_A A/I} I^nA_{\mathfrak{p}} \cap A$. It is also important to set $F_n = \overline{I^n}$, where $\overline{I^n}$ denotes the integral closure of I^n.

We denote by $\ell(\mathcal{F})$ the Krull dimension of the ring $A/\mathfrak{m} \otimes_A \mathrm{R}(\mathcal{F})$ and call it the analytic spread of $\mathcal{F}$ (cf. [13]). If $\mathcal{F}$ is the ideal adic filtration of an ideal I, then $\ell(\mathcal{F})$ is just the analytic spread of I in the sense of Northcott and Rees. It is easy to see that, similarly as the case of ideals, the inequality $\ell(\mathcal{F}) \geq \mathrm{ht}_A F_1$ always hold (here it should be noticed that $\mathrm{ht}_A F_1 = \mathrm{ht}_A F_n$ for all $n \geq 1$). So, following Huckaba and Huneke [8], we set $\mathrm{ad}(\mathcal{F}) := \ell(\mathcal{F}) - \mathrm{ht}_A F_1$ and call it the analytic deviation of $\mathcal{F}$. For example, if $\mathcal{F}$ is the symbolic filtration of an ideal I, then $\mathrm{ad}(\mathcal{F}) < d - \mathrm{ht}_A I$. On the other hand, if

$I^n \subseteq F_n \subseteq \overline{I^n}$ for all n, then $\mathrm{ad}(\mathcal{F})$ coincides with the analytic deviation of I in the sense of Huckaba and Huneke.

The main result of this report is a characterization of the Cohen-Macaulay property of the form ring associated to a filtration having small analytic deviation. If $\mathcal{F}$ is a filtration with $\ell(\mathcal{F}) = \ell$, we can choose elements $a_1, \cdots, a_\ell$ in A so that $a_1 \in F_{k_1}, \cdots, a_\ell \in F_{k_\ell}$ for some positive integers $k_1, \cdots, k_\ell$ and $F_n = \sum_{i=1}^{\ell} a_i F_{n-k_i}$ for all $n \gg 0$. We will show that, in the case where $\mathrm{ad}(\mathcal{F}) = 0$, $\mathrm{G}(\mathcal{F})$ is Cohen-Macaulay if and only if $A/(a_1, \cdots, a_\ell) + F_n$ is Cohen-Macaulay for finite number of n and $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Assh}_A A/F_1$, where $\mathcal{F}_\mathfrak{p}$ is the filtration $\{F_n A_\mathfrak{p}\}_{n \in \mathbb{Z}}$ of $A_\mathfrak{p}$ and $\mathrm{Assh}_A A/F_1 = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid F_1 \subseteq \mathfrak{p}$ and $\dim A/\mathfrak{p} = \dim A/F_1\}$. This characterization was already proved by Goto in the case where $\mathcal{F}$ is the symbolic filtration of a prime ideal $\mathfrak{p}$ such that $\dim A/\mathfrak{p} = 1$ (cf. [3]). We will also discuss the case where $\mathrm{ad}(\mathcal{F}) = 1$. Although the statement is rather complicated, a condition for $\mathrm{G}(\mathcal{F})$ to be Cohen-Macaulay will be given similarly as the case where $\mathrm{ad}(\mathcal{F}) = 0$.

Throughout this report A is a d-dimensional Noetherian local ring with the maximal ideal $\mathfrak{m}$. and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration of A such that $\mathrm{R}(\mathcal{F})$ is a $d+1$-dimensional Noetherian ring. In the case where $\mathcal{F}$ is the symbolic filtration of I, we write $\mathrm{R}_s(I)$, $\mathrm{R}'_s(I)$, and $\mathrm{G}_s(I)$ instead of $\mathrm{R}(\mathcal{F})$, $\mathrm{R}'(\mathcal{F})$, and $\mathrm{G}(\mathcal{F})$.

2 Preliminaries

Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ be a filtration of A. We begin with the following.

Lemma 2.1 $\dim \mathrm{R}(\mathcal{F})/\mathfrak{m}\mathrm{R}(\mathcal{F}) = \dim \mathrm{G}(\mathcal{F})/\mathfrak{m}\mathrm{G}(\mathcal{F})$.

Definition 2.2 *The analytic spread* $\ell(\mathcal{F})$ *is defined to be* $\dim \mathrm{R}(\mathcal{F})/\mathfrak{m}\mathrm{R}(\mathcal{F})$, *which is equal to* $\dim \mathrm{G}(\mathcal{F})/\mathfrak{m}\mathrm{G}(\mathcal{F})$ *by 2.1*

Definition 2.3 *Let* $a_1, \cdots, a_r$ *be a system of elements of* A. *If* $a_1 \in F_{k_1}, \cdots, a_r \in F_{k_r}$ *for some positive integers* $k_1, \cdots, k_r$ *and* $F_n = \sum_{i=1}^{r} a_i F_{n-k_i}$ *for all* $n \gg 0$, *we say that* $a_1, \cdots, a_r$ *is a reduction of* $\mathcal{F}$.

Let $\ell = \ell(\mathcal{F})$. If $a_1, \cdots, a_r$ is a reduction of $\mathcal{F}$ stated in 2.3, then we have $\ell \leq r$ since the ring $\mathrm{R}(\mathcal{F})/(a_1 T^{k_1}, \cdots, a_r T^{k_r})\mathrm{R}(\mathcal{F}) + \mathfrak{m}\mathrm{R}(\mathcal{F})$ is Artinian. On the other hand, we can always choose $a_1, \cdots, a_\ell$ in A so that $a_1 \in F_{k_1}, \cdots, a_\ell \in F_{k_\ell}$ for some positive integers $k_1, \cdots, k_\ell$ and $a_1T^{k_1}, \cdots, a_\ell T^{k_\ell}$ form an sop for $\mathrm{R}(\mathcal{F})/\mathfrak{m}\mathrm{R}(\mathcal{F})$. Then $a_1, \cdots, a_\ell$ is a reduction of $\mathcal{F}$. When this is the case, we have $k_i = \max\{n \mid a_i \in F_n\}$ for all $1 \le i \le \ell$, because $a_1T^{k_1}, \cdots, a_\ell T^{k_\ell}$ is an sop also on $\mathrm{G}(\mathcal{F})/\mathfrak{m}\mathrm{G}(\mathcal{F})$.

Lemma 2.4 $\mathrm{ht}_A F_1 \le \ell(\mathcal{F}) \le d$.

Definition 2.5 *We denote by* $\mathrm{ad}(\mathcal{F})$ *the difference* $\ell(\mathcal{F}) - \mathrm{ht}_A F_1$ *and call it the analytic deviation of* $\mathcal{F}$. *In particular,* $\mathcal{F}$ *is said to be equimultiple, if* $\mathrm{ad}(\mathcal{F}) = 0$.

Example 2.6 *Let* I *be an ideal of* A *with* $\mathrm{ht}_A I = s$. *Let* $F_n = I^{(n)}$ *for all* n. *Then* $\mathrm{ad}(\mathcal{F}) < d - s$. *In particular,* $\mathcal{F}$ *is equimultiple, if* $s = d - 1$.

Example 2.7 *Let* I *be an ideal with a reduction* $J = (a_1, \cdots, a_r)A$. *Let* $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ *be a filtration such that* $I^n \subseteq F_n \subseteq \overline{I^n}$ *for all* n. *Then* $a_1, \cdots, a_r$ *is a reduction of* $\mathcal{F}$. *In particular, if* I *is generated by an regular sequence, then* $\mathcal{F}$ *is equimultiple.*

Finally we state about localization of filtration. For a prime ideal $\mathfrak{p}$, we set $\mathcal{F}_\mathfrak{p} = \{(F_n)_\mathfrak{p}\}_{n\in\mathbb{Z}}$. Notice that $\mathcal{F}_\mathfrak{p}$ is a filtration of $A_\mathfrak{p}$. We always have $\ell(\mathcal{F}_\mathfrak{p}) \le \ell(\mathcal{F})$ because $\mathcal{F}$ has a reduction consisting of $\ell(\mathcal{F})$ elements and it is a reduction of $\mathcal{F}_\mathfrak{p}$. Because $A_\mathfrak{p} \otimes_A \mathrm{G}(\mathcal{F}) \cong \mathrm{G}(\mathcal{F}_\mathfrak{p})$, once $\mathrm{G}(\mathcal{F})$ is Cohen-Macaulay, then so is $\mathrm{G}(\mathcal{F}_\mathfrak{p})$.

3 Equimultiple filtration

Let $\mathcal{F}$ be a filtration of A. The following theorem is a characterization of the Cohen-Macaulay property of the form ring associated to an equimultiple filtration of a Cohen-Macaulay ring. It was already proved by Goto [3, Theorem(1.2)] in the case where $\mathcal{F}$ is the symbolic filtration of a prime ideal $\mathfrak{p}$ with $\dim A/\mathfrak{p} = 1$.

Theorem 3.1 *Let* A *be a Cohen-Macaulay ring and* $\mathrm{ht}_A F_1 = s$. *Let* $a_1, \cdots, a_s$ *be elements in* A *such that* $a_1 \in F_{k_1}, \cdots, a_s \in F_{k_s}$ *for some positive integers* $k_1, \cdots, k_s$ *and* $F_n = \sum_{i=1}^s a_i F_{n-k_i}$ *for* $n \gg 0$. *Set* $N = \sum_{i=1}^s k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Assh}_A A/F_1\}$. *Then the following conditions are equivalent.*

(1) $\mathrm{G}(\mathcal{F})$ *is a Cohen-Macaulay ring.*

(2) $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \mathrm{Assh}_A A/F_1$ *and* $A/(a_1, \cdots, a_s) + F_n$ *is Cohen-Macaulay for all* $1 \le n \le N$.

When this is the case, A/F_n is Cohen-Macaulay for any $n \geq 1$, $F_n = \sum_{i=1}^{s} a_i F_{n-k_i}$ for any $n > N$, and $\mathrm{a}(\mathrm{G}(\mathcal{F})) = \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Assh}_A A/F_1\}$.

4 The case where $\mathrm{ad}(\mathcal{F}) = 1$

Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of A with $\mathrm{ht}_A F_1 = s < d$. In this section, we would like to deal with the case where $\mathrm{ad}(\mathcal{F}) = 1$. However, even if we can easily check that $\mathrm{ad}(\mathcal{F}) \leq 1$, it may be difficult to see that $\mathrm{ad}(\mathcal{F})$ is surely 1. So we will develop our argument assuming that $\mathcal{F}$ has a reduction consisting of $s+1$ elements, which is equivalent to the condition that $\mathrm{ad}(\mathcal{F})$ is at most 1.

Throughout this section we always assume that $a_1, \cdots, a_s, a_{s+1}$ are elements in A such that $a_1 \in F_{k_1}, \cdots, a_s \in F_{k_s}, a_{s+1} \in F_{k_{s+1}}$ for some positive integers $k_1, \cdots, k_s, k_{s+1}$ and $F_n = \sum_{i=1}^{s+1} a_i F_{n-k_i}$ for $n \gg 0$. Moreover, we assume that, $a_1, \cdots, a_s$ is an A-regular sequence and if $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$, then $F_n A_\mathfrak{q} = \sum_{i=1}^{s} a_i F_{n-k_i} A_\mathfrak{q}$ for $n \gg 0$. It should be noticed that we can always find such $a_1, \cdots, a_s, a_{s+1}$ if $\ell(\mathcal{F}) \leq s+1$. We put

$$\begin{aligned}
\mathcal{P} &= \{\mathfrak{p} \in \mathrm{Spec}\, A \mid F_1 \subseteq \mathfrak{p} \text{ and } \mathrm{ht}_A \mathfrak{p} \leq s+1\}, \\
\alpha &= \sum_{i=1}^{s} k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) \mid \mathfrak{q} \in \mathrm{Assh}_A A/F_1\} + 1, \\
\beta &= \sum_{i=1}^{s+1} k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathcal{P}\}, \text{ and} \\
K &= (a_1, \cdots, a_s)A.
\end{aligned}$$

We will denote a_{s+1} (resp. k_{s+1}) by b (resp. k).

Theorem 4.1 *Let $\mathrm{G}(\mathcal{F})$ be a Cohen-Macaulay ring. Then we have the following assertions.*

(1) $\mathrm{depth}\, A/K + F_n \geq d - s - 1$ *for all $n > 0$.*

(2) *bT^k is a non-zero-divisor on $\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha}$, where $\overline{\mathcal{F}}$ is the filtration $\{K + F_n/K\}_{n\in\mathbb{Z}}$ of A/K.*

(3) $\mathrm{depth}\, A/K + bF_\alpha + F_n \geq d - s - 1$ *for any $n > 0$.*

(4) $F_n = \sum_{i=1}^{s+1} a_i F_{n-k_i}$ *for any $n > \beta$.*

(5) $\mathrm{a}(\mathrm{G}(\mathcal{F})) = \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathcal{P}\}$.

Theorem 4.2 *Let A be a Cohen-Macaulay ring. Let $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ be a Cohen-Macaulay ring for any $\mathfrak{p} \in \mathcal{P}$ and $\operatorname{depth} A/K + bF_\alpha + F_n \geq d - s - 1$ for all $1 \leq n \leq \beta$. Then we have the following assertions.*

(1) $\operatorname{depth} A/F_n \geq d - s - 1$ *for any* $n > 0$.

(2) *If $A/K+F_n$ is Cohen-Macaulay for any $1 \leq n \leq \alpha$, then $\mathrm{G}(\mathcal{F})$ is a Cohen-Macaulay ring.*

5 Applications

Let A be the formal power series ring $K[[X, Y, Z, W]]$ over a field K. Let I be the ideal of A generated by the maximal minors of the matrix

$$M = \begin{pmatrix} X & Y & Z & W^m \\ Y & Z & W & X \\ Z & W & X & Y^m \end{pmatrix},$$

where m is a positive integer. Then A/I is a Cohen-Macaulay ring with $\dim A/I = 2$. In the following, applying the results in previous sections, we will compute the symbolic powers of I.

For $1 \leq i \leq 4$, let a_i be the minor corresponding to the matrix derived from M deleting the i-th column. Usually, we denote a_1, a_2, a_3, and a_4 by a, b, c, and d, respectively. Then we have the following relations:

$$(\#_1) \qquad Xa - Yb + Zc - W^m d = 0,$$

$$(\#_2) \qquad Ya - Zb + Wc - Xd = 0, \text{ and}$$

$$(\#_3) \qquad Za - Wb + Xc - Y^m d = 0.$$

Lemma 5.1 *Let $\mathfrak{p} \in \operatorname{Assh}_A A/I$. Then $IA_\mathfrak{p} = (a, d)A_\mathfrak{p}$. Hence $I^{(n)}A_\mathfrak{p} = I^n A_\mathfrak{p}$ for all n and $\mathrm{G}(IA_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(IA_\mathfrak{p})) = -2$.*

Proof. Let $\mathfrak{q}$ be the ideal of A generated by the maximal minors of the mtrix

$$\begin{pmatrix} Y & Z \\ Z & W \\ W & X \end{pmatrix}.$$

Then $I \not\subseteq \mathfrak{q}$ as $\mathfrak{q} \subseteq (Y, Z, W)A$ and $b \equiv -X^3 \mod (Y, Z, W)A$. It follows that $\mathfrak{q} \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Assh}_A A/I$ as $\mathfrak{q}$ is a prime ideal with $\operatorname{ht}_A \mathfrak{q} = 2$. Because $\mathfrak{q}I \subseteq (a, d)A$, $IA_\mathfrak{p} \subseteq (a, d)A_\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Assh}_A A/I$. Thus we get the assertion.

Theorem 5.2 *Let $m = 1$. Then there exists $e \in I^{(2)} \setminus I^2$ such that $\mathrm{R}_s(I) = A[IT, eT^2]$. When this is the case, $\mathrm{R}_s(I)$ is a Gorenstein ring.*

Proof. Set

$$u = XZ - Y^2, \quad v = X^2 - YW, \quad w = XW - YZ,$$
$$f = XY - ZW, \quad g = YW - Z^2, \quad h = XZ - W^2.$$

Then we have the following relations:

$$(\#_3) \qquad v(c^2 - bd) = u(b^2 - ac),$$

$$(\#_4) \qquad w(c^2 - bd) = u(bc - ad),$$

$$(\#_5) \qquad f(c^2 - bd) = u(ab - cd),$$

$$(\#_6) \qquad g(c^2 - bd) = u(ac - d^2),$$

$$(\#_7) \qquad h(c^2 - bd) = u(a^2 - bd).$$

Because u, v is a regular sequence, by $(\#_3)$ there exists $e \in A$ such that $ue = c^2 - bd$ and $ve = b^2 - ac$. Moreover, by $(\#_4)$ we have

$$\begin{aligned} we(c^2 - bd) &= ue(bc - ad) \\ &= (c^2 - bd)(bc - ad), \end{aligned}$$

and so $we = bc - ad$. Similarly, using $(\#_5)$, $(\#_6)$, and $(\#_7)$, we get $fe = ab - cd$, $ge = ac - d^2$, and $he = a^2 - bd$. Hence $e \in I^2 : \mathfrak{A}$, where $\mathfrak{A} = (u, v, w, f, g, h)A$. This implies $e \in I^{(2)}$ as $\mathrm{ht}_A \mathfrak{A} \geq 3$. We have $e \notin I^2$ since $(Y, Z, W)A + I^2 = (X^6, Y, Z, W)A$ and $e \equiv X^4 \mod (Y, Z, W)A$.

We set

$$F_n = \begin{cases} \displaystyle\sum_{\substack{i,j \geq 0 \\ 2i + j = n}} e^i I^j & \text{if } n \geq 0 \\ A & \text{if } n < 0. \end{cases}$$

Then $F_1 = I$, $F_2 = I^2 + eA$, and $I^n \subseteq F_n \subseteq I^{(n)}$ for all n. Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$. It is easy to see that $\mathcal{F}$ is a filtration such that $F_n = aF_{n-1} + eF_{n-2}$ for all $n \geq 2$. Hence $\mathcal{F}$ is an equimultiple filtration and a, e is a reduction of $\mathcal{F}$.

Let $\mathfrak{p} \in \mathrm{Assh}_A A/F_1$. Then by 5.1 $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ $(= \mathrm{G}(I_\mathfrak{p}))$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) = -2$, and so $1 + 2 + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Assh}_A A/F_1\} = 1$. Notice that

A/F_1 is a Cohen-Macaulay ring. Therefore by 3.1 we see that $G(\mathcal{F})$ is Cohen-Macaulay and A/F_n is Cohen-Macaulay for all $n \geq 1$. Now, by [4, Theorem 1.2] it follows that $G(\mathcal{F})$ is a Gorenstein ring with $a(G(\mathcal{F})) = -2$. Then [5, Corollary 1.4] implies that $R(\mathcal{F})$ is a Gorenstein ring. Let n be a positive integer. Since A/F_n is Cohen-Macaulay, $F_n \subseteq I^{(n)}$, and $F_nA_\mathfrak{p} = I^{(n)}A_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/F_n = \mathrm{Assh}_A A/I$, we get $F_n = I^{(n)}$. Therefore $R(\mathcal{F}) = R_s(I)$ and the proof is completed.

In the case where $m \geq 2$, using the results stated in section 4, we get the following result.

Theorem 5.3 *Let $m \geq 2$. Then there exists $e \in I^{(3)} \setminus I^3$ such that $R_s(I) = A[IT, eT^3]$. When this is the case, $R_s(I)$ is a Gorenstein ring.*

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MONOMIALIZATION OF MORPHISMS

S. DALE CUTKOSKY

1. Monomialization along a valuation

Suppose that

$$R \subset S \tag{1}$$

is a local homomorphism of local domains, essentially of finite type over a field k. The structure of such an extension is extremely complicated, even when R and S are regular.

The simplest examples of an extension (1) are monomial mappings and monoidal transforms.

$R \to S$ is a monomial mapping if R has regular parameters $(y_1, \ldots, y_m)$ and there exists an etale extension of S which has regular parameters $(x_1, \ldots, x_n)$, and a matrix a_{ij} such that

$$\begin{aligned} y_1 &= x_1^{a_{11}} \cdots x_n^{a_{1n}} \\ &\vdots \\ y_m &= x_1^{a_{m1}} \cdots x_m^{a_{mn}} \end{aligned}$$

$R \to S$ is a monoidal transform if there exists a regular prime $P \subset R$ and $0 \neq y \in P$ such that $S = R[\frac{P}{y}]_m$ where m is a prime which contracts to the maximal ideal of R. If R is regular, then there exists a regular system of parameters $(y_1, \ldots, y_n)$ in R, and $r \leq n$ such that

$$S = R\left[\frac{y_2}{y_1}, \ldots, \frac{y_r}{y_1}\right]_m .$$

Suppose that V is a valuation ring of the quotient field K of S, such that V dominates S. Then we can ask if there are sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R', and $R' \to S'$ has an especially good structure.

$$\begin{array}{ccc} R' & \to & S' \subset V \\ \uparrow & & \uparrow \\ R & \to & S \end{array} \tag{2}$$

An example of a theorem of this kind is Zariski's Local Uniformization Theorem.

Theorem 1.1. *(Zariski's Local Uniformization Theorem [20]) Suppose that k has characteristic zero, R is a local domain essentially of finite type over k, and V is a valuation ring of the quotient field of R dominating R. Then there exists a regular local ring R', essentially of finite type over k such that $R \subset R' \subset V$.*

Using the resolution theorems of Hironaka [13], we can perform monoidal transforms on R and S in (2) to reduce to the case where R and S are regular.

It is possible to obtain a diagram (2) making $R' \to S'$ a monomial mapping whenever the quotient field of S is a finite extension of the quotient field of R, and the characteristic of k is 0.

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Theorem 1.2. *(Monomialization, Theorem 1.1* [9]*) Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R.*

Let V be a valuation ring of K which dominates S. Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R' and $R' \to S'$ is a monomial mapping.

In fact, we show that there are regular parameters $(x_1, ..., x_n)$ in R', $(y_1, ..., y_n)$ in S', units $\delta_1, \ldots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that $\text{Det}(a_{ij}) \neq 0$ and

$$\begin{aligned} x_1 &= y_1^{a_{11}} .....y_n^{a_{1n}} \delta_1 \\ &\vdots \\ x_n &= y_1^{a_{n1}} .....y_n^{a_{nn}} \delta_n. \end{aligned} \tag{3}$$

By extracting t-th roots of the δ_i where $t = \text{Det}(a_{ij})$, we obtain an etale extension of S' realizing $(x_1, \ldots, x_n)$ as monomials in a regular system of parameters.

The standard theorems on resolution of singularities allow one to easily find R' and S' such that (3) holds, but without the essential condition that $\text{Det}(a_{ij}) \neq 0$. The difficulty of the proof is to acheive this condition.

The most difficult case of the proof of Theorem 1.2 is when V is a nondiscrete rational rank 1 valuation. Such valuations are not Noetherian.

Under very mild assumptions, valuations can be represented by "formal" Puiseux series (Kaplansky [14]). Let k be a field, Γ be an ordered abelian group. Define $S(\Gamma)$ to be the set of series

$$\sum a_i t^{\sigma_i}$$

such that $a_i \in k$, $\sigma_i \in \Gamma$ and $\{\sigma_i\}$ is well ordered. $S(\Gamma)$ is a field, with valuation $\nu(f) = \sigma_1$ if $f = a_1 t^{\sigma_1} +$ higher order terms, with $a_1 \neq 0$. If K is an algebraic function field, with an embedding $\tau : K \to S(\Gamma)$, then ν restricts to a valuation ν' of K. We can obtain interesting value groups such as $\mathbf{Q}$ in this way.

Theorem 1.2 is false in positive characteristic p. A simple counterexample is the morphism of curves

$$y = x^p + x^{p+1}.$$

Extracting a p-th root of $1 + x$ is inseparable. However, it may be possible to obtain a diagram (3) in positive characteristic. This would be sufficient to prove Theorem 1.3 below in positive characteristic.

¿From Theorem 1.2, we obtain an affirmative answer to two conjectures of Abhyankar.

The first conjecture of Abhyankar is that simultaneous resolution from above along a valuation (proved in 2 dimensional function fields by Abhyankar [1]) is true for arbitrary function fields. The statement of the Theorem, giving an affirmative answer in characteristic zero, follows.

Theorem 1.3. *(Theorem 1.1* [10]*) Let k be a field of characteristic zero, L/k an algebraic function field, K a finite algebraic extension of L, ν a valuation of K/k, and (R, M) a regular local ring with quotient field K, essentially of finite type over k, such that ν dominates R. Then for some sequence of monodial transforms $R \to R^*$ along ν, there exists a normal local ring S^* with quotient field L, essentially of finite type over k, such that R^* is the localization of the integral closure T of S^* in K at a maximal ideal of T.*

Abhyankar has shown [3] that the conjecture is false if we ask for S^* to be regular. Simultaneous resolution from above is a key step in generalizing Abhyankar's proof of local uniformization for surfaces to arbitrary dimensions.

If (1) is a birational extension, (1) has a factorization by quadratic transforms if dim $R =$ dim $S = 2$, and R, S are regular (Zariski [22], [23], Abhyankar [2]) .

However, if dim $R =$ dim $S \geq 3$, with R, S regular, there does not in general exist a factorization of $R \rightarrow S$ by monoidal transforms (Hironaka [12], Sally [16], Shannon [17]).

Thus if $R \subset S$ is birational, the strongest possible conclusion is the existence, for any valuation ring V of the quotient field of S, of a diagram (2) such that $R' = S'$. This has been conjectured by Abhyankar [4]. We prove this second conjecture in characteristic zero.

Theorem 1.4. *(Local Factorization)(Theorem A* [8], *Theorem 1.9* [9]*) Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, with a common quotient field K. Let V be a valuation ring of K which dominates S. Then there exists a regular local ring T, with quotient field K, such that T dominates S, V dominates T, and the inclusions $R \rightarrow T$ and $S \rightarrow T$ can be factored by sequences of monoidal transforms.*

$$\begin{array}{ccccc} & & V & & \\ & & \uparrow & & \\ & & T & & \\ & \nearrow & & \nwarrow & \\ R & & \longrightarrow & & S \end{array}$$

Local factorization along a valuation of maximal rank in dimension 3 was proven by Christensen [7]. The conjecture was proven in dimension 3, and characteristic zero, in [8]. The conjecture is proven in all dimensions, and characteristic zero in [9]. Theorem 1.2 allows us to reduce the proof of Abhyankar's conjecture (in characteristic zero) to the special case of a monomial mapping. Local factorization of a monomial mapping follows from a theorem of Morelli [15] on "Strong factorization" of toric (locally monomial) mappings.

Recently, Wlodarczyk [18], [19] and Abramovich, Karu, Matsuki and Wlodarczyk [5] have proven the "Weak Factorization Conjecture" for birational morphisms. They have shown that a birational morphism of nonsingular projective varieties, over a field of characteristic zero, can be factored by alternating sequences of blowups and blowdowns.

2. Monomialization of proper morphisms

We will now look at some global analogs of the preceeding theory.

Hironaka (Chapter 0 of [13]) has defined "complete" varieties. A complete k-variety is an integral scheme X of finite type over k, such that for any valuation ring V of the function field $k(X)$ of X, there exists a morphism from spec(V) into X which lifts the embedding of V into $k(X)$. If the morphism is always unique, X is proper. A complete variety satisfies the existence part of the valuative criterion of properness. Thus if a variety is both separated and complete then it is proper. As pointed out by Hironaka in [13], from Zariski's Local Uniformization Theorem, and his theorem [21] on the quasi compactness of the Zariski Riemann manifold of valuations of a function field, it follows that singularities can be resolved in characteristic zero by a complete morphism.

Theorem 2.1. *(Zariski) If X is a proper k-variety (of characteristic zero) then there exists a complete nonsingular k-variety Y, and a birational morphism $Y \to X$.*

The existence of a resolution of singularites of X by a proper morphism is proven by Hironaka in [13].

Now suppose that $\Phi : X \to Y$ is a morphism of nonsingular proper k-varieties, where k is a field of characteristic 0.

Definition 2.2. *Φ is* **locally monomial** *if for every $p \in X$ there exist regular parameters $(y_1, \ldots, y_m)$ in $\mathcal{O}_{Y,\Phi(p)}$, and an etale cover U of an affine neighborhood of p, uniformizing parameters $(x_1, \ldots, x_n)$ on U and a matrix a_{ij} such that*

$$\begin{aligned} y_1 &= x_1^{a_{11}} \cdots x_n^{a_{1n}} \\ &\vdots \\ y_m &= x_1^{a_{m1}} \cdots x_n^{a_{mn}} \end{aligned}$$

Definition 2.3. *A morphism $\Psi : X_1 \to Y_1$ is a* **global monomialization** *of Φ if there are sequences of monoidal transforms $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$, and a morphism $\Psi : X_1 \to Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \overset{\Psi}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \overset{\Phi}{\to} & Y \end{array}$$

commutes, and Ψ is a locally monomial morphism.

An extremely interesting question is if every morphism $\Phi : X \to Y$ of proper varieties (over a field of characteristic 0) has a global monomialization.

Recall that there exist morphisms over a field k of positive characteristic which do not have a global monomialization.

The construction of a monomialization by complete varieties follows from Theorem 1.2.

Theorem 2.4. *(Theorem 1.2 [9]) Let k be a field of characteristic zero, $\Phi : X \to Y$ a generically finite morphism of nonsingular proper k-varieties. Then there are birational morphisms of nonsingular complete k-varieties $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$, and a locally monomial morphism $\Psi : X_1 \to Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \overset{\Psi}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \overset{\Phi}{\to} & Y \end{array}$$

commutes and α and β are locally products of blowups of nonsingular subvarieties. That is, for every $z \in X_1$, there exist affine neighborhoods V_1 of z, V of $x = \alpha(z)$, such that $\alpha : V_1 \to V$ is a finite product of monoidal transforms, and there exist affine neighborhoods W_1 of $\Psi(z)$, W of $y = \alpha(\Psi(z))$, such that $\beta : W_1 \to W$ is a finite product of monoidal transforms.

A monoidal transform of a nonsingular k-scheme S is the map $T \to S$ induced by an open subset T of $\mathrm{Proj}(\oplus \mathcal{I}^n)$, where $\mathcal{I}$ is the ideal sheaf of a nonsingular subvariety of S.

If $\Phi : X \to C$ is a morphism from a projective variety to a curve, the existence of a global monomialization follows immediately from resolution of singularities. In the case of a morphism of complex projective surfaces, a proof of the existence of a global monomialization follows from results of Akbulut and King (Chapter 7 of [6]).

We have proven that a global monomialization exists for a morphism from a 3-fold to a surface [11].

Theorem 2.5. *(Theorem 1* [11]*) Let* $\Phi : X \to S$ *be a proper dominant morphism from a 3-fold* X *to a surface* S*, over an algebraically closed field* k *of characteristic zero. Then* ϕ *has a global monomialization.*

¿From our proof, we can in fact put the mapping in a more combinatorial form.

Theorem 2.6. *(Theorem 2* [11]*) Let* $\Phi : (U_X \subset X) \to (U_S \subset S)$ *be a dominant proper morphism of toroidal embeddings from a 3-fold* X *to a surface* S*. Then* ϕ *has a global monomialization such that* $\Psi : (\alpha^{-1}(U_X) \subset X_1) \to (\beta^{-1}(U_S) \subset S)$ *is a toroidal morphism.*

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ASYMPTOTIC REGULARITY

S. DALE CUTKOSKY

1. Asymptotic regularity of ideals

Let $A = k[x_1, \dots, x_r]$ be a polynomial ring over a field k. Let m be the maximal graded ideal. Suppose that $I \subset A$ is a homogeneous ideal. let

$$0 \to \cdots \to F_1 \to F_0 \to I \to 0$$

be a graded free resolution. Let

$$a_j = \max \{\text{degrees of generators of } F_j\}.$$

The regularity of I is defined to be

$$\begin{aligned} \operatorname{reg}(I) &= \max \{n \mid \text{ there exists } j \text{ such that } H^j_m(I)_{n-j} \neq 0\} \\ &= \max \{n \mid \text{ there exists } j \text{ such that } \operatorname{Tor}^A_j(k, I)_{n+j} \neq 0\} \\ &= \max \{a_j - j \mid j \geq 0\}. \end{aligned}$$

The equivalence of these definitions can be found in Eisenbud and Goto [7], Bayer and Mumford [1].

In general one expects that invariants of an ideal I behave better after taking a sufficiently high power of I. It is of interest to understand the asymptotic behaviour of the regularity of high powers of I.

Irena Swanson [10] has shown that there exists a bound D such that

$$\operatorname{reg}(I^n) \leq nD$$

for all $n \geq 0$.

Let I be any homogeneous ideal of A. In Theorem 1.1 of [5], which is joint work with Jürgen Herzog and Trung, it is shown that $\operatorname{reg}(I^n)$ is a linear polynomial for all n large enough. Let $d(I)$ be the maximal degree of a minimal set of generators of I.

Theorem 1.1. *(Theorem 1.1* [5]*) Let I be an arbitrary homogeneous ideal. Let $d(I)$ denote the maximum degree of the homogeneous generators of I. Then*

1. *There is a number e such that reg $(I^n) \leq nd(I) + e$ for all $n \geq 1$.*
2. *reg (I^n) is a linear polynomial for all n large enough.*

Similar results have been obtained independently by Vijay Kodiyalam [8].

Since we have bounds $d(I^n) \leq \operatorname{reg}(I^n)$ for all n, it follows from that the asypmptotic regularity

$$\lim \frac{\operatorname{reg}(I^n)}{n} = \lim \frac{d(I^n)}{n}$$

always exists, and is a natural number.

We will give an outline of the proof of Theorem 1. We will first discuss the proof of 1.

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Let $R = \bigoplus_{n\geq 0} I^n t^n$ be the Rees algebra of I. Let $I = (f_1, \dots, f_s)$ where the f_j are homogeneous of degree d_j. R is a bigraded $S = k[x_1, \dots, x_r, y_1, \dots, y_s]$ module, with the grading $\deg(y_j) = (d_j, 1)$, $\deg(x_j) = (1, 0)$. Let $N = (x_1, \dots, x_r)$.

$$H^i_N(S)_{(a,n)} = \begin{cases} 0 & i \neq r \\ 0 & a \geq nd(I) - r + 1, i = r \end{cases}$$

Consider a bigraded free resolution of R.

$$\cdots \to F^j = \bigoplus_t S(-a_{t_j}, -b_{t_j}) \to \cdots \to \bigoplus_t S(-a_{t_0}, -b_{t_0}) \to R \to 0$$

Set

$$c_j = \max_t\{a_{t_j} - b_{t_j} d(I)\},$$
$$e = \max\{c_j - j\}.$$

Since $H_N(F^{r-i})_{(m-i,n)} = 0$ implies $H^i_m(I^n)_{m-i} = H^i_N(R)_{(m-i,n)} = 0$, we get that $\mathrm{reg}(I^n) \leq nd(I) + e$.

Now we will give discuss the idea of the proof of 2. of Theorem 1.1.

$$\mathrm{Tor}^A_i(k, I^n)_a = \mathrm{Tor}^S_i(S/mS, R)_{(a,n)}.$$

Define

$$\mathrm{reg}_i(I^n) = \max\{a \mid \mathrm{Tor}^S_i(S/mS, R)_{(a,n)} \neq 0\} - i.$$

Then

$$\mathrm{reg}(I^n) = \max\{\mathrm{reg}_i(I^n), i \geq 0\}$$

Each $\mathrm{Tor}^S_i(S/mS, R)$ is a finitely generated bigraded module over $k[y_1, \dots, y_s]$. It follows that $\mathrm{reg}_i(I^n)$ is linear for $n >> 0$.

2. Asymptotic regularity of ideal sheaves

Let $\mathcal{I}$ be an ideal sheaf on $\mathbf{P}^r$. The regularity $\mathrm{reg}(\mathcal{I})$ of $\mathcal{I}$ is defined to be the least integer t such that $H^i(\mathbf{P}^r, \mathcal{I}(t-i)) = 0$ for all $i \geq 1$ (c.f. Mumford, Lecture 14 [9]).

Suppose that $\mathcal{I}$ is the sheafification of a homogeneous ideal I of A. The saturation of I is defined by

$$\widetilde{I} = \{f \in A \mid \text{ for each } 0 \leq i \leq n, \text{ there exists } n_i > 0 \text{ such that } X_i^{n_i} f \in I\}$$

The local cohomology of I is related to the global cohomology of $\mathcal{I}$ by the following well known exact sequences.

$$0 \to I \to \widetilde{I} = \bigoplus_n H^0(\mathbf{P}^r, \mathcal{I}(n)) \to H^1_m(I) \to 0$$

$$H^i(\mathbf{P}^r, \mathcal{I}(n)) \cong H^{i+1}_m(I)_n$$

for $i \geq 0$.

Thus $\mathrm{reg}(\mathcal{I}) = \mathrm{reg}(\widetilde{I})$.

It follows from 1. of Theorem 1.1 that if $\mathcal{I}$ is generated scheme theoretically in degree d, then there exists a constant e such that

$$\mathrm{reg}(\mathcal{I}^n) \leq nd + e$$

for all $n \geq 0$. This result was proven for nonsingular complex varieties by Bertram, Ein and Lazarsfeld [2].

Let H be a hyperplane section of $\mathbf{P}^n$. let

$$f : B = \mathrm{proj}(\bigoplus_{n\geq 0} \mathcal{I}^n) \to \mathbf{P}^r$$

be the blowup of $\mathcal{I}$, with exceptional divisor

$$E = \mathrm{proj}(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}).$$

Lawrence Ein and Robert Lazarsfeld [6] have defined the Seshadri constant $\epsilon(\mathcal{I})$ of $\mathcal{I}$ to be

$$\epsilon(\mathcal{I}) = \sup\{\eta \in \mathbf{Q} | f^*(H) - \eta E \text{ is ample }\}.$$

Lazarsfeld and Ein have proven that

$$\lim \frac{\mathrm{reg}(\mathcal{I}^n)}{n} = \frac{1}{\epsilon(\mathcal{I})},$$

the reciprical of the Seshadri constant of $\mathcal{I}$.

We give examples in in Theorem 10 [3] of ideal sheaves $\mathcal{I}$ of nonsingular complex space curves whose asymptotic regularity

$$\lim \frac{\mathrm{reg}(\mathcal{I}^n)}{n}$$

is irrational. In particular, 2. of Theorem 1.1 is not true for ideal sheaves.

These examples also give smooth space curves which have irrational Seshadri constants.

A particular realization of this example satisfiew

$$\mathrm{reg}(\mathcal{I}^r) = [r(9+\sqrt{2})] + 1 + \sigma(r)$$

for $r > 0$, where $[r(9+\sqrt{2})]$ is the greatest integer in $r(9+\sqrt{2})$,

$$\sigma(r) = \begin{cases} 0 & \text{if } r = q_{2n} \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

q_m is defined recursively by $q_0 = 1, q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$. Note that r such that $\sigma(r) = 0$ are quite sparse, as

$$q_{2n} \geq 3^n.$$

Example 10 of [3] uses the method of [4], which is to find a rational line which intersects the boundary of the cone of effective curves on a projective surface in irrational points.

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Toric varieties and schemes based on semigroups

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Introduction

Let N be a free $\mathbf{Z}$-module of rank $r \geq 0$. A subset $\sigma \subset N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ is said to be a **convex polyhedral cone** or simply a cone if $\sigma = \mathbf{R}_0 x_1 + \cdots + \mathbf{R}_0 x_s$ for a finite subset $\{x_1, \ldots, x_s\} \subset N_{\mathbf{R}}$, where $\mathbf{R}_0 := \{c \in \mathbf{R} \,;\, c \geq 0\}$. The cone σ is said to be **rational** if x_i's are in N, and **strongly convex** if $\sigma \cap (-\sigma) = \{0\}$. A subset η of a convex polyhedral cone σ is said to be a **face** and we write $\eta \prec \sigma$ if there exists a linear function l of $N_{\mathbf{R}}$ such that $\sigma \subset (l \geq 0)$ and $\eta = \sigma \cap (l = 0)$. A set X of strongly convex rational polyhedral cones in $N_{\mathbf{R}}$ is said to be a **fan** or rational fan if (1) X is nonempty, (2) $\sigma \in X$ and $\eta \prec \sigma$ imply $\eta \in X$ and (3) if $\sigma, \tau \in X$ then $\sigma \cap \tau$ is a common face of σ and τ. A **real fan** is defined to be a fan without the rationality condition of the cones.

The theory of toric varieties is based on the fact that each toric variety of dimension r corresponds to a fan in $\mathbf{R}^r$. In this note, we give a scheme theoretic definition of fans. Then we see that a fan itself is a kind of variety, and the associated toric variety is its base extension. This remark is not essentially new, however I cannot give a good reference now.

We have already had the theory of intersection complexes on "rational" fans as well as dualizing complexes and de Rham complexes. One of our purpose is to generalize these theories for "real" fans. We define a dualizing functor on a category of complexes on a real fan. A kind of Serre duality and Poincaré duality hold for this fonctor. Still we have not enough results for real fans.

1 Fan is a kind of scheme

By a semigroup we always mean a subsemigroup with 0 of a torsion free additive group.

Let S be a semigroup. A subset $I \subset S$ is said to be an **ideal** if $I + S \subset I$. An ideal P of S is an **prime ideal** if $P \neq S$ and $x, y \in S \setminus P$ implies $x + y \in S \setminus P$. The empty subset of S is a prime ideal of S by definition.

For a semigroup S, we denote by $\mathbf{1}$-Spec S the set of prime ideals of S.

Let $X := \mathbf{1}\text{-Spec}\, S$. For an element $m \in S$, we set $X_m := \{P \in X \; ; \; m \notin P\}$. The topology of X is defined by the open basis $\{X_m \; ; \; m \in S\}$. The structure sheaf $\mathcal{S}_X$ of X is a sheaf of semigroups such that $\mathcal{S}_X(X_m) = S + \mathbf{Z}m$ for every $m \in S$. We call the pair $(X, \mathcal{S}_X)$ an **affine 1-scheme**.

Example 1.1 For the semigroup $S = \{0\}$, we denote $\mathbf{1} := \mathbf{1}\text{-Spec}\, S$, and call it **the zerodimensional torus**.

Let $\phi : T \to S$ be a homomorphism of semigroups. If $P \subset S$ is a prime ideal, then the pull-back $\phi^{-1}(P)$ is a prime ideal of T. Hence, this correspondence $P \mapsto \phi^{-1}(P)$ defines a map

$$^a\phi : X = \mathbf{1}\text{-Spec}\, S \longrightarrow Y = \mathbf{1}\text{-Spec}\, T \; .$$

For each $m' \in T$, we have $^a\phi^{-1}(Y_{m'}) = X_{\phi(m')}$. In particular, $^a\phi$ is a continuous map. The homomorphism of sheaves $\phi' : {}^a\phi^{-1}\mathcal{S}_Y \to \mathcal{S}_X$ is defined so that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\phi} & S \\ \downarrow & & \downarrow \\ {}^a\phi^{-1}\mathcal{S}_Y(X) & \xrightarrow{\phi'(X)} & \mathcal{S}_X(X) \end{array}$$

commutes.

A general $\mathbf{1}$-scheme is defined as follows. The pair $(X, \mathcal{S}_X)$ of a topological space X and a sheaf $\mathcal{S}_X$ of semigroups is said to be a **1-scheme**, if X is covered by open subsets U such that $(U, \mathcal{S}_X|U)$ is isomorphic to an affine $\mathbf{1}$-scheme. Such an open set U is called an **affine open set** of X. We often denote the $\mathbf{1}$-scheme $(X, \mathcal{S}_X)$ simply by X.

Let $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ be $\mathbf{1}$-schemes. A pair $f = (f_0, \phi)$ of a continuous map $f_0 : X \to Y$ and a homomorphism $\phi : f_0^{-1}\mathcal{S}_Y \to \mathcal{S}_X$ of semigroup sheaves is said to be a **regular morphism** if it is locally a morphism of affine **1**-schemes at each point of X.

Let S and T be semigroups. A natural bijiction

$$\mathbf{1}\text{-Spec}\, S \times \mathbf{1}\text{-Spec}\, T \longrightarrow \mathbf{1}\text{-Spec}(S \oplus T)$$

is defined by $(P_1, P_2) \mapsto (P_1 \oplus T) \cup (S \oplus P_2)$.

Let X, Y be **1**-schemes. We denote by $X \times_{\mathbf{1}} Y$ the set theoretical product $X \times Y$ with the product topology. We define the **1**-scheme structure on $X \times_{\mathbf{1}} Y$ as follows. For affine open subsets $U = \mathbf{1}\text{-Spec}\, S \subset X$ and $V = \mathbf{1}\text{-Spec}\, T \subset Y$, we define

$$\mathcal{S}_{X \times_{\mathbf{1}} Y}(U \times V) = S \oplus T\,.$$

Then we get a product **1**-scheme $(X \times_{\mathbf{1}} Y, \mathcal{S}_{X \times_{\mathbf{1}} Y})$. In particular, $\mathbf{1}\text{-Spec}(S \oplus T) = \mathbf{1}\text{-Spec}\, S \times_{\mathbf{1}} \mathbf{1}\text{-Spec}\, T$.

For a cone σ in $N_{\mathbf{R}} \simeq \mathbf{R}^r$, the **dual cone** $\sigma^\vee \subset M_{\mathbf{R}} \simeq \mathbf{R}^r$ is defined by

$$\sigma^\vee := \{x \in M_{\mathbf{R}}\,;\, \langle x, u\rangle \geq 0, \forall u \in \sigma\}\,.$$

It is a cone in $M_{\mathbf{R}}$. The dual cone of a cone in $M_{\mathbf{R}}$ is defined in $N_{\mathbf{R}}$ similarly, and the equality $(\sigma^\vee)^\vee = \sigma$ holds. We also define

$$\sigma^\perp := \{x \in M_{\mathbf{R}}\,;\, \langle x, u\rangle = 0, \forall u \in \sigma\}$$

which is a linear subspace of $M_{\mathbf{R}}$. Let π be a cone of $N_{\mathbf{R}}$. Then $\pi^\vee \cap \sigma^\perp$ is a face of $\pi^\vee$ for every face σ of π.

Let X be a fan of $N_{\mathbf{R}}$. We regard X a **1**-scheme as follows. For each element $\sigma \in X$, we denote by $F(\sigma)$ the set of faces of σ. A bijection $\phi_\sigma : F(\sigma) \to \mathbf{1}\text{-Spec}(M \cap \sigma^\vee)$ is defined by $\phi_\sigma(\eta) := M \cap (\sigma^\vee \backslash \eta^\perp)$. By this bijection, we introduce an affine **1**-scheme structure on $F(\sigma)$. These **1**-scheme structures of $F(\sigma)$ for $\sigma \in X$ are naturally patched together as open subschemes, and we get a **1**-scheme structure on X.

A **1**-scheme X is said to be **locally of finite type** if $\mathcal{S}_X(U)$ is finitely generated for every affine open set $U \subset X$. We say that X is **of finite type** if it is locally of finite type and is covered by a finite number of affine open subsets.

Lemma 1.2 *Let X be a **1**-scheme locally of finite type. For each point $x \in X$, there exists a unique affine open set $U = \mathbf{1}\text{-Spec}\, S$ of X such that x is a closed point of U.*

We denote by $\mathcal{S}_{X,x}$ the semigroup S in the above lemma. $\mathcal{S}_{X,x}$ is called the **local semigroup** of X at x. The local semigroups are defined also for points in **1**-schemes which are not necessary locally of finite type.

Let x, x' be points of **1**-scheme X. x' is said to be a **specialization** of x if x' is contained in the closure of $\{x\}$ in X. x is said to be a **generalization** of x'.

Let S be a semigroup. Then the semigroup ring $A[S]$ is defined for any commutative ring A. $A[S]$ is defined as the free A-module $\bigoplus_{m \in S} A\mathbf{e}(m)$ with the basis $\{\mathbf{e}(m)\ ;\ m \in S\}$. The multiplications in $A[S]$ is defined by $\mathbf{e}(m)\mathbf{e}(m') = \mathbf{e}(m + m')$ and $\mathbf{e}(0) = 1$ in $A[S]$. We denote by ϕ_A the map $S \to A[S]$ defined by $\phi_A(m) := \mathbf{e}(m)$. If $P \subset A[S]$ is a prime ideal, then $\phi_A^{-1}(P)$ is a prime ideal of the semigroup S. Hence we get a map ${}^a\phi_A : \operatorname{Spec} A[S] \to \mathbf{1}\text{-}\operatorname{Spec} S$. The morphism of sheaves ${}^a\phi_A^* \mathcal{S}_{\mathbf{1}\text{-}\operatorname{Spec} S} \to \mathcal{O}_{\operatorname{Spec} A[S]}$ is defined naturally.

Let X be a **1**-scheme. For a commutative ring A, an A-scheme X_A is defined as follows. For each affine open subset $U = \mathbf{1}\text{-}\operatorname{Spec} S \subset X$, we set $U_A = \operatorname{Spec} A[S]$. Then these affine A-schemes U_A are naturally patched together to an A-scheme X_A. We also write it by $X \times_{\mathbf{1}} \operatorname{Spec} A$, and call it the **base extension** of X to A.

A homomorphism $\phi : T \to S$ of semigroups is said to be **finite** if there exist finite elements $x_1, \ldots, x_s \in S$ such that

$$S = (x_1 + \phi(T)) \cup \cdots \cup (x_s + \phi(T)) .$$

For any nontrivial commutative ring A, it is easy to see that ϕ is finite if and only if $A[S]$ is a finitely generated $A[T]$-module with respect to the associated ring homomorphism $A[T] \to A[S]$.

A morphism $f : X \to Y$ of **1**-schemes is said to be **finite** if, for any affine open subset $V \subset Y$, the pull-back $U = f^{-1}(V)$ is an affine open subset of X, and the associated homomorphism $\mathcal{S}_Y(V) \to \mathcal{S}_X(U)$ of semigroups is finite.

For a **1**-scheme X, the diagonal morphism $\Delta_X : X \to X \times_{\mathbf{1}} X$ is defined naturally. A **1**-scheme is said to be **separated** if the diagonal morphism is finite.

For affine open subsets U, V of X, the pull-back $\Delta^{-1}(U \times V)$ is equal to $U \cap V$.

For a finitely generated semigroup S, we denote by $M(S)$ the minimal additive group containing S. We denote by $C(S)$ the cone generated by S in the real space $M(S)_{\mathbf{R}} := M(S) \otimes_{\mathbf{Z}} \mathbf{R}$. The dual $\mathbf{Z}$-module of $M(S)$ is denoted by $N(S)$.

Lemma 1.3 *Let $\phi : T \to S$ be an injective homomorphism of finitely generated semigroups. Then ϕ is finite if and only if $\phi_{\mathbf{R}}(C(T)) = C(S)$, where $\phi_{\mathbf{R}} : M(T)_{\mathbf{R}} \to M(S)_{\mathbf{R}}$ is the linear map extending ϕ.*

Proof. Since the extended homomorphism $\phi_{\mathbf{Z}} : M(T) \to M(S)$ is injective, we may assume that $M(T)$ is a submodule of $M(S)$ and $\phi_{\mathbf{R}}$ is an inclusion map. Clearly the cone $C(T)$ is contained in $C(S)$.

Assume that ϕ is finite. There exists a nonempty finite set $\{x_1, \ldots, x_s\}$ of S such that

$$S = (x_1 + T) \cup \cdots \cup (x_s + T) .$$

It is sufficient to show the dual cones $C(T)^\vee$ and $C(S)^\vee$ in $N(S)_{\mathbf{R}}$ are equal. The inclusion $C(T) \subset C(S)$ implies $C(S)^\vee \subset C(T)^\vee$. Let u be an element of $N(S)_{\mathbf{R}} \setminus C(S)^\vee$. Then there exist $x \in S$ with $\langle x, u \rangle < 0$. We take sufficiently large integer $a > 0$ such that $\langle ax, u \rangle < \langle x_i, u \rangle$ for $i = 1, \ldots, s$. Since $as \in S$, there exist i and $y \in T$ with $ax = x_i + y$. Then $\langle y, u \rangle = \langle ax, u \rangle - \langle x_i, u \rangle < 0$. Hence u is not in $C(T)^\vee$. This implies $C(T)^\vee \subset C(S)^\vee$.

Conversely, assume $C(T) = C(S)$. Let k be an arbitrary field. The assumption implies that, for any element $x \in T$, there exists a positive integer d with $dx \in S$. Hence the semigroup ring $k[S]$ is integral over the subring $k[T]$. Since $k[S]$ is finitely generated, it is an $k[T]$-module of finte type. Hence the morphism of the semigroups is finite.

q.e.d.

Let X be a $\mathbf{1}$-scheme locally of finite type. X is said to be **irreducible** if there exists a point $\mathbf{0} \in X$ such that X is the closure of $\{\mathbf{0}\}$. Then $\mathbf{0}$ is called the **generic point** of X. If $\mathbf{0}$ is the generic point of X, then $\mathcal{S}_{X,\mathbf{0}}$ is a free

$\mathbf{Z}$-module of finite rank. We call this group the **base group** of X. In this case, $\mathcal{S}_X(U)$ is a subsemigroup of $\mathcal{S}_{X,\mathbf{0}}$ for any affine open subset $U \subset X$.

Theorem 1.4 *Let M and N be mutually dual free $\mathbf{Z}$-modules of rank $r \geq 0$. A $\mathbf{1}$-scheme X is a separated normal irreducible $\mathbf{1}$-scheme locally of finite type with the base group M if and only if X is a fan of $N_{\mathbf{R}}$.*

Proof. For each element $x \in X$, we set $\sigma_x := C(\mathcal{S}_x)^\vee \subset N_{\mathbf{R}}$.

Let x, y be elements of X. Then, for the diagonal morphism $\Delta : X \to X \times X$, we have

$$\Delta^{-1}(F(\sigma_x) \times F(\sigma_y)) = \{z \in X \; ; x, y \in \overline{\{z\}}\};.$$

Since X is separated, this is an affine open subset of X. Hence this is equal to $F(\sigma_w)$ for some $w \in X$. Since the morphism $F(\sigma_w) \to F(\sigma_x \times \sigma_y)$ is finite, the inclusion map $\mathcal{S}_x + \mathcal{S}_y \subset \mathcal{S}_w$ is finite. By Lemma 1.3, the cone $C(\mathcal{S}_w)$ is equal to $C(\mathcal{S}_x) + C(\mathcal{S}_y)$. By taking dual cones, we have $\sigma_w = \sigma_x \cap \sigma_y$. Hence σ_w is a common face of σ_x and σ_y.

In particular, if $\sigma_x = \sigma_y$, then we have $\sigma_w = \sigma_x$ and hence $w = x$ since w is a generalization of x. Similarly we have $w = y$. Hence $\sigma_x = \sigma_y$ implies $x = y$. This means we may regard X a set of cones in $N_{\mathbf{R}}$. Then X is a fan by the above observation.

q.e.d.

Let X be a fan. Then the associated toric variety over a field k is the base extension $X_k = X \times_{\mathbf{1}} \operatorname{Spec} k$.

2 Real fans

Let M be a free $\mathbf{Z}$-module of rank $r \geq 0$.

A real fan X in $M_{\mathbf{R}}$ is regarded as a $\mathbf{1}$-scheme as follows. The topology of X is defined similarly as "rational" fans. Namely, $\{F(\sigma) \; ; \sigma \in X\}$ is an open basis of X. For each $\sigma \in X$, we set $\mathcal{S}_X(F(\sigma)) := \sigma^\vee$, which is not finitely generated in general.

In [1], we studied on the categories of complexes of graded modules on rational fans. De Rham complexes and the intersection complexes are described in this category. We intend to generalize this theory for real fans. Up to now, we could define the category of graded exterior modules on a real fan, and define a dualizing functor of it.

References

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Hilbert-Kunz multiplicity, McKay correspondence and Good ideals in 2-dimensional Rational Singularities

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§1. Introduction

Hilbert-Kunz multiplicity is known to be a very mysterious invariant of a ring or an ideal. We will show a very beautiful formula on Hilbert-Kunz multiplicity for integrally closed ideals in 2-dimensional Gorenstein rational singularities. In the proof, McKay correspondence and "Riemann-Roch formula" play essential roles. Also this formula gives a new significance to "good ideals".

In this talk, $(A, \mathfrak{m}, k)$ is a Noetherian local ring of characteristic $p > 0$ with $d = \dim A > 0$ and the residue field k is algebraically closed.

First we recall the definition of Hilbert-Kunz multiplicity. This notion was defined implicitly by Kunz ([Ku1],1969) using the Frobenius morphism in characteristic $p > 0$ and formulated explicitly by Monsky ([Mo],1983).

Definition (1.1). ([Ku1,Ku2,Mo]) Let A be as above. Then the *Hilbert-Kunz multiplicity* of an $\mathfrak{m}$-primary ideal I is defined as

$$e_{HK}(I) := \lim_{e\to\infty} \frac{l_A(A/I^{[q]})}{q^d} = \lim_{e\to\infty} \frac{l_A(A^{1/q}/I.A^{1/q})}{q^d},$$

where $I^{[q]}$ ($q = p^e$) is the ideal generated by the q-th powers of all elements of I. By definition the Hilbert–Kunz multiplicity of A is $e_{HK}(A) = e_{HK}(\mathfrak{m})$.

(1.2) Let $e(I)$ denote the usual multiplicity of I. Then inequalities $\frac{e(I)}{d!} \le e_{HK}(I) \le e(I)$ hold; see [Hu]. Suppose that A is Cohen-Macaulay. Then it is well-known that $e(I) \ge l_A(A/I)$ and equality holds if and only if I is generated by an A-regular sequence. On the other hand, Dutta [Du,Theorem (1.9)] proved that if A is a complete intersection local ring then

(1.2.1) $e_{HK}(I) \ge l_A(A/I)$.
(1.2.2) If, in addition, $\mathrm{pd}_A\, A/I < \infty$, then equality holds in (1.2.1).[1]

[1] After this symposium, Kurano and Hashimoto informed us that if A is a Cohen-Macaulay Roberts ring (e.g. complete intersections, quotient singularities etc.), a local ring with $\dim A \le 2$ or a Gorenstein local ring of dimension 3 then equality (1.2.2) holds for any $\mathfrak{m}$-primary ideal of finite projective dimesion.

But, it is an open question whether the inequality $e_{HK}(I) \geq l_A(A/I)$ always holds or not. It is important to investigate this inequality (and equality) in studying the relationship between Hilbert-Kunz multiplicity and the other invariants of a local ring; see [WY1,WY2]. Furthermore, our main theorem in this talk gives a new significance to the difference "$e_{HK}(I) - l_A(A/I)$" in the theory of singularities.

In general, it is not so easy to compute $e_{HK}(I)$; see e.g. [HM,BCP]. In fact, we do not know even whether $e_{HK}(I)$ is a rational number or not. But, in case of quotient singularities, we can calculate $e_{HK}(I)$ by the following formula.

Proposition A. ([BCP,WY]) *Let $(A, \mathfrak{m}) \subset (B, \mathfrak{n})$ be an extension of local domains where B is a finite A-module of rank r and $A/\mathfrak{m} = B/\mathfrak{n}$. Then for every $\mathfrak{m}$-primary ideal I, $e_{HK}(I, A) = \frac{1}{r} e_{HK}(IB, B)$.*

In particular, if B is regular, then $e_{HK}(I) = \frac{1}{r} l_B(B/IB)$.

Next, we recall the notion of "good ideals" in a rational singularity of dimension 2.

Let A be a two-dimensional normal local ring. Then A is said to be a *rational singularity* if there exists a resolution of singularities[2] $f : X \to \operatorname{Spec} A$ such that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Remark. ([Li]) If A is a two-dimensional excellent normal local ring, then there exists a resolution of singularities $f : X \to \operatorname{Spec} A$ (even if characteristic of A is $p > 0$).

In the following, assume that A is a rational singularity of dimension 2 and $f : X \to \operatorname{Spec} A$ is a resolution of singularities with $E := \cup_{i=1}^r E_i = f^{-1}(\mathfrak{m})$ the exceptional divisor on X, where E_i's are the irreducible components of E. Then an ideal I in A is said to be *represented on* X if the sheaf $I\mathcal{O}_X$ is invertible and $I = H^0(X, I\mathcal{O}_X)$.

If an ideal I is represented on some resolution X, then it is integrally closed. Conversely, any integrally closed ideal can be represented on some resolution X. In fact, Giraud [Gi] showed the following

Theorem B. ([Li, §18], [Gi]) *Assume that A is a rational singularity of dimension 2. Then there is a one-to-one correspondence between the set of integrally closed $\mathfrak{m}$-primary ideals I in A that are represented on X and the set of effective anti-nef divisors $Z = \sum_{i=1}^r a_i E_i$ on X (i.e. $a_i \geq 0$, $ZE_i \leq 0$ for all i). The correspondence is given by $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = \mathrm{H}^0(X, \mathcal{O}_X(-Z))$.*

A resolution $f : X \to \operatorname{Spec} A$ is *minimal* if X contains no (-1) curves.[3] In case of two-dimensional local rings, such a minimal resolution is unique up to isomorphism.

Suppose that $f : X \to \operatorname{Spec} A$ is a minimal resoulution. In the set $\mathcal{C}$ of cycles supported on $f^{-1}(\mathfrak{m})$, we can define a partial order $\leq$ as follows: for Z, $Z' \in \mathcal{C}$, $Z \leq Z'$ if $Z' - Z$ is an effective divisor on X. Then the *fundamental cycle* Z_0 is the minimum element with respect to $\leq$ among all positive anti-nef divisors.

Definition (1.3). ([GIW]) Assume that A is a rational singularity of dimension 2. An ideal I of A is called *good* if I is integrally closed and represented on the minimal resolution $f : X \to \operatorname{Spec} A$.

[2] $f : X \to \operatorname{Spec} A$ is a projective and birational morphism and X is a regular scheme.

[3] A curve C is said to be (-1) *curve* if $C \cong \mathbb{P}_k^1$ and $C^2 = -1$.

In [GIW], Goto et.al. defined the notion of good ideals for any Gorenstein local ring as follows: Let I be an $\mathfrak{m}$-primary ideal of a Gorenstein local ring A. Then I is "*good*" if the associated graded ring $gr_I(A)$ is a Gorenstein ring with $a(gr_I(A)) = 1 - d$.

Further, the following fact has been proved in [GIW], which justifies our definition for good ideals in (1.3).

Theorem C. (cf. [GIW]) *Suppose that A is Gorenstein. Then I is good if and only if one of the following conditions holds.*

(1) *I is stable and $e(I) = 2 \cdot l_A(A/I)$.*

(2) *$I^2 = JI$ and $I = J : I$ for some (any) minimal reduction J of I.*

Furthermore, if A is a rational singularity of dimension 2, then the following condition is also equivalent to the above ones.

(3) *I is an integrally closed ideal represented on the minimal resolution.*

We are now ready to state our main result in this talk.

Main Theorem (1.4). *Let A be a two-dimensional rational Gorenstein local ring and assume that A is a pure subring of a regular ring B, which is a finite A-module of rank N.*

(1) *Let $I = \mathrm{H}^0(X, \mathcal{O}_X(-Z))$ be a good ideal such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and put $Z = \sum_{i=1}^r a_i E_i$, where $E_1, \ldots, E_r$ are irreducible exceptional curves on a minimal resolution $X \to \operatorname{Spec} A$. Then we have*

$$e_{HK}(I) - l_A(A/I) = \frac{\sum_{i=1}^r a_i n_i}{N},$$

where n_i is determined by the fundamental cycle $Z_0 = \sum_{i=1}^r n_i E_i$.

(2) *Let I be arbitrary integrally closed ideal of A and let $\widetilde{I}$ be the "good closure" of I (that is, $\widetilde{I}$ is the minimum good ideal with $\widetilde{I} \supset I$). Then we have*

$$e_{HK}(I) - l_A(A/I) = e_{HK}(\widetilde{I}) - l_A(A/\widetilde{I}).$$

Also, if A is a rational singularity which may not be Gorenstein, we have the following

Proposition (1.5). *Assume that A is a rational singularity of dimension 2 and let $I \subseteq I'$ be integrally closed ideals in A. Then*

$$e_{HK}(I) - l_A(A/I) \geq e_{HK}(I') - l_A(A/I')$$

and equality holds if I and I' have the same good closures. In particular, $e_{HK}(I) > l_A(A/I)$ for integrally closed ideals.

As an application of the above proposition, we can show the following

Corollary (1.6). *Let A be a rational singularity of dimension 2. Put*

$$\nu(n) := \sup \left\{ \frac{e_{HK}(I)}{e(I)} \;\middle|\; I \text{ is an integrally closed ideal with } I \subseteq \mathfrak{m}^n \right\} \quad \text{for each } n \geq 1.$$

Then we have $\lim_{n \to \infty} \nu(n) = \dfrac{1}{2}$.

Remark (1.7). *Let A be a two-dimensional local ring of characteristic $p > 0$. If there exists an $\mathfrak{m}$-primary integrally closed ideal I of finite projective dimension, then A is regular.*[4]

In fact, we take an ideal I mentioned as above. Since such an ideal I is tightly closed, we have that A is F-rational by Aberbach [Ab,Theorem (1.1)]. Thus Proposition (1.5) implies that $e_{HK}(I) > l_A(A/I)$ unless A is regular. On the other hand, since I has a finite projective dimension, we have $e_{HK}(I) = l_A(A/I)$ by Dutta [Du]; see (1.2.2). Hence A must be regular. □

Example (1.8). Let A be a two-dimensional complete local ring which is isomorphic to $\widehat{k[x,y]}^G$, where G is a finite subgroup of $SL(2,k)$. Then A is F-rational and Gorenstein.

Moreover, since $e(A) = 2$, the maximal ideal $\mathfrak{m}$ is a good ideal. In fact, $\mathfrak{m} = \mathrm{H}^0(X, \mathcal{O}_X(-Z_0))$, where $Z_0 = \sum_{i=1}^r n_i E_i$ is the fundamental cycle of a minimal resolution X. Thus Theorem (1.4) implies that

$$e_{HK}(\mathfrak{m}) = l_A(A/\mathfrak{m}) + \frac{1}{|G|}\sum_{i=1}^r n_i^2 = 1 + \frac{|G|-1}{|G|} = 2 - \frac{1}{|G|};\ \text{see also [WY1, §5].}$$

§2. Proof of the Main Theorem

The tools of the proof of the Main theorem are so-called "McKay correspondence" and "Riemann-Roch formula".

First, we recall the McKay correspondence.

Proposition D. (McKay correspondence) ([AV,Yo]) *Let $B = k[[X,Y]]$ and let G be a subgroup of $SL(2,k)$ such that $|G| \neq 0$ in k and $A = B^G$, the invariant subring of B. Then we have the following facts.*

(1) *As an A-module, $B = \bigoplus_{i=0}^r M_i^{n_i}$, where $\{M_i\}$ coincides with the set of isomorphism classes of indecomposable maximal Cohen-Macaulay A-modules with $n_i = \operatorname{rank} M_i$ and $M_0 = A$.*

(2) *There is a one-to-one correspondence between the set $\{M_i\}$ and the set of isomorphism classes of irreducible representations of G via the group algebra $B[G]$.*

(3) *Let $f : X \to \operatorname{Spec} A$ be the minimal resolution of A and let $Z = \sum_{i=1}^r n_i E_i$ be the fundamental cycle on X. Then if we choose the indices suitably, then $c(\mathcal{M}_i).E_j = \delta_{ij}$ and $\operatorname{rank} M_i = n_i$ (where $c(*)$ denotes the Chern class and $\mathcal{M}_i = f^*(M_i)/torsion$).*

Next, we recall the *Kato's Riemann-Roch formula*, which plays the central role in the proof of our Main Theorem and (1.5).

Theorem E (Kato's Riemann-Roch formula). ([Li]),[Ka]) *Let $f : X \to \operatorname{Spec} A$ be a resolution of singularities of A, where A is any normal local ring of dimension 2. Let $\mathcal{F}$ be a locally free sheaf on X of rank r and let Z be a cycle supported on $f^{-1}(\mathfrak{m})$. If we put*

$$\chi(\mathcal{F}) := l_A\left(\frac{\mathrm{H}^0(X - E, \mathcal{F})}{\mathrm{H}^0(X, \mathcal{F})}\right) + l_A(\mathrm{H}^1(X, \mathcal{F})),$$

[4] Recently, this result was generalized by Goto to arbitrary rings of arbitrary dimension.

then we have

$$\chi(\mathcal{F}(Z)) - \chi(\mathcal{F}) = r \cdot \frac{Z \cdot (K_X - Z)}{2} - c(\mathcal{F}) \cdot Z,$$

where K_X denotes the canonical divisor on X and $c(\mathcal{F})$ is the Chern class of $\mathcal{F}$ defined by

$$c(\mathcal{F}) \cdot E_i = \deg(\mathcal{F}|_{E_i}) = \chi(E_i, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{E_i}) - r \cdot \chi(E_i, \mathcal{O}_{E_i})$$

for every irreducible component E_i of E.

Corollary (2.1). *Assume that A is a rational singularity, I is an integrally closed* $\mathfrak{m}$*-primary ideal and $f : X \to \operatorname{Spec} A$ is a resolution of its singularity such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ is invertible, where Z is an effective anti-nef divisor supported on $f^{-1}(\mathfrak{m})$.*

Also, let M be a finitely generated reflexive A-module of $\operatorname{rank} M = r$, *and let $\mathcal{M} = f^*(M)/torsion$.*[5] *Then we have*

(1) $\chi(\mathcal{M}(-Z)) = l_A(M/IM) = -r \cdot \dfrac{Z \cdot (K_X + Z)}{2} + c(\mathcal{M}) \cdot Z.$

In particular,

(2) $\chi(\mathcal{O}_X(-Z)) = l_A(A/I) = -\dfrac{Z \cdot (K_X + Z)}{2}.$

Proof. ([Li]) Since A is a rational singularity, $\mathrm{H}^1(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X generated by its global sections. In particular, we have $\mathrm{H}^1(X, \mathcal{M}) = 0$ and $\mathrm{H}^1(X, \mathcal{O}_X(-Z)) = 0$. Also, since M is reflexive, we have $\mathrm{H}^0(X - E, \mathcal{M}(-Z)) = \mathrm{H}^0(X - E, \mathcal{M}) = \mathrm{H}^0(X, \mathcal{M}) = M$. Hence $\chi(\mathcal{M}) = 0$. Also, let $a_1, \dots, a_r$ be a system of generators of I, and let $\pi : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X(-Z)$ be the surjection defined by $a_1, \dots, a_r$ and put $\mathcal{F} := \operatorname{Ker} \pi$. From the exact sequence

(2.1.1) $$0 \longrightarrow \mathcal{F} \to \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{O}_X(-Z) \longrightarrow 0$$

we have the long exact sequence of cohomology modules

$$\mathrm{H}^0(X, \mathcal{O}_X)^{\oplus r} \longrightarrow I = \mathrm{H}^0(X, \mathcal{O}_X(-Z)) \longrightarrow \mathrm{H}^1(X, \mathcal{F}) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X)^{\oplus r} = 0.$$

Since the first map is surjective, we have $\mathrm{H}^1(X, \mathcal{F}) = 0$ and also $\mathrm{H}^1(X, \mathcal{F} \otimes \mathcal{M}) = 0$ since $\mathcal{M}$ is generated by its global sections. Now, tensoring $\mathcal{M}$ with π, we get an exact sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{M} \longrightarrow \mathcal{M}^{\oplus r} \longrightarrow \mathcal{M}(-Z) \longrightarrow 0.$$

Taking the global sections and noting that $\mathrm{H}^1(X, \mathcal{F} \otimes \mathcal{M}) = 0$, we have the surjection $M^{\oplus r} = \mathrm{H}^0(X, \mathcal{M})^{\oplus r} \longrightarrow \mathrm{H}^0(X, \mathcal{M}(-Z))$ sending $(x_1, \dots, x_r)$ to $\sum_{i=1}^r a_i x_i$. This shows that $\mathrm{H}^0(X, \mathcal{M}(-Z)) = IM$ and $\chi(\mathcal{M}(-Z)) = l_A(M/IM)$. Now our results follows from Theorem E taking $\chi(\mathcal{M}(-Z)) - \chi(\mathcal{M})$. $\square$

[5] $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module generated by its global sections (cf. [AV]).

Remark (2.2). Note that $c(\mathcal{F}) \cdot E_i \geq 0$ if $\mathcal{F}$ is generated by its global sections. In particular, $c(\mathcal{M}) \cdot Z \geq 0$ in (2.1). This fact is essential in the proof of (1.5).

Although Theorem E is contained essentially in [Li] or [Ka], it is stated in a different form in these literatures. So, for the convenience of the readers, we put a proof here. The key of the proof is in the following Lemma.

Lemma (E-1). *Let $f : X \to \operatorname{Spec} A$ be as in Theorem E and let Z, Z' be cycles supported on $f^{-1}(\mathfrak{m}) = \bigcup_{i=1}^{r} E_i$ such that $Z' = Z + E_i$ for some i. Then*

$$\chi(\mathcal{F}(Z')) - \chi(\mathcal{F}(Z)) = -\chi(E_i, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}).$$

where $\chi(E, \mathcal{L}) := l_A(\mathrm{H}^0(E, \mathcal{L})) - l_A(\mathrm{H}^1(E, \mathcal{L}))$ denotes the "Euler characteristic of $\mathcal{L}$" for any coherent sheaf $\mathcal{L}$ over a curve E.

Proof. From the exact sequence

$$0 \to \mathcal{F}(Z') \otimes \mathcal{O}_X(-E_i) = \mathcal{F}(Z) \to \mathcal{F}(Z') \to \mathcal{F}(Z') \otimes \mathcal{O}_{E_i} \to 0,$$

we get the following long exact sequence of cohomology modules

$$\begin{aligned} 0 \to \mathrm{H}^0(X,\, \mathcal{F}(Z)) &\xrightarrow{\alpha_1} \mathrm{H}^0(X,\, \mathcal{F}(Z')) \xrightarrow{\alpha_2} \mathrm{H}^0(E_i,\, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}) \\ \to \mathrm{H}^1(X,\, \mathcal{F}(Z)) &\xrightarrow{\alpha_3} \mathrm{H}^1(X,\, \mathcal{F}(Z')) \xrightarrow{\alpha_4} \mathrm{H}^1(E_i,\, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}) \to 0. \end{aligned}$$

On the other hand, for any cycle Z on E, we have

$$0 \to \mathrm{H}^0(X, \mathcal{F}(Z)) \to \mathrm{H}^0(X - E, \mathcal{F}(Z)) \to C \to 0$$

for some C. Thus we obtain the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{F}(Z)) & \longrightarrow & \mathrm{H}^0(X - E, \mathcal{F}(Z)) & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & & \\ 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{F}(Z')) & \longrightarrow & \mathrm{H}^0(X - E, \mathcal{F}(Z')) & \longrightarrow & C' & \longrightarrow & 0, \end{array}$$

where the second vertical map β_2 is isomorphism. Therefore we get

$$\begin{aligned} &\dim_k \mathrm{H}^1(X, \mathcal{F}(Z')) - \dim_k \mathrm{H}^1(X, \mathcal{F}(Z)) \\ &= \dim_k \mathrm{H}^1(E_i, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}) - \dim_k \mathrm{H}^0(E_i, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}) \\ &\quad + \dim_k H^0(X, \mathcal{F}(Z')) - \dim_k \mathrm{H}^0(X, \mathcal{F}(Z)) \\ &= -\chi(E_i, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}) + \dim_k \operatorname{Cok} \beta_1 \\ &= -\chi(E_i, \mathcal{F}(Z') \otimes \mathcal{O}_{E_i}) + \dim_k C - \dim_k C'. \end{aligned}$$

This yields the required assertion. □

For the convenience of the readers, we recall the Riemann-Roch formula on a curve E.

Lemma (E-2). (cf. [Li, §10]) *Let E be an irreducible curve on the non-singular surface X, and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of* $\operatorname{rank}\mathcal{F} = r$. *Then we have*

$$\chi(E, \mathcal{F}\otimes\mathcal{O}_E) = r\cdot\chi(E,\mathcal{O}_E) + \deg(\mathcal{F}\otimes\mathcal{O}_E).$$

Remark (E-3). Under the same notation as in Lemma (E-2), we have

$$\chi(E,\mathcal{O}_E) = -\frac{E\cdot(K_X+E)}{2}.$$

Furthermore, if $E\cong\mathbb{P}^1$, then $\chi(\mathcal{O}_E)=1$.

Proof of Theorem E. Let $Z=\sum_{i=1}^r n_iE_i$ and we prove by induction on $N(Z):=\sum_{i=1}^r |n_i|$. If $N(Z)=0$, then $Z=0$ and statement is trivial. Then put $Z'=Z+E_i$ as in Lemma E-1. Now, by Lemma E-1, $\chi(\mathcal{F}(Z'))-\chi(\mathcal{F}(Z)) = -\chi(E_i,\mathcal{F}(Z')\otimes\mathcal{O}_{E_i})$ and the latter is equal to $-\deg(\mathcal{F}(Z')\otimes\mathcal{O}_{E_i})-r\chi(\mathcal{O}_{E_i}) = -(rZ'+c(\mathcal{F}))\cdot E_i + r\dfrac{(K_X+E_i)\cdot E_i}{2}$ (cf. [Li, §10]). Now, $N(Z')$ is equal to either $N(Z)+1$ or $N(Z)-1$. Assume that Theorem E is true for Z and we will show for Z'. Then

$$\begin{aligned}\chi(\mathcal{F}(Z'))-\chi(\mathcal{F}) &= \left\{\frac{rZ\cdot(K_X-Z)}{2} - c(\mathcal{F})Z\right\} - \chi(E_i,\mathcal{F}(Z')\otimes\mathcal{O}_{E_i})\\ &= \left\{\frac{rZ\cdot(K_X-Z)}{2} - c(\mathcal{F})Z\right\} - (rZ'+c(\mathcal{F}))\cdot E_i + r\frac{(K_X+E_i)\cdot E_i}{2}\\ &= \frac{rZ'\cdot(K_X-Z')}{2} - c(\mathcal{F})Z',\end{aligned}$$

which proves Theorem E in this case and the same argument shows also the case $N(Z') = N(Z)-1$. □

Outline of the proof of Theorem (1.4) and Proposition (1.5). Let A be a two-dimensional rational Gorenstein local ring, and let $f: X\to\operatorname{Spec}A$ be the minimal resolution. Put $Z_0=\sum_{i=1}^r n_iE_i$ be the fundamental cycle on X. Also, let I be a good ideal and take the corresponding anti-nef divisor $Z=\sum_{i=1}^r a_iE_i$ such that $I = \mathrm{H}^0(X,\mathcal{O}_X(-Z))$. Then we must show $N\cdot e_{HK}(I) = N\cdot l_A(A/I)+\sum_{i=1}^r a_in_i$.

(1) Since A is a finite subring of a regular local ring B, we have $N\cdot e_{HK}(I) = l_B(B/IB)$ by Proposition A. Further, from the McKay correspondence, the ring B can be written as

$$B = \bigoplus_{r=0}^{r} M_i^{\oplus n_i} = A\bigoplus\bigoplus_{r=1}^{r} M_i^{\oplus n_i},$$

where $\{M_0(=A), M_1,\ldots,M_r\}$ is the set of indecomposable maximal Cohen-Macaulay A-modules such that $\operatorname{rank}M_i=n_i$ for all $i=0,1,\ldots,r$.

Put $\mathcal{M}_i = f^*(M_i)/\text{torsion}$ for each i. Then by (2.1), we have

$$l_A(M_i/I.M_i) = \chi(\mathcal{M}_i(-Z)) = \operatorname{rank}(\mathcal{M}_i)\frac{-Z^2-K_X\cdot Z}{2} + c(\mathcal{M}_i)\cdot Z.$$

Furthermore, the McKay correspondence implies that

$$c(\mathcal{M}_i)\cdot E_j = \deg(\mathcal{M}_i|_{E_j}) = \begin{cases} 1 & (j=i) \\ 0 & (j\neq i). \end{cases}$$

Therefore we get

$$\begin{aligned} l_B(B/IB) &= \sum_{i=0}^{r} n_i \cdot l_A(M_i/I.M_i) \\ &= \sum_{i=0}^{r} n_i \left[n_i \frac{-Z^2 - K_X\cdot Z}{2} + c(\mathcal{M}_i)\cdot Z \right] \\ &= \sum_{i=0}^{r} n_i^2 \cdot l_A(A/I) + \sum_{i=1}^{r} a_i n_i \qquad (\because c(\mathcal{M}_0)=0) \\ &= N\cdot l_A(A/I) + \sum_{i=1}^{r} a_i n_i. \end{aligned}$$

(2) Let I be an integrally closed $\mathfrak{m}$-primary ideal and $\widetilde{I}$ the good closure of I. Let $f' : X' \to \operatorname{Spec} A$ be a resolution on which I is represented and let $f : X \to \operatorname{Spec} A$ be the minimal resolution so that there is a morphism $g : X' \to X$ such that $f' = f\circ g$. In this situation we have $I\mathcal{O}_{X'} = \mathcal{O}_{X'}(-Z')$ and $\widetilde{I} = \mathrm{H}^0(X, \mathcal{O}_X(-\overline{Z}))$ where $\overline{Z} = g_*(Z')$. Now, since $Z' = g^*\overline{Z} + Y$, where Y is an effective cycle on X' which is contracted to points on X by g, and since $c(\mathcal{M})$ is defined on minimal resolution for any reflexive A-module M, we have $c(\mathcal{M})\cdot Y = 0$. This shows (1.4) (2).

(3) To prove (1.5), we consider a locally free sheaf $\mathcal{F}_q = f^*(A^{1/q})/\text{torsion}$ of rank q^2. If $I = \mathrm{H}^0(X, \mathcal{O}_X(-Z))$, then by the same argument as is (1), we have

$$l_A(A^{1/q}/I.A^{1/q}) = \chi(\mathcal{F}_q(-Z)) = q^2\,\frac{-Z^2 - K_X\cdot Z}{2} + c(\mathcal{F}_q)\cdot Z = q^2\cdot l_A(A/I) + c(\mathcal{F}_q)\cdot Z.$$

Now, let $\widetilde{I}$ be the good closure of I and let $X, X', \overline{Z}, Z', Y$ be as in (2). Then since $e_{HK}(I) - l_A(A/I) = \lim_{q\to\infty}(c(\mathcal{F}_q)\cdot Z)/q^2$ and $c(\mathcal{F}_q)\cdot Y = 0$ (note that $c(\mathcal{F}_q)$ is defined on X), we have $e_{HK}(I) - l_A(A/I) = e_{HK}(\widetilde{I}) - l_A(A/\widetilde{I})$. Also, if $I\subseteq I'$ are good ideals of A represented by anti-nef cycles Z, Z' respectively, with $Z\geq Z'$, then $c(\mathcal{F}_q)\cdot(Z-Z')\geq 0$ since $\mathcal{F}_q$ is generated by its global sections (recall (2.2)). This shows the inequality in (1.5). □

Remark. Let A be a rational singularity of dimension 2 and $I\subseteq I'$ be integrally closed ideals in A. We believe that if we have equality in (1.5), then I and I' have same good closure although we have no proof yet.

§3. Examples

The list of F-rational Gorenstein rings. *$A = k[[x,y,z]]/(f(x,y,z))$, where k is a field of characteristic p.*

type	equation	char A	$e_{HK}(A)$	
(A_n)	$f = xy + z^{n+1}$	$p \geq 2$	$2 - 1/(n+1)$	$(n \geq 1)$
(D_n)	$f = x^2 + yz^2 + y^{n-1}$	$p \geq 3$	$2 - 1/4(n-2)$	$(n \geq 4)$
(E_6)	$f = x^2 + y^3 + z^4$	$p \geq 5$	$2 - 1/24$	
(E_7)	$f = x^2 + y^3 + yz^3$	$p \geq 5$	$2 - 1/48$	
(E_8)	$f = x^2 + y^3 + z^5$	$p \geq 7$	$2 - 1/120$	

Example (3.1). Let $A = k[(s,t)^e]$, the e^{th} Veronese subring of $k[s,t]$. Then there exists a minimal resolution $f : X \to \operatorname{Spec} A$ such that $f^{-1}(\mathfrak{m}) = E \cong \mathbb{P}^1$ with $E^2 = -e$. Thus the set of good ideals in A is $\{\mathfrak{m}^n\}_{n\geq 1}$. Further, we have

$$l_A(A/\mathfrak{m}^n) = \frac{e}{2}n^2 + \left(1 - \frac{e}{2}\right)n \quad \text{and} \quad e_{HK}(\mathfrak{m}^n) - l_A(A/\mathfrak{m}^n) = n \cdot \frac{e-1}{2}.$$

Let us explain the details in the following. Let $N = \mathbb{Z}_{\geq 0}\overrightarrow{e_1} \oplus \mathbb{Z}_{\geq 0}\overrightarrow{e_2}$ be a lattice of rank 2, and let $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice with dual pairing denoted $\langle\, , \rangle$. Now consider the following three cones in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$;

$$\begin{aligned} \sigma &:= \mathbb{R}_{\geq 0}\overrightarrow{e_1} + \mathbb{R}_{\geq 0}(-\overrightarrow{e_1} + e\overrightarrow{e_2}), \\ \sigma_1 &:= \mathbb{R}_{\geq 0}\overrightarrow{e_1} + \mathbb{R}_{\geq 0}\overrightarrow{e_2}, \\ \sigma_2 &:= \mathbb{R}_{\geq 0}\overrightarrow{e_2} + \mathbb{R}_{\geq 0}(-\overrightarrow{e_1} + e\overrightarrow{e_2}). \end{aligned}$$

Then $A \cong k[y, xy, x^2y, \ldots, x^e y] = k[\sigma^\vee \cap M]$ is the affine coordinate ring of the cyclic quotient singularity $T_N(\Delta)$ corresponding to the finite fan (N, Δ) where $\Delta = \{\sigma\}$.

Moreover, the toric singularity $T_N(\Delta')$ corresponding to the finite fan (N, Δ'), where $\Delta' = \{\sigma_1, \sigma_2\}$ as the subdivision of Δ gives a minimal resolution $f : X := T_N(\Delta') \to \operatorname{Spec} A$ of A with exceptional divisor $E \cong \mathbb{P}^1$ and $E^2 = -e$.

From Remark (E-3) and $E^2 = -e$, we obtain $K_X \cdot E = e - 2$. Thus by Kato's Riemann-Roch formula, we get

$$l_A(A/\mathfrak{m}^n) = -\frac{(nE)^2 + (nE)\cdot K_X}{2} = -\frac{-n^2e + n(e-2)}{2} = \frac{(n^2 - n)e}{2} + n.$$

On the other hand, we put $I_i = (x^i y, \cdots, x^e y)A$ for all $i = 1, 2, \ldots, e$. Then each I_i is an indecomposable maximal Cohen-Macaulay A-module of rank 1. Further, we have $B := k[s,t] \cong I_1 \oplus \cdots \oplus I_{e-1} \oplus I_e$ as A-modules.

Put $\mathcal{I}_i = f^*(I_i)$ for each i. Then since $\mathcal{I}_i$ has no torsion, it is a locally free $\mathcal{O}_X$-module of rank 1. Moreover, since $I_iA_1 = x^i y A_1$ and $I_iA_2 = x^e y A_2$, where $A_1 := k[\sigma_1^\vee \cap M] = k[x,y]$ and $A_2 := k[\sigma_2^\vee \cap M] = k[x^{-1}, x^e y]$, we have that $c(\mathcal{I}_i) \cdot E = e - i$ for each i. Hence by the similar argument as in the proof of the main theorem, we get

$$e_{HK}(\mathfrak{m}^n) - l_A(A/\mathfrak{m}^n) = n \cdot \frac{1}{e}\sum_{i=1}^{e}(e-i) = n \cdot \frac{e-1}{2}; \quad \text{see also [WY2].}$$

Example (3.2). Let k be an algebraically closed field of characteristic $p \geq 3$. Suppose that $G := \left\langle \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$ naturally acts on $B = k[s,t]$. Then the invariant subring is $A := B^G = k[x,y,z]$, where $x = st(s^4 - t^4)$, $y = s^2t^2$, $z = s^4 + t^4$.

Put $A = k[[X,Y,Z]]/(X^2 - YZ^2 + 4Y^3) \cong \widehat{B^G}$. Then A is an F-rational singularity of type (D_4).

The graph of the set of exceptional divisors of the minimal resolution $f : X \to \operatorname{Spec} A$ of A can be written as

$$\begin{array}{ccccc} E_1 & — & E_0 & — & E_2 \\ & & | & & \\ & & E_3 & & \end{array}$$

with $E_0^2 = E_1^2 = E_2^2 = E_3^2 = -2$ and $E_0 \cdot E_1 = E_0 \cdot E_2 = E_0 \cdot E_3 = 1$.

For any effective divisor $Z = \sum_{i=0}^{3} a_i E_i$ supported on $f^{-1}(\mathfrak{m})$, Z is anti-nef if and only if the following inequalities hold:

$$\textbf{(3.2.1)} \qquad \begin{cases} 2a_0 & \geq a_1 + a_2 + a_3 \\ 2a_1 & \geq a_0 \\ 2a_2 & \geq a_0 \\ 2a_3 & \geq a_0 \end{cases}$$

Put $\overrightarrow{z_0} = (2,1,1,1)$, $\overrightarrow{z_1} = (2,2,1,1)$, $\overrightarrow{z_2} = (2,1,2,1)$, $\overrightarrow{z_3} = (2,1,1,2)$ and $\overrightarrow{z_4} = (3,2,2,2)$. Then for any vector $\overrightarrow{z} = (a_0, a_1, a_2, a_3)$ in the semigroup

$$\left\langle (a_0, a_1, a_2, a_3) \in \mathbb{Z}_{\geq 0}^4 \;\middle|\; (3.2.1) \text{ holds for } a_i \ (i = 0,1,2,3) \right\rangle,$$

if a_0 is even (resp. odd), then we can write as

$$\overrightarrow{z} = \sum_{i=0}^{3} c_i \overrightarrow{z_i} \quad \left(\text{resp.} \quad \overrightarrow{z} = \sum_{i=0}^{3} c_i \overrightarrow{z_i} + \overrightarrow{z_4}\right) \qquad \text{for some } c_i \in \mathbb{Z}_{\geq 0}.$$

Thus the set of anti-nef divisors on X forms a semigroup generated by Z_0, Z_1, Z_2, Z_3, and Z_4, where

$$\begin{aligned} Z_0 &= 2E_0 + E_1 + E_2 + E_3, \text{ (the fundamental cycle)} \\ Z_1 &= 2E_0 + 2E_1 + E_2 + E_3, \\ Z_2 &= 2E_0 + E_1 + 2E_2 + E_3, \\ Z_3 &= 2E_0 + E_1 + E_2 + 2E_3, \\ \text{and } Z_4 &= 3E_0 + 2E_1 + 2E_2 + 2E_3. \end{aligned}$$

Further, we put $a = 2y - z, b = 2y + z$ and $c = 2x$. Then the corresponding good ideal I_i $(i = 0,1,2,3,4)$ to the anti-nef divisor Z_i $(i = 0,1,2,3,4)$ can be found in the list below.

divisor	good ideal I	reduction J	$e_{HK}(I)$
Z_0	$I_0 = \mathfrak{m} = (a,b,c)$	$J_0 = (a,b)$	$1+7/8$
Z_1	$I_1 = (a^2,b,c)$	$J_0 = (a^2,b)$	$2+8/8$
Z_2	$I_2 = (a,b^2,c)$	$J_0 = (a,b^2)$	$2+8/8$
Z_3	$I_3 = (a+b,a^2,c)$	$J_0 = (a+b,a^2)$	$2+8/8$
Z_4	$I_4 = (c,a^2,ab,b^2)$	$J_0 = (c,(a-b)^2)$	$3+12/8$

Thus any good ideal in A can be written as a product $\mathfrak{m}^{n_0} I_1^{n_1} I_2^{n_2} I_3^{n_3} I_4^{n_4}$ ($n_4 \in \{0,1\}$) in a unique manner.

Remark (3.3). (cf. [WY2]) If A is a two-dimensional Cohen-Macaulay local ring and if I is stable, then

$$e_{HK}(I^n) - l_A(A/I^n) = n \cdot \{e_{HK}(I) - l_A(A/I)\}.$$

For example, if A is a rational singularity of dimension 2 and if I is an integrally closed ideal, then we can apply this formula; see also [WY2].

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A Note on Column Invariants of Local Rings

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In this note, we summarize some of the facts on various invariants of Noetherian local rings introduced in [KL1-2]. We also define some other invariants related to columns of the presenting matrix of module, and discuss some of their properties.

1. Basic Definitions and Facts

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a commutative Noetherian local ring, and all modules are unitary.

We first state the following result in [KL1], on which the definitions of our invariants are based:

Theorem. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then there is an integer $t \geq 1$ such that for each finitely generated A-module M of infinite projective dimension, the ideal generated by the entries of the map φ_i is not contained in $\mathfrak{m}^t$ for all $i > 1 + \operatorname{depth} A$ ($i > \operatorname{depth} A$ if A is Cohen-Macaulay), where

$$(F_\bullet, \varphi_\bullet) : \cdots \longrightarrow A^{n_{j+1}} \xrightarrow{\varphi_{j+1}} A^{n_j} \xrightarrow{\varphi_j} A^{n_{j-1}} \longrightarrow \cdots \longrightarrow A^{n_0} \longrightarrow 0$$

is a minimal (free) resolution of M.

Some previously known results in commutative ring theory are slightly improved from this fact: One direction of Kunz's characterization of regular local rings in characteristic $p > 0$ ([Ku]) can be easily explained - if the Frobenius endomorphism is flat and some finitely generated A-module M is of infinite projective dimension, then every map in the minimal resolution of ${}^eM$ (M with the Frobenius map applied e-times) would have all of its entries in $\mathfrak{m}^{p^e}$. This contradicts the above theorem.

One can similarly explain, and slightly improve, Herzog's characterization ([H2]) of the modules of finite projective and injective dimension in characteristic $p > 0$ ([KL1, Corollary 2.8]). The theorem also explains the results of Dutta and Eisenbud ([Du,Ei]) on the existence of free summands in certain syzygy modules ([KL1, Proposition 2.2]) because an A-summand of a module is characterized by a column of zeros in its presenting matrix.

Also, using the argument of the theorem, some new numerical invariants of local rings were introduced in [KL1]. They are $col(A)$ and $row(A)$ associated with the columns and rows, respectively, of the maps in infinite minimal resolutions. In [KL2], two more invariants were defined, say $crs(A)$ and $drs(A)$, which are associated with the Cyclic modules determined by Regular Sequences and their Matlis duals, respectively. We remark here that the 'dual' invariants $row(A)$ and $drs(A)$, which correspond to $col(A)$ and $crs(A)$ respectively, can be defined using the finitely generated modules of finite injective dimension and the vanishing of Ext-modules.

More precisely, we have the following definitions :

Definition. In defining $row(A)$ and $drs(A)$ below, we assume that A is Cohen-Macaulay. We denote by $\varphi_i(M)$ the i-th map in a minimal resolution of a finitely generated A-module M. We also use the usual notation $Soc(M) := \mathrm{Hom}_A(A/\mathfrak{m}, M)$ to denote the socle of M.

i) $col(A) :=$ inf $\{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each column of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i > 1 +$ depth A $\}$.

$row(A) :=$ inf $\{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each row of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i >$ depth A $\}$.

ii) $crs(A) :=$ inf $\{t \geq 1 : Soc(A/(\mathbf{x})) \not\subset \mathfrak{m}^t(A/(\mathbf{x}))$ for some maximal regular sequence $\mathbf{x} = x_1, \cdots, x_n\}$.

$drs(A) :=$ inf $\{t \geq 1 : Soc((A/(\mathbf{x}))^\vee) \not\subset \mathfrak{m}^t((A/(\mathbf{x}))^\vee)$ for some system of parameters $\mathbf{x} = x_1, \cdots, x_d\}$.

When A is regular local, we interpret the above definition as $col(A) = row(A) = 1$. These invariants are related as follows:

(1) i) $1 \leq col(A) \leq crs(A) < \infty$.
ii) If A is Cohen-Macaulay, then $1 \leq row(A) \leq drs(A) < \infty$.
iii) A is a regular local ring if and only if any, equivalently all, of the invariants in i) and ii) is 1. ([KL2, Proposition 1.3])

(2) If A is Gorenstein, then
i) $col(A) = row(A)$ and ii) $crs(A) = drs(A)$. ([KL2, Proposition 4.1])

In view of the above fact (1), it seems natural to consider the following conjecture:

Conjecture. ([KL2]) Let $(A, \mathfrak{m})$ be a Noetherian local ring such that some system of parameters is a reduction of $\mathfrak{m}$. Then
(i) $col(A) = crs(A)$,
(ii) if A is Cohen-Macaulay, $row(A) = drs(A)$.

We first remark that the additional assumption on the system of parameters is essential: For the hypersurface ring $A = F[\![x, y]\!]/(xy(x + y))$ where F is a field with two elements ([HS, Example 3.2]), $col(A) = mult(A) = 3$ but $crs(A) = drs(A) = 4$.

We list some of facts which show that the conjecture is in the affirmative.

(3) (i) If depth $A = 0$, then $col(A) = crs(A)$.
(ii) If dim $A = 0$, then $row(A) = drs(A)$. ([KL2, proposition 1.7])

(4) Let $(A, \mathfrak{m})$ be a non-regular Cohen-Macaulay local ring such that some system of parameters is a reduction of $\mathfrak{m}$. Then A is of minimal multiplicity, i.e., $mult(A) = 1 + \text{edim}\, A - \dim A$, if and only if all of its four invariants are equal to 2. ([KL2, Corollary3.7])

(5) If $(A, \mathfrak{m})$ is a hypersurface ring, then $col(A) = mult(A)$. Moreover, if some system of parameters is a reduction of $\mathfrak{m}$, then all four invariants of A are equal to $mult(A)$. ([KL2, Theorem 4.3])

(6) Let $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$, where k is a field and $e \geq 4$. (It is known that R is an 1-dimensional Cohen-Macaulay local ring, but it is not Gorenstein.) Then
(i) $col(R) = 2 = crs(R)$,
(ii) $row(R) = e - 1 = drs(R)$. ([L2, Theorem 2.6])

We close this section with a very useful fact ([KL2, Proposition 1.4]) that for a Cohen-Macaulay local ring A, $drs(A)$ can be described as the generalized Loewy length $\ell\ell(A)$, i.e., $drs(A) = \ell\ell(A)$, where $\ell\ell(A)$ was defined in [D3] as the infimum of positive integer t such that $\mathfrak{m}^t \subseteq (\mathbf{x})$ for some system of parameters $\mathbf{x} = x_1, \cdots, x_n$ of A.

2. More Column Invariants and Questions

In this section, we define some invariants related to the columns of the presenting matrix of module, and discuss their behavior. We also ask some questions on the invariants defined in this paper.

Definition. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, and M a maximal Cohen-Macaulay module without free summands. We define $col_{CM}(M)$ to be the smallest c such that each column of the presenting matrix of M contains an element outside $\mathfrak{m}^c$. We now define:

$col_{CM}(A) := \sup\{col_{CM}(M) : M$ is a maximal Cohen-Macaulay module without free summands. $\}$.

$col_{CM}(A)$ is not necessarily finite while $col(A)$ is always finite. Noting that $col(A)$ is determined by the minimal presenting matrices of the (1 + depth A)-th syzygy modules, we extend the definition of $col(A)$ as follows:

Definition. For a nonnegative integer j,

$col_j(A) := \inf\{t \geq 1$: for each j-th syzygy module without free summands of infinite projective dimension, each column of its minimal presenting matrix contains an element outside $\mathfrak{m}^t\}$.

The following is immediate from the above definitions:

Proposition. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then
(1) $col(A) = col_{1+\mathrm{depth}\,(A)}(A)$.
(2) $col_j(A) \geq col_{j+1}(A)$ for all j.
(3) $col_j(A) \leq col(A) < \infty$ for all $j \geq 1 + \mathrm{depth}\, A$.

The following theorem shows the relationship between invariants $col_{CM}(A)$ and $col_j(A)$.

Theorem. Let A be a Cohen-Macaulay local ring of dimension d. Then

$$col_d(A) = col_{CM}(A).$$

We sketch the proof of the above theorem, which will appear in [L1].

We assume that $col_{CM}(A)$ and $col_d(A)$ are finite. One inequality $col_{CM}(A) \geq col_d(A)$ follows immediately from the definition of $col_{CM}(A)$. For the other inequality, let $t = col_{CM}(A)$. Then there exists a maximal Cohen-Macaulay module M with no free summands such that the presenting matrix of M has a column which is contained in $\mathfrak{m}^{t-1}$. If φ denotes the presenting matrix of M, then φ has a column contained in $\mathfrak{m}^{t-1}$. Using a mapping cone of complexes, we construct a d-th syzygy module, which has no free summands and its presenting matrix is of the form

$$\begin{pmatrix} * & 0 \\ * & (\pm 1)\varphi \end{pmatrix}.$$

This gives the other inequality $col_d(A) \geq t = col_{CM}(A)$. ■

To relate two invariants $col(\text{-})$ and $index(\text{-})$, we recall the definition of $index(\text{-})$: In [D2] Ding defined the index of a Gorenstein local ring $(R, \mathfrak{m})$ as $index(R) := \inf\{t \geq 1 : \delta(R/\mathfrak{m}^t) > 0\}$, and later considered the same definition for Cohen-Macaulay local rings A with canonical modules in [D1]. For this case, it is shown in [KL3] that $col_{CM}(A) = index(A)$. Moreover, if A is Gorenstein then $col(A) = index(A)$ since $col_{CM}(A) = col(A)$ ([KL3]).

Next we put forth some of questions about the invariants defined in this paper.

Question 1. (Ding's Conjecture, [D3]) Let R be a Gorenstein local ring. Is $index(R)$ the same as $\ell\ell(R)$?

In [D2], Ding shows that his conjecture holds if R is a hypersurface ring with the infinite residue field, and that in that case $index(R)$ is equal to the multiplicity of R. In [H1] J. Herzog answers Ding's conjecture in the affirmative for homogeneous Gorenstein k-algebras. On the other hand, Ding ([D3]) has generalized this result to the case when the associated graded ring $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay. However, M. Hashimoto and A. Shida recently ([HS]) showed that Ding's conjecture may not hold unless the residue field of R is infinite; their example is $R = F[\![x, y]\!]/(xy(x + y))$, where F is a field with two elements.

Question 2. (Conjecture, [KL2]) Let A be a Noetherian local ring with the minimal reduction. Then $col(A) = crs(A)$, and if A is Cohen-Macaulay then $row(A)$ $= drs(A)$?

¿From the argument before Question 1 and the fact $\ell\ell(A) = drs(A)$, we know that the positive answer to Question 2 implies Ding's conjecture.

Question 3. Let A be a Cohen-Macaulay local ring. Then $col(A) \leq row(A)$?

We note that $crs(A) \leq drs(A)$, and if R is Gorenstein then $col(R) = row(R)$ and $crs(R) = drs(R)$.

Question 4. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Is $col(A)$ the same as $col(A/\mathfrak{m})$? If A is Cohen-Macaulay, $row(A) = row(A/\mathfrak{m})$?

For a finitely generated A-module M, we define $\mathrm{col}(M)$ [resp. $\mathrm{row}(M)$]:= 1 if $\mathrm{projdim}\, M < \infty$. Otherwise, we define $\mathrm{col}(M)$ [resp. $\mathrm{row}(M)$] to be the smallest c

such that for each $i > 1 + \operatorname{depth} A$ [resp. $i > \operatorname{depth} A$], each column [resp. row] of φ_i contains an element outside $\mathfrak{m}^c$, where $(F_\bullet, \varphi_\bullet)$ is a minimal resolution of M.

Using a mapping cone of complexes, we can show that $col(A)$ [resp. $row(A)$] has the same value as $col(M)$ [resp. $row(M)$] for some finitely generated A-module M of depth 0.

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ON ARITHMETIC MACAULAYFICATION OF NOETHERIAN RINGS

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1. Introduction

Let A be a commutative ring with identity and $\mathfrak{b}$ an ideal in A. The Rees algebra of $\mathfrak{b}$ is the graded ring

$$R(\mathfrak{b}) = \bigoplus_{n\geq 0} (\mathfrak{b}T)^n$$

where T is an indeterminate. The blowing-up of $\operatorname{Spec} A$ with center $\operatorname{Spec} A/\mathfrak{b}$ is $\operatorname{Proj} R(\mathfrak{b})$. In the present paper we consider the existence of Cohen-Macaulay Rees algebras. We say that A has an arithmetic Macaulayfication if there is an ideal $\mathfrak{b}$ of positive height such that $R(\mathfrak{b})$ is Cohen-Macaulay.

Main theorems of this paper are the following.

Theorem 1. *Let A be a Noetherian local ring of positive dimension. Then the following statements are equivalent*:

(A) *A has an arithmetic Macaulayfication*;

(B) *A is unmixed and all the forward fibers are Cohen-Macaulay.*

Theorem 2. *Let A be a Noetherian ring possessing a dualizing complex. If the codimension function of A is a constant on the associated primes of A, then A has an arithmetic Macaulayfication.*

Here a Noetherian local ring A is said to be unmixed if $\dim \hat{A}/\mathfrak{p} = \dim \hat{A}$ for every associated prime $\mathfrak{p}$ of A. A formal fiber of A are the rings $\hat{A}_\mathfrak{p}/\mathfrak{p}\hat{A}_\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal in A.

The studies on the Cohen-Macaulay property of Rees algebras started from Barshy [2]. He gave the defining ideal of $R(\mathfrak{b})$ and its free resolution if $\mathfrak{b}$ is generated by a regular sequence. Around 1980, Goto and Shimoda studied several properties of $R(\mathfrak{b})$ in the case where A is a local ring and $\mathfrak{b}$ is a parameter ideal. See [5], [6], [7], and [13].

Summarizing these investigations, Goto and Yamagishi [8] established the theory of u.s.d-sequences. Their theory contains the existence of an arithmetic Macaulayfication in the case where A is an unmixed and Spec $\hat{A}$ is Cohen-Macaulay except for the closed point. See also Brodmann [3] and Schenzel [12]. Recently Kurano [11] proved that a Noetherian local ring A containing a finite field has an arithmetic Macaulayfication if the non F-rational locus of A is of dimension 1. Independently this was also done by Aberbach [1]. Theorem 1 gives a necessary and sufficinet condition to exist an arithmetic Macaulayfication.

The theorems above give some consequences.

Corollary 3. *A Noetherian local ring is a homomorphic image of a Cohen-Macaulay local ring if and only if it is universally catenary and all the formal fibers of it are Cohen-Macaulay. An excellent local ring is a homomorphic image of a Cohen-Macaulay, excellent local ring.*

Corollary 4. *A Noetherian ring has a dualizing complex if and only if it is a homomorphic image of a finite-dimensional Gorenstein ring.*

2. Sketch of the proof of Theorem 1

In the rest of this paper we sketch the proof of Theorem 1. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. The implication (A)$\Rightarrow$(B) is easy and the converse is also if $\dim A = 1$. Therefore we see the proof of the implication (B)$\Rightarrow$(A) in the case where $\dim A > 1$.

First we describe the choice of center. For a finitely generated A-module M, let $\mathfrak{a}^p(M)$ denote the annihilator of the p^{th} local cohomology module $H^p_{\mathfrak{m}}(M)$ of M and let $\mathfrak{a}(M) = \prod_{p<\dim M} \mathfrak{a}^p(M)$.

Definition 5. Let M be a finitely generated A-module of dimension $d > 0$ and $0 \leq s < d$ an integer. A p-standard system of parameters of type s for M is a system of parameters $x_1, \ldots, x_d$ for M such that

- $x_{s+1}, \ldots, x_d \in \mathfrak{a}(M)$ and
- $x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$ for $i \leq s$.

This notion was given by N. T. Cuong [4]. If the condition (B) holds, then we can find a p-standard system of parameters.

Proposition 6. *Assume that A satifies* (B) *and $d = \dim A \geq 2$. Then there exist an integer $0 \leq s < d-1$ and a p-standard system of parameters of type s for A.*

The implication (B)$\Rightarrow$(A) comes from the following proposition.

Proposition 7. *Assume that $d = \dim A \geq 2$ and there exists a p-standard system of parameters $x_1, \ldots, x_d$ of type $s \leq d-2$ for A. Let $\mathfrak{q}_i = (x_i, \ldots, x_d)$ for $1 \leq i \leq s+1$ and $\mathfrak{b} = \mathfrak{q}_1 \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1}$. If $0 : x_d = 0$, then the Rees algebra $R(\mathfrak{b})$ is Cohen-Macaulay.*

Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$ be ideals in A. The mult-Rees algebra of them is defined to be

$$R(\mathfrak{b}_1, \ldots, \mathfrak{b}_r) = \bigoplus_{n_1, \ldots, n_r \geq 0} (\mathfrak{b}_1 T_1)^{n_1} \cdots (\mathfrak{b}_r T_r)^{n_r}$$

where $T_1, \ldots, T_r$ are indeterminates. We show that

$$R(\mathfrak{q}_1, \ldots, \mathfrak{q}_s, \underbrace{\mathfrak{q}_{s+1}, \ldots, \mathfrak{q}_{s+1}}_{d-s-1 \text{ times}})$$

is Cohen-Macaulay because Hyry [9] gave the following theorem.

Theorem 8 (Hyry). *Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$ be ideals in A of positive height. If $R(\mathfrak{b}_1, \ldots, \mathfrak{b}_r)$ is Cohen-Macaulay, then $R(\mathfrak{b}_1 \cdots \mathfrak{b}_r)$ is also.*

A p-standard system of parameters satifies assumptions (1)–(5) of the following theorem. Therefore we obtian Proposition 7.

Proposition 9. *Let $x_t, \ldots, x_d$ be a sequence in A and $\mathfrak{q}_i = (x_i, \ldots, x_d)$ for $t \leq i \leq d$. We fix integers $t \leq s+1 < d$, $\alpha_t, \ldots, \alpha_s > 0$, and $\alpha_{s+1} \geq d-s-1$ and put*

$$S = R(\underbrace{\mathfrak{q}_t, \ldots, \mathfrak{q}_t}_{\alpha_t \text{ times}}, \ldots, \underbrace{\mathfrak{q}_{s+1}, \ldots, \mathfrak{q}_{s+1}}_{\alpha_{s+1} \text{ times}}).$$

If the sequence $x_t, \ldots, x_d$ satifies the following six conditions:

(1) $x_i, \ldots, x_d$ *is a d-sequence on* $A/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)$ *for all* $t \leq i \leq s+1$, $n_t, \ldots, n_{i-1} > 0$, *and* $\Lambda \subset \{t, \ldots, i-1\}$;

(2) $x_i, \ldots, x_{d-1}$ *is a d-sequence on* $A/(\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_d)$ *for all* $t \leq i \leq s+1$, $n_t, \ldots, n_{i-1} > 0$, *and* $\Lambda \subset \{t, \ldots, i-1\}$;

(3) $x_{s+1}, \ldots, x_d$ *is a u.s.d-sequence on* $A/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)$ *for all* $n_t, \ldots, n_s > 0$ *and* $\Lambda \subset \{t, \ldots, s\}$;

(4)

$$(\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1}) : x_l \cap [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}}]$$
$$= (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_{s+1}^{n_{s+1}}$$

for all $t \leq i \leq k \leq s+1$, $k \leq l \leq d$, $n_t, \ldots, n_{i-1}, n_k > 0$, $n_i, \ldots, n_{k-1}, n_{k+1}, \ldots, n_{s+1} \geq 0$, *and* $\Lambda \subset \{t, \ldots, i-1\}$;

(5)

$$[(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}}] : x_{i-1}^{n_{i-1}} = [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda) + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}}] : \mathfrak{q}_{i-1}$$

for any $t < i \leq s+1$, $n_t, \ldots, n_{i-1} > 0$, $n_i, \ldots, n_{s+1} \geq 0$, *and* $\Lambda \subset \{t, \ldots, i-2\}$;

(6) $0 : x_d \subset 0 : x_t$,

then

$$H^0_{\mathfrak{q}_t S+S_+}(S) = 0 : x_d,$$
$$H^p_{\mathfrak{q}_t S+S_+}(S) = 0 \quad \textit{for } p \neq 0,\ d-t+1+\alpha_t+\cdots+\alpha_{s+1}$$

and

$$[H^{d-s+1+\alpha_t+\cdots+\alpha_{s+1}}_{\mathfrak{q}_t S+S_+}(S)]_{(n_1,\ldots,n_{\alpha_t+\cdots+\alpha_{s+1}})} = 0$$

unless $n_1, \ldots, n_{\alpha_t+\cdots+\alpha_{s+1}} < 0$.

The proof is the descending induction on t. If $t = s+1$, then the assertion comes from the theory of u.s.d-sequences.

Assume that $t < s+1$. We may assume that $\alpha_t = 1$ by using the general theory of mult-Rees algebras. Let $\bar{A} = A/x_tA$ and $\bar{\mathfrak{q}}_i = \mathfrak{q}_i\bar{A}$ for $t+1 \leq i \leq s+1$. Then the sequence $x_{t+1}, \ldots, x_d$ in $\bar{A}$ satifies (1)–(6). Therefore

$$\bar{S} = R(\underbrace{\bar{\mathfrak{q}}_{t+1}, \ldots, \bar{\mathfrak{q}}_{t+1}}_{\alpha_{t+1}+1 \text{ times}}, \underbrace{\bar{\mathfrak{q}}_{t+2}, \ldots, \bar{\mathfrak{q}}_{t+2}}_{\alpha_{t+2} \text{ times}}, \ldots, \underbrace{\bar{\mathfrak{q}}_{s+1}, \ldots, \bar{\mathfrak{q}}_{s+1}}_{\alpha_{s+1} \text{ times}})$$

is ALMOST COHEN-MACAULAY. Furthermore (4) gives an ALMOST EXACT sequence

$$0 \longrightarrow S(-1,0,\ldots,0) \xrightarrow{x_tT_1} S \longrightarrow \bar{S} \longrightarrow 0.$$

Hence S is ALMOST COHEN-MACAULAY.

The precise proof is too long to submit this proceedings. Please refer the preprint [10].

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Three theorems on Gorensteinness in associated graded rings

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1 Introduction.

This is a joint work with Professer Shiro Goto ([GI]). In this paper we are going to develop a theory of Gorensteinness in graded rings associated to certain of ideals in Gorenstein local rings. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\dim A = d$. We assume the field $A/\mathfrak{m}$ is infinite and A is a homomorphic image of a Gorenstein local ring. For each ideal I in A we put $\mathcal{R}(I) = \sum_{i\geq 0} I^i t^i \subseteq A[t]$, $\mathcal{R}'(I) = \sum_{i\in\mathbb{Z}} I^i t^i \subseteq A[t, t^{-1}]$, and $\mathcal{G}(I) = \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I)$ and call them respectively the Rees algebra, the extended Rees algebra, and the associated graded ring of I. Let $a = \mathrm{a}(\mathcal{G}(I))$ denote the a-invariant of $\mathcal{G}(I)$. The purpose of this paper is to prove the following three theorems. The first one is a slight generalization of a theorem due to [Hy].

Theorem 1.1. *Let I ($\neq A$) be an ideal in A and let $k \geq 1$ be an integer with $I^k \neq (0)$. Suppose that $G = \mathcal{G}(I)$ is a Cohen-Macaulay ring. Then the following conditions are equivalent.*

(1) *G is a Gorenstein ring and $a \equiv -1 \bmod k$.*

(2) *$\mathcal{G}(I^k)$ is a Gorenstein ring and $I^{a-i+1} \supseteq (0) : I^i$ for all $1 \leq i \leq a$.*

When this is the case, we have $\mathrm{a}(\mathcal{G}(I^k)) = \dfrac{a+1}{k} - 1$.

We will prove our theorem 1.1 in Section 3. Next we will investigate a certain special kind of ideals. Let I ($\neq A$) be an ideal in A of height s. Let J be a minimal reduction of I. Hence $J \subseteq I$ and $I^{n+1} = JI^n$ for some $n \geq 0$. We put $r = r_J(I) := \min\{n \geq 0 \mid I^{n+1} = JI^n\}$ and call it the reduction number of I with respect to J. Let $\lambda(I)$ denote the analytic spread of I, that is $\lambda(I) = \dim A/\mathfrak{m} \otimes_A \mathcal{G}(I)$. We put $\ell = \lambda(I)$. Then J is minimally generated by ℓ-elements ([NR] or [S], Ch.2), whence $\ell \geq s$. Let $\mathrm{ad}(I) = \ell - s$ and call it the analytic deviation of I (cf. [HH]). Ideals having analytic deviation 0 are said to be equimultiple. With this notation our second main result is stated as follows, which we will prove in Section 4.

Theorem 1.2. *Assume that A is a Gorenstein local ring and let I be an equimultiple ideal in A with $\mathrm{ht}_A I \geq 1$. Then the following two conditions are equivalent.*

(1) *$\mathcal{G}(I)$ is a Gorenstein ring.*

(2) *$\mathcal{G}(I)$ is a Cohen-Macaulay ring and $I^r = J^r : I^r$.*

Lastly we shall explore in Section 5 the ideals I with $\mathrm{ad}(I) = 1$. Let

$$U = \bigcap_{\mathfrak{p}\in \mathrm{Assh}_A A/I} (IA_{\mathfrak{p}} \cap A)$$

where $\mathrm{Assh}_A A/I = \{\mathfrak{p} \in \mathrm{V}(I) \mid \dim A/I = \dim A/\mathfrak{p}\}$. Then we have the following.

Theorem 1.3. *Assume that* $\mathrm{ad}(I) = 1$ *and* $I_\mathfrak{p}$ *is generated by an* $A_\mathfrak{p}$*-regular sequence for all* $\mathfrak{p} \in \mathrm{Assh}_A A/I$*. Then the following two conditions are equivalent.*

(1) $\mathcal{G}(I)$ *is a Gorenstein ring.*

(2) (a) A *is a Gorenstein ring,*

(b) $\operatorname{depth} A/I \geq d - s - 1,$

(c) $r_J(I) \leq 1$, *and*

(d) $I = (JU : I) \cap U.$

Our theorem 1.3 is based on Theorem (1.1) of [GNa] in the case where $d = 1$ and $s = 0$. With the same hypothesis as in Theorem 1.3 the first author and Huckaba [GH] studied the Gorenstein property of $\mathcal{R}(I)$ and essentially proved in the case where $s \geq 2$ that $\mathcal{R}(I)$ is a Gorenstein ring if and only if $\mathcal{G}(I)$ is a Gorenstein ring and $s = 2$ (cf. [GH], Corollary 2.12). And summarizing their result with our theorem 1.3 we have

Corollary 1.4. *Suppose that* $I_\mathfrak{p}$ *is generated by an* $A_\mathfrak{p}$*-regular sequence for all* $\mathfrak{p} \in \mathrm{Assh}_A A/I$*. Let* $\mathrm{ad}(I) = 1$ *and* $s \geq 2$*. Then the following two conditions are equivalent.*

(1) $\mathcal{R}(I)$ *is a Gorenstein ring.*

(2) (a) A *is a Gorenstein ring,*

(b) $s = 2,$

(c) $\operatorname{depth} A/I \geq d - 3,$

(d) $r_J(I) \leq 1$, *and*

(e) $I = (JU : I) \cap U.$

The defining ideals of projective monomial varieties of codimension 2 always satisfy conditions 2 stated in Corollary 1.4 ([BM]), whence their Rees algebras are necessarily Gorenstein rings.

Before entering into details, let us fix again the standard notation in this paper. Throughout let $(A, \mathfrak{m})$ denote a Cohen-Macaulay local ring with $\dim A = d$ and assume that A is a homomorphic image of a Gorenstein local ring. We denote by K_A the canonical module of A. Let I ($\neq A$) be an ideal in A of height s and J a minimal reduction of I. We put $R' = \mathcal{R}'(I)$ and $G = \mathcal{G}(I)$. Let $a = \mathrm{a}(G)$ and let$r = \mathrm{r}_J(I)$. We denote by $K_{R'}$ and K_G the graded canonical modules of R' and G respectively.

2 Preliminarites.

In this section we summarize some facts which we will frequently use in this paper. Firstly we shall give a result originated in an argument given by Herzog, Simis, and Vasconcelos [HSV], concerning the structure of canonical modules of extended Rees algebras. Namely we have the following result, which is due to [TVZ] in the case where $\mathcal{R}(I)$ is a Cohen-Macaulay ring with $\mathrm{ht}_A I \geq 1$.

Fact 2.1. *There exists a unique family* $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ *of* A*-submodules of* K_A *satisfying the following four conditions.*

(a) $\omega_{i+1} \subseteq \omega_i$ *for all* $i \in \mathbb{Z}$,

(b) $\omega_i = \mathrm{K}_A$ *for all* $i \ll 0$,

(c) $I^i\omega_j \subseteq \omega_{i+j}$ *for all* $i, j \in \mathbb{Z}$, *and*

(d) $\mathrm{K}_{R'} \cong \sum_{i\in\mathbb{Z}} \omega_i t^i$ *as graded* R'*-modules.*

If G *is a Cohen-Macaulay ring, we have* $\omega_i = \mathrm{K}_A \supsetneq \omega_{-a}$ *for all* $i \leq -a-1$ *and get an isomorphism*

$$\mathrm{K}_G \cong \bigoplus_{i\geq -a} \omega_{i-1}/\omega_i$$

of graded G*-modules.*

Let us refer to this family $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ as the canonical I-filtration of K_A. We furthermore have

Fact 2.2. *Let* $\mathrm{ad}(I) = 0$ *and* $s \geq 1$. *Suppose* G *is a Cohen-Macaulay ring. Then*

(1) $\omega_i = J^{i+r-s+1}\mathrm{K}_A :_{\mathrm{K}_A} I^r$ *for all* $i \in \mathbb{Z}$.

(2) $\omega_i = I^{i-s+1}\omega_{s-1} = J^{i-s+1}\omega_{s-1}$ *for all* $i \geq s-1$.

The next criterion will play a key role in this paper.

Fact 2.3. *Assume that* A *is a Gorenstein local ring and* G *is a Cohen-Macaulay ring. Let* $\mathcal{A} = \{\mathfrak{p} \in V(I) \mid \mathrm{ht}_A \mathfrak{p} = \dim G_\mathfrak{p}/\mathfrak{p}G_\mathfrak{p}\}$. *Then* $\mathcal{A} = \mathrm{Ass}_A G$ *and the following three conditions are equivalent to each other.*

(1) G *is a Gorenstein ring.*

(2) *There is a monomorphism* $G(a) \hookrightarrow \mathrm{K}_G$ *of graded* G*-modules.*

(3) $G_\mathfrak{p}$ *is a Gorenstein ring with* $a = \mathrm{a}(G_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathcal{A}$.

3 The proof of Theorem 1.1.

In this section we assume that A is a Gorenstein local ring and G is a Cohen-Macaulay ring. Let ω denote the canonical I-filtration of $\mathrm{K}_A = A$. Let $k \geq 1$ be an integer. Then $\mathcal{G}(I^k)$ is also a Cohen-Macaulay ring with the canonical I^k-filtration $\omega^{(k)} = \{\omega_{kj}\}_{j\in\mathbb{Z}}$ of $\mathrm{K}_A = A$. If G is a Gorenstein ring and $a \equiv -1 \mod k$, we have $\mathrm{K}_{\mathcal{R}'(I^k)} \cong \mathcal{R}'(I^k)(\dfrac{a+1}{k})$ since $\mathrm{K}_{R'} \cong R'(a+1)$, so that $\mathcal{G}(I^k)$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I^k)) = \dfrac{a+1}{k} - 1$. The converse is also true as we shall show in this section.

To begin with we note

Lemma 3.1. *Suppose that* $I^k \neq (0)$ *and* $\mathcal{G}(I^k)$ *is a Gorenstein ring. Then*

(1) $a \equiv -1 \mod k$,

(2) $\mathrm{a}(\mathcal{G}(I^k)) = \dfrac{a+1}{k} - 1$, *and*

(3) $I^{kj} = \omega_{-a+kj-1}$ *for all* $j \in \mathbb{Z}$.

Proof. Let $b = \mathrm{a}(\mathcal{G}(I^k))$. Then $\{\omega_{ki}\}_{i\in\mathbb{Z}}$ is the canonical I^k-filtration of A and $\omega_{k(-b-1)} = A$ (cf. Fact 2.1). Hence $\omega_{ki} = I^{k(i+b+1)}$ for all $i \in \mathbb{Z}$. Consequently $\omega_{k(-b)} = I^k \subsetneq A = \omega_{-a-1}$ so that we have $k(-b) > -a-1$. Let $n = k(-b)+a+1 \geq 1$. Then $I^n = I^n\omega_{-a-1} \subseteq \omega_{-a+n-1} = I^k$, whence $n \geq k$ as $I^k \neq (0)$. On the other hand, because $-a+n-1-k = k(-b-1)$ and $\omega_{k(-b-1)} = A$, we have $-a+n-1-k \leq -a-1$ (recall that $\omega_{-a-1} = A \supsetneq \omega_{-a}$), whence $n \leq k$ so that we have $n = k$. Therefore $a \equiv -1 \mod k$ and $b = \dfrac{a+1}{k} - 1$, because $k = k(-b)+a+1$. Hence $k(-b+j-1) = -a+kj-1$ for all $j \in \mathbb{Z}$ and so $\omega_{-a+kj-1} = I^{kj}$ for all $j \in \mathbb{Z}$. $\square$

Identify $G = \bigoplus_{i\in\mathbb{Z}} I^i/I^{i+1}$ and $\mathrm{K}_G = \bigoplus_{i\in\mathbb{Z}} \omega_{i-1}/\omega_i$. Then we set $\varphi : G(a) \to \mathrm{K}_G$ be the homomorphism of graded G-modules defined by

$$\varphi(x \mod I^{i+a+1}) = x \mod \omega_i$$

for all $x \in I^{i+a}$ and $i \in \mathbb{Z}$. Let $\mathcal{K} = \ker\varphi$. Then $\mathcal{K}_i = [I^{i+a} \cap \omega_i]/I^{i+a+1}$ for all $i \in \mathbb{Z}$.

Lemma 3.2. *Suppose that $\mathcal{G}(I^k)$ is a Gorenstein ring. Then $\mathcal{K}$ is a finitely graded G-module.*

Proof. Since $\mathrm{K}_{R'}$ is the finitely generated graded R'-module, we have an integer α such that $\omega_{i+1} = I\omega_i$ for all $i \geq \alpha$, while by Lemma 3.1 (3) we see $\omega_m = I^{a+m+1}$ for some $m \geq \alpha$. Hence $\omega_j = I^{a+j+1}$ for all $j \geq m$, because $\omega_{i+1} = I\omega_i$ for all $i \geq \alpha$. This forces $\mathcal{K}_j = (0)$ if $j \geq m$, whence $\mathcal{K}$ is a finitely graded G-module. $\square$

The next result is due to [Hy]. Let us give a brief proof in our context.

Corollary 3.3 ([Hy], Theorem 2.4). *Let* $\mathrm{ht}_A I \geq 1$. *The following conditions are equivalent.*

(1) *G is a Gorenstein ring and* $a \equiv -1 \mod k$.

(2) *$\mathcal{G}(I^k)$ is a Gorenstein ring.*

Proof. It suffices to show the implication (2) ⇒ (1). We look at the homomorphism $\varphi : G(a) \to \mathrm{K}_G$. Then by Lemma 3.2 $\mathcal{K}$ is finitely graded, which forces $\mathcal{K} = (0)$ since $G_+ \subseteq \sqrt{(0) :_G \mathcal{K}}$ and $\mathrm{grade}\, G_+ = \mathrm{ht}_A I \geq 1$ (notice that G is a Cohen-Macaulay ring). Hence the map φ is a monomorphism and so G is a Gorenstein ring by Fact 2.3. $\square$

Lemma 3.4. *Let* $\mathrm{ht}_A I = 0$. *Then* $I^{a-i+1} \supseteq (0) : I^i$ *for every* $1 \leq i \leq a$ *if G is a Gorenstein ring.*

Proof. Induction on d. We have $a = \mathrm{a}(G_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{V}(I)$. By [O], Theorem 1.6 we may assume that $d \geq 1$ and our assertion holds true for $d-1$. Let $1 \leq i \leq a$ and put $Z = (0) :_A I^i$. Then $Z \neq A$. We must show $Z \subseteq I^{a-i+1}$. Let $\varepsilon : G \to \mathcal{G}((I+Z)/Z)$ be the canonical epimorphism of associated graded rings and put $L = \ker\varepsilon$. Then L is a graded ideal in G with $L_m = (I^m \cap Z)/(I^{m+1} \cap Z)$ for each $m \in \mathbb{Z}$. Notice that $\ell_A(L_m) < \infty$ for all $m < a-i+1$, where $\ell_A(L_m)$ denotes the length of L_m. In fact, suppose $(L_m)_\mathfrak{p} \neq (0)$ for some

$\mathfrak{p} \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}$. Then by the hypothesis of induction we see $Z_\mathfrak{p} \subseteq I_\mathfrak{p}^{a-i+1} \subseteq I_\mathfrak{p}^{m+1}$ so that $(L_m)_\mathfrak{p} = \left(I_\mathfrak{p}^m \cap Z_\mathfrak{p}\right) / \left(I_\mathfrak{p}^{m+1} \cap Z_\mathfrak{p}\right) = (0)$, which is absurd. Hence $\ell_A(L_m) < \infty$ if $m < a - i + 1$. Notice that $Z \cap I^{a+1} = (0)$, because for each $\mathfrak{p} \in \operatorname{Ass} A$ we have $Z_\mathfrak{p} = (0)$ if $\mathfrak{p} \not\supseteq I$ and $I_\mathfrak{p}^{a+1} = (0)$ otherwise. Thus L is a finitely *graded* G-module, whence $\mathrm{H}_\mathfrak{m}^0(L) = \bigoplus_{j\in\mathbb{Z}} \mathrm{H}_\mathfrak{m}^0(L_j) = \mathrm{H}_\mathfrak{M}^0(L) = (0)$ since G is a Cohen-Macaulay ring with $d = \dim G \geq 1$. Therefore $L_m = (0)$ so that $Z \cap I^m = Z \cap I^{m+1}$ for all $m < a - i + 1$. Thus $Z \subseteq I^{a-i+1}$ as was claimed. □

The next result will provide for the lack of criteria similar to Corollary 3.3 in the case where $\operatorname{ht}_A I = 0$.

Theorem 3.5. *Suppose* $\operatorname{ht}_A I = 0$ *and* $I^{a-i+1} \supseteq (0) : I^i$ *for all* $1 \leq i \leq a$. *Let* $I^k \neq (0)$. *Then the following conditions are equivalent.*

(1) *G is a Gorenstein ring and* $a \equiv -1 \mod k$.

(2) $\mathcal{G}(I^k)$ *is a Gorenstein ring.*

When this is the case, we have $\mathrm{a}(\mathcal{G}(I^k)) = \dfrac{a+1}{k} - 1$.

Proof. By Lemma 3.1 we have only to show that G is a Gorenstein ring if so is $\mathcal{G}(I^k)$. We may assume $k \geq 2$. Hence $1 \leq k - 1 \leq a$ because $a \equiv -1 \mod k$ by Lemma 3.1 (1). We begin with the following.

Claim 1. $a = \mathrm{a}(G_\mathfrak{p})$ *for all* $\mathfrak{p} \in V(I)$.

Proof. Let $\mathfrak{p} \in V(I)$. Then $a \geq \mathrm{a}(G_\mathfrak{p})$. We have the equality if $\omega_{-a} \subseteq \mathfrak{p}$. Suppose that $\omega_{-a} \not\subseteq \mathfrak{p}$. Then $I^k = \omega_{-a+k-1} \supseteq I^{k-1}\omega_{-a}$ by Lemma 3.1 (3), whence $I_\mathfrak{p}^k = I_\mathfrak{p}^{k-1}$. Therefore $I_\mathfrak{p}^{k-1} = (0)$, while we get $(0) : I^{k-1} \subseteq I^{a-k+2} \subseteq \mathfrak{p}$ by our standard assumption. This is impossible. □

To complete the proof it suffices by Fact 2.3 to show that $G_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathcal{A} = \{\mathfrak{p} \in V(I) \mid \operatorname{ht}_A \mathfrak{p} = \lambda(I_\mathfrak{p})\}$. Let $\mathfrak{p} \in \mathcal{A}$ and put $\ell = \operatorname{ht}_A \mathfrak{p}$. By induction on ℓ we will show that $G_\mathfrak{p} = \mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring. Let $\ell = 0$. Then since $a = \mathrm{a}(G_\mathfrak{p})$ by Claim 1, we have $I_\mathfrak{p}^{a+1} = (0)$ so that $I_\mathfrak{p}^{a-i+1} = (0) : I_\mathfrak{p}^i$ for all $1 \leq i \leq a$. Thus $G_\mathfrak{p}$ is a Gorenstein ring by [O], Theorem 1.6. Assume that $\ell \geq 1$ and $G_\mathfrak{q}$ is a Gorenstein ring for any $\mathfrak{q} \in \mathcal{A}$ with $\operatorname{ht}_A \mathfrak{q} < \ell$. We look at the homomorphism $\varphi : G(a) \to \mathrm{K}_G$. The purpose is to show $A_\mathfrak{p} \otimes_A \varphi$ is a monomorphism. Let $\mathcal{K} = \ker \varphi$. Suppose that $\mathcal{K}_\mathfrak{p} \neq (0)$ and choose any $\mathfrak{q} \in \operatorname{Ass}_A \mathcal{K}$ so that $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\mathcal{K}_\mathfrak{q} \neq (0)$ and $\mathfrak{q} \in \mathcal{A}$ since $\mathcal{A} = \operatorname{Ass}_A G$ (Fact 2.3). If $\mathfrak{q} \subsetneq \mathfrak{p}$, then by the hypothesis on $\ell = \operatorname{ht}_A \mathfrak{p}$ the ring $G_\mathfrak{q} = \mathcal{G}(I_\mathfrak{q})$ is Gorenstein with $a = \mathrm{a}(\mathcal{G}(I_\mathfrak{q}))$, so that $A_\mathfrak{q} \otimes_A \varphi$ is an isomorphism (recall that $\omega_\mathfrak{q} = \{\omega_{i\mathfrak{q}}\}_{i\in\mathbb{Z}}$ is the canonical $I_\mathfrak{q}$-filtration of $A_\mathfrak{q}$), which is absurd. Thus $\operatorname{Ass}_{A_\mathfrak{p}} \mathcal{K}_\mathfrak{p} = \{\mathfrak{p}A_\mathfrak{p}\}$, whence $\ell_{A_\mathfrak{p}}(\mathcal{K}_\mathfrak{p}) < \infty$ since $\mathcal{K}$ is a finitely graded G-module by Lemma 3.2. Therefore $\operatorname{depth} G_\mathfrak{p} = 0$ so that we have $\dim G_\mathfrak{p} = 0$, whence $\ell = \operatorname{ht}_A \mathfrak{p} = 0$. This is the required contradiction and we conclude that $A_\mathfrak{p} \otimes_A \varphi$ is a monomorphism. Hence $G_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathcal{A}$ by Fact 2.3 and Claim 1. Thus G is a Gorenstein ring. □

As a special case of Theorem 3.5 we have the following.

Corollary 3.6. *Let* $\operatorname{ht}_A I = 0$ *and put* $n = \min\{n \in \mathbb{Z} \mid I_\mathfrak{p}^{n+1} = (0)$ *for all* $\mathfrak{p} \in \operatorname{Assh}_A A/I\}$. *Then the following conditions are equivalent.*

(1) *G is a Gorenstein ring.*

(2) $\mathcal{G}(I^{n+1})$ *is a Gorenstein ring and* $I^{n-i+1} \supseteq (0) : I^i$ *for all* $1 \leq i \leq n$.

When this is the case, we have $a = n$ *and* $\mathrm{a}(\mathcal{G}(I^{n+1})) = 0$.

Proof. By Proposition 2.4 and Theorem 1.1 it suffices to show $a = n$ in any case. This is certainly true if $I^{n+1} = (0)$. Assume that $I^{n+1} \neq (0)$ and choose $\mathfrak{p} \in \mathrm{Assh}_A A/I$ so that $I_\mathfrak{p}^n \neq (0)$. Then $n = \mathrm{a}(G_\mathfrak{p})$ since $I_\mathfrak{p}^{n+1} = (0)$. Therefore we get $a = n$ if G is a Gorenstein ring (recall that $a = \mathrm{a}(G_\mathfrak{p})$). Suppose $\mathcal{G}(I^{n+1})$ is a Gorenstein ring. Then $\mathrm{a}(\mathcal{G}(I^{n+1})) = 0$ since $\mathrm{a}(\mathcal{G}(I_\mathfrak{p}^{n+1})) = 0$, so that we have $a = n$ by Lemma 3.1 (2). □

4 The proof of Theorem 1.2.

In this section let I be an equimultiple ideal of height s and it's reduction J generated by elements $a_1, a_2, \cdots, a_s$. Hence $a_1, a_2, \ldots, a_s$. The purpose of this section is to prove the following.

Theorem 4.1. *Let* $s \geq 1$. *Then* G *is a Gorenstein ring if and only if the following four conditions are simultaneously satisfied.*

(1) *A is a Gorenstein ring.*

(2) $A/(I^i + J)$ *is a Cohen-Macaulay ring for all* $1 \leq i \leq r$.

(3) $J \cap I^i = JI^{i-1}$ *for all* $1 \leq i \leq r$.

(4) $I^r = J^r : I^r$.

When this is the case, we have $I^i = J^i : I^r$ *for all* $i \in \mathbb{Z}$.

Proof. We may assume A is Gorenstein. Let $f_i = a_i t$ $(1 \leq i \leq s)$. Then $\{f_i\}_{1 \leq i \leq s}$ is a subsystem of homogeneous parameters for G. Therefore the sequence $\{f_i\}_{1 \leq i \leq s}$ is G-regular if G is Cohen-Macaulay. By [VV] we furthermore have condition (3) is equivalent to saying that $f_1, f_2, \cdots, f_s$ is G-regular. Hence in order to prove our theorem we may always assume condition (3) is satisfied. Then since

$$G/(f_1, f_2, \cdots, f_s)G \cong \mathcal{G}(I/J)$$

as graded A-algebras, passing to the ring $\overline{A} := A/J$, we find G is a Cohen-Macaulay ring if and only if condition (2) is satisfied (use the exact sequencese $0 \to I^i\overline{A}/I^{i+1}\overline{A} \to \overline{A}/I^{i+1}\overline{A} \to \overline{A}/I^i\overline{A} \to 0$). When this is the case, by Fact 2.2 we get $\omega_{s-1} = J^r : I^r$ and $\omega_i = I^{i-s+1}\omega_{s-1}$ for all $i \geq s-1$. Therefore condition (4) is satisfied and $I^i = J^i : I^r$ for all $i \in \mathbb{Z}$, once G is a Gorenstein ring. Conversely assume that G is a Cohen-Macaulay ring and $I^r = J^r : I^r$. Then $\omega_i = I^{i+r-s+1}$ for all $i \geq s-1$, since $\omega_{s-1} = J^r : I^r = I^r$ and $\omega_i = I^{i-s+1}\omega_{s-1}$. Therefore the kernel $\mathcal{K}$ of the homomorphism φ given in Section 3 is a finitely graded G-module, whence $\mathcal{K} = (0)$ because $G_+ \subseteq \sqrt{(0) :_G \mathcal{K}}$ and $\operatorname{grade} G_+ = \mathrm{ht}_A I \geq 1$. Thus by Fact 2.3 G is a Gorenstein ring. □

5 The proof of Theorem 1.3.

In this section we assume that the field $A/\mathfrak{m}$ is infinite. Let I $(\neq A)$ be an ideal in A with $\mathrm{ht}_A I = s$ and $\lambda(I) = s+1$. Assume that $I_\mathfrak{p}$ is generated by an $A_\mathfrak{p}$-regular sequence for all $\mathfrak{p} \in \mathrm{Assh}_A A/I$. Let J be a minimal reduction of I with $r = r_J(I)$. Then since $A/\mathfrak{m}$ is infinite, we may choose a system $a_1, a_2, \cdots, a_{s+1}$ of generators for J so that the sequence $a_1, a_2, \cdots, a_s$

is A-regular and $I_{\mathfrak{p}} = K_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Assh}_A A/I$ where $K = (a_1, a_2, \cdots, a_s)$ (cf. [AHT]). Let $U = U(I) := \bigcap_{\mathfrak{p} \in \mathrm{Assh}_A A/I} (IA_{\mathfrak{p}} \cap A)$ denote the unmixed component of I. We put $R = \mathcal{R}(I)$, $R' = \mathcal{R}'(I)$, and $G = \mathcal{G}(I)$. Let $a = \mathrm{a}(G)$. To begin with, let us summarize the following.

Lemma 5.1 ([GNN2], Lemma 2.8 and Proposition 2.9). *Suppose that G is a Gorenstein ring. Then $r_J(I) \le 1$ and $a = -s$.*

Lemma 5.2. $(JU : I) \cap U = [(JU + K) : I] \cap U$.

Proof. Let $x \in [(JU + K) : I] \cap U$. Then $xI \subseteq (JU + K) \cap U^2 = JU + (K \cap U^2)$. Hence to see $(JU : I) \cap U = [(JU + K) : I] \cap U$ it suffices to show that $KU = K \cap U^2$. It is enough to show $K_{\mathfrak{p}} U_{\mathfrak{p}} = K_{\mathfrak{p}} \cap U_{\mathfrak{p}}^2$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/KU$. We may assume $U \subseteq \mathfrak{p}$. Look at the exact sequence $0 \to K/KU \to A/KU \to A/K \to 0$ of A-modules. Then since $K/KU \cong (A/U) \otimes_{A/K} (K/K^2) \cong (A/U)^s$, we have $\mathrm{Ass}_A A/KU \subseteq \mathrm{Ass}_A A/U \cup \mathrm{Ass}_A A/K$. Therefore $\mathrm{ht}_A \mathfrak{p} = s$ whence $U_{\mathfrak{p}} = K_{\mathfrak{p}}$. □

The next result is a slight modification of the theorem given by [GNa]. In the original statement condition (3) is that $I = (JU : I) \cap (I : \mathfrak{m})$. Let us check these two conditions are equivalent to each other under the assumption that $d = 1$, $s = 0$, and $r_J(I) \le 1$. Since $I : \mathfrak{m} \subseteq U$, we get $I = (JU : I) \cap (I : \mathfrak{m})$ if $I = (JU : I) \cap U$. Assume that $I = (JU : I) \cap (I : \mathfrak{m})$ and take $x \in (JU : I) \cap U$. Then since $x \in U$, $\mathfrak{m}^{\ell} x \subseteq I$ for some $\ell \ge 1$. We take such an integer $\ell \ge 1$ as small as possible. Then $\ell = 1$. In fact, assume $\ell \ge 2$ and we have $\mathfrak{m}^{\ell-1} x \subseteq (JU : I) \cap (I : \mathfrak{m}) = I$, which contradicts the minimality of ℓ. Hence $\mathfrak{m} x \subseteq I$ so that $x \in (JU : I) \cap (I : \mathfrak{m}) = I$. Therefore we have $I = (JU : I) \cap U$ since $I \subseteq (JU : I) \cap U$ (recall that $I^2 = JI$).

Proposition 5.3 ([GNa], Theorem (1.1)). *Let $d = 1$ and $s = 0$. Assume that $I_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \mathrm{Assh}_A A/I$. Then G is a Gorenstein ring if and only if* (1) *A is a Gorenstein ring,* (2) *$r_J(I) \le 1$, and* (3) *$I = (JU : I) \cap U$.*

We are now ready to prove Theorem 1.3.

Proof. If conditions (a), (b), and (c) are satisfied, then G is a Cohen-Macaulay ring and $a = -s$ ([GNN1] and [GH], Proposition 2.4). Conversely, if G is a Gorenstein ring, then $a = -s$ and conditions (a), (b), and (c) are satisfied by Lemma 5.1 together with Burch's inequality ([B]). Therefore, to prove the theorem we may assume without loss of generality that conditions (a), (b), and (c) are satisfied. Then because G is a Cohen-Macaulay ring, the sequence $a_1 t, a_2 t, \cdots, a_s t$ is G-regular ([GNN2], Corollary 2.3 (2)) and so we get the isomorphism $G/(a_1 t, a_2 t, \cdots, a_s t)G \cong \mathcal{G}(I/K)$ of graded A-algebras. Besides, we have $r_J(I) = r_{J/K}(I/K)$ ([GNN2], Lemma 2.4 (1)) and $(JU : I) \cap U = [(JU + K) : I] \cap U$ by Lemma 5.2. Hence passing to the ring A/K, we may furthermore assume that $s = 0$. We put $b = a_1$. Hence $J = bA$. We must check that G is a Gorenstein ring if and only if $I = (JU : I) \cap U$. Firstly suppose that $I = (JU : I) \cap U$. Let $\mathfrak{p} \in \mathcal{A} = \mathrm{Ass}_A G$ (cf. Fact 2.3). We want to show that $\mathcal{G}(I_{\mathfrak{p}})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I_{\mathfrak{p}})) = 0$. Notice that $\mathrm{ht}_A \mathfrak{p} = \lambda(I_{\mathfrak{p}}) \le 1$ because $J_{\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$. Hence $\mathrm{ht}_{A_{\mathfrak{p}}} I_{\mathfrak{p}} \le 1$. If $\mathrm{ht}_{A_{\mathfrak{p}}} I_{\mathfrak{p}} = 1$, we have $\mathrm{ht}_A \mathfrak{p} = 1$ whence $I_{\mathfrak{p}}$ is an $\mathfrak{p}A_{\mathfrak{p}}$-primary ideal in A_p. Therefore $\mathfrak{p} \not\supset U$ so that we have $I_{\mathfrak{p}} = J_{\mathfrak{p}} : I_{\mathfrak{p}}$. Hence $\mathcal{G}(I_{\mathfrak{p}})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I_{\mathfrak{p}})) = 0$ by Theorem 1.2. Suppose that $\mathrm{ht}_{A_{\mathfrak{p}}} I_{\mathfrak{p}} = 0$. If $\mathrm{ht}_A \mathfrak{p} = 0$, then $I_{\mathfrak{p}} = (0)$ and we have nothing to prove. Assume that $\mathrm{ht}_A \mathfrak{p} = 1$. Then $J_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$ since $\mathrm{ht}_A \mathfrak{p} = \lambda(I_\mathfrak{p}) = 1$ and we have $I_\mathfrak{p} = [J_\mathfrak{p} \cdot U(I_\mathfrak{p}) : I_\mathfrak{p}] \cap U(I_\mathfrak{p})$. Therefore thanks to Proposition 5.3, we get $\mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I_\mathfrak{p})) = 0$. Thus by Fact 2.3 G is a Gorenstein ring.

Conversely suppose that G is a Gorenstein ring. Then $a = 0$. We want to show $I = (JU : I) \cap U$. Let $L = (JU : I) \cap U$. Then $I \subseteq L$ since $I^2 = JI$. Assume that $I \subsetneq L$ and choose $\mathfrak{p} \in \mathrm{Ass}_A A/I$ so that $I_\mathfrak{p} \subsetneq L_\mathfrak{p}$. Then $\mathfrak{p} \in \mathcal{A} = \mathrm{Ass}_A G$. Hence $\mathrm{ht}_A \mathfrak{p} = \lambda(I_\mathfrak{p}) \leq 1$, because $J_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$. If $\mathrm{ht}_{A_\mathfrak{p}} I_\mathfrak{p} = 1$, then $I_\mathfrak{p}$ is an $\mathfrak{p}A_\mathfrak{p}$-primary ideal of $A_\mathfrak{p}$ and $J_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$. Hence $I_\mathfrak{p} = J_\mathfrak{p} : I_\mathfrak{p}$ by Theorem 1.2, because $\mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I_\mathfrak{p})) = 0$. Therefore we have $I_\mathfrak{p} = L_\mathfrak{p}$ since $\mathfrak{p} \not\supseteq U$, which is impossible. Thus $\mathrm{ht}_{A_\mathfrak{p}} I_\mathfrak{p} = 0$. If $\mathrm{ht}_A \mathfrak{p} = 0$, then $I_\mathfrak{p} = (0)$ so that $U_\mathfrak{p} = U(I_\mathfrak{p}) = (0)$. Hence $L_\mathfrak{p} = (0)$, which is impossible. Thus $\mathrm{ht}_A \mathfrak{p} = 1$ and so $J_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$. Therefore by Proposition 5.3 we get $I_\mathfrak{p} = (J_\mathfrak{p} \cdot U(I_\mathfrak{p}) : I_\mathfrak{p}) \cap U(I_\mathfrak{p}) = L_\mathfrak{p}$. This is a final contradiction, which completes the proof of Theorem 1.3. □

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COMPRESSED POLYTOPES

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This is a brief draft of the paper [3]. A convex polytope $\mathcal{P} \subset \mathbb{R}^n$ is called integral if all vertices of $\mathcal{P}$ belong to $\mathbb{Z}^n$. Moreover, a convex

polytope $\mathcal{P} \subset \mathbb{R}^n$ is said to be a $(0,1)$-polytope if each of its vertices belongs to $\{0,1\}^n$. Let $\mathcal{P} \subset \mathbb{R}^n$ be an integral convex polytope with the vertices $\delta_1, \delta_2, \ldots, \delta_q$ and $K[y_1, y_2, \ldots, y_q]$ the polynomial ring in q variables over a field K. The *toric ideal* of $\mathcal{P}$ is the ideal $I_{\mathcal{P}} \subset K[y_1, y_2, \ldots, y_q]$ generated by all homogeneous binomials $f = \prod_{\ell=1}^{N} y_{t_\ell} - \prod_{\ell=1}^{N} y_{s_\ell}$, where $N = 2, 3, \ldots$, with $\sum_{\ell=1}^{N} \delta_{t_\ell} = \sum_{\ell=1}^{N} \delta_{s_\ell}$. A *compressed polytope* [5, p. 337] is an integral convex polytope $\mathcal{P} \subset \mathbb{R}^n$ such that the initial ideal of $I_{\mathcal{P}}$ with respect to any reverse lexicographic monomial order on $K[y_1, y_2, \ldots, y_q]$ is generated by squarefree monomials. If $\mathcal{P} \subset \mathbb{R}^n$ is compressed, then all faces of $\mathcal{P}$ are again compressed. It turns out [7, Corollary 8.9] that an integral convex polytope is compressed if and only if any of its "pulling triangulations" (e.g., [5]) is unimodular.

It is known [5, Example 2.4 (b)] that the convex polytope of all $n \times n$ doubly stochastic matrices is compressed. Moreover, it is proved [7, Theorem 14.8] that all reverse lexicographic initial ideals of its toric ideal are generated by squarefree monomials of degree at most n. The purpose of the present paper is to discuss the technique appearing in the proof of [7, Theorem 14.8] in a much more general situation and to show that the convex polytope determined by a certain system of linear inequalities is compressed provided that the polytope is a $(0,1)$-polytope. The explicit statement will appear in Theorem 1. Our situation is particularly nice if the coefficient matrix of the system of linear inequalities is totally

unimodular, because the required assumption that the polytope is a $(0,1)$-polytope is automatically satisfied if the coefficient matrix is totally unimodular (Corollary 2). Example 3 says that the class of compressed $(0,1)$-polytopes includes (i) hypersimplices, (ii) order polytopes of finite partially ordered sets, and (iii) stable polytopes of perfect graphs.

We fix integers a_{ij}, b_i and ε_i, $1 \leq i \leq m$, $1 \leq j \leq n$, with each $\varepsilon_i \in \{0,1\}$, and suppose that the set of all solutions

$$x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$$

of the system of linear inequalities

$$b_i \leq \sum_{j=1}^{n} a_{ij} x^{(j)} \leq b_i + \varepsilon_i, \qquad 1 \leq i \leq m; \tag{1}$$

$$0 \leq x^{(j)} \leq 1, \qquad 1 \leq j \leq n \tag{2}$$

1

is nonempty. Thus the system of linear inequalities (1) and (2) determines a convex polytope $\mathcal{P} \subset \mathbb{R}^n$. The question

whether $\mathcal{P}$ is a $(0,1)$-polytope or not seems to be rather difficult. However, once the convex polytope $\mathcal{P}$ turns out to be a $(0,1)$-polytope, we immediately conclude that $\mathcal{P}$ is compressed.

Theorem 1. *Suppose that the convex polytope $\mathcal{P} \subset \mathbb{R}^n$ determined by the system of linear inequalities* (1) *and* (2) *is a $(0,1)$-polytope. Then $\mathcal{P}$ is compressed.*

Theorem 1 is particularly nice if the coefficient matrix $(a_{ij})_{1\le i\le m; 1\le j\le n}$ of the system of linear inequalities (1) is totally unimodular, because the required assumption that $\mathcal{P}$ is a $(0,1)$-polytope is automatically satisfied if $(a_{ij})_{1\le i\le m; 1\le j\le n}$ is totally unimodular.

A matrix is called *totally unimodular* if each of its subdeterminants belongs to $\{0,+1,-1\}$. It follows from Hoffman and Kruskal [2] that if an $m \times n$ matrix $(a_{ij})_{1\le i\le m; 1\le j\le n}$ with each $a_{ij} \in \mathbb{Z}$ is totally unimodular, then, for arbitrary integers b_i and ε_i, $1 \le i \le m$, with each $\varepsilon_i \in \{0,1\}$, all of the vertices of the convex polytope in $\mathbb{R}^n$ determined by the system of linear inequalities (1) and (2) belong to $\mathbb{Z}^n$. Hence

Corollary 2. *Let an $m \times n$ matrix $(a_{ij})_{1\le i\le m; 1\le j\le n}$ be totally unimodular. Then, for arbitrary integers b_i and ε_i, $1 \le i \le m$, with each $\varepsilon_i \in \{0,1\}$, the convex polytope determined by the system of linear inequalities* (1) *and* (2) *is compressed.*

Examples of totally unimodular matrices contain vertex–edge incidence matrices of finite bipartite graphs. Consult, e.g., [4] for the detailed information about totally unimodular matrices.

Example 3. (a) One of the most direct applications of Corollary 2 concerns the hypersimplex. Let $2 \le d < n$ be integers. The d-th *hypersimplex* in $\mathbb{R}^n$ is the convex polytope $\Delta(n;d) \subset \mathbb{R}^n$ which is the convex hull of all $(0,1)$-vectors $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ with $x^{(1)} + x^{(2)} + \cdots + x^{(n)} = d$. Since the $1 \times n$ matrix $[1,1,\ldots,1]$ is totally unimodular, it follows that $\Delta(n;d)$ is compressed.

(b) Let $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite partially ordered set and $\mathcal{O}_P$ the *order polytope* [6, Example 4.6.34] of P. Thus $\mathcal{O}_P$ is the convex polytope in $\mathbb{R}^n$ whose vertices are the $(0,1)$-vectors $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ satisfying that if $x^{(s)} = 1$

and if $\alpha_t \le \alpha_s$ in P then $x^{(t)} = 1$. In particular, the origin of $\mathbb{R}^n$ is a vertex of $\mathcal{O}_P$.

Since the system of linear inequalities (i) $x^{(s)} \le x^{(t)}$ for all s and t with $\alpha_t \le \alpha_s$ and (ii) $0 \le x^{(j)} \le 1$ for all $1 \le j \le n$ determines $\mathcal{O}_P$, it follows from Theorem 1 that $\mathcal{O}_P$ is compressed.

(c) Let G be a finite graph on the vertex set $V(G) = \{1,2,\ldots,n\}$ having no loop and no multiple edge, and $E(G)$ the edge set of G. We associate each subset $W \subset V(G)$ with the $(0,1)$-vector $\rho(W) = \sum_{j\in W} \mathbf{e}_j \in \mathbb{R}^n$. Here $\mathbf{e}_j$ is the j-th unit coordinate vector in $\mathbb{R}^n$. In particular, $\rho(\emptyset)$ is the origin of $\mathbb{R}^n$. A subset $W \subset V(G)$ is called stable if $\{i,j\} \not\in E(G)$ for any $i,j \in W$ with $i \ne j$. Note that the empty set and all single-element subsets of $V(G)$ are stable. Let $S(G)$ denote the set of all stable sets of G. The *stable polytope* of G is the $(0,1)$-polytope in $\mathbb{R}^n$ which is the convex hull of $\{\rho(W)\,;\, W \in S(G)\}$. It then follows from Chvátal [1] together with Theorem 1 that the stable polytope of G is compressed if G is a perfect graph. (A finite graph G is called perfect if, for all induced subgraphs H of G including G itself, the chromatic number of H is equal to the maximal degree of complete subgraphs contained in H.)

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Existence of homogeneous prime ideals fitting into long Bourbaki sequences

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§1. Introduction

Let p, r be integers with $2 \leq p \leq r-2$. Given a homogeneous ideal I of height p in a polynomial ring $R := k[x_1, \ldots, x_r]$, there is a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq p-1$ that fits into an exact sequence of the form

$$(*) \qquad 0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules (see e.g. [2], [13]). Conversely, as proved in our previous paper [3], given a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq p-1$, there is a homogeneous ideal I of height p in R fitting into an exact sequence of the above form. But, unfortunately, the residue class rings defined by the ideals constructed by our method of [3] are not even reduced in general.

When does there exist a prime I fitting into $(*)$? The aim of this talk is to give an answer to this question. In fact, there is a homogeneous prime ideal I of height p fitting into an exact sequence of the form $(*)$ for some c and $S_0, \ldots, S_{p-1}$, if the given module M as above is reflexive . Since M is reflexive if R/I is equidimensional by the local version of [13, Corollary 1.20], it turns out that the reflexiveness of M is equivalent to the existence of such a prime I. This theorem is known well for the classical case where $p = 2$ (see e.g. [8], [10]).

To prove our theorem, as in the proof of the main theorem of [3], we make full use of the minimal free complex $F_\bullet$ bounded on both sides with differentials $\partial^F_\bullet$ defined by the conditions $M = \mathrm{Coker}(\partial^F_1)$, $H_i(F_\bullet) = 0$ for $i > 0$, and $H^i(F^\vee_\bullet) = 0$ for $i \leq 0$. The way we use it in our new proof, however, is very different from that in our previous one.

This time, we make a kind of decomposition of $F_\bullet$ regarding its homologies as in [1], in order to construct a finitely generated torsion-free graded module $\tilde{M}$ over the integral normal domain $A := R/(f_1, \ldots, f_{p-2})$ defined by a homogeneous R-regular sequence $f_1, \ldots, f_{p-2}$ such that there is a homomorphism $\varphi : M \longrightarrow \tilde{M}$ over R inducing an isomorphism $H^i_{\mathfrak{m}}(\varphi) : H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(\tilde{M})$ for all $0 \leq i < r-p+2$. Then, applying to the A-module $\tilde{M}$ the well-known determinantal method for constructing two-codimensional subschemes, we obtain our main results. The same method has already been proposed in [12] to treat the simplest case where $F_\bullet$ is the direct sum of the free complexes giving the minimal free resolutions of graded modules of finite length over R.

§2. Descent of graded modules via free complexes and existence of homogeneous prime ideals

Let $R := k[x_1, \ldots, x_r]$ be a polynomial ring in r indeterminates $x_1, \ldots, x_r$ over an infinite field k, $\mathfrak{m} := \bigoplus_{i>0}[R]_i$ the irrelevant maximal ideal in R, and A the graded residue class ring of R defined by a homogeneous ideal in R. We assume that A is Cohen-Macaulay. For a complex $C_\bullet$ of finitely generated graded free modules over A, we say that $C_\bullet$ is *minimal* if $\mathrm{Im}(\partial^C_i) \subset \mathfrak{m}C_{i-1}$ for all $i \in \mathbf{Z}$, where ∂^C_i $(i \in \mathbf{Z})$ are the differentials of $C_\bullet$. If there are a minimal complex $C'_\bullet$ and a split exact complex $C''_\bullet$, of finitely generated graded free modules over A, such that $C_\bullet = C'_\bullet \oplus C''_\bullet$, then we will denote $C'_\bullet$ (resp. $C''_\bullet$) by $\mathrm{min}(C_\bullet)_\bullet$ (resp. $\mathrm{se}(C_\bullet)_\bullet$). Further, in this case, $\mathrm{min}(C_\bullet)_\bullet$ (resp. $\mathrm{se}(C_\bullet)_\bullet$) will be called the *minimal* (resp. *split exact*) *part* of $C_\bullet$ (see [1, (1.1) and (1.2)]). Given a chain map $\mu_\bullet : C_\bullet \longrightarrow D_\bullet$ of complexes, its mapping cone will be denoted by $\mathrm{con}(\mu_\bullet)_\bullet$.

Definition 1. Let $L_\bullet$ be a complex of finitely generated graded free modules over A, $\mathfrak{a}$ a homogeneous ideal in A, and m an integer. We say that a subcomplex $L'_\bullet$ of $L_\bullet$ is a quasi-direct summand of $L_\bullet$ up to $(\mathfrak{a}, m)$ if it satisfies the following conditions.

(i) $L'_\bullet$ and $L''_\bullet := L_\bullet/L'_\bullet$ are free complexes.

(ii) There is a chain map $\mu_\bullet : L''_\bullet \longrightarrow L'[-1]_\bullet$ satisfying $\mathrm{Im}(\mu_i) \subset \mathfrak{a}L'_{i-1}$ for all $i \leq m$ such that $L_\bullet = \mathrm{con}(\mu_\bullet)_\bullet$.

Lemma 2. *Let V be a finitely generated graded module over A,*

$$\cdots \longrightarrow P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\delta^P} V \longrightarrow 0$$

a minimal free resolution of V over A, and $\mathfrak{a}$ a homogeneous ideal in A annihilating V. Let further $n \geq 0$ and m be integers. Then, there is an integer l such that, for an arbitrary homogeneous A-regular element f of $\mathfrak{a}^l$, there is a free resolution

$$\cdots \longrightarrow \tilde{P}_2 \longrightarrow \tilde{P}_1 \longrightarrow \tilde{P}_0 \longrightarrow V \longrightarrow 0$$

of V over $\bar{A} := A/(f)$ such that

(i) $\bar{P}_\bullet := P_\bullet / f P_\bullet$ *is a quasi-direct summand of* $\tilde{P}_\bullet$ *up to* $(\mathfrak{a}^n, m)$, *where* $P_i = 0$ *and* $\tilde{P}_i = 0$ *for* $i < 0$,

(ii) *the canonical homomorphism* $H_i(\alpha_\bullet) : H_i(P_\bullet) \longrightarrow H_i(\tilde{P}_\bullet)$, *induced from the chain map* $\alpha_\bullet : P_\bullet \longrightarrow \tilde{P}_\bullet$ *over* A *obtained by composing the natural surjection* $P_\bullet \twoheadrightarrow \bar{P}_\bullet$ *and the injection* $\bar{P}_\bullet \hookrightarrow \tilde{P}_\bullet$, *is an isomorphism for all* $i \in \mathbf{Z}$.

Lemma 3. *Let* a, l, m, n, s_i $(a+2 \le i \le m)$ *be integers with* $n \ge 0$, $m \ge a+1$, $0 \le s_i \le l$ $(a+2 \le i \le m)$ *and* $\mathfrak{a}$ *be a homogeneous ideal in* A. *Let further* $L_\bullet$, $L'_\bullet$, $G_\bullet$, *and* $G'_\bullet$ *be complexes of finitely generated graded free modules over* A *such that* $L'_\bullet$ *(resp.* $G'_\bullet$*) is a quasi-direct summand of* $L_\bullet$ *(resp.* $G_\bullet$*) up to* $(\mathfrak{a}^l, m)$ *(resp.* $(\mathfrak{a}^l, m+1)$*). Suppose* $G_i = 0$ *for* $i < a+1$, $H_i(G_\bullet) = 0$ *for* $i > a+1$, $\mathfrak{a}H_{a+1}(G_\bullet) = 0$, *and*

$$(3.1)\quad \mathrm{Im}(\partial_i^{G'}) \cap \mathfrak{a}^s G'_{i-1} \subset \mathfrak{a}^{s-s_i}\,\mathrm{Im}(\partial_i^{G'}) \quad \textit{for all } s,\ i \textit{ with } s_i \le s \le l,\ a+2 \le i \le m.$$

If there is a chain map $\lambda'_\bullet : L'_\bullet \longrightarrow G'_\bullet$ *and* $l \ge n+1+\sum_{j=a+2}^m s_j$, *then there is a chain map* $\lambda_\bullet : L_\bullet \longrightarrow G_\bullet$ *such that*

(i) $\lambda_\bullet|_{L'_\bullet} = \iota_\bullet \circ \lambda'_\bullet$,

(ii) $\mathrm{con}(\lambda'_\bullet)_\bullet$ *is a quasi-direct summand of* $\mathrm{con}(\lambda_\bullet)_\bullet$ *up to* $(\mathfrak{a}^n, m)$,

where $\iota_\bullet : G'_\bullet \longrightarrow G_\bullet$ *denote the injection and we understand* $\sum_{j=a+2}^m s_j = 0$ *in case* $m = a+1$.

Lemma 4 (cf. [1, (1.5)]). *Let* a_0 *and* a *be integers with* $a \ge a_0$, *and let* $F_\bullet$ *be a minimal complex of finitely generated graded free modules over* A *such that* $F_i = 0$ *for* $i < a_0$ *and* $H_i(F_\bullet) = 0$ *for* $i > a$. *Let further*

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow H_a(F_\bullet) \longrightarrow 0$$

be a minimal free resolution of $H_a(F_\bullet)$ *over* A *and* $G_\bullet := P[-a-1]_\bullet$. *Then there exist a minimal complex* $L_\bullet$ *of finitely generated graded free modules over* A *and a chain map* $\mu_\bullet : L_\bullet \longrightarrow G_\bullet$ *such that* $F_\bullet$ *is the minimal part of* $\mathrm{con}(\mu_\bullet)_\bullet$, *where* $L_i = 0$ *for* $i < a_0$ *and* $H_i(L_\bullet) = 0$ *for* $i \ge a$.

Applying the above three lemmas repeatedly, we obtain

Lemma 5. *Let* $a_0 < 0$ *be an integer,* $d := \dim(A)$, *and* $F_\bullet$ *a minimal complex of finitely generated graded free modules over* A *such that* $F_i = 0$ *for* $i < a_0$, $\dim(H_i(F_\bullet)) < d-2+i$ *for* $a_0 \le i \le -1$, *and* $H_i(F_\bullet) = 0$ *for* $i \ge 0$. *Let further* $\mathfrak{a}$ *be a homogeneous ideal in* A *annihilating all* $H_i(F_\bullet)$ $(a_0 \le i \le -1)$ *and* $n \ge 0$ *an integer. Then, there is a positive integer* n_0 *such that, for an arbitrary homogeneous* A*-regular element* f *of* $\mathfrak{a}^{n_0}$, *there are a complex* $D_\bullet$ *of finitely generated graded free modules over* A *and a complex* $\tilde{D}_\bullet$ *of finitely generated graded free modules over* $\bar{A} := A/(f)$, *satisfying the following conditions.*

(i) $F_\bullet$ *is the minimal part of* $D_\bullet$.

(ii) $\bar{D}_\bullet := D_\bullet / f D_\bullet$ *is a quasi-direct summand of* $\tilde{D}_\bullet$ *up to* $(\mathfrak{a}^n, 0)$.

(iii) $D_i = 0$ *and* $\tilde{D}_i = 0$ *for* $i < a_0$.

(iv) *The canonical homomorphism* $H_i(\nu_\bullet) : H_i(D_\bullet) \longrightarrow H_i(\tilde{D}_\bullet)$, *induced from the chain map* $\nu_\bullet : D_\bullet \longrightarrow \tilde{D}_\bullet$ *over* A *obtained by composing the natural surjection* $D_\bullet \twoheadrightarrow \bar{D}_\bullet$ *and the injection* $\bar{D}_\bullet \hookrightarrow \tilde{D}_\bullet$, *is an isomorphism for all* $i \in \mathbf{Z}$.

(v) *The canonical homomorphism*

$$H^i_{\mathfrak{m}}(\nu_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^D_1)) \longrightarrow H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{D}}_1))$$

induced from ν_0 *is an isomorphism for all* $0 \leq i < d-1$ *with*

$$H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^D_1)) \cong H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{D}}_1)) = 0.$$

Theorem 6. *Let* a_0, p *be integers with* $a_0 < 0$, $2 \leq p \leq \dim(A)$ *and* $F_\bullet$ *a minimal complex of graded free modules over* A *such that* $F_i = 0$ *for* $i < a_0$, $\dim(H_i(F_\bullet)) \leq \dim(A) - p + i$ *for* $a_0 \leq i \leq -1$, *and* $H_i(F_\bullet) = 0$ *for* $i \geq 0$. *Let further* $\mathfrak{a}$ *be a homogeneous ideal in* A *of grade larger than or equal to* $p-2$ *annihilating all* $H_i(F_\bullet)$ $(a_0 \leq i \leq -1)$. *Then, there are a homogeneous* A*-regular sequence* $f_1, \ldots, f_{p-2}$ *with* $f_i \in \mathfrak{a}$ *for all* $1 \leq i \leq p-2$, *a minimal complex* $\tilde{F}_\bullet$ *of finitely generated graded free modules over* $A/(f_1, \ldots, f_{p-2})$, *and a chain map* $\tau_\bullet : F_\bullet \longrightarrow \tilde{F}_\bullet$ *over* A *satisfying the following conditions.*

(i) $\tilde{F}_i = 0$ *for* $i < a_0$.

(ii) *The canonical homomorphism*

$$H^i_{\mathfrak{m}}(\tau_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^F_1)) \longrightarrow H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{F}}_1))$$

induced from τ_0 *is an isomorphism for all* $0 \leq i < \dim(A) - p + 2$.

(iii) *The canonical homomorphism* $H_i(\tau_\bullet) : H_i(F_\bullet) \longrightarrow H_i(\tilde{F}_\bullet)$ *induced from* $\tau_\bullet$ *is an isomorphism for all* $i \in \mathbf{Z}$.

Moreover we can choose $f_1, \ldots, f_{p-2}$ *so that* $\mathrm{Proj}(A/(f_1, \ldots, f_{p-2}))$ *is smooth in the outside of the union of* $\mathrm{Proj}(A/\mathfrak{a})$ *and the singularity of* $\mathrm{Proj}(A)$.

Borrowing the idea of orientation from [5], [6] and [9], we will say that a coherent sheaf $\mathcal{M}$ of rank v on a projective scheme $X \hookrightarrow \mathrm{Proj}(R)$ is *orientable* on a Zariski open subset U of X, if $\bigwedge^v \mathcal{M}|_U \cong \mathcal{O}_U(n)$ for some integer n.

Let M be a finitely generated graded module over R with no free direct summand,

$$\cdots \xrightarrow{\partial_{p+1}} F_p \xrightarrow{\partial_p} F_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial'_0} M \longrightarrow 0$$

a minimal free resolution of M over R, and

$$0 \longrightarrow F''_{a_0} \xrightarrow{\partial''_{a_0-1}} \cdots \xrightarrow{\partial''_{-2}} F''_{-2} \xrightarrow{\partial''_{-1}} F''_{-1} \xrightarrow{\partial''_0} M^\vee \longrightarrow 0$$

a minimal free resolution of $M^\vee$ over R, where $a_0 < 0$. Let further $F_i := F_i''^\vee$ and $\partial_i = \partial_i''^\vee$ for $i < 0$, and $\partial_0 := (\partial_0'^\vee \circ \partial_0'')^\vee$. Connecting the former resolution to the dual of the latter with the use of ∂_0, we obtain a complex

$$F_\bullet : \cdots \xrightarrow{\partial_{p+1}} F_p \xrightarrow{\partial_p} F_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} \cdots \xrightarrow{\partial_{a_0-1}} F_{a_0} \longrightarrow 0$$

bounded on both sides (cf. [1], [3]) such that $H_i(F_\bullet) = 0$ for $i > 0$ and $H^i(F_\bullet^\vee) = 0$ for $i \le 0$, where $F_i = F_i^{\vee\vee}$ and $\partial_i = \partial_i^{\vee\vee}$ for $i < 0$. We will denote this complex by $\mathrm{cpx}(M)_\bullet$.

Applying the above theorem to $\mathrm{cpx}(M)_\bullet$, we obtain the following

Theorem 7. *Let p, u be integers with $2 \le p \le r$, $u > p$, $\mathfrak{a}$ a homogeneous ideal in R of height larger than or equal to u, and M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}_R^i(M, R) = 0$ for $1 \le i \le p-1$ such that $M_\mathfrak{p}$ is free for all homogeneous prime ideals $\mathfrak{p} \subset R$ not containing $\mathfrak{a}$. Then, there are a homogeneous R-regular sequence $f_1, \dots, f_{p-2}$ with $f_i \in \mathfrak{a}$ for all $1 \le i \le p-2$, a finitely generated graded module $\tilde{M}$ over $\bar{R} := R/(f_1, \dots, f_{p-2})$, and a homomorphism $\varphi : M \longrightarrow \tilde{M}$ over R satisfying the following conditions.*

(i) *The canonical homomorphism $H_\mathfrak{m}^i(\varphi) : H_\mathfrak{m}^i(M) \longrightarrow H_\mathfrak{m}^i(\tilde{M})$ induced from φ is an isomorphism for all $0 \le i < r - p + 2$ and $H_\mathfrak{m}^{r-p+1}(\tilde{M}) = 0$.*

(ii) *The scheme $X :=$ $\mathrm{Proj}(\bar{R})$ is an integral normal scheme which is smooth in outside of its subscheme of codimension $u - p + 2$.*

(iii) *The sheaf $\tilde{\mathcal{M}}$ on X that $\tilde{M}$ defines is locally free and orientable on the outside of a subscheme of X of codimension $u - p + 2$.*

(iv) *$\tilde{M}$ is torsion-free over the integral domain $\bar{R}$.*

Lemma 8. *Let $s \ge 4$ be an integer, X the projective scheme* $\mathrm{Proj}(A)$, *Z a subscheme of X of codimension larger than or equal to s. Assume that X is an integral normal scheme which is smooth in the outside of Z. Let $\mathcal{M}$ be a torsion-free coherent sheaf on X of rank $t + 1 \ge 2$ which is locally free and orientable on the outside of Z and m an integer such that $\mathcal{M}(m)$ is generated over $\mathcal{O}_X$ by its global sections. Let further n be the integer such that $\bigwedge^{t+1} \mathcal{M}|_{X \setminus Z} \cong \mathcal{O}_{X \setminus Z}(n)$. Then for all integers $m_1, \dots, m_t$ larger than m, there exists a two-codimensional closed subscheme Y of X smooth in the outside of a subscheme of X of codimension not less than* $\min(s, 6)$, *whose ideal sheaf $\mathcal{I}_Y$ fits into a Bourbaki sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{t} \mathcal{O}_X(-m_i) \longrightarrow \mathcal{M} \longrightarrow \mathcal{I}_Y(c) \longrightarrow 0,$$

where $c = n + \sum_{i=1}^{t} m_i$.

Lemma 9. *Let p, q be integers with $2 \leq p \leq r$, $1 \leq q < p$, M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq p-1$, $f_1, \ldots, f_q$ homogeneous polynomials of R forming an R-regular sequence, A the factor ring $R/(f_1, \ldots, f_q)$, $\tilde{M}$ a finitely generated graded module over A, $\varphi : M \longrightarrow \tilde{M}$ a homomorphism over R such that $H^i_{\mathfrak{m}}(\varphi) : H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(\tilde{M})$ is an isomorphism for all $i \leq r-p$. Suppose there exists a homogeneous ideal $\tilde{I}$ of height $p-q$ in A and a homomorphism $\tilde{\psi} : \tilde{M} \longrightarrow \tilde{I}(c)$ ($c \in \mathbf{Z}$) such that $H^i_{\mathfrak{m}}(\tilde{\psi}) : H^i_{\mathfrak{m}}(\tilde{M}) \longrightarrow H^i_{\mathfrak{m}}(\tilde{I}(c))$ is an isomorphism for all $i \leq r-p$. Then there is a homogeneous ideal I of height p in R containing $f_1, \ldots, f_q$ with $I/(f_1, \ldots, f_q) = \tilde{I}$ and a homomorphism $\psi : M \longrightarrow I(c)$ such that $H^i_{\mathfrak{m}}(\psi) : H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(I(c))$ is an isomorphism for all $i \leq r-p$.*

We need the following property that one can prove applying the local version of [13, Corollary 1.20] to the graded case.

Lemma 10. *Let $I \in R$ be a homogeneous ideal of height $p \geq 2$ and M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq p-1$. Assume that they fit into an exact sequence of the form*

$$0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free modules over R. Then R/I is equidimensional if and only if M is reflexive.

Main Theorem. *Let p be an integer with $2 \leq p \leq r-2$ and M a finitely generated torsion-free graded reflexive module over R with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq p-1$. Then, there is a homogeneous prime ideal I of height p which fits into an exact sequence of the form*

$$0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free modules over R.

Outline of the proof of Main Theorem.

(1) Let $F_\bullet := \mathrm{cpx}(M)_\bullet$, a_0 be a negative integer such that $F_i = 0$ for all $i < a_0$, and $\mathfrak{a}$ be the product of $\mathrm{ann}(H_i(F_\bullet))$ $(a_0 \leq i \leq -1)$. Then $H_{-1}(F_\bullet) = 0$, $\dim(H_i(F_\bullet)) = \dim(\mathrm{Ext}^{p-i}_R(\mathrm{Coker}(\partial^{F\vee}_p), R)) \leq r - p + i$ for all $i < p$, and $\mathrm{ht}(\mathfrak{a}) \geq p+2$.

(2) There are a homogeneous R-regular sequence $f_1, \ldots, f_{p-2}$ with $f_i \in \mathfrak{a}$ for all $1 \leq i \leq p-2$, a finitely generated graded module $\tilde{M}$ over $\bar{R} := R/(f_1, \ldots, f_{p-2})$, and a homomorphism $\varphi : M \longrightarrow \tilde{M}$ over R satisfying the conditions (i), (ii), (iii), (iv) stated in Theorem 7 with $u = p+2$.

(3) By Lemma 8, there is a two-codimensional closed subscheme Y of $X := \mathrm{Proj}(\bar{R})$ smooth in the outside of a subscheme, say Z', of X of codimension not less than $\min(4, 6) = 4$, whose ideal sheaf $\tilde{\mathcal{I}}_Y$ fits into an exact sequence of the form

$$0 \longrightarrow \bigoplus_{i=1}^{t} \mathcal{O}_X(-m_i) \longrightarrow \tilde{\mathcal{M}} \longrightarrow \tilde{\mathcal{I}}_Y(c) \longrightarrow 0.$$

Let $\tilde{I}$ be the saturated homogeneous ideal of Y in $\bar{R}$. Then, we have a Bourbaki sequence

$$0 \longrightarrow \bigoplus_{i=1}^{t} \bar{R}(-m_i) \longrightarrow \tilde{M} \longrightarrow \tilde{I}(c) \longrightarrow 0.$$

(4) With the use of Lemma 9, we can show that there is a homogeneous ideal I in R of height p containing $f_1, \ldots, f_{p-2}$ with $I/(f_1, \ldots, f_{p-2}) = \tilde{I}$ which fits into the long Bourbaki sequence (✱).

(5) By Lemma 10, Y is equidimensional and hence is reduced.

(6) Since $F_\bullet$ is exact at F_0 and F_{-1}, we can prove that $Y \backslash Z'$ is connected if m_i $(1 \leq i \leq t)$ are large enough by showing $H^0(\mathcal{O}_{Y \backslash Z'}) \cong k$.

(7) Now Y is an equidimensional reduced scheme such that $Y \backslash Z'$ is smooth and connected. Since $\dim(Y) > \dim(Z')$, Y can have only one irreducible component. Hence the saturated ideal I is prime.

See [4] for the detail.

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Dutta multiplicities and test modules

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We shall define Dutta multiplicities and test modules. Dutta multiplicity is an invariant for a bounded finite free complex with homology of finite length. It is slightly different from the alternating sum of lengths of homologies as below. By the theory of Dutta multiplicities, P. Roberts proved the vanishing theorem of intersection multiplicities [13] (he used a vanishing of Dutta multiplicities under some condition) and the New Intersection Theorem in the mixed-characteristic case [14] (he used a positivity of Dutta multiplicities under some condition in the case of positive characteristic). The positivity of Dutta mulitplicity in the case of mixed-characteristic is still a open problem. It is deeply related to Serre's positivity conjecture of intersection multiplicities as below. A test module is a maximal Cohen-Macaulay module with some additional condition. Once a test module exists, then it helps calculation of Dutta multiplicities and the positivity conjecture of Dutta multiplicities is true. We shall show that the small Macaulay modules conjecture implies the existence of test modules. We shall give some examples of test modules.

Before defining Dutta multiplicities, let's see a relation between Dutta multiplicites and alternating sum of lengths of homologies.

Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ and alternating sum of lengths of homologies

Let (A, m) be a homomorphic image of a regular local ring. Let

$$\mathbb{F}. \ : \ 0 \to F_s \to \cdots \to F_0 \to 0$$

be a perfect A-complex with support in $\{m\}$, i.e., $\mathbb{F}.$ is a bounded complex of finitely generated A-free modules such that $\mathbb{F}. \otimes_A A_P$ is exact for any prime ideal $P \neq m$.

Then, the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ is a ratonal number (that is defined later) satisfying:

Fact 1 *With notation as above, if one of the following two conditions is satisfied, then the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ coincides with $\sum_i (-1)^i \ell_A(H_i(\mathbb{F}.))$.*

(1) There exist a regular local ring S and an S-free complex $\mathbb{G}.$ such that (i) A is a homomorphic image of S, (ii) $\mathbb{F}. = \mathbb{G}. \otimes_S A$. (We say that $\mathbb{F}.$ is liftable to a regular local ring *if the condition is satisfied.)*

(2) A is a Roberts ring.

There exists an example $\mathbb{F}.$ that satisfies $\chi_\infty(\mathbb{F}.) \neq \sum_i (-1)^i \ell_A(H_i(\mathbb{F}.))$ (over a Gorenstein ring [12]). (Therefore, such a complex $\mathbb{F}.$ never satisfies the condition (1) as above.)

The author does not know any example of a complex that its Dutta multiplicity is not an integer.

Let $\mathbb{K}.$ be the Koszul complex of a system of parameters $\underline{x}$ of A. Then, $\mathbb{K}.$ satisfies the condition (1) as above. Therefore we have

$$\chi_\infty(\mathbb{K}.) = \sum_i (-1)^i \ell_A(H_i(\mathbb{K}.)) = e(\underline{x}, A) > 0.$$

There are a lot of examples of Roberts rings, e.g., complete intersections, quotient singularities, Galois extensions of regular local rings, an associated graded ring of a regular local ring (with respect to any filtration of ideals that makes the ring Noetherian), simplicial semi-group rings.

Notation

Let A be a commutative Noetherian ring with 1 that is a homomorphic image of a regular ring.

$\mathrm{K}_0(A)$ denotes the Grothendieck group of finitely generated A-modules. For an A-module M, $[M]$ denotes the element in $\mathrm{K}_0(A)$ corresponding to M.

If a ring homomorphism $g : A \to B$ is finite (as a module), we have the induced homomorphism $g^* : \mathrm{K}_0(B) \to \mathrm{K}_0(A)$ of additive groups defined by $g^*([M]) = [{}_gM]$ for an B-module M, where ${}_gM$ is an A-module M whose A-module structure is given through g.

$\mathrm{A}_*(A)$ is the Chow group of the affine scheme $\operatorname{Spec} A$. For a prime ideal $\mathfrak{p}$ of A of $\dim A/\mathfrak{p} = i$, $[\operatorname{Spec} A/\mathfrak{p}]$ denotes the cycle in $\mathrm{A}_i(A)$ corresponding to the closed subscheme $\operatorname{Spec} A/\mathfrak{p}$. If a ring homomorphism $g : A \to B$ is finite, we have the induced homomorphism $g^* : \mathrm{A}_*(B) \to \mathrm{A}_*(A)$ by $g^*([\operatorname{Spec} B/P]) = [Q(B/P) : Q(A/P \cap A)][\operatorname{Spec} A/P \cap A]$, where $Q(\ )$ is the field of fractions. (See 1.4 in Fulton [5]. g^* is the push-forward of cycles for the proper morphism $\operatorname{Spec} B \to \operatorname{Spec} A$.)

For an additive group N, $N_{\mathbb{Q}}$ denotes $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let (A, m) be a local ring. Let

$$\mathbb{F}. : 0 \to F_s \to \cdots \to F_0 \to 0$$

be a perfect A-complex with support in $\{m\}$. We shall refer s as the length of the complex $\mathbb{F}.$ if F_s is not 0. We define the Euler characteristic map $\chi_{\mathbb{F}.} : \mathrm{K}_0(A) \to \mathbb{Z}$ (or $\chi_{\mathbb{F}.} : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathbb{Q}$) to be

$$\chi_{\mathbb{F}.}([M]) = \sum_t (-1)^t \ell_A(H_t(\mathbb{F}. \otimes_A M)).$$

Definition of the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$

Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Here, we shall give three kind of definitions of the Dutta multiplicity of a complex $\mathbb{F}.$. (Of course they are equivalent to each others.)

(I) We assume that (A, m) is a d-dimensional complete local ring containing a field of characteristic $p > 0$ with perfect residue class field A/m.

Let $f : A \to A$ be the Frobenius map, i.e., $f^e(x) = x^{p^e}$. Note that f^e is finite for every $e \geq 0$. Then the Dutta multiplicity of the complex $\mathbb{F}.$ is defined to be

$$\chi_\infty(\mathbb{F}.) = \lim_{e\to\infty} \frac{\chi_{\mathbb{F}.}([{}_{f^e}A])}{p^{de}} \in \mathbb{Q}.$$

(This definition is due to Dutta [2].)

(II) Here, assume that (A, m) is a Nagata local domain with Noether normalization S, i.e., S is a regular local subring of A such that the inclusion $S \to A$ is finite.

Take any finite normal extension L of $Q(S)$ containing $Q(A)$. Let B_L be the integral closure of A in L. (B_L is a finitely generated A-module, but it may not be a local ring.)

Then we have

$$\chi_\infty(\mathbb{F}.) = \frac{\chi_{\mathbb{F}.}([B_L])}{[L : Q(A)]} \in \mathbb{Q}.$$

(It does not depend on the choice of L.)

(III) Let A be a homomorphic image of a regular local ring and put $d = \dim A$. Then, we define Dutta multiplicity as

$$\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(\tau_A^{-1}([\operatorname{Spec} A]_d)) \in \mathbb{Q}.$$

Here, $\tau_A : \mathrm{K}_0(A)_\mathbb{Q} \to \mathrm{A}_*(A)_\mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-vector spaces given by the singular Riemann-Roch theory (Fulton [5]), and we put

$$[\operatorname{Spec} A]_d = \sum_{\dim A/P = d} \ell_{A_P}(A_P)[\operatorname{Spec} A/P] \in \mathrm{A}_d(A)_\mathbb{Q}.$$

It is known that $\tau_A([A]) = [\operatorname{Spec} A]_d +$ (lower dimensional terms). Here, we say that A is a Roberts ring if $\tau_A([A]) = [\operatorname{Spec} A]_d$ is satisfied. Therefore, if A is a Roberts ring, then we have $\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}([A])$.

We can also define the Dutta multiplicity [11] by using Adams operations of compleces defined by Gillet-Soulé [6].

Positivity of Dutta multiplicities

Let $\mathbb{K}.$ be the Koszul complex of a system of parameters $\underline{x}$ of A. Then, we have $\chi_\infty(\mathbb{K}.) = \chi_{\mathbb{K}.}([A]) = e(\underline{x}, A) > 0$. The following conjecture is a natural generalization of the positivity. (So, $\chi_\infty(\mathbb{F}.)$ is a generalization of the usual multiplicity.)

Conjecture 2 *Let A be a homomorphic image of a regular local ring and put $d = \dim A$. Let*

$$\mathbb{F}. \ : 0 \to F_d \to \cdots \to F_0 \to 0$$

be a perfect A-complex with support in $\{m\}$. If $H_0(\mathbb{F}.) \neq 0$, then $\chi_\infty(\mathbb{F}.) > 0$.

Remark 3 (1) With notation as in the conjecture, there exists an example such that $\chi_{\mathbb{F}.}([A]) < 0$ (Roberts [15]).

If A is a Cohen-Macaulay ring, then we have $\chi_{\mathbb{F}.}([A]) = \ell_A(H_0(\mathbb{F}.)) > 0$.

Therefore, if A is a Cohen-Macaulay Roberts ring, then we have $\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}([A]) = \ell_A(H_0(\mathbb{F}.)) > 0$ and the conjecture is true for A.

(2) Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$. Then, it is easy to see $\chi_\infty(\mathbb{F}.[-1]) = -\chi_\infty(\mathbb{F}.)$, where $\mathbb{F}.[-1]$ is the shifted complex. If $\mathbb{K}.$ is the Koszul complex of a system of parameters, then we have $\chi_\infty(\mathbb{K}.[-1]^{\oplus 2} \oplus \mathbb{K}.) = 2\chi_\infty(\mathbb{K}.[-1]) + \chi_\infty(\mathbb{K}.) = -\chi_{\mathbb{K}.}([A]) < 0$.

Therefore, we can not remove the assumption on the length of the complex in the conjecture.

The New Intersection Theorem implies that the length of the complex in the conjecture is just d.

(3) Roberts proved the conjecture if A is of positive characteristic. It played an essential role in his proof of the New Intersection Theorem in the mixed-characteristic case.

If A contains a field of characteristic 0, then the conjecture is also true (Kurano-Roberts [11]).

The conjecture is open even if A is Gorenstein.

(4) The conjecture is deeply related to Serre's positivity conjecture of intersection multiplicities [16]. Pricisely speaking, the conjecture in the case where $\mathbb{F}.$ is liftable to a regular local ring (see Fact 1 (1)) is equivalent to Serre's positivity conjecture of intersection multiplicities in the case where one of two modules is a Cohen-Macauly module (not necessary maximal). They are still open problems.

Let $\mathbb{F}.$ be a complex as in Conjecture 2. If $\mathbb{F}.$ is liftable to a regular local ring that is equi-characteristic or unramified of mixed-characteristic, then the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ is positive by Serre's positivity theorem in those cases [16]. If $\mathbb{F}.$ is liftable to a regular local ring that is ramified of mixed-characteristic, then the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ is non-negative by Gabber's non-negativity theorem [1].

Test modules

Here we define the notion of test modules. Using test modules, we prove that the small Macaulay modules conjecture (Hochster [7]) implies Conjecture 2.

Definition 4 Let A be a homomorphic image of a regular local ring and put $d = \dim A$. A finitely generated A-module M is called a *test module* for A if M is a (non-zero) maximal Cohen-Macaulay module with $\tau_A([M]) \in \mathrm{A}_d(A)_\mathbb{Q}$.

Assume that (A, m) is a d-dimensional complete local ring containing a field of characteristic $p > 0$ with perfect residue class field A/m. Let f be the Frobenius map. Then $\tau_A([M]) \in \mathrm{A}_d(A)_\mathbb{Q}$ if and only if $[{}_fM] = p^d[M]$ is satisfied in $\mathrm{K}_0(A)_\mathbb{Q}$.

Remark that A itself is a test module for A if and only if A is a Cohen-Macaulay Roberts ring.

Proposition 5 *Let (A, m) be a d-dimensional integral domain that is a homomorphic image of a regular local ring. Assume that A has a test module M. Then, for a perfect A-complex $\mathbb{F}.$ with support in $\{m\}$, the following are satisfied:*

(1) $\chi_\infty(\mathbb{F}.) = \dfrac{\chi_{\mathbb{F}.}([M])}{\operatorname{rank}_A M}$

(2) If the length of $\mathbb{F}.$ *is equal to d, then we have* $\chi_\infty(\mathbb{F}.) = \dfrac{\ell_A(H_0(\mathbb{F}. \otimes_A M))}{\operatorname{rank}_A M}$. *Therefore, if* $\mathbb{F}.$ *is a complex as in Conjecture 2, then we have* $\chi_\infty(\mathbb{F}.) > 0$.

Outline of a proof. Put $r = \operatorname{rank}_A M$. Since M is a test module, we have $\tau_A([M]) = r[\operatorname{Spec} A]$.

Therefore we have

$$\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(\tau_A^{-1}([\operatorname{Spec} A])) = \chi_{\mathbb{F}.}([M]/r) = \chi_{\mathbb{F}.}([M])/r.$$

Since M is a maximal Cohen-Macaulay module, we have $\chi_{\mathbb{F}.}([M]) = \ell_A(H_0(\mathbb{F}. \otimes_A M))$ if the length of $\mathbb{F}.$ is equal to d. **q.e.d.**

By Proposition 5, a test module makes calculation of Dutta multiplicities easier.

Furthermore, if a local domain A has a test module, then Conjecture 2 is true for A.

It is known that Conjecture 2 is reduced to the case where A is a complete local domain. Therefore, if every complete local domain has a test module, then Conjecture 2 is true.

By the next theorem, the small Macaulay modules conjecture (Hochster [7]) implies Conjecture 2.

Theorem 6 *If every complete local domain has a maximal Cohen-Macaulay module, then every complete local domain has a test module.*

Outline of a proof. Let (A, m) be a complete local domain. Take a Noether normalization S of A.

For the simplicity, assume that the field extension $Q(A)/Q(S)$ is separable.

Take a finite Galois extension L over $Q(S)$ that contains $Q(A)$. Let B_L be the integral closure of A in L. (B_L is a complete local domain.) Let G be a Galois group of $L/Q(S)$. Then, every $g \in G$ gives an S-automorphism $g : B_L \to B_L$. Therefore, every $g \in G$ induces $g^* : \mathrm{K}_0(B_L)_\mathbb{Q} \to \mathrm{K}_0(B_L)_\mathbb{Q}$ and $g^* : \mathrm{A}_*(B_L)_\mathbb{Q} \to \mathrm{A}_*(B_L)_\mathbb{Q}$. So, G acts on both $\mathrm{K}_0(B_L)_\mathbb{Q}$ and $\mathrm{A}_*(B_L)_\mathbb{Q}$.

Let N be a maximal Cohen-Macaulay B_L-module. Put $M = \bigoplus_{g \in G} {}_gN$. Then, it is easy to see that M is a maximal Cohen-Macaulay B_L-module with ${}_gM \simeq M$ for any $g \in G$. Then $[M] \in \mathrm{K}_0(B_L)_\mathbb{Q}$ is an G-invariant. Hence so is $\tau_{B_L}([M]) \in \mathrm{A}_*(B_L)_\mathbb{Q}$. On the other hand, we have $(\mathrm{A}_*(B_L)_\mathbb{Q})^G \simeq \mathrm{A}_*(B_L^G)_\mathbb{Q} = \mathrm{A}_*(S)_\mathbb{Q} = \mathrm{A}_d(S)_\mathbb{Q}$ (Example 1.7.6 in Fulton [5]). Therefore we have $\tau_{B_L}([M]) \in \mathrm{A}_d(B_L)_\mathbb{Q}$. Then we obtain $\tau_A([M]) \in \mathrm{A}_d(B_L)_\mathbb{Q}$ and, therefore, M is a test module for A. **q.e.d.**

Remark 7 (1) A complete intersection is a Cohen-Macaulay Roberts ring. Therefore, if A is a complete intersection, then A itself is a test module for A.

It is not known whether test modules exist or not even if A is an equi-characteristic complete Gorenstein normal domain.

(2) Let A is a d-dimensional Cohen-Macaulay ring. Put $\tau_A([A]) = q_d + \cdots + q_0$, where $q_i \in \mathrm{A}_i(A)_{\mathbb{Q}}$. Then it is known that $\tau_A([K_A]) = q_d - q_{d-1} + \cdots + (-1)^i q_{d-i} + \cdots$.

Therefore, if $q_i = 0$ for $i \leq d-2$, then $A \oplus K_A$ is a test module for A.

(3) Let (S, n) be a Cohen-Macaulay Roberts ring. Assume that A is a subring of S satisfying the following two conditions; (1) A is a homomorphic image of a regular local ring, (2) the inclusion $A \to S$ is finite. Then S is a test module for A.

Example 8 (1) Let k be an algebraically closed field of characteristic 0. Let $T = k[x_1, \ldots, x_n]$ be the graded polynomial ring over k with $\deg(x_1) = \cdots = \deg(x_n) = 1$ and I a homogeneous prime ideal of T with $I \subseteq (x_1, \ldots, x_n)^2$. Put $A = (T/I)_{(x_1,\ldots,x_n)}$.

Suppose that A is a Cohen-Macaulay ring of minimal multiplicity, i.e., $e(m, A) = \dim_{A/m} m/m^2 - \dim A + 1$. Then, A has a test module. (Use Bertini's classification, e.g., p166 in Yoshino [17].)

(2) Let k be an algebraically closed field of characteristic 0. Let $T = k[x_1, \ldots, x_n]$ be the graded polynomial ring over k with $\deg(x_1) = \cdots = \deg(x_n) = 1$ and I a homogeneous ideal of T. Put $A = \widehat{(T/I)_{(x_1,\ldots,x_n)}}$, that is the completion of $(T/I)_{(x_1,\ldots,x_n)}$.

Assume that A is a Cohen-Macaulay ring with only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay A-modules.

Then A has a test module. (Use Eisenbud-Herzog's classification [4]).

(3) Let k be a field and m, n positive integers at least 2. Put $T = k[x_{ij} \mid 1 \leq i \leq m; 1 \leq j \leq n]$. Put $A = (T/I_2(x_{ij}))_{(\underline{x})}$.

If $m = 2$, then A has a test module.

If $m + n \leq 16$, then A has a test module.

If $m = n = 8$, then $\dim A = 15$ and we have

$$\tau_A([A]) = q_{15} + q_{13} + q_{11} + q_9,$$

where $0 \neq q_i \in \mathrm{A}_i(A)_{\mathbb{Q}}$ for $i = 9, 11, 13, 15$.

(4) Let k be a field and A the completion of the $(2,1)$-scroll, i.e.,

$$A = k[[s_0, s_1, s_2, t_0, t_1]] \Big/ I_2 \begin{pmatrix} s_0 & s_1 & t_0 \\ s_1 & s_2 & t_1 \end{pmatrix}.$$

Then, A is a 3-dimensional Cohen-Macaulay normal domain. We shall describe all test modules for A. We refer the reader to Yoshino [17] for the theory of maximal Cohen-Macaulay modules.

It is known that A has just 5 kinds of (isomorphism classes of) indecomposable maximal Cohen-Macaulay modules as follows:

$$A, \ K_A, \ S_{-2}, \ S_1, \ M$$

Put $\alpha = [A]$, $\beta = [K_A]$, $\gamma = [S_{-2}]$, $\delta = [S_1]$, $\epsilon = [M]$ in $\mathrm{K}_0(A)$. Then, it is known that $\mathrm{K}_0(A) = \mathbb{Z}^2 = \mathbb{Z}\alpha + \mathbb{Z}\beta$, and $\gamma = 2\beta - \alpha$, $\delta = 2\alpha - \beta$, $\epsilon = \alpha + \beta$.

Here, we have an isomorphism $\tau_A : \mathrm{K}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$. Put $\tau_A(\alpha) = q_3 + q_2 + q_1$, where $q_i \in \mathrm{A}_i(A)_{\mathbb{Q}}$. (Note that $\mathrm{A}_0(A)_{\mathbb{Q}} = 0$.) Then, $\tau_A(\beta) = q_3 - q_2 + q_1$ is satisfied. Then, it is easy to see that $\mathrm{A}_3(A)_{\mathbb{Q}} = \mathbb{Q}q_3 \simeq \mathbb{Q}$, $\mathrm{A}_2(A)_{\mathbb{Q}} = \mathbb{Q}q_2 \simeq \mathbb{Q}$, $\mathrm{A}_1(A)_{\mathbb{Q}} = \mathrm{A}_0(A)_{\mathbb{Q}} = 0$, $(q_1 = 0)$.

Therefore, we have $\tau_A(\alpha) = q_3 + q_2$, $\tau_A(\beta) = q_3 - q_2$, $\tau_A(\gamma) = q_3 - 3q_2$, $\tau_A(\delta) = q_3 + 3q_2$, $\tau_A(\epsilon) = 2q_3$.

Hence,

$$A^{n_1} \oplus K_A^{n_2} \oplus S_{-2}^{n_3} \oplus S_1^{n_4} \oplus M^{n_5}$$

is a test module for A if and only if $n_1 - n_2 - 3n_3 + 3n_4 = 0$. ($n_1, \ldots, n_5$ are non-negative integers such that at least one of them is positive.)

Let $a_1, \ldots, a_5$ be non-negative rational numbers (at least one of them is positive) and put

$$b_3 q_3 + b_2 q_2 = a_1 \tau_A(\alpha) + a_2 \tau_A(\beta) + a_3 \tau_A(\gamma) + a_4 \tau_A(\delta) + a_5 \tau_A(\epsilon).$$

(Note that, if b_3, b_2 are rational numbers such that $0 \neq 3b_3 \geq |b_2|$, then we can find $a_1, \ldots, a_5$ satisfying the above equation.)

Let $\mathbb{F}.$ be a perfect A-complex with support in $\{m\}$ of length 3 and assume that it is not exact. For any maximal Cohen-Macaulay module N, we have $\chi_{\mathbb{F}.}([N]) > 0$. Therefore, we obtain

$$b_3 \chi_{\mathbb{F}.}(\tau_A^{-1}(q_3)) + b_2 \chi_{\mathbb{F}.}(\tau_A^{-1}(q_2)) = a_1 \chi_{\mathbb{F}.}(\alpha) + a_2 \chi_{\mathbb{F}.}(\beta) + a_3 \chi_{\mathbb{F}.}(\gamma) + a_4 \chi_{\mathbb{F}.}(\delta) + a_5 \chi_{\mathbb{F}.}(\epsilon) > 0.$$

That is, for any rational numbers b_3, b_2 satisfying $0 \neq 3b_3 \geq |b_2|$, we have $b_3 \chi_{\mathbb{F}.}(\tau_A^{-1}(q_3)) + b_2 \chi_{\mathbb{F}.}(\tau_A^{-1}(q_2)) > 0$. Therefore, we obtain

$$\chi_\infty(\mathbb{F}.) = \chi_{\mathbb{F}.}(\tau_A^{-1}(q_3)) > 3|\chi_{\mathbb{F}.}(\tau_A^{-1}(q_2))|.$$

If a complex $\mathbb{F}.$ is liftable to a regular local ring, then $\chi_{\mathbb{F}.}(\tau_A^{-1}(q_2)) = 0$ is satisfied.

By using an example dut to Dutta-Hochster-MacLaughlin [3], we can construct a complex $\mathbb{F}.$ such that $\chi_\infty(\mathbb{F}.) = 60$ and $\chi_{\mathbb{F}.}(\tau_A^{-1}(q_2)) = 1$.

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EQUIVARIANT TWISTED INVERSE PSEUDO-FUNCTORS WITHOUT EQUIVARIANT COMPACTIFICATION

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1 Introduction

Throughout these notes, a scheme means a separated noetherian scheme, for simplicity. We fix a base scheme B. Let G be a flat B-group scheme of finite type. We denote by $\mathcal{A}_G$ the category of G-schemes of finite type and G-morphisms.

The purpose of these notes is to discuss an equivariant version of Grothendieck's duality theorem. The original Grothendieck's duality theorem is as follows, see [5].

Theorem 1.1 *Let $f : X \rightarrow Y$ be a proper morphism between schemes. Then, 'the canonical' natural map*

$$\rho(f) : Rf_* R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathbb{F}, f^!\mathbb{G}) \rightarrow R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_Y}(Rf_*\mathbb{F}, \mathbb{G}) \qquad (1)$$

is an isomorphism for $\mathbb{F} \in D^-_{\mathrm{Coh}}(X)$ and $\mathbb{G} \in D^+_{\mathrm{Qco}}(Y)$, where $f^!$ denotes the right adjoint functor of

$$Rf_* : D^+_{Qco}(X) \rightarrow D^+_{Qco}(Y)$$

(called the twisted inverse functor).

Problem 1.2 *Let $f : X \rightarrow Y$ be a proper morphism in $\mathcal{A}_G$. Formulate and prove an "equivariant version" of Grothendieck's duality theorem.*

An equivariant version of Grothendieck's duality may not be unique in its formulation. In fact in [6], the associated action of a generalized hyperalgebra of G is used to establish an equivariant duality theorem. In order to formulate an equivariant version of the duality theorem, the category $\mathrm{Qco}(G, X)$ of G-linearlized quasi-coherent $\mathcal{O}_X$-modules in [GIT] is a good substitute of

$\mathrm{Qco}(X)$, the category of quasi-coherent sheaves over X, in the equivariant case. There is no difference with [6] concerning with this point.

Even for the non-equivariant case, it is convenient to embed $\mathrm{Qco}(X)$ into $\mathrm{Mod}(X)$, the category of $\mathcal{O}_X$-module sheaves. This is mainly because $\mathrm{Qco}(X)$ is not closed under $\underline{\mathrm{Hom}}$ operation. So we need to embed $\mathrm{Qco}(G, X)$ into some larger category $\mathrm{Mod}(G, X)$ which is closed under various operations necessary to formulate and prove a duality theorem.

However, the framework of [6] is far from satisfaction. We list the problems in [6, section 2]. First, unless $B = \operatorname{Spec} k$, we do not know how to find a generalized hyperalgebra of G in general. So the duality is not applicable to a quite general G which is flat of finite type. Secondly, the category $\mathrm{Qco}(U, X)$ is not closed under $\underline{\mathrm{Hom}}$. The reason why (6) in page 191 in [6] is so restrictive (i.e., x is required to be in $D^-(\mathrm{Coh}(U, X))$) comes from this problem. We are not able to discuss $R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(D_X, D_X)$ in this framework (where D_X is the equivariant dualizing complex) appropriately. Moreover, $\mathrm{Qco}(G, X)$ is not a thick full subcategory of $\mathrm{Qco}(U, X)$ in general, and we can not use $D_{\mathrm{Qco}(G,X)}(\mathrm{Qco}(U, X))$.

Although there is no new application which is not discussed in [6] here, these are the reason why we are going to discuss a new approach.

Problem 1.3 *Establish an equivariant version of the theory of twisted inverse for morphisms in $\mathcal{A}_G$, and define "the" equivariant dualizing complex for objects in $\mathcal{A}$ when the base scheme B is Gorenstein of finite Krull dimension.*

This problem is related to the problem of equivariant compactifications. Let $f : X \to Y$ be a morphism in $\mathcal{A}_G$. We say that a sequence of morphisms

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

is a G-equivariant compactification of f if it is a sequence of morphisms in $\mathcal{A}_G$, the morphism i is an open immersion, and p is proper. If there exists an equivariant compactification $f = pi$ of f, then we may well define $f^! = i^* p^!$, where $p^!$ is the right adjoint of Rp_* as before, and i^* is the restriction (i.e., the inverse image via the open immersion i). Twisted inverse was defined thus in the non-equivariant case by P. Deligne [2].

If G is a trivial group, then it is the usual compactification. M. Nagata [9] proved that there is a compactification of a (separated of finite type) morphism between noetherian schemes. His proof was valuation-theoretic, and a more scheme-theoretic proof is available in [7]. On the other hand, the problem of equivariant compactifications is open (however, see [13]). In these notes, we solve Problem 1.3 avoiding this open problem. We extend the category $\mathcal{A}_G$, and we look for a good substitute of a compactification in this enlarged category. The extended category is not a category of single schemes, but is a category of diagrams of several schemes.

The use of diagrams of schemes is a standard technique in descent theory and it has been known that it is also useful in studying equivariant sheaves, see for example [11, 3, 1]. In particular, there is nothing (essentially) new in sections 2 and 3. Probably what is essentially new here is only Theorem 4.5 and the results that follow. The details of these notes will appear elsewhere.

2 Diagrams of schemes

Let I be a small category, and $\mathcal{P} = \mathcal{P}(I, \mathrm{Sch}/B)$ be the functor category $\mathrm{Func}(I^{\mathrm{op}}, \mathrm{Sch}/B)$, where I^{op} is the opposite category of I, and Sch/B is the category of B-schemes. Let $X_\bullet \in \mathcal{P}$. For $i \in \mathrm{ob}(I)$, we denote the B-scheme $X_\bullet(i)$ by X_i. For $\phi \in \mathrm{Mor}(I)$, we denote the morphism of B-schemes $X_\bullet(\phi)$ by X_ϕ.

For $X_\bullet \in \mathcal{P}$, we define a small category $\mathrm{Zar}(X_\bullet)$ as follows. An object of $\mathrm{Zar}(X_\bullet)$ is a pair (i, U) such that i is an object of I, and U a Zariski open subset of X_i. A morphism from (j, V) to (i, U) is a pair (ϕ, h) such that $\phi : i \to j$ is a morphism of I such that $V \subset X_\phi^{-1}(U)$, and $h : V \to U$ is the restriction of X_ϕ.

We define a pretopology of $\mathrm{Zar}(X_\bullet)$ so that $\mathrm{Zar}(X_\bullet)$ is a site. A family of morphisms

$$\{(\phi_\lambda, h_\lambda) : (j_\lambda, V_\lambda) \to (i, U)\}_\lambda$$

is said to be a covering of (i, U) if $j_\lambda = i$ and $\phi_\lambda = 1_i$ for any λ and $\bigcup_\lambda V_\lambda = U$.

Letting $\Gamma((i,U), \mathcal{O}_{X_\bullet}) := \Gamma(U, \mathcal{O}_{X_i})$, the sheaf of commutative rings $\mathcal{O}_{X_\bullet}$ is defined, and $\mathrm{Zar}(X_\bullet)$ is a ringed site. The restriction map is given by

$$\Gamma((i,U), \mathcal{O}_{X_\bullet}) = \Gamma(U, \mathcal{O}_{X_i}) \xrightarrow{u} \Gamma(X_\phi^{-1}(U), \mathcal{O}_{X_j}) \xrightarrow{\mathrm{res}} \Gamma(V, \mathcal{O}_{X_j}) = \Gamma((j,V), \mathcal{O}_{X_\bullet})$$

for any $(\phi, h) : (j, V) \to (i, U)$, where u is the map induced by the unit of adjunction $\mathcal{O}_{X_i} \to (X_\phi)_* X_\phi^* \mathcal{O}_{X_i} = (X_\phi)_* \mathcal{O}_{X_j}$.

We denote the category of $\mathcal{O}_{X_\bullet}$-modules over $\mathrm{Zar}(X_\bullet)$ by $\mathrm{Mod}(X_\bullet)$. The category $\mathrm{Mod}(X_\bullet)$ is a Grothendieck category with projective limits. Various operations on module sheaves over a ringed sites are discussed in [4]. In particular, for two objects $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X_\bullet)$, the tensor product $\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{N}$ and the hom-sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{X_\bullet}}(\mathcal{M}, \mathcal{N})$ are defined.

Let $X_\bullet \in \mathcal{P}$ and J a subcategory of I. Then, we have a canonical restriction of $X_\bullet$ to J^{op}, which we denote by X_J. There is an obvious inclusion functor $Q_J : \mathrm{Zar}(X_J) \hookrightarrow \mathrm{Zar}(X_\bullet)$, which is continuous and cocontinuous (see [14]) almost by the definition of the topology of $\mathrm{Zar}(X_\bullet)$. Moreover, we have that the restriction of $\mathcal{O}_{X_\bullet}$ to $\mathrm{Zar}(X_J)$ is nothing but $\mathcal{O}_{X_J}$. The pull-back by Q_J is denoted by $(?)_J : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(X_J)$.

For $i \in \mathrm{ob}(I)$, we denote the subcategory of I with the object set $\{i\}$ with a single morphism $\{1_i\}$ by $\langle i\rangle$. The category $\mathrm{Zar}(X_{\langle i\rangle})$ and $\mathrm{Zar}(X_i)$ are canonically identified. So we have a canonical restriction functor

$$(?)_i = (?)_{\langle i\rangle} : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(X_i).$$

Thus, for $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$, we have a collection of sheaves $(\mathcal{M}_i)$. However, it is unreasonable to expect that we can retrieve $\mathcal{M}$ from the collection $(\mathcal{M}_i)$. We need some more information which give connections among these sheaves.

Let $\phi : i \to j$ be a morphism in I. Then, there is a natural map $\beta_\phi : \mathcal{M}_i \to (X_\phi)_*\mathcal{M}_j$ given by

$$\Gamma(U,\mathcal{M}_i) = \Gamma((i,U),\mathcal{M}) \xrightarrow{\mathrm{res}} \Gamma((j,X_\phi^{-1}),\mathcal{M}) = \Gamma(U,(X_\phi)_*\mathcal{M}_j).$$

It is easy to check the following.

Lemma 2.1 *For $i \in \mathrm{ob}(I)$, the canonical map $\mathcal{M}_i \cong (\mathrm{id}_{X_i})_*\mathcal{M}_i = (X_{1_i})_*\mathcal{M}_i$ agrees with β_{1_i}. For two composable morphisms $\phi : i \to j$ and $\psi : j \to k$, the composite map*

$$\mathcal{M}_i \xrightarrow{\beta_\phi} (X_\phi)_*\mathcal{M}_j \xrightarrow{(X_\phi)_*\beta_\psi} (X_\phi)_*(X_\psi)_*\mathcal{M}_k \cong (X_{\psi\phi})_*\mathcal{M}_k$$

agrees with $\beta_{\psi\phi}$.

Conversely, if $((\mathcal{M}_i)_{i\in\mathrm{ob}(I)},(\beta_\phi)_{\phi\in\mathrm{Mor}(I)})$ is a collection such that $\mathcal{M}_i \in \mathrm{Mod}(X_i)$ for $i \in \mathrm{ob}(I)$, $\beta_\phi \in \mathrm{Mod}(X_i)(\mathcal{M}_i,(X_\phi)_*\mathcal{M}_j)$ for $\phi \in I(i,j)$, and the conditions in the lemma above are satisfied, then the sheaf $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is retrieved from these data.

For a morphism $\phi : i \to j$ of I, we define α_ϕ to be the composite map

$$(X_\phi)^*\mathcal{M}_i \xrightarrow{(X_\phi)^*\beta_\phi} (X_\phi)^*(X_\phi)_*\mathcal{M}_j \xrightarrow{\varepsilon} \mathcal{M}_j,$$

where ε is the counit of adjunction.

Definition 2.2 (see [1]) We say that $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is *equivariant* if α_ϕ is an isomorphism for each $\phi \in \mathrm{Mor}(I)$. We say that $\mathcal{M}$ is *locally quasi-coherent* (resp. locally coherent) if $\mathcal{M}_i$ is quasi-coherent (resp. coherent) for each $i \in \mathrm{ob}(I)$. We say that $\mathcal{M}$ is *quasi-coherent* (resp. coherent) if $\mathcal{M}$ is both equivariant and locally quasi-coherent (resp. locally coherent).

We denote the full subcategory of $\mathrm{Mod}(X_\bullet)$ consisting of equivariant (resp. locally quasi-coherent, quasi-coherent, and coherent) sheaves by $\mathrm{Eq}(X_\bullet)$ (resp. $\mathrm{LQco}(X_\bullet)$, $\mathrm{Qco}(X_\bullet)$, and $\mathrm{Coh}(X_\bullet)$).

Let $\mathbb{P}$ be a property of schemes. We say that $X_\bullet$ satisfies $\mathbb{P}$ if X_i satisfy $\mathbb{P}$ for all $i \in \mathrm{ob}(I)$. Let $\mathbb{Q}$ be a property of morphisms. We say that $X_\bullet$ satisfies $\mathbb{Q}$ if all the structure morphisms $X_i \to B$ satisfy $\mathbb{Q}$. We say that $X_\bullet$ has $\mathbb{Q}$-arrows if X_ϕ satisfy $\mathbb{Q}$ for all $\phi \in \mathrm{Mor}(I)$. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism. We say that $f_\bullet$ satisfies $\mathbb{Q}$ if f_i satisfies $\mathbb{Q}$ for each $i \in \mathrm{ob}(I)$.

The following is checked easily, using the five lemma.

Lemma 2.3 *Let $X_\bullet \in \mathcal{P}$. Then,* $\mathrm{LQco}(X_\bullet)$ *is a thick subcategory of* $\mathrm{Mod}(X_\bullet)$*. If moreover, $X_\bullet$ has flat arrows, then* $\mathrm{Eq}(X_\bullet)$ *and* $\mathrm{Qco}(X_\bullet)$ *are also thick subcategories of* $\mathrm{Mod}(X_\bullet)$*.*

We say that a morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a morphism *of fiber type*, if

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & Y_j \\ \downarrow X_\phi & & \downarrow Y_\phi \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

is a fiber square for each morphism $\phi : i \to j$ of I.

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$. Then, a continuous functor $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ is given by $f_\bullet^{-1}((i, U)) = (i, f_i^{-1}(U))$. It actually gives a morphism of ringed sites

$$f_\bullet : (\mathrm{Zar}(X_\bullet), \mathcal{O}_{X_\bullet}) \to (\mathrm{Zar}(Y_\bullet), \mathcal{O}_{Y_\bullet})$$

in an obvious way. Thus, the direct image functor $(f_\bullet)_* : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(Y_\bullet)$ and the inverse image functor $f_\bullet^* : \mathrm{Mod}(Y_\bullet) \to \mathrm{Mod}(X_\bullet)$ are defined in a natural way.

3 The simplicial object associated to a group action

Let Δ denotes the category of standard simplices. The object set $\mathrm{ob}(\Delta)$ is $\{[0], [1], [2], \ldots\}$, where $[n]$ is the ordered set $\{0 < 1 < \cdots < n\}$. Morphisms of Δ are monotone maps. An object of $\mathcal{P}(\Delta, \mathrm{Sch}/B)$ is called a simplicial B-scheme.

Let $X \in \mathrm{ob}(\mathcal{A}_G)$. We define $B(G, X)$ as follows. We define $B(G, X)_{[n]}$ to be $G^n \times_B X$ for $n \geq 0$. We define $B(G, X)_{\delta_i^n} : B(G, X)_{[n+1]} \to B(G, X)_{[n]}$ by

$$B(G,X)_{\delta_i^n}(g_{n+1}, \ldots, g_1, x) := \begin{cases} (g_{n+1}, \ldots, g_2, g_1 x) & (i = 0) \\ (g_{n+1}, \ldots, g_{i+1} g_i, \ldots, g_1, x) & (0 < i < n) \\ (g_n, \ldots, g_1, x) & (i = n+1), \end{cases}$$

where $\delta_i^n : [n] \to [n+1]$ is the unique injective monotone map such that $i \notin \mathrm{Im}\, \delta_i$. We define $B(G, X)_{\sigma_i^n} : B(G, X)_{[n]} \to B(G, X)_{[n+1]}$ by

$$B(G, X)_{\sigma_i^n}(g_n, \ldots, g_1, x) := (g_n, \ldots, g_{i+1}, e, g_i, \ldots, g_1, x)$$

for $i = 0, 1, \ldots, n$, where $\sigma_i^n : [n+1] \to [n]$ is the unique surjective monotone map such that $\#(\sigma_i^n)^{-1}(i) = 2$, and e denotes the unit element of G. This gives the definition of $B(G, X)$, see [8, (VII.5)]. Note that $B(G, ?)$ is a functor from $\mathcal{A}_G$ to $\mathcal{P}(\Delta, \mathrm{Sch}/B)$.

We define the subcategory Δ_M of Δ by

$$\mathrm{ob}(\Delta_M) := \{[0], [1], [2]\}, \quad \mathrm{Mor}(\Delta_M) := \{\text{injective monotone maps}\}.$$

We denote the restriction $B(G, X)_{\Delta_M}$ of $B(G, X)$ by $B_M(G, X)$.

Lemma 3.1 *The following hold:*

1 *There is an equivalence of categories* $\mathrm{Qco}(G,X) \cong \mathrm{Qco}(B(G,X))$.

2 *The restriction induces an equivalence* $\mathrm{Qco}(B(G,X)) \cong \mathrm{Qco}(B_M(G,X))$.

3 *Let* $\mathbb{Q}$ *be a property of morphisms of noetherian schemes such that*

- **i** *Any isomorphism satisfies* $\mathbb{Q}$.
- **ii** *A composition of two morphisms which satisfy* $\mathbb{Q}$ *again satisfies* $\mathbb{Q}$.
- **iii** $\mathbb{Q}$ *is stable under base change.*
- **iv** *The structure map* $G \to B$ *satisfies* $\mathbb{Q}$.

Then, $B_M(G,X)$ *has* $\mathbb{Q}$*-arrows. In particular,* $B_M(G,X)$ *has flat, of finite type arrows.*

4 *Let* $f : X \to Y$ *be a morphism in* $\mathcal{A}_G$. *Then,* $B(G,f)$ *is a morphism of fiber type. In particular,* $B_M(G,f) : B_M(G,X) \to B_M(G,Y)$ *is a morphism of fiber type. If* $\mathbb{Q}$ *is a property of morphisms stable under base change and* f *satisfies* $\mathbb{Q}$, *then* $B_M(G,f)$ *satisfies* $\mathbb{Q}$.

4 Equivariant twisted inverse

A diagram of schemes is required to be noetherian and separated. Thanks to the development of the theory of unbounded derived category and derived functors [12, 10], the existence of the right adjoint of the derived functor $R(f_\bullet)_*$ is fairly easy.

Theorem 4.1 (Neeman) *Let* $F : \mathcal{S} \to \mathcal{T}$ *be a triangulated functor between triangulated categories. If* $\mathcal{S}$ *is compactly generated and* F *respects coproducts, then* F *has a right adjoint.*

Utilizing the theorem, the following is proved without difficulty.

Theorem 4.2 *Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a morphism in* $\mathcal{P}$. *Then,*

$$R(f_\bullet)_* : D_{\mathrm{LQco}}(X_\bullet) \to D_{\mathrm{LQco}}(Y_\bullet)$$

has a right adjoint $f_\bullet^!$.

This gives a definition of $f^!$ for a *proper* morphism $f : X \to Y$ in $\mathcal{A}_G$. We define the equivariant $f^!$ to be $B_M(G,f)^!$. We also denote $R(B_M(G,f)_*)$ simply by f_*. As it is easy to see that f_* is way-out both (after all, we only consider finitely many quasi-compact separated schemes), we have that $f^!$ is way-out right. In particular, we have

Lemma 4.3 $f^!(D^+_{\mathrm{LQco}}(Y_\bullet)) \subset D^+_{\mathrm{LQco}}(X_\bullet)$.

Although it is a difficult problem to find an equivariant compactification of f, it is easy to find a 'fake compactification' of $B_M(G, f)$.

Proposition 4.4 *Let $f : X \to Y$ be a morphism in $\mathcal{A}_G$. Then, there is a factorization of $B_M(G, f)$*

$$B_M(G, X) \xrightarrow{i_\bullet} \bar{X}_\bullet \xrightarrow{p_\bullet} B_M(G, Y)$$

such that $p_\bullet$ is proper and $i_\bullet$ is an image-dense open immersion.

This is an immediate consequence of Nagata's (non-equivariant) compactification theorem. Utilizing a fake compactification, we can define $f^!$ for a non-proper morphism f in $\mathcal{A}_G$.

Theorem 4.5 *Let $B_M(G, f) = p_\bullet i_\bullet$ be a fake compactification, and $n \in \{0, 1, 2\}$. Then, there is a 'canonical' isomorphism*

$$(?)_n i_\bullet^* p_\bullet^! \cong i_n^* p_n^! (?)_n$$

between functors from $D^+_{\mathrm{LQco}}(B_M(G, Y))$ to $D^+_{\mathrm{LQco}}(B_M(G, X))$.

As a corollary, the following follow. Let the notation be as above.

- $i_\bullet^* p_\bullet^!$ maps $D^+_{\mathrm{Qco}}(B_M(G, Y))$ to $D^+_{\mathrm{Qco}}(B_M(G, X))$.
- "Flat base change" of $i_\bullet^* p_\bullet^!$
- The composite functor $f^! := i_\bullet^* p_\bullet^!$ is independent of the choice of fake compactification.
- For any composable $f, g \in \mathrm{Mor}(\mathcal{A}_G)$, there is a canonical isomorphism $(gf)^! \cong f^! g^!$ such that $(hgf)^! \cong (gf)^! h^! \cong f^! g^! h^!$ agrees with $(hgf)^! \cong f^!(hg)^! \cong f^! g^! h^!$.

Thus, a pseudofunctor $(?)^!$ over $\mathcal{A}_G$ is defined, which is compatible with the forgetful functor (which forgets the G-equivariant structures)

$$(?)_{[0]} : D^+_{\mathrm{LQco}}(B_M(G, X)) \to D^+_{\mathrm{Qco}}(X).$$

Let $f : X \to Y$ be a proper morphism in $\mathcal{A}_G$. Then, the canonical isomorphism (1) is an isomorphism in $D^+_{\mathrm{Qco}}(G, Y)$ for $D^-_{\mathrm{Coh}}(G, X)$ and $D^+_{\mathrm{Qco}}(G, Y)$.

What we miss here is, an explicit description of $f^!$ for the case f is finite or smooth. We only remark that there are obvious analogies of the explicit constructions of $f^!$ found in [5] for those two cases.

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A NOTE ON THE CONORMAL MODULE OF AN IDEAL

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ABSTRACT. This note is to study the conormal module M of an ideal $\bar{\mathfrak{g}}$ in an analytic algebra $\mathcal{O}/\mathfrak{h}$, where $\mathcal{O}$ is the convergenet power series over the complex numbers. The torsion part $T(M)$ of M and the torsion free module $M/T(M)$ are expressed by the relative primitive ideal. Two characterizations for $M/T(M)$ to be free are proved. Some immediate applications are worked out.

1. INTRODUCTION

Let $\mathcal{O}$ be the stalk over 0 of the structure sheaf of $\mathbb{C}^m$. Let $\Sigma \subset X$ be reduced analytic space germs at 0 locally embedded in $(\mathbb{C}^m, 0)$ defined by two radical ideals $\mathfrak{g} \subset \mathfrak{h}$ of $\mathcal{O}$ respectively. Let $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{h}$ be the image of $\mathfrak{g}$ in $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$ under the canonical projection. The main interest of this article is to study the conormal module $M := \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^2$ by the relative primitive ideal $\int_X \mathfrak{g}$ of $\mathfrak{g}$. The primitive ideal of $\mathfrak{g}$ (relative to X) was introduced by Siersma-Pellikaan [10, 11] and generalized to relative version in [5, 6].

In general, $\mathcal{O}_X$ is not regular, so the $\mathcal{O}_\Sigma := \mathcal{O}/\mathfrak{g}$ module M is neither free nor torsion free even if both X and Σ are complete intersections. Especially, Σ is not a complete intersection in X. Then the following questions would be interesting.

a) Find descriptions of the torsion part $T(M)$ of M, calculate the length (when it is finite) of $T(M)$;
b) Find descriptions of the torsion free module $N := M/T(M)$ and conditions on the freeness of N.

The motivation to these questions is the studying of functions with non-isolated singularities on singular spaces. The primitive ideal of $\mathfrak{g}$ collects all the functions whose zero level hypersurfaces pass through Σ and are tangent to the regular part X_{reg} of X along $\Sigma \cap X_{\text{reg}}$. If we supply X with the so called *logarithmic stratification* [12], then the primitive ideal of $\mathfrak{g}$ consists of exactly all the functions from $\mathfrak{g}$ whose stratified critical loci on X contain Σ (cf. [5]). Hence, locally the primitive ideal plays a similar role to the second power of the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^m,0}$ in singularity theory. In order to study the topology of the Milnor fibre F_f of a function f with singular locus Σ, we use a good deformation (the Morsification) f_s of f. This f_s has relatively simpler singularities than f. The existence of the good deformation and related invariants (both topological and algebraic) have close relationship with M, $T(M)$ and N. Roughly speaking, the freeness of N implies the existence of the good deformation [5]. The length of torsion module $T(M)$ (when it is finite) gives some information on how Σ sits in X (cf. [7]).

Under some conditions, we answer the questions a) and b). More precisely, after some descriptions of $T(M)$ and N, we mainly prove the following (see also Remark 9)

Main Theorem *Let $\Sigma \subset X$ be reduced complete intersection germs of pure dimension, defined by two ideals $\mathfrak{g} \subset \mathfrak{h}$ of $\mathcal{O}$ respectively. Assume that X_{sing} does not contain any*

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irreducible components of Σ. The $\mathcal{O}_\Sigma$-module N is free if and only if there exists an $\mathcal{O}$-regular sequence $g_1, \dots, g_n$ generating $\mathfrak{g}$, such that

$$\int_X \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2,$$

where $p := \operatorname{codim} X$, $n := \operatorname{codim} \Sigma$.

As an application, in the last section we study lines on a variety with isolated complete intersection singularity. More applications to the general deformation theory of non-isolated singularities on singular spaces will be given in the sequel papers.

2. Primitive ideals

Let $\mathcal{O}_{\mathbb{C}^m}$ be the structure sheaf of $\mathbb{C}^m$. The stalk over 0 is denoted by $\mathcal{O}_m$ or $\mathcal{O}$. Let $\operatorname{Der}(\mathcal{O})$ denote the $\mathcal{O}$-module of all the $\mathbb{C}$-derivations of $\mathcal{O}$. Let $(X, 0)$ be a reduced space germ defined by a radical ideal $\mathfrak{h} \subset \mathcal{O}$. Define

$$\operatorname{Der}_X(\mathcal{O}) := \{\xi \in \operatorname{Der}(\mathcal{O}) \mid \xi(\mathfrak{h}) \subset \mathfrak{h}\}.$$

Let $(\Sigma, 0) \subset (X, 0)$ be a subspace defined by an ideal $\mathfrak{g} \subset \mathcal{O}$.

Definition 1. The *primitive ideal of* $\mathfrak{g}$ relative to X is

$$\int_X \mathfrak{g} := \{f \in \mathfrak{g} \mid \xi(f) \in \mathfrak{g} \text{ for any } \xi \in \operatorname{Der}_X(\mathcal{O})\}.$$

Remarks 2. (1) When X is smooth this definition was given by Pellikaan [10, 11]. It is straightaway to verify that $\int_X \mathfrak{g}$ is an ideal of $\mathcal{O}$, and $\mathfrak{g}^2 + \mathfrak{h} \subset \int_X \mathfrak{g} \subset \mathfrak{g}$ always holds. And for $\mathfrak{g}_i \supset \mathfrak{h}$ $(i = 1, 2)$, we have $\int_X \mathfrak{g}_1 \cap \int_X \mathfrak{g}_2 = \int_X (\mathfrak{g}_1 \cap \mathfrak{g}_2)$;

(2) Geometrically, the relative primitive ideal collects all the functions whose zero level surfaces pass through Σ and are tangent to the regular part X_{reg} of X along $\Sigma \cap X_{\text{reg}}$.

(3) The relative primitive ideals have been generalized to higher relative primitive ideals in [6]. Under the assumption that $\mathfrak{h}$ is pure dimension, $\mathfrak{g}$ is radical, and the Jacobian ideal of $\mathfrak{h}$ is not contained in any associated prime of $\mathfrak{g}$, it was proved that the primitive ideal $\int_X \mathfrak{g}$ is the inverse image in $\mathcal{O}$ of the second symbolic power of the quotient ideal $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{h}$ of $\mathcal{O}_X$. Remark that the results in [6] generalized the results of [13, 10, 11].

3. Conormal module: the torsion part

Let $\mathfrak{h} \subset \mathfrak{g}$ be radical ideals of $\mathcal{O}$, $X = \mathcal{V}(\mathfrak{h})$ and $\Sigma = \mathcal{V}(\mathfrak{g})$. Denote by X_{sing} the singular locus of X and $X_{\text{reg}} = X \setminus X_{\text{sing}}$. Denote by $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{h}$, the quotient ideal in $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$. The $\mathcal{O}_\Sigma := \mathcal{O}/\mathfrak{g}$-module $M := \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^2 \simeq \mathfrak{g}/\mathfrak{g}^2 + \mathfrak{h}$ is called the conormal module of $\bar{\mathfrak{g}}$.

Proposition 3. *Let $\Sigma \subset X$ be defined by radical ideals $\mathfrak{g} \supset \mathfrak{h}$ of $\mathcal{O}$. If X_{sing} does not contain any irreducible components of Σ, then we have*

$$T(M) = T := \frac{\int_X \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}}.$$

Consequently, we have the following exact sequence

$$0 \longrightarrow T(M) \xrightarrow{\imath} M \xrightarrow{\pi} N \longrightarrow 0 \tag{3.1}$$

where $N := \mathfrak{g}/\int_X \mathfrak{g}$.

Proof. Let U be an open neighborhood of 0 in $\mathbb{C}^m$ in which X and Σ are defined. Let $V = \Sigma \setminus (\Sigma_{\text{sing}} \cup X_{\text{sing}})$. Then V is an open dense subset of Σ by the assumption. Then $T = 0$ on V since V is a reduced local complete intersection in X_{reg}. Hence $T \subset T(M)$.

For any $\bar{a} \in T(M)$, let $\bar{\beta} \in \mathcal{O}_\Sigma$ be a non-zero divisor such that $\bar{\beta}\bar{a} = 0$. By taking representatives we have $\beta a \in \mathfrak{g}^2 + \mathfrak{h}$. So for any $\xi \in D_X$, we have $\xi(\beta a) \equiv \beta\xi(a) \mod \mathfrak{g}$. But $\xi(\beta a) \in \mathfrak{g}$, hence $\xi(a) \in \mathfrak{g}$. This tells us that $a \in \int \mathfrak{g}$. □

Proposition 4. *Let Σ be a reduced complete intersection in $\mathbb{C}^m$. Assume that X_{sing} does not contain any irreducible components of Σ. If one can choose an $\mathcal{O}$-regular sequence $g_1, \dots, g_n$ which is the minimal generating set of $\mathfrak{g}$, such that*

a) *there exists an integer $0 \le t \le n$ such that the images of $g_1, \dots, g_t$ are in $T(M)$;*
b) *the germs $(\bar{g}_{t+1})_z, \dots, (\bar{g}_n)_z$ at $z \in X_{reg}$ generate $\bar{\mathfrak{g}}_z$ and form an $\mathcal{O}_{X,z}$-regular sequence, where $\mathcal{O}_{X,z}$ is the localization of $\mathcal{O}_X$ at z,*

then

1) *$\int \mathfrak{g} = (g_1, \dots, g_t) + (g_{t+1}, \dots, g_n)^2$;*
2) *N is free $\mathcal{O}_\Sigma$-module of rank $n - t$, (3.1) splits and $M = N \oplus T(M)$.*

Proof. 1) If $f = a_1 g_1 + \dots + a_n g_n \in \mathfrak{g}$ and $\xi \in D_X$ then

$$\xi(f) \equiv a_{t+1}\xi(g_{t+1}) + \dots + a_n\xi(g_n) \quad (\bmod \mathfrak{g}).$$

$f \in \int_X \mathfrak{g}$ if and only if $\xi(f) \equiv 0(\bmod \mathfrak{g})$ for any $\xi \in D_X$. By the assumption on $g_{t+1}, \dots, g_n$, we know that $(\xi(g_{t+1}), \dots, \xi(g_n)) = d(g_{t+1}, \dots, g_n)$ is injective at every point on $\Sigma \setminus (X_{\text{sing}} \cup \Sigma_{\text{sing}})$, hence $a_{t+1}, \dots a_n \in \mathfrak{g}$.

2) Suppose that there exist $\bar{\beta}_{t+1}, \dots, \bar{\beta}_n \in \mathcal{O}_\Sigma$ such that $\bar{a} = \bar{\beta}_{t+1}\bar{g}_{t+1} + \dots + \bar{g}_n\bar{\beta}_n = 0$ in N. By taking the representatives we have

$$a = \beta_{t+1}g_{t+1} + \dots + \beta_n g_n \in (g_1, \dots, g_t) \cap (g_{t+1}, \dots, g_n) + (g_{t+1}, \dots, g_n)^2.$$

Let $a = \beta_1 g_1 + \dots + \beta_t g_t + G$, where $G \in (g_{t+1}, \dots, g_n)^2$, then

$$-\beta_1 g_1 - \dots - \beta_t g_t + \beta_{t+1}g_{t+1} + \dots + \beta_n g_n \in (g_{t+1}, \dots, g_n)^2.$$

Since $g_1, \dots, g_n$ are $\mathcal{O}$-regular, $\beta_j \in \mathfrak{g}$. Hence $\bar{g}_{t+1}, \dots, \bar{g}_n$ form a free basis of N. □

Let $\mathfrak{g}$ be generated by an $\mathcal{O}$-regular sequence: $g_1, \dots, g_n$, and let there exist an integer $0 \le t \le n$ and non-zero divisors $\bar{\lambda}_1, \dots, \bar{\lambda}_t \in \mathcal{O}_\Sigma$ such that $\bar{\lambda}_1\bar{g}_1, \dots, \bar{\lambda}_t\bar{g}_t$ are zero in M as $\mathcal{O}_\Sigma$-module. Namely

$$\lambda_1 g_1, \dots, \lambda_t g_t \in \mathfrak{g}^2 + \mathfrak{h}$$

where $\lambda_i g_i$ is the representative of $\bar{\lambda}_i\bar{g}_i$.

Let $h_1, \dots, h_p$ form a minimum generating set of $\mathfrak{h}$. Denote $h = (h_1, \cdots, h_p)^T$, $g = (g_1, \cdots, g_n)^T$, $G = (G_1, \cdots, G_t)^T$, and $\Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_t\}$, where T means the transposition of the matrix indicated. Let A and $B = (B_1 \vdots B_2)$ be the matrices such that $\Lambda (g_1, \cdots, g_t)^T = Ah + G$, $h = Bg$ where A is a $t \times p$ matrix, B a $p \times n$ matrix, and B_1 is a $p \times t$ matrix, B_2 is a $p \times (n-t)$ matrix, $G_i \in \mathfrak{g}^2$.

Let $C_1 = AB_1, C_2 = AB_2$, by (2.1.5.1), we have

$$(\Lambda - C_1)(g_1, \cdots, g_t)^T - C_2 (g_{t+1}, \cdots, g_n)^T \equiv 0 \mod \mathfrak{g}^2)$$

Note that $\mathfrak{g}/\mathfrak{g}^2$ is a free $\mathcal{O}_\Sigma$-module. we have $\bar{\Lambda} = \bar{C}_1$, $\bar{C}_2 = 0$ in $\mathcal{O}_\Sigma$. ¿From this we obtain the following lemma similar to the implicit function theorem.

Lemma 5. *Let Σ a complete intersection in $\mathbb{C}^m$ defined by $\mathfrak{g}$ as above. There is an open and dense subset Σ^0 of $\Sigma \setminus X_{sing}$ such that for each $z \in \Sigma^0$,*

$$\det C_1 = \det \Lambda(z) = \lambda_1(z) \cdots \lambda_t(z) \neq 0,$$

and consequently $\text{rank}(B_1(P)) \ge t$ and $t \le p$. Hence from $h_1 = \dots = h_p = 0$, one can express $g_1, \dots, g_t$ as functions of $g_{t+1}, \dots, g_n$ in a neighborhood of every point $P \in \Sigma^0$. □

4. Freeness of N and the primitive ideal

Let $\Sigma \subset X$ be reduced complete intersections in $\mathbb{C}^m$ of pure dimensions. Assume that X_{sing} does not contain any irreducible components of Σ. If $\mathfrak{g}$ is generated by an $\mathcal{O}$-regular sequence $g_1, \dots, g_n$, we can choose the generating set $\{h_1, \dots, h_p\}$ of $\mathfrak{h}$ such that (with changing of the generators of $\mathfrak{g}$ if necessary):

$$h_i \equiv \sum_{j=1}^{t} b_{ij} g_j \mod \mathfrak{g}^2 \quad 1 \le i \le p \tag{4.1}$$

where t is an integer $0 \le t \le n$, $b_{ij} \notin \mathfrak{g} \setminus 0$, and for each j, $(b_{1j}, \dots, b_{pj}) \neq 0$ in $(\mathcal{O}_\Sigma)^p$, (otherwise one could lower t). Denote $B := (b_{ij})$.

Lemma 6. *Under the assumptions above, we have*

(1) *$t \ge p$;*
(2) *There exists at least one maximal minor of B which is non-zero divisor in $\mathcal{O}_\Sigma$.*

Proof. Since X is a complete intersection of pure dimension, X_{sing} can be defined by the Jacobian $\mathcal{J}(\mathfrak{h})$ of $\mathfrak{h}$, which can be generated by $\mathfrak{h}$ and the $p \times p$ minors of the Jacobian matrix $J(\mathfrak{h})$ of $\mathfrak{h}$. Since each of these minors, say $\Delta_{j_1,\dots,j_p}$, is the determinant of $BG_{j_1,\dots,j_p}$ modulo $\mathfrak{g}$, where $G_{j_1,\dots,j_p}$ is the $t \times p$ submatrix of $J(\mathfrak{g})$, consisting of the $0 \le j_1 < \cdots j_p \le m$ columns of $J(\mathfrak{g})$, the Jacobian matrix of $\mathfrak{g}$. Suppose $t < p$, there would be, $\det(BG_{j_1,\dots,j_p}) \equiv 0 \mod \mathfrak{g}$. This is impossible since we assume that X_{sing} does not contain any irreducible components of Σ. This proves (1).

(2) Suppose that all the $p \times p$ minors of B are zero divisor in $\mathcal{O}_\Sigma$. Then there exists $0 \neq a \in \mathcal{O}_\Sigma$ such that $ab_1 \wedge b_2 \wedge \cdots \wedge b_p = 0$ in $\bigwedge^p(\mathcal{O}_\Sigma)^t$, where $b_i \in (\mathcal{O}_\Sigma)^t$ is the image of the i-th row vector of B. Hence $ab_1, b_2, \dots, b_p$ are linearly dependent in $(\mathcal{O}_\Sigma)^t$. Then there are $a_1, \dots, a_p \in \mathcal{O}_\Sigma$ which are not all zero, such that $a_1 b_1 + \cdots + a_p b_p = 0 \in (\mathcal{O}_\Sigma)^t$. Hence $a_1 h_1 + \cdots + a_p h_p \equiv 0 \mod \mathfrak{g}^2$. ¿From this we have $\mathcal{J}(\mathfrak{h}) \subset \mathfrak{g}$, a contradiction. □

Proposition 7. *Let $\mathfrak{h} \subset \mathfrak{g} \subset$ define two reduced complete intersections $X \supset \Sigma$ of pure dimension. Assume that X_{sing} does not contain any irreducible components of Σ. If in (4.1) we have $t = p = \operatorname{codim} X$, then $b := \det(b_{ij})$ is a non-zero divisor in $\mathcal{O}_\Sigma$, and*

1) *the images $\hat{g}_1, \dots, \hat{g}_p$ of $g_1, \dots, g_p$ generate $T(M)$ over $\mathcal{O}_\Sigma$;*
2) *the images $\hat{g}_{p+1}, \dots, \hat{g}_n$ of $g_{p+1}, \dots, g_n$ generate N freely over $\mathcal{O}_\Sigma$, so $M \simeq T(M) \oplus N$, and $\operatorname{rank}(M) = \operatorname{rank}(N) = \dim X - \dim \Sigma = n - p$;*
3) *For each $z \in X_{\text{reg}} \cap \Sigma$, the germs $(\bar{g}_{p+1})_z, \dots, (\bar{g}_n)_z$ form an $\mathcal{O}_{X,z}$-regular sequence and generate $\bar{\mathfrak{g}}_z$;*
4) *$\int_X \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2$;*
5) *there is a length formula if it is finite*

$$\lambda(\Sigma, X) := l_{\mathcal{O}_\Sigma}(T(M)) = l_{\mathcal{O}_\Sigma}\left(\frac{\mathcal{O}}{(b) + \mathfrak{g}}\right).$$

We call $\lambda(\Sigma, X)$ the *torsion number* of the pair (Σ, X). When Σ and X are clear from the context, we write λ for $\lambda(\Sigma, X)$.

Proof. Since $t = p$ and b is a non-zero divisor, one can see that $\hat{g}_1, \dots, \hat{g}_p \in T(M)$ by multiplying B^* to the both sides of (4.1) , where B^* is the adjoint matrix of B.

Since $\hat{g}_1, \dots, \hat{g}_n$ generate M over $\mathcal{O}_\Sigma$ and (3.1) is exact, $\pi(\hat{g}_{p+1}), \dots, \pi(\hat{g}_n)$ generate N. If there is a relation: $\bar{\beta}_{p+1}\pi(\hat{g}_{p+1}) + \cdots + \bar{\beta}_n \pi(\hat{g}_n) = 0 \in N$, then $\bar{\beta}_{p+1}\hat{g}_{p+1} + \cdots + \bar{\beta}_n \hat{g}_n \in T(M)$. This means that there is a non-zero divisor $\bar{\beta} \in \mathcal{O}_\Sigma$ such that $\bar{\beta}(\bar{\beta}_{p+1}\hat{g}_{p+1} + \cdots + \bar{\beta}_n \hat{g}_n) = 0 \in M$. By taking representatives, this simply means $\beta\beta_{p+1}g_{p+1} + \cdots + \beta\beta_n g_n \in \mathfrak{g}^2 + \mathfrak{h}$. Hence there are $\mu_1, \dots \mu_p \in \mathcal{O}$ such that

$$\mu_1 h_1 + \cdots + \mu_p h_p + \beta\beta_{p+1} g_{p+1} + \cdots + \beta\beta_n g_n \in \mathfrak{g}^2.$$

By (4.1), this becomes

$$\mu'_1 g_1 + \cdots + \mu'_p g_p + \beta\beta_{p+1} g_{p+1} + \cdots + \beta\beta_n g_n \in \mathfrak{g}^2,$$

where $\begin{pmatrix}\mu'_1 & \cdots & \mu'_p\end{pmatrix} = \begin{pmatrix}\mu_1 & \cdots & \mu_p\end{pmatrix} B$. Since $g_1, \dots, g_n$ form an $\mathcal{O}$-regular sequence, we have $\bar{\beta}\bar{\beta}_j = 0$ in $\mathcal{O}_\Sigma$. Note that $\bar{\beta}$ is a non-zero divisor, hence $\bar{\beta}_j = 0$ in $\mathcal{O}_\Sigma$. This proves 1) and 2).

For each prime $z \in X_{\text{reg}} \cap \Sigma$, N_z is also free with the images of $(\bar{g}_{p+1})_z, \dots, (\bar{g}_n)_z$ as basis. Then $\left(\int_X \bar{\mathfrak{g}}\right)_z = \bar{\mathfrak{g}}_z^2$ since (Σ, z) is a reduced complete intersection in the regular space (X, z). Since $\mathcal{O}_{X,z}$ is regular, by Vasconcelos' Theorem [15], $(\bar{g}_{p+1})_z, \dots, (\bar{g}_n)_z$ is an $\mathcal{O}_{X,z}$-regular sequence, and they generated $\bar{\mathfrak{g}}_z$ by Nakayama lemma;

4) follows from Proposition 4.

For the length formula, note that

$$T(M) = \frac{\int_X \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}} \cong \frac{(g_1, \dots, g_p)}{(g_1, \dots, g_p)^2 + (g_1, \dots, g_p)(g_{p+1}, \dots, g_n) + (h_1, \dots, h_p)}.$$

It is easy to see that

$$M_1 := \frac{(g_1, \dots, g_p)}{(g_1, \dots, g_p)^2 + (g_1, \dots, g_p)(g_{p+1}, \dots, g_n)}$$

is a free $\mathcal{O}_\Sigma$-module. Since b is a non-zero divisor, the following sequence is exact

$$0 \longrightarrow M_1 \xrightarrow{\phi_B} M_1 \longrightarrow T(M) \longrightarrow 0,$$

where $\phi_B(\bar{g}_i) := \sum_{j=1}^{p} \bar{b}_{ij}\bar{g}_j$. By [2, A.2.6], we have the length formula of $T(M)$. □

Note that in the following , we do not assume (4.1).

Proposition 8. *Let $\Sigma \subset X$ be complete intersection germs of pure dimensions defined by radical ideals $\mathfrak{g} \supset \mathfrak{h}$ of $\mathcal{O}$. Let* $\operatorname{codim} X = p$ *and* $\operatorname{codim} \Sigma = n$. *Assume that X_{sing} does not contain any irreducible components of Σ. If N is a free $\mathcal{O}_\Sigma$-module, then*

1) *there exists an $\mathcal{O}$-regular sequence $g_1, \dots, g_n$, generating $\mathfrak{g}$, such that*
   - *the images $\hat{g}_1, \dots, \hat{g}_p$ of $g_1, \dots, g_p$ generate $T(M)$;*
   - *the images $\hat{g}_{p+1}, \dots, \hat{g}_n$ of $g_{p+1}, \dots, g_n$ form a basis of N;*
   - *rank(M)=rank(N) $= n - p = \dim X - \dim \Sigma$;*
2) $\int_X \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2$.
3) *we can choose the generators $h_1, \dots, h_p$ of $\mathfrak{h}$ such (4.1) holds with $t = p$ and b a non-zero divisor in $\mathcal{O}_\Sigma$;*

Proof. Let the images $\hat{g}_{t+1}, \dots, \hat{g}_n$ of $g_{t+1}, \dots, g_n \in \mathfrak{g}$ generate N over $\mathcal{O}_\Sigma$, where $t := n - \operatorname{rank} N$.

For any $z \in X_{\text{reg}} \cap \Sigma$, by the assumption, N_z is a free $\mathcal{O}_{\Sigma,z}$ module with the images of $(\bar{g}_{t+1})_z, \dots, (\bar{g}_n)_z$ as basis. By Vasconcelos' theorem [15], the germs $(\bar{g}_{t+1})_z, \dots, (\bar{g}_n)_z$ in $\mathcal{O}_{X,z}$ form an $\mathcal{O}_{X,z}$-regular sequence. And

$$\bar{\mathfrak{g}}_z = (\bar{g}_{t+1}, \dots, \bar{g}_n)_z + \left(\frac{\int_X \mathfrak{g}}{\mathfrak{h}}\right)_z \tag{4.2}$$

Hence the germs $(h_1)_z, \dots, (h_p)_z, (g_{t+1})_z, \dots, (g_n)_z$ in $\mathcal{O}_{\mathbb{C}^m,z}$ form an $\mathcal{O}_{\mathbb{C}^m,z}$-regular sequence, where $h_1, \dots, h_p$ form a minimal generating set of $\mathfrak{h}$. However, since $\mathfrak{h}_z \subset \mathfrak{g}_z$, we have $n - t + p = \operatorname{grade}(h_1, \dots, h_p, g_{t+1}, \dots, g_n)_z \leq \operatorname{grade}(\mathfrak{g})_z = n$. Hence $t \geq p$.

Extend $g_{t+1}, \dots, g_n$ to an $\mathcal{O}$-regular sequence: $g_1, \dots, g_n$, such that they generate $\mathfrak{g}$. Then $\hat{g}_1, \dots, \hat{g}_n$ generate M over $\mathcal{O}_\Sigma$. We look for the generator set of $T(M)$. Let

$$\pi(\hat{g}_i) = \bar{c}_{it+1}\pi(\hat{g}_{t+1}) + \cdots + \bar{c}_{in}\pi(\hat{g}_n), \quad i = 1, \dots, t.$$

Hence by (3.1), $\hat{g}'_i := -\hat{g}_i + \bar{c}_{it+1}\hat{g}_{t+1} + \cdots + \bar{c}_{in}\hat{g}_n \in T(M), i = 1, \dots, t$. Taking representatives, denote $g'_i := -g_i + c_{it+1}g_{t+1} + \cdots + c_{in}g_n, i = 1, \dots, t$, $g'_{t+j} = g_{t+j}$, $j = 1, \dots, n-t$. Then $\mathfrak{g} = (g'_1, \dots, g'_n)$, with $\hat{g}'_1, \dots, \hat{g}'_t \in T(M)$. By Lemma 5, $t \preceq p$. We have proved 1).

Note that $\bar{\mathfrak{g}}_z \subset \mathcal{O}_{X,z}$ defines also a complete intersection (Σ, z) in the regular space germ (X, z), and we have $p = t$ in (4.2). By actually [10, 11], $\left(\frac{\int_X \mathfrak{g}}{\mathfrak{h}}\right)_z = \bar{\mathfrak{g}}_z^2$. By Nakayama lemma and (4.2), we have $\bar{\mathfrak{g}}_z = (g_{p+1}, \dots, g_n)_z$. By Proposition 4, we have 2).

Since $\mathfrak{h} \subset \mathfrak{g}$, we have $h_i = b_{i1}g'_1 + \cdots + b_{ip}g'_p + b_{ip+1}g'_{p+1} + \cdots + b_{in}g'_n, i = 1, \dots, p$. For any $\xi \in \mathrm{Der}_X(\mathcal{O})$, we have $b_{ip+1}\xi(g'_{p+1}) + \cdots + b_{in}\xi(g'_n) \equiv 0 \mod \mathfrak{g}, i = 1, \dots, p$. Hence $b_{ip+1}g'_{p+1} + \cdots + b_{in}g'_n \in \int_X \mathfrak{g}$, which implies that $b_{ip+1}, \dots, b_{in} \in \mathfrak{g}$ for $i = 1, \dots, p$. It is obvious that b is a non-zero divisor in $\mathcal{O}_\Sigma$. □

Remark 9. Combining the conclusions in Proposition 4, Proposition 7 and 8, one sees that the Main Theorem is proved. Moreover, either of the equivalent conditions in the Main Theorem is equivalent to 3) in Proposition 8.

Example 10. Let $\mathfrak{h}$ be defined by $h := x^3 + xy^3 + 2x^2z + 2z^2 = 0$, $\mathfrak{g}$ be defined by $g_1 := x^2 + y^3 = 0, g_2 := z = 0$. Thus $\mathfrak{h} = (h), \mathfrak{g} = (g_1, g_2)$. Notice that h is not weighted homogeneous. So it is not easy to find the generator set of $\mathrm{Der}_X(\mathcal{O})$. Then we have the same problem for $\int_X \mathfrak{g}$. If we denote $g'_1 = g_1 + 2xg_2 + g_2^2$, then $h = xg'_1 + (2-x)g_2^2$, where x is a non-zero divisor in $\mathcal{O}_\Sigma$. By Proposition 7, we have:

- $T(M)$ is generated by g'_1 over $\mathcal{O}_\Sigma$
- $\int_X \mathfrak{g} = (g'_1, g_2^2) = (x^2 + y^3 + 2xz, z^2)$
- $N = (g_2)/(g'_1 g_2, g_2^2)$ is a free $\mathcal{O}_\Sigma$-module.

The following example shows that it is not necessary for $T(M)$ to be generated by $\bar{g}_i$ when $t > p$.

Example 11. Let $\mathfrak{g} = (g_1, g_2)$ with $g_1 = xy, g_2 = z$ and $\mathfrak{h} = (h)$ with $h = x^2y + yz + z^2 = xg_1 + yg_2 + g_2^2$. Then $\mathcal{O}_\Sigma \cong \mathbb{C}\{x, y\}/(xy)$, $\int_X \mathfrak{g} = (x^2y, yz, z^2)$ and $\mathfrak{g}^2 + \mathfrak{h} = (x^2y^2, xyz, z^2, x^2y + yz)$. So $T(M) \cong \mathbb{C}x^2y$. And N is not a free $\mathcal{O}_\Sigma$-module.

5. Lines on spaces with isolated complete intersection singularities

We include some applications of the theory to lines on a space germ with isolated complete intersection singularity. Let Σ be the germ of a smooth curve in $(\mathbb{C}^{n+1}, 0)$ defined by $\mathfrak{g}$. We call such a Σ a line since locally it is biholomorphic to a line. Define ${}_\Sigma\mathcal{K} := \mathcal{R}_\Sigma \rtimes \mathcal{C}$, the semi-product of $\mathcal{R}_\Sigma$ with the contact group $\mathcal{C}$ (cf. [8]), where $\mathcal{R}_\Sigma$ is a subgroup of $\mathcal{R} := \mathrm{Aut}(\mathbb{C}^m, 0)$ consisting of all the $\varphi \in \mathcal{R}$ preserving $\mathfrak{g}$. This group has an action on the space $\mathfrak{m}\mathfrak{g}\mathcal{O}^p$. For $h = (h_1, \dots, h_p) \in \mathfrak{m}\mathfrak{g}\mathcal{O}^p$, there is an ideal $\mathfrak{h}$ generated by $h_1, \dots, h_p$, and a germ $X = \mathcal{V}(\mathfrak{h})$. The image of the differential of h: $\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p$ is denoted by $\mathrm{th}(h)$. Define a ${}_\Sigma\mathcal{K}$-invariant

$$\tilde{\lambda} := \tilde{\lambda}(\Sigma, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}^p}{\mathrm{th}(h) + \mathfrak{g}\mathcal{O}^p}.$$

Choose Σ as the x-axis. Then Σ can be defined by $\mathfrak{g} = (y_1, \dots, y_n)$.

Proposition 12. *Let Σ be a line on a space germ X with isolated complete intersection singularity of codimension p. Then h is ${}_\Sigma\mathcal{K}$-equivalent to an $\tilde{h}$ with components $\tilde{h}_i \equiv b_iy_i$ mod $\mathfrak{g}^2$, where $b_i \notin \mathfrak{g}$, $i = 1, \dots p$. Moreover $\lambda(\Sigma, X) = \tilde{\lambda}(\Sigma, X) = \dim_{\mathbb{C}} \mathcal{O}/(b + \mathfrak{g}) = \sum_{k=1}^p l_i$. where $b := b_1 \cdots b_p$, and l_i is the valuation of $\bar{b}_i$ in $\mathcal{O}_\Sigma$.*

Proof. Since $\Sigma \subset X$, for a given generator set $\{h_1, \dots, h_p\}$ of $\mathfrak{h}$, we have $h_i \equiv \sum \bar{b}_{ij} y_j \mod \mathfrak{g}^2, i = 1, \dots, p$, where $\bar{b}_{ij} \in \mathcal{O}_\Sigma$, and for fixed i, $\bar{b}_{ij}$'s are not all zero since X is complete intersection and $X_{\text{sing}} = \{0\} \subsetneq \Sigma$. Since $\mathcal{O}_\Sigma$ is a principal ideal domain, by changing the indices, we can assume that $\bar{b}_{11} \mid \bar{b}_{ij}$. Let $y_1' = y_1 + \sum_{j=2}^n \frac{\bar{b}_{1j}}{\bar{b}_{11}} y_j$. Then $h_1 \equiv \bar{b}_{11} y_1' \mod \mathfrak{g}^2$. Let $h_i' = h_i - \frac{\bar{b}_{i1}}{\bar{b}_{11}} h_1, i = 2, \dots, p$. Repeat the above argument will prove the first part of the proposition.

Consider the exact sequence

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p \longrightarrow \operatorname{coker}(dh^*) \longrightarrow 0.$$

By tensoring with $\mathcal{O}_\Sigma$, we have the exact sequence

$$\mathcal{O}_\Sigma^{n+1} \xrightarrow{d\bar{h}^*} \mathcal{O}_\Sigma^p \longrightarrow \operatorname{coker}(d\bar{h}^*) \longrightarrow 0.$$

However by the expression of h_i's above, this is just

$$\mathcal{O}_\Sigma^p \xrightarrow{d\bar{h}^*} \mathcal{O}_\Sigma^p \longrightarrow \frac{\mathcal{O}^p}{\operatorname{th}(h) + \mathfrak{g}\mathcal{O}^p} \longrightarrow 0.$$

Since $\bar{b} \neq 0$, by [2, A.2.6], we have the formula for $\tilde{\lambda}$. □

Corollary 13. *Let X be a space germ with isolated complete intersection singularity of codimension p in $(\mathbb{C}^{n+1}, 0)$, and Σ is a line in X defined by $\mathfrak{g}$. Then we can choose the coordinates of $(\mathbb{C}^{n+1}, 0)$ such that $\mathfrak{g} = (y_1, \dots, y_n)$, $\hat{y}_1, \dots, \hat{y}_p \in T(M)$ and $\hat{y}_{p+1}, \dots, \hat{y}_n$ generate N which is free of rank $n-p$, and*

$$\int_X \mathfrak{g} = (y_1, \dots, y_p) + (y_{p+1}, \dots, y_n)^2$$ □

Corollary 14. (Due to Pellikaan) *Let X be the germ of a space with isolated complete intersection singularity of codimension p in $(\mathbb{C}^{n+1}, 0)$, and Σ a line in X, defined by $\mathfrak{g} = (y_1, \dots, y_n)$. Then the Second Exact Sequence* [9, 3] *is exact on the left also:*

$$0 \longrightarrow M \xrightarrow{\delta} \Omega_X^1 \otimes \mathcal{O}_\Sigma \longrightarrow \Omega_\Sigma^1 \longrightarrow 0.$$

Furthermore it is splitting and

$$T(\Omega_X^1 \otimes \mathcal{O}_\Sigma) = T(M), \quad \operatorname{rank}(\Omega_X^1 \otimes \mathcal{O}_\Sigma) = n - p - 1.$$

These tell us that the torsion number $\lambda(\mathfrak{h}, \mathfrak{g})$ is independent of the choice of the generator sets of $\mathfrak{g}$ and $\mathfrak{h}$.

Proof. We have the following presentation:

$$\mathcal{O}_X^p \xrightarrow{dh} \mathcal{O}_X^{n+1} \longrightarrow \Omega_X^1 \longrightarrow 0.$$

Tensoring with $\mathcal{O}_\Sigma$, we have the exact sequence

$$\mathcal{O}_\Sigma^p \xrightarrow{\overline{dh}} \mathcal{O}_\Sigma^{n+1} \longrightarrow \Omega_X^1 \otimes \mathcal{O}_\Sigma \longrightarrow 0.$$

Remark that the map $\overline{dh}$ is equivalent to a map defined by the matrix $(\bar{b}_{ij})$. Hence

$$\Omega_X^1 \otimes \mathcal{O}_\Sigma \cong \frac{\mathcal{O}_\Sigma^{n+1}}{\operatorname{im}\overline{dh}} \cong \mathcal{O}_\Sigma^{n-p+1} \oplus \frac{\mathcal{O}_\Sigma^p}{\operatorname{im}(\bar{b}_{ij})} \cong \mathcal{O}_\Sigma \oplus N \oplus T(M) \cong \mathcal{O}_\Sigma \oplus M.$$

Then exact is

$$0 \longrightarrow M \longrightarrow \frac{\mathcal{O}_\Sigma^{n+1}}{\operatorname{im}\overline{dh}} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0.$$

Since Ω_Σ^1 is free $\mathcal{O}_\Sigma$-module of rank 1, by [2, A.2.2], we have the exact sequence. □

Remark 15. In general, given an analytic space germ $(X, 0)$, one cannot find a smooth curve $L \subset X$ that passes through and is not contained in X_{sing}. However, if there are smooth curves on X in the above sense, how to distinguish them is a problem. We found that the torsion number λ is a nice candidate for this purpose [7]. In studying the Euler-Poincaré characteristic $\chi(F)$ of the Milnor fibre F of a function with singular locus a smooth curve on a singular space, we found that this λ also appears in $\chi(F)$. Note also that the torsion number was generalized to " higher torsion numbers" in [6, 7].

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SHEAVES ON A FINITE POSET AND LOCAL COHOMOLOGY MODULES WITH SUPPORTS IN MONOMIAL IDEALS

KOHJI YANAGAWA

1. Introduction

The local cohomology module $H_I^i(R)$ of a noetherian commutative ring R with supports in a (non-maximal) ideal I has been studied by a number of authors. But it is still very mysterious. When R is a regular local ring containing a field, some remarkable results on the minimal injective resolution of $H_I^i(R)$ were obtained by Huneke-Sharp [7] and Lyubeznik [8, 9]. Among other things, they proved that the Bass numbers of these local cohomology modules are always finite.

Very recently, a combinatorial description of the local cohomology module $H_I^i(S)$ with supports in a monomial ideal I of a polynomial ring $S = k[x_1, \dots, x_n]$ was obtained by Mustaţă [10] and Terai [13]. Since $H_I^i(S) = H_{\sqrt{I}}^i(S)$, we may assume that I is the Stanley-Reisner ideal I_Δ of a simplicial complex $\Delta \subset 2^{\{1,\dots,n\}}$.

In this note, we will study a minimal injective resolution of $H_{I_\Delta}^i(S)$. We will see that Bass numbers $\mu^i(P, H_{I_\Delta}^j(S))$ are equal to certain values of the ($\mathbb{Z}^n$-graded) Hilbert function of $\mathrm{Ext}_S^l(\mathrm{Ext}_S^j(S/I_\Delta, S), S)$ for some l.

Next, we will study the local cohomology module $H_I^i(R)$ of a normal Gorenstein semigroup ring $R = k[C]$, $C \subset \mathbb{Z}^n \subset \mathbb{R}^n$, with supports in a monomial ideal (i.e., a $\mathbb{Z}^n$-graded ideal) I. We say R is *simplicial* if the cone $\mathbb{R}_+C$ can be spanned by $\dim(\mathbb{R}_+C)$ elements of $\mathbb{R}^n$. If R is simplicial and Gorenstein, the Bass number $\mu^i(P, H_I^j(R))$ is always finite. And it is equal to a certain value of the ($\mathbb{Z}^n$-graded) Hilbert function of $\mathrm{Ext}_R^l(\mathrm{Ext}_R^j(R/I, R), R)$ as in the polynomial ring case.

2. Squarefree Modules and Straight Modules

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Consider a natural $\mathbb{Z}^n$-grading on S. An ideal of S is $\mathbb{Z}^n$-graded if and only if it is a monomial ideal. Let $\mathfrak{m} := (x_1, \dots, x_n)$ be the graded maximal ideal. For a $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_\mathbf{a}$ and $\mathbf{b} \in \mathbb{Z}^n$, we denote by $M(\mathbf{b})$ the $\mathbb{Z}^n$-graded S-module which coincides with M as the underlying S-module and whose grading is given by $[M(\mathbf{b})]_\mathbf{a} = M_{\mathbf{a}+\mathbf{b}}$. Let $\omega_S := S(-1, \dots, -1)$ be the canonical module of S. For a subset $F \subset [n] := \{1, \dots, n\}$, P_F denotes the monomial prime ideal $(x_i \mid i \notin F)$ of S. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, denote the monomial $\prod x_i^{a_i}$ by $\mathbf{x}^\mathbf{a}$.

We denote the category consisting of all the S-modules (resp. $\mathbb{Z}^n$-graded S-modules) and their S-homomorphisms (resp. $\mathbb{Z}^n$-graded S-homomorphisms) by **Mod** (resp. ***Mod**). Here, we say an S-homomorphisms of $\mathbb{Z}^n$-graded modules $f : N \to M$ is $\mathbb{Z}^n$*-graded*, if $f(N_\mathbf{a}) \subset M_\mathbf{a}$ for all $\mathbf{a} \in \mathbb{Z}^n$. Let M, N be $\mathbb{Z}^n$-graded

S-modules. Recall that if N is finitely generated then $\mathrm{Ext}^i_S(N,M) \in \mathbf{Mod}$ has natural $\mathbb{Z}^n$-grading (cf. [2, 4]).

In this note, $\mathbb{N}$ means the set of non-negative integers. A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_{\mathbf{a}}$ is called $\mathbb{N}^n$*-graded* if $M_{\mathbf{a}} = 0$ for all $\mathbf{a} \notin \mathbb{N}^n$. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, set $\mathrm{supp}(\mathbf{a}) := \{i \mid a_i > 0\} \subset [n]$. We say $\mathbf{a} \in \mathbb{Z}^n$ is *squarefree* if $a_i = 0, 1$ for all $i \in [n]$. When $\mathbf{a} \in \mathbb{Z}^n$ is squarefree, we identify $\mathbf{a}$ with $F := \mathrm{supp}(\mathbf{a})$. For example, we denote $M_{\mathbf{a}}$ by M_F and $M(\mathbf{a})$ by $M(F)$ for $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_{\mathbf{a}} \in {}^*\mathbf{Mod}$.

Let $\Delta \subset 2^{[n]}$ be a simplicial complex, i.e., if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$. The *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal $I_\Delta := (\mathbf{x}^F \mid F \notin \Delta)$ of S. Any squarefree monomial ideal is the Stanley-Reisner ideal I_Δ for some Δ.

Definition 2.1 ([14]). We say an $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ is *squarefree*, if M is finitely generated and the multiplication map $M_{\mathbf{a}} \ni y \mapsto \mathbf{x}^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\mathrm{supp}(\mathbf{a}+\mathbf{b}) = \mathrm{supp}(\mathbf{a})$.

It is easy to see that a Stanley-Reisner ring S/I_Δ is a squarefree module. The free module $S(-F)$ is squarefree for all $F \subset [n]$, in particular, ω_S is squarefree. And $S/P_F(-G)$ is squarefree if and only if $G \subset F$. If M is squarefree then $\dim_k M_{\mathbf{a}} = \dim_k M_{\mathrm{supp}(\mathbf{a})} < \infty$ for all $\mathbf{a} \in \mathbb{N}^n$, and we have $\dim_S M = \max\{\, |F| \mid M_F \neq 0 \,\}$.

We denote by $\mathbf{Sq}_S$ (or simply $\mathbf{Sq}$) the full subcategory of ${}^*\mathbf{Mod}$ consisting of all squarefree modules. $\mathbf{Sq}$ is an abelian category admitting the Jordan-Hörder Theorem. Moreover, $\mathbf{Sq}$ has enough projectives and enough injectives. An indecomposable projective (resp. injective) object in $\mathbf{Sq}$ is isomorphic to $S(-F)$ (resp. S/P_F) for some $F \subset [n]$. If M is squarefree, then both $\mathrm{Syz}_i(M)$ and $\mathrm{Ext}^i_S(M, \omega_S)$ are also for all $i \geq 0$ ([14, Corollary 2.4 and Theorem 2.6]).

Recently, monomial ideals of an exterior algebra $E = k\langle x_1, \dots, x_n\rangle$ are studied by several authors (e.g., [1, 3, 12]). Römer [12] defined squarefree modules over E. A monomial ideal of E is always squarefree, and the category $\mathbf{Sq}_E$ of squarefree E-modules is equivalent to the category $\mathbf{Sq}_S$ of squarefree modules over $S = k[x_1, \dots, x_n]$, and an indecomposable projective object in $\mathbf{Sq}_E$ is a certain monomial ideal of E. A projective object in $\mathbf{Sq}_E$ has a simple and explicit (but infinite) E-free resolution. The notion of a squarefree module is also useful in the study of an exterior algebra, see [12, 3].

Definition 2.2. A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_{\mathbf{a}}$ is called *straight*, if the following two conditions are satisfied.

(a) $\dim_k M_{\mathbf{a}} < \infty$ for all $\mathbf{a} \in \mathbb{Z}^n$.
(b) The multiplication map $M_{\mathbf{a}} \ni y \mapsto \mathbf{x}^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbf{b} \in \mathbb{N}^n$ with $\mathrm{supp}(\mathbf{a}+\mathbf{b}) = \mathrm{supp}(\mathbf{a})$.

A finitely generated S-module M is straight if and only if M is a direct sum of finitely many copies of ω_S. The injective hull ${}^*E(S/P_F)$ of S/P_F in ${}^*\mathbf{Mod}$ (we will say ${}^*E(S/P_F)$ is the **injective hull* of S/P_F) is straight. Recall that ${}^*E(S/P_F)$ is not injective in $\mathbf{Mod}$, if $P_F \neq \mathfrak{m}$.

Denote by $\mathbf{Str}$ the full subcategory of ${}^*\mathbf{Mod}$ consisting of all straight modules. For a $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_{\mathbf{a}}$, we call the $\mathbb{N}^n$-graded submodule $\bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ the $\mathbb{N}^n$*-graded part* of M, and denote it by $\bar{M}$. It is easy to see that if

M is straight, then $\bar{M}$ is squarefree. For example, the $\mathbb{N}^n$-graded part of $^*E(S/P_F)$ is isomorphic to S/P_F. Conversely, for a squarefree module N, there is a unique (up to isomorphism) straight module $\hat{N}$, whose $\mathbb{N}^n$-graded part is isomorphic to N. We say $\hat{N}$ is the *straight hull* of N. Summing up, we have the following.

Proposition 2.3 ([15, Proposition 2.7]). *The functors* $\mathbf{Sq} \ni N \mapsto \hat{N} \in \mathbf{Str}$ *and* $\mathbf{Str} \ni M \mapsto \bar{M} \in \mathbf{Sq}$ *give the category equivalence* $\mathbf{Sq} \cong \mathbf{Str}$.

3. Injective Resolutions

The abelian category **Str** has enough projectives and enough injectives. An injective object of **Str** is a direct sum of finitely many copies of $^*E(S/P_F)$ for various $F \subset [n]$. So we have the following.

Proposition 3.1 ([15, Theorem 3.2]). *Let M be a straight S-module, and* $^*E^\bullet : 0 \to M \to {}^*E^0 \to {}^*E^1 \to \cdots$ *a minimal injective resolution of M in the category* ***Mod** *(we say $^*E^\bullet$ a minimal *injective resolution of M). Then $^*E^i$ is straight for all $i \geq 0$. More precisely,* $^*E^i \cong \bigoplus_{F \subset [n]} {}^*E(S/P_F)^{\mu^i(P_F,M)}$ *(the degree shifting does not occur). And we have* $^*\mathrm{inj.dim}_S M \leq \mathrm{inj.dim}\, M = \dim_S \mathrm{Supp}(M) = \dim_S \bar{M}$.

Let M, N be $\mathbb{Z}^n$-graded S-modules, and $^*E^\bullet$ a minimal *injective resolution of M. If N is finitely generated, then $H^i(\mathrm{Hom}_S(N, {}^*E^\bullet)) \cong \mathrm{Ext}^i_S(N, M)$. Since $H^i_I(M) = \varinjlim \mathrm{Ext}^i_S(S/I^l, M)$, we have $H^i_I(M) = H^i(\Gamma_I({}^*E^\bullet))$, where $\Gamma_I(-) := \varinjlim \mathrm{Hom}_R(R/I^l, -)$ is the functor from ***Mod** to itself. So $H^i_I(M)$ has a natural $\mathbb{Z}^n$-grading.

Corollary 3.2 ([15, Corollary 3.4]). *If M is straight, so is $H^i_{I_\Delta}(M)$.*

Proof. Let $^*E^\bullet$ be a minimal *injective resolution of M. By Proposition 3.1, the complex $^*E^\bullet$ consists of straight modules. Hence so does $\Gamma_{I_\Delta}({}^*E^\bullet)$. □

Since the canonical module ω_S is straight, a minimal *injective resolution $^*D^\bullet$ of ω_S consists of $^*E(S/P_F)$'s (without degree shifting) by Theorem 3.1.

Corollary 3.3 (Mustaţă [11] and Terai [13]). *For all $i \geq 0$, the local cohomology module* $H^i_{I_\Delta}(\omega_S) \cong H^i_{I_\Delta}(S)(-1, \cdots, -1)$ *is a straight module whose $\mathbb{N}^n$-graded part is isomorphic to* $\mathrm{Ext}^i_S(S/I_\Delta, \omega_S)$.

Proof. Since ω_S is straight, so is $H^i_{I_\Delta}(\omega_S)$ by Corollary 3.2. Observe that the $\mathbb{N}^n$-graded part of $\mathrm{Hom}_S(S/I_\Delta, {}^*E(S/P_F))$ is isomorphic to S/P_F if $I_\Delta \subset P_F$, and 0 if not. Thus the $\mathbb{N}^n$-graded part of $\mathrm{Hom}_S(S/I_\Delta, {}^*D^\bullet)$ and that of $\Gamma_{I_\Delta}({}^*D^\bullet)$ are isomorphic. Recall that $\mathrm{Ext}^i_S(S/I_\Delta, \omega_S)$ is squarefree. So the $\mathbb{N}^n$-graded part of $H^i_{I_\Delta}(\omega_S)$ is isomorphic to $\mathrm{Ext}^i_S(S/I_\Delta, \omega_S)$. □

We have an explicit formula for the Bass numbers of straight modules.

Theorem 3.4 ([15, Theorem 3.5]). *Let M be a straight S-module, and $\bar{M}$ its $\mathbb{N}^n$-graded part. Then,* $\mu^i(P_F, M) = \dim_k[\mathrm{Ext}^{n-i-|F|}_S(\bar{M}, \omega_S)]_F$.

Corollary 3.5 ([15, Corollary 3.6]). *For all i, j, we have*

$$\begin{aligned}\mu^i(P_F, H^j_{I_\Delta}(S)) &= \dim_k[\operatorname{Ext}_S^{n-i-|F|}(\operatorname{Ext}_S^j(S/I_\Delta, \omega_S), \omega_S)]_F \\ &= \dim_k[\operatorname{Ext}_S^{n-i-|F|}(\operatorname{Ext}_S^j(S/I_\Delta, S), S)]_F.\end{aligned}$$

We can also compute the Bass numbers at an arbitrary prime ideal P, since we have $\mu^i(P, H^i_{I_\Delta}(S)) = \mu^{i-t}(P^*, H^i_{I_\Delta}(S))$ by [4, Theorem 1.2.3], where P^* is the largest monomial prime ideal contained in P and $t := \dim(S_P/P^*S_P)$.

Set $d := \dim S/I_\Delta$. If $\operatorname{Ext}_S^{n-d}(S/I_\Delta, S)$ is Cohen-Macaulay (e.g., S/I_Δ itself is Cohen-Macaulay), a minimal injective resolution of $H^{n-d}_{I_\Delta}(S)$ is naturally "visualized" using Δ, see [15].

Let I be an arbitrary ideal of S, and $x = \{x_1, \dots, x_n\}$ a generating set of I. Set $C^\bullet := \varinjlim K^\bullet(x^l)$ be the *Čech complex* with respect to x. Let M be an S-module. It is well-known that $H^i_I(M) = H^i(M \otimes_S C^\bullet)$ for all $i \geq 0$. Note that $C^\bullet$ is a complex of flat S-modules. Thus $H^i_I(M)$ is the ith cohomology group of the derived tensor product of $C^\bullet \otimes^{\mathbf{L}}_S M$. From now on, we assume that I is a squarefree monomial ideal. Let $\mathbb{F}_\bullet$ be a $\mathbb{Z}^n$-graded minimal free resolution of S/I over S. Set $\mathbb{G}^\bullet := \operatorname{Hom}^\bullet_S(\mathbb{F}_\bullet, \omega_S)$. Since $\mathbb{G}^\bullet$ is a direct sum of the copies of squarefree modules $S(-F)$, we have the straight hull $\mathbb{E}^\bullet := \hat{\mathbb{G}}^\bullet$ of $\mathbb{G}^\bullet$, which consists of flat S-modules. By Corollary 3.3, we have $H^i(\mathbb{E}^\bullet) = H^i_I(\omega_S)$. But we can prove more. By an argument similar to the proof of Corollary 3.3, we see that the Čech complex $C^\bullet$ and $\mathbb{E}^\bullet$ represent the same object in the derived category $D^b(\mathbf{Mod})$ of $\mathbf{Mod}$. Thus we get a new proof of a result of E. Miller.

Theorem 3.6 (Miller [10]). *With the above notation, we have*

$$H^i_I(M) = H^i(\mathbb{E}^\bullet \otimes_S M)$$

for a (not necessarily $\mathbb{Z}^n$-graded) module M. If M is $\mathbb{Z}^n$-graded, $H^i_I(M) \otimes_S \omega_S = H^i(\mathbb{E}^\bullet \otimes_S M)$ as $\mathbb{Z}^n$-graded modules.

4. Sheaves on Finite Posets

The results of the remaining part of this note are from [16].

In this section, we study sheaves on a partially ordered set (*poset* for short). Let P be a finite poset. We regard P as a category. The objects of this category are the elements of P, and the morphism from x to y in P is relation of the form $x \leq y$, that is, $\operatorname{Hom}_P(x, y)$ is either $\{x \leq y\}$ or $\emptyset$. Let k be a field, and **vect** the category of *finite dimensional* k-vector spaces and their linear maps. A *sheaf* on P (with value in **vect**) is a covariant functor $\mathcal{F} : P \to \mathbf{vect}$. For $x, y \in P$ with $x \leq y$, we call the corresponding linear map $\mathcal{F}(x) \to \mathcal{F}(y)$ the *restriction map*. A morphism of two sheaves $\mathcal{F}$ and $\mathcal{G}$ on P is a natural transform. Denote the category of sheaves on P by $\mathbf{Sh}(P)$. It is easy to see that $\mathbf{Sh}(P)$ is an abelian category admitting the Jordan-Hörder Theorem and the Krull-Schmidt Theorem.

For $x \in P$, we define the sheaves $\mathcal{J}_x$ and $\mathcal{E}_x$ as follows:

$$\mathcal{J}_x(y) = \begin{cases} k & \text{if } y \geq x, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{E}_x(y) = \begin{cases} k & \text{if } y \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

where the restriction map of $\mathcal{J}_x$ (resp. $\mathcal{E}_x$) is always injective (resp. surjective).

Proposition 4.1. *(a) The sheaf $\mathcal{J}_x$ (resp. $\mathcal{E}_x$) is a projective (resp. injective) object in* $\mathbf{Sh}(P)$ *for all $x \in P$. And any projective (resp. injective) object in* $\mathbf{Sh}(P)$ *is a direct sum of finitely many copies of $\mathcal{J}_x$ (resp. $\mathcal{E}_x$) for various $x \in P$.*

(b) The category $\mathbf{Sh}(P)$ *has enough projectives and enough injectives. For all $\mathcal{F} \in \mathbf{Sh}(P)$, both* $\operatorname{proj.dim}_{\mathbf{Sh}(P)} \mathcal{F}$ *and* $\operatorname{inj.dim}_{\mathbf{Sq}(P)} \mathcal{F}$ *are at most* $\operatorname{rank}(P)$. *In particular, both of them are finite.*

5. Squarefree Modules over a Normal Semigroup Ring

Let $C \subset \mathbb{Z}^d \subset \mathbb{R}^d$ be an affine semigroup (i.e., a finitely generated semigroup containing 0), and $R := k[C] = k[\mathbf{x}^c \mid c \in C] \subset k[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}]$ its semigroup ring over a filed k. A $\mathbb{Z}^d$-graded ideal of R is called a *monomial ideal*, since it is generated by monomials. *In this note, we always assume that C is normal and positive.* In other words, R is normal, and $\mathfrak{m} := (\mathbf{x}^c \mid 0 \neq c \in C) \subset R$ is the graded maximal ideal. Since R is normal, it is Cohen-Macaulay (c.f. [2, Theorem 6.3.5]). We also assume that $\mathbb{Z}C = \mathbb{Z}^d$. Hence we have $\dim R = d$.

Let $\mathbb{R}_+$ be the set of non-negative real numbers. Consider the rational polyhedral cone $\mathbb{R}_+C := \{\sum r_i c_i \mid r_i \in \mathbb{R}_+,\ c_i \in C\} \subset \mathbb{R}^d$. There are vectors $a_i \in \mathbb{Z}^d \subset \mathbb{R}^d$, $1 \leq i \leq n$, such that $\mathbb{R}_+C = \{z \in \mathbb{R}^d \mid \langle a_i, z\rangle \geq 0 \text{ for all } i\}$. Here $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ is the inner product of two vectors $x, y \in \mathbb{R}^d$. For $z \in \mathbb{R}^d$, set $\operatorname{supp}(z) := \{i \mid \langle a_i, z\rangle > 0\} \subset [n]$. It is easy to see that $z \in \mathbb{R}^d$ is contained in the relative interior of $\mathbb{R}_+C$ if and only if $\operatorname{supp}(z) = [n]$.

We say R is *simplicial* if $\mathbb{R}_+C$ can be spanned by d elements of $\mathbb{R}^d$. In this case, we can take $n = d$. A polynomial ring $S = k[x_1, \ldots, x_n] = k[\mathbb{N}^n]$ is a simplicial semigroup ring. In this case, $\operatorname{supp}(z)$ for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is given by $\{i \mid z_i > 0\} \subset [n]$.

We denote the set of non-empty faces of $\mathbb{R}_+C$ by L. The order by inclusion makes L a finite poset. The rank of L as a poset is d. Note that R is simplicial if and only if L is isomorphic to the boolean lattice B_n as a poset. For $z \in \mathbb{R}_+C$, there is a unique face $s(z) \in L$ such that z is contained in the relative interior of $s(z)$. For $z, z' \in \mathbb{R}_+C$, $\operatorname{supp}(z) = \operatorname{supp}(z')$ if and only if $s(z) = s(z')$.

Next we will define a squarefree C-graded module over $R = k[C]$, $C \subset \mathbb{Z}^d$. A $\mathbb{Z}^d$-graded R-module $M = \bigoplus_{a \in \mathbb{Z}^d} M_a$ is called *C-graded* if $M_a = 0$ for all $a \notin C$.

Definition 5.1. A C-graded R-module M is called *squarefree*, if M is finitely generated and the multiplication map $M_a \ni y \mapsto \mathbf{x}^b y \in M_{a+b}$ is bijective for all $a, b \in C$ with $\operatorname{supp}(a + b) = \operatorname{supp}(a)$.

When R is a polynomial ring, the above definition coincides with the previous definition of a squarefree module over a polynomial ring.

We denote the category consisting of all the R-modules (resp. $\mathbb{Z}^d$-graded R-modules) and their R-homomorphisms (resp. $\mathbb{Z}^d$-graded R-homomorphisms) by **Mod** (resp. ***Mod**). We denote the full subcategory of ***Mod** consisting of all squarefree modules by $\mathbf{Sq}_R$, or simply **Sq**. It is easy to see that **Sq** is abelian.

Theorem 5.2. *The categories* $\mathbf{Sq}_R$ *and* $\mathbf{Sh}(L)$ *are equivalent.*

It is obvious that R itself is a squarefree module. Since the canonical module ω_R of R is isomorphic to the ideal $(\mathbf{x}^c \mid c \in C \text{ with } \mathrm{supp}(c) = [n])$, c.f. [2, Therorem 6.3.5], ω_R is also squarefree. Let $\Delta \subset L$ be an order ideal (i.e, if $F \in \Delta$, $G \in L$, and $G \subset F$, then $G \in \Delta$). Set $I_\Delta := (\mathbf{x}^c \mid c \in C \text{ and } s(c) \notin \Delta)$ to be a radical monomial ideal of R. A radical monomial ideal I of R is I_Δ for some Δ. Both I_Δ and R/I_Δ are squarefree. When R is a polynomial ring (i.e., $C \cong \mathbb{N}^d$ and $L \cong B_n$ as a poset), the above definition of I_Δ coincides with that of the Stanley-Reisner ideal I_Δ of a simplicial complex Δ. Note that an order ideal $\Delta \subset B_n$ can be regarded as a simplicial complex whose vertex set is (a subset of) $[n]$.

For a face $F \in L$, set $P_F := (\mathbf{x}^c \mid c \in C \text{ with } s(c) \not\subset F) \subset R$ be a monomial ideal. Since R/P_F can be regarded as the normal semigroup ring $k[C \cap F]$, P_F is a prime ideal and R/P_F is Cohen-Macaulay. Conversely, any monomial prime ideal of R is of the form P_F for some $F \in L$. Observe that $\dim(R/P_F) = \dim F$, where $\dim F$ is the dimension as a face of $\mathbb{R}_+C$.

As in the polynomial ring case, we can prove that if M is a squarefree module then so is $\mathrm{Ext}^i_R(M, \omega_R)$ for all $i \geq 0$.

The category **Sq** has enough projectives and enough injectives. For a face $F \in L$, we denote the radical monomial ideal $(\mathbf{x}^c \mid c \in C \text{ and } s(c) \supset F)$ by J_F. It is easy to see that J_F is a squarefree module corresponding to the sheaf $\mathcal{J}_F \in \mathbf{Sh}(L)$. Thus an indecomposable projective object in **Sq** is isomorphic to J_F for some F. Similarly, an indecomposable injective object in **Sq** is isomorphic to R/P_F for some F.

When $R = S = k[x_1, \dots, x_n]$ is a polynomial ring, J_F is nothing other than the free module $S(-F)$. If R is not a polynomial ring, J_F is not free for some F. For example, ω_R is always projective in $\mathbf{Sq}_R$. A projective object in $\mathbf{Sq}_R$ is always a maximal Cohen-Macaulay R-module. All squarefree maximal Cohen-Macaulay R-module is projective in $\mathbf{Sq}_R$ if and only if R is simplicial.

6. Local Cohomology Modules with Supports in a Monomial Ideal (Normal Semigroup Ring Case)

Definition 6.1. We say a $\mathbb{Z}^d$-graded R-module $M = \bigoplus_{a \in \mathbb{Z}^d} M_a$ is *straight*, if

(i) $\dim_k M_a < \infty$ for all $a \in \mathbb{Z}^d$.
(ii) The multiplication map $M_a \ni y \mapsto \mathbf{x}^b y \in M_{a+b}$ is bijective for all $a \in \mathbb{Z}^d$ and $b \in C$ with $\mathrm{supp}(a+b) = \mathrm{supp}(a)$.

A finitely generated module M is straight if and only if $M \cong \omega_R^{\oplus t}$ for some $t \in \mathbb{N}$. The injective hull ${}^*E(R/P_F)$ of R/P_F in ***Mod** is a straight module.

Denote by **Str** the full subcategory of ***Mod** consisting of all straight modules. **Str** is an abelian category (even if R is not simplicial). Let $M = \bigoplus_{a \in \mathbb{Z}^d} M_a$ be a

$\mathbb{Z}^d$-graded R-module. We call the submodule $\bar{M} := \bigoplus_{a \in C} M_a$ the *C-graded part* of M. It is easy to see that if M is straight then $\bar{M}$ is squarefree.

Proposition 6.2. *Suppose that R is simplicial. For a squarefree module N, there is a unique straight module M such that $\bar{M} \cong N$. From this correspondence, the categories* **Sq** *and* **Str** *are equivalent.*

If R is not simplicial, there is a non-zero straight module M with $\bar{M} = 0$.

If R is simplicial, **Str** is an abelian category with enough projectives and injectives. An indecomposable injective object in **Str** is ${}^*E(R/P_F)$ for some $F \in L$.

Theorem 6.3. *Assume that R is simplicial, and M is a straight R-module. For a face $F \in L$ and $c \in C$ with $s(c) = F$, we have*

$$\mu^i(P_F, M) = \dim_k[\operatorname{Ext}_R^{d-i-\dim F}(\bar{M}, \omega_R)]_c.$$

Here $\bar{M}$ is the C-graded part of M.

As in the polynomial ring case, we can compute $\mu^i(P, M)$ at an arbitrary prime ideal P, using Theorem 6.3 and a principle of [4].

If R is simplicial, the local cohomology $H_I^i(M)$ of a straight module M with supports in a monomial ideal I is straight again, as in the polynomial ring case.

Even if R is not simplicial, a $\mathbb{Z}^d$-graded minimal injective resolution of ω_R consists of straight modules ${}^*E(R/P_F)$ (without degree shifting), see (6.1) below. By the same argument to the polynomial ring case, we have the following.

Theorem 6.4. *Let I be a radical monomial ideal of R. Then $H_I^i(\omega_R)$ is a straight module whose C-graded part is isomorphic to $\operatorname{Ext}_R^i(R/I, \omega_R)$.*

By Theorems 6.3 and 6.4, we have the following.

Corollary 6.5. *Suppose that R is simplicial and I is a radical monomial ideal of R. For all $F \in L$ and $c \in C$ with $s(c) = F$, we have*

$$\mu^i(P_F, H_I^j(\omega_R)) = \dim_k[\operatorname{Ext}_R^{d-i-\dim F}(\operatorname{Ext}_R^j(R/I, \omega_R), \omega_R)]_c < \infty.$$

If R is not simplicial, Corollary 6.5 is not true. In fact, if $I := (x, y) \subset R := k[x, y, z, w]/(xz - yw)$, then $\mu^0(\mathfrak{m}, H_I^2(S)) = \infty$ for $\mathfrak{m} := (x, y, z, w)$, see [6, §3]. Note that R is a normal semigroup ring, but not simplicial.

Finally, we will give a formula on the Hilbert function of $H_{I_\Delta}^i(\omega_R)$. Recall that a $\mathbb{Z}^d$-graded minimal injective resolution ${}^*D^\bullet$ of ω_R is given by

$$(6.1) \qquad 0 \to {}^*D^0 \to {}^*D^1 \to \cdots \to {}^*D^d \to 0,$$

$${}^*D^i = \bigoplus_{\substack{F \in L \\ \dim F = d-i}} {}^*E(R/P_F).$$

Let T be a cross-section of the cone $\mathbb{R}_+C$. Then T is a polygon of dimension $d-1$. For a face $F \in L$ of $\mathbb{R}_+C$, $\bar{F}$ denotes the face $T \cap F$ of T. By this correspondence, the face lattice $\bar{L}$ of T is isomorphic to L as a poset. We can regard $\bar{L}$ as a *finite regular cell complex* (c.f. [2, §6.2]). Let $\tilde{C}$ be the augmented oriented chain complex of $\bar{L}$. For an order ideal $\Delta \subset L$ and $a \in \mathbb{Z}^d$, set $\tilde{\mathcal{C}}(\Delta) := \bigoplus_{F \in \Delta} k\bar{F}$ be a subcomplex of $\tilde{\mathcal{C}}$, and

$$\tilde{\mathcal{C}}(\Delta_a) := \bigoplus_{\substack{F \in \Delta \\ \mathrm{supp}(F) \not\supset \mathrm{supp}(a)}} k\bar{F}$$

be a subcomplex of $\tilde{\mathcal{C}}(\Delta)$. We also set $\tilde{\mathcal{C}}(\Delta, \Delta_a) := \tilde{\mathcal{C}}(\Delta)/\tilde{\mathcal{C}}(\Delta_a)$.

Theorem 6.6. *With the above notation, we have* $H^i_{I_\Delta}(\omega_R)_a = H_{d-i-1}(\tilde{\mathcal{C}}(\Delta, \Delta_a))$ *for all* $i \geq 0$ *and* $a \in \mathbb{Z}^d$.

If R is simplicial, the chain complex $\tilde{\mathcal{C}}(\Delta, \Delta_a)$ is isomorphic to the augmented chain complex of $\mathrm{lk}_F \Delta$ for some $F \in L$. Thus, when R is a polynomial ring, the above formula is nothing other than Terai's formula in [13].

Using Theorem 6.6, we can get a simple combinatorial proof of "Hartshorne-Lichtenbaum vanishing theorem" [5, Theorem 3.1] for a normal semigroup ring and a monomial ideal (when R is Gorenstein).

Corollary 6.7. *If* $I_\Delta \neq \mathfrak{m}$ *(equivalently* $\Delta \neq \{\phi\}$*), then* $H^d_{I_\Delta}(\omega_R) = 0$.

Proof. If $\Delta \neq \{\phi\}$, then $H_{-1}(\tilde{\mathcal{C}}(\Delta)) = H_{-1}(\tilde{\mathcal{C}}(\Delta, \Delta_a)) = 0$ for all $a \in \mathbb{Z}^d$. □

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MUTATIONS OF EXCEPTIONAL SEQUENCES ON GRADED GORENSTEIN RINGS

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The concept of an exceptional sequence on the category of coherent sheaves was developed in [2] and generalized in [1]. We define it in the context of commutative algebra similarly on the category of coherent sheaves.

Let R be an **N**-graded Gorenstein local ring and $R_0 = k$ is an algebraically closed field.

We denote by $\operatorname{mod} R$ the category of finitely generated **Z**-graded R-modules with morphisms that are degree preserving maps, by $\operatorname{CM} R$ the full subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay modules, and by $\mathfrak{D}^b(\operatorname{mod} R)$ the derived category of bounded complexes.

Definition 1. For $E^\cdot \in \mathfrak{D}^b(\operatorname{mod} R)$, $E^\cdot$ is called *exceptional* provided $\operatorname{RHom}(E^\cdot, E^\cdot)$ is isomorphic to k.

A sequence of exceptional complexes $\varepsilon = (\cdots, E^\cdot_{i-1}, E^\cdot_i, E^\cdot_{i+1}, \cdots)$ is called an *exceptional sequence* provided $\operatorname{RHom}(E^\cdot_i, E^\cdot_j) = 0$ for $i > j$.

Example 2.
1. Let $R = k[x]$ be a polynomial ring in one variable. Then any indecomposable module is exceptional, and $\varepsilon = (\cdots, R(-1), R, R(1), \cdots)$ is an exceptional sequence.
2. Let R be a 1-dimensional **N**-graded Gorenstein local ring of finite Cohen-Macaulay type, then any indecomposable maximal Cohen-Macaulay module is exceptional.

Definition 3. For $E^\cdot$ and $F^\cdot \in \mathfrak{D}^b(\operatorname{mod} R)$, there exists a canonical map from $\operatorname{RHom}(E^\cdot, F^\cdot) \otimes_k E^\cdot$ to $F^\cdot$. The left mutation $\mathfrak{L}_{E^\cdot} F^\cdot$ of $F^\cdot$ by $E^\cdot$ is defined by the triangle;

$$\operatorname{RHom}(E^\cdot, F^\cdot) \otimes_k E^\cdot \to F^\cdot \to \mathfrak{L}_{E^\cdot} F^\cdot$$

Dually,the right mutation $\mathfrak{R}_{F^\cdot} E^\cdot$ of $E^\cdot$ by $F^\cdot$ is defined by the triangle;

$$\mathfrak{R}_{F^\cdot} E^\cdot \to E^\cdot \to \operatorname{D}\operatorname{RHom}(E^\cdot, F^\cdot) \otimes_k F^\cdot$$

where D is the k-dual.

Let B be the braid group on infinite strings, having generators σ_i ($i \in \mathbf{Z}$), with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all i and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j-i| \geq 2$. Then B acts on the set of exceptional sequences by

$$\sigma_i\varepsilon = (\cdots, \mathrm{E}^{\cdot}_{i-1}, \mathrm{E}^{\cdot}_{i+1}, \mathfrak{R}_{\mathrm{E}^{\cdot}_{i+1}}\mathrm{E}^{\cdot}_{i}, \mathrm{E}^{\cdot}_{i+2}\cdots)$$
$$\sigma_i^{-1}\varepsilon = (\cdots, \mathrm{E}^{\cdot}_{i-1}, \mathfrak{L}_{\mathrm{E}^{\cdot}_{i}}\mathrm{E}^{\cdot}_{i+1}, \mathrm{E}^{\cdot}_{i}, \mathrm{E}^{\cdot}_{i+2}\cdots)$$

A direct sum $\mathbf{Z}^\infty$ of $\mathbf{Z}$ also acts on the set of exceptional sequences by

$$e_i\varepsilon = (\cdots, \mathrm{E}^{\cdot}_{i-1}, \mathrm{E}^{\cdot}_{i}[1], \mathrm{E}^{\cdot}_{i+1}, \cdots)$$

where e_i is the natural basis on $\mathbf{Z}^\infty$. So, if G is the semi direct product $B \ltimes \mathbf{Z}^\infty$, then G acts on the set of exceptional sequences.

Definition 4. Let ε be an exceptional sequence.

1. We say that an exceptional module E is generated by ε provided there exists σ in G such that E is contained in $\sigma\varepsilon$.
2. ε is called *generating* provided any exceptional module is generated by ε.

Problem 5. Does there exist a generating exceptional sequence ?

Answer 6. 1. It is yes if $R = k[x]$ is a polynomial ring. In this case $(\cdots, R(-1), R, R(1), \cdots)$ is a generating exceptional sequence.

2. If R is a 1-dimensional $\mathbf{N}$-graded Gorenstein local ring of finite Cohen-Macaulay type, then there exists a generating exceptional sequence.

I will give an explanation on this second answer in my lecture.

Theorem 7. *If R is a 1-dimensional $\mathbf{N}$-graded Gorenstein local ring of finite Cohen-Macaulay type, then there exists a generating exceptional sequence.*

Since R is a 1-dimensional $\mathbf{N}$-graded Gorenstein local ring of finite Cohen-Macaulay type, R is isomorphic to one of following types (c.f.[4]);

$$\begin{array}{ll} (A_n) & R = k[x,y]/(y^2 - x^n) \\ (D_n) & R = k[x,y]/(xy^2 - x^n) \\ (E_6) & R = k[x,y]/(x^3 + y^4) \\ (E_7) & R = k[x,y]/(x^3 + xy^3) \\ (E_8) & R = k[x,y]/(x^3 + y^5) \end{array}$$

The Auslander-Reiten quiver of CM R for each type are shown in figures (1)-(7). We take the exceptional sequence ε_0 as in figures (1)-(7).

Claim 8. *The exceptional sequence ε_0 is generating.*

To show this claim, we need two lemmas.

Lemma 9. *If* $0 \to X \to \oplus_{i=1}^{m} Y_i^{r_i} \to Z \to 0$ *is Auslander-Reiten sequence in* CM R, *then* $\mathfrak{R}_{Y_1}\mathfrak{R}_{Y_2}\cdots\mathfrak{R}_{Y_m}X$ *is isomorphic to* $Z[-1]$ *and* $\mathfrak{L}_{Y_1}\mathfrak{L}_{Y_2}\cdots\mathfrak{L}_{Y_m}Z$ *is isomorphic to* $X[1]$.

Lemma 10. *Let* E *be an exceptional module. If* E *is not a maximal Cohen-Macaulay module, then there exists a maximal Cohen-Macaulay module* X *and integer* n *such that* $0 \to X \to R(n)^r \to E \to 0$ *is an exact squence where* $r = \dim_k \mathrm{Hom}(X, R(n)) = \dim_k \mathrm{RHom}(X, R(n))$, *i.e.* $E \cong \mathfrak{R}_{R(n)}X[1]$

By lemma 9, any exceptinal maximal Cohen-Macaulay modules are generated by ε_0, and by lemma 10, any exceptinal non maximal Cohen-Macaulay modules are generated by ε_0. Therefore ε_0 is generating.

Conjecture 11. Let ε be an exceptional sequence. The following claims would be true.

1. ε is generating if and only if for any exceptinal maximal Cohen-Macaulay modules are generated by ε.
2. G acts transitively on the set of generating exceptional sequences.
3. ε is generating if and only if it is maximal. Here we say that ε is *maximal* provided there is no exceptional module that we can add into ε as an exceptional sequence.

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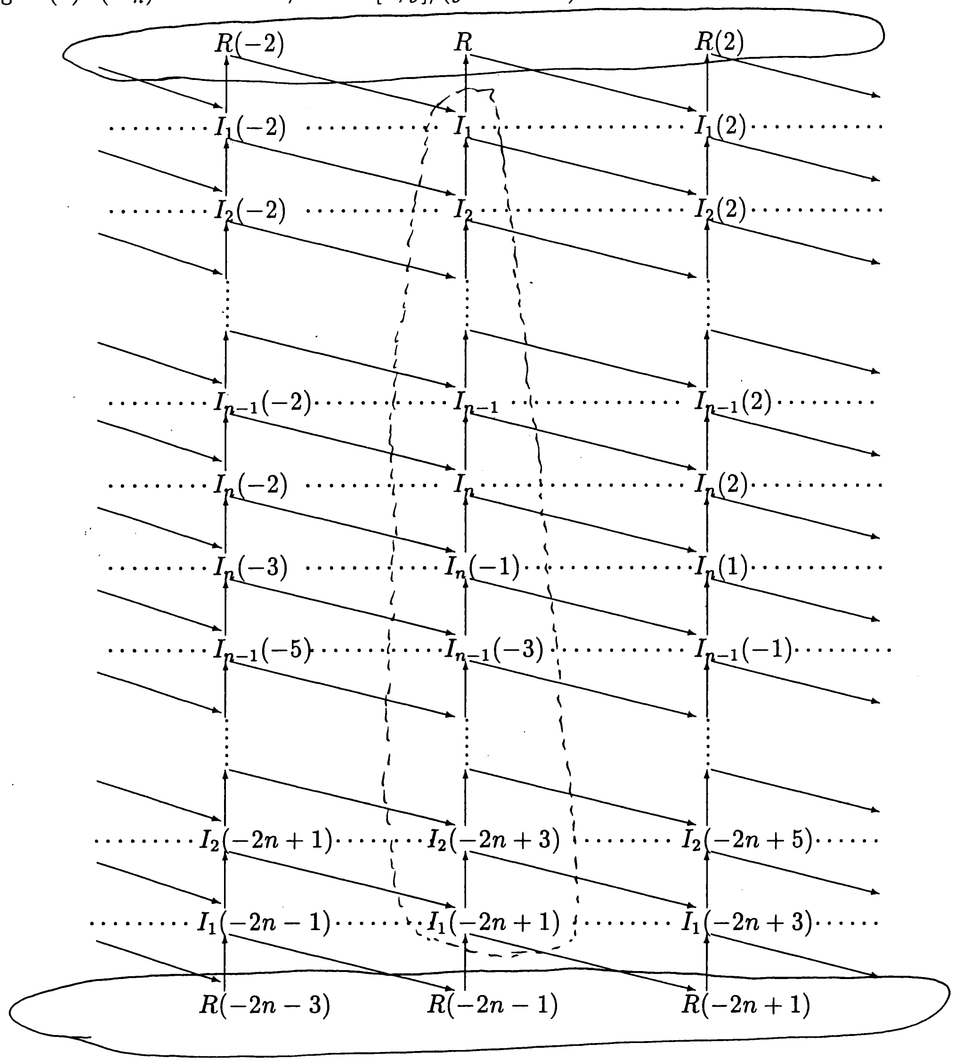
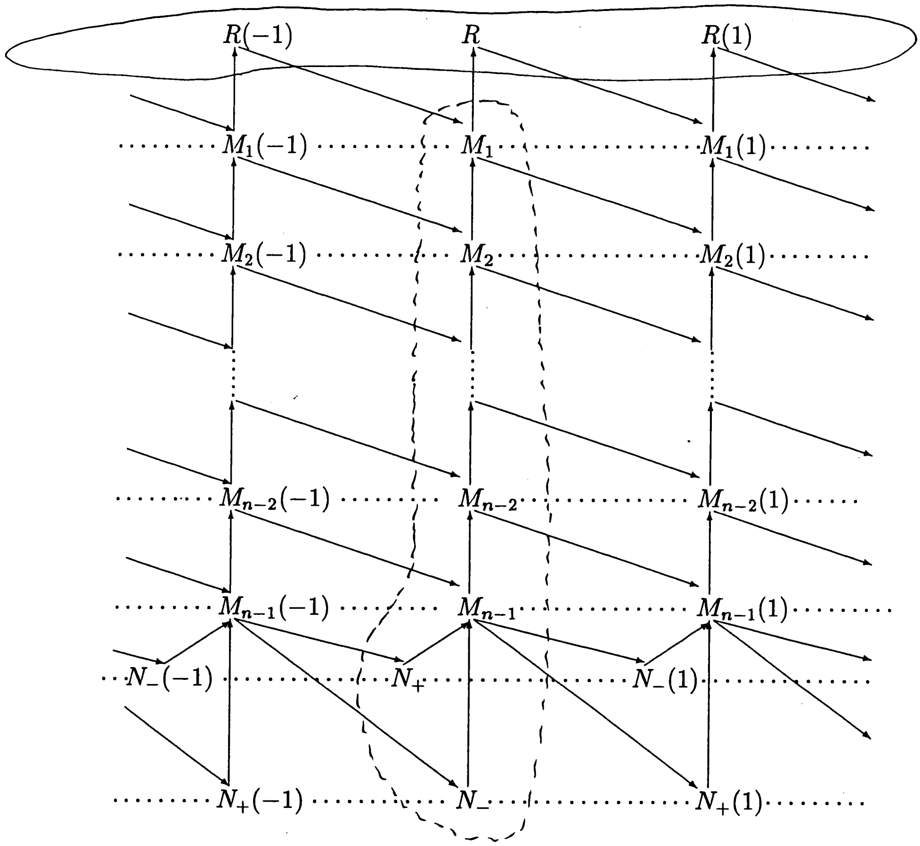
figure(1): $(\mathbf{A}_n)$ *with n odd,* $R = k[x,y]/(y^2 - x^{2n+1})$



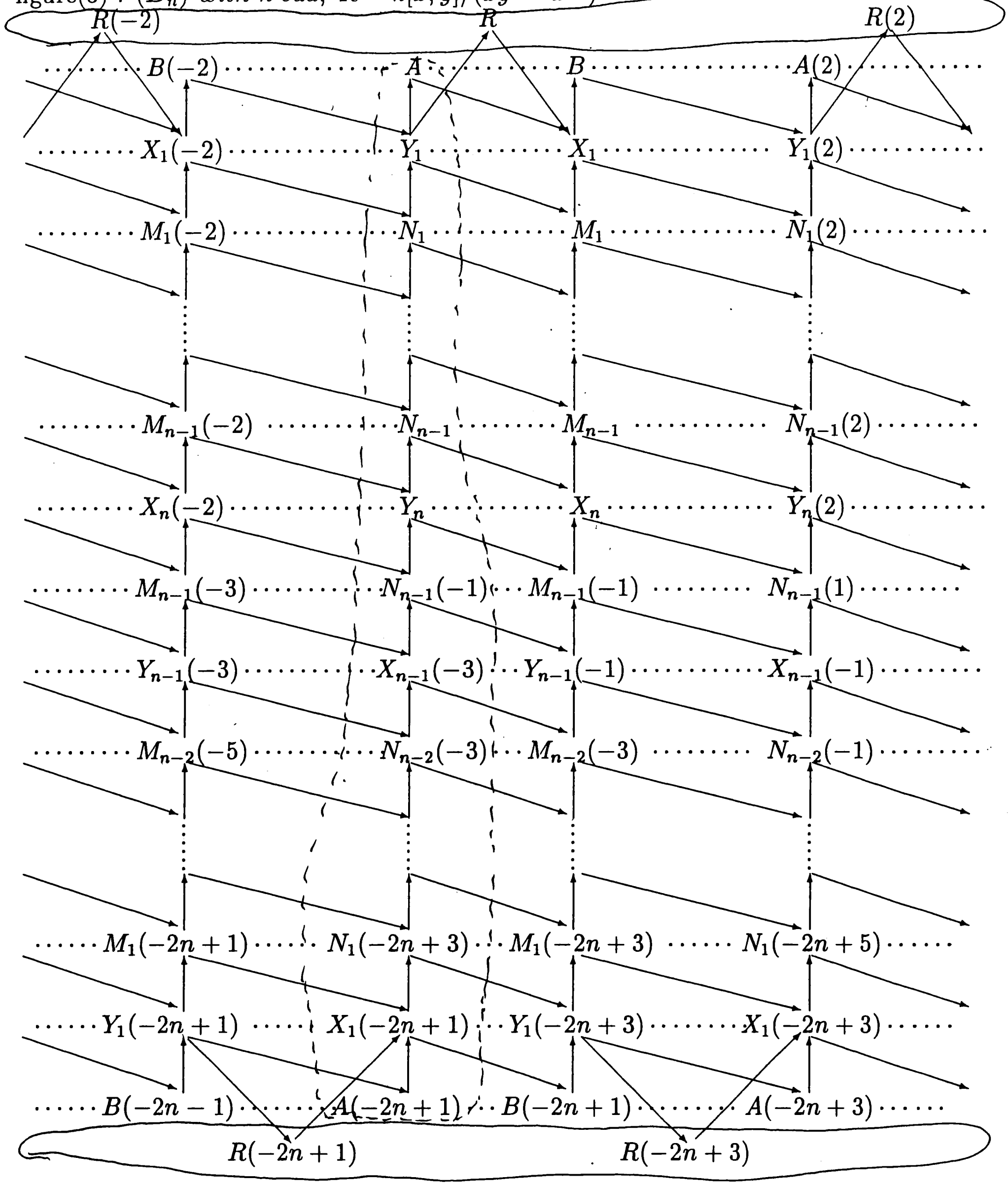
$$\varepsilon_0 = (\cdots, R(-2n-3), R(-4), R(-2n-1), R(-2), I_1(-2n+1), I_2(-2n+3), \cdots, I_n(-1), I_n, \cdots, I_2, I_1, R(-2n+1), R, R(-2n+3), R(2), \cdots)$$

figure(2) : $(\mathbf{A}_n)$ *with n even*, $R = k[x,y]/(y^2 - x^{2n})$



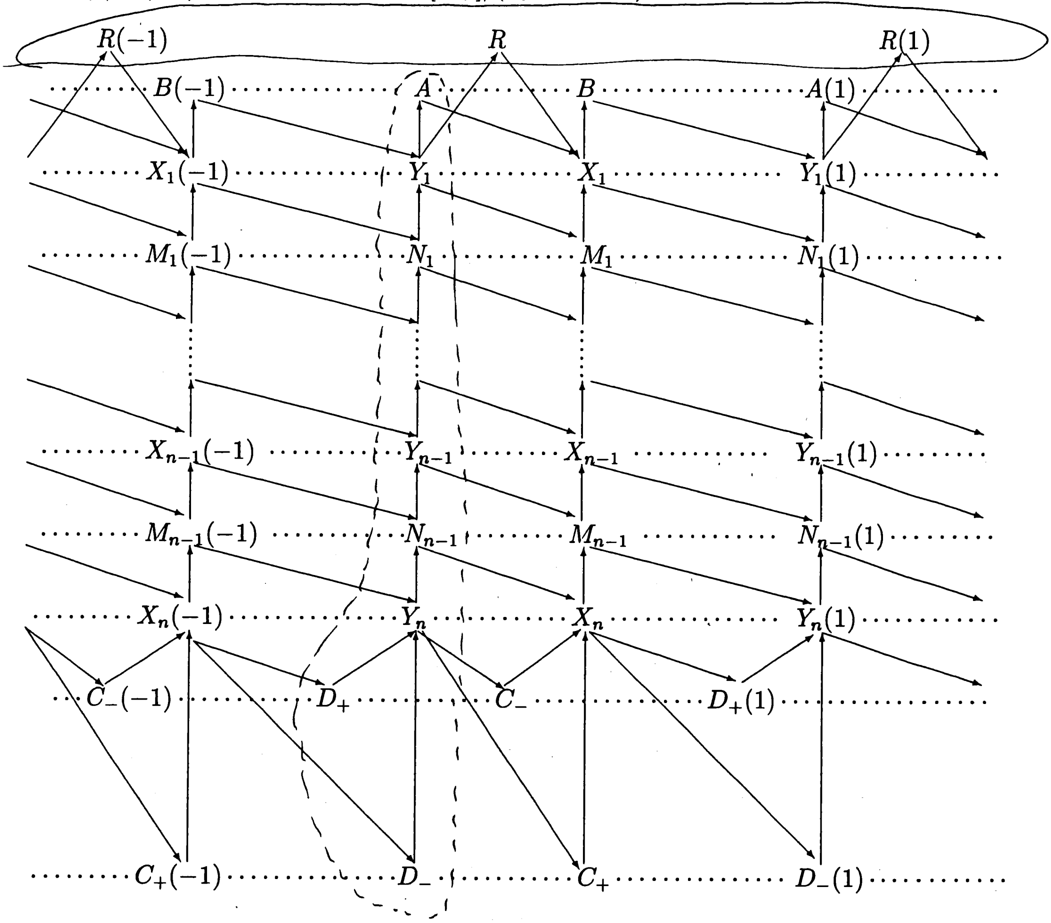
$$\varepsilon_0 = (\cdots, R(-2), R(-1), N_+, N_-, M_{n-1}, M_{n-2}, \cdots, M_1, R, R(1), \cdots)$$

figure(3) : $(\mathbf{D}_n)$ *with n odd*, $R = k[x,y]/(xy^2 - x^{2n})$



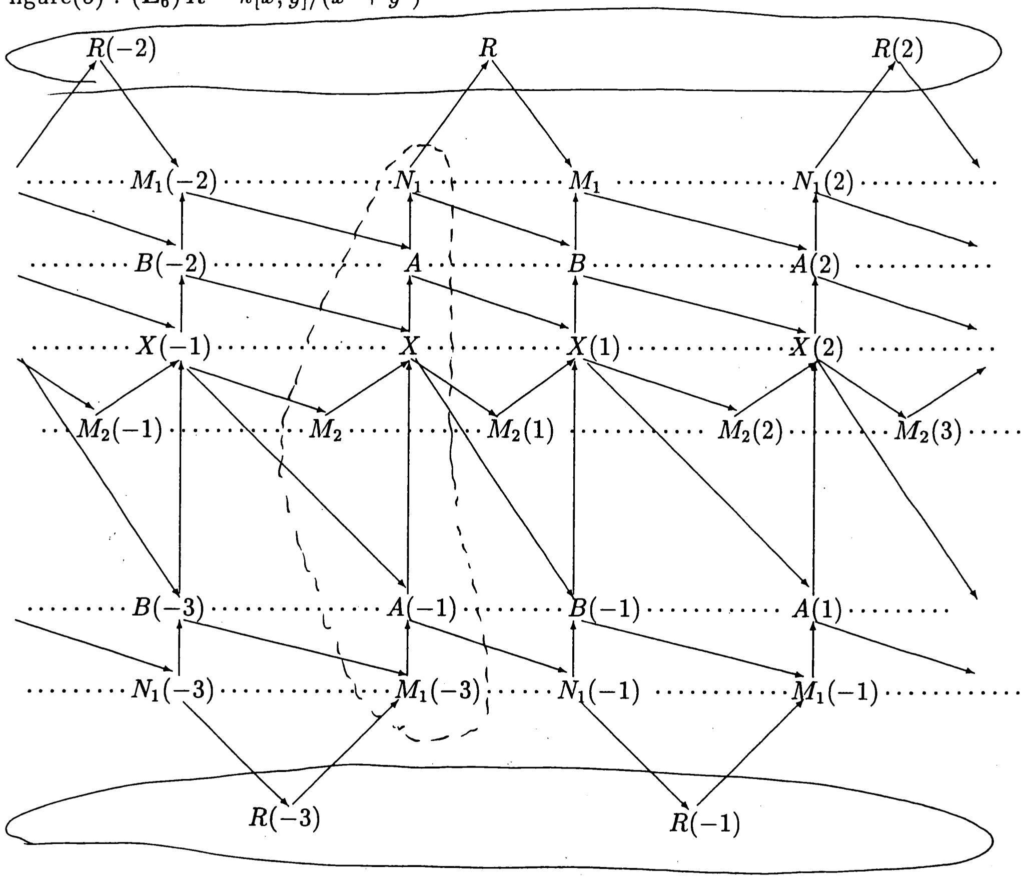
$$\varepsilon_0 = (\cdots, R(-2), R(-2n+1), A(-2n+1), X_1(-2n+1), \cdots, A, R, R(-2n+3), R(2), \cdots)$$

figure(4) : $(\mathbf{D}_n)$ *with n even,* $R = k[x,y]/(xy^2 - x^{2n+1})$



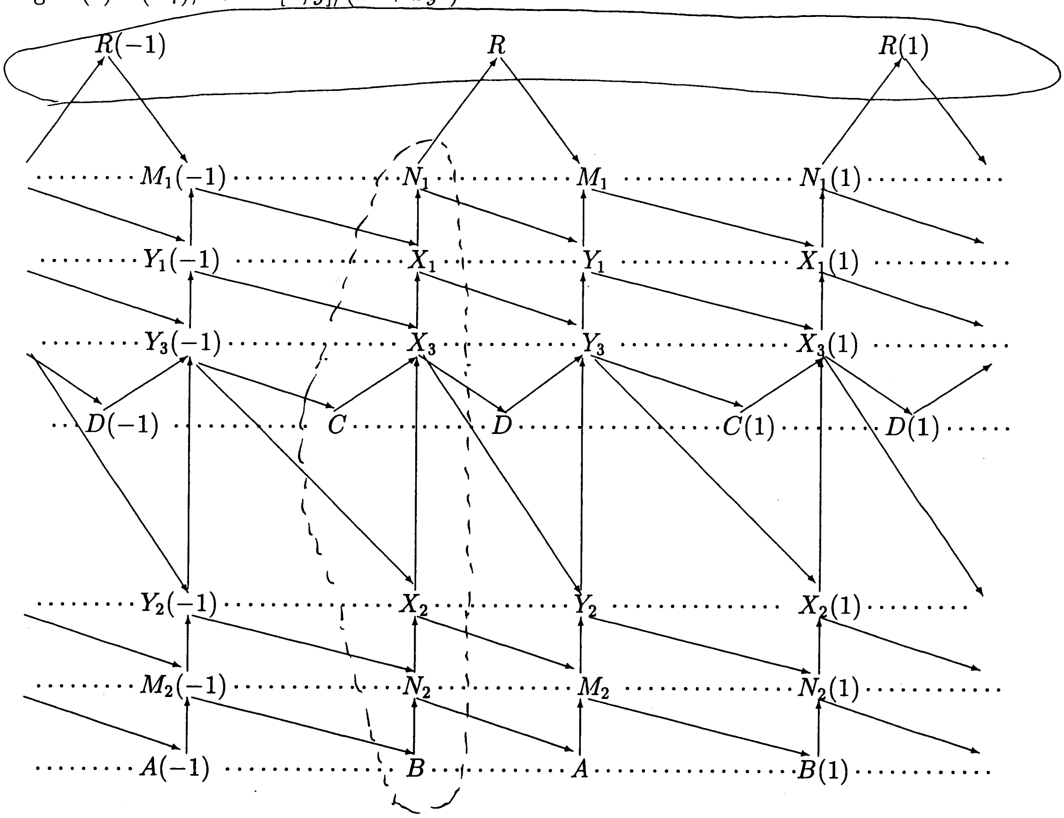
$$\varepsilon_0 = (\cdots, R(-2), R(-1), D_+, D_-, Y_n, N_{n-1}, \cdots, Y_1, A, R, R(1), \cdots)$$

figure(5) : $(\mathbf{E}_6)\, R = k[x,y]/(x^3+y^4)$



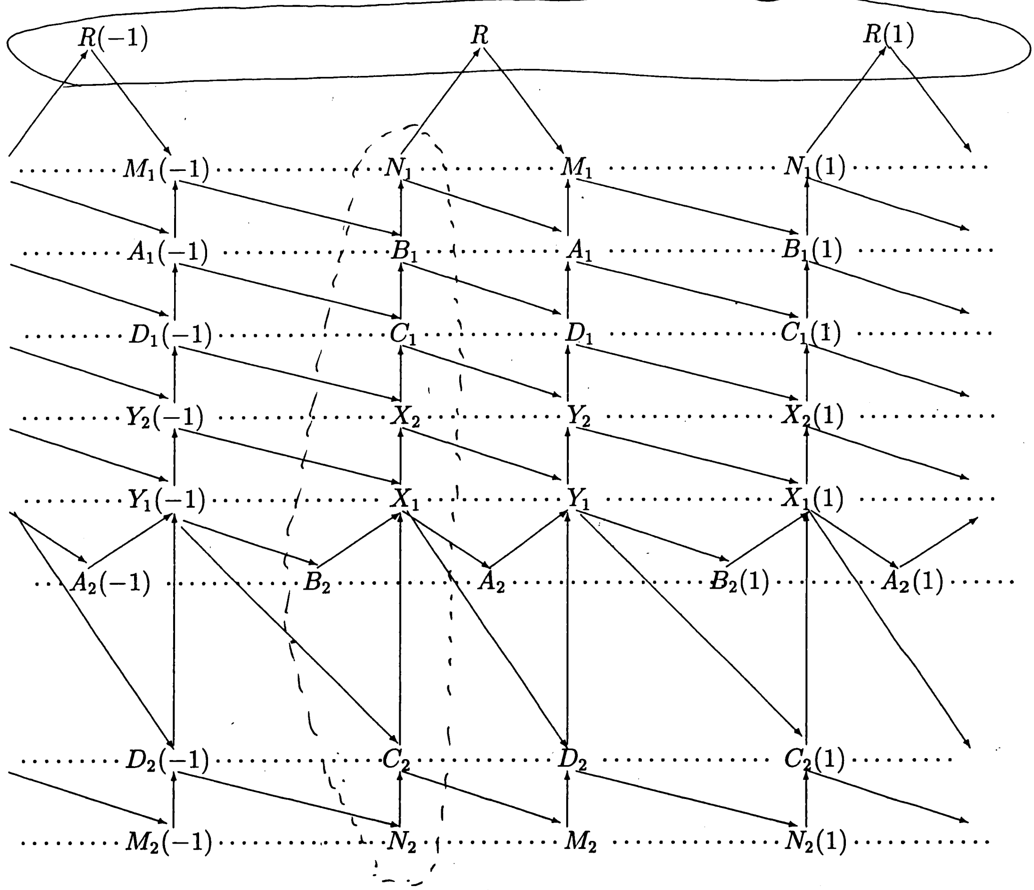
$$\varepsilon_0 = (\cdots, R(-5), R(-2), R(-3), M_1(-3), A(-1), M_2, X, A, N_1, R, R(-1), R(2), \cdots)$$

figure(6) : $(\mathbf{E}_7)$, $R = k[x,y]/(x^3 + xy^3)$



$$\varepsilon_0 = (\cdots, R(-2), R(-1), B, N_2, X_2, C, X_3, X_1, N_1, R, R(1), \cdots)$$

figure(7) : $(\mathbf{E}_8)$, $R = k[x,y]/(x^3+y^5)$



$$\varepsilon_0 = (\cdots, R(-2), R(-1), N_2, C_2, B_2, X_1, X_2, C_1, B_1, N_1, R, R(1), \cdots).$$

A note on graded Betti numbers of Gorenstein rings

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This note is a brief summary of a joint work [7, Sections 6~8] with Professors A. V. Geramita and Y. S. Shin. We study graded Betti numbers of some Artinian Gorenstein rings obtained from a construction of linkage theory.

1. PRELIMINARIES

1.1. Let A be a (*standard*) *graded ring* over a field k, namely, A is a graded ring $\oplus_{i\geq 0}A_i$ satisfying $A_0 = k$, $A = k[A_1]$ and $\dim_k A_1 < \infty$. This means that there is an integer $n > 0$ such that $A = R/I$ where $R = k[x_0, x_1, \ldots, x_n] = \oplus_{i\geq 0}R_i$ is the polynomial ring with the standard grading, i.e., each $\deg x_i = 1$, and $I = \oplus_{i\geq 0}I_i$ is a homogeneous ideal of R.

The *Hilbert function* of A is defined by the numerical function $\mathbf{H}(A, -) : \mathbf{N} \longrightarrow \mathbf{N}$ with

$$\mathbf{H}(A, i) := \dim_k A_i = \dim_k R_i - \dim_k I_i$$

for all $i \geq 0$, in particular $\mathbf{H}(A, 0) = 1$. When $A = \bigoplus_{i\geq 0} A_i$ is Artinian, we put

$$s(A) = \mathrm{Max}\{i \mid A_i \neq (0)\}$$

and we call $s(A)$ the *socle degree* of A. In this case we denote by the finite sequence

$$\mathbf{H}(A) = (\mathbf{H}(A, 0), \mathbf{H}(A, 1), \ldots, \mathbf{H}(A, s(A)))$$

the Hilbert function of A. A finite sequence $(h_0, h_1, \ldots, h_s)$ is *unimodal* if there is an integer ℓ such that

$$h_0 \leq h_1 \leq \cdots \leq h_\ell \geq h_{\ell+1} \geq \cdots \geq h_s.$$

The *Hilbert series* of A is defined by the formal power series

$$F(A, \lambda) = \sum_{i=0}^{\infty} H(A, i)\lambda^i \in \mathbf{Z}[[\lambda]].$$

Let

$$0 \longrightarrow \bigoplus_{j=1}^{b_r} R(-j)^{\beta_{r,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} R(-j)^{\beta_{1,j}} \longrightarrow R \longrightarrow A \longrightarrow 0$$

be a graded minimal free resolution of A as a graded module over R. The numbers $\{\beta_{i,j}\}$ are uniquely determined by A, namely,

$$\beta_{i,j} = \dim_k \mathrm{Tor}_i^R(A, k)_j$$

for all (i, j). So we call $\beta_{i,j}$ the (i, j)-th graded *Betti number* of A. The Betti numbers of A determine the Hilbert series (i.e., the Hilbert function) of A by

$$F(A, \lambda) = \frac{1 + \sum_{i=1}^{r} \sum_{j=1}^{b_i} (-1)^i \lambda^{\beta_{i,j}}}{(1-\lambda)^{n+1}}.$$

The converse is not necessarily true. However we notice from many examples that the Hilbert function of an algebra gives some restrictions on the Betti numbers of the algebra.

1.2. Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in the projective space $\mathbb{P}^n := \mathbb{P}^n(k)$ (where $k = \bar{k}$ is an algebraically closed field). Then $P_i \leftrightarrow \wp_i = (L_{i1}, \ldots, L_{in}) \subset R = k[x_0, x_1, \ldots, x_n]$ where the L_{ij}, $j = 1, \ldots, n$, are n linearly independent linear forms and $\wp_i$ is the (homogeneous) prime ideal of R generated by all the forms which vanish at P_i. The ideal

$$I = I_{\mathbb{X}} := \wp_1 \cap \cdots \cap \wp_s$$

is the ideal generated by all the forms which vanish at all the points of $\mathbb{X}$.

The Hilbert function of $\mathbb{X}$ is defined by

$$\mathbf{H}_{\mathbb{X}}(i) := \mathbf{H}(R/I, i)$$

and we put

$$\sigma(\mathbb{X}) := \mathrm{Min}\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mid \mathbb{X} \mid\}$$

where $\mid \mathbb{X} \mid$ denote the number of points on $\mathbb{X}$. Since A is 1-dimensional Cohen-Macaulay, we can check that

$$\mathbf{H}_{\mathbb{X}}(0) < \cdots < \mathbf{H}_{\mathbb{X}}(\sigma(\mathbb{X}) - 1) = \mathbf{H}_{\mathbb{X}}(\sigma(\mathbb{X})) = \cdots = \mid \mathbb{X} \mid .$$

Let $\mathbb{Z}$ be a finite set of points in $\mathbb{P}^n$. We say that $\mathbb{Z}$ is a *complete intersection of type* $(a_1, \ldots, a_n)$ if $I_{\mathbb{Z}} = (f_1, \ldots, f_n)$ for some $f_i \in R_{a_i}$ $(1 \leq i \leq n)$. In this case we have $\sigma(\mathbb{Z}) = a_1 + \cdots + a_n - n + 1$ and $\mid \mathbb{X} \mid = \prod_{i=1}^{n} a_i$.

2. Hilbert functions and Betti numbers of some Artinian Gorenstein rings

First we give some description of Hilbert functions and Betti numbers of Artinian Gorenstein graded rings obtained from the construction of the following lemma which is a special case of Remark 1.4 in [17].

Lemma 1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be finite sets of points in $\mathbb{P}^n$ such that $\mathbb{X} \cap \mathbb{Y} = \emptyset$ and $\mathbb{X} \cup \mathbb{Y}$ is complete intersection. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinian Gorenstein graded ring.*

Theorem 1 (Theorems 6.1, 6.5 and 7.1, [7]). *With the same notation as in Lemma 1, we have*

1) $s := s(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{X} \cup \mathbb{Y}) - 2$ *and*
2) *the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is unimodal.*

Moreover, assume that $2\sigma(\mathbb{X}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$. Then we get the following.

3)

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), i) = \begin{cases} \mathbf{H}(\mathbb{X}, i), & \text{for all } i = 0, 1, \ldots, [s/2], \\ \mathbf{H}(\mathbb{X}, s-i), & \text{for all } i = [s/2]+1, \ldots, s. \end{cases}$$

4)

$$[\mathrm{Tor}_i^R(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), k)]_j \cong [\mathrm{Tor}_i^R(R/I_{\mathbb{X}}, k)]_j \oplus [\mathrm{Tor}_{n+1-i}^R(R/I_{\mathbb{X}}, k)]_{s+1+n-j}$$

for all i, j, that is,

$$\beta_{ij}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \beta_{ij}(\mathbb{X}) + \beta_{n+1-i\ s+1+n-j}(\mathbb{X}).$$

Remark. The assertion 4) on Betti numbers is a slight generation of a wonderful result, first discovered by M. Boij [3].

Remark. The equality 3) on Hilbert functions is not necessarily hold without assuming $2\sigma(\mathbb{X}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$. For example, we can show it by taking the following two basic configurations as the sets $\mathbb{X}$ and $\mathbb{Y}$.

• • • * * * $\mathbb{Y}$
• • • * * *
$\mathbb{X}$ • • • * * *

Then we can check that $\sigma(\mathbb{X}) = \sigma(\mathbb{Y}) = 5, \sigma(\mathbb{X} \cup \mathbb{Y}) = 8$ and

$$\mathbf{H}(\mathbb{X}) = \mathbf{H}(\mathbb{Y}) \ : \ 1 \ \ 3 \ \ 6 \ \ 8 \ \ 9 \ \ 9 \ \ 9 \ \ 9 \ \ \cdots$$

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \ : \ 1 \ \ 3 \ \ 6 \ \ 7 \ \ 6 \ \ 3 \ \ 1 \ \ 0 \ \ \cdots,$$

i.e., $\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), 3) \neq \mathbf{H}(\mathbb{X}, 3)$. However we do not know such an example for the equality 4). So we have a question.

Question. Does the equality 4) on Betti numbers hold without assuming $2\sigma(\mathbb{X}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$?

Next, taking care to control the sets $\mathbb{X}$ and $\mathbb{Y}$ in Theorem 1, we would like to discuss a problem on maximal Betti numbers of Artinian Gorenstein rings.

Definition (n-type vectors). We do this inductively.

1) A 0-*type vector* is defined to be $\mathcal{T} = 1$. It is the only 0-*type vector.* Then we define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.

2) A 1-*type vector* is a vector of the form $\mathcal{T} = (d)$ where d is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d$ and $\sigma(\mathcal{T}) = d$.

3) A 2-*type vector* is an ordered collection

$$\mathcal{T} = ((d_1), (d_2), \ldots, (d_u))$$

of 1-type vectors $(d_1, \ldots, d_u)$ such that $\sigma(d_i) < \alpha(d_{i+1})$, i.e., $d_i < d_{i+1}$ for $i = 1, \ldots, u-1$. For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = u$ and $\sigma(\mathcal{T}) = \sigma((d_u)) = d_u$.

4) Now let $n \geq 3$. An n-*type vector* is an ordered collection

$$\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$$

of $(n-1)$-type vectors $\mathcal{T}_1, \ldots, \mathcal{T}_u$ such that $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \ldots, u-1$. For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = u$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_u)$.

Definition (k-configuration).

1) Let $\mathcal{T}$ be a 0-type vector. A single point in $\mathbb{P}^0$ is a *k-configuration in $\mathbb{P}^0$ of type $\mathcal{T}$.*

2) Let $\mathcal{T} = (d)$ be a 1-type vector. A set of d distinct points in $\mathbb{P}^1$ is called a *k-configuration in $\mathbb{P}^1$ of type $\mathcal{T}$.*

3) Let $\mathcal{T} = ((d_1), \ldots, (d_u))$be a 2-type vector. A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a k-configuration in $\mathbb{P}^2$ of type $\mathcal{T}$ if there exist subsets $\mathbb{X}_1, \ldots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct u lines $\mathbb{L}_1, \ldots, \mathbb{L}_u$ in $\mathbb{P}^2$ such that
   i) $\mathbb{X} = \cup_{i=1}^{u} \mathbb{X}_i$,
   ii) $\mathbb{X}_i \subset \mathbb{L}_i$ ($\cong \mathbb{P}^1$) is a k-configuration in $\mathbb{P}^1$ of type (d_i) for all $1 \leq i \leq u$,
   iii) Every $\mathbb{L}_i$ does not contain any point of $\mathbb{X}_j$ for all $j < i$.

4) Now suppose that we have defined a k-configuration in $\mathbb{P}^{n-1}$ of type $\widetilde{\mathcal{T}}$, where $\widetilde{\mathcal{T}}$ is an $(n-1)$-type vector.

   Let $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$ be a n-type vector. A finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ is called a k-configuration in $\mathbb{P}^n$ of type $\mathcal{T}$ if there exist subsets $\mathbb{X}_1, \ldots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct u hyperplanes $\mathbb{H}_1, \ldots, \mathbb{H}_u$ in $\mathbb{P}^n$ such that
   i) $\mathbb{X} = \cup_{i=1}^{u} \mathbb{X}_i$,
   ii) $\mathbb{X}_i \subset \mathbb{H}_i$ ($\cong \mathbb{P}^{n-1}$) is a k-configuration in $\mathbb{P}^{n-1}$ of type $\mathcal{T}_i$ for all $1 \leq i \leq u$,
   iii) Every $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for all $j < i$.

Remark. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$. Then the Hilbert function and the Betti numbers of $R/I_{\mathbb{X}}$ are completely determined by *Type*($\mathbb{X}$). Hence, it follows from Theorem 1 that the Hilbert function and the Betti numbers of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are also completely determined by *Type*($\mathbb{X}$). (For the details, please see [7].)

3. Maximal Betti numbers and the weak Lefschetz property

Let $\mathcal{H}_n$ be the collection of all the possible Hilbert functions of Artinian Gorenstein rings of codimension $n+1$, and for a $\mathbf{H} \in \mathcal{H}_n$, we consider the collection $\mathcal{B}_n(\mathbf{H})$ of all the possible Betti sequences of Artinian Gorenstein rings with Hilbert function $\mathbf{H}$. We put a partial ordering on this set $\mathcal{B}_n(\mathbf{H})$ by saying that $\{\beta_{ij}\} \geq \{\gamma_{ij}\}$ if $\beta_{ij} \geq \gamma_{ij}$ for all i, j.

Definition. Let $\mathbf{H} \in \mathcal{H}_n$. A Betti sequence $\{\beta_{ij}\} \in \mathcal{B}_n(\mathbf{H})$ is *maximal* if $\{\beta_{ij}\} \geq \{\gamma_{ij}\}$ for all $\{\gamma_{ij}\} \in \mathcal{B}_n(\mathbf{H})$.

Problem. Let $\mathbf{H} \in \mathcal{H}_n$.

1) Does there exist the maximal Betti sequence in $\mathcal{B}_n(\mathbf{H})$?

2) If there exists, then find the unique maximal Betti sequence and show an example of an Artinian Gorenstein ring with that maximal Betti sequence for Hilbert function $\mathbf{H}$?

When $n = 1$ the answer is easy. There is only one Betti sequence for a given $\mathbf{H} \in \mathcal{H}_1$! When $n = 2$ the answer is not easy, but both a maximum and a minimum exist for every possible $\mathbf{H} \in \mathcal{H}_2$ (see [5]). Unfortunately, when $n > 2$ it seems nothing is known. However, since our linkage method we discussed above can be carried in any codimension, we have (rashly?) conjectured the following.

Conjecture. Let $\mathbb{X}$ be a *k-configuration* in $\mathbb{P}^n$ and $\mathbb{Y}$ a finite set of points such that $\mathbb{X} \cap \mathbb{Y} = \emptyset$, $\mathbb{X} \cup \mathbb{Y}$ is complete intersection and $2\sigma(\mathbb{X}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$. Let $\mathbf{H}$ be the Hilbert function of $R/I_{\mathbb{X}} + I_{\mathbb{Y}}$. Then the Betti sequence of $R/I_{\mathbb{X}} + I_{\mathbb{Y}}$ is maximal in $\mathcal{B}_n(\mathbf{H})$.

Our interest in this problem comes from the following fact for codimension 3.

Theorem 2. *Let* $n = 2$.

1) *Any Hilbert function in the set* $\mathcal{H}_2$ *actually occurs for some Artinian Gorenstein ring obtained from the construction of Conjecture* (see [13]).
2) *Let* $\mathbf{H}$ *be a possible Hilbert function in* $\mathcal{H}_2$. *The Betti sequence of an Artinian Gorenstein ring, with Hilbert function* $\mathbf{H}$, *obtained from the construction of Conjecture is maximal in* $\mathcal{B}_2(\mathbf{H})$ (see [11]). *Namely, the conjecture is true in the case* $n = 2$.

To give some evidence for our conjecture, we prove it, with an additional assumption, for the family of Artinian Gorenstein graded rings having the weak Lefschetz property.

Definition ([21]). Let $A = \oplus_{i=0}^{s} A_i$ be an Artinian ring. We say that A has the *weak Lefschetz property* if there is an element $g \in A_1$ such that $g : A_i \to A_{i+1}$ (multiplication by g) is injective or surjective for every i.

We notice by [14] that all the Artinian Gorenstein rings obtained from the construction of Conjecture have this property.

Theorem 3 (Theorem 8.2, [7]). *Let $R/I_{\mathbb{X}} + I_{\mathbb{Y}}$ be an Artinian Gorenstein ring obtained from the construction of Conjecture, and let* $\mathbf{H}$ *be the Hilbert function of the ring. Here we put an additional assumption,*

$$\sigma(\mathbb{X} \cup \mathbb{Y}) - 2\sigma(\mathbb{X}) \geq n.$$

Then

$$\beta_{ij}(R/I_{\mathbb{X}} + I_{\mathbb{Y}}) \geq \beta_{ij}(B)$$

for all i, j, where B is any Artinian Gorenstein ring with the weak Lefschetz property which have Hilbert function $\mathbf{H}$.

As J. Watanabe showed in [21] that almost all Artinian Gorenstein rings have the weak Lefschetz property, so this result makes our conjecture more interesting.

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EXTREMAL MINIMAL RESOLUTIONS AND DECOMPOSITIONS OF A FINITE SET OF POINTS IN $\mathbb{P}^n$

YONG SU SHIN

This is a part of a joint work with Professor A.V. Geramita and Professor T. Harima ([3], [4], and [5]).

Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct points in the projective space $\mathbb{P}^n(k)$ (where $k = \overline{k}$ is an algebraically closed field). Then $P_i \leftrightarrow \wp_i = (L_{i1}, \dots, L_{in}) \subset R = k[x_0, x_1, \dots, x_n]$ where the L_{ij}, $j = 1, \dots, n$ are n linearly independent linear forms and $\wp_i$ is the (homogeneous) prime ideal of R generated by all the forms which vanish at P_i. The ideal

$$I = I_{\mathbb{X}} := \wp_1 \cap \dots \cap \wp_s$$

is the ideal generated by all the forms which vanish at all the points of $\mathbb{X}$.

Since $R = \oplus_{i=0}^{\infty} R_i$ (R_i the vector space of dimension $\binom{i+n}{n}$ generated by all the monomials in R having degree i) and $I = \oplus_{i=0}^{\infty} I_i$ we get that

$$A = R/I = \oplus_{i=0}^{\infty}(R_i/I_i) = \oplus_{i=0}^{\infty} A_i$$

is a graded ring. The numerical function

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}(A, t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the set $\mathbb{X}$ (or of the ring A).

Let $I = I_{\mathbb{X}} \subset R = k[x_0, \dots, x_n]$ and let

$$\mathcal{F}_I : 0 \to \mathcal{F}_n \to \dots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. We write

$$\mathcal{F}_i = \bigoplus_{j=1}^{r_i} R(-a_{ij}) \text{ where } a_{i1} \le \dots \le a_{ir_i}$$

where the numbers a_{ij} are the *graded Betti numbers* of I. We set

$$\mathcal{B}_i := \{a_{i1}, \dots, a_{ir_i}\}$$

and call it the i-th *Betti numbers* of I and call

$$\mathcal{B}(I) := \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$$

the *Betti multiset* of I.

Now fix a Hilbert function $\mathbf{H}$ and consider

$$\mathcal{H}_{\mathbf{H}} := \{I \subset R \mid \mathbf{H}(R/I, t) = \mathbf{H}(t) \text{ for all } t\}$$

Let I and J be in $\mathcal{H}_{\mathbf{H}}$. We say

$$\mathcal{B}(I) \leq \mathcal{B}(J) \Leftrightarrow \mathcal{B}_i(I) \subseteq \mathcal{B}_i(J)$$

for all $1 \leq i \leq n$.

Theorem 1 ([6]). *If we follow the procedure above, then we always get the maximum Betti multiset for a codimension 3 Gorenstein Hilbert function.*

Question. What about codimension > 3?

In this case we have neither Stanley's result [11] nor that of Buchsbaum and Eisenbud [2]. However, we have the theorem of Bigatti-Hulett-Pardue which gives the maximum Betti multiset for points ([1], [9], and [10]).

In order to use points to make the maximum Betti multiset we devised in the papers [3] and [5], we define a ***character*** called the ***n-type vector*** which generalizes to $\mathbb{P}^n$ the ***numerical character*** of Gruson-Peskine [8] to $\mathbb{P}^2$.

We now recall some definitions from [3].

Definition 2. 1) A 0-*type vector* will be defined to be $\mathcal{T} = 1$. It is the only 0-*type vector*. We shall define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.

2) A 1-*type vector* is an object of the form $\mathcal{T} = (d)$ where $d \geq 1$ is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d = \sigma(\mathcal{T})$.

3) A 2-*type vector*, $\mathcal{T}$, is

$$\mathcal{T} = ((d_1), (d_2), \ldots, (d_m))$$

where $m \geq 1$, and the (d_i) are 1-*type vectors*. We also insist that $\sigma(d_i) = d_i < \alpha(d_{i+1}) = d_{i+1}$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = m$ and $\sigma(\mathcal{T}) = \sigma((d_m)) = d_m$.

Clearly, $\alpha(\mathcal{T}) \leq \sigma(\mathcal{T})$ with equality if and only if $\mathcal{T} = ((1), (2), \ldots, (m))$.

Remark: For simplicity in the notation we usually rewrite the 2-*type vector* $((d_1), \ldots, (d_m))$ as $(d_1, \ldots, d_m)$.

4) Now let $n \geq 2$. An *n-type vector*, $\mathcal{T}$, is an ordered collection of $(n-1)$-*type vectors*, $\mathcal{T}_1, \ldots, \mathcal{T}_s$, i.e.

$$\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s)$$

for which $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \ldots, s-1$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = s$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$.

Example 3. Consider the following sequence which is the Hilbert function of 16 points in $\mathbb{P}^3$

$$\mathbf{H} = 1 \;\; 4 \;\; 9 \;\; 14 \;\; 16 \;\; \rightarrow$$

We associate to it the 3-type vector

$$\mathbf{H} \leftrightarrow \mathcal{T} = (\mathcal{T}_1 = (1,3), \;\; \mathcal{T}_2 = (1,2,4,5))$$

where

$$\mathcal{T}_1 = (1,3) \quad \leftrightarrow \quad \mathbf{H}_1 : 1 \;\; 3 \;\; 4 \;\; \rightarrow$$
$$\mathcal{T}_2 = (1,2,4,5) \quad \leftrightarrow \quad \mathbf{H}_2 : 1 \;\; 3 \;\; 6 \;\; 10 \;\; 11 \;\; \rightarrow$$

two Hilbert functions of points in $\mathbb{P}^2$. In turn $\mathbf{H}_1$ decomposes into

$$\mathbf{H}_{11} : 1 \to \quad \text{and} \quad \mathbf{H}_{12} : 1\ 2\ 3\ \to$$

and $\mathbf{H}_2$ into

$$\mathbf{H}_{21} : 1 \to \ ; \quad \mathbf{H}_{22} : 1\ 2\ \to ;$$
$$\mathbf{H}_{23} : 1\ 2\ 3\ 4\ \to \text{ and } \mathbf{H}_{24} : 1\ 2\ 3\ 4\ 5\ \to$$

all these last Hilbert functions of points on lines.

Interestingly enough, one can reverse the process and thus construct a set of points which realizes the given Hilbert function. A set of points constructed in this way is called a *k-configuration.*

Theorem 4 ([3]). *Let $\mathcal{S}_n$ is the collection of Hilbert functions of all sets of points in $\mathbb{P}^n$. There is a 1-1 correspondence*

$$\mathcal{S}_n \leftrightarrow \{\ n\text{-type vectors}\ \}$$

where if $\mathbf{H} \in \mathcal{S}_n$ and $\mathbf{H} \leftrightarrow \mathcal{T}$ then $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$ and $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$.

Theorem 5 ([4], [7]). *All k-configurations associated with the same Hilbert function $\mathbf{H}$ have the same minimal free resolution. It is the "worst" minimal free resolution associated to Cohen-Macaulay rings having Hilbert function $\mathbf{H}$.*

Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^n$ and let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ which is contained in $\mathbb{Z}$. Let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$.

Theorem 6 ([4]). *Let $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$ be as above, let $R = k[x_0, \ldots, x_n]$, $\mathbf{H} = \mathbf{H}(R/I_{\mathbb{X}} + I_{\mathbb{Y}}, t)$. Suppose that $\sigma(\mathbb{Z}) \geq n + 2\sigma(\mathbb{X})$. Let $\mathcal{B}$ be the Betti multiset of $I_{\mathbb{X}} + I_{\mathbb{Y}}$. If J is any other Gorenstein ideal of R such that*

(1) $\mathbf{H}(R/J, t) = \mathbf{H}$; and

(2) R/J has the Weak Lefschetz Property,

then $\mathcal{B} \geq \mathcal{B}(J)$.

Remark 7. Let $\mathbf{H} = \{b_i\}_{i\geq 0}$ be a 0-dimensional differentiable O-sequence with $b_i = \binom{n+i}{i}$ for $i \leq d$ and let $\{c_i\}_{i\geq 0}$ be the sequence obtained from $\mathbf{H}$ by subtracting the Hilbert function $\{e_i\}_{i\geq 0}$ of the coordinate ring $k[x_0, \ldots, x_n]/(Q)$ where Q is a form of degree d.

$$\begin{array}{cccccccc|cc}
 & \overset{b_1}{\shortparallel} & & \overset{b_d}{\shortparallel} & & & & & & \\
1 & \binom{n+1}{1} & \cdots\cdots & \binom{n+d}{d} & & \cdots\cdots & b_{h+d-1} & & b_{h+d} & \cdots\cdots \\
1 & \binom{n+1}{1} & \cdots\cdots & \binom{n+d}{d} & -1 & \cdots\cdots & e_{h+d-1} & & e_{h+d} & \cdots\cdots \\
 & \underset{e_1}{\shortparallel} & & \underset{e_d}{\shortparallel} & & & & & & \\
\hline
 & & & 1 & & \cdots\cdots & c_{h-1} & & c_h & \cdots\cdots
\end{array}$$

Then there is an integer h for which

$$1 = c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_{h-1} \text{ and } c_{h-1} > c_h.$$

Let $\mathbf{H}_1$ denote the sequence $c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_{h-1} \quad \rightarrow$.

Theorem 8 ([5]). *Let* $\mathbf{H}_1 = \{c_i\}_{i \geq 0}$ *be as above. Then* $\mathbf{H}_1$ *is a 0-dimensional differentiable O-sequence.*

Theorem 9 ([5]). *Let* $\mathbf{H} = \{b_i\}_{i \geq 0}$, Q, $\{e_i\}_{i \geq 0}$, *and* h *be as above. Define a new sequence* $\mathbf{H}'_1 = \{c'_i\}_{i \geq 0}$ *as follows:*

$$c'_i = \begin{cases} e_i, & \text{for } i \leq h+d-1, \\ b_i - c_{h-1}, & \text{for } i \geq h+d-1. \end{cases}$$

Then $\mathbf{H}'_1$ *is a 0-dimensional differentiable O-sequence.*

Example 10. Consider the 0-dimensional differentiable O-sequence

$$\mathbf{H} \ : \ 1 \quad 4 \quad 10 \quad 20 \quad 34 \quad 50 \quad 67 \quad 84 \quad 102 \quad 122 \quad \rightarrow .$$

Let Q be a form of degree 2 in $R = k[x, y, z, w]$. Then the Hilbert function of the coordinate ring $R/(Q)$ is

$$\mathbf{H}(R/(Q), t) = t^2 + 2t + 1 \quad \text{for} \quad t \geq 0.$$

Proceeding as in Remark 7 we obtain:

	$\mathbf{H}$:	1	4	10	20	34	50	67	84	102	122	122	$\rightarrow$
		1	4	9	16	25	36	49	64	81	100	121	
yielding				1	4	9	14	18	20	21	22	1	

$$\mathbf{H}_1 \ : \ 1 \quad 4 \quad \mathbf{9} \quad 14 \quad 18 \quad 20 \quad 21 \quad 22 \quad \rightarrow$$
$$\mathbf{H}'_1 \ : \ 1 \quad 4 \quad \mathbf{9} \quad 16 \quad 25 \quad 36 \quad 49 \quad 64 \quad 81 \quad 100 \quad \rightarrow .$$

Thus $\mathbf{H}$ is the Hilbert function of the union of : 100 points on a conic C_1 and 22 points on a conic C_2 where C_1 and C_2 can be chosen to be any two distinct conics and no point chosen on C_2 lies on C_1.

Theorem 11 ([5]). *Let* $\mathbb{X}$ *be a finite set of points in* $\mathbb{P}^n$ *which has Hilbert function* $\mathbf{H}$ *and let* $\mathbf{H}'_1 = \{c'_i\}_{i \geq 0}$ *be as in Theorem 9. Then, for every hypersurface* C *in* $\mathbb{P}^n$ *of degree* $d \geq 1$,

$$\mathbf{H}(\mathbb{X} \cap C, t) \leq \mathbf{H}'_1(t)$$

for every $t \geq 0$.

Example 12. Let $\mathbf{H}$ be as in Example 10 and let $\mathbb{X}$ be a set of 122 points in $\mathbb{P}^3$ with Hilbert function $\mathbf{H}$.

Theorem 11 says that if $\mathbb{Y}$ is a subset of $\mathbb{X}$ and $\mathbb{Y}$ lies on a conic hypersurface in $\mathbb{P}^3$, then the Hilbert function of $\mathbb{Y}$, $\mathbf{H}_{\mathbb{Y}}$, satisfies $\mathbf{H}_{\mathbb{Y}} \leq \mathbf{H}'$ where

$$\mathbf{H}' \ : \ 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad 36 \quad 49 \quad 64 \quad 81 \quad 100 \quad \rightarrow .$$

In other words, for every integer t, $\mathbf{H}_{\mathbb{Y}}(t) \leq \mathbf{H}'(t)$.

Moreover, let $\mathbb{Z}$ be a k-configuration in $\mathbb{P}^3$ of type

$$((1,3), (2,3,5,8), (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9,10)).$$

Then the Hilbert function of $\mathbb{Z}$ is $\mathbf{H}$. Let $\mathbb{Z}_1$ be a subset of $\mathbb{Z}$ and a k-configuration in $\mathbb{P}^3$ of type $((1,2,3,4,5,6,7,8,9),(1,2,3,4,5,6,7,8,9,10))$. Then the Hilbert function of $\mathbb{Z}_1$ is $\mathbf{H}'$. Hence, we can achieve the bound $\mathbf{H}'$ from a k-configuration.

There is one more observation we would like to make about sets of points $\mathbb{X} \subset \mathbb{P}^n$ which have Hilbert function $\mathbf{H}$ where $\mathbf{H} \leftrightarrow \mathcal{T}$, $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$, an *n-type vector.*

Let $\mathbb{X}$ be a finite set of points with Hilbert function $\mathbf{H}$ such that $\mathbf{H}(\mathbb{X}, t) = \mathbf{H}(\mathcal{T}, t)$ where $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$.

Proposition 13 ([5]). *Let $\mathbb{X}$, $\mathbf{H}$ and $\mathcal{T}$ be as above and let $\mathbb{U} \subset \mathbb{X}$ be such that the Hilbert function of $\mathbb{U}$, $\mathbf{H}_{\mathbb{U}}$, satisfies $\mathbf{H}_{\mathbb{U}} = \mathbf{H}(\mathcal{T}_{r-(d-1)}, \dots, \mathcal{T}_r)$.*

Then, if we let $\mathcal{T}' = (\mathcal{T}_1, \dots, \mathcal{T}_{r-d})$ and $\mathbb{X}' = \mathbb{X} - \mathbb{U}$ then $\mathbf{H}_{\mathbb{X}'} = \mathbf{H}(\mathcal{T}')$.

Continuing with Example 10, let $\mathbb{X}$ be a set of 122 points in $\mathbb{P}^3$ with Hilbert function

$$\mathbf{H} \;:\; 1 \quad 4 \quad 10 \quad 20 \quad 34 \quad 50 \quad 67 \quad 84 \quad 102 \quad 122 \rightarrow$$

Suppose, in addition, that $\mathbb{Y}$ is a subset of $\mathbb{X}$ with Hilbert function

$$\mathbf{H}' \;:\; 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad 36 \quad 49 \quad 64 \quad 81 \quad 100 \rightarrow$$

Then Proposition 13 says that $\mathbb{Z} := \mathbb{X} - \mathbb{Y}$ has Hilbert function

$$\mathbf{H}_{\mathbb{Z}} \;:\; 1 \quad 4 \quad 9 \quad 14 \quad 18 \quad 20 \quad 21 \quad 22 \rightarrow .$$

In other words, $\mathbb{X}$ must be the union of $\mathbb{Y}$ and a set $\mathbb{Z}$ with Hilbert function $\mathbf{H}_{\mathbb{Z}}$.

Now I would like to show 0-dimensional differentiable O-sequences which can be from only k-configurations. To show that, we shall classify the minimal free resolution of the ideal in $\mathbb{P}^3$ with Hilbert function

$$\mathbf{H} \leftrightarrow \mathcal{T} = (((1)),((1),(d))).$$

- Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(((1)),((1),(2)))$. Then the Hilbert function of $\mathbb{X}$ is

$$1 \quad 4 \quad 4 \rightarrow,$$

  which is generic and hence the minimal free resolution of the ideal I of any set of points with Hilbert function $\mathbf{H}_{\mathbb{X}}$ is

$$0 \rightarrow R^3(-6) \rightarrow R^8(-5) \rightarrow R^6(-4) \rightarrow I \rightarrow 0 .$$

- Let $\mathbb{X}$ be a set of 5-points in $\mathbb{P}^3$ with Hilbert function

$$\mathbf{H} \;:\; 1 \quad 4 \quad 5 \quad 5 \rightarrow .$$

– If $\mathbb{X}$ is a k-configuration in $\mathbb{P}^3$ of type $(((1)),((1),(3)))$, then the Hilbert function of $I_{\mathbb{X}}$ is $\mathbf{H} = \mathbf{H}_{\mathbb{X}}$ and a minimal free resolution of $I_{\mathbb{X}}$ is

$$\begin{aligned} 0 \;\to\; & R^2(-4) \oplus R(-5) \;\to\; R^6(-3) \oplus R^2(-4) \\ \to\; & R^5(-2) \oplus R(-3) \;\to\; I_{\mathbb{X}} \to 0 \,. \end{aligned}$$

– Let $\mathbb{Y}$ be a set of five points in $\mathbb{P}^3$ such that four points lie on a hyperplane and are a complete intersection in that hyperplane and the remaining point is not on that hyperplane. Then the Hilbert function of $I_{\mathbb{Y}}$ is $\mathbf{H} = \mathbf{H}_{\mathbb{Y}}$ and a minimal free resolution of $I_{\mathbb{Y}}$ is

$$\begin{aligned} 0 \;\to\; & R(-4) \oplus R(-5) \to R^5(-3) \oplus R^1(-4) \\ \to\; & R^5(-2) \to I_{\mathbb{Y}} \to 0 \,. \end{aligned}$$

– Let $\mathbb{Z}$ be a set of five points in $\mathbb{P}^3$ such that any four points among them do not lie on a hyperplane. Then the Hilbert function of $I_{\mathbb{Z}}$ is $\mathbf{H} = \mathbf{H}_{\mathbb{Z}}$ and a minimal free resolution of $I_{\mathbb{Y}}$ is

$$0 \;\to\; R(-5) \;\to\; R^5(-3) \;\to\; R^5(-2) \;\to\; I_{\mathbb{Z}} \;\to\; 0 \,.$$

Hence the Hilbert functions $\mathbf{H}_{\mathbb{X}}$, $\mathbf{H}_{\mathbb{Y}}$, and $\mathbf{H}_{\mathbb{Z}}$ are the same, however, the ideals $I_{\mathbb{X}}$, $I_{\mathbb{Y}}$, and $I_{\mathbb{Z}}$ have all different kinds of minimal free resolutions. This means that we have three kinds of minimal free resolutions with Hilbert function $\mathbf{H} \;:\; 1 \;\; 4 \;\; 5 \;\to$.

Theorem 14. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(((1)),((1),(d)))$ with $d \geq 4$. Then the Hilbert function $\mathbf{H}_{\mathbb{X}}$ can be from only k-configuration in $\mathbb{P}^3$ of type $(((1)),((1),(d)))$.*

In particular, the minimal free resolution of the ideal I of any set of points in $\mathbb{P}^3$ with Hilbert function $\mathbf{H}_{\mathbb{X}}$ is

$$\begin{aligned} 0 \;\to\; & R^2(-4) \oplus R(-(d+2)) \;\to\; R^6(-3) \oplus R^2(-(d+1)) \\ \to\; & R^5(-2) \oplus R(-d) \;\to\; I \to 0 \,. \end{aligned}$$

Theorem 15 ([3]). *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of type $(e_1, \ldots, e_r)$. Suppose that $e_{i+1} - e_i > 2$ for all $i = 1, \ldots, r-1$. Then the Hilbert function $\mathbf{H}_{\mathbb{X}}$ can be from only a k-configuration in $\mathbb{P}^2$.*

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Properties of the discrete counterpart of an algebra with straightening laws

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1 Introduction

DeConcini, Eisenbud and Procesi defined the notion of a Hodge algebra in their article [DEP] and proved many properties of Hodge algebras. They also showed that many algebras appearing in algebraic geometry and commutative ring theory have structures of Hodge algebras. In fact, the theory of Hodge algebras is an abstraction of combinatorial arguments that are used to study those rings.

A Hodge algebra is an algebra with relations which satisfy certain laws regulated by combinatorial data. It is possible to exist many Hodge algebras with the same combinatorial data. And there is the simplest Hodge algebra with given combinatorial data, called the discrete Hodge algebra. For a given Hodge algebra, we call the discrete Hodge algebra with the same combinatorial data the discrete counterpart of it.

Among the most important facts of DeConcini, Eisenbud and Procesi's results are

- A Hodge algebra and its discrete counterpart have the same dimension.
- The depth of the discrete counterpart is not greater than the depth of the original Hodge algebra.

It is known that there is a Hodge algebra whose discrete counterpart has strictly smaller depth than the original one [Hib]. And we note in Section 3 that there is a series of examples of Cohen-Macaulay Hodge algebras of dimension n whose discrete counterparts have depth 0, where n runs over the set of all positive integers. So there is no hope to restrict the difference of the depth of a Hodge algebra and that of the discrete counterpart.

But if we restrict our attention to ordinal Hodge algebras (algebras with straightening laws, ASL for short), the influence of the combinatorial data to the ring theoretical properties become greater. So there may be a restriction to the combinatorial data by the ring theoretical properties of an ASL.

In this article, we study the poset which generate a Cohen-Macaulay ASL. Since it is equivalent to study the properties of combinatorial data of an ASL, i.e., the properties of the partially ordered set (poset for short) generating the ASL, and to study the properties of the discrete counterpart, our results are sometimes written in the language of posets and sometimes in the language of commutative rings.

2 Preliminaries

In this article all rings and algebras are commutative with identity. For two sets X and Y, we denote by $X \setminus Y$ the set $\{x \in X | x \notin Y\}$. Standard terminology on Hodge algebras and Stanley-Reisner rings are used freely. See [DEP], [BV], [Sta, Chapter II], [BH, Chapter 5] and [Hoc] for example. However, we use the term "algebra with straightening laws" (ASL for short) to mean an ordinal Hodge algebra.

In addition we use the following notation and convention.

- For a poset P, we define the order complex $\Delta(P)$ of P by

$$\Delta(P) := \{\sigma \subseteq P \mid \sigma \text{ is a chain}\},$$

  where a chain stands for a totally ordered subset. We also define the reduced Euler characteristic $\tilde{\chi}(P)$ of P by

$$\tilde{\chi}(P) := \tilde{\chi}(\Delta(P)).$$

- When considering a poset, we denote by ∞ (or by $-\infty$ resp.) a new element which is larger (smaller resp.) than any other element.

- If P is a poset and x, $y \in P \cup \{\infty, -\infty\}$ with $x < y$, we define

$$(x, y)_P := \{z \in P \mid x < z < y\}.$$

- If A is a Hodge algebra over k generated by H governed by Σ, we denote by A_{dis} the discrete Hodge algebra over k generated by H governed by Σ and call it the discrete counterpart of A.

Now we recall the notion of a standard subset [Miy2].

Definition 2.1. Let A be a Hodge algebra over k generated by H governed by Σ. A subset Ω of H is called a standard subset of H if for any element $x \in \Omega A$ and for any standard monomial M_i appearing in the standard representation

$$x = \sum_i b_i M_i \qquad (0 \neq b_i \in k,\ M_i \text{ is standard})$$

of x, $\operatorname{supp} M_i$ meets Ω.

Note that if Ω is a standard subset of H, then $A/\Omega A$ is a Hodge algebra over k generated by $H \setminus \Omega$ governed by Σ/Ω.

Like [Sta, II.5], we make the following

Definition 2.2. For a Hodge algebra A over k generated by H governed by Σ, we define

$$\begin{aligned} \operatorname{core} H &:= \bigcup_{N \text{ is a generator of } \Sigma} \operatorname{supp} N, \\ \operatorname{core} \Sigma &:= \{\mu \in \Sigma \mid \operatorname{supp} \mu \subseteq \operatorname{core} H\}, \\ \operatorname{core} A &:= A/(H \setminus \operatorname{core} H)A. \end{aligned}$$

It is obvious that if $\Omega = \{x_1, \ldots, x_t\}$ is a subset of H such that $\Omega \cap \operatorname{core} H = \emptyset$, then Ω is a standard subset of H and $x_1, \ldots, x_t$ is an A-regular sequence. In particular,

Lemma 2.3. $\operatorname{core} A$ *is a Hodge algebra generated by* $\operatorname{core} H$ *governed by* $\operatorname{core} \Sigma$. *Furthermore, if* $H \setminus \operatorname{core} H = \{x_1, \ldots, x_t\}$, *then* $x_1, \ldots, x_t$ *is an* A*-regular sequence and* $\operatorname{core} A = A/(x_1, \ldots, x_t)$.

Moreover, it is easily verified that

$$(\operatorname{core} A)_{\text{dis}} = \operatorname{core}(A_{\text{dis}}).$$

So we denote both sides by $\operatorname{core} A_{\text{dis}}$.

3 Examples

In this section we give a series of examples of Hodge algebras whose discrete counterparts have strictly smaller depths.

Let k be a field and $y_1, \ldots, y_n$ be indeterminates over k, where n is an integer greater than 1. Set $A = k[y_1, \ldots, y_n]^{(2)}$, the second Veronese subring of the polynomial ring $k[y_1, \ldots, y_n]$. We define a Hodge algebra structure of A by the following way.

Set $H = \{x_{ij} \mid 1 \le i \le j \le n\}$ and define the order on H by $x_{11} < x_{12} < \cdots < x_{1n} < x_{22} < x_{23} < \cdots < x_{nn}$. Embed H in A by $\phi\colon x_{ij} \longmapsto y_i y_j$. Let Σ be the ideal of monomials on H generated by $\{x_{ij}x_{kl} \mid j > k, l > i\}$. Then it is verified that A is a Hodge algebra over k generated by H governed by Σ. It is known that A is a Cohen-Macaulay ring with $\dim A = n$.

In order to study the depth of A_{dis}, we use the technique of polarization (see [DEP, §4]). Let X_{ij} $(1 \le i \le j \le n)$ and Y_{ij} $(1 \le i < j \le n)$ be indeterminates. Then A_{dis} is isomorphic to $k[X]/I$, where I is the monomial ideal generated by $\{X_{ij}X_{kl} \mid j > k, l > i\}$, and the polarization J of I is the square-free monomial ideal of $k[X, Y]$ generated by

$$\{X_{ij}Y_{ij} \mid 1 \le i < j \le n\} \cup \{X_{ij}X_{kl} \mid (i,j) \ne (k,l),\ j > k,\ l > i\}.$$

It is known that $X_{12}-Y_{12}, X_{13}-Y_{13}, \ldots, X_{1n}-Y_{1n}, X_{23}-Y_{23}, \ldots, X_{n-1,n}-Y_{n-1,n}$ is a $k[X,Y]/J$-regular sequence (see [DEP, Proposition 4.3]) and

$$k[X,Y]/(J+(X_{ij}-Y_{ij}|1 \le i < j \le n)) \simeq k[X]/I.$$

So

$$\dim A_{\mathsf{dis}} - \operatorname{depth} A_{\mathsf{dis}} = \dim(k[X,Y]/J) - \operatorname{depth}(k[X,Y]/J).$$

Since J is a square-free monomial ideal, it corresponds to a simplicial complex by the theory of Stanley-Reisner rings. It is easily verified that

$$\{X_{11}, X_{1n}, X_{nn}\} \cup \{Y_{ij} \mid 1 \le i < j \le n,\ (i,j) \ne (1,n)\}$$

and

$$\{X_{ij} \mid 1 \le i \le j \le n,\ j \le i+1\} \cup \{Y_{ij} \mid 1 \le i,\ i+2 \le j \le n\}$$

are facets of this simplicial complex. So it has facets of dimensions $\binom{n+1}{2}-1$ and $\binom{n}{2}+1$. Since $(\binom{n+1}{2}-1)-(\binom{n}{2}+1) = n-2$, we see, by the theory of Stanley-Reisner rings, that

$$\dim(k[X,Y]/J) - \operatorname{depth}(k[X,Y]/J) \ge n-2$$

(see e.g. [Miy1, p. 370] or [BH, Theorem 5.1.4]). Therefore

$$\operatorname{depth} A_{\mathsf{dis}} \le 2$$

since $\dim A_{\mathsf{dis}} = n$.

On the other hand, since no generator of I involve X_{11} and X_{nn}, we see that X_{11}, X_{nn} is an $k[X]/I$-regular sequence. So

$$\operatorname{depth} A_{\mathsf{dis}} = 2.$$

Summing up, A is a Cohen-Macaulay homogeneous Hodge algebra of dimension n and the discrete counterpart A_{dis} has depth 2. And, since $H \setminus \operatorname{core} H = \{x_{11}, x_{nn}\}$, $\operatorname{core} A$ is a Cohen-Macaulay homogeneous Hodge algebra of dimension $n-2$ and $\operatorname{depth}(\operatorname{core} A_{\mathsf{dis}}) = 0$.

4 Stepping stones

In the following of this article, we restrict our attention to ASL and consider the following

Problem 4.1. *If there is a Cohen-Macaulay ASL over k generated by a poset P, what can be said about P? In particular, is P Cohen-Macaulay over k?*

To tackle this problem, we state two stepping stones and consider the following four conditions.

(i) P is a poset.

(ii) P is a pure poset.

(iii) P is a Buchsbaum poset.

(iv) P is a Cohen-Macaulay poset.

The implications (iv)⇒(iii)⇒(ii)⇒(i) are well known. And in the following, we state that, under the assumption that there is a Cohen-Macaulay ASL generated by P, (iii)⇒(iv) and (i)⇒(ii) are also valid.

First we state the following result of [Miy3].

Theorem 4.2. *Let A be a Cohen-Macaulay square-free Hodge algebra over a field k. Suppose that* core A_{dis} *is Buchsbaum. Then A_{dis} is Cohen-Macaulay. In particular, if P is a Buchsbaum poset which generate a Cohen-Macaulay ASL, then P is Cohen-Macaulay.*

Let A be the Hodge algebra, of the case $n = 3$, considered in the last section. It is verified that core A_{dis} is Buchsbaum and, as is stated in the last section, core A is a Cohen-Macaulay ring of dimension 1 and depth(core A_{dis}) $= 0$. So the square-free assumption in the theorem above is essential.

Now let k be a field and $x_1, \ldots, x_n$ be indeterminates over k and $S = k[x_1, \ldots, x_n]$ a polynomial ring. Assume that S is given a graded ring structure such that $S_0 = k$ and each x_i is a homogeneous element of positive degree.

For a graded ideal I of S, Hartshorne [Har], Sturmfels-Trung-Vogel [STV] defined the notion of geometric degree geom-deg I and arithmetic degree arith-deg I of I. By definition

$$\deg I \leq \text{geom-deg}\, I \leq \text{arith-deg}\, I$$

and

$$\deg I = \text{geom-deg}\, I \iff S/I \text{ is equidimensional,}$$

$$\text{geom-deg}\, I = \text{arith-deg}\, I \iff S/I \text{ has no embedded prime ideals.}$$

Assume a monomial order on S is settled and let $\text{in}(I)$ be the initial ideal of I with respect to this monomial order. It is well known that $\deg I = \deg(\text{in}(I))$. And Hartshorne [Har], Sturmfels-Trung-Vogel [STV] showed that

$$\text{geom-deg}(\text{in}(I)) \leq \text{geom-deg}\, I$$

and

$$\text{arith-deg}(\text{in}(I)) \geq \text{arith-deg}\, I.$$

So if $S/\text{in}(I)$ has no embedded prime ideals, then

$$\text{geom-deg}(\text{in}(I)) = \text{geom-deg}\, I = \text{arith-deg}\, I = \text{arith-deg}(\text{in}(I))$$

and S/I is equidimensional if and only if $S/\operatorname{in}(I)$ is equidimensional.

Assume that A is a graded Hodge algebra over k. Then it is well known that there is a polynomial ring with monomial order and a graded ideal I of S such that

$$A \simeq S/I \quad \text{and} \quad A_{\text{dis}} \simeq S/\operatorname{in}(I).$$

Assume further that A is square-free. Then it is known [DEP, Proposition 5.1] that A and A_{dis} are reduced rings. So by the arguments above, we see the following

Proposition 4.3. *Let A be a square-free graded Hodge algebra over a field. Then A is equidimensional if and only if A_{dis} is equidimensional. In particular, if A is a graded ASL generated by a poset P, then A is equidimensional if and only if P is pure.*

Since a Cohen-Macaulay ring is equidimensional, we see the following

Corollary 4.4. *Let P be a poset. If there is a Cohen-Macaulay ASL generated by P, then P is pure.*

5 Rees algebras

As is noted below, a Rees algebra of an ASL have a structure of an ASL under certain conditions. In this section we study the relation between the Cohen-Macaulay property of such a Rees algebra and the property of the discrete counterpart of it.

First we recall the definition of a straightening closed ideal.

Definition 5.1. Let A be a graded ASL over a field k generated by a poset P and Q a poset ideal of P. If every standard monomial μ_i appearing in the standard representation

$$\alpha\beta = \sum_i r_i\mu_i, \qquad 0 \neq r_i \in k$$

of $\alpha\beta$ with α, $\beta \in Q$ and $\alpha \not\sim \beta$, has at least two factors in Q, we say Q (or the ideal QA of A) is straightening closed.

Note that if the ASL is discrete, then any poset ideal is straightening closed.

Now let P be a poset and Q a poset ideal of P. We define the poset $P \uplus Q$ as follows (cf. [BV, Section 9]). Denote a copy of Q by Q^* and the element corresponding to $x \in Q$ by $x^* \in Q^*$. Set $P \uplus Q = P \cup Q^*$ as the underlying set. And for α, $\beta \in P \uplus Q$, we define $\alpha < \beta$ if and only if one of the following three conditions is satisfied.

- α, $\beta \in P$ and $\alpha < \beta$ in P.
- $\alpha = x^*$, $\beta = y^*$ with x, $y \in Q$ and $x < y$ in P.

- $\alpha = x^*$ with $x \in Q$, $\beta \in P$ and $x \leq \beta$ in P.

With this notation, we recall the following fact (see [DEP, 10d] or [BV, (9.13)]).

Proposition 5.2. *Let A be a graded ASL over a field k generated by a poset P. Suppose that Q is a straightening closed poset ideal of P and $I = QA$. Then*

(i) *The Rees algebra R with respect to I is a graded ASL over k generated by $P \uplus Q$.*

(ii) *The associated graded ring G is a graded ASL over k generated by P such that $\operatorname{ind} G \subseteq \operatorname{ind} A$. In particular, if A is the discrete ASL, then so is G.*

Now we state a result about Cohen-Macaulay property of $P \uplus Q$.

Theorem 5.3. *Let P be a Cohen-Macaulay poset over a field k and Q a poset ideal of P. If*

$$\tilde{\chi}((-\infty, x)_P) = 0 \quad \text{for any } x \in (P \cup \{\infty\}) \setminus Q \tag{5.1}$$

then $P \uplus Q$ is also Cohen-Macaulay over k.

See [Miy3] for the proof.

It follows directly from Theorem 5.3 the following

Corollary 5.4. *If P is a Cohen-Macaulay poset over a field k with unique minimal element and Q is a poset ideal of P, then $P \uplus Q$ is also Cohen-Macaulay over k.*

Note the posets considered by Bruns-Vetter in [BV, Section 9] are Cohen-Macaulay posets with unique minimal element. So Corollary 5.4 gives another proof of [BV, (9.4) Theorem (b)].

Now let P be a Cohen-Macaulay poset over a field k, A an ASL generated by P, Q a straightening closed poset ideal of P and $I = QA$. Denote the Rees algebra with respect to I by R and the associated graded ring by G. Then G is Cohen-Macaulay by Proposition 5.2 and by the argument of [Miy3], the condition (5.1) is equivalent to $a(G) < 0$, where $a(-)$ is the a-invariant of Goto-Watanabe [GW]. Therefore, by the result of Trung-Ikeda [TI], we see the following

Theorem 5.5. *In the setting above, R is Cohen-Macaulay if and only if $P \uplus Q$ is Cohen-Macaulay over k.*

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