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序

この冊子は第 22 回可換環論シンポジウムの報告集です.

このシンポジウムは 2000 年 11 月 6 日から 11 月 9 日にかけてインテック大山研修センター（富山県上新川郡大山町）において開催されました. 国内の多数の研究者・大学院生の他に, Tata Institute から S. M. Bhatwadekar 氏の参加もあり, 合計 20 の興味深い講演が行われました.

なお, このシンポジウムを開催するにあたり, 文部省科学研究費の基盤研究 (B)(2) （課題番号 11440006 代表者 : 梅村浩）および基盤研究 (B)(1) （課題番号 12440001 代表者 : 森田康夫, 分担者 : 渡辺敬一）からの援助を頂きました. ここにあらためてお礼申し上げます.

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目次

Algebras generated by idempotents and group rings

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In the following, all rings and algebras are commutative with identity and all groups are abelian with operation written multiplicatively.

Notation. Let R be a ring and let G be a group.
(i) $RG =$ the group ring of G over R.
(ii) If A is an R-algebra,
$\Delta_R(A)$ = the R-subalgebra of A generated by all the idempotents.
(iii) $U(R) =$ the units of R.
(iv) $\phi_n(X) =$ the n-th cyclotomic polynomial.

The following theorem motivates this report.

Theorem ([1, Chap.6, Theorem 4.1 and Corollary 4.2], [3]). *Let K be an algebraically closed field and let A be a K-algebra. Assume that the following conditions hold.*
(1) $\Delta_K(A) = A$.
(2) *A contains no primitive idempotents.*
(3) *A is countably infinite dimensional over K.*

Then, $A \cong KG$ as K-algebras for any countably infinite torsion group G having no element whose order is equal to $\mathrm{char}\, K$, the characteristic of K.

We are concerned with the condition (1) of the preceding theorem. The relationship between an algebra generated by idempotents over an algebraically closed field and group rings is given by the following.

Theorem 1. *Let K be an algebraically closed field and let A be a K-algebra. Then the following are equivalent :*
(1) $\Delta_K(A) = A$;
(2) *there exists a surjective K-algebra homomorphism $KG \to A$ for a torsion group G having no element whose order is equal to $\mathrm{char}\, K$.*

Proof. See [2, Section 2]. □

If we cinsider the above conditions (1) and (2) without the assumption that K is algebraically closed, then neither '(1)⇒(2)' nor '(2)⇒(1)' necessarily holds.

Example 1 ((1)$\not\Rightarrow$(2), [5, the last Remark]). Let $\mathbb{F}_2$ be the field of two elements and let $A = \mathbb{F}_2 \times \mathbb{F}_2$. Suppose $\varphi : \mathbb{F}_2 G \to A$ is a $\mathbb{F}_2$-algebra homomorphism for a group G. Then, $\varphi(G) \subseteq U(A) = \{ (1, 1) \}$. Hence $\varphi(\mathbb{F}_2 G) \subsetneq A$.

Example 2 ((1)$\not\Rightarrow$(2)). Let $K = \mathbb{F}_2(X)$, the field of rational functions over $\mathbb{F}_2$. Let $A = K \times K$. Suppose $\varphi : KG \to A$ is a K-algebra homomorphism for a torsion group G. Then, $\varphi(G) = \{ (1, 1) \}$ since $U(A)$ is torsion-free. Therefore we have $\varphi(KG) \subsetneq A$.

Example 3 ((2)$\not\Rightarrow$(1)). Let G be the cyclic group of order 3 with a generator g. Let $A = \mathbb{F}_2G$. We can take the identity map of A as a $\mathbb{F}_2$-algebra homomorphism required in (2). However, the set of idempotents of A is $\{\ 0,\ 1,\ g+g^2,\ 1+g+g^2\ \}$. So, it follows that

$$\Delta_{\mathbb{F}_2}(A) = \mathbb{F}_2[g+g^2] = \mathbb{F}_2 + \mathbb{F}_2(g+g^2).$$

Hence $g \notin \Delta_{\mathbb{F}_2}(A)$, and $\Delta_{\mathbb{F}_2}(A) \subsetneq A$.

Example 4 ((2)$\not\Rightarrow$(1)). Let G be the cyclic group of order 3 and let $\mathbb{Q}$ be the rational numbers. Let $A = \mathbb{Q}G$ and let ζ be a primitive 3-rd root of 1. Then, as Example 3, (2) holds. However, we have $A \cong \mathbb{Q} \times \mathbb{Q}[\zeta]$ as $\mathbb{Q}$-algebras. So, $\Delta_{\mathbb{Q}}(A) \subsetneq A$.

For algebras over rings, we have the following.

Theorem 2. *Let R be a ring and let A be an R-algebra. Suppose that a prime number p is a unit of R and that $\phi_p(X) = 0$ has a root in R. Then the following are equivalent :*
(1) $\Delta_R(A) = A$;
(2) *there exists a surjective R-algebra homomorphism $RG \to A$ for a group G satisfying $G^p = 1$.*

Proof. For (1)$\Rightarrow$(2), let ζ be an element of R satisfying $\phi_p(\zeta) = 0$ and let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a set of idempotents of A satisfying $A = R[\{e_\lambda\}_{\lambda\in\Lambda}]$. For each $\lambda \in \Lambda$, define

$$a_\lambda = (1-\zeta)e_\lambda + \zeta. \tag{2.1}$$

We have

$$a_\lambda^p = \{e_\lambda + \zeta(1-e_\lambda)\}^p = 1 \tag{2.2}$$

since $\zeta^p = 1$. Let G be the subgroup of $U(A)$ generated by $\{a_\lambda\}_{\lambda \in \Lambda}$. Then $G^p = 1$ from (2.2). By [2, Lemma 2.1], we know $\prod_{i=1}^{p-1}(1-\zeta^i) = p$. Hence our assumption $p \in U(R)$ implies $1-\zeta \in U(R)$. So we can rewrite (2.1) as follows :

$$e_\lambda = (1-\zeta)^{-1}(a_\lambda - \zeta) \quad (\lambda \in \Lambda).$$

Therefore A is an image of RG.

For (2)⇒(1), we have $\Delta_R(RG) = RG$ by [4]. Hence the equality $\Delta_R(A) = A$ holds. □

In the three theorems above, every group ring that appears in the statements is generated by idempotents. For the study of group rings generated by idempotents, some results are given by Professor Onoda and the author in [4].

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On maps of Grothendieck groups induced by completion

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This is a joint work with Yuji Kamoi (Meiji University).

1 Introduction

For a scheme X that is of finite type over a regular scheme S, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{X/S} : \mathrm{G}_0(X)_{\mathbb{Q}} \to \mathrm{A}_*(X)_{\mathbb{Q}}$$

by the singular Riemann-Roch theorem (Chapter 18 and 20 in Fulton [1]), where $\mathrm{G}_0(X)$ (resp. $\mathrm{A}_*(X)$) denotes the Grothendieck group of coherent $\mathcal{O}_X$-modules (resp. Chow group of X). It is the natural generalization of Grothendieck-Riemann-Roch theorem to singular schemes. The construction of the map $\tau_{X/S}$ depend not only on X but also on S (see Section 4).

Here, let T be a regular local ring and let A be a homomorphic image of T. Since A is of finite type over T, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{\mathrm{Spec}\,A/\mathrm{Spec}\,T} : \mathrm{G}_0(\mathrm{Spec}\,A)_{\mathbb{Q}} \to \mathrm{A}_*(\mathrm{Spec}\,A)_{\mathbb{Q}}$$

by the singular Riemann-Roch theorem as above. We denote $\tau_{\mathrm{Spec}\,A/\mathrm{Spec}\,T}$, $\mathrm{G}_0(\mathrm{Spec}\,A)$ and $\mathrm{A}_*(\mathrm{Spec}\,A)$ simply by $\tau_{A/T}$, $\mathrm{G}_0(A)$ and $\mathrm{A}_*(A)$, respectively. In 1985, P. Roberts [9] proved that the vanishing theorem holds for a local ring A that satisfies $\tau_{A/T}([A]) \in \mathrm{A}_{\dim A}(A)_{\mathbb{Q}}$, where we say that the vanishing theorem holds for A if $\sum_i (-1)^i \ell_A(\mathrm{Tor}_i^A(M,N)) = 0$ is satisfied for two finitely generated A-modules M and N that satify; (1) both of them have finite projective dimension, (2) $\dim M + \dim N < \dim A$, (3) $M \otimes_A N$ is of finite length. (It contains an affirmative answer to a conjecture proposed by Serre [11]. The conjecture was independently solved by Roberts, Gillet and Soulé [2].)

Inspired by the result of Roberts, the author defined the notion of Roberts rings as below and studied them in [3].

Definition 1.1 A local ring A is said to be a *Roberts* ring if there exists a regular local ring T such that A is a homomorphic image of T and $\tau_{A/T}([A]) \in \mathrm{A}_{\dim A}(A)_{\mathbb{Q}}$ is satisfied.

The category of Roberts rings contains complete intersections, quotient singularities, Galois extensions of regular local rings. Normal Roberts rings are $\mathbb{Q}$-Gorenstein. There are examples of Gorenstein normal non-Roberts rings. If A is a Roberts ring, then so is the completion $\hat{A}$.

Here, it seems to be natural to consider the following conjecture and ask the following question.

Conjecture 1.2 Let A be a local ring that is a homomorphic image of a regular local ring T. Then, the Riemann-Roch map $\tau_{A/T}$ as above is independent of the choice of T.

Question 1.3 Let A be a local ring that is a homomorphic image of a regular local ring. Assume that the completion $\hat{A}$ is a Roberts ring. Then, is A a Roberts ring, too?

Conjecture 1.2 is affirmatively solved [3] if A is a complete local ring or A is essentially of finite type over either a field or the ring of integers.

There is a deep connection between the conjecture and the question. As we shall see in Section 4, Question 1.3 is true for any A if and only if Question 1.4 as below is true for any A. Furthermore, if Question 1.4 is true for a local ring A, then Conjecture 1.2 is true for the local ring A.

Question 1.4 Let A be a local ring that is a homomorphic image of a regular local ring. Then, is the map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ (induced by the natural map $A \xrightarrow{f} \hat{A}$) injective?

As we shall see in Proposition 4.1, Question 1.4 is equvalent to that the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{g_*} \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective for any étale local homomorphism $A \xrightarrow{g} B$ that is essentially of finite type.

In section 3, we shall prove the following theorem:

Theorem 1.5 *Let A be a homomorphic image of an excellent regular local ring. If A satisfies one of the following three conditions, then the natual map $\mathrm{G}_0(A) \xrightarrow{f_*} \mathrm{G}_0(\hat{A})$ is injective.*

(1) A is a henselian local ring.

(2) $A = S_M$, where $S = \oplus_{n\geq 0} S_n$ is a Noetherian positively graded ring over a henselian local ring (S_0, m_0), and $M = m_0 S_0 + S_+$. (Here $S_+ = \oplus_{n>0} S_n$.)

(3) A has at most isolated singularity.

In the proof of Claim 4.3 in [3], the injectivity was proved in the case (1) as above. Note that the category of rings in (1) is contained in that in (2). In a proof in the case (1), Popescue-Ogoma's approximation theorem ([6], [7]) is used. In the case of (2), we use a method similar to "deformation to normal cones" in Chapter 5 in Fulton [1]. We use the localization sequence in K-theory due to Thomason and Trobaugh [13] in the case of (3).

We shall give some applications of Theorem 1.5 in Section 4.

The next section is devoted to preliminaries.

2 Preliminaries

Throughout the article, a local ring is always assumed to be a homomorphic image of a regular local ring.

First of all, let us define the Grothendieck group and the Chow group of a ring A.

Definition 2.1 For a ring A, let $\mathrm{G}_0(A)$ be the Grothendieck group of finitely generated A-modules, i.e.,

$$\mathrm{G}_0(A) = \frac{\bigoplus_{M\,:\,\text{a finitely generated } A\text{-molule}} \mathbb{Z}\cdot[M]}{\langle [M]-[L]-[N] \mid 0\to L\to M\to N\to 0 \text{ is exact}\rangle}.$$

Let $\mathrm{A}_i(A)$ be the i-th Chow group of A, i.e.,

$$\mathrm{A}_i(A) = \frac{\bigoplus_{P\in\operatorname{Spec} A,\ \dim A=i} \mathbb{Z}\cdot[\operatorname{Spec} A/P]}{\langle \operatorname{div}(Q,x) \mid Q\in\operatorname{Spec} A,\ \dim A/Q=i+1,\ x\in A\setminus Q\rangle},$$

where

$$\operatorname{div}(Q,x) = \sum_{P\in\operatorname{Min}_A A/(Q,x)} \ell_{A_P}(A_P/(Q,x)A_P)[\operatorname{Spec} A/P].$$

The Chow group of A is defined to be $\mathrm{A}_*(A) = \oplus_{i=0}^{\dim A} A_i(A)$.

For an abelian group M, $M_{\mathbb{Q}}$ denotes $M\otimes_{\mathbb{Z}}\mathbb{Q}$.

Definition 2.2 (1) Let $g : A\to B$ be a flat ring homomorphism. Then, we have the induced homomorphism $g_* : \mathrm{G}_0(A)\to\mathrm{G}_0(B)$ by $g_*([M]) = [M\otimes_A B]$.

(2) Let (A,m) be a local ring. For each i, the natural map $A\xrightarrow{f}\hat{A}$ ($\hat{A}$ is the completion of A in the m-adic topology) induces the natural map $\mathrm{A}_i(A)\xrightarrow{f_*}\mathrm{A}_i(\hat{A})$ by

$$f_*([\operatorname{Spec} A/P]) = \sum \ell_{\hat{A}_{\mathfrak{p}}}(\hat{A}_{\mathfrak{p}}/P\hat{A}_{\mathfrak{p}})[\operatorname{Spec}\hat{A}/\mathfrak{p}],$$

where the sum is taken over all minimal prime ideals of $\hat{A}/P\hat{A}$ as an $\hat{A}$-module. Here, note that $\hat{A}/P\hat{A}$ is equi-dimensional since A is universally catenary (Theorem 31.7 in Matsumura [5]).

Remark 2.3 Assume that (A,m) is a d-dimensional excellent normal local ring. Then $\hat{A}$ is also normal and the natural map $\mathrm{Cl}(A)\to\mathrm{Cl}(\hat{A})$ is injective, where $\mathrm{Cl}(A)$ is the divisor class group of A.

On the other hand, it is well known that $\mathrm{A}_{d-1}(A)$ coincides with $\mathrm{Cl}(A)$. Thus, we know that $f_* : \mathrm{A}_{d-1}(A)\to\mathrm{A}_{d-1}(\hat{A})$ is injective if A is normal.

Then, we have

Proposition 2.4 *Let (A,m) be a local ring. Then, the following conditions are equivalent:*

1. $\mathrm{G}_0(A)_{\mathbb{Q}}\xrightarrow{f_*}\mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ *is injective.*

2. $\mathrm{A}_i(A)_{\mathbb{Q}}\xrightarrow{f_*}\mathrm{A}_i(\hat{A})_{\mathbb{Q}}$ *is injective for all i.*

Proof. Take a regular local ring T such that A is a homomorphic image of T. Then, by the singular Riemann-Roch theorem, we have isomorphisms of $\mathbb{Q}$-vector spaces $\tau_{A/T}$:

$\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ and $\tau_{\hat{A}/\hat{T}} : \mathrm{G}_0(\hat{A})_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ such that the following diagram is commutative (Lemma 4.1 (c) in [3]):

$$\begin{array}{ccc} \mathrm{G}_0(A)_{\mathbb{Q}} & \stackrel{\tau_{A/T}}{\longrightarrow} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ \downarrow f_* & & \downarrow f_* \\ \mathrm{A}_*(\hat{A})_{\mathbb{Q}} & \stackrel{\tau_{\hat{A}/\hat{T}}}{\longrightarrow} & \mathrm{A}_*(\hat{A})_{\mathbb{Q}} \end{array} \qquad (1)$$

Here $f_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ is the direct sum of $\{f_* : \mathrm{A}_i(A)_{\mathbb{Q}} \to \mathrm{A}_i(\hat{A})_{\mathbb{Q}} \mid i = 0, 1, \ldots, \dim A\}$. Therefore we know immediately that the two conditions in the proposition are equivalent. **q.e.d.**

By the proposition, we know that Question 1.4 is a natural generalization of the injectivity of divisor class groups (Remark 2.3) in a sense.

3 The proof of Theorem 1.5

We shall give a proof to Theorem 1.5 in the section.

First, assume that A is an excellent henselian local ring. In the case, the result was found in [3]. Here, we shall give an outline of the proof.

Take $\alpha \in \mathrm{G}_0(A)$ such that $f_*(\alpha) = 0$. We want to show $\alpha = 0$ in $\mathrm{G}_0(A)$. By the assumption, we have short exact sequences of $\hat{A}$-modules that make $f_*(\alpha)$ vanish. Thanks to Popescue-Ogoma's approximation theorem ([6], [7]), we can find short exact sequences of A-modules that make α vanish.

Next put $A = S_M$, where $S = \oplus_{n \geq 0} S_n$ is a Noetherian positively graded ring over a henselian local ring (S_0, m_0), and $M = m_0 S_0 + S_+$.

Put $B = \prod_{n \geq 0} S_n$. It is the S_+A-adic completion of A. Therefore, B is flat over A and $\hat{A} = \hat{B}$ is satisfied. Let $g : A \to B$ be the natural map. Since A is an excellent local ring, so is B by Theorem 3 in Rotthaus [10]. Furthermore, since S_0 is a henselian local ring, so is B. Thus, B is an excellent henselian local ring. By (1) in Theorem 1.5, we know that $\mathrm{G}_0(B) \to \mathrm{G}_0(\hat{B}) = \mathrm{G}_0(\hat{A})$ is injective. Therefore, in order to show the injectivity of $\mathrm{G}_0(A) \to \mathrm{G}_0(\hat{A})$, we have only to prove that of $\mathrm{G}_0(A) \to \mathrm{G}_0(B)$.

Put

$$F_i = \begin{cases} (\oplus_{n \geq i} S_n)A & \text{if } i \geq 0 \\ A & \text{if } i < 0. \end{cases}$$

Then, $F = \{F_i\}_{i \in \mathbb{Z}}$ is a filtration of ideals of A, that is, it satisfies; (1) $F_i \supseteq F_{i+1}$ for any $i \in \mathbb{Z}$, (2) $F_0 = A$, (3) $F_i F_j \subseteq F_{i+j}$ for any $i, j \in \mathbb{Z}$. Similarly put

$$\hat{F}_i = \begin{cases} \prod_{n \geq i} S_n & \text{if } i \geq 0 \\ B & \text{if } i < 0. \end{cases}$$

Then, $\hat{F} = \{\hat{F}_i\}_{i \in \mathbb{Z}}$ is a filtration of ideals of B. Note that $\hat{F}_i$ coincides with $F_i B = F_i \otimes_A B$ for each i. Put

$$\begin{aligned} R(F) &= \oplus_{i \in \mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}] \\ G(F) &= R(F)/(t^{-1})R(F) = \oplus_{i \geq 0} F_i / F_{i+1} \\ R(\hat{F}) &= \oplus_{i \in \mathbb{Z}} \hat{F}_i t^i \subseteq B[t, t^{-1}] \\ G(\hat{F}) &= R(\hat{F})/(t^{-1})R(\hat{F}) = \oplus_{i \geq 0} \hat{F}_i / \hat{F}_{i+1}, \end{aligned}$$

where t is an indeterminate. Note that $R(F) \otimes_A B = R(\hat{F})$, $G(F) \otimes_A B = G(\hat{F})$ and $S = G(F) = G(\hat{F})$.

Flat homomorphisms $A \xrightarrow{\alpha} A[t, t^{-1}]$ and $R(F) \xrightarrow{\beta} R(F)[(t^{-1})^{-1}] = A[t, t^{-1}]$ induce the natural maps $\mathrm{G}_0(A) \xrightarrow{\alpha_*} \mathrm{G}_0(A[t, t^{-1}])$ and $\mathrm{G}_0(R(F)) \xrightarrow{\beta_*} \mathrm{G}_0(A[t, t^{-1}])$, respectively (see Definition 2.2). Since $R(F) \xrightarrow{\gamma} G(F)$ is finite, we have the induced map $\mathrm{G}_0(G(F)) \xrightarrow{\gamma^*} \mathrm{G}_0(R(F))$ by $\gamma([M]) = [M]$ for each finitely generated $G(F)$-module M. Thus we have the following diagram:

$$\begin{array}{ccccccc} & & & & \mathrm{G}_0(A) & & \\ & & & & \downarrow \alpha_* & & \\ \mathrm{G}_0(G(F)) & \xrightarrow{\gamma^*} & \mathrm{G}_0(R(F)) & \xrightarrow{\beta_*} & \mathrm{G}_0(A[t, t^{-1}]) & \longrightarrow & 0 \end{array}$$

It is known that the horizontal sequence in the above diagram is exact. We refer the basic facts on algebraic K-theory to Quillen [8] or Srinivas [12]. (The horizontal exact sequence is called the localization sequence induced by a localization of a category.)

On the other hand, we have the map $\gamma_* : \mathrm{G}_0(R(F)) \to \mathrm{G}_0(G(F))$ by $\gamma_*([N]) = [N/(t^{-1})N] - [0 :_N t^{-1}]$ for each finitely generated $R(F)$-module N. It is easy to see $\gamma_*\gamma^* = 0$. Hence, we obtain the induced map $\overline{\gamma_*} : \mathrm{G}_0(A[t, t^{-1}]) \to \mathrm{G}_0(G(F))$ that satisfies $\overline{\gamma_*}\beta_* = \gamma_*$ because of the exactness of the localization sequence.

Similarly we have the diagram

$$\begin{array}{ccccccc} & & & & \mathrm{G}_0(B) & & \\ & & & & \downarrow \hat{\alpha}_* & & \\ \mathrm{G}_0(G(\hat{F})) & \xrightarrow{\hat{\gamma}^*} & \mathrm{G}_0(R(\hat{F})) & \xrightarrow{\hat{\beta}_*} & \mathrm{G}_0(B[t, t^{-1}]) & \longrightarrow & 0 \end{array}$$

and the induced map $\overline{\hat{\gamma}_*} : \mathrm{G}_0(B[t, t^{-1}]) \to \mathrm{G}_0(G(\hat{F}))$.

Let $h : S \to A$ be the localization. Put $g_1 = g \otimes 1 : A[t, t^{-1}] \to A[t, t^{-1}] \otimes_A B = B[t, t^{-1}]$ and $g_2 = g \otimes 1 : G(F) \to G(F) \otimes_A B = G(\hat{F})$. Here, g_2 is an isomorphism.

Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} \mathrm{G}_0(S) & \xrightarrow{h_*} & \mathrm{G}_0(A) & \xrightarrow{\alpha_*} & \mathrm{G}_0(A[t, t^{-1}]) & \xrightarrow{\overline{\gamma_*}} & \mathrm{G}_0(G(F)) & & \\ & & \downarrow g_* & & \downarrow g_{1*} & & \downarrow g_{2*} & & \\ & & \mathrm{G}_0(B) & \xrightarrow{\hat{\alpha}_*} & \mathrm{G}_0(B[t, t^{-1}]) & \xrightarrow{\overline{\hat{\gamma}_*}} & \mathrm{G}_0(G(\hat{F})) & = & \mathrm{G}_0(S) \end{array}$$

We denote by $\varphi : \mathrm{G}_0(S) \to \mathrm{G}_0(S)$ the composite map as above.

Here, we need to show the following claim:

Claim 3.1 *φ is the identity map.*

We shall give an outline of a proof of the claim later.

Is is easy to see that h_* is surjective since h is a localization. Then, by the claim, we know that h_* is an isomorphism and $g_* : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ is injective.

Proof of Claim 3.1. It is easily verified that $\mathrm{G}_0(S)$ is generated by

$$\{[S/P] \mid P \text{ is a homogeneous prime ideal of } S\}.$$

Therefore, we have only to show $\varphi([S/P]) = [S/P]$ for a homogeneous prime ideal P of S. We leave it to the reader.

We have completed the proof in the case (2).

Next, we shall prove the following Claim:

Claim 3.2 *Let A be an excellent local ring and let I be an ideal of A such that* $\operatorname{Spec} A \setminus \operatorname{Spec} A/I$ *is a regular scheme. Let B be the I-adic completion of A. Then, the induced map* $\mathrm{G}_0(A) \to \mathrm{G}_0(B)$ *is injective.*

Note that, when I is the maximal ideal of A, the claim is equivalent to the case (3) in Theorem 1.5.

Now we start to prove Claim 3.2.

We put $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, $U = X \setminus Y$, $\hat{X} = \operatorname{Spec} B$, $\hat{Y} = \operatorname{Spec} B/IB$, $\hat{U} = \hat{X} \setminus \hat{Y}$. Then the natural map $\hat{Y} \to Y$ is an isomorphism and we have the fibre squares:

$$\begin{array}{ccccc} \hat{Y} & \longrightarrow & \hat{X} & \longleftarrow & \hat{U} \\ \| & & \downarrow & & \downarrow \\ Y & \longrightarrow & X & \longleftarrow & U \end{array}$$

Since the map $\hat{X} \to X$ is regular, so is $\hat{U} \to U$. Therefore, since U is a regular scheme, so is $\hat{U}$.

If U is empty, the assertion is obvious. Suppose that U is not empty.

Here we have the following commutative diagram:

$$\begin{array}{ccccccccccc} \mathrm{G}_1(X) & \xrightarrow{p} & \mathrm{G}_1(U) & \longrightarrow & \mathrm{G}_0(Y) & \longrightarrow & \mathrm{G}_0(X) & \longrightarrow & \mathrm{G}_0(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow r & & \downarrow s & & \downarrow t & & \downarrow u & & \\ \mathrm{G}_1(\hat{X}) & \xrightarrow{q} & \mathrm{G}_1(\hat{U}) & \longrightarrow & \mathrm{G}_0(\hat{Y}) & \longrightarrow & \mathrm{G}_0(\hat{X}) & \longrightarrow & \mathrm{G}_0(\hat{U}) & \longrightarrow & 0 \end{array}$$

Here, for a scheme W, $\mathrm{G}_i(W)$ denotes the i-th K-group of the exact category of coherent $\mathcal{O}_W$-modules. Horizontal sequences are exact (see Quillen [8] or Srinivas [12]). Vertical maps are induced by flat morphisms.

We denote by C (resp. $\hat{C}$) the cokernel of p (resp. q). Let $v : C \to \hat{C}$ be the induced map by r. In order to prove the injectivity of $t : \mathrm{G}_0(X) \to \mathrm{G}_0(\hat{X})$, we have only to show the following:

- $u : \mathrm{G}_0(U) \to \mathrm{G}_0(\hat{U})$ is injective.
- $v : C \to \hat{C}$ is surjective.

(Remember that $s : \mathrm{G}_0(Y) \to \mathrm{G}_0(\hat{Y})$ is an isomorphism.)

On the other hand, thanks to Thomason and Trobaugh [13], we have the localization sequence in K-theory, that is, we have the following commutative diagram:

$$\begin{array}{ccccccccccc} \mathrm{K}_1(X) & \xrightarrow{p'} & \mathrm{K}_1(U) & \to & \mathrm{K}_0(X \text{ on } Y) & \to & \mathrm{K}_0(X) & \xrightarrow{\alpha} & \mathrm{K}_0(U) & \to & \mathrm{K}_{-1}(X \text{ on } Y) \\ \downarrow & & \downarrow r' & & \downarrow s' & & \downarrow t' & & \downarrow u' & & \downarrow w' \\ \mathrm{K}_1(\hat{X}) & \xrightarrow{q'} & \mathrm{K}_1(\hat{U}) & \to & \mathrm{K}_0(\hat{X} \text{ on } \hat{Y}) & \to & \mathrm{K}_0(\hat{X}) & \xrightarrow{\beta} & \mathrm{K}_0(\hat{U}) & \to & \mathrm{K}_{-1}(\hat{X} \text{ on } \hat{Y}) \end{array}$$

Here, for a scheme W, $\mathrm{K}_i(W)$ denotes the i-th K-group of the category of locally free $\mathcal{O}_W$-modules of finite rank. We denotes by $\mathrm{K}_i(X \text{ on } Y)$ the i-th K-group of the derived category of perfect $\mathcal{O}_X$-complexes with support in Y (see Thomason and Trobaugh [13]).

Let W be a scheme. Then, by definition, we have the natural map $\xi_W : \mathrm{K}_i(W) \to \mathrm{G}_i(W)$ for each i. Furthermore, they are isomorphisms if W is a regular scheme (see 27p in Quillen [8]). Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_0(U) & \xrightarrow{\xi_U} & \mathrm{G}_0(U) \\ \downarrow u' & & \downarrow u \\ \mathrm{K}_0(\hat{U}) & \xrightarrow{\xi_{\hat{U}}} & \mathrm{G}_0(\hat{U}) \end{array}$$

Since both of U and $\hat{U}$ are regular schemes, both of ξ_U and $\xi_{\hat{U}}$ are isomorphisms. Thus, we know that u is injective if and only if so is u'.

On the other hand, since $\hat{X} \to X$ is flat and $Y = \hat{Y}$, we know that the natural map $\mathrm{K}_i(X \text{ on } Y) \to \mathrm{K}_i(\hat{X} \text{ on } \hat{Y})$ is isomorphism for each $i \in \mathbb{Z}$ by Theorem 7.1 in Thomason and Trobaugh [13]. In particular, s' and w' are isomorphisms. Furthermore, since A and B are local rings, we have $\mathrm{K}_0(A) = \mathrm{K}_0(B) = \mathbb{Z}$ and t' is an isomorphism. Since U and $\hat{U}$ are not empty, α and β are injective. Therefore, u' is injective.

Since both α and β are injective and s' is an isomorphism, we obtain that the cokernel of p' is isomorphic to that of q'. Therefore we have

$$\mathrm{K}_1(\hat{U}) = \mathrm{Im}(q') + \mathrm{Im}(r'). \tag{2}$$

On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{K}_1(U) & & \\
 & & \downarrow r' & \searrow \xi_U & \\
\mathrm{K}_1(\hat{X}) & \xrightarrow{q'} & \mathrm{K}_1(\hat{U}) & & \mathrm{G}_1(U) \\
 & \searrow \xi_{\hat{X}} & & \searrow \xi_{\hat{U}} & \downarrow r \\
 & & \mathrm{G}_1(\hat{X}) & \xrightarrow{q} & \mathrm{G}_1(\hat{U})
\end{array}$$

Since $\hat{U}$ is a regular scheme, $\xi_{\hat{U}}$ is an isomorphism. By the equation (2), we immediately obtain

$$\mathrm{G}_1(\hat{U}) = \mathrm{Im}(q) + \mathrm{Im}(r).$$

Therefore, the map $v : C \to \hat{C}$ is surjective. We have completed the proof of Theorem 1.5.

Remark 3.3 If a local ring satisfies one of (1), (2) and (3) in Theorem 1.5, we know, by Proposition 2.4, that $\mathrm{A}_i(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{A}_i(\hat{A})_{\mathbb{Q}}$ is injective for all i.

If a local ring satisfies one of (1) and (2) in Theorem 1.5, we can prove that $\mathrm{A}_i(A) \xrightarrow{f_*} \mathrm{A}_i(\hat{A})$ is injective for all i by the same method as in the proof of Theorem 1.5.

4 Motivation and Application

Here, we shall see motivation (or application) of Question 1.4.

(I) Let X be a scheme of finite type over a regular scheme S. Then, the singular Riemann-Roch theorem says that there exists an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{X/S} \;:\; \mathrm{G}_0(X)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(X)_{\mathbb{Q}}$$

satisfying several good properties (Chapter 18 in Fulton [1]). Remember that the construction of the map $\tau_{X/S}$ depends not only on X but also on S.

In fact, there are examples that the map $\tau_{X/S}$ actually depends on the choice of a regular base scheme S. Let k be an arbitrary field. Put $X = \mathbb{P}^1_k$ and $S = \operatorname{Spec} k$. Then, we have $\tau_{X/X}(\mathcal{O}_X) = [X]$ by the construction of $\tau_{X/X}$. On the other hand, by Hirzebruch-Riemann-Roch theorem, we obtain $\tau_{X/S}(\mathcal{O}_X) = [X] + \chi(\mathcal{O}_X)[t]$, where t is a rational point of X. It is well known that $\chi(\mathcal{O}_X) = 1$ and $[t] \neq 0$ in $\mathrm{A}_*(\mathbb{P}^1_k)_{\mathbb{Q}}$.

Let T be a regular local ring and let A be a homomorphic image of T. Then, by the singular Riemann-Roch theorem as above, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{A/T} \ : \ \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

determined by both of A and T.

Here, it seems to be natural to consider Conjecture 1.2. In fact, for many important local rings, the conjecture is true. (Conjecture 1.2 is affirmatively solved [3] if A is a complete local ring or A is essentially of finite type over either a field or the ring of integers.)

Here, look at the diagram (1). The bottom of the diagram (1) is independent of the choice of T since $\hat{A}$ is complete. Therefore if vertical maps in the diagram (1) are injective, $\tau_{A/T}$ is independent of the choice of T. Hence, we know that *if Question 1.4 is true for a local ring A, then Conjecture 1.2 is true for the local ring A.*

(II) Let A and T be rings as above and put $d = \dim A$. Put

$$\tau_A([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0, \quad (\tau_i \in \mathrm{A}_i(A)_{\mathbb{Q}}).$$

These τ_i's enjoy interesting properties as follows (see Proposition 3.1 in [4]):

(a) If A is a Cohen-Macaulay ring, then

$$\tau_A([\omega_A]) = \tau_d - \tau_{d-1} + \tau_{d-2} - \cdots + (-1)^i \tau_{d-i} + \cdots$$

is satisfied, where ω_A denotes the canonical module of A.

(b) If A is a Gorenstein ring, then we have $\tau_{d-i} = 0$ for each odd i.

(c) If A is a complete intersection, then we have $\tau_i = 0$ for $i < d$.

(d) τ_d is equal to $[\mathrm{Spec}\, A]_d$, where

$$[\mathrm{Spec}\, A]_d = \sum_{\substack{P \in \mathrm{Spec}\, A \\ \dim A/P = d}} \ell_{A_P}(A_P)[\mathrm{Spec}\, A/P] \in \mathrm{A}_d(A)_{\mathbb{Q}}.$$

In particular, we have $\tau_d \neq 0$.

(e) Assume that A is normal. Let $\mathrm{cl}(\omega_A) \in \mathrm{Cl}(A)$ be the isomorphism class containing ω_A. Then, we have $\tau_{d-1} = \mathrm{cl}(\omega_A)/2$ in $\mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$.

Here, we define the notion of *Roberts rings* as in Definition 1.1.

The category of Roberts rings contains complete intersections (see (c) as above), quotient singularities and Galois extensions of regular local rings. There are examples of Gorenstein non-Roberts rings. (It is proved that

$$\frac{k[x_{ij} \mid i = 1, \ldots, m \ ; \ j = 1, \ldots, n]_{(\{x_{ij}\})}}{I_t(x_{ij})}$$

is a Roberts ring if and only if it is a complete intersection.) In 1985, P. Roberts [9] proved that the vanishing property of intersection multiplicity is satisfied for Roberts rings. We refer the reader to basic facts and examples of Roberts rings to [3] and [4].

By the diagram (1), we immediately obtain that, if A is a Roberts ring, then so is the completion $\hat{A}$. (By the commutativity of diagram (1), we have $\tau_{\hat{A}/\hat{T}}([\hat{A}]) = f_*(\tau_{A/T}([A]))$. Note that the map $\mathrm{A}_*(A)_{\mathbb{Q}} \xrightarrow{f_*} \mathrm{A}_*(\hat{A})_{\mathbb{Q}}$ in the diagram (1) is graded.) Furthermore, if $\hat{A}$ is a Roberts ring and f_* is injective, then we know that A is a Roberts ring.

Furthermore, we can prove the following:

Proposition 4.1 *The following conditions are equivalent:*

(i) The induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{A})_{\mathbb{Q}}$ is injective for any local ring A, that is, Question 1.4 is true.

(ii) The induced map $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim A-1}(\hat{A})_{\mathbb{Q}}$ is injective for any local ring A.

(iii) The following is satisfied for any local ring A; if $\hat{A}$ is a Roberts ring, then so is A. (That is to say, Question 1.3 is true.)

(iv) For any étale local homomorphism $A \xrightarrow{g} B$ that is essentially of finite type, the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{g_*} \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective.

(v) For any étale local homomorphism $A \xrightarrow{g} B$ that is essentially of finite type, the induced map $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim A-1}(B)_{\mathbb{Q}}$ is injective.

(vi) For any étale local homomorphism $A \xrightarrow{g} B$ that is essentially of finite type, A is a Roberts ring if and only if so is B.

(vii) For any reduced equi-dimensional local ring (A, m), the following is satisdied: Let n be a positive integer and let $a_1, \ldots, a_n$ be elements in A. Assume that $a_n \in m$ and $a_{n-1} \notin m$. Put $B = A[x]_{(m,x)}/(x^n + a_1x^{n-1} + \cdots + a_n)$. (Note that B is étale over A.) Then, the induced map $\mathrm{G}_0(A)_{\mathbb{Q}} \xrightarrow{g_*} \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective.

(viii) Let A and B be rings that satisfy the same assumptions as in (vii). Then, the induced map $\mathrm{A}_{\dim A-1}(A)_{\mathbb{Q}} \to \mathrm{A}_{\dim A-1}(B)_{\mathbb{Q}}$ is injective.

(ix) Let A and B be rings that satisfy the same assumptions as in (vii). Then, A is a Roberts ring if and only if so is B.

By Theorem 1.5, we can prove that some of assertions in Proposition 4.1 are true if A has at most isolated singularity. For example, we have the folloing theorem:

Theorem 4.2 *Let $(A, m) \to (B, n)$ be an étale local homomorphism that is essentially of finite type. Assume that A is an excellent local ring that has at most isolated singularity. Then, we obtain the following:*

1. *The natural map $\mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective.*

2. *A is a Roberts ring if and only if so is B.*

Remark 4.3 Here assume that Question 1.4 is true for all local rings. Then, the following is satisfied:

Let $(A, m) \to (B, n)$ be a flat local homomorphism with closed fibre complete intersection. Suppose that the extension of the residue class fields is finitely generated (as a field) separable. Then, we have the following:

1. The natural map $\mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{G}_0(B)_{\mathbb{Q}}$ is injective.

2. A is a Roberts ring if and only if so is B.

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THE CONDUCTOR OF A k-CONFIGURATION IN $\mathbb{P}^n$

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This is a part of a joint work [4] with Professors A. V. Geramita and Y. S. Shin.

In [1], we gave an easily describable family of sets of points in $\mathbb{P}^n$ with a given Hilbert function, and called the sets of this family *k-configurations* in $\mathbb{P}^n$. Moreover, from [1, 2, 3], we see that k-configurations (among all sets of points with a fixed Hilbert function) have some extremal properties concerning the graded Betti numbers of their coordinate rings, the Hilbert functions of hyperplane sections of the sets and the graded Betti numbers of artinian Gorenstein rings associated to the sets, etc.. In this note, we show that k-configurations have a new extremal property with respect to the conductors of their coordinate rings. This answers (in the affirmative) a question raised by A. V. Geramita, P. Maroscia and L. Roberts in [5] for points in $\mathbb{P}^n$.

1. Preliminary

Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in $\mathbb{P}^n_k$, where $k = \overline{k}$ is an algebraically closed field, and let $A = \oplus_{i\geq 0} A_i$ be the (homogeneous) coordinate ring of $\mathbb{X}$. The numerical function $\mathbf{H}_{\mathbb{X}}(i) = \mathbf{H}(A, i) := \dim_k A_i$ is called the *Hilbert function* of the set $\mathbb{X}$ (or of the ring A) and put $\sigma(\mathbb{X}) := \max\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}$. The integral closure of A in its total ring of quotients is of the form $\overline{A} = \prod_{i=1}^{s} k[t_i]$ (where $k[t_i]$ is isomorphic to the coordinate ring of P_i). In [7], F. Orecchia showed that, as an ideal of $\overline{A}$, the *conductor* of $\mathbb{X}$ (or of A), $C_{\mathbb{X}} := \{a \in \overline{A} \mid a\overline{A} \subset A\}$, is of the form $C_{\mathbb{X}} = \prod_{i=1}^{s} t_i^{d_i} k[t_i]$, where d_i

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is the least degree of any hypersurface which passes through all of $\mathbb{X}$ except for P_i. We call d_i the *degree of conductor* of P_i in $\mathbb{X}$ and write $d_{\mathbb{X}}(P_i)$ for d_i. Then we have $d_{\mathbb{X}}(P) \leq \sigma(\mathbb{X}) - 1$ for all $P \in \mathbb{X}$ and it is known ([5, 6]) that $\mathbb{X}$ always has at least one point $P \in \mathbb{X}$ such that $d_{\mathbb{X}}(P) = \sigma(\mathbb{X}) - 1$ (the maximum possible). If all points of $\mathbb{X}$ have $d_{\mathbb{X}}(P) = \sigma(\mathbb{X}) - 1$, then $\mathbb{X}$ is called a *Cayley-Bacharach set of points* in $\mathbb{P}^n$. By relabelling if necessary, we always assume $d_1 \leq \cdots \leq d_s$ and we write $C_{\mathbb{X}} =< d_1, \cdots, d_s >$ as a short form for $\prod_{i=1}^{s} t_i^{d_i} k[t_i]$. Let $\mathbf{H}$ be a numerical function which can be the Hilbert function of a set of points in $\mathbb{P}^n$. We set $\mathcal{C}(\mathbf{H}) := \{< d_1, \cdots, d_s > \mid \prod_{i=1}^{s} t_i^{d_i} k[t_i]$ is the conductor of a set of points with Hilbert function $\mathbf{H}\}$ and put a partial ordering on $\mathcal{C}(\mathbf{H})$ by saying that $< d_1, \cdots, d_s > \leq < d'_1, \cdots, d'_s >$ if $d_i \leq d'_i$ for all $i = 1, \ldots, s$.

2. The conductor of a k-configuration in $\mathbb{P}^n$

Definition 1 (n-type vectors). We do this inductively.

1) A 0-*type vector* is defined to be $\mathcal{T} = 1$. It is the only 0-*type vector.* Then we define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.

2) A 1-*type vector* is a vector of the form $\mathcal{T} = (d)$ where d is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d$ and $\sigma(\mathcal{T}) = d$.

3) A 2-*type vector* is an ordered collection

$$\mathcal{T} = ((d_1), (d_2), \ldots, (d_u))$$

of 1-type vectors $(d_1, \ldots, d_u)$ such that $\sigma(d_i) < \alpha(d_{i+1})$, i.e., $d_i < d_{i+1}$ for $i = 1, \ldots, u-1$. For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = u$ and $\sigma(\mathcal{T}) = \sigma((d_u)) = d_u$.

4) Now let $n \geq 3$. An n-*type vector* is an ordered collection

$$\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$$

of $(n-1)$-type vectors $\mathcal{T}_1, \ldots, \mathcal{T}_u$ such that $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \ldots, u-1$. For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = u$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_u)$.

Definition 2 (k-configurations).

1) Let $\mathcal{T}$ be a 0-type vector. A single point in $\mathbb{P}^0$ is a *k-configuration in $\mathbb{P}^0$ of type $\mathcal{T}$.*

2) Let $\mathcal{T} = (d)$ be a 1-type vector. A set of d distinct points in $\mathbb{P}^1$ is called a *k-configuration in $\mathbb{P}^1$ of type $\mathcal{T}$.*

3) Let $\mathcal{T} = ((d_1), \dots, (d_u))$be a 2-type vector. A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a k-configuration in $\mathbb{P}^2$ of type $\mathcal{T}$ if there exist subsets $\mathbb{X}_1, \dots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct u lines $\mathbb{L}_1, \dots, \mathbb{L}_u$ in $\mathbb{P}^2$ such that
   i) $\mathbb{X} = \cup_{i=1}^{u} \mathbb{X}_i$,
   ii) $\mathbb{X}_i \subset \mathbb{L}_i$ ($\cong \mathbb{P}^1$) is a k-configuration in $\mathbb{P}^1$ of type (d_i) for all $1 \leq i \leq u$,
   iii) Every $\mathbb{L}_i$ does not contain any point of $\mathbb{X}_j$ for all $j < i$.

4) Now suppose that we have defined a k-configuration in $\mathbb{P}^{n-1}$ of type $\widetilde{\mathcal{T}}$, where $\widetilde{\mathcal{T}}$ is an $(n-1)$-type vector.
   Let $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_u)$ be a n-type vector. A finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ is called a k-configuration in $\mathbb{P}^n$ of type $\mathcal{T}$ if there exist subsets $\mathbb{X}_1, \dots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct u hyperplanes $\mathbb{H}_1, \dots, \mathbb{H}_u$ in $\mathbb{P}^n$ such that
   i) $\mathbb{X} = \cup_{i=1}^{u} \mathbb{X}_i$,
   ii) $\mathbb{X}_i \subset \mathbb{H}_i$ ($\cong \mathbb{P}^{n-1}$) is a k-configuration in $\mathbb{P}^{n-1}$ of type $\mathcal{T}_i$ for all $1 \leq i \leq u$,
   iii) Every $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for all $j < i$.

Proposition 3. Let $\mathbb{X} = \cup_{i=1}^{u} \mathbb{X}_i$ be a k-configuration in $\mathbb{P}^n$. Then

$$d_{\mathbb{X}}(P) = d_{\mathbb{X}_i}(P) + (u - i)$$

for all $P \in \mathbb{X}_i$ $(1 \leq i \leq u)$.

Proof. Please see [4].

Example 4. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^1$ of type (d). Then, one can easily check that

$$d_{\mathbb{X}}(P) = d - 1$$

for all $P \in \mathbb{X}$.

Example 5. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of type $(d_1, \ldots, d_m)$. Then, it follows by Proposition 3 and Example 4 that

$$d_{\mathbb{X}}(P) = d_i - 1 + m - i$$

for all $P \in \mathbb{X}_i$, where $1 \le i \le m$.

Example 6. Let $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ be a 3-type vector, where $\mathcal{T}_1 = (1,2), \mathcal{T}_2 = (1,3,4), \mathcal{T}_3 = (2,4,5,6,10)$. Let $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup \mathbb{X}_3$ be a k-configuration in $\mathbb{P}^3$ of type $\mathcal{T}$, and we write

$$\begin{aligned} \mathbb{X}_1 &= \mathbb{X}_{11} \cup \mathbb{X}_{12}, \\ \mathbb{X}_2 &= \mathbb{X}_{21} \cup \mathbb{X}_{22} \cup \mathbb{X}_{23} \\ \text{and} \quad \mathbb{X}_3 &= \mathbb{X}_{31} \cup \mathbb{X}_{32} \cup \mathbb{X}_{33} \cup \mathbb{X}_{34} \cup \mathbb{X}_{35}, \end{aligned}$$

where $\mathbb{X}_{ij}$ are sub-k-configurations of $\mathbb{X}_i$ $(1 \le i \le 3)$. We consider the rooted tree $T(\mathcal{T})$ associated to $\mathcal{T}$ and recall that every node of $T(\mathcal{T})$ corresponds to the unique sub-i-type vector of $\mathcal{T}(i = 1,2,3)$ and in particular every leave of $T(\mathcal{T})$ corresponds to the unique sub-1-type vector of $\mathcal{T}$. Furthermore, by the same way as in [1], we place the numbers on all the edges and the leaves, and $T(\mathcal{T})$ is labeled as follows:

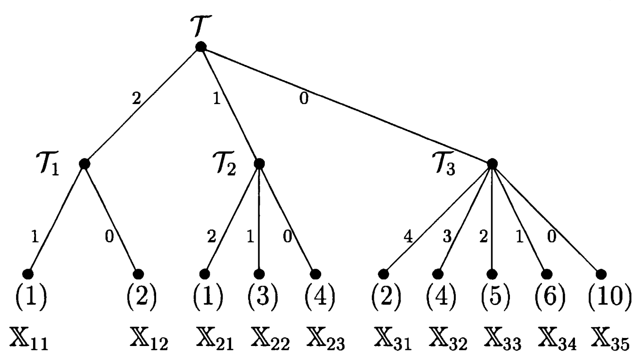


FIGURE 1

The number d on a leave in $T(\mathcal{T})$ is the number of points in the sub-k-configuration $\mathbb{X}_{ij}$ of $\mathbb{X}$ corresponding to the sub-1-type vector (d) of $\mathcal{T}$.

We can use this tree $T(\mathcal{T})$ to find the degrees of conductors of points in $\mathbb{X}$. By Proposition 3, we see that, for all $P \in \mathbb{X}_{ij}$,

$$\begin{aligned} d_{\mathbb{X}}(P) &= \text{(the number on the leave corresponding to } \mathbb{X}_{ij}) - 1 \\ &\quad + \sum \text{(the numbers on 2-edges to get to the root} \\ &\qquad\qquad \text{from the leave corresponding to } \mathbb{X}_{ij}). \end{aligned}$$

For example,

$$d_{\mathbb{X}}(P) = 2 - 1 + 4 + 0 = 5$$

for all $P \in \mathbb{X}_{31}$, and we have that

$$d_{\mathbb{X}}(P) = \begin{cases} 3 & P \in \mathbb{X}_{11} \cup \mathbb{X}_{12} \cup \mathbb{X}_{21}, \\ 4 & P \in \mathbb{X}_{22} \cup \mathbb{X}_{23}, \\ 5 & P \in \mathbb{X}_{31}, \\ 6 & P \in \mathbb{X}_{32} \cup \mathbb{X}_{33} \cup \mathbb{X}_{34}, \\ 9 & P \in \mathbb{X}_{35}. \end{cases}$$

3. Main theorem and comments

Theorem 7. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ with Hilbert function* $\mathbf{H}$. *Then $C_{\mathbb{X}} \le C_{\mathbb{X}'}$ for all $C_{\mathbb{X}'} \in \mathcal{C}(\mathbf{H})$.*

Proof. Please see [4].

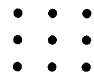
Problem (Problem 5.12, [5]). Does the construction of Theorem 4.1 in [5] always produce a set $\mathbb{X}$ of points for which $\dim(A/C_{\mathbb{X}})$ is as small as possible among those sets of points with Hilbert function $\mathbf{H}$?

Remark 8. We can observe that every set of points in $\mathbb{P}^n$ produced by the construction of Theorem 4.1 is a k-configuration. Hence we see that our theorem answers the question above.

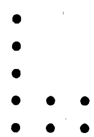
Example 9 ([8]). The following $\mathbb{X}_1, \ldots, \mathbb{X}_4$ are examples of sets of 9 points in $\mathbb{P}^2$ with Hilbert function $\mathbf{H}$,

$$\mathbf{H}: \ 1 \ \ 3 \ \ 6 \ \ 8 \ \ 9 \ \rightarrow .$$

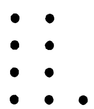
The permissible values for $\mathbf{H}$ are $\ell = 2, 3, 4$.



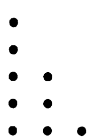
$$C_{\mathbb{X}_1} = (t_3^4 k[t_3])^9$$



$$C_{\mathbb{X}_2} = (t_2^3 k[t_2])^4 \times (t_3^4 k[t_3])^5$$



$$C_{\mathbb{X}_3} = (t_1^2 k[t_1]) \times (t_3^4 k[t_3])^8$$



$$C_{\mathbb{X}_4} = (t_1^2 k[t_1]) \times (t_2^3 k[t_2])^3 \times (t_3^4 k[t_3])^5$$

Since the set $\mathbb{X}_4$ is a k-configuration, we have that $C_{\mathbb{X}_4} \leq C_{\mathbb{X}'}$ for all $C_{\mathbb{X}'} \in \mathcal{C}(\mathbf{H})$, and $\dim_k A/C_{\mathbb{X}_4} = 31 \leq \dim_k A/C_{\mathbb{X}'}$. However, in this case, we can observe from [8] that these examples are all possible conductors of sets with Hilbert function $\mathbf{H}$, and $|\ \mathcal{C}(\mathbf{H})\ | = 4$. In [8], A. Sodhi gave an algorithm of describing all possible conductors of sets of points with the Hilbert function of a complete intersection in $\mathbb{P}^2$. This problem is open in general case.

Problem. Determine all possible conductors of sets of points in $\mathbb{P}^n$ with a given Hilbert function $\mathbf{H}$.

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On a certain upper bound for the regularity of monomial ideals

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序

極小自由分解の理論が進むにつれ、regularity という不変量の重要性が認識され、近年その研究が活発に行われてきた。特に、その上限を他の不変量で評価することが、その興味の中心になっている様に思われる。単項式イデアルの regularity に関しては、[Ho-Tr] において、様々な興味深い評価が与えられている。彼らは、単項式イデアルの 算術的次数や、極小生成系の次数による上限を与えている。[Ho-Tr] のおいては、代数的手法が主に用いられているが、[Fr-Te] や [Te2] においては、Hochster の公式 に基づいて、組合せ論的、位相幾何学的手法がとられている。本小論では、それらについて 解説したい。

多項式環の中の一般の次数付きイデアルの regularity に関して言えば、次の Eisenbud-Goto 予想が、その研究のためのモチヴェーションを与え続けてきた。

Eisenbud-Goto 予想 ([Ei-Go, Introduction]). $A = k[x_1, x_2, \ldots, x_n]$ を n 変数の体 k 上の多項式環とし、P を A の次数付き素イデアルで 1 次式を含まないものとする。このとき、

$$\mathrm{reg}\ P \leq \deg A/P - \mathrm{codim} A/P + 1。$$

この予想は、今も 可換環論や、代数幾何学で活発に研究されている。興味のある人は、例えば、[Kw] や [Mi-Vo] 及び そこに挙げられている参考文献を見られたい。

本小論においては、Eisenbud により予想された単項式版の Eisenbud-Goto 予想について考察する。

定理 0.1 (Eisenbud の予想)(cf. [Fr-Te], [Te2]). k を体とし、Δ を pure で strongly connected な単体的複体とする。このとき、

$$\mathrm{reg}\ I_\Delta \leq \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1。$$

この定理から Gröbner 基底の理論を用いてもともとの Eisenbud-Goto 予想に対しては次のことが言える。

系 0.2. $A = k[x_1, x_2, \ldots, x_n]$ を n 変数の体 k 上の多項式環とし、P を A の次数付き素イデアルで 1 次式を含まないものとする。さらに $A/\mathrm{in}P$ は reduced であると仮定する。ここで、$\mathrm{in}P$ は、ある項順序に関する P の initial ideal とする。このとき、

$$\mathrm{reg}\ P \leq \deg A/P - \mathrm{codim} A/P + 1。$$

次に Eisenbud-Goto の不等式において等号が成立する場合について考察する。このとき、次の定理が成立する。

定理 0.3 ([Te2]). k を体とし、Δ を pure で strongly connected な $(d-1)$ 次元の単体的複体とする。$r = \mathrm{reg} I_\Delta$ とおく。このとき、

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1$$

であるための必要十分条件は、Δ が次の条件を満たすこと。
(1) もし、$r = 2$ ならば、Δ は、$(d-1)$-tree であること。ただし、$(d-1)$ 単体ではないこと。
(2) もし、$r = 3$ ならば、ある $(d-1)$-tree Δ' とある separated な $v, w \in V(\Delta')$ に対して $\Delta = \Delta'(v \to w)$ となること。
(3) もし、$r \geq 4$ ならば、$\Delta \cong \partial\Delta(r) * \Delta(d-r+1) + ((d-1)\text{-branches})$。

定義されていない用語に対しては、§1 及び §4 を見られたい。

1. PRELIMINARIES

We first fix notation. Let $\mathbf{N}$(resp.$\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). Let $\mid S \mid$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings. We refer the reader to, e.g., [Br–He], [Hi], [Hoc] and [St] for the detailed information about combinatorial and algebraic background.

A *(abstract) simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. The vertex set of Δ is denoted by $V(\Delta)$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $|F| = i+1$ and we call a maximal face a *facet*. Let F be a face but not a facet. We call F *free* if there is a unique facet G such that $F \subset G$. If $\{x_i\}$ is free, we call x_i free.

We define the *dimension* of $F \in \Delta$ to be $\dim F = |F| - 1$ and the *dimension* of Δ to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. We say that Δ is *pure* if all facets have the same dimension. In a pure $(d-1)$-dimensional complex Δ, we call $(d-2)$- face a *subfacet*. We say that a pure complex Δ is *strongly connected* if for any two facets F and G, there exists a sequence of facets

$$F = F_0, F_1, \ldots, F_m = G$$

such that $F_{i-1} \cap F_i$ is a subfacet for $i = 1, 2, \ldots, m$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d-1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

Next we summarize basic facts on the Hilbert series. Let k be a field and R a homogeneous k-algebra. By a homogeneous k-algebra R we mean a noetherian graded ring $R = \bigoplus_{i \geq 0} R_i$ generated by R_1 with $R_0 = k$. Let M be a graded R-module with $\dim_k M_i < \infty$ for all $i \in \mathbf{Z}$, where $\dim_k M_i$ denotes the dimension of M_i as a k-vector space. The *Hilbert series* of M

is defined by

$$F(M,t) = \sum_{i\in\mathbf{Z}} (\dim_k M_i)t^i.$$

It is well known that the Hilbert series $F(R,t)$ of R can be written in the form

$$F(R,t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{\dim R}},$$

where $h_0(=1), h_1, \ldots, h_s$ are integers with $\deg R := h_0 + h_1 + \cdots + h_s \geq 1$, which is called the *degree* of R. The vector $h(R) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R. We consider $k[\Delta]$ as the graded algebra $k[\Delta] = \bigoplus_{i\geq 0} k[\Delta]_i$ with $\deg x_j = 1$ for $1 \leq j \leq n$. The Hilbert series $F(k[\Delta],t)$ of a Stanley-Reisner ring $k[\Delta]$ can be written as follows:

$$\begin{aligned} F(k[\Delta],t) &= 1 + \sum_{i=1}^{d} \frac{f_{i-1}t^i}{(1-t)^i} \\ &= \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}, \end{aligned}$$

where $\dim \Delta = d-1$, $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of Δ, and $(h_0, h_1, \ldots, h_d)$ is the h-vector of Δ. It is easy to see $\deg k[\Delta] = f_{d-1}$. On the other hand, the *arithmetic degree* of $k[\Delta]$ is defined to be the number of facets in Δ, which is denoted by a-deg$k[\Delta]$. See, e.g., [Ho-Tr] for the definition of the arithmetic degree of a general ring R.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ over a field k. Let $M(\neq 0)$ be a finitely generated graded A-module and let

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A. The length h of this resolution is called the *projective dimension* of M and denoted by $h = \mathrm{pd}M$. We call $\beta_i(M) = \sum_{j\in\mathbf{Z}} \beta_{i,j}(M)$ the i-th *Betti number* of M over A. We define the *Castelnuovo-Mumford regularity* reg M of M by

$$\mathrm{reg}\ M = \max\ \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

See, e.g., [Ei] for further information on regularity. We define the *initial degree* indeg M of M by

$$\mathrm{indeg}\ M = \min\ \{i \mid M_i \neq 0\} = \min\ \{j \mid \beta_{0,j}(M) \neq 0\}.$$

Let l be a natural number. We say that M satisfies (N_l) condition if $\beta_{i,i+s}(M) = 0$ for $i < l$, $s \neq \operatorname{indeg} M$.

We denote the number of generators of M by $\mu(M) = \beta_0(M)$.

The following two theorem are a starting point for our study.

THEOREM 1.1 (Hochster's formula on the Betti numbers [Hoc, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F \subset [n],\ |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

It is easy to see:

COROLLARY 1.2.
$\operatorname{reg} I_\Delta = \max\ \{i + 2 \mid \tilde{H}_i(\Delta_F; k) \neq 0 \text{ for some } F \subset V\}$.

If F is a face of Δ, then we define a subcomplex $\operatorname{link}_\Delta F$ by

$$\operatorname{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

THEOREM 1.3 (Hochster's formula on the local cohomology modules (cf. [St, Theorem 4.1])).

$$F(H^i_{\boldsymbol{m}}(k[\Delta]),\ t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-|F|-1}(\operatorname{link}_\Delta F; k) \left(\frac{t^{-1}}{1 - t^{-1}}\right)^{|F|}.$$

where $H^i_{\boldsymbol{m}}(k[\Delta])$ denote the i-th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $\boldsymbol{m}$.

COROLLARY 1.4.
$\operatorname{reg} I_\Delta = \max\ \{i + 2 \mid \tilde{H}_i(\operatorname{link}_\Delta F; k) \neq 0 \text{ for some } F \in \Delta\}$.

Next we recall the definition of Alexander dual complexes. For a simplicial complex Δ on the vertex set V, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset V : V \setminus F \notin \Delta\}.$$

THEOREM 1.5 [Te1, Corollary 0.3]. *Let k be a field. Let Δ be a simplicial complex. Then*

$$\text{reg } I_\Delta = \text{ pd } k[\Delta^*].$$

2. REGULARITY OF THE SUM OF IDEALS

In this section we give a upper bound for the sums of sqare-free monomial ideals.

In the rest of the paper we always assume that k is a fixed field.

First we prove the following proposition. It seems to be known, but we cannot find it in literature.

PROPOSITION 2.1. *Let I be a monomial ideal in the polynomial ring $A = k[x_1, x_2, \ldots, x_n]$ and m a monomial in A. Then*

$$\text{pd}A/(I + (m)) \leq \text{ pd}A/I + 1.$$

For the regularity of the sum of square-free monomial ideals, we have the following conjecture:

CONJECTURE 2.2. *Let $\Delta_i(\neq \emptyset)$ be a simplicial complex for $i = 1, 2$. Then we have*

$$\text{reg}(I_{\Delta_1} + I_{\Delta_2}) \leq \text{reg } I_{\Delta_1} + \text{reg } I_{\Delta_2} - 1.$$

If I_{Δ_1} and I_{Δ_2} are complete intersections, then the above inequality holds. The next theorem gives a weaker upper bound.

THEOREM 2.3. *Let $\Delta_i(\neq \emptyset)$ be a simplicial complex for $i = 1, 2$. Then we have*

$$\text{reg}(I_{\Delta_1} + I_{\Delta_2}) \leq \min\{\text{reg } I_{\Delta_1} + \text{a-deg}k[\Delta_2],\ \text{reg } I_{\Delta_2} + \text{a-deg}k[\Delta_1]\} - 1.$$

REMARK. Since the inequality reg $I_\Delta \leq$ a-deg$k[\Delta]$ holds (cf. [Ho-Tr] and [Fr-Te]), Theorem 2.3 is weaker than Conjecture 2.2.

3. EISENBUD-GOTO INEQUALITY

The theme of this section is Eisenbud-Goto inequality for Stanley-Reisner rings of pure and strongly connected simplicial complexes.

First we give a lemma which is necessary for inductive argument.

LEMMA 3.1. *Let Δ be a pure and strongly connected simplicial complex. Then there exists a facet $F \in \Delta$ such that*

$$\Delta' := \{H \in \Delta \mid H \subset G \text{ for some facet } G(\neq F) \in \Delta\}$$

is pure and strongly connected.

Now we give the main result in this section.

THEOREM 3.2(cf. [Fr-Te, Theorem 4.1]). Let Δ be a pure and strongly connected simplicial complex. Then we have

$$\mathrm{reg} I_\Delta \leq \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

COROLLARY 3.3. *Let Δ be a simplicial complex such that* $\mathrm{codim} k[\Delta] \geq 2$. *Assume I_Δ satisfies* (N_2) *condition. Then we have*

$$\mathrm{pd} k[\Delta] \leq \mu(I_\Delta) - \mathrm{indeg} I_\Delta + 1.$$

4. EQUALITY CASE

In this section, we classify pure and strongly conneceted simplicial complexes Δ which satisfy $\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1$, and give some characterization for such complexes.

First we introduce some notation. Put $[m] = \{1, 2, \ldots, m\}$. We denote the elementary $(m-1)$-simplex by $\Delta(m) = \mathbf{2}^{[m]}$ and put $\Delta(0) = \{\emptyset\}$. We put $\partial\Delta(m) = \mathbf{2}^{[m]} \setminus \{[m]\}$, which is the boundary complex of $\Delta(m)$.

Let Δ_i be a $(d-1)$-dimensional pure simplicial complex for $i = 1, 2$. If $\Delta_1 \cap \Delta_2 = \mathbf{2}^F$ for some F with $\dim F = d - 2$, we write $\Delta_1 \cup_F \Delta_2$ for $\Delta_1 \cup \Delta_2$. We sometimes write $\Delta_1 \cup_* \Delta_2$ for $\Delta_1 \cup_F \Delta_2$ if we do not need to express F explicitly.

We define a $(d-1)$-tree inductively as follows.
(1)$\Delta(d)$ is a $(d-1)$-tree.
(2)if Υ is a $(d-1)$-tree, then so is $\Upsilon \cup_* \Delta(d)$.

If $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$ are $(d-1)$-trees, we abbreviate $\Delta \cup_* \Upsilon_1 \cup_* \Upsilon_2 \cup_* \cdots \cup_* \Upsilon_m$ as $\Delta + ((d-1)\text{-branches})$.

Let Δ be a $(d-1)$-dimensional pure and strongly connected complex. Take $v,\ w \in V(\Delta)$. We say v and w are *separated* in Δ if $\{v,\ w\} \notin \Delta$ and that there exists no subfacet F in Δ with $\{v\} \cup F,\ \{w\} \cup F \in \Delta$. If v and w are separated in Δ, We denote $\Delta(v \to w)$ for the abstract simplicial complex which is obtained by substitution of w for every v in Δ. The vertex set of $\Delta(v \to w)$ is $V(\Delta) \setminus \{v\}$.

By Lemma 3.1 we know that every $(d-1)$-dimensional pure and strongly connected simplicial complex can be constructed from the $(d-1)$-dimensional elementary simplex $\Delta(d)$ by a succession

$$\Delta(d) = \Delta_1 \to \Delta_2 \to \cdots \to \Delta_{f_{d-1}}$$

of either of the following operations :
(1)$\Delta_{i+1} = \Delta_i \cup_{F'} \mathbf{2}^F$, where $x \notin V(\Delta_i)$, F' is a subfacet of Δ_i and $F = F' \cup \{x\}$.
(2)$\Delta_{i+1} = (\Delta_i \cup_{F'} \mathbf{2}^F)(x \to y)$, where $x \notin V(\Delta_i)$, F' is a subfacet of Δ_i and $y \in V(\Delta_i)$ such that x and y are separated and $F = F' \cup \{x\}$.

Let Δ_i be a simplicial complex for $i = 1, 2$ such that $V(\Delta_1) \cap V(\Delta_2) = \emptyset$. We define the simplicial join $\Delta_1 * \Delta_2$ of Δ_1 and Δ_2 by

$$\Delta_1 * \Delta_2 = \{F \cup G \mid F \in \Delta_1, G \in \Delta_2\}.$$

LEMMA 4.1. *Let Δ be a $(d-1)$-dimensional pure and strongly connected complex. We assume*

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1 = 3.$$

Then Δ can be expressed as follows:

$$\Delta \cong \Delta'(x \to y) * \Delta(d-s) + ((d-1)\text{-branches})$$

for some $(s-1)$-tree Δ' and for some separated $x,\ y \in V(\Delta')$ with $\tilde{H}_1(\Delta'(x \to y); k) \neq 0$.

THEOREM 4.2. *Let Δ be a $(d-1)$-dimensional pure and strongly connected complex. We put $r = \mathrm{reg} I_\Delta$. Then*

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

if and only if Δ *satisfies the following condition:*
(1) Δ *is a* $(d-1)$*-tree which is not the* $(d-1)$*-simplex if r=2.*
(2) $\Delta = \Delta'(v \to w)$ *for some* $(d-1)$*-tree* Δ' *and for some separated* $v, w \in V(\Delta')$ *if r=3.*
(3)$\Delta \cong \partial\Delta(r) * \Delta(d-r+1) + ((d-1)\text{-branches})$ *if* $r \geq 4$.

COROLLARY 4.3. *Let* Δ *be a* $(d-1)$*-dimensional pure and strongly connected complex on the vertex set* $[n]$*. Assume* $r := \mathrm{reg} I_\Delta \geq 4$*. Then the following conditions are equivalent:*
(1)

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

(2)

$$\Delta \cong \partial\Delta(r) * \Delta(d-r+1) + ((d-1)\text{-branches}).$$

(3) $k[\Delta]$ *is Cohen-Macaulay with h-vector* $(1, n-d, 1, \ldots, 1(= h_{r-1}))$.
(4)

$$\beta_{i,i+j}(k[\Delta]) = \begin{cases} 1, & \text{for } i = j = 0 \\ (n-d-1)\binom{n-d}{i} - \binom{n-d-1}{i+1}, & \text{for } j = 1, i = 1, 2, \ldots, n-d \\ \binom{n-d-1}{i-1}, & \text{for } j = r-1, i = 1, 2, \ldots, n-d \\ 0, & \text{otherwise.} \end{cases}$$

(5)

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \begin{cases} 0, & \text{for } i \neq d \\ \frac{t^{-d+r-1}+t^{-d+r-2}+\cdots+t^{-d+2}+(n-d)t^{-d+1}+t^{-d}}{(1-t^{-1})^d}, & \text{for } i = d. \end{cases}$$

COROLLARY 4.4. *Let* Δ *be a* $(d-1)$*-dimensional pure and strongly connected complex on the vertex set* $[n]$*. Assume* $\mathrm{reg} I_\Delta = 3$ *and* $k[\Delta]$ *satisfies* (S_2) *condition. Then the following conditions are equivalent:*
(1)

$$\mathrm{reg} I_\Delta = \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

(2)

$$\Delta = \Delta(l\text{-gon}) * \Delta(d-2) + ((d-1)\text{-branches})$$

for some $l \geq 3$*, where* $\Delta(l\text{-gon})$ *is the boundary complex of the* l*-gon.*
(3) $k[\Delta]$ *is Cohen-Macaulay with h-vector* $(1, n-d, 1)$.

(4)

$$\beta_{i,i+j}(k[\Delta]) = \begin{cases} 1, & \text{for } i=j=0 \\ \frac{i(n-d-i)}{n-d+i}\binom{n-d+2}{i+1} + \binom{n-d-l+2}{i-l+1}, & \text{for } j=1,\ i=1,2,\ldots,n-d \\ \binom{n-d-l+2}{i-l+2}, & \text{for } j=2,\ i=1,2,\ldots,n-d \\ 0, & \text{otherwise} \end{cases}$$

for some $l \geq 3$.

(5)

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \begin{cases} 0, & \text{for } i \neq d \\ \frac{t^{-d+2}+(n-d)t^{-d+1}+t^{-d}}{(1-t^{-1})^d}, & \text{for } i = d. \end{cases}$$

REMARK. A Cohen-Macaulay homogeneous ring R with h-vector $h(R) = (1, h_1, 1, 1, \ldots, 1)$ is called a *stretched* Cohen-Macalay ring (cf.[Oo]). These corollaries also give the classification of stretched Cohen-Macaulay Stanley-Reisner rings.

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Gorenstein associated graded rings of analytic deviation two ideals

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1 Introduction.

Let A be a Gorenstein local ring with the maximal ideal $\mathfrak{m}$ and $\dim A = d$. We assume that the field $A/\mathfrak{m}$ is infinite. This paper studies the question of when the associated graded ring $\mathcal{G}(I) := \bigoplus_{n \geq 0} I^n/I^{n+1}$ of an ideal I in A is a Gorenstein ring. We shall give a characterization of Gorenstienness in the graded ring associated to certain ideal having analytic deviation two. Our main result implies, for example, that if A is a regular local ring, $\mathfrak{p}$ is a prime ideal in A with $\dim A/\mathfrak{p} = 2$, and $A/\mathfrak{p}$ is a complete intersection in codimension one, then the associated graded ring $\mathcal{G}(\mathfrak{p})$ is Gorenstein if and only if the reduction number of $\mathfrak{p}$ is at most 1.

Before stating our main result, let us fix some notation. Let $I(\neq A)$ be an ideal in A of height s. We put $\ell = \lambda(I) := \dim A/\mathfrak{m} \otimes_A \mathcal{G}(I)$ and call it the analytic spread of I. Let $\mathrm{ad}(I) := \ell - s$ that we call the analytic deviation of I ([HH]). Let J be a minimal reduction of I. Hence $J \subseteq I$ and $I^{n+1} = JI^n$ for some $n \geq 0$. We put $\mathrm{r}_J(I) := \min\{n \geq 0 \mid I^{n+1} = JI^n\}$ and call it the reduction number of I with respect to J. Suppose that $a_1, a_2, \ldots, a_\ell$ is a minimal system of generators for the minimal reduction J of I satisfying the following two conditions.

(*) $J_i A_\mathfrak{p}$ is a reduction of $IA_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{V}(I)$ with $i = \mathrm{ht}_A \mathfrak{p} \leq \ell$.

(**) $a_i \notin \mathfrak{p}$ if $\mathfrak{p} \in \mathrm{Ass}_A A/J_{i-1} \setminus \mathrm{V}(I)$ for any $1 \leq i \leq \ell$.

Here $\mathrm{V}(I)$ denotes the set of prime ideals in A containing I, and we set $J_i = (a_1, a_2, \cdots, a_i)$ for $0 \leq i \leq \ell$. According to [GNN1], 2.1, there always exists a minimal system of generators $a_1, a_2, \cdots, a_\ell$ for J satisfying conditions (*) and (**). We put

$$\mathrm{r}_i = \max\{\mathrm{r}_{J_{i\mathfrak{p}}}(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{V}(I) \text{ and } \mathrm{ht}_A \mathfrak{p} = i\} \text{ for any } s \leq i < \ell.$$

Our ideal I is said to be generically a complete intersection if $\mathrm{r}_s = 0$. Let

$$U = \mathrm{U}(I) := \bigcap_{\mathfrak{p} \in \mathrm{Assh}_A A/I} (IA_\mathfrak{p} \cap A),$$

where $\mathrm{Assh}_A A/I = \{\mathfrak{p} \in \mathrm{V}(I) \mid \dim A/I = \dim A/\mathfrak{p}\}$. If $I = U$, then we say I is unmixed. We denote the a-invariant of $\mathcal{G}(I)$ by $\mathrm{a}(\mathcal{G}(I))$ ([GW], 3.1.4). With this notation the main result of this paper is stated as follows.

Theorem 1.1. *Assume that* $\mathrm{ad}(I) \leq 2$ *and* $\mathrm{ht}_A \mathfrak{p} < s+2$ *for any* $\mathfrak{p} \in \mathrm{Ass}_A A/I$. *Suppose that* $\mathcal{G}(I)$ *is a Cohen-Macaulay ring. Then the following two conditions are equivalent.*

(1) $\mathcal{G}(I)$ *is a Gorenstein ring and* $\mathrm{a}(\mathcal{G}(I)) = -s$.

(2) (a) $\mathrm{r}_J(I) \leq \mathrm{ad}(I)$,

(b) $\mathrm{r}_i \leq i - s$ *for any* $s \leq i < \ell$, *and*

(c) $I = J_{s+1}U :_U (J_{s+1}I :_I I)$.

When this is the case, if $\mathrm{ad}(I) = 2$, *then the following two assertions are satisfied:*

(i) $I = U$ *if and only if* $\mathrm{r}_{s+1} = 0$;

(ii) $\mathrm{r}_{J_\mathfrak{p}}(I_\mathfrak{p}) \leq 1$ *for any* $\mathfrak{p} \in \mathrm{V}(I)$ *such that* $\mathrm{ht}_A \mathfrak{p} = \ell$.

The Cohen-Macaulay property of the associated graded ring $\mathcal{G}(I)$ closely studied, and we have a satisfactory criterion given by the first author, Y. Nakamura, and K. Nishida ([GNN1]). If $\mathcal{G}(I)$ is a Cohen-Macaulay ring, then we have $\operatorname{depth} A/I^n \geq d - \ell$ for all $n \geq 1$, so that $\mathrm{ht}_A \mathfrak{p} \leq \ell$ for any $\mathfrak{p} \in \mathrm{Ass}_A A/I^n$. Hence when $\mathrm{ad}(I) \leq 1$, the Cohen-Macaulayness of $\mathcal{G}(I)$ require the condition that $\mathrm{ht}_A \mathfrak{p} < s+2$ for any $\mathfrak{p} \in \mathrm{Ass}_A A/I$.

Many authors have studied the Gorensteinness of the associated graded rings $\mathcal{G}(I)$. However, almost all authors assumed the ring A/I is Cohen-Macaulay and we lack satisfactory references analyzing ideals for which the rings A/I are not necessarily Cohen-Macaulay. In our theorem 1.1 we have assumed that $\mathrm{ht}_A \mathfrak{p} < s+2$ for any $\mathfrak{p} \in \mathrm{Ass}_A A/I$ but the assumption that the ring A/I is Cohen-Macaulay is removed. The first author and Y. Nakamura [GNa1] explored generically complete intersection ideals I (which are not necessarily Cohen-Macaulay ideals) with $\mathrm{ad}(I) = 1$ in a Gorenstein local ring A of $\dim A = 1$ and gave in that case a criterion for the Gorensteinness in $\mathcal{G}(I)$. We have recently succeed in generalizing their result to the case where the dimension of A is arbitrary ([GI]). By our theorem 1.1 we have overcome the assumption that $\mathrm{ad}(I) = 1$ in [GI] also, and get a characterization in the case where $\mathrm{ad}(I) = 2$.

We now briefly mention the contents of the paper. The proof of Theorem 1.1, which will be given in Section 3, is based on the case where $d = \ell = 2$ and $s = 0$. In Section 2 we shall devote Theorem 1.1 in such a case. In Section 4 we will summarize some corollaries for unmixed ideals derived from our theorem.

Before entering into details, let us fix again the standard notation in this paper. Throughout let $(A, \mathfrak{m})$ denote a Gorenstein local ring with $\dim A = d$ and assume that the field $A/\mathfrak{m}$ is infinite. Let I $(\neq A)$ be an ideal in A with a minimal reduction J. We put $s = \mathrm{ht}_A I$ and $\ell = \lambda(I)$. Suppose that a minimal system of generators $a_1, a_2, \cdots, a_\ell$ of J satisfy conditions $(*)$ and $(**)$. We set $J_i = (a_1, a_2, \cdots, a_i)$. Let $U = \mathrm{U}(I)$ that is the unmixed component of I. We put $\mathcal{R}'(I) = A[It, t] \subseteq A[t, t^{-1}]$ (t is an indeterminate over A) and $\mathcal{G}(I) = \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I)$. We denote $\mathcal{G}(I)$ simply by G and put $a = \mathrm{a}(G)$, which denotes a-invariant of G. Let $\mathfrak{M} = \mathfrak{m}G + G_+$. We denote by $\mathrm{H}^i_\mathfrak{M}(*)$ $(i \in \mathbb{Z})$ the $i^{\underline{\mathrm{th}}}$ local cohomology functor of G with respect to $\mathfrak{M}$. For each graded G-module E, let $[E]_n$ stand for the homogeneous component of E of degree n and let $\mathrm{a}_i(E) = \sup\{n \in \mathbb{Z} \mid [\mathrm{H}^i_\mathfrak{M}(E)]_n \neq (0)\}$ $(i \in \mathbb{Z})$. Let $\mathcal{A} = \{\mathfrak{p} \in \mathrm{V}(I) \mid \mathrm{ht}_A \mathfrak{p} = \dim G_\mathfrak{p}/\mathfrak{p}G_\mathfrak{p}\}$. We denote by K_G the graded canonical modules of G. We shall freely refer to [BH], [GN], [HK], and [HIO] for details of the theory on canonical modules.

2 The case where $d = \ell = 2$ and $s = 0$.

Throughout this section we always assume that $d = \ell = 2$ and $s = 0$. Let I be generically a complete intersection. The purpose of this section is to prove the next proposition.

Proposition 2.1. *Assume that* $\operatorname{depth} A/I > 0$. *Then the following two conditions are equivalent.*

(1) *G is a Gorenstein ring.*

(2) (i) $\mathrm{r}_J(I) \leq 1$,

(ii) $\mathrm{r}_1 \leq 1$, *and*

(iii) $I = a_1U :_U (a_1I :_I I)$.

When this is the case, $I = U$ *if and only if* $\mathrm{r}_1 = 0$.

When the conditions (i) and (ii) are satisfied, we get G is a Cohen-Macaulay ring if $\operatorname{depth} A/I > 0$ (see [GNN1]). Then from the a-invariant formula:

$$\mathrm{a}(G) = \max\{\{\mathrm{r}_i - i \mid s \leq i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}\}$$

(cf. [U], 1.4) we obtain that $a = 0$. Conversely when G is a Gorenstein ring, we have $a = \mathrm{a}(G_\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{V}(I)$, so that $a = 0$ (recall that $\mathrm{r}_0 = 0$). Therefore we get $\mathrm{r}_1 \leq 1$ by the a-invariant formula. Hence in the rest of this section we consider the case where $\operatorname{depth} A/I > 0$, $\mathrm{r}_1 \leq 1$, and G is a Cohen-Macaulay ring with $a = 0$. Before proving Proposition 2.1, let us summarize the following. Let $\mathfrak{a} = (0) : I$ and let $\overline{A} = A/\mathfrak{a}$. We have $U \cap \mathfrak{a} = (0)$ and hence a_1 is $\overline{A}$-regular element (see [GNa1], 2.1). Let $\mathrm{Q}(\overline{A})$ denote the total quotient ring of $\overline{A}$. We consider a commutative A-algebra

$$B := I\overline{A} :_{\mathrm{Q}(\overline{A})} I\overline{A}$$

that is finite as A-module. We have $\operatorname{depth} I\overline{A} = 2$ because $\operatorname{depth} A/I > 0$ and $I\overline{A} \cong I$ (recall that $I \cap \mathfrak{a} = (0)$). Therefore the ring B is a maximal Cohen-Macaulay A-module.

Lemma 2.2. $\mathrm{K}_B \cong a_1U :_U (a_1I :_I I)$.

Proof. Let $D = a_1I\overline{A} :_{I\overline{A}} I\overline{A}$ and let $\alpha = \overline{a_1}$, which denotes the coset of $\mathfrak{a}$ containing a_1. Then we have

$$B = \frac{D}{\alpha} := \left\{ \frac{\beta}{\alpha} \mid \beta \in D \right\}$$

in $\mathrm{Q}(\overline{A})$. In fact, it is routine to check $B \supseteq \dfrac{D}{\alpha}$. Conversely, we take $\gamma \in B$. Then $\alpha\gamma \in I\overline{A}$ and hence $\gamma = \beta/\alpha$ for some $\beta \in I\overline{A}$. Since $I\overline{A} \cdot \beta = (I\overline{A} \cdot \gamma) \cdot \alpha \subseteq I\overline{A} \cdot \alpha$ we have $\beta \in D$. Therefore $\gamma \in \dfrac{D}{\alpha}$.

We obtain that

$$\begin{aligned} \mathrm{K}_B := U\overline{A} :_{\mathrm{Q}(\overline{A})} B &= U\overline{A} :_{\overline{A}} \frac{D}{\alpha} \\ &= \alpha U\overline{A} :_{\overline{A}} D = \frac{(a_1U + \mathfrak{a}) :_A ((a_1I + \mathfrak{a}) :_I I)}{\mathfrak{a}}. \end{aligned}$$

Since $I \cap \mathfrak{a} = (0)$, we have $a_1U :_A (a_1I :_I I) = (a_1U + \mathfrak{a}) :_A ((a_1I + \mathfrak{a}) :_I I)$ and hence

$$\mathrm{K}_B = \frac{a_1U :_A (a_1I :_I I)}{\mathfrak{a}}.$$

And we obtain that the natural homomorphism

$$a_1U :_U (a_1I :_I I) \to \frac{a_1U :_A (a_1I :_I I)}{\mathfrak{a}}$$

is bijective. In fact, since $U \cap \mathfrak{a} = (0)$, we have the map φ is injective. Let $x \in a_1U :_A (a_1I :_I I)$. We get $a_1x \in a_1U$, as $a_1 \in a_1I :_I I$. Hence $x \in U + \mathfrak{a}$ because $\mathfrak{a} = (0) : a_1$ (see [GNa1], 2.1). Therefore we get $a_1U :_A (a_1I :_I I) \subseteq U + \mathfrak{a}$ and hence it is surjective. □

We put $T = \mathcal{G}(I\overline{A})$ and $S = \mathcal{G}(IB)$ for short. Look at the natural exact sequence $0 \to \overline{A} \to B \to B/\overline{A} \to 0$. Let $C = B/\overline{A}$. Then $\dim C \leq 1$. Since $I\overline{A} = IB$, we get the exact sequence

$$0 \to T \overset{\varphi}{\to} S \to C \to 0 \qquad (\sharp)$$

of graded G-modules. Moreover we have the exact sequence

$$0 \to \mathfrak{a} \to G \overset{\varepsilon}{\to} T \to 0 \qquad (\sharp\sharp)$$

of graded G-modules by [GNa1], 2.3. We note a_1 is B-regular element and hence $\lambda(IB_{\mathfrak{n}}) > 0$ for all maximal ideal $\mathfrak{n}$ in B. Let $\widehat{A}$ denote the $\mathfrak{m}$-adic completion of A. Notice that $\widehat{A} \otimes_A S \cong \prod_{j=1}^{n} \mathcal{G}(IB_j)$ is the direct product of associated graded rings $S_j := \mathcal{G}(IB_j)$ of ideals IB_j (with positive analytic spread) in Cohen-Macaulay local rings B_j, which are finite as $\widehat{A}$-modules.

Lemma 2.3. *S is a maximal Cohen-Macaulay G-module.*

Proof. Since $\operatorname{depth} \mathfrak{a} = 2$, we get $\operatorname{depth} T > 0$ by ($\sharp\sharp$). We apply the local cohomology functors $\mathrm{H}^i_{\mathfrak{M}}(*)$ ($i \in \mathbb{Z}$) to the graded exact sequences ($\sharp$) and ($\sharp\sharp$). Then by the resulting graded exact sequence

$$0 \to \mathrm{H}^1_{\mathfrak{M}}(T) \to \mathrm{H}^2_{\mathfrak{M}}(\mathfrak{a}) \to \mathrm{H}^2_{\mathfrak{M}}(G) \to \mathrm{H}^2_{\mathfrak{M}}(T) \to 0$$

of local cohomology modules from ($\sharp$), we have $\mathrm{H}^1_{\mathfrak{M}}(T) = [\mathrm{H}^1_{\mathfrak{M}}(T)]_0$ and $\mathrm{a}(T) \leq 0$ because $\mathrm{H}^2_{\mathfrak{M}}(\mathfrak{a}) = [\mathrm{H}^2_{\mathfrak{M}}(\mathfrak{a})]_0$ (see [GH], 2.2) and $a = 0$. And by the resulting graded exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{M}}(S) \to \mathrm{H}^0_{\mathfrak{M}}(C) \to \mathrm{H}^1_{\mathfrak{M}}(T) \to \mathrm{H}^1_{\mathfrak{M}}(S) \to \mathrm{H}^1_{\mathfrak{M}}(C) \to \mathrm{H}^2_{\mathfrak{M}}(T) \to \mathrm{H}^2_{\mathfrak{M}}(S) \to 0$$

of local cohomology modules from (♯♯), we have $\mathrm{H}^i_{\mathfrak{M}}(S) = [\mathrm{H}^i_{\mathfrak{M}}(S)]_0$ for any integers $i = 0, 1$ and $\mathrm{a}_2(S) \leq 0$ because $\mathrm{H}^1_{\mathfrak{M}}(T)$, $\mathrm{H}^0_{\mathfrak{M}}(C)$, and $\mathrm{H}^1_{\mathfrak{M}}(C)$ are concentrated in degree 0 (see [GH], 2.2) and $\mathrm{a}(T) \leq 0$.

Now assume that S is not a Cohen-Macaulay G-module. Let $t = \operatorname{depth} S$. Then $t = 0$ or 1. Because $\widehat{A} \otimes_A \mathrm{H}^t_{\mathfrak{M}}(S) \cong \bigoplus_{j=1}^n \mathrm{H}^t_{\widehat{A} \otimes_A \mathfrak{M}}(S_j)$ as graded $\widehat{A} \otimes_A G$-modules, we can find $1 \leq j \leq n$ such that $(0) \neq \mathrm{H}^t_{\widehat{A} \otimes_A \mathfrak{M}}(S_j) = [\mathrm{H}^t_{\widehat{A} \otimes_A \mathfrak{M}}(S_j)]_0$. From [KN], 3.1 we obtain that $\mathrm{a}_t(S_j) < \mathrm{a}_{t+1}(S_j)$. However this is impossible since $\mathrm{a}_t(S_j) = 0$ and $\mathrm{a}_{t+1}(S_j) \leq \mathrm{a}_{t+1}(S) \leq 0$. □

Apply the functor $\mathrm{Hom}_G(*, \mathrm{K}_G)$ to the graded exact sequences (♯) and (♯♯), and we get the following commutative and exact diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathrm{Ext}^1_G(C, \mathrm{K}_G) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_S & \longrightarrow & \mathrm{K}_G & \longrightarrow & \operatorname{Coker} \varepsilon^* \circ \varphi^* & \longrightarrow & 0 \\
 & & {\scriptstyle \varphi^*}\downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_T & \xrightarrow{\varepsilon^*} & \mathrm{K}_G & \longrightarrow & \mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G) & \longrightarrow & \mathrm{Ext}^1_G(T, \mathrm{K}_G) \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{Ext}^1_G(C, \mathrm{K}_G) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

of graded G-modules where K_S denotes formally $\mathrm{Hom}_G(S, \mathrm{K}_G)$. Notice that $\mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G)$ and $\mathrm{Ext}^1_G(C, \mathrm{K}_G)$ are concentrated in degree 0 (see [BH], 3.6.19 and [GH], 2.2). Now let $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ stand for the canonical I-filtration of A ([GI]). Hence we have $I^{i+1} \subseteq \omega_i$ for all $i \in \mathbb{Z}$ and $\mathrm{K}_G \cong \bigoplus_{i \geq 0} \omega_{i-1}/\omega_i$ as graded G-modules.

Lemma 2.4. $\operatorname{Coker} \varepsilon^* \cong A/U$.

Proof. We put $Z = \operatorname{Coker} \varepsilon^*$. Since $Z \subseteq \mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G)$, we have Z is concentrated in degree 0. Therefore $Z \cong A/E$ for some ideal E in A because we have a surjective homomorphism $A/\omega_0 \cong [\mathrm{K}_G]_0 \twoheadrightarrow Z$. Hence $U \subseteq E$, as $\mathfrak{a} = (0) : U$. Assume $U \subsetneq E$ and choose $\mathfrak{p} \in \mathrm{Ass}_A\, A/U$ so that $U_{\mathfrak{p}} \subsetneq E_{\mathfrak{p}}$. Since $\mathfrak{p} \not\supseteq \mathfrak{a}$, we have $T_{\mathfrak{p}} = (0)$, so that $(0) = \mathrm{K}_{T_{\mathfrak{p}}} \cong [\mathrm{K}_T]_{\mathfrak{p}}$. Thus we get $[\mathrm{K}_G]_{\mathfrak{p}} \cong [A/E]_{\mathfrak{p}}$ by the diagram above. From $[\mathrm{K}_G]_{\mathfrak{p}} \cong \mathrm{K}_{G_{\mathfrak{p}}} = A_{\mathfrak{p}}$ we obtain $E_{\mathfrak{p}} = (0)$, which is a contradiction. □

Lemma 2.5. *Suppose* $\mathfrak{p} \in \mathrm{V}(I)$ *such that* $\mathrm{ht}_A\, \mathfrak{p} = 1$ *and* $\mathfrak{p} \supseteq \mathfrak{a}$. *Then* $IB_{\mathfrak{p}} = a_1 B_{\mathfrak{p}}$.

Proof. We have $I\overline{A}_\mathfrak{p}$ is $\mathfrak{p}\overline{A}_\mathfrak{p}$-primary ideal with $(I\overline{A}_\mathfrak{p})^2 = a_1 I\overline{A}_\mathfrak{p}$ because $\mathrm{r}_1 \le 1$. From [GIW], 4.1 we obtain that $[I\overline{A}]_\mathfrak{p} = a_1 B_\mathfrak{p}$. Since $I\overline{A} = IB$, we get $IB_\mathfrak{p} = a_1 B_\mathfrak{p}$. □

We put $X = \mathrm{Coker}\,\varepsilon^* \circ \varphi^*$. Since $\mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G)$ and $\mathrm{Ext}^1_G(C, \mathrm{K}_G)$ are concentrated in degree 0, we get $X = [X]_0$. Hence there exists an ideal F in A such that $X \cong A/F$ (recall that $A/\omega_0 \cong [\mathrm{K}_G]_0 \twoheadrightarrow X$).

Lemma 2.6. *Suppose $\omega_0 = I$. Then $I = F$ and hence $\mathrm{r}_J(I) \le 1$.*

Proof. We have a surjective homomorphism $A/I \twoheadrightarrow X$ and hence $I \subseteq F$. Assume $I \subsetneq F$ and choose $\mathfrak{p} \in \mathrm{Ass}_A A/I$ so that $I_\mathfrak{p} \subsetneq F_\mathfrak{p}$. We have $\mathfrak{p} \ne \mathfrak{m}$ because $\mathrm{depth}\, A/I > 0$. If $\mathfrak{p} \not\supseteq \mathfrak{a}$, then $S_\mathfrak{p} = (0)$, so that $(0) = \mathrm{K}_{S_\mathfrak{p}} \cong [\mathrm{K}_S]_\mathfrak{p}$. Thus we get $[\mathrm{K}_G]_\mathfrak{p} \cong [A/F]_\mathfrak{p}$ by the diagram above. Since $[\mathrm{K}_G]_0 \cong A/\omega_0 = A/I$, we have $[A/I]_\mathfrak{p} \cong [A/F]_\mathfrak{p}$. Therefore $I_\mathfrak{p} = F_\mathfrak{p}$, which is impossible. If $\mathfrak{p} \supseteq \mathfrak{a}$, then $\mathrm{ht}_A \mathfrak{p} = 1$ and hence $S_\mathfrak{p} = \mathcal{G}(a_1 B_\mathfrak{p})$ is the polynomial ring in one variable by Lemma 2.5. Then we have $\mathrm{a}_2(S_\mathfrak{p}) = -1$ and hence $(0) = [\mathrm{K}_{S_\mathfrak{p}}]_0 \cong [[\mathrm{K}_S]_\mathfrak{p}]_0$ (see [BH], 3.6.19). Therefore we have $[A/I]_\mathfrak{p} \cong [A/F]_\mathfrak{p}$, whence $I_\mathfrak{p} = F_\mathfrak{p}$. This is a contradiction. Thus we get $I = F$.

Then we get the exact sequence

$$0 \to \mathrm{K}_S \to \mathrm{K}_G \to A/I \to 0$$

of graded G-modules. Since $[\mathrm{K}_G]_0 \cong A/I$, we have $[\mathrm{K}_S]_0 = (0)$ and hence $\mathrm{a}_2(S) < 0$ (see [BH], 3.6.19). Let $\mathfrak{n}$ be an maximal ideal in B. Then $S_\mathfrak{n} = \mathcal{G}(IB_\mathfrak{n})$ is a Cohen-Macaulay ring with $\mathrm{a}(S_\mathfrak{n}) < 0$ (recall that $\mathrm{a}_2(S) \ge \mathrm{a}(S_j) = \mathrm{a}(S_\mathfrak{n})$ for some $j = 1, 2, \ldots, n$). We have $\lambda(IB_\mathfrak{n}) > 0$. Suppose $\lambda(IB_\mathfrak{n}) = 1$. There is an element $b \in JB_\mathfrak{n}$ such that $bB_\mathfrak{n}$ is a minimal reduction of $IB_\mathfrak{n}$. Thanks to the a-invariant formula, we get $\mathrm{r}_{bB_\mathfrak{n}}(IB_\mathfrak{n}) \le \mathrm{a}(S_\mathfrak{n}) + 1 \le 0$ and hence $bB_\mathfrak{n} = JB_\mathfrak{n} = IB_\mathfrak{n}$. Suppose $\lambda(IB_\mathfrak{n}) = 2$. Then we obtain $\mathrm{r}_{JB_\mathfrak{n}}(IB_\mathfrak{n}) \le \mathrm{a}(S_\mathfrak{n}) + 2 \le 1$ from the a-invariant formula. Therefore in any case we get $\mathrm{r}_{JB_\mathfrak{n}}(IB_\mathfrak{n}) \le 1$, whence $I^2 B = JIB$. So $I^2\overline{A} = JI\overline{A}$, as $IB = I\overline{A}$. Then $I^2 \subseteq JI + \mathfrak{a}$, and hence we have $I^2 = JI$ because $I \cap \mathfrak{a} = (0)$. □

We now come to the proof of our proposition.

Proof of Proposition 2.1. (1) ⇒ (2). Assume G is a Gorenstein ring. Then $\omega_0 = I$, so that $\mathrm{r}_J(I) \le 1$ by Lemma 2.6. We must show that $I = a_1 U :_U (a_1 I :_I I)$. Let $L = a_1 U :_U (a_1 I :_I I)$. Then we have $I \subseteq L$. Let $\mathfrak{p} \in \mathrm{Ass}_A A/I$. It is enough to prove $I_\mathfrak{p} = L_\mathfrak{p}$. We have $\mathfrak{p} \in \mathcal{A}$ (see, e.g., [GI], 3.1) and $\mathfrak{p} \ne \mathfrak{m}$, as $\mathrm{depth}\, A/I > 0$. If $\mathrm{ht}_A \mathfrak{p} = 0$, then $U_\mathfrak{p} = (0)$. Hence we may assume $\mathrm{ht}_A \mathfrak{p} = 1$. Then $a_1 A_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$ with $\mathrm{r}_{a_1 A_\mathfrak{p}}(I_\mathfrak{p}) \le 1$ because $\mathfrak{p} \in \mathcal{A}$ and $\mathrm{r}_1 \le 1$. So $I_\mathfrak{p} = a_1 I_\mathfrak{p} :_{I_\mathfrak{p}} I_\mathfrak{p}$. If $\mathrm{ht}_A I_\mathfrak{p} = 0$, then $I_\mathfrak{p}$ is generically a complete intersection and hence we get $I_\mathfrak{p} = a_1 \mathrm{U}(I_\mathfrak{p}) :_{\mathrm{U}(I_\mathfrak{p})} I_\mathfrak{p}$ by [GI], 6.3. Therefore $I_\mathfrak{p} = L_\mathfrak{p}$, as $\mathrm{U}(I_\mathfrak{p}) = \mathrm{U}(I)_\mathfrak{p}$. If $\mathrm{ht}_A I_\mathfrak{p} = 1$, then $I_\mathfrak{p}$ is an $\mathfrak{p}A_\mathfrak{p}$-primary ideal of $A_\mathfrak{p}$ with $\mathrm{r}_{a_1 A_\mathfrak{p}}(I_\mathfrak{p}) = 1$ because $\mathrm{r}_{a_1 A_\mathfrak{p}}(I_\mathfrak{p}) = \mathrm{a}(\mathcal{G}(I_\mathfrak{p})) + 1$ and $\mathrm{a}(\mathcal{G}(I_\mathfrak{p})) = \mathrm{a}(\mathcal{G}(I)_\mathfrak{p}) = a = 0$. Hence, according to [GI], 1.4, we get $I_\mathfrak{p} = a_1 A_\mathfrak{p} : I_\mathfrak{p}$. Therefore $I_\mathfrak{p} = L_\mathfrak{p}$, as $\mathfrak{p} \not\supseteq U$.

(2) ⇒ (1). First of all, we obtain that $G_\mathfrak{p} = \mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(G_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/I$. In fact, let $\mathfrak{p} \in \mathrm{Ass}_A A/I$. If $\mathrm{ht}_A \mathfrak{p} = 0$, then $G_\mathfrak{p} = A_\mathfrak{p}$, so that

we have nothing to prove. Hence we may assume $\mathrm{ht}_A\, \mathfrak{p} = 1$. Then $a_1A_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$ with $\mathrm{r}_{a_1A_\mathfrak{p}}(I_\mathfrak{p}) \leq 1$, whence $I_\mathfrak{p} = a_1I_\mathfrak{p} :_{I_\mathfrak{p}} I_\mathfrak{p}$. When $\mathrm{ht}_A\, I_\mathfrak{p} = 0$, we get $I_\mathfrak{p} = a_1\mathrm{U}(I_\mathfrak{p}) :_{\mathrm{U}(I_\mathfrak{p})} I_\mathfrak{p}$. And when $\mathrm{ht}_A\, I_\mathfrak{p} = 1$, we get $I_\mathfrak{p} = a_1A_\mathfrak{p} : I_\mathfrak{p}$ (recall that $\mathfrak{p} \not\supseteq U$) and hence $\mathrm{r}_{a_1A_\mathfrak{p}}(I_\mathfrak{p}) = 1$. Consequently, from [GI], 6.3 and 1.4 we obtain that $G_\mathfrak{p}$ is a Gorenstein ring with $\mathrm{a}(G_\mathfrak{p}) = 0$ in any case.

Then we have $[\omega_0]_\mathfrak{p} = I_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A\, A/I$, therefore $\omega_0 = I$ (recall that $\omega_0 \supseteq I$). According to Lemma 2.6, we have $I = F$. Hence we get the exact sequence

$$0 \to \mathrm{K}_S \to \mathrm{K}_G \to A/I \to 0$$

of graded G-modules. Since $[\mathrm{K}_G]_0 = A/I$, we get $[\mathrm{K}_S]_0 = (0)$.

Claim 2.7. *The graded G-module K_S is generated by elements of degree 1.*

Proof. Since a_1 is $I\overline{A}$-regular element, a_1 is B-regular element. We put $\overline{B} = B/a_1B$ and $\overline{S} = \mathcal{G}(I\overline{B})$. We have $B/IB = \overline{B}/I\overline{B}$ is a Cohen-Macaulay ring and $\mathrm{ht}_B\, IB = 1$, as B and IB are maximal Cohen-Macaulay A-modules. From Lemma 2.5 we obtain that $I\overline{B}_\mathfrak{Q} = (0)$ for any $\mathfrak{Q} \in \mathrm{V}(I\overline{B})$ such that $\mathrm{ht}_{\overline{B}}\, \mathfrak{Q} = 0$. Thus we have $I\overline{B}$ is not nilpotent since $I\overline{B} \neq (0)$ and $I\overline{B}_\mathfrak{Q} = (0)$ for all $\mathfrak{Q} \in \mathrm{Ass}_{\overline{B}}\, \overline{B}/I\overline{B}$. We have $I^2\overline{B} = a_2I\overline{B}$, therefore a_2t is $\overline{S}_+ := \bigoplus_{i>0}[\overline{S}]_i$-regular element because a_2 is $I\overline{B}$-regular element (see [GNa1], 2.1). Thus we get the graded exact sequence $0 \to \overline{S}_+(-1) \xrightarrow{a_2t} \overline{S}_+ \to I\overline{B}/I^2\overline{B} \to 0$. Apply the local cohomology functors $\mathrm{H}^i_\mathfrak{M}(*)$ ($i \in \mathbb{Z}$) to this, and we have the graded exact sequence $0 \to \mathrm{H}^0_\mathfrak{M}(I\overline{B}/I^2\overline{B}) \to \mathrm{H}^1_\mathfrak{M}(\overline{S}_+)(-1) \xrightarrow{a_2t} \mathrm{H}^1_\mathfrak{M}(\overline{S}_+)$ of local cohomology modules. And furthermore, applying the functor $\mathrm{Hom}_G(G/\mathfrak{M}, *)$ to this, we get isomorphism

$$\mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^0_\mathfrak{M}(I\overline{B}/I^2\overline{B})) \cong \mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(\overline{S}_+))(-1)$$

of graded G-modules because $a_2tG \subseteq \mathfrak{M}$, and hence $\mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(\overline{S}_+))$ is concentrated in degree 0 because $\mathrm{H}^0_\mathfrak{M}(I\overline{B}/I^2\overline{B}) = [\mathrm{H}^0_\mathfrak{M}(I\overline{B}/I^2\overline{B})]_1$ (see [GH], 2.2). We apply the local cohomology functors $\mathrm{H}^i_\mathfrak{M}(*)$ ($i \in \mathbb{Z}$) to the graded exact sequence $0 \to \overline{S}_+ \to \overline{S} \to B/IB \to 0$. Then by the resulting graded exact sequence $0 \to \mathrm{H}^1_\mathfrak{M}(\overline{S}_+) \to \mathrm{H}^1_\mathfrak{M}(\overline{S}) \to \mathrm{H}^1_\mathfrak{M}(B/IB) \to 0$ of local cohomology modules, we get the exact sequence

$$0 \to \mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(\overline{S}_+)) \to \mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(\overline{S})) \to \mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(B/IB))$$

of graded G-modules. We have $\mathrm{H}^1_\mathfrak{M}(B/IB) = [\mathrm{H}^1_\mathfrak{M}(B/IB)]_0$, hence we obtain that $\mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(\overline{S}))$ is concentrated in degree 0. We note a_1t is S-regular element (see [GNN2], 2.3). Then we have the sequence $0 \to S(-1) \xrightarrow{a_1t} S \to \overline{S} \to 0$ of graded G-modules, as $\overline{S} \cong S/a_1tS$ (see [VV]). Applying the local cohomology functors $\mathrm{H}^i_\mathfrak{M}(*)$ ($i \in \mathbb{Z}$) to this, we have the graded exact sequence $0 \to \mathrm{H}^1_\mathfrak{M}(\overline{S}) \to \mathrm{H}^2_\mathfrak{M}(S)(-1) \xrightarrow{a_1t} \mathrm{H}^2_\mathfrak{M}(S)$ of local cohomology modules. Therefore we get the isomorphism

$$\mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^1_\mathfrak{M}(\overline{S})) \cong \mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^2_\mathfrak{M}(S))(-1)$$

of graded G-modules, and hence $\mathrm{Hom}_G(G/\mathfrak{M}, \mathrm{H}^2_\mathfrak{M}(S))$ is concentrated in degree -1. This means that the graded G-module K_S is generated by elements of degree 1. □

By the condition (iii) and Lemma 2.2 we may assume $\mathrm{K}_B = I$. Using Claim 2.7 and [HSV], 2.4 we obtain $\mathrm{K}_S = G_+$ because S is a Cohen-Macaulay ring. Therefore we have the exact sequence

$$0 \to G_+ \to \mathrm{K}_G \to A/I \to 0$$

of graded G-modules. Look at the homogeneous components

$$\begin{array}{ccccccc} & 0 & \to & A/\omega_0 & \to & A/I & \to 0 \\ 0 \to & I/I^2 & \to & \omega_0/\omega_1 & \to & 0 & \\ 0 \to & I^2/I^3 & \to & \omega_1/\omega_2 & \to & 0 & \\ & & & \vdots & & & \end{array}$$

of above, where $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ is the canonical I-filtration of A ([GI]). By induction on i, we see that $\omega_i = I^{i+1}$ for all integers $i \geq 0$. In fact, we have $\omega_0 = I$. Let $i > 0$ and assume $\omega_{i-1} = I^i$. We note that $\omega_i \supseteq I^{i+1}$. From bijections above we obtain that $I^i/I^{i+1} \cong \omega_{i-1}/\omega_i = I^i/\omega_i$, and hence the natural surjective map $I^i/I^{i+1} \to I^i/\omega_i$ is bijective. Thus $\omega_i = I^{i+1}$ for all $i \geq 0$. This means G is a Gorenstein ring.

Now let us check the last assertion. Suppose $I = U$. We take any $\mathfrak{p} \in \mathrm{V}(I)$ such that $\mathrm{ht}_A \mathfrak{p} = 1$. If $\mathfrak{p} \not\supseteq \mathfrak{a}$, then $I_\mathfrak{p} = (0)$. And if $\mathfrak{p} \supseteq \mathfrak{a}$, then $B_\mathfrak{p} = \overline{A}_\mathfrak{p}$ because $\overline{A}_\mathfrak{p}$ is a Cohen-Macaulay ring with $\mathrm{K}_{\overline{A}_\mathfrak{p}} \cong I_\mathfrak{p}$. Therefore we have $I\overline{A}_\mathfrak{p} = a_1\overline{A}_\mathfrak{p}$ by Lemma 2.5, so that $IA_\mathfrak{p} \subseteq a_1A_\mathfrak{p} + \mathfrak{a}_\mathfrak{p}$. Since $IA_\mathfrak{p} \cap \mathfrak{a}_\mathfrak{p} = (0)$, we get $IA_\mathfrak{p} = a_1A_\mathfrak{p}$. Thus in any case we obtain $\mathrm{r}_{a_1A_\mathfrak{p}}(IA_\mathfrak{p}) = 0$. This establishes that $\mathrm{r}_1 = 0$.

Conversely, assume $I \subsetneq U$ and choose $\mathfrak{p} \in \mathrm{Ass}_A A/I$ so that $I_\mathfrak{p} \subsetneq U_\mathfrak{p}$. Then we have $\mathrm{ht}_A \mathfrak{p} = 1$ because $U_\mathfrak{p} \neq (0)$ and $\mathrm{depth}\, A/I > 0$, and hence $I\overline{A}_\mathfrak{p} = a_1\overline{A}_\mathfrak{p}$ since $\mathrm{r}_1 = 0$. Take the G-dual of the sequence (♯♯), and we have the exact sequence

$$0 \to \mathrm{K}_T \to G \to A/U \to 0$$

of graded G-modules by Lemma 2.4, and hence $[\mathrm{K}_T]_0 \cong U/I$. So $[\mathrm{K}_{T_\mathfrak{p}}]_0 \cong U_\mathfrak{p}/I_\mathfrak{p} \neq (0)$. However, we have $\mathrm{a}(T_\mathfrak{p}) = 1$ because $T_\mathfrak{p} = \mathcal{G}(a_1\overline{A}_\mathfrak{p})$ is a polynomial ring in one variable. This is a contradiction, which completes the proof of Proposition 2.1. □

We would like to close this section the following example of certain mixed ideal I satisfying the conditions (i), (ii), and (iii) in Proposition 2.1 and whose the associated ring is a Gorenstein ring.

Let $k[[X,Y,Z,W]]$ be the formal power series ring in 4 variables over an infinite field k. We put $A = k[[X,Y,Z,W]]/(X(Y+W),(X+Z)(Y+Z)W)$ that is a Gorenstein local ring with dimension 2. We denote by x, y, z, and w the reduction of X, Y, Z, and W mod $(X(Y+W),(X+Z)(Y+Z)W)$. Let $I := (x,z) \cap (x,y,w) = (x,yz,zw)$. Then I is generically a complete intersection with height 0. Let $a_1 = x + yz + zw$ and $a_2 = zw$. We put $J = (a_1,a_2)$ that is a reduction of I with $\mathrm{r}_J(I) = 1$, as $I^2 = JI + (x^2)$ and $x^2 = a_1x$. Moreover we get J is minimal because a_1, a_2 is d-sequence, and hence $\lambda(I) = 2$. It is routine to check that a_1, a_2 satisfy (∗) and (∗∗) stated in Section 1 and $\mathrm{r}_1 = 1$. We put $L = a_1U :_U (a_1I :_I I)$. Then $I \subseteq L$. Assume that $I \subsetneq L$ and take $\mathfrak{p} \in \mathrm{Ass}_A A/I$ so that $I_\mathfrak{p} \subsetneq L_\mathfrak{p}$. Then $\mathfrak{p} = (x,y,w)$, which is height one. We have $\mathrm{r}_{a_1A}(I_\mathfrak{p}) = 1$. Consequently, $L_\mathfrak{p} = a_1A_\mathfrak{p} :_{A_\mathfrak{p}} I_\mathfrak{p} \neq A_\mathfrak{p}$. Since $I_\mathfrak{p}$ is the maximal ideal in $A_\mathfrak{p}$, we have $I_\mathfrak{p} = L_\mathfrak{p}$. Therefore $I = L$. Hence we get $\mathcal{G}(I)$ is a Gorenstein ring by Proposition 2.1.

3 The proof of Theorem 1.1.

In this section we prove Theorem 1.1. To do this, we may assume that I is a generically a complete intersection and $a = -s$ because when G is Cohen-Macaulay, we have $a = -s$ if and only if the conditions (a) and (b) stated in our theorem 1.1 are satisfied (use the a-invariant formula: $\mathrm{a}(G) = \max\{\{\mathrm{r}_i - i \mid s \le i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}\}$). If $\mathrm{ad}(I) = 1$, then Theorem 1.1 is covered by [GI], 1.6. Hence we may assume furthermore $\mathrm{ad}(I) = 2$. Thus it is enough to show the following.

Theorem 3.1. *Assume that I is generically a complete intersection with $\mathrm{ad}(I) = 2$ and $\mathrm{ht}_A \mathfrak{p} < s+2$ for any $\mathfrak{p} \in \mathrm{Ass}_A A/I$. Suppose that G is a Cohen-Macaulay ring. Then the following two conditions are equivalent.*

(1) *G is a Gorenstein ring.*

(2) (a) $\mathrm{r}_J(I) \le 2$,

(b) $\mathrm{r}_{s+1} \le 1$, *and*

(c) $I = J_{s+1}U :_U (J_{s+1}I :_I I)$.

When this is the case $\mathrm{a}(G) = -s$, and the following two assertions are satisfied:
(i) $I = U$ *if and only if* $\mathrm{r}_{s+1} = 0$;
(ii) $\mathrm{r}_{J_\mathfrak{p}}(I_\mathfrak{p}) \le 1$ *for any* $\mathfrak{p} \in \mathrm{V}(I)$ *such that* $\mathrm{ht}_A \mathfrak{p} = s+2$.

Proof. If G is a Gorenstein ring, then $a = -s$ (recall that $\mathrm{r}_s = 0$ and $a = \mathrm{a}(G(I_\mathfrak{q}))$ for all $\mathfrak{q} \in \mathrm{V}(I)$) and hence conditions (a) and (b) stated in Theorem 3.1 are satisfied (use the a-invariant formula). Therefore, to prove Theorem 3.1, we may assume that conditions (a) and (b) are satisfied. Then we obtain that $a = -s$ (use a-invariant formula again). Here we put $K = (a_1, a_2, \cdots, a_s)$. Since G is a Cohen-Macaulay ring, we get the sequence $a_1t, a_2t, \cdots, a_st$ is G-regular (see [GNa2], 3.3), so that

$$G/(a_1t, a_2t, \ldots, a_st)G \cong \mathcal{G}(I/K)$$

of graded A-algebras ([VV]). We note the following.

Claim 3.2. $J_{s+1}U :_U (J_{s+1}I :_I I) = (J_{s+1}U + K) :_U ((J_{s+1}I + K) :_I I)$.

Proof. We put $D = J_{s+1}I :_I I$, and then $D \subseteq (J_{s+1}I + K) :_I I$. Let $x \in (J_{s+1}I + K) :_I I$. Then $xI \subseteq (J_{s+1}I + K) \cap I^2 = J_{s+1}I + K \cap I^2$. Since $a_1t, a_2t, \cdots, a_st$ is G-regular, we get $K \cap I^2 = KI$, and hence $D = (J_{s+1}I + K) :_I I$. Moreover we have $J_{s+1}U :_U D \subseteq (J_{s+1}U + K) :_U D$. Let $x \in (J_{s+1}U + K) :_U D$. Then $xD \subseteq (J_{s+1}U + K) \cap IU = J_{s+1}U + K \cap IU$. Hence it suffices to show that $K \cap IU = KU$. Let $\mathfrak{p} \in \mathrm{Ass}_A /KU$. Then it is enough to prove $K_\mathfrak{p} \cap I_\mathfrak{p}U_\mathfrak{p} = K_\mathfrak{p}U_\mathfrak{p}$. We may assume $U \subseteq \mathfrak{p}$. Look at the exact sequence $0 \to K/KU \to A/KU \to A/K \to 0$ of A-modules. Then since $K/KU \cong (A/U) \otimes_{A/K} (K/K^2) \cong (A/U)^s$, we have $\mathrm{Ass}_A A/KU \subseteq \mathrm{Ass}_A A/U \cup \mathrm{Ass}_A A/K$. Therefore $\mathrm{ht}_A \mathfrak{p} = s$, whence $I_\mathfrak{p} = U_\mathfrak{p} = K_\mathfrak{p}$. □

According to [GNN1], 3.4 together with Claim 3.2, passing to the ring A/K, we may assume furthermore that $s = 0$. We must check that G is a Gorenstein ring if and only if $I = a_1U :_U (a_1I :_I I)$. Assume G is a Gorenstein ring. Then in the same way of the proof of Proposition 2.1, (1) $\Rightarrow$ (2), we get $I = a_1U :_U (a_1I :_I I)$.

Conversely, we take any $\mathfrak{p} \in \mathcal{A}$. It suffices to show that $\mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I_\mathfrak{p})) = 0$ (see [GI], 1.2). Notice that $\mathrm{ht}_A\,\mathfrak{p} = \lambda(I_\mathfrak{p}) \leq 2$ because $J_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$. If $\mathrm{ht}_A\,\mathfrak{p} = 0$, then $I_\mathfrak{p} = (0)$ and hence we have nothing to prove.

Assume that $\mathrm{ht}_A\,\mathfrak{p} = 1$. Then $a_1A_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$ with $\mathrm{r}_{a_1A_\mathfrak{p}}(I_\mathfrak{p}) \leq 1$ because $\mathrm{ht}_A\,\mathfrak{p} = \lambda(I_\mathfrak{p}) = 1$ and $\mathrm{r}_1 \leq 1$, so that $I_\mathfrak{p} = a_1I_\mathfrak{p} :_{I_\mathfrak{p}} I_\mathfrak{p}$. If $\mathrm{ht}_{A_\mathfrak{p}}\,I_\mathfrak{p} = 1$, then $I_\mathfrak{p}$ is a $\mathfrak{p}A_\mathfrak{p}$-primary ideal in $A_\mathfrak{p}$. Therefore $\mathfrak{p} \not\supset U$. So we have $I_\mathfrak{p} = a_1A_\mathfrak{p} : I_\mathfrak{p}$ and hence $\mathrm{r}_{a_1A_\mathfrak{p}}(I_\mathfrak{p}) = 1$. If $\mathrm{ht}_{A_\mathfrak{p}}\,I_\mathfrak{p} = 0$, then $\mathrm{U}(I)_\mathfrak{p} = \mathrm{U}(I_\mathfrak{p})$ and hence we have $I_\mathfrak{p} = [a_1\mathrm{U}(I_\mathfrak{p}) : I_\mathfrak{p}] \cap \mathrm{U}(I_\mathfrak{p})$. Consequently, thanks to [GI], 1.4 and [GI], 6.3, we get $\mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathcal{G}(I_\mathfrak{p})) = 0$.

Suppose that $\mathrm{ht}_{A_\mathfrak{p}}\,\mathfrak{p} = 2$. Then $J_\mathfrak{p}$ is a minimal reduction of $I_\mathfrak{p}$ with $\mathrm{r}_{J_\mathfrak{p}}(I_\mathfrak{p}) \leq 2$ because $\mathrm{ht}_A\,\mathfrak{p} = \lambda(I_\mathfrak{p}) = 2$ and $\mathrm{r}_J(I) \leq 2$. If $\mathrm{ht}_{A_\mathfrak{p}}\,I_\mathfrak{p} = 2$, then $\mathfrak{p}$ is a minimal prime ideal of I. So we have $\mathfrak{p} \in \mathrm{Ass}_A\,A/I$, however this is a contradiction to the standard assumption. Hence we get $\mathrm{ht}_A\,I_\mathfrak{p} \leq 1$. Let $\mathrm{ht}_{A_\mathfrak{p}}\,I_\mathfrak{p} = 1$. Hence $\mathrm{ad}(I) = 1$. Since $\mathfrak{p} \notin \mathrm{Ass}_A\,A/I$ we have $A_\mathfrak{p}/I_\mathfrak{p}$ is a Cohen-Macaulay ring. Thanks to [GNN2], it is enough to check the following three assertions:

1. $\mathrm{r}_{[a_1A_\mathfrak{p}]_Q}([I_\mathfrak{p}]_Q) = 1$ for all $Q \in \mathrm{Ass}_{A_\mathfrak{p}}\,A_\mathfrak{p}/I_\mathfrak{p}$;
2. $a_1A_\mathfrak{p} :_{A_\mathfrak{p}} I_\mathfrak{p} \subseteq I_\mathfrak{p}$;
3. $\mathrm{r}_{a_1A_\mathfrak{p}}(I_\mathfrak{p}) \leq 1$.

Since $\mathrm{ht}_{A_\mathfrak{p}}\,I_\mathfrak{p} = 1$, we have $\mathfrak{p} \not\supseteq U$ and hence $I_\mathfrak{p} = a_1A_\mathfrak{p} :_{A_\mathfrak{p}} (a_1I_\mathfrak{p} :_{I_\mathfrak{p}} I_\mathfrak{p}) \supseteq a_1A_\mathfrak{p} :_{A_\mathfrak{p}} I_\mathfrak{p}$. Thus we get the assertion 2. Let $Q \in \mathrm{Ass}_{A_\mathfrak{p}}\,A_\mathfrak{p}/I_\mathfrak{p}$. Then $\mathrm{ht}_{A_\mathfrak{p}}\,Q = 1$ because the ring $A_\mathfrak{p}/I_\mathfrak{p}$ is Cohen-Macaulay. Take $\mathfrak{q} \in \mathrm{V}(I)$ such that $Q = \mathfrak{q}A_\mathfrak{p}$, and we have $\mathrm{r}_{a_1A_\mathfrak{q}}(I_\mathfrak{q}) \leq 1$, as $\mathrm{ht}_A\,\mathfrak{q} = 1$. Since $a_1A_\mathfrak{q} :_{A_\mathfrak{q}} I_\mathfrak{q} \subseteq I_\mathfrak{q} \neq A_\mathfrak{q}$, we get $\mathrm{r}_{a_1A_\mathfrak{q}}(I_\mathfrak{q}) = 1$. We must show the assertion 3. Notice that $\mathrm{a}(G_\mathfrak{p}) = 0$ (use the a-invariant formula: $\mathrm{a}(G_\mathfrak{p}) = \max\{\mathrm{r}_1(I_\mathfrak{p}) - 1, \mathrm{r}_{J_\mathfrak{p}}(I_\mathfrak{p}) - 2\}$). Since $G_\mathfrak{p}$ is a Cohen-Macaulay ring, we have a_1t is $G_\mathfrak{p}$-regular element (see [GNN2], 2.3). Hence, passing to the ring $A_\mathfrak{p}/a_1A_\mathfrak{p}$, it suffices to show the following claim, which is due to [GNN2] (see [GNN1], 3.4).

Claim 3.3 ([GNN2]). *Let A be a Gorenstein local ring. Let I be an ideal in A with $\mathrm{ht}_A = 0$ and bA a minimal reduction of I. Assume that G is a Cohen-Macaulay ring with $\mathrm{a}(G) = 1$. Then $\mathrm{r}_{bA}(I) \leq 1$ if A/I is Cohen-Macaulay and $(0) :_A I \subseteq I$.*

Proof. We put $\mathfrak{a} = (0) :_A I$, $\overline{A} = A/\mathfrak{a}$ and $T = \mathcal{G}(I\overline{A})$. According to [GNN2], 4.5, we have the exact sequence

$$0 \to \mathrm{K}_T \to \mathrm{K}_G \to A/I(1) \to 0$$

of graded G-modules. Let $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ be the canonical I-filtration of A ([GI]), hence we have $\omega_i = A$ for $i \leq -2$ and $\mathrm{K}_G \cong \bigoplus_{i\geq -1} \omega_{i-1}/\omega_i$ as graded G-modules. Look at the homogeneous component $0 \to [\mathrm{K}_T]_{-1} \to A/\omega_{-1} \to A/I \to 0$ of degree -1 in the exact sequence above, and we get $\omega_{-1} = I$ because $\omega_{-1} \supseteq I$ and hence $[\mathrm{K}_T]_{-1} = (0)$. Therefore

$a(T) = 1$. So $r_{b(\overline{A})}(I\overline{A}) \leq 1$ by [GNN2], 4.4. Then $I^2 \subseteq bI + \mathfrak{a}$, so that $I^2 = bI$ because $\mathfrak{a} \cap I^2 = (0)$ (see [GNN2], 4.2 (1)). □

Now let $\mathrm{ht}_A I_\mathfrak{p} = 0$. To prove that $\mathcal{G}(I_\mathfrak{p})$ is a Gorenstein ring with $a(\mathcal{G}(I_\mathfrak{p})) = 0$, we must show $r_{J_\mathfrak{p}}(I_\mathfrak{p}) \leq 1$ (see Proposition 2.1). It is enough to show $\omega_{0\mathfrak{p}} = I_\mathfrak{p}$ by Lemma 2.6. We have $\omega_{0\mathfrak{p}} \supseteq I_\mathfrak{p}$. Let $\mathfrak{q} \in \mathrm{Ass}_A A_\mathfrak{p}/I_\mathfrak{p}$. Then $\mathrm{ht}_A \mathfrak{q} \leq 1$. We have discussed earlier that $\mathcal{G}(I_\mathfrak{q})$ is a Gorenstein ring with $a(\mathcal{G}(I_\mathfrak{q})) = 0$. Therefore we get $\omega_{0\mathfrak{q}} = I_\mathfrak{q}$, so that $\omega_{0\mathfrak{p}} = I_\mathfrak{p}$.

Because $G_\mathfrak{p}$ is Gorenstein for any $\mathfrak{p} \in V(I)$ such that $\mathrm{ht}_A \mathfrak{p} = 2$, the last assertion follows from Proposition 2.1. This completes the proof of Theorem 3.1. □

As consequence of Theorem 3.1 with [GNN1], 6.6, we get the following.

Corollary 3.4. *Assume that I is generically a complete intersection with* $\mathrm{ad}(I) = 2$. *Suppose that* $\mathrm{depth}\, A/I \geq d - s - 1$. *Then the following two conditions are equivalent.*

(1) *G is a Gorenstein ring*

(2) (a) $r_J(I) \leq 2$,

(b) $r_{s+1} \leq 1$,

(c) $I = J_{s+1}U :_U (J_{s+1}I :_I I)$, *and*

(c) $\mathrm{depth}\, A/I^2 \geq d - s - 2$.

When this is the case, $a(G) = -s$, *and the following two assertions are satisfied:*

(i) $I = U$ *if and only if* $r_{s+1} = 0$;

(ii) $r_{J_\mathfrak{p}}(I_\mathfrak{p}) \leq 1$ *for any* $\mathfrak{p} \in V(I)$ *such that* $\mathrm{ht}_A \mathfrak{p} = s + 2$.

4 The case where I is unmixed.

In this section we summarize some corollaries derived from Theorem 1.1. When I is an unmixed ideal, we can simplify these conditions in 1.1 and readily have the following.

Corollary 4.1. *Assume that* $\mathrm{ad}(I) \leq 2$ *and I is an unmixed ideal. Suppose* $\mathcal{G}(I)$ *is a Cohen-Macaulay ring. Then the following two conditions are equivalent.*

(1) *$\mathcal{G}(I)$ is a Gorenstein ring and $a(\mathcal{G}(I)) = -s$.*

(2) (a) $r_J(I) \leq ad(I)$,

(b) $\mu_{A_\mathfrak{p}}(I_\mathfrak{p}) \leq \mathrm{ht}_A \mathfrak{p}$ *for any* $\mathfrak{p} \in V(I)$ *such that* $\mathrm{ht}_A \mathfrak{p} < \ell$.

Here let $\mu(*)$ stand for the number of generators. The condition (b) in Corollary 4.1 is equivalent to saying that our ideal I has a certain special reduction (cf. [N], (2.2)).

Next, let us consider the case where $d = \ell$. Then we get the following, which is due to [GNa1] when $\mathrm{ad}(I) = 1$.

Corollary 4.2. *Assume that* $\mathrm{ad}(I) \leq 2$ *and* I *is an unmixed ideal. Suppose* $d = \ell$*. Then the following two conditions are equivalent.*

(1) $\mathcal{G}(I)$ *is a Gorenstein ring and* $a(\mathcal{G}(I)) = -s$.

(2) (a) $\mathrm{r}_J(I) \leq \max\{0, ad(I) - 1\}$,

(b) $\mu_{A_\mathfrak{p}}(I_\mathfrak{p}) \leq \mathrm{ht}_A \mathfrak{p}$ *for any* $\mathfrak{p} \in \mathrm{V}(I)$ *such that* $\mathrm{ht}_A \mathfrak{p} < \ell$.

Proof. We may assume G is Cohen-Macaulay by [GNN1], 6.5. Hence the sequence $a_1t, a_2t, \cdots, a_st$ is G-regular (see [GNN2], 2.3 and [GNa2], 3.3) and hence we may assume furthermore $s = 0$. Then by Proposition 2.1 with [GNa1], 2.11, we get Corollary 4.2. □

Let us note the following corollary.

Corollary 4.3. *Let* $\mathfrak{p}$ *be a prime ideal in a regular local ring* A *with* $\dim A/\mathfrak{p} = 2$*. Assume that* $A/\mathfrak{p}$ *is a complete intersection in codimension one, then the associated graded ring* $\mathcal{G}(\mathfrak{p})$ *is Gorenstein if and only if there is a minimal reduction* J *with* $\mathrm{r}_J(\mathfrak{p}) \leq 1$.

We conclude this paper with the following example of the case where $\mathrm{ad}(\mathfrak{p}) = 2$. Let $k[X_1, X_2, X_3, Y_1, Y_2]$ and $k[s,t]$ be the polynomial rings over an infinite field k. Let $n \geq 4$ be an integer such that $3 \nmid n$ and let $\phi : k[X_1, X_2, X_3, Y_1, Y_2] \to k[s,t]$ be the homomorphism of k-algebras defined by $\phi(X_1) = s, \phi(X_2) = st, \phi(X_3) = st^2, \phi(Y_1) = t^3, \phi(Y_2) = t^n$. We put $S = k[X_1, X_2, X_3, Y_1, Y_2]$ and $A = S_\mathfrak{M}$, where $\mathfrak{M} = (X_1, X_2, X_3, Y_1, Y_2)S$. Let $\mathfrak{P} = \ker \phi$ and $\mathfrak{p} = \mathfrak{P}A$. We have the ring $A/\mathfrak{p}$ is a complete intersection in codimension one. Let $k > 0$. Then the ideal $\mathfrak{P}$ is minimally generated by the following seven elements:

$$f_1 = X_2^2 - X_1X_3, \qquad f_2 = X_2X_3 - X_1^2Y_1,$$

$$f_3 = X_3^2 - X_1X_2Y_1, \qquad f_4 = \begin{cases} X_1Y_2 - X_2Y_1^k & (n = 3k+1) \\ X_1Y_2 - X_3Y_1^k & (n = 3k+2), \end{cases}$$

$$f_5 = \begin{cases} X_2Y_2 - X_3Y_1^k & (n = 3k+1) \\ X_2Y_2 - X_1Y_1^{k+1} & (n = 3k+2), \end{cases} \qquad f_6 = \begin{cases} X_3Y_2 - X_1Y_1^{k+1} & (n = 3k+1) \\ X_3Y_2 - X_2Y_1^{k+1} & (n = 3k+2), \textit{and} \end{cases}$$

$$g = Y_2^3 - Y_1^n.$$

When $n = 3k + 1$, they satisfy the relations:

$$\begin{cases} f_1g = -Y_1^{k+1}f_4^2 + Y_2f_5^2 - Y_2f_4f_6 + Y_1^kf_5f_6 \\ f_2g = -Y_1Y_2f_4^2 + Y_1^kf_6^2 - Y_1^{k+1}f_4f_5 + Y_2f_5f_6 \\ f_1f_6 = f_2f_5 - f_3f_4 \end{cases}$$

and hence, letting $\mathfrak{J} = (f_3, f_4, f_5, f_2 + g, f_1 + f_6)S$, we get $\mathfrak{P}^2 = \mathfrak{J}\mathfrak{P}$. When $n = 3k + 1$, they satisfy the relations:

$$\begin{cases} f_1g = -Y_2f_5^2 + Y_1^kf_6^2 - Y_1^{k+1}f_4f_5 + Y_2f_4f_6 \\ f_2g = -Y_1Y_2f_4^2 + Y_1^{k+1}f_5^2 - Y_1^{k+1}f_4f_6 + Y_2f_5f_6 \\ f_2f_6 = f_3f_5 - Y_1f_1f_4 \end{cases}$$

and hence, letting $\mathfrak{J} = (f_3, f_4, f_5, f_1 + g, f_2 + f_6)S$, we get $\mathfrak{P}^2 = \mathfrak{J}\mathfrak{P}$. Hence by Corollary 4.3, the ring $\mathcal{G}(\mathfrak{p})$ is a Gorenstein ring.

In particular, we have $\mathrm{ad}(\mathfrak{p}) = 2$. In fact, assume $\mathfrak{P}^{(2)} = \mathfrak{P}^2$. Then $\mathfrak{P}^2 \ni f_1 f_3 - f_2^2 = X_1(3X_1X_2X_3Y_1 - X_2^3Y_1 - X_3^3 - X_1^3Y_1^2)$, so that $3X_1X_2X_3Y_1 - X_2^3Y_1 - X_3^3 - X_1^3Y_1^2 \in \mathfrak{P}^2$. However this is impossible. Therefore, thanks to [N], we have $\lambda(\mathfrak{p}) = 5$ and hence $\mathrm{ad}(\mathfrak{p}) = 2$.

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ON THE FIBER CONES OF BUCHSBAUM MODULES

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1. MAIN RESULTS

The aim of this note is to investigate the behaviours of the fiber cones. Let $(A, \mathfrak{m})$ be a Noetherian local ring and E a finitely generated A-module. For an ideal $\mathfrak{a}$, we denote by $\mathrm{F}_{\mathfrak{a}}(E)$ the fiber cone of E with respect to $\mathfrak{a}$, namely,

$$\mathrm{F}_{\mathfrak{a}}(E) := \bigoplus_{n \geq 0} \mathfrak{a}^n E / \mathfrak{a}^n \mathfrak{m} E.$$

There were already given several works on behaviours, especially on the Cohen-Macaulayness, of the fiber cones, see [S] etc. But, concerning on the Buchsbaumness of them, we had only quite few results unfortunately, cf., see [G2]. Thus, we are here very interested in the Buchsbaumness of the fiber cones and we particularly calculate their local cohomology modules. Namely, we try to find an answer to the following question:

Problem. How does one the local cohomology modules of the fiber cone $\mathrm{F}_{\mathfrak{a}}(E)$ calculate? Moreover, when does it obtain the Buchsbamness?

In particular, here we mainly discuss the case where the equality

$$\mathbb{I}(\mathrm{G}_{\mathfrak{a}}(E)) = \mathbb{I}(E)$$

holds and the reduction numbers of such ideals $\mathfrak{a}$ are at most one. Here we define an invariant of E, written $\mathbb{I}(E)$, as follows:

$$\mathbb{I}(E) := \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot h^i(E),$$

where we denote by $h^i(E)$ the length of the ith local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(E)$ of E, i.e., $h^i(E) := l_A(\mathrm{H}^i_{\mathfrak{m}}(E))$, and write $s := \dim_A E$. Recall that $\mathrm{G}_{\mathfrak{a}}(E) := \bigoplus_{n \geq 0} \mathfrak{a}^n E / \mathfrak{a}^{n+1} E$ denotes the associatde graded module of E with respect to an ideal $\mathfrak{a}$ and $\mathrm{R}_{\mathfrak{a}}(E) := \bigoplus_{n \geq 0} \mathfrak{a}^n E$ denotes the Rees module of E associated to an ideal $\mathfrak{a}$. Moreover, $\mathrm{R} := \mathrm{R}_{\mathfrak{a}}(A)$ denotes the Rees algebra of $\mathfrak{a}$ and also $\mathfrak{N}$ denotes the unique homogeneous maximal ideal of R, i.e., $\mathfrak{N} := \mathfrak{m}\mathrm{R} + \mathrm{R}_+$. Then we have the only partial answer as follows.

Theorem 1. *Suppose that the following conditions are fulfilled:*

(i) *E is a Buchsbaum A-module of dimensin $s > 0$:*
(ii) *the equality $\mathbb{I}(\mathrm{G}_{\mathfrak{a}}(E)) = \mathbb{I}(E)$ holds:*
(iii) *$\mathfrak{a}^2 E = \mathfrak{q}\mathfrak{a}E$ holds for some minimal reduction $\mathfrak{q}$ of $\mathfrak{a}$ with respect to E:*
(iv) *$\mathfrak{m}E$ is also a Buchsbaum A-module:*
(v) *the equality $\mathbb{I}(\mathrm{G}_{\mathfrak{a}}(\mathfrak{m}E)) = \mathbb{I}(\mathfrak{m}E)$ also holds.*

Then, $\mathrm{F}_{\mathfrak{a}}(E)_+ := \bigoplus_{n>0} \mathfrak{a}^n E/\mathfrak{a}^n \mathfrak{m}E$ is a Buchsbaum module over R, and moreover the fiber cone $\mathrm{F}_{\mathfrak{a}}(E)$ itself is so if and only if $\mathfrak{m}\mathfrak{a}^2 E : \mathfrak{a}^2 = \mathfrak{m}\mathfrak{a}E : \mathfrak{a}$ holds.

As an application of Theorem 1, we can state a generalization of Theorem 5 described below in the following.

Corollary 2. *Suppose that the following conditions are fulfilled:*

(i) *E is a Buchsbaum A-module of dimensin $s > 0$:*
(ii) *the equality $\mathbb{I}(\mathrm{G}_{\mathfrak{a}}(E)) = \mathbb{I}(E)$ holds:*
(iii) *$\mathfrak{m}E$ is also a Buchsbaum A-module:*
(iv) *$\mathfrak{a}\mathfrak{m}E = \mathfrak{q}\mathfrak{m}E$ holds for some minimal reduction $\mathfrak{q}$ of $\mathfrak{a}$ with respect to E.*

Then the fiber cone $\mathrm{F}_{\mathfrak{a}}(E)$ is a Buchsbaum module such that

$$h^p(\mathrm{F}_{\mathfrak{a}}(E)) = \begin{cases} h^0(E) - h^0(\mathfrak{m}E) + h^0(\mathfrak{a}E) & (p = 0) \\ l_A(\,[\mathrm{H}^p_{\mathfrak{N}}(\mathrm{G}_{\mathfrak{a}}(E))]_{1-p}\,) & (1 \leq p < s). \end{cases}$$

Moreover, $\mathrm{F}_{\mathfrak{a}}(E)$ is a Cohen-Macaulay module, if E is so.

In 1999 S. Goto [G2] just brought us an epoch-making work. He introcduced a new notion called an ($\mathfrak{m}$-primary) ideal of minimal multiplicity (in Cohen-Macaulay rings) and studied the Buchsbaumness of Rees algebras (and also the associated graded rings and the fiber cones) associated to such ideals of minimal multiplicity. This notion is naturally extended for Buchsbaum rings. Namely, for an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in a Buchsbaum ring A of dimension $d > 0$, we say that $\mathfrak{a}$ possesses *minimal multiplicity* if the following equality holds:

$$e_{\mathfrak{a}}(A) = \mu_A(\mathfrak{a}) + l_A(A/\mathfrak{a}) - d - \mathbb{I}(A).$$

In general, the term appeared in the right hand side makes a lower bound of the multiplicity of $\mathfrak{a}$. So this is just a natural generalization of the notion in Cohen-Macaulay (local) rings introduced by S. Goto, and moreover it is easy to see that this is equivalent to saying that $\mathfrak{a}\mathfrak{m} \subseteq \mathfrak{q}$ (and hence the equality $\mathfrak{a}\mathfrak{m} = \mathfrak{q}\mathfrak{m}$) holds for some (and hence every) minimal reduction $\mathfrak{q}$ of $\mathfrak{a}$. Then we also have the following as an corollary of our theorem.

Corollary 3. *Let A be a Buchsbaum ring and $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal of A. Suppose that the equality $\mathbb{I}(\mathrm{G}_{\mathfrak{a}}(A)) = \mathbb{I}(A)$ holds and $\mathfrak{a}$ possesses minimal multiplicity. Then the fiber cone $\mathrm{F}_{\mathfrak{a}}(A)$ is a Buchsbaum ring.*

Nextly, in the one dimensional case, we also have the following results.

Theorem 4. *Let E be a one-dimesional Buchsbaum A-module and $\mathfrak{a}$ an ideal of A such that $\mathfrak{a}^2E = a\mathfrak{a}E$ holds for some minimal reduction (a) of $\mathfrak{a}$ with respect to E. Then the following coditions are equivalent.*

(1) *The firber cone $\mathrm{F}_{\mathfrak{a}}(E)$ is a Buchsbaum module such that*

$$\mathbb{I}(\mathrm{F}_{\mathfrak{a}}(E)) = e_{\mathfrak{a}}(E) - l_A(E/\mathfrak{a}E) - \mu_A(\mathfrak{a}E) + \mu_A(E) + \mathbb{I}(E) + \mathbb{I}(\mathfrak{a}E).$$

(2) $\mathfrak{a}\mathfrak{m}E \subseteq aE$ *for all minimal reducition (a) of $\mathfrak{a}$ with respect to E.*

Theorem 5. *Under the same situation as in Theorem 4 above, suppose that $\mathfrak{a}\mathfrak{m}E \subseteq aE$ holds for some minimal reducition (a) of $\mathfrak{a}$ with respect to E. Then the following conditions are equivalent.*

(1) *The firber cone $\mathrm{F}_{\mathfrak{a}}(E)$ is a Buchsbaum module such that*

$$\mathbb{I}(\mathrm{F}_{\mathfrak{a}}(E)) = \mathbb{I}(E) - \mathbb{I}(\mathfrak{m}E) + \mathbb{I}(\mathfrak{a}E).$$

(2) $\mathfrak{a}\mathfrak{m}E = a\mathfrak{m}E$ *holds.*

(3) $e_{\mathfrak{a}}(E) = l_A(E/\mathfrak{a}E) + \mu_A(\mathfrak{a}E) - \mu_A(E) - \mathbb{I}(\mathfrak{m}E)$ *holds.*

2. Outline of the proof of Theorem 1

For simplicity, we usually denote by $\mathrm{F}(E)$, $\mathrm{R}(E)$, $\mathrm{G}(E)$ etc. omitting the letter of an odeal $\mathfrak{a}$ from our notation defined above.

In this section we assume that the following conditions are fulfilled:

(i) E is a Buchsbaum A-module of dimensin $s > 0$:
(ii) the equality $\mathbb{I}(\mathrm{G}(E)) = \mathbb{I}(E)$ holds:
(iii) $\mathfrak{a}^2E = \mathfrak{q}\mathfrak{a}E$ holds for some minimal reduction $\mathfrak{q}$ of $\mathfrak{a}$ with respect to E:

Then, we begin with recalling the following facts.

Lemma 6 ([N], see also [SY]). *The following statements are true.*

(1) $\mathrm{G}(E)$ *is a Buchsbaum* R*-module such that*

$$[\mathrm{H}^p_{\mathfrak{M}}(\mathrm{G}(E))]_n = (0) \quad (n \neq -p, 1-p)$$

for $0 \leq p < s$ and $a(\mathrm{G}(E)) \leq 1 - s$.

(2) *For any minimal reduction of $\mathfrak{a}$ with respect to E, say $\mathfrak{r} := (b_1, b_2, \ldots, b_s)$, the equalities*

$$\mathfrak{a}^2E = \mathfrak{r}\mathfrak{a}E \quad \textit{and} \quad (b_i^{n_i} \mid i \in I)E \cap \mathfrak{a}^nE = \sum_{i \in I} b_i^{n_i}\mathfrak{a}^{n-n_i}E$$

hold, where $I \subseteq [1, s]$, $n_i > 0$ and $n \in \mathbb{Z}$.

Lemma 7 ([Y3]). *The following statements are true.*

(1)

$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}(E))]_n = \begin{cases} \mathrm{H}^0_{\mathfrak{m}}(E) & (n=0) \\ \mathfrak{a}E \cap \mathrm{H}^0_{\mathfrak{m}}(E) & (n=1) \\ (0) & (else). \end{cases}$$

(2)

$$[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}(E))]_n = \begin{cases} [\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}(E))]_0 & (n=0) \\ (0) & (else). \end{cases}$$

(3) $\mathrm{H}^2_{\mathfrak{N}}(\mathrm{R}(E)) = (0)$ *if* $s \geq 2$.

(4) *If* $3 \leq p \leq s$, *then*

$$[\mathrm{H}^p_{\mathfrak{N}}(\mathrm{R}(E))]_n = \begin{cases} \mathrm{H}^{p-1}_{\mathfrak{m}}(E) & (n \in [3-p, -1]) \\ [\mathrm{H}^{p-1}_{\mathfrak{N}}(\mathrm{G}(E))]_{1-p} & (n = 2-p) \\ (0) & (else). \end{cases}$$

Furthermore we assume that the following conditions are also fulfilled:

(iv) $\mathfrak{m}E$ is also a Buchsbaum A-module:

(v) the equality $\mathbb{I}(\mathrm{G}(\mathfrak{m}E)) = \mathbb{I}(\mathfrak{m}E)$ also holds.

Then, as the same as in Lemma 7, we also have the following.

Lemma 8. *The following statements are true.*

(1)

$$[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E))]_n = \begin{cases} \mathfrak{m}E \cap \mathrm{H}^0_{\mathfrak{m}}(E) & (n=0) \\ \mathfrak{a}E \cap \mathrm{H}^0_{\mathfrak{m}}(E) & (n=1) \\ (0) & (else). \end{cases}$$

(2)

$$[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E))]_n = \begin{cases} [\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}(\mathfrak{m}E))]_0 & (n=0) \\ (0) & (else). \end{cases}$$

(3) $\mathrm{H}^2_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E)) = (0)$ *if* $s \geq 2$.

(4) *If* $3 \leq p \leq s$, *then*

$$[\mathrm{H}^p_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E))]_n = \begin{cases} \mathrm{H}^{p-1}_{\mathfrak{m}}(E) & (n \in [3-p, -1]) \\ [\mathrm{H}^{p-1}_{\mathfrak{N}}(\mathrm{G}(\mathfrak{m}E))]_{1-p} & (n = 2-p) \\ (0) & (else). \end{cases}$$

Now consider the following exact sequence of graded R-modules:

$$0 \longrightarrow \mathrm{R}(\mathfrak{m}E) \xrightarrow{\sigma} \mathrm{R}(E) \longrightarrow \mathrm{F}(E) \longrightarrow 0. \qquad (\sharp\ 1)$$

Then we have the long exact sequence of local cohomology modules of $\mathrm{F}(E)$ as follows:

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E)) \xrightarrow{\sigma^0} \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}(E)) \longrightarrow \mathrm{H}^0_{\mathfrak{N}}(\mathrm{F}(E)) \longrightarrow \cdots \\ &\longrightarrow \mathrm{H}^p_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E)) \xrightarrow{\sigma^p} \mathrm{H}^p_{\mathfrak{N}}(\mathrm{R}(E)) \longrightarrow \mathrm{H}^p_{\mathfrak{N}}(\mathrm{F}(E)) \longrightarrow \cdots \quad . \end{aligned} \qquad (\sharp\ 2)$$

According to Lemmas 6-8, the long exact sequence of local cohomology modules ($\sharp$ 2) is divided into several parts as follows:

Lemma 9. *The following statements are true.*

(1) *The following sequences of local cohomology modules are exact:*

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E)) \xrightarrow{\sigma^0} \mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}(E)) \longrightarrow \mathrm{H}^0_{\mathfrak{N}}(\mathrm{F}(E)) \\ &\longrightarrow \mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E)) \xrightarrow{\sigma^1} \mathrm{H}^1_{\mathfrak{N}}(\mathrm{R}(E)) \longrightarrow \mathrm{H}^1_{\mathfrak{N}}(\mathrm{F}(E)) \longrightarrow 0; \end{aligned} \qquad (\sharp\ 3)$$

and for each $2 \le p < s$

$$0 \longrightarrow \mathrm{H}^p_{\mathfrak{N}}(\mathrm{F}(E)) \longrightarrow \mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E)) \xrightarrow{\sigma^{p+1}} \mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(E)) \longrightarrow 0. \qquad (\sharp\ 4)$$

(2) *For each $2 \le p < s$, the homogeneous component of σ^{p+1} of degree n*

$$[\sigma^{p+1}]_n : [\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E))]_n \longrightarrow [\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(E))]_n$$

is isomorphic over A for all $2 - p \le n \le -1$.

Proof. Look for the following commutative diagram of graded R-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{R}(\mathfrak{a}E) & \longrightarrow & \mathrm{R}(E) & \longrightarrow & \mathrm{G}(E) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{R}(\mathfrak{m}E) & \xrightarrow{\sigma} & \mathrm{R}(E) & \longrightarrow & \mathrm{F}(E) & \longrightarrow & 0 \end{array}$$

Let $2 \le p < s$. By Lemmas 6 and 7, we see that the canonical map

$$\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(\mathfrak{a}E)) \longrightarrow \mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(E))$$

is surjective. Hence this implies that σ^{p+1} is so. Next look for the following:

$$[\sigma^{p+1}]_n : [\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E))]_n \longrightarrow [\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(E))]_n.$$

By Lemmas 7 and 8, we know that

$$l_A([\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(\mathfrak{m}E))]_n) = h^p(E) = l_A([\mathrm{H}^{p+1}_{\mathfrak{N}}(\mathrm{R}(E))]_n)$$

and hence the surjective map $[\sigma^{p+1}]_n$ must be isomorphic.

Combining these observations we get the following.

Proposition 10. *The following statements are true.*

(1) $[\mathrm{H}^0_{\mathfrak{M}}(\mathrm{F}(E))]_n = (0)$ *for all* $n \neq 0, 1$.

(2) *If* $1 \leq p < s$, *then* $[\mathrm{H}^p_{\mathfrak{M}}(\mathrm{F}(E))]_n = (0)$ *for all* $n \neq 1 - p$. *In particular, there exists the following exact sequence of A-modules:*

$$0 \longrightarrow [\mathrm{H}^p_{\mathfrak{M}}(\mathrm{F}(E))]_{1-p} \longrightarrow [\mathrm{H}^p_{\mathfrak{M}}(\mathrm{G}(\mathfrak{m}E))]_{-p} \longrightarrow [\mathrm{H}^p_{\mathfrak{M}}(\mathrm{G}(E))]_{-p} \longrightarrow 0$$

for each $2 \leq p < s$.

(3) $a(\mathrm{F}(E)) \leq 1 - s$.

Look for the following exact sequence of graded R-modules.

$$0 \longrightarrow \mathrm{F}(E)_+ \longrightarrow \mathrm{F}(E) \longrightarrow E/\mathfrak{m}E \longrightarrow 0. \qquad (\sharp\ 5)$$

By Proposition 10 we konw that

$$[\mathrm{H}^p_{\mathfrak{M}}(\mathrm{F}(E)_+)]_n = (0) \quad \text{for all } n \neq 1 - p,$$

where $0 \leq p < s$. According to [G1, Proposition (3.1)] it is easy to check that $\mathrm{F}(E)_+$ is Buchsbaum over R. Moreover it is also easy to see that the fiber cone $\mathrm{F}(E)$ itself is Buchsbaum if and only if $\mathfrak{m}\mathfrak{a}^2 E : \mathfrak{a}^2 = \mathfrak{m}\mathfrak{a}E : \mathfrak{a}$ holds. This finishes the proof of Theorem 1.

3. Outline of the proofs of other results

In this section, we assume that the following two conditions are fulfilled:

(i) E is a one-dimensional Buchsbaum A-module:

(ii) $\mathfrak{a}^2 E = a\mathfrak{a}E$ holds for some minimal reduction (a) of $\mathfrak{a}$ with respect to E:

We also use the same notation $\mathrm{F}(E)$, $\mathrm{R}(E)$ etc. as in the previous section. Moreover we usually regard the Rees algebra R as the graded A-subalgebra of $A[t]$, the polynomial ring over A with an indeterminate t, namely

$$\mathrm{R} = \sum_{n \geq 0} \mathfrak{a}^n t^n \subset A[t].$$

Let us consider the linear form $at \in [\mathrm{R}]_1$. Then

Proposition 11. *The following statements are true.*

(1) *The following inequality*

$$l_{\mathrm{R}}([0 \underset{\mathrm{F}(E)}{:} at]) \leq e_{\mathfrak{a}}(E) - l_A(E/\mathfrak{a}E) - \mu_A(\mathfrak{a}E) + \mu_A(E) + \mathbb{I}(E) + \mathbb{I}(\mathfrak{a}E)$$

holds. Moreover, the equality holds if and only if $\mathfrak{a}\mathfrak{m}E \subseteq aE$ *holds.*

(2) at *is a d-sequence (cf. see [H]) on* $\mathrm{F}(E)$ *if* $\mathfrak{a}\mathfrak{m}E \subseteq aE$ *holds.*

Lemma 12. *The following equality holds:*

$$e_{\mathfrak{a}}(E) = \left\{l_A(E/\mathfrak{a}E) + \mu_A(\mathfrak{a}E) - \mu_A(E) - \mathbb{I}(\mathfrak{m}E)\right\} + l_A(\mathfrak{a}\mathfrak{m}E/a\mathfrak{m}E),$$

where (a) is a minimal reduction of $\mathfrak{a}$ with respect to E.

Note that the following formula holds in general:

$$\begin{aligned} & e_{\mathfrak{a}}(E) - l_A(E/\mathfrak{a}E) - \mu_A(\mathfrak{a}E) + \mu_A(E) + \mathbb{I}(E) + \mathbb{I}(\mathfrak{a}E) \\ & = \mathbb{I}(E) - \mathbb{I}(\mathfrak{m}E) + \mathbb{I}(\mathfrak{a}E) + l_A(\mathfrak{a}\mathfrak{m}E/a\mathfrak{m}E). \end{aligned} \qquad (\sharp\ 6)$$

Combining these observations, our Theorems 4 and 5 follow at once.

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On Chow groups of G-graded rings

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1 Introduction

In this paper, we study Chow groups of Noetherian rings graded by a finitely generated Abelian group. It is well known that a class group of a normal graded ring is determined by homogeneous height one prime ideals and homogeneous principal divisors. We will give a generalization of this statement for Chow groups of Noetherian rings graded by a finitely generated Abelian group.

Let G be a finitely generated Abelian group (not necessary a torsion free) and $A = \bigoplus_{g \in G} A_g$ be the Noetherian G-graded ring. We call that a G-graded ideal $\mathfrak{p} \subset A$ is G-prime, if every homogeneous element of $A/\mathfrak{p}$ is a nonzero divisor. If G has no torsion, then a G-prime ideal is a G-graded prime ideal. Otherwise a G-prime ideal is not necessary a prime ideal and these ideals involve an enough information to determine some property among G-graded rings. We use these graded ideals to describe a Chow group of A. One can define a group $\mathrm{A}^G_{\bullet}(A) = \mathrm{Z}^G_{\bullet}(A)/\mathrm{Rat}^G_{\bullet}(A)$, where $\mathrm{Z}^G_{\bullet}(A)$ is the free Abelian group generated by $[A/\mathfrak{p}]$ for each G-prime $\mathfrak{p}$ and $\mathrm{Rat}^G_{\bullet}(A)$ is a subgroup of $\mathrm{Z}^G_{\bullet}(A)$ which is a graded analogue of rational equivalence determined by homogeneous elements and G-prime (see (2.5)). Then our main result is stated as follows.

Theorem 1.1 *Let* $W = \{P \in \mathrm{Spec}(A) \mid P$ *is a minimal prime of some* G*-prime ideal*$\} \subset \mathrm{Spec}(A)$ *and* $\mathrm{Z}_{\bullet}(W) = \bigoplus_{P \in W} \mathbb{Z}[A/P] \subset \mathrm{Z}_{\bullet}(A)$. *If* $G = \mathbb{Z}^m \oplus T$ *with* $|T| < \infty$, *then there is a canonical map* $\phi : \mathrm{A}^G_{\bullet}(A) \longrightarrow \mathrm{A}_{\bullet}(A)$ *satisfying following conditions*

(1) $\mathrm{Coker}(\phi) \cong \mathrm{Z}_{\bullet}(W)/\mathrm{Z}^G_{\bullet}(A) + \mathrm{Rat}_{\bullet}(A) \cap \mathrm{Z}_{\bullet}(W)$

(2) $|T|\mathrm{Ker}(\phi) = 0$.

If G is torsion free, i.e. $T = (0)$, then G-prime coincide with G-graded prime. Thus W in Theorem 1.1 is the set of all G-graded prime ideals and $\mathrm{Z}_{\bullet}(W) = \mathrm{Z}^G_{\bullet}(A)$ (and of course, $|T| = 1$). Hence the next corollary is direct consequence of Theorem 1.1.

Corollary 1.2 *If* G *is torsion free, then* $\mathrm{A}^G_{\bullet}(A)$ *is isomorphic to* $\mathrm{A}_{\bullet}(A)$.

2 Definition of $A^G_{\cdot}(A)$

Let A be a Noetherian ring of finite type over a regular domain R. We treat a Chow group of A using relative dimension instead of Krull dimension (Chap. 20 in Fulton[1]). Relative dimension $\dim_R(A/P)$ is defined as $\dim_R(A/P) = \mathrm{tr.deg}(k(P)/k(R\cap P)) - \mathrm{ht}_R(R\cap P)$ for every $P \in \mathrm{Spec}(A)$. Note that $\dim_R(A/P) = \dim_R(A/Q) + \mathrm{ht}_{A/P}(Q/P)$ for $P \subset Q \in \mathrm{Spec}(A)$. For a finitely generated A-module M, we set

$$\begin{aligned} \dim_R(M) &= \sup\{\dim_R(A/P) \mid P \in \mathrm{Supp}_A(M)\} \\ \mathrm{Assh}_R(M) &= \{P \in \mathrm{Supp}_A(M) \mid \dim_R(M) = \dim_R(A/P)\}. \end{aligned}$$

The i-th cycle $\mathrm{Z}_i(A)$ of A is a free Abelian group generated by $[A/P]$ for every $P \in \mathrm{Spec}(A)$ with $\dim_R(A/P) = i$. $\mathrm{Rat}_i(A)$ is a subgroup of $\mathrm{Z}_i(A)$ generated by $\mathrm{div}(Q, a)$ for every $Q \in \mathrm{Spec}(A)$, $\dim_R(A/Q) = i + 1$ and for every $a \in A \setminus Q$, where

$$\mathrm{div}(Q, a) = \sum_{P \in \mathrm{Min}_A(A/(a,Q))} \ell_{A_P}(A_P/(a, Q)A_P)[A/P].$$

The i-th Chow group $\mathrm{A}_i(A)$ is defined to be a quotient group $\mathrm{Z}_i(A)/\mathrm{Rat}_i(A)$. We define the Chow group (resp. cycles, rational equivalence) of A by $\mathrm{A}_{\cdot}(A) = \oplus_{i\geq 0}\mathrm{A}_i(A)$ (resp. $\mathrm{Z}_{\cdot}(A) = \oplus_{i\geq 0}\mathrm{Z}_i(A)$, $\mathrm{Rat}_{\cdot}(A) = \oplus_{i\geq 0}\mathrm{Rat}_i(A)$).

The goal of this setion is to define a similar notion of the Chow group for graded rings and to difine a natural map from this group to the ordinary Chow group. Let $(G, +)$ be a finitely generated Abelian group. We call that a ring A is a G-graded ring, if there exist a family $\{A_g\}_{g\in G}$ of subgroups of A such that $A = \bigoplus_{g\in G} A_g$ and $A_gA_h \subset A_{g+h}$ for every $g, h \in G$. Similarly, a G-graded A-module is a A-module M with a family $\{M_g\}_{g\in G}$ of subgroups of M such that $M = \bigoplus_{g\in G} M_g$ and $A_gM_h \subset M_{g+h}$ for every $g, h \in G$. The subgroup M_g is called a degree g part of M. Also, an element $x \in M_g \setminus \{0\}$ is called a homogeneous element of degree g and we denote by $\deg x = g$.

Henceforth, we assume that A is a Noetherian G-graded ring, its degree 0 part A_0 is of finite type over R and all A-modules are finitely generated. By the result of Goto-Yamagishi[3], A is finitely generated over the degree 0 part A_0. Hence A is of finite type over R. Let us recall some definition of G-graded rings from [4]. Defining the analogy of Chow group, we will use the following notion.

Definition 2.1 A G-graded ideal $\mathfrak{p}$ of A is said to be G-prime, if every homogeneous element of $A/\mathfrak{p}$ is not a divisor of zero. We denote the set of all G-prime ideals by $\mathrm{Spec}^G(A)$.

Remark 2.2 If G is torsion free, then G-prime ideals are nothing but G-graded prime ideals and $\mathrm{Spec}^G(A) \subset \mathrm{Spec}(A)$. However, if G has torsion, then G-prime ideals are not necessary prime. For example, Let $\mathbb{Z}[x] = \mathbb{Z}[X]/(X^2 - 1)$. We consider $\mathbb{Z}[x]$ as a $\mathbb{Z}/(2)$-graded ring by $\deg x = \bar{1} \in \mathbb{Z}/(2)$. Then $\mathbb{Z}[x]$ has no graded prime ideals and $\mathrm{Spec}^G(\mathbb{Z}[x]) = \{(0)\}$. G-prime ideals give a lot of information of G-graded ring and G-graded modules. It plays a role of prime ideals in the category of G-graded ring (and the category of G-graded modules). See, for example, [4], [5].

For an arbitrary ideal $P \subset A$, we put $P^* = \oplus_{g \in G} P \cap A_g$ the maximal graded ideal contained in P. If P is prime, then P^* is G-prime. Conversely, if $\mathfrak{p} \in \mathrm{Spec}^G(A)$, then $P^* = \mathfrak{p}$ for every $P \in \mathrm{Ass}_A(A/\mathfrak{p})$ ((2,2) in [4]) and $\mathrm{Spec}^G(A) = \{P^* \mid P \in \mathrm{Spec}(A)\}$. Furthermore, we have $\mathrm{Ass}_A(A/\mathfrak{p}) = \mathrm{Min}_A(A/\mathfrak{p})$ since $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$ is G-simple (see (2.6) below).

Let $\mathfrak{p} \subset A$ be a G-prime and M be a G-graded A-module. We define a homogeneous localization $M_{(\mathfrak{p})}$ of M at $\mathfrak{p}$ by $M_{(\mathfrak{p})} = S^{-1}M$, where S is the set of all homogeneous elements of $A \setminus \mathfrak{p}$. We denote by $\mathrm{Supp}_A^G(M) = \{\mathfrak{p} \in \mathrm{Spec}^G(A) \mid M_{(\mathfrak{p})} \neq 0\}$. Note that $P \in \mathrm{Supp}_A(M)$ if and only if $P^* \in \mathrm{Supp}_A^G(M)$. We denote by $\mathrm{Min}_A^G(M)$ (resp. $\mathrm{Assh}_A^G(M)$, $\mathrm{Assh}_R^G(M)$) the set of minimal G-prime ideals in $\mathrm{Supp}_A^G(M)$ (the set of G-prime $\mathfrak{p} \in \mathrm{Supp}_A^G(M)$ with $\dim M = \dim A/\mathfrak{p}$, the set of G-prime $\mathfrak{p} \in \mathrm{Supp}_A^G(M)$ with $\dim_R(M) = \dim_R(A/\mathfrak{p})$).

Definition 2.3 We denote by $\mathrm{Z}_i^G(A)$ a free Abelian group with basis $[A/\mathfrak{p}]$ consisting of all G-prime $\mathfrak{p}$ such that $\dim_R(A/\mathfrak{p}) = i$. The G-cycles of A is defined by the direct sum of $\mathrm{Z}_i^G(A)$ over all i and denote by $\mathrm{Z}_\bullet^G(A)$.

For a finitely generated G-graded A-module M, M has a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of G-graded submodules of M such that $M_{i+1}/M_i \cong A/\mathfrak{p}_i(g_i)$ for some G-prime $\mathfrak{p}_i$ and for some $g_i \in G$. Note that $\mathrm{Min}_A^G(M) \subset \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. For $\mathfrak{p} \in \mathrm{Min}_A^G(M)$, we denote the number of copies of $A/\mathfrak{p}(g)$ (for some $g \in G$) appearing in these subquotients by $\ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})$. Then the number $\ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})$ does not depend on the choice of the filtration of M. In fact, $\ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})$ is coincides with the length of the maximal chain of G-graded submodules of $M_{(\mathfrak{p})}$. We denote by $[M] = \sum_{\mathfrak{p} \in \mathrm{Assh}_R^G(M)} \ell_{A_{(\mathfrak{p})}}^G(M_{(\mathfrak{p})})\, [A/\mathfrak{p}] \in \mathrm{Z}_{\dim_R(M)}^G(A)$.

Definition 2.4 Let $\mathfrak{p} \in \mathrm{Spec}^G(A)$ and let $a \in A \setminus \mathfrak{p}$ be a homogeneous element We denote by

$$\mathrm{div}^G(\mathfrak{p},\ a) = \sum_{\mathfrak{q} \in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \ell_{A_{(\mathfrak{p})}}^G(A_{(\mathfrak{q})}/(a,\mathfrak{p})A_{(\mathfrak{q})})\, [A/\mathfrak{q}]\,.$$

We define $\mathrm{Rat}_i^G(A)$ by a subgroup of $\mathrm{Z}_\bullet^G(A)$ generated by $\mathrm{div}^G(\mathfrak{p},\ a)$ for all $\mathfrak{p} \in \mathrm{Spec}^G(A)$ with $\dim_R(A/\mathfrak{p}) = i+1$ and for all homogeneous element $a \in A \setminus \mathfrak{p}$. We put $\mathrm{Rat}_\bullet^G(A) = \sum_{i \geq 0} \mathrm{Rat}_i^G(A) \subset \mathrm{Z}_\bullet^G(A)$ and call it the G-rational equivalence of A.

Later, we will see that $\mathrm{div}^G(\mathfrak{p},\ a) = [A/(a,\mathfrak{p})]$ (Lemma 2.9). Hence we have $\mathrm{Rat}_i^G(A) \subset \mathrm{Z}_i^G(A)$ for all i and $\mathrm{Rat}_\bullet^G(A) = \bigoplus_{i \geq 0} \mathrm{Rat}_i^G(A) \subset \mathrm{Z}_\bullet^G(A)$.

Definition 2.5 The i-th G-Chow group of A is defined by $\mathrm{A}_i^G(A) = \mathrm{Z}_i^G(A)/\mathrm{Rat}_i^G(A)$. We define the G-Chow group of A by $\mathrm{A}_\bullet^G(A) = \mathrm{Z}_\bullet^G(A)/\mathrm{Rat}_\bullet^G(A)$.

Definition-Proposion 2.6 ((1.6) in [4]) A G-graded ring A is said to be G-simple, if A has no proper G-graded ideal. If A is G-simple and $G' = \{g \in G \mid A_g \neq 0\}$, then A_0 is field and A is a twisted group ring $A_0^t[G']$ of G' over the field A_0. In particular, A is complete intersection.

Now, we have the following relative dimesion formula for G-prime ideals.

Lemma 2.7 *Let* $\mathfrak{p} \in \operatorname{Spec}^G(A)$ *and* $P \in \operatorname{Spec}(A)$ *with* $P^* = \mathfrak{p}$. *Then we have*

$$\dim_R(A/P) = \dim_R(A/\mathfrak{p}) - \dim A_P/\mathfrak{p}A_P.$$

Particularly, $\dim_R(A/\mathfrak{p}) = \dim_R(A/Q)$ *for all* $Q \in \operatorname{Ass}_A(A/\mathfrak{p})$, *i.e.* $\operatorname{Assh}_R(A/\mathfrak{p}) = \operatorname{Ass}_A(A/\mathfrak{p})$.

Proof. We put $K = [A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}]_0$ and $G(\mathfrak{p}) = \{g \in G \mid [A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}]_g \neq 0\}$. Then we have

$$\begin{aligned}\dim_R(A/P) &= \operatorname{tr.deg}(k(P)/K) + \operatorname{tr.deg}(K/k(R \cap \mathfrak{p}_0)) - \operatorname{ht}_R(R \cap \mathfrak{p}_0) \\ &= \dim K^t[G(\mathfrak{p})]/PK^t[G(\mathfrak{p})] + \operatorname{tr.deg}(K/k(R \cap \mathfrak{p}_0)) - \operatorname{ht}_R(R \cap \mathfrak{p}_0) \\ &= \dim A_{(\mathfrak{p})}/PA_{(\mathfrak{p})} + \operatorname{tr.deg}(K/k(R \cap \mathfrak{p}_0)) - \operatorname{ht}_R(R \cap \mathfrak{p}_0)\end{aligned}$$

If $Q \in \operatorname{Ass}_A(A/\mathfrak{p})$, then $\dim A_{(\mathfrak{p})}/QA_{(\mathfrak{p})} = \dim A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$ by (2.6). Hence we have $\dim_R(A/\mathfrak{p}) - \dim_R(A/P) = \dim A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})} - \dim A_{(\mathfrak{p})}/PA_{(\mathfrak{p})} = \dim A_P/\mathfrak{p}A_P$. □

Lemma 2.8 *(1)* $\operatorname{Min}_A(M) = \bigcup_{\mathfrak{p} \in \operatorname{Min}_A^G(M)} \operatorname{Ass}_A(A/\mathfrak{p})$.

(2) $\operatorname{Assh}_A(M) = \bigcup_{\mathfrak{p} \in \operatorname{Assh}_A^G(M)} \operatorname{Ass}_A(A/\mathfrak{p})$.

(3) $\operatorname{Assh}_R(M) = \bigcup_{\mathfrak{p} \in \operatorname{Assh}_R^G(M)} \operatorname{Ass}_A(A/\mathfrak{p})$.

Proof. (1) Let $P \in \operatorname{Min}_A(M)$ and $\mathfrak{p} = P^*$. If $\mathfrak{q} \in \operatorname{Min}_A^G(M)$ and $\mathfrak{q} \subset \mathfrak{p}$, then $\mathfrak{q} \subset P$. Since $\operatorname{Ass}_A(A/\mathfrak{q}) \subset \operatorname{Supp}_A(M)$, we have $P \in \operatorname{Min}_A(A/\mathfrak{q}) = \operatorname{Ass}_A(A/\mathfrak{q})$ and $\mathfrak{q} = P^* = \mathfrak{p}$. Thus $\operatorname{Min}_A(M) \subset \bigcup_{\mathfrak{p} \in \operatorname{Min}_A^G(M)} \operatorname{Ass}_A(A/\mathfrak{p})$. Conversely, let $\mathfrak{p} \in \operatorname{Min}_A^G(M)$ and $P \in \operatorname{Ass}_A(A/\mathfrak{p})$. If $Q \in \operatorname{Min}_A(M)$ with $Q \subset P$, then $Q^* \in \operatorname{Min}_A^G(M)$ as above. Also, we have $Q^* \subset P^* = \mathfrak{p}$ and, by the minimality of $\mathfrak{p}$, $Q^* = \mathfrak{p}$. Since P is minimal prime of $A/\mathfrak{p}$, we have $Q = P$. This complete the proof of (1).

(3) Let $\mathfrak{p} \in \operatorname{Assh}_R^G(M)$ and $P \in \operatorname{Assh}_R(M)$. Then, by definition, $\dim_R(A/P) \geq \dim_R(A/\mathfrak{p}) \geq \dim_R(A/P^*)$. On the other hand, we have $\dim_R(A/P) \leq \dim_R(A/P^*)$ by (2.7). Hence $\dim_R(A/P) = \dim_R(A/\mathfrak{p}) = \dim_R(A/P^*)$. This implies that $P^* \in \operatorname{Ass}_R^G(M)$, $P \in \operatorname{Ass}_A(A/P^*)$ and $\operatorname{Ass}_A(A/\mathfrak{p}) \subset \operatorname{Assh}_R(M)$ (again by (2.7)). Thus $\operatorname{Assh}_R(M) = \bigcup_{\mathfrak{p} \in \operatorname{Assh}_R^G(M)} \operatorname{Ass}_A(A/\mathfrak{p})$.

The assertion (2) is proved by the same way as (3). □

Lemma 2.9 *Let* $\mathfrak{p} \in \operatorname{Spec}^G(A)$ *and* $a \in A \setminus \mathfrak{p}$ *is a homogeneous element. Then we have the following.*

(1) $\operatorname{Assh}_R^G(A/(a,\mathfrak{p})) = \operatorname{Min}_A^G(A/(a,\mathfrak{p}))$ *and* $\operatorname{Assh}_R(A/(a,\mathfrak{p})) = \operatorname{Min}_A(A/(a,\mathfrak{p}))$.

(2) $\dim_R(A/\mathfrak{q}) = \dim_R(A/\mathfrak{p}) - 1$ *for all* $\mathfrak{q} \in \operatorname{Min}_A^G(A/(a,\mathfrak{p}))$ *and* $\dim_R(A/Q) = \dim_R(A/\mathfrak{p}) - 1$ *for all* $Q \in \operatorname{Min}_A(A/(a,\mathfrak{p}))$.

Proof. The assertion (1) follows from the assertion (2).

(2) Let $\mathfrak{q} \in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))$ and $Q \in \mathrm{Ass}_A(A/\mathfrak{q})$. If $P \in \mathrm{Ass}_A(A/\mathfrak{p})$ with $Q \supset P$, then Q is minimal prime of $A/(a,P)$. Indeed, if we let $Q_1 \in \mathrm{Min}_A(A/(a,P))$ such that $Q \supset Q_1$, then $\mathfrak{q} = Q^* \supset Q_1^* \supset (a,P)^* \supset (a,\mathfrak{p})$. Then, by the minimality of $\mathfrak{q}$, we have $\mathfrak{q} = Q^* = Q_1^*$ and, by the minimality of Q, $Q = Q_1$. Since $P^* = \mathfrak{p}$, we have $a \notin P$ and $\dim_R(A/\mathfrak{q}) = \dim_R(A/Q) = \dim_R(A/P) - 1 = \dim_R(A/\mathfrak{p}) - 1$. This completes the proof of Lemma. □

Now we define a group homomorphism from $\mathrm{Z}_{\bullet}^G(A)$ to $\mathrm{Z}_{\bullet}(A)$ as follows;

$$\begin{array}{cccc} \phi & : & \mathrm{Z}_{\bullet}^G(A) & \longrightarrow & \mathrm{Z}_{\bullet}(A) \\ & & [A/\mathfrak{p}] & \longmapsto & \sum_{P\in \mathrm{Ass}_R(A/\mathfrak{p})} \ell_{A_P}(A_P/\mathfrak{p}A_P)[A/P]. \end{array}$$

Then, by (2.7), ϕ is a graded group homomorphism. Namely, $\phi(\mathrm{Z}_i^G(A)) \subset \mathrm{Z}_i(A)$. Sometime, we consider that $\mathrm{Z}_{\bullet}^G(A)$ is a subgroup of $\mathrm{Z}_{\bullet}(A)$ via ϕ. Let $\mathfrak{p} \in \mathrm{Spec}^G(A)$ and $a \in \bigcup_{g\in G} A_g \setminus \mathfrak{p}$. Then, by Lemma 2.9, we have

$$\begin{aligned} \phi(\mathrm{div}^G(\mathfrak{p},\ a)) &= \textstyle\sum_{\mathfrak{q}\in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \ell^G_{A_{(\mathfrak{q})}}(A_{(\mathfrak{q})}/(a,\mathfrak{p})A_{(\mathfrak{q})})\phi([A/\mathfrak{q}]) \\ &= \textstyle\sum_{\mathfrak{q}\in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \sum_{Q\in \mathrm{Ass}_A(A/\mathfrak{q})} \ell^G_{A_{(\mathfrak{q})}}(A_{(\mathfrak{q})}/(a,\mathfrak{p})A_{(\mathfrak{q})})\ell_{A_Q}(A_Q/\mathfrak{q}A_Q)[A/Q] \\ &= \textstyle\sum_{\mathfrak{q}\in \mathrm{Min}_A^G(A/(a,\mathfrak{p}))} \sum_{Q\in \mathrm{Ass}_A(A/\mathfrak{p})} \ell_{A_Q}(A_Q/(a,\mathfrak{p})A_Q)[A/Q] \\ &= \textstyle\sum_{Q\in \mathrm{Min}_A(A/(a,\mathfrak{p}))} \ell_{A_Q}(A_Q/(a,\mathfrak{p})A_Q)[A/Q] \\ &= \textstyle\sum_{Q\in \mathrm{Assh}_R(A/(a,\mathfrak{p}))} \ell_{A_Q}(A_Q/(a,\mathfrak{p})A_Q)[A/Q]. \end{aligned}$$

Since a is a nonzero divisor of $A/\mathfrak{p}$, $\phi(\mathrm{div}^G(a,\ \mathfrak{p})) = \sum_{Q\in \mathrm{Assh}_R(A/(a,\mathfrak{p}))} \ell(A_Q/(a,\mathfrak{p})A_Q)[A/Q]$ belongs to $\mathrm{Rat}_{\bullet}(A)$. Hence ϕ induces a graded group homomorphism $\phi : \mathrm{A}_{\bullet}^G(A) \longrightarrow \mathrm{A}_{\bullet}(A)$. Henceforth we call this ϕ the natural homomorphism from $\mathrm{A}_{\bullet}^G(A)$ to $\mathrm{A}_{\bullet}(A)$.

In the same way as ordinary Chow groups, we have the following.

Lemma 2.10 *(1) If $f : A \longrightarrow B$ is a flat G-graded ring homomorphism of relative dimension k , then the map from $\mathrm{Z}_i^G(A)$ to $\mathrm{Z}_{i+k}^G(B)$ that sends $[A/\mathfrak{p}]$ to $[B/\mathfrak{p}B]$ induces a map on G-Chow groups from $\mathrm{A}_i^G(A)$ to $\mathrm{A}_{i+k}^G(B)$.*

(2) Let S be a multiplicatively closed subset of A consisting from homogeneous elements. For each G-prime $S^{-1}\mathfrak{p}$ of $S^{-1}A$, we define $\dim_R(S^{-1}A/S^{-1}\mathfrak{p}) = \dim_R(A/\mathfrak{p})$. Let $\mathrm{Z}_{\bullet}^G(S,A)$ denote the subgroup of $\mathrm{Z}_{\bullet}^G(A)$ generated by $[A/\mathfrak{p}]$ such that $\mathfrak{p}\cap S \neq \phi$. Then the inclusion $\mathrm{Z}_{\bullet}^G(S,A) \hookrightarrow \mathrm{Z}_{\bullet}^G(A)$ induces an exact sequence

$$\mathrm{Z}_{\bullet}^G(S,A) \longrightarrow \mathrm{A}_{\bullet}^G(A) \longrightarrow \mathrm{A}_{\bullet}^G(S^{-1}A) \longrightarrow 0.$$

(3) A finite G-graded ring homomorphism $g : A \longrightarrow B$ induces the map $g_* : \mathrm{A}_{\bullet}^G(B) \longrightarrow \mathrm{A}_{\bullet}^G(A)$ such that $g_*([B/\mathfrak{P}]) = \ell^G_{A_{(\mathfrak{p})}}(B_{(\mathfrak{P})}/\mathfrak{P}B_{(\mathfrak{P})})[A/\mathfrak{p}]$ $(\dim_R(B/\mathfrak{P}) = \dim_R(A/\mathfrak{p}), \mathfrak{p} = A\cap\mathfrak{P})$, or 0 (otherwise).

□

Similar to section 2.3, 2.4 of [1], we have an intersection with divisors on G-Chow groups. The following lemma will play an important role in the proof of our main result.

Lemma 2.11 *Let a be a homogeneous element of A such that $a \notin \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Assh}_R^G(A)$. We define a map* $\cap\mathrm{div}^G(a)$ *from* $\mathrm{Z}_{\bullet}^G(A)$ *to* $\mathrm{Z}_{\bullet}^G(A/(a))$ *by*

$$\begin{array}{ccc} \mathrm{Z}_{\bullet}^G(A) & \xrightarrow{\cap\mathrm{div}^G(a)} & \mathrm{Z}_{\bullet}^G(A/(a)) \\ [A/\mathfrak{p}] & \longmapsto & \begin{cases} [A/(a,\mathfrak{p})] & (a \notin \mathfrak{p}) \\ 0 & (a \in \mathfrak{p}). \end{cases} \end{array}$$

Then the map $\mathrm{div}^G(a)$ *induces* $\mathrm{A}_{\bullet}^G(A) \longrightarrow \mathrm{A}_{\bullet}^G(A/(a))$ *such that*

$$\begin{array}{ccc} \mathrm{A}_i^G(A) & \xrightarrow{\cap\mathrm{div}^G(a)} & \mathrm{A}_{i-1}^G(A/(a)) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathrm{A}_i(A) & \xrightarrow{\cap\mathrm{div}(a)} & \mathrm{A}_{i-1}(A/(a)). \end{array}$$

is commutative.

□

The next proposition is follows from (2.10) and (2.11).

Lemma 2.12 *Let $A[t]$ be the polynomial ring over the G-graded ring A. We regard $A[t]$ as a G-graded ring by putting $\deg t = g$ for some $g \in G$. Then the following two maps are isomorphism;*

(1) $\mathrm{A}_{\bullet}^G(A[t]) \xrightarrow{f^*} \mathrm{A}_{\bullet}^G(A[t,t^{-1}])$ *induced from* $A[t] \xrightarrow{f} A[t,t^{-1}]$,

(2) $\mathrm{A}_i^G(A[t]) \xrightarrow{\cap\mathrm{div}^G(t)} \mathrm{A}_{i-1}^G(A)$.

In particular, $[A[t]/\mathfrak{p}A[t]] \cap \mathrm{div}^G(t) = [A/\mathfrak{p}] \in \mathrm{A}_{\bullet}^G(A)$ *for* $\mathfrak{p} \in \mathrm{Spec}^G(A)$.

□

3 Proof of Theorem 1.1

(1) Since ϕ factor through an inclusion $\mathrm{Z}_{\bullet}(W)/\mathrm{Z}_{\bullet}(W) \cap \mathrm{Rat}_{\bullet}(A) \hookrightarrow \mathrm{A}_{\bullet}(A)$, it is enough to proved that $\mathrm{Z}_{\bullet}(W)/\mathrm{Z}_{\bullet}(W) \cap \mathrm{Rat}_{\bullet}(A) \hookrightarrow \mathrm{A}_{\bullet}(A)$ is surjective.

Let $P \in \mathrm{Spec}(A)$ with $\dim A_P/P^*A_P = d$. We prove that $[A/P] \in \mathrm{A}_{\bullet}(A)$ comes from $\mathrm{Z}_{\bullet}(W)$ by induction on d. Suppose that $d > 0$ and $Y = W \cap \mathrm{Spec}(A/P^*)$. Then the following diagram

$$\begin{array}{ccc} \frac{\mathrm{Z}_{\bullet}(W)}{\mathrm{Z}_{\bullet}(W)\cap\mathrm{Rat}_{\bullet}(A)} & \hookrightarrow & \mathrm{A}_{\bullet}(A) \\ \uparrow & \circlearrowleft & \uparrow \\ \frac{\mathrm{Z}_{\bullet}(Y)}{\mathrm{Z}_{\bullet}(Y)\cap\mathrm{Rat}_{\bullet}(A/P^*)} & \hookrightarrow & \mathrm{A}_{\bullet}(A/P^*). \end{array}$$

is commutative. To prove our assertion, it is only need to show that $[A/P]$ is contained in $\mathrm{Im}[\mathrm{Z}_{\bullet}(Y)/\mathrm{Z}_{\bullet}(Y) \cap \mathrm{Rat}_{\bullet}(A/P^*) \to \mathrm{A}_{\bullet}(A/P^*)]$. Thus we may assume that $P^* = 0$, (namely, any nonzero homogeneous elements are nonzero divisors). Consider the localization sequence

$$\mathrm{A}_{\dim_R(A/P)}(S,A) \longrightarrow \mathrm{A}_{\dim_R(A/P)}(A) \longrightarrow \mathrm{A}_{\dim_R(A/P)}(A_{(0)}) \to 0$$

where S is the set of all nonzero homogeneous elements of A. Since $A_{(0)}$ is G-simple (or $A_{(0)}$ is a twisted group ring over the field $[A_{(0)}]_0$), $\mathrm{A}_.(A_{(0)}) \cong \mathrm{A}_{\dim_R(A_{(0)})}(A_{(0)}) = \mathrm{A}_{\dim_R(A)}(A_{(0)})$. On the other hand, $d = \dim A_P = \dim_R(A) - \dim_R(A/P) > 0$ and $\dim_R(A) > \dim_R(A/P)$. Hence $[A/P]$ is rationally equivalent to some cycle $\sum_s n_s[A/P_s]$ such that $\dim_R(A/P_s) = \dim_R(A/P)$ and P_s contains a nonzero homogeneous element. Since $P_s^* \neq 0$ and $\dim_R(A/P_s^*) < \dim_R(A)$, $\dim A_{P_s}/P_s^* A_{P_s} = \dim_R(A/P_s^*) - \dim_R(A/P_s) < \dim_R(A) - \dim_R(A/P) = d$. Then, by induction hypothsis, $[A/P_s] \in \mathrm{Im}[\mathrm{Z}_.(W)/\mathrm{Z}_.(W) \cap \mathrm{Rat}_.(A) \to \mathrm{A}_.(A)]$ and so is $[A/P]$.

(2) In order to prove $|T|\mathrm{Ker}(\phi) = 0$, we construct $\psi : \mathrm{A}_.(A) \longrightarrow \mathrm{A}_.^G(A)$ such that $\psi\phi([A/\mathfrak{p}]) = |T|[A/\mathfrak{p}]$ for every $\mathfrak{p} \in \mathrm{Spec}^G(A)$.

First, we convert prime ideals of A to G-prime using group ring extension. Let $A[G] = \bigoplus_{g \in G} Ae_g$ be a group ring over A and $A[G]$ regards as a G-graded ring by $\deg(ae_g) = \deg(a) + g$ for every homogeneous element $a \in A$.

Claim. A flat ring homomorphism $f : A \longrightarrow A[G]$ such that $f(a) = ae_{-g}$ for $a \in A_g$, $g \in G$ induces an isomorphism $f^* : \mathrm{A}_i(A) \cong \mathrm{A}_{i+m}^G(A[G])$. (Here m is a rank of the free part of G, i.e. $m = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$.

Proof of Claim. Since $A[G]_0 = \oplus_{g \in G} A_g e_{-g}$ and $A[G]$ is also a group ring over $A[G]_0$, we have the following bijective correspondence between $\mathrm{Spec}(A[G]_0)$ and $\mathrm{Spec}^G(A[G])$:

$$\begin{array}{ccc} \mathrm{Spec}(A[G]_0) & \longleftrightarrow & \mathrm{Spec}^G(A[G]) \\ P & \longrightarrow & PA[G] \\ \mathfrak{P}_0 & \longleftarrow & \mathfrak{P}. \end{array}$$

Furthermore, this bijection gives isomorphism $\mathrm{Z}_i(A[G]_0) \cong \mathrm{Z}_{i+m}^G(A[G])$. Also this isomorphism inudeces $\mathrm{Rat}_i(A[G]_0) \cong \mathrm{Rat}_{i+m}^G(A[G])$ and $\mathrm{A}_i(A[G]_0) \cong \mathrm{A}_{i+m}^G(A[G])$. Since A is isomorphic to $A[G]_0$ via f, we have $f^* : \mathrm{A}_i(A) \cong \mathrm{A}_{i+m}^G(A[G])$.

Next, we regard $A[G]$ as a Laurent polynomial ring $A[G] = A[T][x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ with variables $x_1, \cdots, x_m$. Then we have an isomorphism $g^* : \mathrm{A}_.^G(A[T][x_1, \cdots, x_m]) \cong \mathrm{A}_.^G(A[T][x_1^{\pm 1}, \cdots, x_m^{\pm 1}]) = \mathrm{A}_.^G(A[G])$ induced from $A[T][x_1, \cdots, x_m] \longrightarrow A[T][x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$, by (2.12), (1). Also, by (2.12), (2), we have an isomorphism

$$\mathrm{A}_{i+m}^G(A[T][x_1, \cdots, x_m]) \xrightarrow{\cap \mathrm{div}^G(x_1)} \mathrm{A}_{i+m-1}^G(A[T][x_2, \cdots, x_m]) \longrightarrow \cdots \xrightarrow{\cap \mathrm{div}^G(x_m)} \mathrm{A}_i^G(A[T]).$$

Denote these composition by $\eta : \mathrm{A}_.^G(A[T][x_1, \cdots, x_m]) \longrightarrow \mathrm{A}_.^G(A[T])$. Finally, a finite map $h : A \longrightarrow A[T]$ of G-graded rings induces a homomorphis $h_* : \mathrm{A}_.^G(A[T]) \longrightarrow \mathrm{A}_.^G(A)$.

Now, we define $\psi : \mathrm{A}_.(A) \longrightarrow \mathrm{A}_.^G(A)$ by a composition map of

$$\mathrm{A}_i(A) \xrightarrow{f^*} \mathrm{A}_{i+m}^G(A[G]) \xrightarrow{(g^*)^{-1}} \mathrm{A}_{i+m}^G(A[T][x_1, \cdots, x_m]) \xrightarrow{\eta} \mathrm{A}_i^G(A[T]) \xrightarrow{h_*} \mathrm{A}_i^G(A).$$

By definition of each map, it is easy to see that $\eta(g^*)^{-1}f^*([A/\mathfrak{p}]) = [A[T]/\mathfrak{p}A[T]]$ for $\mathfrak{p} \in$

$\mathrm{Spec}^G(A)$. Hence we have

$$\begin{aligned}
\psi(\phi([A/\mathfrak{p}])) &= h_*([A[T]/\mathfrak{p}A[T]]) \\
&= \sum_{\mathfrak{P}\in \mathrm{Ass}^G_{A[T]}(A[T]/\mathfrak{p}A[T])} \ell^G_{A[T]_{(\mathfrak{P})}}(A[T]_{(\mathfrak{P})}/\mathfrak{p}A[T]_{(\mathfrak{P})}) h_*([A[T]/\mathfrak{P}]) \\
&= \sum_{\mathfrak{P}\in \mathrm{Ass}^G_{A[T]}(A[T]/\mathfrak{p}A[T])} \ell^G_{A[T]_{(\mathfrak{P})}}(A[T]_{(\mathfrak{P})}/\mathfrak{p}A[T]_{(\mathfrak{P})}) \ell^G_{A_{(\mathfrak{p})}}(A[T]_{(\mathfrak{P})}/\mathfrak{P}A[T]_{(\mathfrak{P})})[A/\mathfrak{p}] \\
&= \ell^G_{A_{(\mathfrak{p})}}(A[T]_{(\mathfrak{p})}/\mathfrak{p}A[T]_{(\mathfrak{p})})[A/\mathfrak{p}] \\
&= |T|[A/\mathfrak{p}]
\end{aligned}$$

for every $\mathfrak{p} \in \mathrm{Spec}^G(A)$. This implies that $t\mathrm{Ker}(\phi) = \psi\phi(\mathrm{Ker}(\phi)) = 0$. □

Corollary 3.1 *Let H be a subgroup of G such that G/H is torsion. Then $\mathrm{A}^H_\bullet(A^{(H)})_{\mathbb{Q}}$ is isomorphic to $\mathrm{A}^G_\bullet(A)_{\mathbb{Q}}$, where $A^{(H)} = \bigoplus_{h\in H} A_h$ and $\mathrm{A}^G_\bullet(-)_{\mathbb{Q}} = \mathrm{A}^G_\bullet(-)\otimes_{\mathbb{Z}}\mathbb{Q}$.*

Proof. Since G/H is torsion, we have a bijective correspondence between $\mathrm{Spec}^H(A^{(H)})$ and $\mathrm{Spec}^G(A)$;

$$\begin{array}{ccc}
\mathrm{Spec}^H(A^{(H)}) & \longleftrightarrow & \mathrm{Spec}^G(A) \\
\mathfrak{p} & \longrightarrow & (\sqrt{\mathfrak{p}A})^* \\
\mathfrak{P}^{(H)} & \longleftarrow & \mathfrak{P}.
\end{array}$$

Then we have the following exact sequence

$$0 \to \mathrm{Z}^G_\bullet(A) \to \mathrm{Z}^H_\bullet(A^{(H)}) \to D \to 0$$

such that D is torsion. A cokernel of $\mathrm{Rat}^G_\bullet(A) \to \mathrm{Rat}^H_\bullet(A^{(H)})$ is also torsion. This completes the proof of lemma. □

Corollary 3.2 *Let $A = \bigoplus_{geq0} A_n$ be a ordinary graded ring. Then a $\mathbb{Q}$-Chow group $\mathrm{A}_\bullet(A^{(d)})_{\mathbb{Q}}$ of the d-th Veronese subring $A^{(d)}$ of A is isomorphic to $\mathrm{A}_\bullet(A)_{\mathbb{Q}}$.*

Proof. This is the direct consequence of (1.2) and (3.1). □

Remark 3.3 (1) Corollary 1.2 is essentially contained in the result of Fulton-MacPherson-Sottile-Sturmfels[2]. Their striking result tells us, Chow groups of schemes over a feild with an action of a connected solvable group scheme are isomorphic to its equivariant Chow groups.

(2) As in Corollary 1.2, the inverse map of the canonical map $\phi : \mathrm{A}^G_\bullet(A) \longrightarrow \mathrm{A}_\bullet(A)$ is coincide with the map $\eta(g^*)^{-1}f^*$ of the proof of (1.1). We are able to discribe this inverse map explicitly, when graded rings are standerd. If $A = \bigoplus_{n\geq 0} A_n$, then the map $\eta(g^*)^{-1}f^*$ is determined by

$$\begin{array}{ccc}
\mathrm{A}_\bullet(A) & \xrightarrow{\eta(g^*)^{-1}f^*} & \mathrm{A}^G_\bullet(A) \\
[A/P] & \longmapsto & [A/in(P)],
\end{array}$$

where $in(P)$ is a initial term of homogenization of P.

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On the highest Lyubeznik number of local cohomology modules with Cohen-Macaulay support

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ABSTRACT. We will show the result that the highest Lyubeznik number $\lambda_{d,d}(A)$ is one if A is a Cohen-Macaualy local ring containing a field, where d is the dimension of A (cf. [K2]).

We assume that all rings are commutative and noetherian with identity.

§1. Definition and Questions.

The investigation of the structure of local cohomology modules $H^i_Y(\mathcal{F})$ was initiated by Grothendieck and is a very interesting subject in a field of commutative algebra, where Y is a closed subscheme of a scheme X and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Although several authors have developed very interesting results and deep theories, these modules are still very mysterious. For the finiteness properties of local cohohomology modules, Huneke and Sharp ([HuS]) and Lyubeznik ([L1]) proved remarkable results and further Lyubeznik defined a numerical invariant of local rings with respect to local cohomology modules [L1, Theorem-Definition 4.1]:

Definition 1. Let A be a local ring of dimension d which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where R is a regular local ring of dimension n containing a field. Set $I = \ker \pi$ and let $\mathfrak{m}$ be the maximal ideal of R. Then the Bass number $\mu_p(\mathfrak{m}, H^{n-i}_I(R))$ is finite and depends only on A, i and p, but neither on R nor on π. We denote this invariant by $\lambda_{p,i}(A)$, and we call this number the Lyubeznik number (or the (p, i)-Lyubeznik number).

A complete local ring containing a field is always a surjective image of a regular local ring containing a field. So, if A is a local ring containing a field, but not necessarily a surjective image of a regular local ring containing a field, one can set $\lambda_{p,i}(A) = \lambda_{p,i}(A^\wedge)$, where $A^\wedge$ is the completion of A with respect to the maximal ideal.

Lyubeznik gave the following question [L1, Question 4.5].

Question 1 (Lyubeznik). Is it true that $\lambda_{d,d}(A) = 1$ for all A ?

Recently Walther answered this question negatively [W, Proposition 3.2], using the Brodmann sequence. We also prove similar results using the spectral sequence (cf. [K1]). These rings are not Cohen-Macaulay. So we refine the above question as follows:

Question 2. Is it true that $\lambda_{d,d}(A) = 1$ for Cohen-Macaulay rings A ?

Key words and phrases. local cohomology, Lyubeznik number, spectral sequence, Cohen-Macaualy.

This question is true for Cohen-Macaualy local rings A of characteristic p by the result of Peskine and Szpiro [PS, Proposition 4.1], since the spectral sequence $H^p_{\mathfrak{m}}H^q_I(R) \Longrightarrow H^{p+q}_{\mathfrak{m}}(R)$ degenerates. Our aim in this talk is to answer Question 2 affirmatively, that is we will prove that if A is a Cohen-Macaulay local ring containing a field of dimension d, then the highest Lyubeznik number $\lambda_{d,d}(A)$ is one.

§2. Key Lemmas.

The following result follows immeditately from [EGA]:

Lemma 1. *Let $(A, \mathfrak{m})$ be a local ring with the maximal ideal $\mathfrak{m}$. Let B be the ring $((A^\wedge)^{sh})^\wedge$. Then the following assertions hold:*

(i) *B is a local ring with the maximal ideal $\mathfrak{m}B$;*
(ii) *if A is Cohen -Macaulay (resp. regular), then B is Cohen -Macaulay (resp. regular);*
(iii) *if A is a homomorphic image of a Cohen-Macaulay local ring and satisfies the Serre (S_k)-condition, then B satisfies the Serre (S_k)-condition for a positive integer k;*
(iv) *the natural map $A \longrightarrow B$ is a faithfully flat extension;*
(v) *if $A = R/I$ for some local ring R and an ideal I of R, then it holds that*

$$B = ((R^\wedge)^{sh})^\wedge / I((R^\wedge)^{sh})^\wedge;$$

(vi) *if the dimension of A is equal to d, then the dimension of B is equal to d,*

where $A^\wedge$ is the completion of A with respect to the maximal ideal, and A^{sh} is a strict henselization of A (See [EGA, (18.8), pp.144] *for the definition).*

The key lemma is as follows:

Lemma 2. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing a field, and I an ideal of R of dimension $d > 1$. If R/I satisfies the Serre (S_2)-condition, then the following assertions hold:*

(i) $\text{inj.dim}_R\, H^{n-d}_I(R) = d$;
(ii) $\text{inj.dim}_R\, H^j_I(R) < n - 1 - j$ *if* $j > n - d$,

where $\text{inj.dim}_R\, T$ *is the injective dimension of an R-module T.*

Proof. The statement (i) is straightforward from [L1, iv)], so we only have to prove the assertion (ii).

The assertion (ii) follows from the following fact: If R/I satisfies the Serre (S_2)-condition, the "Second Vanishing Theorem" holds for the local cohomology module $H^j_I(R)$, that is $H^j_I(R) = 0$ for $j \geq \dim R - 1$. (cf. [Sp, p.143, line 15]). For the proof, we use Lemma 1 and the results in [HuL] □

§3. Main Results and their Sketch Proofs.

Proposition 1. *Let R be a regular local ring containing a field of dimension n, I an ideal of R of dimension $d > 1$. If R/I satisfies the Serre (S_2)-condition, then we have $\lambda_{d,d}(R/I) = 1$.*

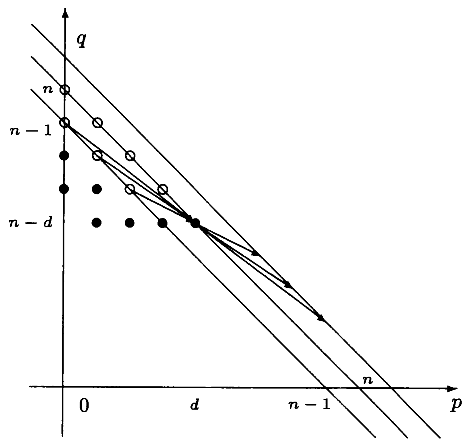
Proof. By Grothendieck's spectral sequence, we obtain the following spectral sequence:

$$E_2^{p,q} = H_{\mathfrak{m}}^p H_I^q(R) \Longrightarrow H^{p+q} = H_{\mathfrak{m}}^{p+q}(R).$$

The spectral sequence has the differentials as follows:

$$E_r^{d-r,n-d-(1-r)} \longrightarrow E_r^{d,n-d} \longrightarrow E_r^{d+r,n-d+(1-r)}.$$

Our aim is to prove that all the differentials that come into and go out of $E_r^{d,n-d}$ are zero for all $r \geq 2$. From Lemma 2, we can express E_2-terms in the diagram below (Figure). The circles mean the vanishing of E_2-terms by Lemma 2. Furthermore all E_2-terms are zero except the black circles.



Figure

The assertion immediately follows from the above diagram.

Therefore the above spectral sequence collapses at $E_2^{d,n-d}$ and we have isomorphisms:

$$\begin{aligned} H_{\mathfrak{m}}^d H_I^{n-d}(R) &= E_2^{d,n-d} \\ &\simeq E_\infty^{d,n-d} \\ &\simeq H^n \\ &= H_{\mathfrak{m}}^n(R). \end{aligned}$$

Since R is a regular local ring, $H_{\mathfrak{m}}^n(R)$ is isomorphic to $E(k)$, where $E(k)$ is the injective hull of k. Since $H_{\mathfrak{m}}^d H_I^{n-d}(R)$ is isomorphic to $E(k)$, it therefore follows from [L1, Lemma 1.4, p.44] that $\lambda_{d,d}(A) = 1$. The proof of the proposition is completed. □

Theorem 1. *Let A be a local ring containing a field with dimension d. If A is Cohen-Macaualy, then we have $\lambda_{d,d}(A) = 1$.*

Proof. Completing the local ring A with respect to the topology defined by the maximal ideal, if necessarily, there is a surjection $R \longrightarrow A^\wedge$ from a regular local ring R containing a fields to $A^\wedge$ by Cohen's structure theorem. We denote its kernel by I and the maximal ideal of R by $\mathfrak{m}$. Since $A^\wedge = R/I$ satisfies the Serre (S_2)-condition, the theorem follows from Proposition 1. □

§4. Monomial Cases.

Recently Yanagawa shows the following result of local cohomology modules with supports in monomial ideals (cf. [Ya]).

Proposition 2 (K. Yanagawa). *Let S be a polynomial ring $k[x_1, \dots, x_n]$. Let I be a squarefree monomial ideal, that is, the Stanley-Reisner ideal I_Δ of a simplicial complex $\Delta \subset 2^{\{1,\dots,n\}}$. Suppose that S/I_Δ is pure d-dimensional. Then $\mu^d(\mathfrak{m}, H^{n-d}_{I_\Delta}(S)) = 1$ if and only if Δ is connected in codimension one.*

Therefore the converse of the theorem unfortunately does not hold in general.

Example 1 (K. Eto). Let S_6 be the localization of $k[x_1, x_2, x_3, x_4, x_5, x_6]$ by an ideal $(x_1, x_2, x_3, x_4, x_5, x_6)$, $I = (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_6) \cap (x_5, x_6, x_1)$ and $P = (x_1, x_2, x_3, x_5, x_6)$. Then it holds that S_6/I is not Cohen-Macaulay. On the other hand, it follows from [Ya, Corollary 3.16] that $\lambda_{3,3}(S_6/I) = 1$. Further one can find the direct proof of the result that $\lambda_{3,3}(S_6/I) = 1$ in [EK], using the spectral sequence.

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On $\mathbf{A}^*$-fibrations

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1 Introduction

This article surveys some recent results (proved in ([BD 2] and [BD 3]) on $\mathbf{A}^*$-fibrations and is based on a talk given at the 22nd Symposium of "Commutative Algebra" held at Toyama (Japan). The author thanks the organisers for the invitation.

Throughout the article, all rings are commutative, noetherian and all modules are finitely generated.

For a ring R, R^* denotes the group of invertible elements of R and $R^{[n]}$ denotes a polynomial ring in n variables over R. For a prime ideal P of R, $k(P)$ denotes the field R_P/PR_P. For an R-algebra A, $\Omega_{A/R}$ is the module of Kahler differentials of A over R.

The notion of $\mathbf{A}^*$-fibration is analogous to that of $\mathbf{A}^1$-fibration. Therefore, we will begin this article by first defining a more general notion of an affine fibration and then defining an $\mathbf{A}^*$-fibration.

Definition 1.1. For a ring R, an R-algebra A is said to be an affine n-fibration over R (denoted by $\mathbf{A}^n$), if the following conditions hold :

(i) A is a finitely generated flat R-algebra.

(ii) For every prime ideal P of R, $A \otimes_R k(P) = k(P)^{[n]}$.

Definition 1.2. For a ring R, an R-algebra B is said to be an $\mathbf{A}^*$-fibration over R, if the following conditions hold :

(i) B is a finitely generated flat R-algebra.

(ii) For every prime ideal P of R, $B \otimes_R k(P)$ is a Laurent polynomial ring $k(P)[T, T^{-1}]$ in one variable T over $k(P)$.

Remark 1.3. Let A be an $\mathbf{A}^n$-fibration over R. Then, since A is a finitely generated flat R-algebra and all fibre are smooth, it is easy to see that $\Omega_{A/R}$ is a finitely generated projective A-module of rank n. Similarly, if B is an $\mathbf{A}^*$-fibration over R, then $\Omega_{B/R}$ a projective B-module of rank 1.

Example 1.4. (1) Let M be a projective module of rank n over a ring R and let $A = \mathrm{Sym}_R(M)$, the symmetric algebra of M over R. Then A is an $\mathbf{A}^n$-fibration over R. We call such an affine fibration *trivial.*

(2) Let L be a projective R-module of rank 1 and let $L^{-1} = \mathrm{Hom}_R(L, R)$. Let $L^{\otimes n} = L \otimes_R \cdots \otimes_R L$ (n-times) if $n \geq 0$ and $L^{\otimes n} = L^{-1} \otimes_R \cdots \otimes_R L^{-1}$ ($-n$-times) if $n \leq 0$. Let $B = \bigoplus_{n \in \mathbf{Z}} L^{\otimes n}$. Then B is an $\mathbf{A}^*$-fibration. We call such an $\mathbf{A}^*$- fibration *trivial.*

Remark 1.5. Let R be a semilocal ring. Then there exists only one trivial $\mathbf{A}^n$-fibration over R viz. $R^{[n]}$. Similarly, the Laurent polynomial ring $R[T, T^{-1}]$ in one variable T is the only trivial $\mathbf{A}^*$-fibration over R.

The general question as to when an $\mathbf{A}^n$-fibration over a ring R is trivial was first raised by Dolgachev and Weisfeiler ([DW]). Though it is easy to give examples of non-trivial $\mathbf{A}^n$-fibrations if R is not regular, the following remarkable theorem of Asanuma gives a complete structure theorem for an $\mathbf{A}^n$-fibration over an arbitrary ring R ([A], Theorem 3.4).

Theorem 1.6. *Let A be an $\mathbf{A}^n$-fibration over a ring R. Then A is (upto an isomorphism) an R-subalgebra of a polynomial ring $R^{[m]}$ for some m such that*

$$A^{[m]} \simeq Sym_{R^{[m]}}(\Omega_{A/R} \otimes_A R^{[m]}).$$

Note that, the above theorem says that if R is a regular local ring, then an $\mathbf{A}^n$-fibration A over R is at least stably a polynomial algebra over R, i.e., $A^{[t]} = R^{[n+t]}$ for some integer $t \geq 0$.

In the next section, we show how this theorem of Asanuma is useful for investigating the problem of triviality of $\mathbf{A}^1$-fibrations.

2 $\mathbf{A}^1$-fibration

Definition 2.1. An integral domain R with the quotient field K is said to be seminormal if $x \in K$ and $x^2, x^3 \in R$ then $x \in R$.

For a ring A, Pic (A) denotes the group of projective A-modules of rank 1. The proof of the following result can be found in ([Sw], Theorem 6.1).

Theorem 2.2. *Let R be a seminormal domain. Then, for every positive integer m, $Pic(R^{[m]}) = Pic(R)$.*

The following result is due to Hamann ([H], Theorem 2.6).

Theorem 2.3. *If R is a seminormal domain, then $R^{[1]}$ is R-invariant (i.e., if $R \subset A$ and $A^{[m]} = R^{[m+1]}$, then $A = R^{[1]}$).*

From theorems (1.6), (2.2) and (2.3) it is easy to deduce:

Theorem 2.4. *Let R be a seminormal, semilocal domain. Let A be an $\mathbf{A}^1$-fibration over R, then $A = R^{[1]}$.*

Thus, using a result of Bass-Connell-Wright ([BCW], Theorem 4.4), one concludes :

Corollary 2.5. *Every* $\mathbf{A}^1$*-fibration over a seminormal domain* R *is trivial.*

As an interesting application of (2.5), Bhatwadekar obtained the the following generalisation ([B], Theorem 3.7) of the famous Abhyankar-Moh/ Suzuki epimorphism theorem.

Theorem 2.6. *Let* R *be a seminormal domain of characteristic zero. Let* I *be an ideal of* $R[X,Y]$ *such that* $R[X,Y]/I = R[T](= R^{[1]})$. *Then* I *is a principal ideal say generated by* F *and* $R[X,Y] = R[F,G]$.

The main ingredient of the proof of (2.6) is the result (under the given assumptions) that $I = (F)$ and $R[X,Y]$ is an $\mathbf{A}^1$-fibration over $R[F]$ with $\Omega_{R[X,Y]/R[F]}$ is a free $R[X,Y]$-module (of rank 1) and then apply (2.5).

Recall that Asanuma's structure theorem (1.6) implies that any $\mathbf{A}^1$-fibration over a ring R is an R-subalgebra of a polynomial algebra $R^{[m]}$ for some m. Therefore it is natural to ask:
What fibre conditions would be sufficient for an R*-subalgebra of* $R^{[m]}$ *to be an* $\mathbf{A}^1$*-fibration over* R *?*

This problem was investigated by Bhatwadekar-Dutta and among other results the following was proved ([BD 1], Theorems 3.10, 3.12) :

Theorem 2.7. *Let* R *be a domain with quotient field* K. *Assume that* R *is normal or* R *contains the field of rationals. Let* A *be an* R*-subalgebra of a polynomial algebra* $R^{[m]}$ *such that*

(i) A *is* R*-flat.*

(ii) $A \otimes_R K$ *is a normal domain of dimension* 1.

(iii) $A \otimes_R k(P)$ *are integral domains for all height one prime ideals* P *of* R.

Then, A *is an* $\mathbf{A}^1$*-fibration over* R.

Subsequently, the following result was obtained by Dutta ([D], Theorem 3.4).

Theorem 2.8. *Let* R *be a normal domain with quotient field* K. *Let* A *be a finitely generated faithfully flat* R*-algebra such that*

(i) $A \otimes_R K = K^{[1]}$.

(ii) For each prime ideal P *of* R *of height* 1, *the fibre ring* $A \otimes_R k(P)$ *is geometrically integral over* $k(P)$.

Then A *is a trivial* $\mathbf{A}^1$*-fibration over* R.

Remark 2.9. Let R be a discrete valuation ring. In this special case, (2.8) was proved earlier by Kambayashi and Miyanishi ([KM], Lemma 1.3).

3 $\mathbf{A}^*$-fibration

Since the notion of $\mathbf{A}^*$-fibrations is analogous to that of $\mathbf{A}^1$-fibrations, it is quite natural to ask whether results similar to (2.5) and (2.8) hold for $\mathbf{A}^*$-fibrations. In this section we address this question.

We begin with the following result ([BD 2], Proposition 3.7) which is an analogue (for an $\mathbf{A}^*$-fibration) of Kambayashi-Miyanishi result mentioned in (2.9).

Proposition 3.1. *Let R be a discrete valuation ring with quotient field K, uniformising parameter π and residue field k. Let B be a finitely generated flat R-algebra such that*

(i) The generic fibre $B \otimes_R K \simeq K[T, T^{-1}]$.

(ii) The closed fibre $B/\pi B$ is geometrically integral over k.

Then there are precisely two possibilities :

(a) If $(B/\pi B)^* \neq k^*$, then $B \simeq R[T, T^{-1}]$

(b) If $(B/\pi B)^* = k^*$, then $B \simeq R[X, Y]/(\pi^m XY + \alpha X + \beta Y + \gamma)$ for some $\alpha, \beta \in R^*$, $\gamma \in R$ and positive integer m. Therefore $B/\pi B = k^{[1]}$.

In particular, every $\mathbf{A}^*$-fibration over a discrete valuation ring is trivial.

The following technical lemma ([BD 2], Lemma 3.1) gives a criterion for an $\mathbf{A}^*$-fibration over an arbitrary domain to be trivial.

Lemma 3.2. *Let R be an integral domain and let B be a flat R-algebra. Suppose that there exist non-zero elements x, y in R such that*

(i) x and y either form an R-sequence or are comaximal in R.

(ii) $B[1/x] \simeq R[1/x][T, T^{-1}]$.

(iii) $B[1/y] \simeq R[1/y][T, T^{-1}]$.

Then B is a trivial $\mathbf{A}^*$-fibration over R.

The following analogue of (2.8) can be easily deduced from (3.1) and (3.2) (see ([BD 2], Theorem 3.11)).

Theorem 3.3. *Let R be a normal domain with quotient field K and let B be a finitely generated flat R-algebra such that*

(i) The generic fibre $K \otimes_R A$ is a Laurent polynomial ring $K[T, T^{-1}]$ in one variable over K.

(ii) For each prime ideal P of R of height one, the fibre $k(P) \otimes_R B$ is geometrically integral and $(k(P) \otimes_R B)^* \neq (k(P))^*$

Then B is a trivial $\mathbf{A}^*$-fibration over R.

As a consequence, we have

Corollary 3.4. *Every $\mathbf{A}^*$-fibration over a normal domain is trivial.*

In view of (3.4) and (2.5), it is natural to ask whether $\mathbf{A}^*$-fibrations over a seminormal domain are trivial. With the help of the following (structure) theorem ([BD 3], Theorem 3.4), one can construct however an explicit example (3.6) of a nontrivial $\mathbf{A}^*$-fibration over a seminormal one-dimensional local domain.

Theorem 3.5. *Let R be a seminormal one-dimensional semilocal domain with Jacobson radical J and quotient field K. Let B be an $\mathbf{A}^*$-fibrations over R. Then, $B \simeq R[X,Y]/(Y^2 - YX - aX^2 - \lambda)$, where $R[X,Y] = R^{[2]}$, $\lambda \in R^*$ and a is an element of J for which there exists $b \in K$ such that $b(b-1) = a$. Moreover, $B \simeq R[T,T^{-1}]$ if and only if $b \in R$.*

Example 3.6. Let k be a field and let $\widetilde{R}$ be a semilocal noetherian normal domain of dimension one with precisely two maximal ideals M_1 and M_2 such that $\widetilde{R}/M_1 = \widetilde{R}/M_2 = k$ and $k \hookrightarrow \widetilde{R}$. (For instance, we may take $\widetilde{R} = S^{-1}k[t]$, where k is a field and $S = k[t] \setminus (I_1 \cup I_2)$, where $I_1 = tk[t]$ and $I_2 = (t-1)k[t]$.) Let $J = M_1 \cap M_2$ and let $R = k + J$. Then J is the conductor ideal of $\widetilde{R}$ in R, $R/J = k$ and $\widetilde{R}/J (= k \oplus k)$ is a finite module over $R/J (= k)$. Therefore, $\widetilde{R}$ is a finite module over R. Hence, as $\widetilde{R}$ is noetherian, by the Eakin-Nagata theorem ([M], 3.7, p.18), R is noetherian. Now it is easy to see that R is a local domain with maximal ideal J and residue field $R/J = k$. Moreover, $\widetilde{R}$ is the normalisation of R and R is semi-normal in $\widetilde{R}$. Since $M_1 + M_2 = \widetilde{R}$, there exists $b \in M_1$ such that $1 - b \in M_2$. Let $a = b(b-1)$. Then $b \notin R$ but $b(b-1) = a \in J \subset R$. Now let $B = R[X,Y]/(Y^2 - YX - aX^2 - 1)$. Then, by (3.5), B is a *non-trivial* $\mathbf{A}^*$-fibration over R.

The following theorem ([BD 3], Theorem 3.8) gives a criterion for an $\mathbf{A}^*$-fibration over a seminormal, one-dimensional, semilocal domain to be trivial.

Theorem 3.7. *Let R be a seminormal one-dimensional semilocal domain and B an $\mathbf{A}^*$-fibration over R. Then B is a trivial $\mathbf{A}^*$-fibration over R if and only if Spec B is an open subscheme of an $\mathbf{A}^1$- fibration over R (or equivalently, Spec B is an open subscheme of Spec $(R^{[1]})$.)*

Remark 3.8. Let R be a seminormal domain. Then, by (2.5), every $\mathbf{A}^1$-fibration over R is trivial. If R is normal, then, by (3.4), every $\mathbf{A}^*$-fibration is trivial. But, if R is not normal, then, as shown by the example (3.6) , there may exist a nontrivial $\mathbf{A}^*$-fibration over R. This anomaly is not so surprising. The plausible explaination (without explicit connection to the problem) is that, if R is normal, then Pic $(R[T,T^{-1}]) =$ Pic $(R[T]) =$ Pic (R). On the other hand, if R is seminormal but not normal, then Pic $(R[T])=$ Pic (R) and it may happen that Pic $(R) \neq$ Pic $(R[T,T^{-1}])$. For example, if R is seminormal, one-dimensional, local, then Pic $(R) \neq$ Pic $(R[T,T^{-1}])$ if and if its normalisation $\overline{R}$ is not local. Therefore, the proof of (3.5) yields the following :

Corollary 3.9. *Let R be a local, one-dimensional, seminormal domain. Then every $\mathbf{A}^*$-fibration over R is trivial if and only if Pic $(R) =$ Pic $(R[T,T^{-1}])$.*

We conclude this article with the following remark which indicates subtle differences between (trivial) $\mathbf{A}^1$ and $\mathbf{A}^*$-fibrations.

Remark 3.10. Let R be a ring. Let A and B denote trivial $\mathbf{A}^1$ and $\mathbf{A}^*$-fibrations respectively over R. Then

(i) R is a retract of A.

(ii) R is a retract of B if and only if B is a Laurent polynomial ring $R[T, T^{-1}]$ in one variable over R.

(iii) $\Omega_{A/R} \simeq A$ if and only if $A \simeq R^{[1]}$.

(iv) $\Omega_{B/R} \simeq B$.

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DERIVED CATEGORY OF SQUAREFREE MODULES

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ABSTRACT. A *squarefree module* over a polynomial ring $S = k[x_1, \dots, x_n]$ is a generalization of a Stanley-Reisner ring, and allows us to apply homological methods to the theory of monomial ideals more systematically.

In the derived category $\mathbf{D}^b(\mathbf{Sq})$ of squarefree modules, we have two duality functors $\mathcal{D}$ and $\mathcal{A}$. The functor $\mathcal{D}$ is the usual one $\mathbf{R}\operatorname{Hom}^\bullet_S(-, \omega_S[n])$, while the *Alexander duality functor* $\mathcal{A}$ is rather combinatorial. We will show a strange relation $\mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A} \cong \mathbf{T}^{2n}$, where $\mathbf{T}$ is the translation functor.

Römer defined a squarefree module over an exterior algebra E. A theorem of Bernstein-Gel'fand-Gel'fand's gives a derived equivalence between finitely generated $\mathbb{Z}$-graded modules over S and those over E. We see that the functors defining this equivalence preserve the squarefreeness, and can be described by our $\mathcal{D}$ and $\mathcal{A}$ in the squarefree case.

1. Squarefree Modules

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Consider an $\mathbb{N}^n$-grading $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_\mathbf{a} = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} k\, x^\mathbf{a}$, where $x^\mathbf{a} = \prod_{i=1}^n x_i^{a_i}$ is the monomial with the exponent $\mathbf{a} = (a_1, \dots, a_n)$. For a $\mathbb{Z}^n$-graded module M and $\mathbf{a} \in \mathbb{Z}^n$, $M_\mathbf{a}$ means the degree $\mathbf{a}$ component of M, and $M(\mathbf{a})$ denotes the shifted module with $M(\mathbf{a})_\mathbf{b} = M_{\mathbf{a}+\mathbf{b}}$. We denote the category of all the S-modules by $\mathbf{Mod}$, and the category of $\mathbb{Z}^n$-graded S-modules by $^*\mathbf{Mod}$. A morphism in $^*\mathbf{Mod}$ is an S-homomorphism $f : M \to N$ with $f(M_\mathbf{a}) \subset N_\mathbf{a}$ for all $\mathbf{a} \in \mathbb{Z}^n$.

For $M, N \in {}^*\mathbf{Mod}$ and $\mathbf{a} \in \mathbb{Z}^n$, set $^*\operatorname{Hom}_S(M, N)_\mathbf{a} := \operatorname{Hom}_{^*\mathbf{Mod}}(M, N(\mathbf{a}))$. Then $^*\operatorname{Hom}_S(M, N) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} {}^*\operatorname{Hom}_S(M, N)_\mathbf{a}$ is a $\mathbb{Z}^n$-graded S-module. If M is finitely generated, $^*\operatorname{Hom}_S(M, N) \cong \operatorname{Hom}_S(M, N)$ as the underlying S-module. Thus, we simply denote $^*\operatorname{Hom}_S(M, N)$ by $\operatorname{Hom}_S(M, N)$ in this case. In the same situation, $\operatorname{Ext}^i_S(M, N)$ also has a $\mathbb{Z}^n$-grading with $\operatorname{Ext}^i_S(M, N)_\mathbf{a} = \operatorname{Ext}^i_{^*\mathbf{Mod}}(M, N(\mathbf{a}))$.

For $\mathbf{a} \in \mathbb{Z}^n$, set $\operatorname{supp}_+(\mathbf{a}) := \{i \mid a_i > 0\} \subset [n] := \{1, \dots, n\}$. We say $\mathbf{a} \in \mathbb{Z}^n$ is *squarefree* if $a_i = 0, 1$ for all $i \in [n]$. When $\mathbf{a} \in \mathbb{Z}^n$ is squarefree, we sometimes identify $\mathbf{a}$ with $\operatorname{supp}_+(\mathbf{a})$.

Definition 1.1 ([12]). We say $M \in {}^*\mathbf{Mod}$ is *squarefree*, if M is finitely generated, $\mathbb{N}^n$-graded (i.e., $M_\mathbf{a} = 0$ for all $\mathbf{a} \notin \mathbb{N}^n$), and the multiplication map $M_\mathbf{a} \ni y \mapsto x^\mathbf{b} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\operatorname{supp}_+(\mathbf{a}+\mathbf{b}) = \operatorname{supp}_+(\mathbf{a})$.

Let $\Delta \subset 2^{[n]}$ be a simplicial complex (i.e., if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$). The *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal $I_\Delta := (x^F \mid F \notin \Delta)$ of S. We say S/I_Δ is the *Stanley-Reisner ring* of Δ. Stanley-Reisner rings and ideals are always squarefree modules. A free module $S(-\mathbf{a})$, $\mathbf{a} \in \mathbb{Z}^n$, is squarefree if and only if $\mathbf{a}$ is squarefree. In particular, the canonical module $\omega_S = S(-\mathbf{1})$ is squarefree, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^n$. For a subset $F \subset [n]$, P_F denotes the monomial prime ideal $(x_i \mid i \notin F)$ of S. Any monomial prime ideal of S is of the form P_F for some $F \subset [n]$. $S/P_F(-G)$ is squarefree if and only if $G \subset F$.

We denote by $\mathbf{Sq}_S$ (or simply $\mathbf{Sq}$) the full subcategory of $^*\mathbf{Mod}$ consisting of squarefree modules. Using the five lemma, we see that $\mathbf{Sq}$ is a subcategory of $^*\mathbf{Mod}$ closed under kernels, cokernels and extensions.

For the study of $\mathbf{Sq}$, the concept of the incidence algebra of a finite partially ordered set (*poset*, for short) is very useful.

Let P be a finite poset. The incidence algebra $A = I(P, k)$ of P over k is the k-vector space with basis $\{e_{x,y} \mid x, y \in P \text{ with } x \leq y\}$. The multiplication defined by $e_{x,y}\, e_{z,w} = \delta_{y,z}\, e_{x,w}$ makes A a finite dimensional associative k-algebra with $1 = \sum_{x \in P} e_{x,x}$. We write e_x for $e_{x,x}$. Note that $e_x\, e_y = \delta_{x,y}\, e_x$. Thus $A \cong \bigoplus_{x \in P} e_x A$ as a right A-module. Denote the category of finitely generated right A-modules by $\mathbf{mod}_A$. Each $e_x A$ is projective in $\mathbf{mod}_A$, and any projective object is a finite direct sum of copies of $e_x A$ for various $x \in P$.

If M is a right A-module, then we have $M = \bigoplus_{x \in P} M e_x$ as a k-vector space. We write M_x for $M e_x$. If $f : M \to N$ is an A-linear map, then $f(M_x) \subset N_x$. Note that $M_x e_{x,y} \subset M_y$ and $M_x e_{y,z} = 0$ for $y \neq x$.

It is easy to see that $[e_x A]_y = k$ if $x \leq y$, and $[e_x A]_y = 0$ otherwise. For each $x \in P$, we can construct an injective object $\bar{E}(x) \in \mathbf{mod}_A$. Let $\bar{E}(x)$ be a k-vector space with basis $\{\bar{e}_y \mid y \leq x\}$. Then we can regard $\bar{E}(x)$ as a right A-module by

$$\bar{e}_y \cdot e_{z,w} = \begin{cases} \bar{e}_w & \text{if } y = z \text{ and } w \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

The following is well-known.

Lemma 1.2. *The category $\mathbf{mod}_A$ has enough projectives and enough injectives. An indecomposable projective (resp. injective) object is isomorphic to $e_x A$ (resp. $\bar{E}(x)$) for some $x \in P$. And*

$$\operatorname{gl.dim} A \leq \max\{t \mid \exists \text{ chain } x_0 < x_1 < \cdots < x_t \text{ in } P\}.$$

The following result was essentially proved in [14] (in a more general situation).

Proposition 1.3. *Let $2^{[n]}$ be the boolean lattice, and $A = I(2^{[n]}, k)$ its incidence algebra. Then we have a category equivalence $\mathbf{Sq} \cong \mathbf{mod}_A$.*

Proof. For $N \in \mathbf{mod}_A$, set $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ to be a k-vector space with $M_{\mathbf{a}} \cong N_{\mathrm{supp}_+(\mathbf{a})}$ for each $\mathbf{a} \in \mathbb{N}^n$ (note that $\mathrm{supp}_+(\mathbf{a}) \subset [n]$ is an element of $2^{[n]}$). Then M has an S-module structure such that the multiplication map $M_{\mathbf{a}} \ni y \mapsto x^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is induced by $N_F \ni y \mapsto y \cdot e_{F,G} \in N_G$, where $F = \mathrm{supp}_+(\mathbf{a})$ and $G = \mathrm{supp}_+(\mathbf{a}+\mathbf{b})$. By an argument similar to the proof of [14, Theorem 3.2], we can see that M is squarefree and the correspondence $\mathbf{mod}_A \ni N \mapsto M \in \mathbf{Sq}$ gives an equivalence $\mathbf{mod}_A \cong \mathbf{Sq}$. □

Let $A = I(2^{[n]}, k)$ and $F \subset [n]$. By the above correspondence, the right A-module $e_F A$ (resp. $\bar{E}(F)$) corresponds to $S(-F)$ (resp. S/P_F). So we have the following.

Corollary 1.4 ([13]). **Sq** *is an abelian category, and has enough projectives and enough injectives. An indecomposable projective (resp. injective) object in* **Sq** *is isomorphic to $S(-F)$ (resp. S/P_F) for some $F \subset [n]$. And*

$$\sup\{\operatorname{proj.dim}_{\mathbf{Sq}} M \mid M \in \mathbf{Sq}\} = \sup\{\operatorname{inj.dim}_{\mathbf{Sq}} M \mid M \in \mathbf{Sq}\} = n.$$

A projective object $S(-F)$ in **Sq** is also projective in ***Mod**. Thus a projective resolution of a squarefree module M in **Sq** is also a projective resolution in ***Mod**. In particular, a $\mathbb{Z}^n$-graded minimal free resolution $P_\bullet$ of $M \in \mathbf{Sq}$ consists of $S(-F)$'s. Since $\operatorname{Hom}_S(S(-F), \omega_S) = S(-F^c)$ with $F^c := [n] \setminus F$, $\operatorname{Hom}_S(P_\bullet, \omega_S)$ is a complex of squarefree modules again. Hence we have the following.

Proposition 1.5 ([12]). *If M is squarefree, so is $\operatorname{Ext}^i_S(M, \omega_S)$ for all $i \geq 0$.*

Remark 1.6. Let $A = I(2^{[n]}, k)$ be the incidence algebra of $2^{[n]}$. The opposite ring A^{op} is isomorphic to A itself by $A^{\mathrm{op}} \ni (e_F)^{\mathrm{op}} \mapsto e_{F^c} \in A$ for $F \subset [n]$. If $N \in \mathbf{mod}_A$, $\operatorname{Ext}^i_A(N, A)$ has a natural left A-module structure. By the identification $A^{\mathrm{op}} \cong A$, $\operatorname{Ext}^i_A(N, A)$ can be seen as a right A-module. If N corresponds to a squarefree module M, then $\operatorname{Ext}^i_A(N, A) \in \mathbf{mod}_A$ corresponds to the squarefree module $\operatorname{Ext}^i_S(M, \omega_S)$.

Definition 1.7 ([13]). A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ is called *straight*, if the following two conditions are satisfied.

(a) $\dim_k M_{\mathbf{a}} < \infty$ for all $\mathbf{a} \in \mathbb{Z}^n$.
(b) The multiplication map $M_{\mathbf{a}} \ni y \mapsto x^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbf{b} \in \mathbb{N}^n$ with $\operatorname{supp}_+(\mathbf{a}+\mathbf{b}) = \operatorname{supp}_+(\mathbf{a})$.

A finitely generated S-module M is straight if and only if M is a direct sum of finitely many copies of ω_S. The injective hull $^*E(S/P_F)$ of S/P_F in ***Mod** is a straight module.

Denote by **Str** the full subcategory of ***Mod** consisting of all the straight modules. For a $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$, we call the $\mathbb{N}^n$-graded submodule $\bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ the *$\mathbb{N}^n$-graded part* of M, and denote it by $\mathcal{N}(M)$. It is easy to see that if M is straight then $\mathcal{N}(M)$ is squarefree. Conversely, for a squarefree module N, there is a unique (up to isomorphism) straight module $\mathcal{Z}(N)$ whose $\mathbb{N}^n$-graded part is isomorphic to N, see [13, §2]. For example, $\mathcal{Z}(S/P_F) \cong {}^*E(S/P_F)$.

Proposition 1.8 ([13, Proposition 2.7]). *The above correspondences give the additive covariant functors $\mathcal{N} : \mathbf{Str} \to \mathbf{Sq}$ and $\mathcal{Z} : \mathbf{Sq} \to \mathbf{Str}$, and they give a category equivalence* $\mathbf{Sq} \cong \mathbf{Str}$.

By Propositions 1.4 and 1.8, **Str** has enough projectives and injectives. An indecomposable projective (resp. injective) object in **Str** is isomorphic to $\mathcal{Z}(S(-F))$ (resp. $^*E(S/P_F)$) for some $F \subset [n]$. Note that $\mathcal{Z}(S(-F))$ is a flat S-module. In fact, $\mathcal{Z}(S(-F))$ is isomorphic to the localization of the free module $S(-F)$ at the monomial x^{F^c}. A minimal injective resolution of a straight module M in **Str** gives a minimal injective resolution in ***Mod**.

Theorem 1.9 (Mustaţă [9] and Terai [11]). *For all $i \geq 0$, the local cohomology module $H^i_{I_\Delta}(\omega_S) \cong H^i_{I_\Delta}(S)(-\mathbf{1})$ is a straight module whose $\mathbb{N}^n$-graded part is isomorphic to* $\operatorname{Ext}^i_S(S/I_\Delta, \omega_S)$.

Theorem 1.9 also holds in the level of complexes. Let $P_\bullet$ be a $\mathbb{Z}^n$-graded minimal free resolution of S/I_Δ. In the argument before Proposition 1.5, we have seen that $\operatorname{Hom}_S(P_\bullet, \omega_S)$ consists of finite direct sums of squarefree modules $S(-F)$. So we have the complex $\mathcal{Z}(\operatorname{Hom}_S(P_\bullet, \omega_S))$ consisting of flat modules $\mathcal{Z}(S(-F))$.

Definition 1.10 (Miller [8]). We say

$$\check{C}^\bullet_{I_\Delta} := \mathcal{Z}(\operatorname{Hom}_S(P_\bullet, \omega_S))(\mathbf{1})$$

is the *canonical Čech complex* of I_Δ.

Note that $\check{C}^\bullet_{I_\Delta}$ is a cochain complex of flat S-modules. Since the i^{th} cohomology of $\operatorname{Hom}_S(P_\bullet, \omega_S)$ is $\operatorname{Ext}^i_S(S/I_\Delta, \omega_S)$, we have

$$H^i(\check{C}^\bullet_{I_\Delta}(-\mathbf{1})) \cong \mathcal{Z}(\operatorname{Ext}^i_S(S/I_\Delta, \omega_S)) \cong H^i_{I_\Delta}(\omega_S),$$

and hence

$$H^i(\check{C}^\bullet_{I_\Delta}) \cong H^i_{I_\Delta}(S).$$

Let $E^\bullet$ be an injective resolution of S in $^*\mathbf{Mod}$. Then $\check{C}^\bullet_{I_\Delta}$ is a flat resolution of $\Gamma_{I_\Delta}(E^\bullet)$, in other words, $\check{C}^\bullet_{I_\Delta}$ and $\Gamma_{I_\Delta}(E^\bullet)$ are isomorphic in the derived category $\mathbf{D}^b(^*\mathbf{Mod})$, see Proposition 2.2 below. Since $H^i_{I_\Delta}(M)$ is the i^{th} cohomology of the derived tensor product $\Gamma_{I_\Delta}(E^\bullet) \otimes^{\mathbb{L}}_S M$, we have the following.

Theorem 1.11 (Miller [8]). *For an arbitrary (not necessarily graded) S-module M, we have*

$$H^i_{I_\Delta}(M) = H^i(\check{C}^\bullet_{I_\Delta} \otimes_S M).$$

Let $C \subset \mathbb{Z}^n \subset \mathbb{R}^n$ be an affine semigroup (i.e., a finitely generated sub-semigroup containing 0) with $\mathbb{Z}C = \mathbb{Z}^n$, and $R := k[C] = k[x^{\mathbf{c}} \mid \mathbf{c} \in C] \subset k[x_1^{\pm}, \dots, x_n^{\pm}]$ its semigroup ring. We assume that R is normal, and $\mathfrak{m} := (x^{\mathbf{c}} \mid 0 \neq \mathbf{c} \in C)$ is the graded maximal ideal. We can define squarefree modules over R, see [14]. As in the polynomial ring case, a radical monomial ideal of R (i.e., a radical $\mathbb{Z}^n$-graded ideal) is squarefree. The category $\mathbf{Sq}_R$ of squarefree R-modules is equivalent to the category $\mathbf{mod}_A$ of finitely generated right A-modules, where $A = I(L, k)$ is the incidence algebra of the face lattice L of an $(n-1)$-polytope obtained as a cross-section of the cone $\mathbb{R}_{\geq 0}C$ ($\subset \mathbb{R}^n$), see [16]. We say R is simplicial if the cone $\mathbb{R}_{\geq 0}C$ is spanned by n vectors. For example, a polynomial ring $S = k[\mathbb{N}^n]$ is simplicial. If R is simplicial, then $L \cong 2^{[n]}$ as a poset and $\mathbf{Sq}_R \cong \mathbf{Sq}_S$.

Let I be a radical monomial ideal of R. From a minimal projective resolution $P_\bullet$ of R/I in $\mathbf{Sq}_R$, we can construct the canonical Čech complex $\check{C}^\bullet_I$ of I. Each $\check{C}^i_I$ is a flat R-module and we have $H^i_I(M) = H^i(\check{C}^\bullet_I \otimes_R M)$ for an R-module M.

Theorem 1.12 ([16]). *If $I \subset R$ is a radical monomial ideal, then*

$$\max\{i \mid H^i_I(R) \neq 0\} = \operatorname{proj.dim}_{\mathbf{Sq}_R}(R/I).$$

For $M \in \mathbf{Sq}_R$, we have $n - \operatorname{depth}_R M \leq \operatorname{proj.dim}_{\mathbf{Sq}_R} M$. If R is simplicial, then the inequality becomes an equality. If R is not simplicial, there is some $M \in \mathbf{Sq}_R$ for which the inequality is strict. The next result states that "the second vanishing theorem" (c.f. [7, Theorem 2.9]) for a regular local ring also holds for a simplicial normal semigroup ring if the support ideal is a monomial ideal.

Corollary 1.13 ([16]). *Assume that R is simplicial. For a radical monomial ideal I, the following conditions are equivalent.*

(1) $H_I^n(R) = H_I^{n-1}(R) = 0$ *(recall that* $\dim R = n$*).*
(2) $\operatorname{Spec}(R/I) \setminus \mathfrak{m}$ *is connected.*

When R is not simplicial, $H_I^{n-1}(R)$ can be non-zero even if $\operatorname{Spec}(R/I) \setminus \mathfrak{m}$ is connected. Set $R = k[x,y,z,w]/(xz - yw)$ and $I = (x, y)$. Then R is a normal semigroup ring, and the corresponding cone $\mathbb{R}_{\geq 0}C$ is the cone over a square. Hence R is not simplicial. Since I is a prime ideal, $\operatorname{Spec}(R/I) \setminus \mathfrak{m}$ is connected. But $H_I^2(R) \neq 0$, while $\dim R = 3$.

2. Functors on the Derived Category of Squarefree Modules

Let $\mathrm{Com}^b(\mathbf{Sq})$ be the category of bounded cochain complexes of squarefree modules, and $\mathbf{D}^b(\mathbf{Sq})$ the bounded derived category of $\mathbf{Sq}$. A squarefree module M can be regarded as a complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M at the 0^{th} place. For a complex $M^\bullet$ and an integer p, let $M^\bullet[p]$ be the p^{th} *translation* of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$ and $d_{M[p]} = (-1)^p d_M$. A complex $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ has a projective (resp. injective) resolution $P^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ (resp. $I^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$). That is, there is a quasi-isomorphism $P^\bullet \to M^\bullet$ (resp. $M^\bullet \to I^\bullet$) and $P^\bullet$ (resp. $I^\bullet$) consists of projective objects $S(-F)$ (resp. injective objects S/P_F) in $\mathbf{Sq}$.

Let $M^\bullet$ and $N^\bullet$ be complexes of S-modules. We define a complex $\operatorname{Hom}_S^\bullet(M^\bullet, N^\bullet)$ by $\operatorname{Hom}_S^i(M^\bullet, N^\bullet) = \prod_{j \in \mathbb{Z}} \operatorname{Hom}_S(M^j, N^{i+j})$ and $d^i(f) = \prod((-1)^i f_{j+1} d_M^j + d_N^{i+j} f_j)$ for $f = (f_j)_{j\in\mathbb{Z}} \in \operatorname{Hom}_S^i(M^\bullet, N^\bullet)$. Note that if $M^\bullet, N^\bullet \in \mathrm{Com}^b({}^*\mathbf{Mod})$ and each M^i is finitely generated then each $\operatorname{Hom}_S(M^j, N^{i+j})$ can be seen as a $\mathbb{Z}^n$-graded module, and hence $\operatorname{Hom}_S^\bullet(M^\bullet, N^\bullet) \in \mathrm{Com}^b({}^*\mathbf{Mod})$.

The following is a key lemma of this section.

Lemma 2.1 ([13, Lemma 3.20]). *For all squarefree module M and all $F \subset [n]$, $\mathcal{N}(\operatorname{Hom}_S(M, {}^*E(S/P_F)))$ is isomorphic to $(M_F)^* \otimes_k (S/P_F)$. Here $(M_F)^*$ is the dual k-vector space of M_F, but we set the degree of $(M_F)^*$ to be 0 (since it is essentially $\operatorname{Hom}_k^\bullet(M_F, [S/P_F]_F)$). In particular, $\mathcal{N}(\operatorname{Hom}_S(M, {}^*E(S/P_F)))$ is squarefree.*

A minimal injective resolution $D^\bullet$ of $\omega_S[n]$ in ${}^*\mathbf{Mod}$ is the following form.

$$D^\bullet : 0 \longrightarrow D^{-n} \longrightarrow D^{-n+1} \longrightarrow \cdots \longrightarrow D^0 \longrightarrow 0, \tag{1}$$

$$D^i = \bigoplus_{\substack{F \subset [n] \\ |F| = -i}} {}^*E(S/P_F),$$

and the differential is composed of $(-1)^{\alpha(j,F)} \cdot \mathrm{nat} : {}^*E(S/P_F) \to {}^*E(S/P_{F\setminus\{j\}})$ for $j \in F$, where $\mathrm{nat} : {}^*E(S/P_F) \to {}^*E(S/P_{F\setminus\{j\}})$ is induced by the natural surjection $S/P_F \to S/P_{F\setminus\{j\}}$, and $\alpha(j,F) := \#\{i \in F \mid i < j\}$. See [3, §5.7].

Proposition 2.2. *Let $M^\bullet \in \mathbf{Com}^b(\mathbf{Sq})$, and $P^\bullet$ its projective resolution. Then the complexes $\mathcal{N}(\mathrm{Hom}^\bullet_S(M^\bullet, D^\bullet))$, $\mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet))$ and $\mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n])$ belong to $\mathbf{Com}^b(\mathbf{Sq})$, and are isomorphic in $\mathbf{D}^b(\mathbf{Sq})$.*

Proof. By Lemma 2.1, the complexes $\mathcal{N}(\mathrm{Hom}^\bullet_S(M^\bullet, D^\bullet))$ and $\mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet))$ belong to $\mathbf{Com}^b(\mathbf{Sq})$. Since $\mathrm{Hom}_S(S(-F), \omega_S) = S(-F^c)$, $\mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n])$ also belongs to $\mathbf{Com}^b(\mathbf{Sq})$. By the usual argument of double complexes, we have a $\mathbb{Z}^n$-graded quasi-isomorphisms

$$\mathrm{Hom}^\bullet_S(M^\bullet, D^\bullet) \to \mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet) \quad \text{and} \quad \mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n]) \to \mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet).$$

Taking the $\mathbb{N}^n$-graded part of these morphisms, we have quasi-isomorphisms

$$\mathcal{N}(\mathrm{Hom}^\bullet_S(M^\bullet, D^\bullet)) \to \mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet))$$

and

$$\mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n]) = \mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n])) \to \mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet)).$$

□

We can check that $\mathcal{D}$ induces a contravariant functor from $\mathbf{D}^b(\mathbf{Sq})$ to itself. We also denote this functor by $\mathcal{D}$. If $P^\bullet$ is a projective resolution of $M^\bullet$, then $\mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, D^\bullet))$ and $\mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n])$ isomorphic to $\mathcal{D}(M^\bullet)$ in $\mathbf{D}^b(\mathbf{Sq})$ by Proposition 2.2. Since $\mathcal{D}(M^\bullet) \cong \mathcal{N}(\mathrm{Hom}^\bullet_S(P^\bullet, \omega_S[n]))$, the next lemma is easy.

Lemma 2.3. *$\mathcal{D}$ is a duality functor, that is, satisfies $\mathcal{D} \circ \mathcal{D} \cong \mathrm{Id}_{\mathbf{D}^b(\mathbf{Sq})}$. And we have $H^i(\mathcal{D}(M^\bullet)) = \mathrm{Ext}^{n+i}_S(M^\bullet, \omega_S)$.*

For a squarefree module M, we can describe $\mathcal{D}(M) = \mathcal{N}(\mathrm{Hom}^\bullet(M, D^\bullet))$ explicitly. By Lemma 2.1, we have

$$(2) \qquad \mathcal{D}(M) : 0 \longrightarrow \mathcal{D}^{-n}(M) \longrightarrow \mathcal{D}^{-n+1}(M) \longrightarrow \cdots \longrightarrow \mathcal{D}^0(M) \longrightarrow 0,$$

$$\mathcal{D}^i(M) = \bigoplus_{\substack{F \subset [n] \\ |F| = -i}} (M_F)^* \otimes_k (S/P_F).$$

As in the lemma, the degree of $(M_F)^*$ is $0 \in \mathbb{Z}^n$. The differential is composed of the maps

$$(-1)^{\alpha(j,F)} \cdot (v_j)^* \otimes_k \mathrm{nat} : (M_F)^* \otimes_k S/P_F \to (M_{F\setminus\{j\}})^* \otimes_k S/P_{F\setminus\{j\}}$$

for $j \in F$. Here $(v_j)^*$ is the k-dual of the multiplication map $v_j : M_{F\setminus\{j\}} \ni y \mapsto x_j y \in M_F$ and "nat" is the natural surjection $S/P_F \to S/P_{F\setminus\{j\}}$. Note that $\mathcal{D}(M)$ is a complex of injective objects in $\mathbf{Sq}$. If $M \in \mathbf{Sq}$ is a Cohen-Macaulay module of dimension d, then $\mathcal{D}(M)$ gives a minimal injective resolution of $\mathrm{Ext}^{n-d}_S(M, \omega_S)$ in $\mathbf{Sq}$ (after suitable translation). Thus, in this case, $\mathcal{Z}(\mathcal{D}(M))$ gives a minimal injective resolution of $\mathcal{Z}(\mathrm{Ext}^{n-d}_S(M, \omega_S))$ in ${}^*\mathbf{Mod}$ ([13, Proposition 3.17]). Since $\mathcal{Z}(\mathrm{Ext}^i_S(S/I_\Delta, \omega_S)) \cong H^i_{I_\Delta}(\omega_S)$, this fact is very useful to the study of the injective resolution of $H^i_{I_\Delta}(S)$. See [13] for detail.

For a complex $M^\bullet = \{M^i, \delta^i\} \in \mathbf{Com}^b(\mathbf{Sq})$, we can also describe the complex $\mathcal{D}(M^\bullet)$ in a similar way. In fact,

$$\mathcal{D}^t(M^\bullet) = \bigoplus_{i-j=t} \mathcal{D}^i(M^j) = \bigoplus_{-|F|-j=t} (M_F^j)^* \otimes_k (S/P_F),$$

and the differential given by

$$\mathcal{D}^t(M^\bullet) \supset (M_F^j)^* \otimes_k (S/P_F) \ni x \otimes y \mapsto d_{\mathcal{D}(M^j)}(x \otimes y) + (-1)^t(\delta^*(x) \otimes y) \in \mathcal{D}^{t+1}(M^\bullet),$$

where $\delta^* : (M_F^j)^* \to (M_F^{j-1})^*$ is induced by the k-dual of $\delta^{j-1} : M^{j-1} \to M^j$. The complex $\mathcal{D}(M^\bullet)$ consists of injective objects, but it is not minimal in general.

If M is straight, then so is $M^\vee := {}^*\operatorname{Hom}_S(M, {}^*E(k))(-\mathbf{1})$. Obviously, $(-)^\vee$ gives an exact duality functor on $\mathbf{Str}$. By the equivalence $\mathbf{Sq} \cong \mathbf{Str}$ of Proposition 1.8, we have an exact duality functor $\mathcal{A}$ on $\mathbf{Sq}$ which corresponds to $(-)^\vee$. More precisely, for $M \in \mathbf{Sq}$, we set

$$\mathcal{A}(M) := \mathcal{N}(\mathcal{Z}(M)^\vee) = \mathcal{N}({}^*\operatorname{Hom}_S(\mathcal{Z}(M), {}^*E(k))(-\mathbf{1})) \in \mathbf{Sq}.$$

We say $\mathcal{A}$ is the *Alexander duality functor* ([8, 10]). It is easy to see that $\mathcal{A}(M)_F$ is the k-dual of M_{F^c}, and the multiplication map $\mathcal{A}(M)_F \ni y \mapsto x_i y \in \mathcal{A}(M)_{F \cup \{i\}}$ for $i \notin F$ is the k-dual of $M_{F^c \setminus \{i\}} \ni y \mapsto x_i y \in M_{F^c}$. For example, $\mathcal{A}(S/P_F) = S(-F^c)$. Hence $\mathcal{A}$ interchanges an injective resolution with a projective resolution. We also have $\mathcal{A}(S/I_\Delta) = I_{\Delta^\vee}$, where $\Delta^\vee := \{F \subset [n] \mid F^c \not\in \Delta\}$ is *(Eagon-Reiner's) Alexander dual simplicial complex* of Δ ([4]).

Remark 2.4. Let $A = I(2^{[n]}, k)$ be the incidence algebra of $2^{[n]}$. Since $\mathbf{Sq} \cong \mathbf{mod}_A$, we have $\mathbf{D}^b(\mathbf{Sq}) \cong \mathbf{D}^b(\mathbf{mod}_A)$. By Remark 1.6, the duality functor $\mathcal{D}$ corresponds to the functor $\mathbf{R}\operatorname{Hom}_A(-, A[n])$ from $\mathbf{D}^b(\mathbf{mod}_A)$ to itself (after the identification $A^{\mathrm{op}} \cong A$ given in Remark 1.6).

Let $J = \bigoplus_{F \subset [n]} \bar{E}(F)$ be the direct sum of indecomposable injective objects. Then $\operatorname{End}(J) \cong A$ and we have an exact contravariant functor $\operatorname{Hom}_A(-, J)$ from $\mathbf{mod}_A$ to $\mathbf{mod}_{A^{\mathrm{op}}}$. Since we have the isomorphism $A \cong A^{\mathrm{op}}$, $\operatorname{Hom}_A(-, J)$ can be seen as a functor from $\mathbf{mod}_A$ to itself. This functor corresponds to $\mathcal{A}$.

Next we will describe the complex

$$\mathcal{F}(M) := \mathcal{A} \circ \mathcal{D}(M) = \mathcal{A}(\mathcal{N}(\operatorname{Hom}^\bullet(M, D^\bullet))),$$

for $M \in \mathbf{Sq}$. For each $F \subset [n]$, $(M_F)^\circ$ denotes a k-vector space with a bijection $\psi_F : M_F \to (M_F)^\circ$. We denote $\psi_F(y) \in (M_F)^\circ$ by y°, and set $\deg(y^\circ) = 0$. Then

$$\mathcal{F}^i(M) = \bigoplus_{|F|=i} (M_F)^\circ \otimes_k S(-F^c)$$

and the differential map is given by

$$d(y^\circ \otimes s) = \sum_{j \notin F} (-1)^{\alpha(j,F)} (x_j y)^\circ \otimes x_j s.$$

For a complex $M^\bullet = \{M^i, \delta^i\} \in \mathrm{Com}^b(\mathbf{Sq})$, we can also describe $\mathcal{F}(M^\bullet) = \mathcal{A} \circ \mathcal{D}(M^\bullet)$ in the following way:

$$\mathcal{F}^t(M^\bullet) = \bigoplus_{i+j=t} \mathcal{F}^i(M^j) = \bigoplus_{|F|+j=t} (M^j_F)^\circ \otimes_k S(-F^c),$$

and the differential is given by

$$\mathcal{F}^t(M^\bullet) \supset (M^j_F)^\circ \otimes_k S(-F^c) \ni y \otimes s \mapsto d_{\mathcal{F}(M^j)}(y \otimes s) + (-1)^t \delta^j_F(y) \otimes s \in \mathcal{F}^{t+1}(M^\bullet).$$

Here $\delta^j_F : (M^j_F)^\circ \to (M^{j+1}_F)^\circ$ is induced by $\delta^j : M^j \to M^{j+1}$.

Theorem 2.5. *We have a natural equivalence*

$$\mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A} \cong \mathbf{T}^{2n}$$

in $\mathbf{D}^b(\mathbf{Sq})$, *where* $\mathbf{T}$ *is the translation functor (i.e.,* $\mathbf{T}^{2n} : M^\bullet \mapsto M^\bullet[2n]$*).*

Proof. For $M^\bullet = \{M^i, \delta^i\} \in \mathrm{Com}^b(\mathbf{Sq})$, the complex $\mathrm{Hom}^\bullet_S(\mathcal{F}(M^\bullet), \omega_S[n])$ is isomorphic to $\mathcal{D} \circ \mathcal{A} \circ \mathcal{D}\,(M^\bullet)$ in $\mathbf{D}^b(\mathbf{Sq})$. We have

$$\begin{aligned}
\mathrm{Hom}^i_S(\mathcal{F}(M^\bullet), \omega_S[n]) &= \mathrm{Hom}_S\Big(\bigoplus_{-i-n=|F|+j} (M^j_F)^\circ \otimes_k S(-F^c),\ \omega_S \Big) \\
&= \bigoplus_{-i-n=|F|+j} (M^j_F)^* \otimes_k S(-F) \\
&= \bigoplus_{i=-n-|F|+j} (M^{-j}_F)^* \otimes_k S(-F).
\end{aligned}$$

Here we simply denote the dual vector space of $(M^{-j}_F)^\circ$ by $(M^{-j}_F)^*$, since $(M^{-j}_F)^\circ = M^{-j}_F$ as k-vector space (but the degree of $(M^{-j}_F)^\circ$ is 0). Also here $\deg(M^{-j}_F)^* = 0$. The differential of $\mathrm{Hom}^\bullet_S(\mathcal{F}(M^\bullet), \omega_S[n])$ is given by

$$(M^{-j}_F)^* \otimes_k S(-F) \ni y \otimes 1 \mapsto (-1)^n \sum_{l \in F} (-1)^{\alpha(l,F)-|F|-j}\, v^*_l(y) \otimes x_l + \delta^*(y) \otimes 1,$$

where $v^*_l : M_F \to M_{F \setminus \{l\}}$ is the k-dual of $v_l : M_{F\setminus\{l\}} \ni z \mapsto x_l z \in M_F$, and $\delta^* : (M^{-j}_F)^* \to (M^{-j-1}_F)^*$ is the k-dual of $\delta^{-j-1} : M^{-j-1}_F \to M^{-j}_F$.

Similarly, $\mathcal{F}(\mathcal{A}(M^\bullet))$ represents $\mathcal{A} \circ \mathcal{D} \circ \mathcal{A}\,(M^\bullet)$ in $\mathbf{D}^b(\mathbf{Sq})$, and we have

$$\begin{aligned}
\mathcal{F}^i(\mathcal{A}(M^\bullet)) &= \bigoplus_{i=|F|+j} (\mathcal{A}(M^j)_F)^\circ \otimes_k S(-F^c) \\
&= \bigoplus_{i=|F|+j} (M^{-j}_{F^c})^* \otimes_k S(-F^c) \\
&= \bigoplus_{i=n-|F|+j} (M^{-j}_F)^* \otimes_k S(-F).
\end{aligned}$$

Also here, we simply denote $(\mathcal{A}(M^{-j})_F)^\circ = ((M^{-j}_{F^c})^*)^\circ$ by $(M^{-j}_{F^c})^*$. The differential of the above complex is given by

$$(M^{-j}_F)^* \otimes_k S(-F) \ni y \otimes 1 \mapsto \sum_{l \in F} (-1)^{\alpha(l,F^c)}\, v^*_l(y) \otimes x_l + (-1)^{|F^c|+j}\, \delta^*(y) \otimes 1.$$

We can check that the multiplication by $(-1)^{\alpha(F,F^c)+j}$ on $(M_F^{-j})^* \otimes_k S(-F)$, which can be regarded as a submodule of both $\mathrm{Hom}^i(\mathcal{F}(M^\bullet), \omega_S[n])$ and $\mathcal{F}^i(\mathcal{A}(M^\bullet))$, induces a quasi-isomorphism between $\mathrm{Hom}^\bullet(\mathcal{F}(M^\bullet), \omega_S[n])$ and $\mathcal{F}(\mathcal{A}(M^\bullet))$. So $\mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \cong \mathbf{T}^{2n} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A}$ as a functor on $\mathbf{D}^b(\mathbf{Sq})$. Since $(\mathcal{A} \circ \mathcal{D} \circ \mathcal{A}) \circ (\mathcal{A} \circ \mathcal{D} \circ \mathcal{A}) \cong \mathrm{Id}_{\mathbf{D}^b(\mathbf{Sq})}$, we get the assertion. □

Example 2.6. Assume $G \subset F$, and set $H := F \setminus G$. Then we have

$$\begin{aligned}
&\mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A}(S/P_F(-G)) \\
&= \mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A} \circ \mathcal{D}(S/P_{G^c}(-F^c)) \\
&= \mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A}((S/P_{G^c})(-G^c + F^c)[n - |G|]) \\
&= \mathcal{D} \circ \mathcal{A} \circ \mathcal{D} \circ \mathcal{A}((S/P_{G^c})(-H)[n - |G|]) \\
&= \mathcal{D} \circ \mathcal{A} \circ \mathcal{D}((S/P_{H^c})(-G)[-n + |G|]) \\
&= \mathcal{D} \circ \mathcal{A}((S/P_{H^c})(G - H^c)[2n - |G| - |H|]) \\
&= \mathcal{D} \circ \mathcal{A}((S/P_{H^c})(-F^c)[2n - |F|]) \\
&= \mathcal{D}(S/P_F(-H)[|F| - 2n]) \\
&= S/P_F(H - F)[3n - |F| - |F^c|] \\
&= S/P_F(-G)[2n].
\end{aligned}$$

3. Relation to the Bernstein-Gel'fand-Gel'fand correspondence

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring as in the previous sections, and $E = k\langle e_1, \dots, e_n\rangle$ an exterior algebra. We regard E as a $\mathbb{Z}^n$-graded ring with $\deg(e_i) = (0, \dots, 0, -1, 0, \dots, 0)$ where -1 is in the i^{th} position. When we regard S and E as $\mathbb{Z}$-graded rings, we set $\deg(x_i) = 1$ and $\deg(e_i) = -1$ for all i. For a $\mathbb{Z}^n$-graded E-module M and $\mathbf{a} \in \mathbb{Z}^n$, $M_{\mathbf{a}}$ means the degree $\mathbf{a}$ component of M, and $M(\mathbf{a})$ is the shifted module with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$ as in the polynomial ring case.

Denote the category of finitely generated $\mathbb{Z}$-graded S-modules (resp. E-modules) by $\mathbf{mod}_S$ (resp. $\mathbf{mod}_E$). Although the categories $\mathbf{mod}_S$ and $\mathbf{mod}_E$ are far from equivalent, a famous theorem of Bernstein-Gel'fand-Gel'fand [2] (see also [1]) states that $\mathbf{D}^b(\mathbf{mod}_S) \cong \mathbf{D}^b(\mathbf{mod}_E)$. We will see that this equivalence also holds in the $\mathbb{Z}^n$-graded context. Denote the category of finitely generated $\mathbb{Z}^n$-graded S-modules (resp. E-modules) by $^*\mathbf{mod}_S$ (resp. $^*\mathbf{mod}_E$). For a functor $\mathcal{R} : \mathbf{D}^b(^*\mathbf{mod}_S) \to \mathbf{D}^b(^*\mathbf{mod}_E)$, we use the convention of [5].

For $M \in {^*\mathbf{mod}_S}$, we define $\mathcal{R}(M) = \mathrm{Hom}_k(E(-\mathbf{1}), M)$ to be a $\mathbb{Z}^n$-graded cochain complex of free E-modules as follows. (The original definition in [5] is $\mathcal{R}(M) = \mathrm{Hom}_k(E, M)$, but we use this grading. We will also shift the grading of $\mathcal{L}(N)$ whose original definition is $S \otimes_k N$.) We can regard $\mathrm{Hom}_k(E(-\mathbf{1}), M_{\mathbf{a}})$ as a free E-module $E(-\mathbf{a})^{\dim_k M_{\mathbf{a}}}$ in natural way. Set the cohomological degree of $\mathrm{Hom}_k(E(-\mathbf{1}), M_{\mathbf{a}})$ to be $||\mathbf{a}|| := \sum_{j \in [n]} a_j$. The differential of $\mathcal{R}(M)$ is defined by

$$\mathrm{Hom}_k(E(-\mathbf{1}), M_{\mathbf{a}}) \ni f \mapsto [e \mapsto \sum_i x_i f(e_i e)] \in \bigoplus_{i \in [n]} \mathrm{Hom}_k(E(-\mathbf{1}), M_{\mathbf{a}+\{i\}}).$$

We can also define the complex $\mathcal{R}(M^\bullet) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_k(E(-\mathbf{1}), M^j)$ for a complex $M^\bullet \in {}^*\mathbf{mod}_S$ in a similar way.

Note that $\mathcal{R}(M)$ is hard to be a bounded complex even for a module M. But, if $M^\bullet$ is a bounded complex, then $\mathcal{R}(M^\bullet)$ has bounded cohomology (i.e., only finitely many non-vanishing cohomology modules). And $\mathcal{R}$ preserves quasi-isomorphisms. Thus $\mathcal{R}$ defines a covariant functor from $\mathbf{D}^b({}^*\mathbf{mod}_S)$ to $\mathbf{D}^b({}^*\mathbf{mod}_E)$.

Next, we will define a left adjoint $\mathcal{L}$ of $\mathcal{R}$. Set $\mathcal{L}(N^\bullet) = \bigoplus_{j\in\mathbb{Z}} S(-\mathbf{1}) \otimes_k N^j$ for a complex $N^\bullet = \{N^i, \delta^i\}$ in ${}^*\mathbf{mod}_E$. The cohomological degree of $\mathcal{L}(N^\bullet)$ is given by $\mathcal{L}^i(N^\bullet) = \bigoplus_{i=j-\|\mathbf{a}\|} S(-\mathbf{1}) \otimes_k N^j_{\mathbf{a}}$. And the differential is defined by

$$\mathcal{L}^i(N^\bullet) \supset S(-\mathbf{1}) \otimes_k N^j_{\mathbf{a}} \ni s \otimes y \mapsto \sum_{i\in[n]} x_i s \otimes e_i y + (-1)^i (s \otimes \delta^j(y)) \in \mathcal{L}^{i+1}(N^\bullet).$$

If $N^\bullet$ is bounded so is $\mathcal{L}(N^\bullet)$. And $\mathcal{L}$ preserves quasi-isomorphisms. Hence $\mathcal{L}$ defines a covariant functor from $\mathbf{D}^b({}^*\mathbf{mod}_E)$ to $\mathbf{D}^b({}^*\mathbf{mod}_S)$. By the same argument to [5, §2], we can prove the following.

Theorem 3.1 (BGG correspondence ($\mathbb{Z}^n$-graded version)). *With the above notation, the functors $\mathcal{R}$ and $\mathcal{L}$ define an equivalence $\mathbf{D}^b({}^*\mathbf{mod}_S) \cong \mathbf{D}^b({}^*\mathbf{mod}_E)$.*

The functors $\mathcal{R}$ and $\mathcal{L}$ are closely related to $\mathcal{D}$ and $\mathcal{A}$ of the previous section. To see this, we recall the definition of a squarefree module over E.

Definition 3.2 (Römer [10]). A $\mathbb{Z}^n$-graded E-module $N = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} N_{\mathbf{a}}$ is *squarefree* if N is finitely generated and $N = \bigoplus_{F\subset[n]} N_{-F}$.

For example, a monomial ideal of E is always squarefree. We denote the full subcategory of ${}^*\mathbf{mod}_E$ consisting of all the squarefree E-modules by $\mathbf{Sq}_E$. There are functors $\mathcal{S} : \mathbf{Sq}_E \to \mathbf{Sq}_S$ and $\mathcal{E} : \mathbf{Sq}_S \to \mathbf{Sq}_E$ defining an equivalence $\mathbf{Sq}_S \cong \mathbf{Sq}_E$. Here $\mathcal{S}(N)_F = N_{-F}$ for $N \in \mathbf{Sq}_E$, and the multiplication map $\mathcal{S}(N)_F \ni y \mapsto x_i y \in \mathcal{S}(N)_{F\cup\{i\}}$ for $i \notin F$ is given by $\mathcal{S}(N)_F = N_{-F} \ni z \mapsto (-1)^{\alpha(i,F)} e_i z \in N_{-(F\cup\{i\})} = \mathcal{S}(N)_{F\cup\{i\}}$. See [10] for further information.

Comparing $\mathcal{L}$ and $\mathcal{F} = \mathcal{A} \circ \mathcal{D}$ defined in the last section, we have the following.

Proposition 3.3. *If $N^\bullet$ is a (bounded) complex of squarefree E-modules, then $\mathcal{L}(N^\bullet) = S(-\mathbf{1}) \otimes_k N^\bullet$ is a (bounded) complex of squarefree S-modules. Hence $\mathcal{L}$ gives a functor from $\mathbf{D}^b(\mathbf{Sq}_E)$ to $\mathbf{D}^b(\mathbf{Sq}_S)$. Moreover, for $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq}_S)$, we have $\mathcal{L} \circ \mathcal{E}(M^\bullet) = \mathcal{A} \circ \mathcal{D}(M^\bullet)$.*

On the other hand, $\mathcal{R}(M)$ is very hard to be a complex of squarefree E-modules. In fact, $\mathcal{R}(M)$ is a complex of free E-modules, but a free E-module $E(-\mathbf{a})$ is squarefree if and only if $\mathbf{a} = 0$. But we have the following.

Proposition 3.4. *If $M^\bullet$ is a bounded complex in $\mathbf{Sq}_S$, then $\mathcal{R}(M^\bullet) \cong \mathcal{E}\circ\mathcal{D}\circ\mathcal{A}(M^\bullet)$ in $\mathbf{D}^b({}^*\mathbf{mod}_E)$.*

We can see that $\mathbf{D}^b_{\mathbf{Sq}_S}({}^*\mathbf{mod}_S) \cong \mathbf{D}^b(\mathbf{Sq}_S)$ and $\mathbf{D}^b_{\mathbf{Sq}_E}({}^*\mathbf{mod}_E) \cong \mathbf{D}^b(\mathbf{Sq}_E)$. If $M^\bullet \in \mathbf{D}^b(\mathbf{Sq}_S)$, then $\mathcal{R}(M^\bullet) \in \mathbf{D}^b_{\mathbf{Sq}_E}({}^*\mathbf{mod}_E)$ by Proposition 3.4. Hence, under the equivalence $\mathbf{D}^b_{\mathbf{Sq}_E}({}^*\mathbf{mod}_E) \cong \mathbf{D}^b(\mathbf{Sq}_E)$, we have $\mathcal{S} \circ \mathcal{R} \cong \mathcal{D} \circ \mathcal{A}$. While Proposition 3.3 states that $\mathcal{L} \circ \mathcal{E} \cong \mathcal{A} \circ \mathcal{D}$.

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F-rationality of Rees algebras

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§1. Introduction

Throughout this talk, let A be an excellent Cohen-Macaulay normal local domain of characteristic $p > 0$ and with $d = \dim A \geq 2$. Further, assume that the residue field $k = A/\mathfrak{m}$ is infinite and perfect. Also, let I be an $\mathfrak{m}$-primary ideal in A. We use the following notation:

$F : A \to A$: the Frobenius map defined by $F(a) = a^p$.

$R(I) = A[It]$ (resp.$R'(I) = A[It, t^{-1}]$), the (resp. the extended) Rees algebra of I.

$gr_I(A) := R(I)/IR(I) \cong R'(I)/t^{-1}R'(I)$, the associated graded ring of I.

We always use the letter q for a power p^e of p. Then we put $I^{[q]} = (a^q \mid a \in I)A$.

Definition (1.1). (cf. [Hu], [FW]) (1) The *tight closure* I^* is the ideal defined by $\alpha \in I^*$ if and only if there exists an element $c \in A^o$ such that for all sufficiently large $q = p^e$, $c\alpha^q \in I^{[q]}$, where $A^o := A \setminus \bigcup\{P \mid P \in \mathrm{Min}(A)\}$.
(2) The *integral closure* $\overline{I}$ is the ideal defined by $\alpha \in \overline{I}$ if and only if there exists a monic polynomial $F(X) = X^n + a_1X^{n-1} + \cdots + a_n \in A[X]$ with $a_i \in I^i$ such that $F(\alpha) = 0$.
(3) A local ring A in which every ideal (resp. parameter ideal) is tightly closed is called *weakly F-regular* (resp. *F-rational*).

The main purpose of this talk is to give a criterion for the Rees algebra $R(I)$ to be F-rational.

In order to explain our motivation, let A be a normal local ring which is essentially of finite type over a field of characteristic zero. Let $f : X \to Y := \mathrm{Spec}\, A$ be a resolution of singularities of Y. The ring A is said to be (or have) a *rational singularity* if $R^j f_*\mathcal{O}_X = 0$ for all $j > 0$. Note that this property does not depend on the choice of a resolution of singularities. Then the following theorem is known.

Theorem (Lipman [Li1]). *If I is an integrally closed ideal in a rational surface singularity, then $R(I)$ is also a rational singularity.*

In the last symposium on Commutative Algebra, Hyry showed the following theorem, which gives a criterion for $R(I)$ to be a rational singularity.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Theorem (Hyry [Hy2]). *Assume that A is regular. Then $R(I)$ is a rational singularity if and only if it is a Cohen-Macaulay normal domain with $\mathfrak{a}(I, J) = \mathrm{adj}(I^{d-1})$ for any minimal reduction J of I, where $\mathfrak{a}(I, J)$ is the largest ideal $\mathfrak{a}$ of A for which $I\mathfrak{a} = J\mathfrak{a}$.*

As is known, the notion of F-rational rings (in positive characteristic) is very closely related to that of rational singularities. In fact, in the above notation, A is a rational singularity if and only if its reduction mod p is F-rational for all sufficiently large prime $p > 0$.

So we shall consider the following questions in positive characteristic. In the following, we use the notation described on the top of the report.

Questions (1.2).

(1) Find a criterion for $R(I)$ to be F-rational.
(2) Assume that $R(I)$ is a Cohen-Macaulay normal domain $(+\alpha)$ over an F-rational ring. Then is $R(I)$ F-rational?
(3) If $R(I)$ is F-rational, then is the base ring A also F-rational?
(4) Both A and $R(I)$ are F-rational if and only if so is $R'(I)$?

In order to state our main theorem, we need the following notion, which has been recently introduced by Hochster.

Definition (1.3) (Hochster [Ho]). Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a finite set of ideals in A. Then $\mathcal{I}$ is the ideal defined by $\alpha \in \mathcal{I}^{\underline{*}}$ if and only if there exists $c \in A^o$ such that $c\alpha^q \in I_1^q + \cdots + I_n^q$ for all sufficiently large $q = p^e$. Then $\mathcal{I}^{\underline{*}}$ is called the *tight integral closure* of $\mathcal{I}$.

Note that an element $x \in A$ is in $\overline{I}$ if and only if there exists an element $c \in A^o$ such that $cx^n \in I^n$ for infinitely many n. Thus $\{I\}^{\underline{*}} = \overline{I}$. Moreover, if an ideal I is generated by elements $\{a_1, \ldots, a_n\}$, then the tight integral closure $\{a_1A, \ldots, a_dA\}^{\underline{*}}$ is equal to the tight closure I^* of I. Thus the notion of tight integral closure can be regarded as a generalization of these two notions. Furthermore, the following fact is important:

$$\left(\overline{I_1} + \cdots + \overline{I_n}\right)^* \subseteq \{I_1, \ldots, I_n\}^{\underline{*}} \subseteq \overline{I_1 + \cdots + I_n}.$$

The next theorem is the main theorem in this talk, which gives a criterion for $R(I)$ to be F-rational.

Theorem (1.4). *Let $(A, \mathfrak{m}, k)$ be an excellent Cohen-Macaulay normal local domain of characteristic $p > 0$. Let I be an $\mathfrak{m}$-primary ideal in A and $J = (f_1, \ldots, f_d)$ a minimal reduction of I. Put $J^{[l]} = (f_1^l, \ldots, f_d^l)$.*

Then $R(I)$ is F-rational if and only if the following statements hold:

(1) *$R(I)$ is Cohen-Macaulay.*
(2) *For all $l \geq 1$ and for $1 \leq s < ld$, $\{I^s, f_1^lA, \ldots, f_d^lA\}^{\underline{*}} = I^s + J^{[l]}$, that is, if there exists an element $c \in A^o$ such that $cx^q \in I^{sq} + J^{[lq]}$ for all sufficiently large $q = p^e$, then $x \in I^s + J^{[l]}$.*

When this is the case, the following statements hold.

(i) *$A_\mathfrak{p}$ is F-rational for all prime $\mathfrak{p} \in \mathrm{Spec}\, A \setminus \{\mathfrak{m}\}$.*
(ii) *$J^* \subseteq J + I^{d-1}$.*

(iii) $\left(J^{[l]}\right)^* = J^{[l]} + f^{l-1}J^*$ *for all* $l \geq 1$.
(iv) $I \subseteq \tau_{par}(A) := \mathrm{ann}(0)^*_{H^d_\mathfrak{m}(A)}$, *the parameter test ideal of* A.

In case of F-rational local rings, the above criterion becomes simpler.

Corollary (1.5). *Assume that* A *is F-rational. Then* $R(I)$ *is F-rational if and only if*
(1) $gr_I(A)$ *is Cohen-Macaulay.*
(2) $\{I^s,\, f_1^l A, \ldots, f_d^l A\}^{\underline{*}} = I^s + J^{[l]}$ *for all* $l,\, s \geq 1$.

Proof. Since excellent F-rational rings are pseudo-rational([Sm]), the assumption (1) implies that $R(I)$ is Cohen-Macaulay ([Li2]). So we may assume that $R(I)$ is Cohen-Macaulay. Then since $R(I^{lq})$ is also Cohen-Macaulay and $J^{[lq]}$ is a minimal reduction of I^{lq}, if $s \geq ld$, then we have

$$I^{sq} \subseteq I^{dlq} = (f_1^{lq}, \ldots, f_d^{lq})I^{lq(d-1)} \subseteq (f_1^{lq}, \ldots, f_d^{lq}).$$

Hence for such integers s, l, we get $\{I^s,\, f_1^l A, \ldots, f_d^l A\}^{\underline{*}} = I^s + (f_1^l, \ldots, f_d^l)$ if and only if $J^{[l]}$ is tightly closed, that is, A is F-rational. Thus the assertion follows from Theorem (1.4). □

The following corollary gives a partial answer to Q.(1.2)(3).

Corollary (1.6). *If* $R(I)$ *is F-rational and* $a(gr_I(A)) \neq -1$, *then* A *is also F-rational.*

Proof. Take any minimal reduction J of I. Since $gr_I(A)$ is Cohen-Macaulay, we get $r_J(I) - d = a(gr_I(A)) \leq -1$. Thus the assumption implies that $r_J(I) \leq d-2$. In particular, $I^{d-1} = JI^{d-2} \subseteq J$. Theorem (1.4) implies that $J^* \subseteq J + I^{d-1} = J$, and so that A is F-rational, as required. □

In the above corollary, the assumption "$a(gr_I(A)) \neq -1$" is not superfluous.

Example (1.7). (See [Ar]) Let k be a field of characteristic 2.
(1) Let $A = k[[x,y,z]]/(z^2 + x^2y + xy^2)$. Then A is pseudo-rational but not F-rational. Further, we have $a(gr_\mathfrak{m}(A)) = -1$ and that $R(\mathfrak{m})$ is F-rational.
(2) Let $A = k[[x,y,z]]/(z^2 + x^3 + xy^3 + xyz)$. Then A is pseudo-rational but not F-rational with $\tau_{par}(A) = \mathfrak{m}$. Furthermore, $R(\mathfrak{m}^r)$ is a Cohen-Macaulay normal domain but not F-rational for every integer $r \geq 1$.

Proof (Sketch):. (1) In order to prove this, it is enough to show that

$$\{\mathfrak{m}^3,\, x^2A,\, y^2A\}^{\underline{*}} = \mathfrak{m}^3 + (x^2,\, y^2) \qquad \text{(see [HYW1] in detail).}$$

Indeed, we can show that any generator $a_1xy + a_2xz + a_3yz$ $(a_i \in k)$ of the socle of $A/\mathfrak{m}^3 + (x^2,\, y^2)$ is not contained in $\{\mathfrak{m}^3,\, x^2A,\, y^2A\}^{\underline{*}}$.
(2) It is enough to show that $(xz)^q \in \mathfrak{m}^{3q} + (x^{2q})$ holds for all $q = 2^e$ by induction on $e \geq 1$. Furthermore, this shows that $(x^{2r-1}y^{r-1}z)^q \in \mathfrak{m}^{3rq} + (x^{2rq})$ for all $q = 2^e$. Thus $R(\mathfrak{m}^r)$ is not F-rational since $x^{2r-1}y^{r-1}z \notin \mathfrak{m}^{3r} + (x^{2r},\, y^{2r})$ for each $r \geq 1$. □

Further, we have

Theorem 1.8. *Let $(A, \mathfrak{m})$ be a local ring of a rational surface singularity of characteristic $p > 0$. Let I be an integrally closed $\mathfrak{m}$-primary ideal of A such that the blowing-up $f: X = \operatorname{Proj} R(I) \to \operatorname{Spec} A$ is a resolution of singularity. Then the Rees algebra $R(I^m)$ is F-rational for all sufficiently large $m \in \mathbb{N}$.*

The above theorem suggests the following conjecture.

Conjecture (1.9). Let A be as in (1.8). Let I be an integrally closed $\mathfrak{m}$-primary ideal of A such that $\operatorname{Proj} R(I)$ is F-rational. Then the Rees algebra $R(I^m)$ is F-rational for all sufficiently large $m \in \mathbb{N}$.

§2. Proof of the Main Theorem

In this section, we give a sketch of the proof of Theorem (1.4).

2.1 *Notation.* In the following, we use the following notation. Let I be an $\mathfrak{m}$-primary ideal of A and $J = (f_1, \dots, f_d)$ its minimal reduction. We put $f = f_1 \cdots f_d$, $g_i = f/f_i$ for all i and $J^{[l]} = (f_1^l, \dots, f_d^l)$ for every $l \geq 1$. Also, we put $\mathcal{R} := R(I) = A[It]$, $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, and $G = \mathcal{R}/I\mathcal{R}$. Moreover, we denote $X = \operatorname{Proj}(\mathcal{R})$ and $\mathcal{O}_X(n) = \mathcal{R}(n)^\sim$ for $n \in \mathbb{Z}$. Let E be the closed fiber of the blow-up $f : X \to \operatorname{Spec} A$ which is induced from the natural ring homomorphism $A \hookrightarrow R(I)$.

We begin with the "*Sancho de Sales sequence*" with respect to the Rees algebra $R(I)$; see e.g. Lipman [Li2].

Lemma (2.2). (Sancho de Sales sequence) *Under the notation as in* (2.1), *we have*

$$\text{(SS)} \qquad \cdots \to H^i_{\mathfrak{M}}(\mathcal{R}) \to \bigoplus_{n \geq 0} H^i_{\mathfrak{m}}(I^n) \to \bigoplus_{n \in \mathbb{Z}} H^i_E(X, \mathcal{O}_X(n)) \to H^{i+1}_{\mathfrak{M}}(\mathcal{R}) \to \cdots$$

Using this, we show the following lemma, which plays a key role in the proof of Theorem (1.4).

Lemma (2.3). *Under the notation as in* (2.1), *suppose that $\mathcal{R}$ is Cohen-Macaulay. Then we have the following exact sequence for all $n \leq -1$:*

$$0 \to H^{d-1}(X, \mathcal{O}_X(n)) \cong \left[H^d_{\mathcal{R}_+}(\mathcal{R})\right]_n \xrightarrow{\phi_n} H^d_{\mathfrak{m}}(A) \xrightarrow{\psi_n} \left[H^{d+1}_{\mathfrak{M}}(\mathcal{R})\right]_n \to 0.$$

Proof. As $E = f^{-1}(\mathfrak{m})$ being closed in $X = \operatorname{Proj}(\mathcal{R})$, there exists the following exact sequence

$$\begin{aligned} 0 = H^{d-1}_E(X, \mathcal{O}_X(n)) &\to H^{d-1}(X, \mathcal{O}_X(n)) \to H^{d-1}(X \setminus E, \mathcal{O}_X(n)) \\ &\to H^d_E(X, \mathcal{O}_X(n)) \to H^d(X, \mathcal{O}_X(n)) = 0. \end{aligned}$$

Since $X \setminus E \cong \operatorname{Spec} A \setminus \{\mathfrak{m}\}$, we have $H^{d-1}(X \setminus E, \mathcal{O}_X(n)) \cong H^d_{\mathfrak{m}}(A)$ (see e.g.[TW, Proposition(1.17)]). Also, under the assumption that $\mathcal{R}$ is Cohen-Macaulay, (SS) implies that $H^d_E(X, \mathcal{O}_X(n)) \cong \left[H^{d+1}_{\mathfrak{M}}(\mathcal{R})\right]_n$ for all $n \leq -1$. Therefore we get the required exact sequence. $\square$

In the following, let ϕ_n and ψ_n denote the homomorphisms described in (2.3). We now consider the condition for $[\frac{a}{f^l}]$ to be included in $\operatorname{Im}(\phi_n)$. In order to do that, we need the following lemma.

Lemma (2.4). *Let A be any Cohen-Macaulay local ring of dimension $d \geq 2$. Let I be an $\mathfrak{m}$-primary ideal of A and $J = (f_1, \cdots, f_d)A$ its minimal reduction. Suppose that $G = gr_I(A)$ is Cohen-Macaulay. Then for all integers l, $r, s \geq 1$ with $1 \leq s < dl$,*

(2.4.1)
$$\left[I^{s+dr} + J^{[l+r]}\right] : (f_1 \cdots f_d)^r \subseteq I^s + J^{[l]}.$$

The following corollary, which will be used in the proof of Theorem (1.4), easily follows from the above lemma. Note that "if part" is trivial but "only if part" is important.

Corollary (2.5). *Under the same notation as in Lemma (2.4), we further suppose that l, n are positive integers such that $dl - n \geq 1$. Then for any element $a \in A$,*

$$\left[\frac{a}{f^l}\right] \in \mathrm{Im}(\phi_{-n}) \quad \textit{if and only if} \quad a \in I^{dl-n} + J^{[l]}.$$

We now prove the main part of Theorem (1.4) by proving the following proposition, which is more practical version.

Proposition (2.6). *Under the same notation as in (2.1), we further assume that $\mathcal{R}$ is Cohen-Macaulay. Put*

$$b(\mathcal{R}) := -\min\left\{n \in \mathbb{Z} \,\middle|\, \left[\mathrm{Soc}\left(H^{d+1}_{\mathfrak{M}}(\mathcal{R})\right)\right]_n \neq 0\right\} (\geq 1).$$

Then $\mathcal{R}$ is F-rational if and only if

(2.6.1)
$$\{I^s, f_1^l A, \cdots, f_d^l A\}^* = I^s + (f_1^l, \cdots, f_d^l)A$$

holds for all $l \geq 1$ and for all s with $\max\{1, dl - b(\mathcal{R})\} \leq s \leq dl - 1$.
When this is the case, (2.6.1) holds for all $l \geq 1$ and $1 \leq s \leq dl - 1$.

Proof (Sketch):. We may assume that $c \in I$ is a test element for parameters in both A and $\mathcal{R}$ by Vélez' theorem([Ve]). Then from (2.3), for all $q = p^e$ and for all $n \leq -1$, we have the following commutative diagram with exact rows:

(2.6.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & [H^d_{\mathcal{R}_+}(\mathcal{R})]_n & \xrightarrow{\phi_n} & H^d_{\mathfrak{m}}(A) & \xrightarrow{\psi_n} & [H^{d+1}_{\mathfrak{M}}(\mathcal{R})]_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle cF^e} & & \downarrow{\scriptstyle cF^e} & & \downarrow{\scriptstyle cF^e} & & \\ 0 & \longrightarrow & [H^d_{\mathcal{R}_+}(\mathcal{R})]_{qn} & \xrightarrow{\phi_{qn}} & H^d_{\mathfrak{m}}(A) & \xrightarrow{\psi_{qn}} & [H^{d+1}_{\mathfrak{M}}(\mathcal{R})]_{qn} & \longrightarrow & 0. \end{array}$$

First, we prove "if part". In order to do that, we show that if $\alpha \in [(0)^*_{H^{d+1}_{\mathfrak{M}}(\mathcal{R})}]_n$ then $\alpha = 0$. We may assume that α is a generator of $\mathrm{Soc}(H^{d+1}_{\mathfrak{M}}(\mathcal{R}))$. Since $\mathrm{Soc}(H^{d+1}_{\mathfrak{M}}(\mathcal{R}))$ is a graded module, we have $\alpha \in \left[H^{d+1}_{\mathfrak{M}}(\mathcal{R})\right]_n$ for some $1 \leq -n \leq b(\mathcal{R})$ (note that $a(\mathcal{R}) = -1$). Since ψ_n is surjective, we can write as $\alpha = \psi_n(\xi)$ for some $\xi \in H^d_{\mathfrak{m}}(A)$. Since $\psi_{nq}(cF^e(\xi)) = cF^e(\alpha) = 0$ by definition, $cF^e(\xi) \in \mathrm{Im}(\phi_{nq})$ for all sufficiently large $q = p^e$. Thus it suffices to show the following claim.

Claim 1:. For any $\xi \in H^d_{\mathfrak{m}}(A)$ and for each $n \leq -1$, if $cF^e(\xi) \in \mathrm{Im}(\phi_{nq})$ for all sufficiently large $q = p^e$, then $\xi \in \mathrm{Im}(\phi_n)$.

Write $\xi = \left[\frac{a}{f^l}\right]$ for some $a \in A$ and $l \geq 1$. By assumption, $cF^e(\xi) = \left[\frac{ca^q}{f^{lq}}\right] \in \mathrm{Im}(\phi_{nq})$ for all sufficiently large $q = p^e$. Thus $ca^q \in I^{dlq+nq} + J^{[lq]}$ by (2.5). It follows from the assumption (2.6.1) that $a \in I^{dl+n} + J^{[l]}$. Then $\xi = \left[\frac{a}{f^l}\right] \in \mathrm{Im}(\phi_n)$ as required. Hence $\mathcal{R}$ is F-rational.

Since the proof of "only if part" is similar, we omit it. □

In order to complete the proof of Theorem (1.4), we prove the following proposition, which gives several necessary conditions for $\mathcal{R}$ to be F-rational. But these conditions are not sufficient even if $\dim A = 2$ (see e.g (1.7)).

Proposition (2.7). *Under the same notation as in* (2.1), *suppose that* $\mathcal{R}$ *is F-rational. Then*

(i) $A_{\mathfrak{p}}$ *is F-rational for all prime* $\mathfrak{p} \in \mathrm{Spec}\, A \setminus \{\mathfrak{m}\}$.
(ii) $J^* \subseteq J + I^{d-1}$.
(iii) $\left(J^{[l]}\right)^* = J^{[l]} + f^{l-1}J^*$ *for all* $l \geq 1$.
(iv) $I \subseteq \tau_{par}(A)$.

Proof. (i) For any prime $\mathfrak{p} \neq \mathfrak{m}$, since $A_{\mathfrak{p}}[t] = \mathcal{R}(I)_{\mathfrak{p}}$ is F-rational, so is $A_{\mathfrak{p}}$. To see (ii) and (iii), we put $s = dl - 1$ in (1.4)(2). Then

$$\left(I^{dl-1} + J^{[l]}\right)^* \subseteq \{I^{dl-1}, f_1^l A, \cdots, f_d^l A\}^{\underline{*}} = I^{dl-1} + J^{[l]}.$$

In particular, $I^{dl-1} + J^{[l]}$ is tightly closed. On the other hand, since $\mathcal{R}$ is Cohen-Macaulay, $I^d = JI^{d-1}$; hence $I^{dl-1} = J^{dl-d}I^{d-1}$. Then since $I^{dl-1} + J^{[l]} = J^{dl-d}I^{d-1} + J^{[l]} = f^{l-1}I^{d-1} + J^{[l]}$, we get

$$\textbf{(2.7.1)} \qquad (J^{[l]})^* \subseteq (f^{l-1}I^{d-1} + J^{[l]})^* = f^{l-1}I^{d-1} + J^{[l]}.$$

Therefore for any $x \in (J^{[l]})^*$, we can write as $x = y + f^{l-1}z$ for some $y \in J^{[l]}$ and $z \in I^{d-1}$. Then $z \in J^*$ since J is generated by a regular sequence. Thus $x \in J^{[l]} + f^{l-1}J^*$. Hence we get the assertion (iii). Moreover, putting $l = 1$ in (2.7.1), we also obtain (ii).

To see (iv), we first prove that $(0)^*_{H^d_{\mathfrak{m}}(A)} \subseteq \mathrm{Im}(\phi_{-1})$. In fact, for any $\xi \in (0)^*_{H^d_{\mathfrak{m}}(A)}$, we write as $\xi = \left[\frac{a}{f^l}\right]$. Since A is Cohen-Macaulay and $cF^e(\xi) = \left[\frac{ca^q}{f^{lq}}\right] = 0$ in $H^d_{\mathfrak{m}}(A)$, we obtain that $ca^q \in J^{[lq]}$ for all sufficiently large $q = p^e$; thus $a \in (J^{[l]})^*$. By (2.7.1), we have $a - f^{l-1}b \in J^{[l]}$ for some $b \in I^{d-1}$. Namely, we have $\left[\frac{a}{f^l}\right] = \left[\frac{b}{f}\right] \in \mathrm{Im}(\phi_{-1})$ in $H^d_{\mathfrak{m}}(A)$, as required.

Then it suffices to show that

$$\textbf{(2.7.2)} \qquad [H^d_{\mathcal{R}_+}(\mathcal{R})]_{-1} \cong [H^d_{\mathfrak{M}}(G)]_{-1} \left(\cong \frac{J + I^{d-1}}{J}\right).$$

In fact, if this is valid, then $I.[H^d_{\mathcal{R}_+}(\mathcal{R})]_{-1} = I.[H^d_{\mathfrak{M}}(G)]_{-1} = 0$, and so $I.(0)^*_{H^d_{\mathfrak{m}}(A)} \subseteq I.\mathrm{Im}(\phi_{-1}) = 0$; hence $I \subseteq \mathrm{ann}_A(0)^*_{H^d_{\mathfrak{m}}(A)} = \tau_{par}(A)$.

In order to see (2.7.2), we consider the following two short exact sequences:

$$0 \to \mathcal{R}_+ \to \mathcal{R} \to A \to 0, \qquad 0 \to \mathcal{R}_+(1) \to \mathcal{R} \to G \to 0.$$

From these sequences, we get

$$0 = H^{d-1}_{\mathfrak{M}}(G) \to H^d_{\mathcal{R}_+}(R_+)(1) \to H^d_{\mathcal{R}_+}(\mathcal{R}) \to H^d_{\mathfrak{M}}(G) \to H^{d+1}_{\mathcal{R}_+}(\mathcal{R})(1) = 0.$$

Thus it is enough to show that $\left[H^d_{\mathcal{R}_+}(\mathcal{R})\right]_0 = 0$. For any element $\beta \in \left[R_{ft^d}\right]_0$, we can write as $\beta = a/f^m$ for some integer $l \geq 1$ and $a \in I^{dl}$. On the other hand, since $I^{dl} = J^{[l]}I^{dl-d}$, we have $\beta \in \operatorname{Im}([\oplus_{i=1}^n \mathcal{R}_{g_i t^{d-1}}]_0 \to [\mathcal{R}_{ft^d}]_0)$. Thus $\left[H^d_{\mathcal{R}_+}(\mathcal{R})\right]_0 = 0$. We have completed the proof of Proposition (2.7), and so Theorem (1.4). □

§3. Applications

The following proposition is an analogy (in positive characteristic) of Lipman's theorem. We were informed by Huneke and Smith that they also have proved this theorem.

Proposition (3.1). *Let A be an F-rational local ring of dimension two. Then for any integrally closed ideal I of A, $R(I)$ is F-rational.*

Proof. Since $R(I)$ is Cohen-Macaulay, it suffices to show that $\{I^s, x^lA, y^lA\}^* = I^s + (x^l, y^l)$ for all $l \geq 1$, $1 \leq s < dl$. Suppose that $z \in \{I^s, x^lA, y^lA\}^*$. Then there exists an element $c \in A^o$ such that $cz^q \in I^{sq} + (x^{lq}, y^{lq})$ for all $q = p^e$.

Since A is a rational surface singularity, one has $I^2 = JI$. Hence for all $q = p^e$, one has that

$$cz^q \in J^{sq-1}I + (x^{lq}, y^{lq}) \subseteq J^{sq-1} + (x^{lq}, y^{lq}) \subseteq \left(J^{s-1} + (x^l, y^l)\right)^{[q]}.$$

Since A is F-rational and J is an ideal generated by monomials in a system of parameters x, y, this implies that

$$z \in \left(J^{s-1} + (x^l, y^l)\right)^* = J^{s-1} + (x^l, y^l) = \left(x^l, x^{l-1}y^{s-l}, \ldots, x^{s-l}y^{l-1}, y^l\right).$$

Hence $z = z' + ax^l + by^l$ for some $z' \in \left(x^{l-1}y^{s-l}, \ldots, x^{s-l}y^{l-1}\right)$ and $a, b \in A$. Similarly, since $cz^q \in J^{sq-1}I + \left(x^{lq}, y^{lq}\right)$, we can also write $cz^q = w + a'x^{lq} + b'y^{lq}$ for some $w \in I \cdot \left(x^{lq-1}y^{q(s-l)}, \ldots, x^{q(s-l)}y^{lq-1}\right)$ and $a', b' \in A$. Therefore, we have that

$$cz'^q - w \in \left(x^{l-1}y^{s-l}, x^{s-l}y^{l-1}\right)^{[q]} \cap \left(x^{lq}, y^{lq}\right) = \left(x^l y^{(s-l)}, x^{(s-l)}y^l\right)^{[q]}.$$

But this implies that $cz'^q \in J^{sq-1}I = I^{sq}$ for all $q = p^e$, so that $z' \in \overline{I^s} = I^s$ by the valuative criterion, where we used the fact that if A is a rational singularity then any power of an integrally closed ideal is also integrally closed. Consequently, we obtain that $z \in I^s + (x^l, y^l)$, as required. □

The following theorem shows that "only if part" of Q.(1.2)(4) is true. But "if part" remains open.

Theorem (3.2). *If A and $R(I)$ are both F-rational, then so is $R'(I)$.*

The above theorem easily follows from the following key lemma and Theorem (1.4).

Lemma (3.3). *Let A be an F-rational local ring, and let I be an integrally closed $\mathfrak{m}$-primary ideal in A. We further suppose that*

(i) *$gr_I(A)$ is Cohen-Macaulay.*

(ii) *There exists a minimal reduction J of I such that*

$$\{I^s, f_1A\cdots, f_dA\}^{\underline{*}} = I^s + J \quad \textit{for all } s = 2, \cdots, r(I)+1.$$

Then $R'(I)$ is F-rational.

Proof. Put $\mathcal{R}' := R'(I) = A[It, t^{-1}]$ and $M = (\mathfrak{m}, It, t^{-1})\mathcal{R}'$. Take a minimal reduction J of I which satisfies (ii). Since $\mathcal{R}'$ is Cohen-Macaulay and (Jt, t^{-1}) is a homogeneous parameter ideal of $\mathcal{R}'$, it is enough to show that (Jt, t^{-1}) is tightly closed to prove that $\mathcal{R}'$ is F-rational.

Now take any non-zero element $c \in I$ and fix it. Then since $\mathcal{R}'_c = A_c[t, t^{-1}]$ is F-rational, c has a power c^n which is a test element for both A and $\mathcal{R}'$. Replacing c with c^n, we may assume that c is a parameter test elment for both A and $\mathcal{R}'$. Fix such an element $c \in I$.

Now suppose that $x \in (Jt, t^{-1})^*$. Since (Jt, t^{-1}) is a homogeneous ideal which contains all of the negative part of $\mathcal{R}'$, we may assume that x can be written as $x = x_k t^k$ for some integer $k \geq 0$ and $x_k \in I^k$. Then for all $q = p^e$, we have

$$cx^q = cx_k^q t^{kq} \in \left(J^{[q]}t^q,\, t^{-q}\right)\mathcal{R}'.$$

Thus we get

$$\textbf{(3.3.1)} \qquad cx_k^q \in \left[t^{-kq}\left(J^{[q]}t^q,\, t^{-q}\right)\mathcal{R}'\right]_0 = J^{[q]}I^{(k-1)q} + I^{(k+1)q} \qquad \text{for all } q = p^e.$$

Under the above notation, we must show that $x_k \in [(Jt, t^{-1})\mathcal{R}']_k = JI^{k-1} + I^{k+1}$ for all $k \geq 0$. In case $k = 0$, by (3.3.1), we have $cx_0^q \in J^{[q]} + I^q = I^q$ for all $q = p^e$. This implies that $x_0 \in \overline{I} = I = [(Jt, t^{-1})\mathcal{R}']_0$.

In case where $1 \leq k \leq r(I)$, we have that $x_k \in J + I^{k+1}$ by (3.3.1) and (ii). On the other hand, $J \cap I^k = JI^{k-1}$ as $gr_I(A)$ is Cohen-Macaulay. Hence

$$x_k \in I^k \cap (J + I^{k+1}) = J \cap I^k + I^{k+1} = JI^{k-1} + I^{k+1} = [(Jt, t^{-1})\mathcal{R}']_k.$$

In case where $k \geq r(I) + 1$, we have $x_k \in I^k = JI^{k-1} + I^{k+1} = [(Jt, t^{-1})\mathcal{R}']_k$ as required. Summing up, we conclude that $x \in (Jt, t^{-1})$ and have completed the proof of the lemma. □

For rational singularities, the similar result as in Theorem (3.2) holds. See [HYW2] in detail.

Theorem (3.4). *Let A be an essentially of finite type over a field k of characteristic zero. Let I be an $\mathfrak{m}$-primary ideal of A. Then $R(I)$ is a rational singularity if and only if so is $R'(I)$.*

Remark. Since A is a rational singularity if and only if it an F-rational type, the above theorem follows from Theorem (3.2), but we have an another proof with geometric flavor; see [HYW2].

In higher dimensional case, there exist many examples of a Gorenstein F-rational local ring whose Rees algebra $R(\mathfrak{m})$ is a Cohen-Macaulay normal domain but not F-rational. Also, one can get many examples of a Cohen-Macaulay normal Rees algebra which is not a rational singularity (in characteristic zero) from this.

Theorem (3.5). *Let $A = k[x_0, x_1, \dots, x_d]/(x_0^2 + x_1^{a_1} + \cdots + x_d^{a_d})$, where a_i are integers with $2 = a_0 \le a_1 \le \cdots \le a_d$. Suppose that $r := \sum_{i=0}^{d} \frac{1}{a_i} > 1$ and $p := char(A) > \frac{a_d r}{r-1}$.*

Then A is a graded F-regular domain with isolated singularity and the following statements hold:

(1) *$R(\mathfrak{m})$ is Cohen-Macaulay.*
(2) *$R(\mathfrak{m})$ is normal if and only if $a_1 = 2, 3$.*
(3) *$R(\mathfrak{m})$ is F-rational if and only if $(a_1, a_2) = (2, n), (3, 3), (3, 4), (3, 5)$, where n is any integer with $n \ge 2$.*

Proof. See [HYW1] in detail. □

Remark. The content of this talk will be contained in [HYW1]; see also [HYW2] in detail.

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REES ALGEBRAS OF F-REGULAR TYPE

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Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 2$ and let I be an $\mathfrak{m}$-primary ideal. The Rees algebra of I is, by definition, the graded ring $R(I) := A[It]$, where the degree of elements of A is zero and t is an indeterminate of degree 1. Then the ring homomorphism $A \hookrightarrow R(I)$ induces a morphism ψ: Proj $R(I) \rightarrow$ Spec A, the blowing-up of Spec A with respect to the ideal I.

Various properties of Rees algebras, e.g., Cohen–Macaulay and Gorenstein properties, has been studied by commutative algebraists. The condition for Rees algebras to have rational singularities has been studied by Lipman [L], Hyry [Hy] and others. Motivated by their works, we studied in [HYW1] the F-rationality of Rees algebras over a local ring of characteristic $p > 0$.

F-rationality is defined via tight closure in rings of characteristic p ([FW], [HH1]) and is known to correspond to rational singularity in characteristic zero via reduction modulo $p \gg 0$ ([H], [MS], [Sm]). Tight closure also enables us to define F-regularity ([HH1]), which is another important class of rings in characteristic $p > 0$ and corresponds to log terminal singularity under $\mathbb{Q}$-Gorenstein property ([H], [HW]).

As far as we know, much is not known about F-regularity of Rees algebras. The purpose of this note is to investigate the F-regularity, and related geometric aspects such as log terminal singularity, of Rees algebras $R(I)$, especially in the case when I is an $\mathfrak{m}$-primary ideal in a 2-dimensional local ring $(A, \mathfrak{m})$.

This note is a summary of [HYW2], and our exposition here is in some sense a geometric counterpart of the preceding talk by Yoshida on [HYW1]. We use various geometric methods such as splitting of the Frobenius map and resolution of singularities. Based on this philosophy, we also prove miscellaneous results concerning singularities of Rees algebras.

1. STRONG F-REGULARITY OF REES ALGEBRAS AND GLOBAL F-REGULARITY

The notion of F-regularity defined for rings of characteristic $p > 0$ has a few variants. Rings in which all ideals are tightly closed are said to be weakly F-regular ([HH1]). In this note we mainly consider strong F-regularity defined as follows, rather than weak F-regularity.

Definition 1.1. ([HR], [HH2]) Let R be a reduced ring of characteristic $p > 0$ which is F-finite (i.e., the inclusion map $R \hookrightarrow R^{1/p}$ is module-finite).

(i) We say that R is *F-pure* if the map $R \hookrightarrow R^{1/p}$ splits as an R-linear map.

(ii) We say that R is *strongly F-regular* if for every element $c \in R$ not in any minimal prime of R, there exists a power $q = p^e$ such that the inclusion map $c^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-linear map.

Strong F-regularity is a priori stronger than weak F-regularity, but they are known to coincide for rings of dimension ≤ 2 ([Wi]) and $\mathbb{Q}$-Gorenstein rings ([Mc]). ($\mathbb{Q}$-Gorenstein means that the canonical class has a finite order in the divisor class group of every local ring.) The following implications hold in characteristic $p > 0$:

$$\begin{array}{ccccccc} \text{regular} & \Rightarrow & \text{strongly/weakly F-regular} & \Rightarrow & \text{F-rational} & \Rightarrow & \text{Cohen–Macaulay, normal.} \\ & & \Downarrow & & & & \\ & & \text{F-pure} & & & & \end{array}$$

Also, a Gorenstein F-rational ring is strongly F-regular.

We can extend these concepts to rings in characteristic zero. Namely, a ring R essentially of finite type over a field of characteristic zero is said to be *F-** type* if its reduction modulo p is F-** for all $p \gg 0$. It is proved that R has only rational singularities (resp. log terminal singularities) if and only if R is of F-rational type (resp. F-regular type and $\mathbb{Q}$-Gorenstein). Also, if R is of F-pure type and $\mathbb{Q}$-Gorenstein, then R has only log canonical singularities. For a proof and generalizations, see [H], [HW], [MS], [Sm].

Next we consider the global version of strong F-regularity and F-purity. Let Y be an F-finite scheme of characteristic $p > 0$, i.e., the Frobenius morphism $F\colon Y \to Y$ is finite. One says that Y is *F-split* if the Frobenius ring homomorphism $F\colon \mathcal{O}_Y \to F_*\mathcal{O}_Y$ splits as an $\mathcal{O}_Y$-module homomorphism ([MR]). We introduce an analoguous notion to study strong F-regularity of Rees algebras.

Definition 1.2. Let Y be a projective scheme over an F-finite ring A of characteristic $p > 0$. We say that Y is globally F-regular if for any effective Cartier divisor D on Y, there exists $e \in \mathbb{N}$ such that the composition map $\mathcal{O}_Y \xrightarrow{F^e} F^e_*\mathcal{O}_Y \hookrightarrow F^e_*\mathcal{O}_Y(D)$ splits as an $\mathcal{O}_Y$-module homomorphism, where $F^e\colon \mathcal{O}_Y \to F^e_*\mathcal{O}_Y$ denotes the e-times iterated Frobenius map.

Note that the above definition includes the case when $Y = \operatorname{Spec} A$. In this case it immediately follows that $Y = \operatorname{Spec} A$ is globally F-regular (resp. F-split) if and only if A is strongly F-regular (resp. F-pure). More generally, we have the following

Proposition 1.3. *Let Y be a projective scheme over an F-finite ring A of characteristic $p > 0$.*

1. *If Y is globally F-regular (resp. F-split), then Y is locally strongly F-regular (resp. F-pure), i.e., the local ring $\mathcal{O}_{Y,y}$ is strongly F-regular (resp. F-pure) for every $y \in Y$. In the case when $Y = \operatorname{Spec} A$, the converse is also true.*
2. *If $f\colon Y \to Z$ is a birational projective morphism with $f_*\mathcal{O}_Y = \mathcal{O}_Z$ and if Y is globally F-regular (resp. F-split), then Z is also globally F-regular (F-split).*

Now we give a criterion for strong F-regularity (resp. F-purity) of Rees algebras in terms of global F-regularity (resp. F-splitting). Although we have criteria for general normal graded rings which are more or less similar to Proposition 1.5 below, we restrict ourselves to the following situation which includes the case of normal Rees algebras with respect to ideal-adic filtrations.

1.4. *Setup.* Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 2$ and let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian normal graded ring with the degree 0 part $R_0 = A$. Let $X = \operatorname{Proj} R$ and denote the closed fiber of $\psi\colon X \to \operatorname{Spec} A$ by $E = \psi^{-1}(\mathfrak{m})$. We also denote $\mathcal{X} = \operatorname{Spec}_X(\bigoplus_{n\geq 0} \mathcal{O}_X(n))$, and let S be the closed subscheme of $\mathcal{X}$ defined by the ideal sheaf $\bigoplus_{n>0} \mathcal{O}_X(n)$ of $\mathcal{O}_{\mathcal{X}}$, where $\mathcal{O}_X(n) = R(n)^\sim$ for $n \in \mathbb{Z}$. Then we have the following "fundamental diagram":

$$\begin{array}{ccccc} S & \hookrightarrow & \mathcal{X} & \xrightarrow{\varphi} & \operatorname{Spec} R \\ & \searrow & \downarrow \pi & & \downarrow \\ & & X & \xrightarrow{\psi} & \operatorname{Spec} A \end{array}$$

We put the following assumption, which is always the case for Rees algebras $R = R(I)$:

1. $\varphi\colon \mathcal{X} \to \operatorname{Spec} R$ is an isomorphism in codimension 1.
2. There is an ample $\mathbb{Q}$-Cartier Weil divisor D (with *integer coefficients*) on X such that $\mathcal{O}_X(n) = \mathcal{O}_X(nD)t^n$ for $n \in \mathbb{N}$, so that

$$R = R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))t^n.$$

The condition (1) is satisfied if $\psi\colon X \to \operatorname{Spec} A$ is a birational morphism. We also note that any normal graded ring is of the form $R(X, D)$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor D on X.

Under the above assumtions, the top local cohomology of the ith symbolic power $\omega_R^{(i)}$ of the canonical module of R is written as

$$H^{d+1}_{\mathfrak{M}}(\omega_R^{(i)}) \cong \bigoplus_{n<i} H^d_E(\omega_X^{(i)}(n))t^n.$$

Note also that the 1-dimensional socle of $H^{d+1}_{\mathfrak{M}}(\omega_R)$ sits in the degree 0 part $H^{d+1}_{\mathfrak{M}}(\omega_R)_0 = H^d_E(\omega_X)$. Hence the following criterion is proved as in [W1].

Proposition 1.5. *With the notation as in* 1.4, *let A be an F-finite ring of characteristic $p > 0$. Then the following conditions are equivalent.*

1. $R = R(X, D)$ *is strongly F-regular (resp. F-pure).*
2. $X = \operatorname{Proj} R$ *is globally F-regular (resp. F-split).*
3. *For every nonzero element $c \in A$, there exists a power $q = p^e$ such that the map $cF^e\colon H^d_E(\omega_X) \to H^d_E(\omega_X^{(q)})$ is injective (resp. the induced Frobenius map $F\colon H^d_E(\omega_X) \to H^d_E(\omega_X^{(p)})$ is injective).*

2. Strong F-regularity of Rees algebras over two-dimensional local rings

In this section, we study the strong F-regularity of Rees algebras of $\mathfrak{m}$-primary ideals in 2-dimensional local rings.

Let $(A, \mathfrak{m})$ be a 2-dimensional local ring of characteristic $p > 0$ and $I \subset A$ an $\mathfrak{m}$-primary ideal. The F-regularity version of Boutot's theorem ([HH1], [HH2]) says that if $R(I)$ is strongly F-regular, then so is A, too. Hence we may assume without loss of generality that A has a rational singularity and I is integrally closed.

In the above situation, we proved in [HYW1] that $R(I)$ is F-rational if A is F-rational. On the other hand, we have examples of integrally closed $\mathfrak{m}$-primary ideals I in a non-F-rational rational surface singularity $(A, \mathfrak{m})$ of any fixed characteristic $p > 0$ such that $R(I)$ is F-rational — in other words, examples in which the F-rationality version of Boutot's theorem breaks down ([W3], [HYW1]).

We now consider a similar question for strong F-regularity. As we mentioned above, A is strongly F-regular if so is $R(I)$. However, contrary to the F-rational case, we cannot expect the converse any more. The strong F-regularity of $R(I)$ imposes somewhat strong conditions on I. If $R(I)$ is strongly F-regular, then $X = \text{Proj } R(I)$ should be globally F-regular (Proposition 1.5), but this is too much to ask in general as we will see in Example 2.3 below.

First we give examples of strongly F-regular Rees algebras. The point of the construction is the following

Lemma 2.1. *Let X be a normal projective surface over an F-finite ring of characteristic $p > 0$ and let $g\colon \tilde{X} \to X$ be the minimal resolution. Then X is globally F-regular (resp. F-split) if and only if $\tilde{X}$ is globally F-regular (F-split).*

Proof. The sufficiency follows from Proposotion 1.3 without assuming that g is the minimal resolution. To prove the necessity, we need the assumption that g is minimal, which implies that $K_{\tilde{X}}$ is g-nef. Then it follows that $g_*\mathcal{O}_{\tilde{X}}(-nK_{\tilde{X}}) = \mathcal{O}_X(-nK_X)$ for $n \geq 0$. Since one has $\mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(F^e_*\mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}) \cong F^e_*\mathcal{O}_{\tilde{X}}((1-p^e)K_{\tilde{X}})$ and a similar isomorphism on X by the adjunction formula ([MR]),

$$\text{Hom}_{\mathcal{O}_{\tilde{X}}}(F^e_*\mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}) \cong \text{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X).$$

Hence an F-splitting $\phi\colon F^e_*\mathcal{O}_X \to \mathcal{O}_X$ of X gives an F-splitting of $\tilde{X}$. The statement for global F-regularity is proved similarly. □

2.2. *Example.* If $(A, \mathfrak{m})$ is a 2-dimensional F-regular ring and if I is an integrally closed $\mathfrak{m}$-primary ideal such that $I\mathcal{O}_{\tilde{X}}$ is an invertible sheaf on the minimal resolution $\tilde{X}$ of Spec A, then $R(I)$ is strongly F-regular. In particular, if $(A, \mathfrak{m})$ is a 2-dimensional F-regular ring, then $R(\mathfrak{m}^r)$ is strongly F-regular for every $r \in \mathbb{N}$.

2.3. *Example.* (2) Let $(A, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. We can successively blow up suitable smooth points to obtain a birational projective morphism $f\colon \tilde{X} \to \text{Spec } A$ with $\tilde{X}$ smooth such that the dual graph of the exceptional set of f is as follows:

$$\begin{array}{ccccc} & & D_2 - E_2 & & \\ & & | & & \\ E_1 - D_1 & - & C & - & D_3 - E_3 \\ & & | & & \\ & & D_4 - E_4 & & \end{array}$$

Here C, D_i and E_i ($i = 1,2,3,4$) denote the exceptional curves ($\cong \mathbb{P}^1$) with self-intersection numbers $C^2 = -5$, $D_i^2 = -2$ and $E_i^2 = -1$, respectively. Consider an anti-f-nef exceptional cycle $Z = 4C + \sum_{i=1}^4 (5D_i + 6E_i)$ and let $I = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z))$. Then $X = \text{Proj } R(I)$ is obtained by contracting $C \cup \bigcup_{i=1}^4 D_i$ in $\tilde{X}$ to a non-log-terminal singularity. Hence X is not even locally F-regular, so $R(I)$ is not strongly F-regular.

2.4. *Notation.* Through the remainder of this section we work under the following notation unless otherwise specified. Let $(A, \mathfrak{m})$ be a 2-dimensional local ring essentially of finite type over a perfect field k with only a rational singularity and let I be an integrally closed $\mathfrak{m}$-primary ideal. Then I^n is integrally closed for all $n \in \mathbb{N}$ and the Rees algebra $R(I) = A[It]$ is normal ([L]). Let $\psi\colon X = \text{Proj } R(I) \to \text{Spec } A$ be the blowing-up with respect to I and let $g\colon \tilde{X} \to X$ be the minimal resolution. Then $f = \psi \circ g\colon \tilde{X} \to \text{Spec } A$ is the smallest resolution such that $I\mathcal{O}_{\tilde{X}}$ is an invertible sheaf. We denote the exceptional set of $f = \psi \circ g$ by $\tilde{E}$. Let $\tilde{E}$ be the exceptional set of f with irreducible components $E_1, \ldots, E_s$.

2.5. *Observation.* Let the notation be as in 2.4. Since $(A, \mathfrak{m})$ is a rational singularity, for any set $\mathcal{A}$ of f-exceptional curves such that $E' = \bigcup_{E_i \in \mathcal{A}} E_i$ is connected, there exists a projective morphism $\tilde{X} \to X'$ which contracts E' to a point $x' \in X'$ with rational singularity and induces an isomorphism $X \setminus E' \cong X' \setminus \{x'\}$ ([A]). In this case, we simply say that a connected subgraph E' of $\tilde{E}$ contracts to a rational singularity (X', x').

Now assume that char $k = p > 0$. Then $R(I)$ is strongly F-regular if and only if $X = \text{Proj } R(I)$ is globally F-regular (Proposition 1.5), or equivalently if $\tilde{X}$ is globally F-regular (Lemma 2.1). If this is the case, then every connected subgraph E' of $\tilde{E}$ contracts to an F-regular singularity by Proposition 1.3.

The following theorem says that the converse of this observation holds true in characteristic zero or $p \gg 0$.

Theorem 2.6. *With the notation as in* 2.4, *let* char $k = 0$. *Then* $R(I)$ *is of strongly F-regular type if and only if every connected subgraph of* $\tilde{E} = \bigcup_{i=1}^{s} E_i$ *contracts to an F-regular type* (*or equivalently, log terminal*) *singularity.*

Outline of the proof. The necessity is already proved in 2.5. We sketch the proof of the sufficiency. Our idea is to consider the "anticanonical ring"

$$R(X, -K_X) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(-nK_X))T^n$$

of X. This is equal to the anticanonical ring $R(\tilde{X}, -K_{\tilde{X}})$ of $\tilde{X}$, since $g \colon \tilde{X} \to X$ is the minimal resolution (cf. 2.1). It is proved that $R(X, -K_X) = R(\tilde{X}, -K_{\tilde{X}})$ is a finitely generated A-algebra and that $Y = \text{Proj } R(X, -K_X)$ is obtained by contracting some subgraph of $\tilde{E} \subset \tilde{X}$ and satisfies the following properties ([Sa]):

1. $R(X, -K_X) = R(Y, -K_Y)$.
2. $-K_Y$ is ample with respect to the natural morphism $Y \to \text{Spec } A$.

Hence we may assume that Y has only log terminal singularities. Also in characteristic $p > 0$, we can apply Proposition 1.5 to prove:

Proposition 2.7. *With the notation as in* 2.4, *let* char $k = p > 0$. *Then* $R(I)$ *is strongly F-regular if and only if the anticanonical ring* $R(X, -K_X)$ *is strongly F-regular.*

This makes the situation better, because $R(X, -K_X)$ is Gorenstein, so that in characteristic zero, $R(X, -K_X)$ has strongly F-regular type if and only if it has log terminal (or equivalently, rational) singularities. ($R(I)$ is not even $\mathbb{Q}$-Gorenstein in general.)

Thus, to complete the proof of Theorem 2.6, it is sufficient to prove that $R := R(X, -K_X) = R(Y, -K_Y)$ has log terminal singularities if so does $Y = \text{Proj } R$. To prove this let $\mathcal{Y} = \text{Spec}_Y(\bigoplus_{n \geq 0} \mathcal{O}_Y(-nK_Y)T^n)$, the "infinite anticanonical covering" of Y, and consider the fundamental diagram for $R = R(Y, -K_Y)$ as in 1.4:

$$\begin{array}{ccccc} S & \hookrightarrow & \mathcal{Y} & \xrightarrow{\varphi} & \text{Spec } R \\ & \searrow & \downarrow \pi & & \downarrow \\ & & Y & \to & \text{Spec } A \end{array}$$

Since R and $\mathcal{Y}$ are Gorenstein and $\varphi \colon \mathcal{Y} \to \text{Spec } R$ is an isomorphism in codimension 1, one has $K_{\mathcal{Y}} = \varphi^* K_R$, so that R is log terminal if and only if so is $\mathcal{Y}$. On the other hand, $\mathcal{Y}$ is log terminal if so is Y, since a finite (anti)canonical covering of Y is a hypersurface in $\mathcal{Y}$ ([E], [KMM], see also [W2]).

Consequently, R is log terminal if so is Y. The theorem is proved. □

2.8. *Remark.* (1) For a fixed prime number $p > 0$, one may ask if the characteristic p version of Theorem 2.6 holds: If char $k = p > 0$ and if every connected subgraph of $\tilde{E}$ contracts to an F-regular singularity, then is $R(I)$ strongly F-regular? We do not know the answer to this question.

(2) Let us say that a resolution graph of a normal surface singularity is a *GFR graph* if every connented subgraph of it contracts to a log terminal singularity. We can check whether or not a given resolution graph is GFR. For example, the graph of the minimal resolution of a log terminal singularity is GFR, while the graph in Example 2.3 is not GFR. Theorem 2.6 asserts that $R(I)$ has strongly F-regular type if and only if I is obtained by $I = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{Z}))$ for some resolution $f\colon \tilde{X} \to \operatorname{Spec} A$ with GFR graph and an anti-(f-)nef exceptional cycle $\tilde{Z}$.

Next we put the assumption that the Rees algebra $R(I)$ is $\mathbb{Q}$-Gorenstein. Note that $\mathbb{Q}$-Gorenstein F-regular rings have log terminal singularities. It turns out that $\mathbb{Q}$-Gorensteinness imposes a very strong restriction on the structure of Rees algebras.

Theorem 2.9. *With the notation as in* 2.4, *let* char $k = p > 0$. *Then the following conditions are equivalent.*

1. *$R(I)$ has log terminal singularities.*
2. *$R(I)$ is Gorenstein and F-regular.*
3. *$(A, \mathfrak{m})$ is a regular local ring and $I = (x, y^m)$ for some regular system of parameters x, y of A and an integer $m \geq 1$.*

Proof. The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are easy.

To prove the implication (1) $\Rightarrow$ (3), let the notation be as in 2.4 and let Z be the ψ-exceptional Cartier divisor with $I\mathcal{O}_X = \mathcal{O}_X(-Z)$. Also, we denote the antidiscrepancies of f and g by $\Delta = f^*K_A - K_{\tilde{X}}$ and $\Delta_g = g^*K_X - K_{\tilde{X}}$, respectively. Then Δ_g an effective divisor since g is the minimal resolution. We preview some results from the next section (Proposition 3.1), which hold true in arbitrary dimension:

(i) If $R(I)$ is $\mathbb{Q}$-Gorenstein, then $\Delta = -g^*Z + \Delta_g$.

(ii) If $R(I)$ is log terminal, then A has a terminal singularity.

Now, if $R(I)$ has log terminal singularities, then A is a regular local ring by (ii), since 2-dimensional terminal singularities are regular. Hence Δ has integer coefficients and we can consider $K_{\tilde{X}} = -\Delta$. Then Δ_g also has integer coefficients by (i).

Let $\mathcal{X} = \operatorname{Spec}_X(\bigoplus_{n\geq 0} \mathcal{O}_X(-nZ)t^n)$ and consider the fundamental diagram for the graded ring $R(I)$ as in 1.4. Since $\pi\colon \mathcal{X} \to X$ has an $\mathbb{A}^1$-bundle structure, $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein as well as X, and X is log terminal if and only if so is $\mathcal{X}$. On the other hand, $\mathcal{X}$ is log terminal if so is $R(I)$, since $\varphi\colon \mathcal{X} \to \operatorname{Spec} R(I)$ is an isomorphism in codimension 1. Hence, if $R(I)$ is log terminal, then so is X, i.e., $\lfloor \Delta_g \rfloor = 0$.

Consequently, our assumption implies that $\Delta_g = \lfloor \Delta_g \rfloor = 0$, so that $-K_{\tilde{X}} = -g^*Z$ is f-nef. Hence every E_i is a (-1)-curve or a (-2)-curve, and we can easily verify that the dual graph of the exceptional divisor $\tilde{E} = \bigcup E_i$ of f must be as follows.

$$\begin{array}{ccccccccc} \circ & - & \circ & - & \cdots & - & \circ & - & \bullet \\ 1 & & 2 & & \cdots & & m-1 & & m \end{array}$$

Here a blank circle $\circ$ (resp. solid circle $\bullet$) denotes a (-2)-curve (resp. (-1)-curve), and the numbers outside of the vertices denote the coefficients of g^*Z. The condition (3) immediately follows from this. $\square$

2.10. *Remark.* We do not have a direct proof of (1) $\Rightarrow$ (2) of Theorem 2.9, but it is likely that $\mathbb{Q}$-Gorensteinness of a Rees algebra implies Gorensteinness under a weak assumption (see the proof of Proposition 3.1). On the other hand, S. Goto and K. Yoshida pointed out that there is a known simple proof of (2) $\Rightarrow$ (3).

3. Miscellaneous results in higher dimension

In this section, we prove miscellaneous results concerning (F-)singularities of Rees algebras of higher dimension.

First we recall the definition of some singularities in characteristic zero. Note that the definition makes sense even in characteristic $p > 0$ if, for example, the singularity under consideration is of dimension 2. A normal $\mathbb{Q}$-Gorenstein variety V is said to have *terminal* (resp. *log terminal*) singularities if for every resolution of singularities $f\colon \tilde{V} \to V$, the coefficient of the discrepancy divisor $K_{\tilde{V}} - f^*K_V$ in each f-exceptional divisor is positive (resp. > -1).

Now we prove the following result, which is used in the previous section.

Proposition 3.1. *Let $(A, \mathfrak{m})$ be a $\mathbb{Q}$-Gorenstein mormal local ring with an isolated singularity, essentially of finite type over a field k, and let I be an $\mathfrak{m}$-primary ideal. Assume that* char $k = 0$ *or* dim $A = 2$. *If the Rees algebra $R(I)$ has only log terminal singularities, then A has only a terminal singularity.*

Proof. Let $\psi\colon X = \operatorname{Proj} R(I) \to \operatorname{Spec} A$ be the blowing-up with respect to I and Z the ψ-exceptional Cartier divisor with $I\mathcal{O}_X = \mathcal{O}_X(-Z)$. Since $R := R(I)$ is $\mathbb{Q}$-Gorenstein, there exist integers $r > 0$ and b such that $\omega_R^{(r)} \cong R(b)$. Hence, comparing

$$\omega_R^{(r)} \cong \bigoplus_{n \geq r} H^0(X, \omega_X^{(r)}(-nZ))t^n$$

with $R(b) = \bigoplus_{n \geq -b} H^0(X, \mathcal{O}_X(-nZ))t^n$, one sees that $r = -b$ and $\omega_X^{(r)} \cong \mathcal{O}_X(rZ)$. So, $K_X \sim_{\mathbb{Q}} Z$ ("$\sim_{\mathbb{Q}}$" denotes $\mathbb{Q}$-linear equivalence), and in particular, X is $\mathbb{Q}$-Gorenstein.

Let $g\colon \tilde{X} \to X$ be a resolution of singularities and let $f = \psi \circ g\colon \tilde{X} \to \operatorname{Spec} A$. We denote the antidiscrepancies of f and g by Δ and Δ_g, respectively. Since $R = R(I)$ is log terminal by the assumption, X is also log terminal, i.e., $\lfloor \Delta_g \rfloor \leq 0$. This follows from an argument using the fundamental diagram (1.4) as in the proof of Theorem 2.9.

Now, it follows from $K_X \sim_{\mathbb{Q}} Z$ that $\Delta \sim_{\mathbb{Q}} \Delta_g - g^*Z$. This implies $\Delta = \Delta_g - g^*Z$, since the both sides are f-exceptional. Hence $\lfloor \Delta \rfloor = \lfloor \Delta_g \rfloor - g^*Z \leq -g^*Z$, since g^*Z has integer coefficients. Since A has an isolated singularity, g^*Z is supported on the exceptional set of f, so the above inequality implies that the coefficient of Δ in every irreducible f-exceptional divisor is negative. Hence A has a terminal singularity. $\square$

3.2. *Remark.* Proposition 3.1 has a partial converse as follows: If $(A, \mathfrak{m})$ is a 3-dimensional Gorenstein terminal singularity of characteristic zero, then the Rees algebra $R(\mathfrak{m})$ has Gorenstein rational (hence log terminal) singularities. We ask if a similar implication for "F-singularities" holds in fixed characteristic $p > 0$. Namely, if $(A, \mathfrak{m})$ is a 3-dimensional Gorenstein F-terminal singularity of characteristic $p > 0$, then is $R(\mathfrak{m})$ F-rational? See [HYW2] for details, and see also [W4] for F-terminal rings.

Let $(A, \mathfrak{m})$ be a local ring and I an $\mathfrak{m}$-primary ideal. Recall that the extended Rees algebra of I is defined to be $R'(I) = A[It, t^{-1}]$. In [HYW1], it is proved that if the Rees algebra $R(I)$ and A are both F-rational (of characteristic $p > 0$), then so is the extended

Rees algebra $R'(I)$, too. This leads us to the following theorem in characteristic zero, since rational singularity is equivalent to F-rational type, and since the rationality of $R(I)$ implies the rationality of A by Boutot's theorem [B].

Theorem 3.3. *Let $(A, \mathfrak{m})$ be a local ring essentially of finite type over a field of characteristic zero and let I be an $\mathfrak{m}$-primary ideal. Then the Rees algebra $R(I)$ has rational singularities if and only if the extended Rees algebra $R'(I)$ has rational singularities.*

We shall give an alternative proof of this theorem with geometric flavor. The proof needs the following lemma, which is proved for rational singularities in characteristic zero by Bingener and Storch [BS], and also for F-rational rings in characteristic $p > 0$ by using the tight closure technique ([HYW2]).

Lemma 3.4. *Let $(A, \mathfrak{m})$ be a local ring essentially of finite type over a field and let $0 \neq f \in \mathfrak{m}$. If A is (F-)rational, then $B = A[x, y]/(xy - f)$ is also (F-)rational.*

Proof of Theorem 3.3. The sufficiency is easy by Boutot's theorem [B]. So let us prove the necessity. Let $\mathcal{O}_X(n) = R(I)(n)^\sim$ on $X = \text{Proj } R(I)$, $Y = \text{Spec}_X(\bigoplus_{n\geq 0} \mathcal{O}_X(n)t^n)$ and let $Y' = \text{Spec}_X\left(\bigoplus_{n\geq 0} \mathcal{O}_X(n)t^n \oplus \bigoplus_{n<0} \mathcal{O}_X t^n\right)$. (Note that we changed the notation partly from the previous sections.) Then the fundamental diagram in 1.4 is extended as follows.

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & \text{Spec } R'(I) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\varphi} & \text{Spec } R(I) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & \text{Spec } A \end{array}$$

Here, φ', φ, ψ are a birational projective morphisms. On the other hand, if we write $\mathcal{O}_X(1)|_U = f\mathcal{O}_U$ for an affine open subset U of X, then $\mathcal{O}_{Y'}|_U = \mathcal{O}_U[ft, t^{-1}] \cong \mathcal{O}_U[x, y]/(xy - f)$. Hence by Lemma 3.4, Y' has rational singularities if so does X.

Now, since $R(I)$ has rational singularities, so does $X = \text{Proj } R(I)$ by [B]. Then Y has rational singularities by [E], and Y' has also rational singularities by the above argument. We have $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$, since Spec $R(I)$ and Y have rational singularities. Hence $H^i(X, \mathcal{O}_X(n)) = 0$ for $i > 0$ and $n \geq 0$, so that $H^i(Y', \mathcal{O}_{Y'}) = 0$.

Consequently, we have $R^i\varphi'_*\mathcal{O}_{Y'} = 0$ for $i > 0$, where $\varphi'\colon Y' \to \text{Spec } R'(I)$ is a birational projective morphism from a variety Y' with only rational singularities. Thus we conclude that $R'(I)$ has rational singularities. □

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Transcendence Degree of a Domain over a Subfield

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The following theorem is well-known.

Theorem 1 *Let k be a field and $k[\underline{a}] = k[a_1, \ldots, a_n]$ an affine domain over k. Then* $\dim k[\underline{a}] = \operatorname{tr.deg}_k k[\underline{a}]$.

Several people tried to generalize this theorem. For example, the following theorems were shown.

Theorem 2 ([4]) *For any subalgebra R with $k \subseteq R \subseteq k[\underline{a}]$,* $\dim R = \operatorname{tr.deg}_k R$.

Theorem 3 ([2]) *For any subalgebra R with $k \subseteq R \subseteq k[\underline{a}]_M$ where $M \in \operatorname{Max} k[\underline{a}]$,* $\dim R = \operatorname{tr.deg}_k R$.

We want to make the same attempt as these theorems. For example, can we change $k[\underline{a}]_M$ to $k[\underline{a}]_P$ where $P \in \operatorname{Spec} k[\underline{a}]$? I was interested in this problem.

Considering the transcendence degree of $k[\underline{a}]_P$, we have to extend the field k. From this reason, we defined before new transcendence degree as follows. For a k-domain R with $\operatorname{tr.deg}_k R < \infty$, we call $\min\{\operatorname{tr.deg}_K R \mid K$ is a subfield of R with $k \subseteq K.\}$ *the transcendence degree of R with respect to k*, and express it by $\operatorname{td}_k R$. And we call a subfield K of R such that $k \subseteq K$ and $\operatorname{td}_k R = \operatorname{tr.deg}_K R$ *a transcendentally maximal subfield* (*a tm-subfield*, for short) *of R over k.*

1. We asked before whether the following equation holds:

$$\operatorname{td}_k R = \max\{\operatorname{td}_k R_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max} R\}.$$

In other words, we asked whether $\mathrm{td}_k R$ has a local property or not. It was shown that if R is semilocal, the answer is affirmative, but in general, I didn't know whether the equation holds or not. But as I have got a counterexample to this problem after that, I show this first.

Example 1 *There is a regular domain R of $\dim R = 1$ containing a field k such that $\mathrm{td}_k R > \max\{\, \mathrm{td}_k R_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}\, R \,\}$.*

Construction of R Let k be any algebraically closed field and X, Y variables over k. For any $a \in k$, we put $f_a(X,Y) = X^3Y^3 + aX^2 + X + a^2Y^2 + Y + a^3$, and for the multiplicatively closed subset S generated by $\{\ f_a(X,Y) \mid a \in k\ \}$, we set $R = S^{-1}k[X,Y]$. Then R is the example which we want.

To show the reason, we first note that $f_a(X,Y)$ is irreducible over k for any $a \in k$, and for any $\mathfrak{m} \in \mathrm{Max}\, k[X,Y]$, there is some $a \in k$ such that $f_a(X,Y) \in \mathfrak{m}$. Therefore $\dim R \leq 1$. On the other hand, since we can show that k is a tm-subfield of R, the proof of which is complicated a little and omitted, we have $\mathrm{td}_k R = \mathrm{tr.deg}_k R = 2$ and $\dim R = 1$.

Now we remember the following theorem which I stated in [5].

Theorem 4 *Let A be a Noetherian domain containing a field k with $\mathrm{tr.deg}_k A < \infty$. Then if $\dim A_{\mathfrak{m}} = \mathrm{td}_k A_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Max}\, A$, the following conditions are equivalent.*

(1) *A is catenary.*

(2) *For any $\mathfrak{m} \in \mathrm{Max}\, A$ and $\mathfrak{p} \in \mathrm{Spec}\, A$ with $\mathfrak{p} \subseteq \mathfrak{m}$,*
   - (a) $\dim A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}} = \mathrm{td}_k A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}$, *and*
   - (b) $\dim A_{\mathfrak{p}} = \mathrm{td}_k A_{\mathfrak{p}}$.

By Theorem 4, $\dim R_{\mathfrak{m}} = \mathrm{td}_k R_{\mathfrak{m}} = 1$ for any $\mathfrak{m} \in \mathrm{Max}\, R$. Hence $\max\{\, \mathrm{td}_k R_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}\, R \,\} = 1 < \mathrm{td}_k R$. □

2. Next, for generalizing Theorem 1, Theorem 2 and Theorem 3, we consider the following situation:

$k \subseteq (R, \mathfrak{m}) \subseteq (A, \mathfrak{n})$, where k is a field, $(R, \mathfrak{m})$ is a quasi-local domain and $(A, \mathfrak{n})$ is a Noetherian local domain with $\dim A = \mathrm{td}_k A$ dominating R.

Then we ask whether the equation $\dim R = \mathrm{td}_k R$ holds.

We consider two cases, that is,

$$(a) \ R \text{ is Noetherian, } \quad \text{and} \quad (b) \ R \text{ is not Noetherian.}$$

For the case (a), I talked before. The answer is affirmative. In this case, A can be easily generalized to the case that it may not be Noetherian.

On the other hand, R cannot be generalized to the non-local case as follows.

Example 2 *There are a Noetherian non-local semi-local domain R containing a field k and a Noetherian local domain $(A, \mathfrak{n})$ with $R \subseteq A$ and $\mathfrak{n} \cap R \in \mathrm{Max}\, R$ which satisify that* $\dim A = \mathrm{td}_k A$ *and* $\dim R < \mathrm{td}_k R$.

We use Nagata's example (cf. [3] Example 2 in Appendix). □

For the case (b), the answer is negative in general as follows.

Example 3 *There are a DVR A and a quasi-local ring R containing a field k with $R \subseteq A$ which satisfy that* $\dim A = \mathrm{td}_k A$ *and* $\dim R < \mathrm{td}_k R$.

Construction of R and A Let k be any field and X, Y variables over k. Set $S = \{ \alpha X \mid \alpha \in k(Y) \}$, then $k \subset k[S] \subset k(Y)[X]$ and $Xk(Y)[X] \cap k[S] = Sk[S]$. Hence $k \subset k[S]_{(S)} \subset k(Y)[X]_{(X)}$, and set $R = k[S]_{(S)}$, $\mathfrak{m} = (S)R$, $A = k(Y)[X]_{(X)}$ and $\mathfrak{n} = XA$. Then $(R, \mathfrak{m})$, $(A, \mathfrak{n})$ are the example which we want.

Because we have that $\dim A = \mathrm{td}_k A = \mathrm{tr.deg}_{k(Y)} A = 1$ and k is a tm-subfield of R. For any $\alpha \in k(Y)$, we have $(\alpha X)^2 = X(\alpha^2 X) \in XR$, so that $\sqrt{XR} = \mathfrak{m}$. Therefore for any $\mathfrak{p} \in \mathrm{Spec}\, R$ with $\mathfrak{p} \subset \mathfrak{m}$, we have $X \notin \mathfrak{p}$. Hence $\mathfrak{p}R[1/X]$ is a proper ideal of $R[1/X]$. Now for any $f \in \mathfrak{n} \cap k[S]$ with $f \neq 0$, we can write $f = \sum_{i=k}^{n} a_i X^i$ $(\forall a_i \in k(Y)\, ,\ a_k \neq 0)$. Then

$$f = a_k X^k (1 + \sum (a_i/a_k) X^{i-k}) \ , \ a_i/a_k \in k(Y).$$

$1+\sum(a_i/a_k)X^{i-k}$ is a unit in R, and by $R[1/X] \supset k(Y)$, a_kX^k is a unit in $R[1/X]$. Hence f is also a unit in $R[1/X]$, so that $\mathfrak{p}=(0)$. Therefore $\dim R = 1 < \mathrm{td}_k R = \mathrm{tr.deg}_k R = 2$. □

On the other hand, if k is a tm-subfield of A and A is catenary, the answer is affirmative, where R need not be local.

Theorem 5 *Let $(A,\mathfrak{n})$ be a Noetherian catenary local domain and R a quasi-local domain containing a field k where k is a tm-subfield of A and* $\dim A = \mathrm{tr.deg}_k A$. *Then $\mathfrak{n}\cap R$ is a maximal ideal of R and* $\dim R = \mathrm{tr.deg}_k R$.

Proof. By Theorem 4, we have $\dim A/\mathfrak{p} = \mathrm{tr.deg}_k A/\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} A$. Therefore we can prove this theorem by the same way as the proof of Theorem 3 which uses Alamelu's theorem (cf. [1]). □

Remark. Let ℓ be a field with $\ell \subseteq k$ and A a localization of an affine domain $\ell[\underline{a}] = \ell[a_1,\dots,a_n]$ over ℓ by a prime ideal P of $\ell[\underline{a}]$. Then it seems that we have a generalization of Theorem 3. But this case arrives at that of Theorem 3. In fact, set $M = P \cap k[\underline{a}]$, then we have $A = k[\underline{a}]_M$. By Theorem 4 we find that $M \in \operatorname{Max} k[\underline{a}]$.

3. For variables $X_1,\dots,X_n$ ($n \in \mathbb{N}$), let M be a semigroup generated by $X_1^{q_1}\cdots X_n^{q_n}$ ($\forall q_i \in \mathbb{Q}$). Then we show that $\dim k[M] = \mathrm{tr.deg}_k k[M]$ for any field k. In particular, we construct a maximal saturated chain of prime ideals. First of all, we define X_i^q for any $q \in \mathbb{Q}$.

Let $T = \{\ X_{iq} \mid i=1,\dots,n\,,\ q \in \mathbb{Q}\ \}$ be a set of variables over k, $G = \langle T\rangle$ a free Abelian group generated by T and $H = \langle X_{iq}^u X_{ir}^{-v} \mid u,v \in \mathbb{Z}\,,\ uq = vr\rangle$ a subgroup of G. I_n denote G/H. Then for any $n \in \mathbb{Z}$, $(X_{iq})^n = X_{i(qn)}$ in I_n. Hence in I_n we express X_{iq} by X_i^q for any $q \in \mathbb{Q}$. Then $I_n = \langle X_i^q \mid i=1,\dots,n\,,\ q\in\mathbb{Q}\rangle$. Then we have the following theorem.

Theorem 6 *Let M be a subsemigroup of I_n. Then* $\dim k[M] = \mathrm{tr.deg}_k k[M]$, *and we can construct a maximal saturated chain of prime ideals. Moreover if $M \subseteq \langle X_i^q \mid i=1,\dots,n\,,\ q\in\mathbb{Q}\,,\ q>0\rangle$, we can construct such a maximal saturated chain descending from $Mk[M]$.*

The proof of Theorem 6 is the same as that in [5].

At the last of my report, we state the following example.

Example 4 *Let k be any field and M a semigroup generated by $\{ XY^q \mid q \in \mathbb{Q}, q \geq 0 \}$ where X, Y are variables over k. Then* $\dim k[M] = \text{tr.deg}_k\, k[M] = 2$, *and we can construct a maximal saturated chain of prime ideals descending from (M) as follows:*

$$(0) \subset (\{ XY^{j+1} - X^2Y^j \mid j \in \mathbb{Q}, j \geq 0 \}) \subset (M)$$

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On Degenerations of Cohen-Macaulay Modules [1]

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1 λ and ρ functions

Let R be a noetherian ring, and we denote by $\mathcal{L}_R$ the set of isomorphism classes of R-modules of finite length. For a given f.g.R-module M, we consider the functions on $\mathcal{L}_R$:

$$\lambda_M(Y) := \text{length}_R(\text{Hom}_R(M,Y)), \quad \rho_M(Y) := \text{length}_R(M \otimes_R Y) \quad \text{for } Y \in \mathcal{L}_R.$$

The first theorem states that these functions determines the local-isomorphism class of a module M.

Theorem 1.1 *The following conditions are equivalent for f.g.R-modules M and N.*
(1) $\lambda_M(Y) = \lambda_N(Y)$ *for any* $Y \in \mathcal{L}_R$.
(2) $\rho_M(Y) = \rho_N(Y)$ *for any* $Y \in \mathcal{L}_R$.
(3) $M_\mathfrak{p} \cong N_\mathfrak{p}$ *for any* $\mathfrak{p} \in \text{Spec}(R)$ *(i.e. M and N are locally isomorphic.)*

This is a generalization of a theorem of Auslander who proved the theorem in artinian case (but non-commutative). See [1]. At this moment I can prove the theorem only in a fairly non-standard way. Actually I need some technics of separated ultraproducts of modules that was developed in [4].

2 Several orderings

In the following we assume that $(R, \mathfrak{m}, k)$ is a local ring. Then Theorem 1.1 says that the λ (or ρ) function determines the isomorphism classes of f.g.modules, but moreover, using the λ function, we can measure the largeness of a module.

[1]This is not the final version of the paper. The final and detailed version will be submitted elsewhere.

Definition 2.1 ("hom "ordering)
For f.g.R-modules M and N, we denote $M \leq_{hom} N$ if $\lambda_M(Y) \leqq \lambda_N(Y)$ for any $Y \in \mathcal{L}_R$.

Theorem 1.1 says that this gives a well-defined partial order on the set of isomorphism classes of f.g.R-modules. It is easy to see that $M \leq_{hom} N$ iff $\rho_M(Y) \leqq \rho_N(Y)$ for any $Y \in \mathcal{L}_R$.

My motivation of this work is to describe this partial ordering for maximal Cohen-Macaulay modules. Before proceeding this we remark that the hom ordering is related to the degeneration problem.

Definition 2.2 ("ext "and "EXT "ordering)
(1) One says that M splittingly degenerates to N if there is an exact sequence $0 \to N_1 \to M \to N_2 \to 0$ such that $N \cong N_1 \oplus N_2$.
(2) $M \leq_{ext} N$ if there is a sequence of splitting degenerations $M = M_0, M_1, \dots, M_n = N$ (i.e. each M_i splittingly degenerates to M_{i+1}).
(3) $\leq_{EXT}$ is a partial order generated by the following rules:
(a) $M \leq_{ext} N \Longrightarrow M \leq_{EXT} N$,
(b) $M \leq_{EXT} N \Longleftrightarrow M \oplus L \leq_{EXT} N \oplus L$,
(c) $M \leq_{EXT} N \Longleftrightarrow M^{(n)} \leq_{EXT} N^{(n)}$.

Definition 2.3 ("deg "and "DEG "ordering)
Suppose that R contains a coefficient field k that is an algebraically closed field.
(1) One says that M degenerates to N if there is a f.g.$R[t]$-module Q which is $k[t]$-flat such that, writing $Q_c := Q \otimes_{k[t]} k[t]/(t-c)$ for $c \in k$, we have $Q_c \cong M$ if $c \neq 0$ and $Q_0 \cong N$.
(2) $M \leq_{deg} N$ if there is a sequence of degenerations $M = M_0, M_1, \dots, M_n = N$ (i.e. each M_i degenerates to M_{i+1}).
(3) $\leq_{DEG}$ is a partial order generated by the following rules:
(a) $M \leq_{deg} N \Longrightarrow M \leq_{DEG} N$
(b) $M \leq_{DEG} N \Longleftrightarrow M \oplus L \leq_{DEG} N \oplus L$
(c) $M \leq_{DEG} N \Longleftrightarrow M^{(n)} \leq_{DEG} N^{(n)}$

One should notice that if M and N are comparable in one of the orders $\leq_{ext}$, $\leq_{EXT}$, $\leq_{deg}$, $\leq_{DEG}$, then M and N have the same multiplicity (or rank). We can easily prove the following implications.

Propositon 2.4 $M \leq_{EXT} N \;\Longrightarrow\; M \leq_{DEG} N \;\Longrightarrow\; M \leq_{hom} N$

3 AR ordering and a main theorem

In the following we assume that R is a Cohen-Macaulay complete local ring with only isolated singularity and we denote by CM(R) the category of maximal Cohen-Macaulay modules over R. In this case it is known that CM(R) admits Auslander-Reiten sequences. See Yoshino [5] for more detail.

Definition 3.1 (AR ordering)
We define the order $\leq_{AR}$ on $\mathrm{CM}(R)$ as the partial order generated by the following rules:

(a) If $0 \to \tau X \to E \to X \to 0$ is an AR-sequence in $\mathrm{CM}(R)$, then $E \leq_{AR} X \oplus \tau X$.

(b) $M \leq_{AR} N \Longleftrightarrow M \oplus L \leq_{AR} N \oplus L$,

(c) $M \leq_{AR} N \Longleftrightarrow M^{(n)} \leq_{AR} N^{(n)}$.

It is obvious that $M \leq_{AR} N \implies M \leq_{EXT} N$. If $\mathrm{CM}(R)$ contains only a finite number of indecomposable Cohen-Macaulay modules, then, since there are only a finite number of AR-sequences, easily in a combinatorial way we can describe the poset structure of $\mathrm{CM}(R)$ in AR-ordering.

The following theorem is the main result of this note.

Theorem 3.2 *Let R be a Cohen-Macaulay complete local ring that is of finite Cohen-Macaulay representation type. And suppose one of the following conditions:*

(1) R is an integral domain of dimension one.

(2) R is of dimension two.

Then, for any M and N in $\mathrm{CM}(R)$ with the same rank, we have the following equivalences.

$$M \leq_{AR} N \iff M \leq_{EXT} N \iff M \leq_{DEG} N \iff M \leq_{hom} N$$

When R is a finite dimensional (noncommutative) algebra, several results similar to our theorem has been known by Bongartz. See [2, 3]. But in a proof of our theorem, one cannot use his method because of the difference of nature between the Auslander-Reiten quivers of Cohen-Macaulay modules and that of finite dimensional algebras. To prove the theorem, it is necessary to have a lot of informations concerning the Cohen-Macaulay approximations of a module of finite length.

For several one-dimensional non-domain cases I can verify the validity of the theorem. But at this moment I have no general proof for this even in dimension one. Therefore it seems to be natural to propose the following

Conjecture 3.3 *The theorem would be valid without any assumption on R but when R is a Cohen-Macaulay local ring of finite Cohen-Macaulay representation type.*

If R is of infinite Cohen-Macaulay representation type, then there is an example that fails the implication $M \leq_{EXT} N \implies M \leq_{AR} N$.

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Stable module therory with kernels

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1 Preliminary

We shall remention the stable module theory of Auslander and Bridger. We extend the notion of stable equivalence, which is the only difference between their original terminology. All the other notions are in [1] and [?].

Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring, and mod R the category consisting of finitely generated R-modules. The projective stabilization $\underline{mod\ R}$ is defined as follows.

- Each object of $\underline{mod\ R}$ is an object of *mod R*.
- For $A, B \in mod\ R$, a set of morphisms from A to B is $\mathrm{Hom}_R(A,B)/\mathcal{P}(A,B)$ where $\mathcal{P}(A,B) := \{f \in \mathrm{Hom}_R(A,B) \mid f$ factors through some projective module$\}$. Each element is denoted as $\underline{f} = f\ mod\ \mathcal{P}(A,B)$.

Definition 1.1 *If $f, f' \in \mathrm{Hom}_R(A,B)$ and $g, g' \in \mathrm{Hom}_R(B,C)$ satisfies $\underline{f} = \underline{f'}$ and $\underline{g} = \underline{g'}$, then $\underline{g \circ f} = \underline{g' \circ f'}$. With this in mind, we may define the composite $\underline{g} \circ \underline{f} = \underline{g \circ f}$.*

Definition 1.2 *Assume $\underline{a} \in \underline{\mathrm{Hom}_R}(A, A')$ is an isomorphism in $\underline{mod\ R}$, that is, $\underline{\alpha} \in \underline{\mathrm{Hom}_R}(A', A)$ exists to make $\underline{a \circ \alpha} = \underline{1_{A'}}$ and $\underline{\alpha \circ a} = \underline{1_A}$. Then we say that $\underline{a}$ is a stable isomorphism, and A is stably equivalent to A' ($A \overset{st}{\cong} A'$).*

A stable module refers to a module without free summand.

Lemma 1.3 (Auslander-Bridger [2]) *The following are equivalent for $\underline{a} \in \mathrm{Hom}_R(A, A')$.*

1) $\underline{a}$ *is a stable isomorphism.*

2) *There exists an isomorphism* $\tilde{a} \in \mathrm{Hom}_R(A \oplus P', A' \oplus P)$ *with some projective modules* P *and* P' *such that* $\underline{t \circ \tilde{a} \circ s} = \underline{a}$ *with some split monomorphism* $s : A \to A \oplus P'$ *and split epimorphism* $t : A' \oplus P \to A'$.

If R *is complete,* A *and* A' *have unique decompositions* $A = A_s \oplus A_p$ *and* $A' = A'_s \oplus A'_p$ *with stable modules* A_s *and* A'_s *and projective ones* A_p *and* A'_p. *Whence we have following condition that is equivalent to* (1) *and* (2).

3) a_{11} *is an isomorphism in mod* R *where*

$$a : A_s \oplus A_p \overset{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}{\longrightarrow} A'_s \oplus A'_p.$$

Here we shall review on the notion of Auslander's transpose which is the functor on $\underline{mod\ R}$. The transpose of modules and morphisms are defined as stable equivalence classes and not as objects nor morphisms of *mod R*. (See [1] (2.5) and (2.6).) Although for the consistency, we would start with defining transpose on *mod R*.

Definition 1.4 *For an R-module M, we define the transpose of M as an R-module* $tr\ M = \mathrm{Coker}\, d_{F_{M1}}{}^*$ *where* $F_{M1} \overset{d_{F_{M1}}}{\to} F_{M0} \to M$ *is the minimal projective presentation of* M.

Since we assume $d_{F_{M1}} \otimes k = 0$, $tr\ M$ is always a stable module; transpose of a projective module is zero. We also observe that $tr\ tr\ M \overset{st}{\cong} M$.

A homomorphism $f \in \mathrm{Hom}_R(A, B)$ induces a chain map $f_\bullet : P_{A\bullet} \to P_{B\bullet}$ where $P_{A\bullet}$ and $P_{B\bullet}$ are minimal free resolutions of A and B respectively. The R-dual $f_\bullet{}^* : P_{B\bullet}{}^* \to P_{A\bullet}{}^*$ again induces a homomorphism $tr\ B \to tr\ A$ which we call $tr\ f$. The map $tr\ f$ is not uniquely determined by f; it depends on the way of lifting up to $f_\bullet$ or projecting down from $f^*{}_\bullet$. Nevertheless we have $\underline{tr\ f} = \underline{tr\ f'}$ for $(tr\ f)'$ induced from f in another way.

Remark 1.5 1) *If* $\underline{f} = \underline{f'}$ *for* $f, f' \in \mathrm{Hom}_R(A, B)$, *then* $\underline{tr\ f} = \underline{tr\ f'}$.

2) $\underline{tr(tr\ f)} = \underline{f}$.

Definition 1.6 *For* $\underline{f} \in \underline{\mathrm{Hom}_R}(A, B)$, *we define* $\underline{tr}\underline{f} := \underline{tr\ f} \in \underline{\mathrm{Hom}_R}(tr\ B, tr\ A)$.

2 kernels and cokernels

Although *mod R* is not an abelian category, we have "something like" kernels of the morphisms.

Lemma 2.1 *[4] For $f \in \mathrm{Hom}_R(A,B)$, there exists a map $\rho \in \mathrm{Hom}_R(P,B)$ from a projective module P such that $\tilde{f} : A \oplus P \xrightarrow{(f\ \rho)} B$ is surjective. Moreover, if $f' \in \mathrm{Hom}_R(A,B)$ with $\underline{f'} = \underline{f}$ and $\rho' \in \mathrm{Hom}_R(P',B)$ with a projective module P' make $\tilde{f'} : A \oplus P' \xrightarrow{(f'\ \rho')} B$ surjective, then $\mathrm{Ker}\,\tilde{f} \overset{st}{\cong} \mathrm{Ker}\,\tilde{f'}$. Hence we would denote $\mathrm{Ker}\,\tilde{f}$ by $\underline{\mathrm{Ker}\,f}$, which is uniquely determined up to stable equivalence.*

Let f, $\tilde{f}$ and P be as in Lemma 2.1. Consider the natural inclusion

$$\mathrm{Ker}\,\tilde{f} \overset{\binom{n_f}{q_f}}{\to} A \oplus P.$$

Together with $\underline{n_f}$, $\mathrm{Ker}\,\tilde{f}$ has the following properties.

Lemma 2.2 *Let the notation as above. We have the following:*

1) $\underline{f} \circ \underline{n_f} = \underline{0}$.

2) *If $\underline{x} \in \underline{\mathrm{Hom}_R}(X,A)$ satisfies $\underline{f} \circ \underline{x} = \underline{0}$, there exists $\underline{h_x} \in \underline{\mathrm{Hom}_R}(X, \mathrm{Ker}\,\tilde{f})$ such that $\underline{x} = \underline{n_f} \circ \underline{h_x}$.*

Proof. We may assume $\tilde{f} = (f\ \rho_B)$. Since $f \circ n_f + \rho_B \circ q_f = 0$, $\underline{f} \circ \underline{n_f} = \underline{0}$. Provided $f \circ x = \rho_X \circ u$ for some $u \in \mathrm{Hom}_R(X, P_B)$, that is, $(f\ \rho_B) \circ \binom{x}{u} = 0$. Then we have some $h \in \mathrm{Hom}_R(X, \mathrm{Ker}\,\tilde{f})$ such that $\binom{x}{u} = \binom{n_f}{q_f} \circ h$, which implies $\underline{n_f} = \underline{g} \circ \underline{h}$. (q.e.d.)

Strictly speaking, $\mathrm{Ker}\,\tilde{f}$ is not the kernel of $\underline{f}$. Because it lacks the uniqueness of h_x in 2) of Lemma 2.2. (See Example 2.3.)

Example 2.3 *Let $R = k[[x,y,z]]/(x^2 - yz)$, $A = R/(yz)$ and $B = R/(yz, y^2, z^2)$. Let $f : A \twoheadrightarrow B$ be the natural map induced from the inclusion $(yz) \subset (yz, y^2, z^2)$. Since f is surjective, $\underline{\mathrm{Ker}\,f} \overset{st}{\cong} \mathrm{Ker}\,f \cong R/(z) \oplus R/(y)$,* *and the sequence* $0 \to \operatorname{Ker} f \xrightarrow{n_f} A \xrightarrow{f} B \to 0$ *is exact. Put* $X = tr\,k$ *and let* $u \in \operatorname{Hom}_R(X, \operatorname{Ker} f)$ *be as follows:*

$$\begin{array}{ccccccccc} 0 & \to & R & \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} & R^3 & \to & X & \to & 0 \\ & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & & \downarrow u & & \\ 0 & \to & R^2 & \xrightarrow{\begin{pmatrix} z & 0 \\ 0 & y \end{pmatrix}} & R^2 & \to & \operatorname{Ker} f & \to & 0. \end{array}$$

Easily we get $\underline{n_f} \circ \underline{u} = \underline{0}_A = \underline{u_f} \circ \underline{0}_K$ *where* $\underline{0}_A = \underline{0} \in \underline{\operatorname{Hom}}_R(X, A)$ *and* $\underline{0}_K = \underline{0} \in \underline{\operatorname{Hom}}_R(X, \operatorname{Ker} f)$. *Also we have* $\underline{u} \neq \underline{0}_K$ *after tedious calculation.*

Definition and Lemma 2.4 *For* $\underline{f} \in \underline{\operatorname{Hom}}_R(A, B)$, *we define* $\underline{\operatorname{Coker} f} \in \underline{mod\ R}$ *as* $\underline{\operatorname{Coker} f} := tr\underline{\operatorname{Ker}\ tr\ f}$. *If we put* $c_f := \underline{tr n_{\underline{tr f}}}$, $(\underline{\operatorname{Coker} f}, \underline{c_f})$ *satisfies the following.*

1) $\underline{c_f} \circ \underline{f} = \underline{0}$.

2) *If* $\underline{y} \in \underline{\operatorname{Hom}}_R(B, Y)$ *satisfies* $\underline{y} \circ \underline{f} = \underline{0}$, *there exists* $\underline{e_y} \in \underline{\operatorname{Hom}}_R(\underline{\operatorname{Coker} f}, Y)$ *such that* $\underline{y} = \underline{e_y} \circ \underline{c_f}$.

Two modules $\operatorname{Ker} f$ and $\underline{\operatorname{Ker} f}$ are not always stably isomorphic. More precisely, we get the following.

Lemma 2.5 1) *There is an exact sequence* $0 \to L \to M \to N \to 0$ *such that* $L \overset{st}{\cong} \operatorname{Ker} f$, $M \overset{st}{\cong} \underline{\operatorname{Ker} f}$ *and* $N \overset{st}{\cong} \Omega^1_R(\operatorname{Coker} f)$.

2) *There is an exact sequence* $0 \to L' \to M' \to N' \to 0$ *such that* $M' \overset{st}{\cong} \underline{\operatorname{Coker} f}$, $N' \overset{st}{\cong} \operatorname{Coker} \underline{f}$ *and* $\Omega^1_R(L')$ *is the surjective image of* $\operatorname{Ker} f$.

Corollary 2.6 1) $\underline{\operatorname{Ker} f} \overset{st}{\cong} \operatorname{Ker} f$ *if* f *is surjective.*

2) $\underline{\operatorname{Coker} f} \overset{st}{\cong} \operatorname{Coker} f$ *if* f *is injective.*

3) $\underline{\operatorname{Coker} f} \overset{st}{\cong} \Omega^{-1}_R(\operatorname{Ker} f)$ *if* $\operatorname{Ext}^1_R(A, R) = 0$.

proof. We shall prove only 3). If $\mathrm{Ext}^1_R(A, R) = 0$, then $\check{f}$ is injective. We have $\Omega^1_R(L') \cong \mathrm{Ker}\, f$ since $\mathrm{Im}\, \check{f} \cong A$. (q.e.d.)

In $\underline{mod\ R}$, by definition, a morphism $\underline{f}$ is injective if and only if $\underline{n_f} = 0$. Hence $\underline{f}$ is injective if $\underline{\mathrm{Ker}\, f} \overset{st}{\cong} 0$. But vanishing of $\underline{\mathrm{Ker}\, f}$ is not a necessary condition for $\underline{f}$ to be injective.

Example 2.7 *Let A,B be two modules with* $\mathrm{pd}(B) \geq 2$. *Let f be a split monomorphism $A \to A \oplus B$. Obviously $\underline{n_f} = 0$ but $\underline{\mathrm{Ker}\, f} \overset{st}{\cong} \Omega^1_R(B)$ is not projective.*

However we have the following:

Proposition 3.2 Suppose $\mathrm{Ext}^1_R(B, R) = 0$. The following are equivalent for $f \in \mathrm{Hom}_R(A, B)$.

1) f is a stable isomorphism.
2) $\underline{n_f} = 0$ and $\underline{c_f} = 0$.
3) $\underline{\mathrm{Coker}\, f} \overset{st}{\cong} 0$ and $\underline{\mathrm{Ker}\, f} \overset{st}{\cong} 0$.

To show this, we need to discuss the complexes.

3 Homotopy classes of complexes

Let A, B be R-modules. Let $F_{A\bullet}$ and $F_{B\bullet}$ be free complexes such that $\tau_0 F_{A\bullet}$, $\tau_0 F_{B\bullet}$, $\tau_1 F_{A\bullet}^*$ and $\tau_1 F_{B\bullet}^*$ are free resolutions of $A, B, \mathrm{tr}\, A$ and $\mathrm{tr}\, B$ respectively. That is, we have exact sequences $\cdots \to F_{A1} \overset{d_{F_{A1}}}{\to} F_{A0} \to$, $0 \leftarrow \mathrm{tr}\, A \leftarrow F_{A1}^* \leftarrow F_{A0}^* \leftarrow \cdots$ and so on. We call $F_{A\bullet}$ a two-sided resolution of A. A homomorphism $f \in \mathrm{Hom}_R(A, B)$ induces a chain map $f_\bullet : F_{A\bullet} \to F_{B\bullet}$. It is not hard to see

$$\underline{\mathrm{Ker}\, f} \overset{st}{\cong} \mathrm{Coker}\, d_{Cone\ (f_\bullet)_1} = \mathrm{Coker}\ \begin{matrix} \\ F_{B1} \\ F_{A0} \end{matrix} \begin{matrix} F_{B2} & F_{A1} \\ \left(\begin{matrix} d_{F_{B2}} \\ 0 \end{matrix}\right. & \left.\begin{matrix} f_1 \\ -d_{F_{A1}} \end{matrix}\right) \end{matrix} \tag{3.1}$$

and

$$\underline{\mathrm{Coker}\, f} \overset{st}{\cong} \mathrm{Coker}\, d_{Cone\ (f_\bullet)_0} = \mathrm{Coker} \begin{array}{c} \\ F_{B0} \\ F_{A-1} \end{array} \begin{array}{c} \begin{array}{cc} F_{B1} & F_{A0} \end{array} \\ \begin{pmatrix} d_{F_{B1}} & f_0 \\ 0 & -d_{F_{A0}} \end{pmatrix} \end{array} \qquad (3.2)$$

These formulae are not surprising because the category of homotopy classes of R-complexes is triangulated. Putting $C_\bullet = Cone(f_\bullet)_{\bullet+1}$, we have a diagram of chain complexes

$$C_\bullet \overset{n_\bullet}{\to} F_{A\bullet} \overset{f_\bullet}{\to} F_{B\bullet} \overset{c_\bullet}{\to} C_{\bullet-1}$$

(For the definition of triangulated category, see [3].) First of all, it is easy to show the following (essentially [5].)

Lemma 3.1 *With notations as above,* $\underline{f} = 0$ *if and only if* $f_\bullet$ *is homotopic to zero.*

Lemma 3.1 is valid only for chain maps between two-sided resolutions. Notice that $C_\bullet = Cone(f_\bullet)_{\bullet+1}$ is not a two-sided resolution any more. (It is if $\mathrm{Ext}^1_R(B, R) = 0$.)

Proposition 3.2 *The following are equivalent for a morphism* f *in mod* R.

1) f *is a stable isomorphism.*

2) $Cone\ (f_\bullet)_\bullet = C_{\bullet-1}$ *is a trivial complex which is a split exact sequence of free modules.*

3) $\underline{\mathrm{Coker}\, f} \overset{st}{\cong} 0$ *and* $\underline{\mathrm{Ker}\, f} \overset{st}{\cong} 0$.

If $\mathrm{Ext}^1_R(B, R) = 0$, *then the following is also equivalent to the above.*

4) $\underline{n_f} = 0$ *and* $\underline{c_f} = 0$.

proof. The implication 2) ⇒ 1) comes from an exact sequence $0 \to F_{A\bullet} \to Cone(n_\bullet)_\bullet \to C_{\bullet-1} \to 0$ where $Cone(n_\bullet)_\bullet$ is a direct sum of $F_{B\bullet}$ and some trivial complex. The equivalence between 2) and 3) is clear because $\tau_0 C_\bullet$ and $\tau_0 C^*{}_\bullet$ are projective resolutions of $\underline{\mathrm{Ker}\, f}$ and $\underline{\mathrm{tr}\, \mathrm{Coker}\, f}$ respectively. The implication 3) ⇒ 4) is obvious, and 4) ⇒ 1) is straightforward from the following Lemma 3.3.

Lemma 3.3 *For $f \in \mathrm{Hom}_R(A,B)$, we have the following. If $\underline{n_f} = 0$, then A' is a direct summand of B' for some $A' \overset{st}{\cong} A$ and B' with $0 \to P \to B' \to B \to 0$. If $\mathrm{Ext}^1_R(B,R) = 0$, then $B \overset{st}{\cong} B'$.*

proof. Let $n_{f_\bullet} : F_{\underline{\mathrm{Ker}} f_\bullet} \to F_{A\bullet}$ be a chain map induced by n_f. We have an exact sequence of complexes

$$0 \to F_{\underline{\mathrm{Ker}} f_\bullet} \to Cyl(n_{f_\bullet})_\bullet \to Cone(n_{f_\bullet})_\bullet \to 0$$

where $Cyl(n_{f_\bullet})_\bullet$ is a direct sum of $F_{A\bullet}$ and some trivial complex. Clearly $\mathrm{Coker}\, d_{Cone(n_{f_\bullet})_1} \cong A \oplus \mathrm{Coker}\, d_{F_{\underline{\mathrm{Ker}} f_0}}$. Since the sequence $0 \to \mathrm{Ker}\,\tilde{f} \overset{\binom{n_f}{q_f}}{\to} A \oplus P \overset{(f *)}{\to} B \to 0$ is exact, we have only to put $B' = \mathrm{Coker}\, d_{Cone(n_{f_\bullet})_1}$. Therefore $B \overset{st}{\cong} A \oplus \mathrm{tr}\,\Omega^1_R \mathrm{tr}\, \underline{\mathrm{Ker}\, f}$. (q.e.d.)

Let $A \in mod\ R$, put $A_\bullet = F_{A\bullet}$ and $A'_\bullet = F_{\Omega^1_R(A)_{\bullet+1}}$. The identity map on $\Omega^1_R(A)$ induces a natural map $u_{A\bullet} : A'_\bullet \to A_\bullet$. The complex $L_{A\bullet} = Cone(u_{A\bullet})_\bullet$ is of the form

$$\cdots \to L_3 \overset{\binom{01}{00}}{\to} L_2 \overset{\binom{01}{00}}{\to} L_1 \to L_0 \to \cdots$$

and satisfies $\mathrm{H}_i(L_{A\bullet\bullet}) = 0\ (i > 0)$ $\quad \mathrm{H}^i(L_{A\bullet}{}^*) = 0 \quad (i > 0)$. Put $B_\bullet$,$B'_\bullet$,and $L_{B\bullet}$, similarly for $B \in mod\ R$. An R-homomorphism $f \in \mathrm{Hom}_R(A,B)$ induces chain maps $f'_\bullet : A_\bullet \to B_\bullet$, $f_\bullet : A'_\bullet \to B'_\bullet$, and $f_{L\bullet} : L_{A\bullet} \to L_{B\bullet}$. We have an diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_{\bullet+1} & \to & L_{B\bullet+1} & \to & B'_\bullet & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_\bullet & \to & C_{L\bullet} & \to & C'_{\bullet-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_\bullet & \to & L_{A\bullet} & \to & A'_{\bullet-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $C_\bullet = Cone(f_\bullet)_{\bullet+1}$, $C_{L\bullet} = Cone(f_{L\bullet})_{\bullet+1}$, and $C'_\bullet = Cone(f'_\bullet)_{\bullet+1}$. This diagram gives us the following two lemmata.

Lemma 3.4 *For given R-homomorphism f, there exist modules $C \overset{st}{\cong} \underline{\mathrm{Coker}\,\Omega_R^1(f)}$, $K \overset{st}{\cong} \underline{\mathrm{Ker}\,f}$, and a projective module K such that $0 \to P \to C \to K \to 0$ is exact.*

Lemma 3.5 *For every integer $r \geq 1$,*

$$\underline{\mathrm{Ker}\,\Omega_R^r(f)} \overset{st}{\cong} \Omega_R^r(\underline{\mathrm{Ker}\,f}).$$

For given $f \in \mathrm{Hom}_R(A, B)$, suppose $\underline{\mathrm{Ker}\,f} \overset{st}{\cong} 0$. Then we have an exact sequence $0 \to Q \to A \oplus P \to B \to 0$ with some projective modules P and Q, which implies that $\Omega_R^1(f)$ is a stable isomorphism. Conversely, in case that $\Omega_R^1(f)$ is a stable isomorphism, $\underline{\mathrm{Ker}\,f}$ is not always free.

Example 3.6 *Let $z \in R$ be an non-zero-divisor of R and M any R-module. Let f be an endomorphism of $M \oplus R/(z^2)$ as $f = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$. Then $\Omega_R^1(f)$ is an endomorphism of $\Omega_R^1(M) \oplus R$ that is a stable isomorphism; $\Omega_R^1(f) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$. But $\underline{\mathrm{Ker}\,f} \overset{st}{\cong} R/(z)$ is not free.*

Lemma 3.7 *Suppose that $\mathrm{Ext}_R^1(A, R) = 0$. For an R-homomorphism $f \in \mathrm{Hom}_R(A, B)$, $\Omega_R^1(f)$ is a stable isomorphism if and only if $\underline{\mathrm{Ker}\,f} \overset{st}{\cong} 0$.*

Proposition 3.8 *Let f be an R-homomorphism $f \in \mathrm{Hom}_R(A, B)$ and r a positive integer.*

1) *If $\mathrm{pd}(\underline{\mathrm{Ker}\,f}) \leq r$, then $\Omega_R^{r+1}(f)$ is a stable isomorphism.*

2) *If $\Omega_R^r(f)$ is a stable isomorphism, then $\mathrm{pd}(\underline{\mathrm{Ker}\,f}) \leq r$.*

proof. If $\Omega_R^r(f)$ is a stable isomorphism, then $\underline{\mathrm{Coker}\,\Omega_R^r(f)}$ is free. From Lemma 3.4, $\mathrm{pd}(\underline{\mathrm{Ker}\,\Omega_R^{r-1}(f)}) \leq 1$. Lemma 3.5 induces $\underline{\mathrm{Ker}\,\Omega_R^{r-1}(f)} \overset{st}{\cong} \Omega_R^{r-1}(\underline{\mathrm{Ker}\,f})$, hence $\mathrm{pd}(\underline{\mathrm{Ker}\,f}) \leq r$.

Suppose $\mathrm{pd}(\underline{\mathrm{Ker}\,f}) \leq r$. Using Lemma 3.4 and Lemma 3.4, we have $\mathrm{pd}(\underline{\mathrm{Ker}\,\Omega_R^r(f)}) = 0$. Thus $\Omega_R^1(\Omega_R^r(f))$ is a stable isomorphism.

4 Derived category

A chain map $f_\bullet : A_\bullet \to B_\bullet$ is called a quasi-isomorphism if $\mathrm{H}_i(f_\bullet)$ is an isomorphism for each i.

Remark 4.1 *If $f \in \mathrm{Hom}_R(A, B)$ is a stable isomorphism, then $f_\bullet : F_{A\bullet} \to F_{B\bullet}$ is a quasi-isomorphism.*

As we see later, the converse of above is not true.

From now on, let us assume that R is Gorenstein. For each $A \in mod\ R$, $0 \to A \to YA \to X^A \to 0$ denotes the finite projective hull of A; that is, $\mathrm{pd}(YA) < \infty$ and X^A is a maximal Cohen-Macaulay module. An R-homomorphism $f : A \to B$ induces the homomorphism $Yf : YA \to YB$.

Proposition 4.2 *The following are equivalent for $f \in \mathrm{Hom}_R(A, B)$.*

1) *$f_\bullet : F_{A\bullet} \to F_{B\bullet}$ is a quasi-isomorphism.*

2) *$Yf : YA \to YB$ is a stable isomorphism.*

3) *$Ytrf : YtrA \to YtrB$ is a stable isomorphism.*

Let $\mathrm{D}(mod\ R)$ denote the derived category of R-complexes, $\mathrm{D}_-(mod\ R)$ ($\mathrm{D}_+(mod\ R)$) the subcategory of complexes $C_\bullet$ with the property that $\mathrm{H}_i(C_\bullet) = 0$ for sufficiently small (or large) i. Consider $\mathrm{F}(mod\ R) = \{C\dot{\in}\mathrm{D}(mod\ R) \mid \mathrm{H}_i(C_\bullet) = 0 \quad (i > 0),\ \mathrm{H}^j(\mathrm{RHom}(C_\bullet, R)) = 0 \quad (i \geq 0)\}$. Yoshino proved the following theorem :

Theorem 4.3 (Yoshino,[5]) *Let R be a (not necessarily Gorenstein) noetherian local ring. There is an equivalence of categories*

$$\mathrm{F}(mod\ R) \cap \mathrm{D}_-(mod\ R) \cong \underline{\mathcal{F}(R)}^{op}$$

where $\mathcal{F}(R)$ denotes the category of finite R-modules with finite projective dimension.

If R is Gorenstein, we don't need the assumption of boundedness. Next theorem is established by Kawasaki's simple proof.

Theorem 4.4 (Kawasaki-K) *Let R be a Gorenstein local ring. There is an equivalence of categories*

$$\mathrm{F}(mod\ R) \cong \underline{\mathcal{F}(R)}^{op}.$$

proof. We have only to show that $C_\bullet \in \mathrm{D}_-(mod\ R)$ if $C_\bullet \in \mathrm{F}(mod\ R)$. Since $C_\bullet \in \mathrm{D}_+(mod\ R)$, $\mathrm{RHom}(C_\bullet, R) \in \mathrm{D}_+(mod\ R) \cap \mathrm{D}_-(mod\ R)$. Hence $\mathrm{RHom}(\mathrm{RHom}(C_\bullet, R), R) \in \mathrm{D}_+(mod\ R) \cap \mathrm{D}_-(mod\ R)$. Because R is Gorenstein, $\mathrm{RHom}(\mathrm{RHom}(C_\bullet, R), R) = C_\bullet$ in $\mathrm{D}(mod\ R)$. (q.e.d.)

This theorem gives a construction of finite-projective hull.

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A NOTE ON NAGATA CRITERION FOR SERRE'S CONDITIONS

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1. Introduction

This is not in a final form. The detailed version of this note will appear in *Math. J. Okayama Univ.* ([10]).

Throughout this note, we assume that all rings are noetherian.

Let A be a ring, and $\mathbb{P}$ be a property of local rings. We denote by $\mathbb{P}(A)$ the set of prime ideals $\mathfrak{p}$ of A such that $A_\mathfrak{p}$ satisfies $\mathbb{P}$, and call it the $\mathbb{P}$-locus of A. The following statement is called the Nagata criterion for $\mathbb{P}$, and we abbreviate it to (NC).

(NC) : For a ring A such that $\mathbb{P}(A/\mathfrak{p})$ contains a non-empty open subset of $\mathrm{Spec}A/\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}A$, $\mathbb{P}(A)$ is open in $\mathrm{Spec}A$.

It is known that (NC) holds for $\mathbb{P}$ = regular ([3][5][6][8]), Cohen-Macaulay ([3][5][6]), Gorenstein ([2][6]), and complete intersection ([2]).

Now we recall Serre's (R_n) and (S_n)-conditions for a ring A. These are defined as follows. Let n be an integer.

(R_n) : For every $\mathfrak{p} \in \mathrm{Spec}A$ with $\mathrm{ht}\mathfrak{p} \leqq n$, $A_\mathfrak{p}$ is regular.

(S_n) : For every $\mathfrak{p} \in \mathrm{Spec}A$, $\mathrm{depth}A_\mathfrak{p} \geqq \inf(n, \mathrm{ht}\mathfrak{p})$.

We easily see that (NC) holds for $\mathbb{P}$ = (integral) domain, coprimary (a ring A is called coprimary if A has just one associated prime), (R_0), (S_1), reduced, and normal. As corollaries of these results, it is easy to see that the $\mathbb{P}$-locus of a homomorphic image of a ring satisfying $\mathbb{P}$ is open for $\mathbb{P}$ = Cohen-Macaulay ([5][6]), Gorenstein([6]), domain, coprimary, (R_0), (S_1), and reduced.

The following theorems are the main results of this note.

Theorem 1. (NC) *holds for* $\mathbb{P} = (\mathrm{S}_n)$.

Theorem 2. (NC) *holds for* $\mathbb{P} = (\mathrm{R}_n)$.

We know that the properties "regular", "Cohen-Macaulay", "reduced", and "normal" are described by using (R_n) and (S_n). So the fact that (NC) holds for each of these four properties is obtained as a corollary of Theorem 1 and 2.

In the next sections, we shall give an outline of the proof of the theorems.

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2. Outline of proof of Theorem 1

We prove it by induction on n. Suppose that a ring A satisfies the assumption in (NC). By induction hypothesis, $\mathrm{S}_{n-1}(A)$ is open in Spec$A$. Hence we can write $\mathrm{S}_{n-1}(A) = \bigcup_{i=1}^{s} \mathrm{D}(f_i)$ with $f_i \in A$. Therefore A_{f_i} satisfies (S_{n-1}) and $\mathrm{S}_n(A) = \bigcup_{i=1}^{s} \mathrm{S}_n(A_{f_i})$. We want to prove that $\mathrm{S}_n(A)$ is open, so replacing A by A_{f_i}, we may assume that A satisfies (S_{n-1}).

There exists a radical ideal I of A such that $\overline{\mathrm{S}_n(A)^c}$ (the closure of the complement set of $\mathrm{S}_n(A)$ in Spec$A$) is equal to $\mathrm{V}(I)$. We can write $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ with $\mathfrak{p}_i \in$ Spec$A$. We may assume that there are no inclusion relations between the $\mathfrak{p}_i$'s and that $\mathrm{ht}\mathfrak{p}_1 \leqq \mathrm{ht}\mathfrak{p}_i$ for all i.

Now we claim that $\mathrm{ht}I \geqq n$. If this is true, we have $\mathrm{S}_n(A)^c = \mathrm{V}(I)$. In fact, suppose that $\mathrm{S}_n(A)^c \subsetneqq \mathrm{V}(I)$ and take $\mathfrak{p} \in \mathrm{V}(I) - \mathrm{S}_n(A)^c$. Then $\mathfrak{p}_k \subseteqq \mathfrak{p}$ for some k, hence $\mathfrak{p}_k \in \mathrm{S}_n(A)$. Since $\mathrm{ht}I \geqq n$, we obtain $\mathrm{ht}\mathfrak{p}_k \geqq n$, hence $\mathrm{depth}A_{\mathfrak{p}_k} \geqq n$. Therefore there exist $x_1, \cdots, x_n \in \mathfrak{p}_k$ and $f \in A - \mathfrak{p}_k$ such that $\mathbf{x}$ is an A_f-sequence and $IA_f = \mathfrak{p}_k A_f$. Since $\mathfrak{p}_k \in \overline{\mathrm{S}_n(A)^c}$, we obtain that $\mathrm{D}(f) \cap \mathrm{S}_n(A)^c \neq \varnothing$. Let $\mathfrak{p}$ be a minimal element of this set. Since $\mathfrak{p} \in \mathrm{V}(I)$, we have $\mathfrak{p}_k A_f \subseteqq \mathfrak{p} A_f$. Therefore $\mathbf{x}$ is an $A_\mathfrak{p}$-sequence in $\mathfrak{p}A_\mathfrak{p}$. It follows that $\mathrm{depth}A_\mathfrak{p} \geqq n$. For any $\mathfrak{q} \in$ Spec$A$ with $\mathfrak{q} \subsetneqq \mathfrak{p}$, we have $\mathfrak{q} \in \mathrm{S}_n(A)$ by the minimality of $\mathfrak{p}$. Thus, $A_\mathfrak{p}$ satisfies (S_n), contrary to the choice of $\mathfrak{p}$.

Now it only remains to prove the claim. It is enough to show that $\mathrm{ht}\mathfrak{p}_1 \geqq n$. Suppose that $l := \mathrm{ht}\mathfrak{p}_1 \leqq n-1$. Then $\mathrm{depth}A_{\mathfrak{p}_1} \geqq l$, hence there exist $x_1, \cdots, x_l \in \mathfrak{p}_1$ and $f \in A - \mathfrak{p}_1$ such that $\mathbf{x}$ is an A_f-sequence, $(\mathbf{x})A_f$ is $\mathfrak{p}_1 A_f$-primary, $IA_f = \mathfrak{p}_1 A_f$,and $\mathrm{D}(f) \cap \mathrm{V}(\mathfrak{p}_1) \subseteqq \mathrm{S}_n(A/\mathfrak{p}_1)$. Replacing A by A_f, we may assume that $\mathbf{x}$ is an A-sequence, $(\mathbf{x})$ is $\mathfrak{p}_1$-primary, $I = \mathfrak{p}_1$, and $A/\mathfrak{p}_1$ satisfies (S_n). Hence $\mathfrak{p}_1^r \subseteqq (\mathbf{x})$ for some $r \geqq 1$.

Let $-$ denote modulo $(\mathbf{x})$. By the generic freeness, replacing A by its localization, we may assume that $\overline{\mathfrak{p}_1}^i/\overline{\mathfrak{p}_1}^{i+1}$ is $\overline{A}/\overline{\mathfrak{p}_1}$-free for all $1 \leqq i < r$. Let $\mathfrak{p} \in \mathrm{S}_n(A)^c$ and $\mathfrak{p}' \in$ Spec$A$ with $\mathfrak{p}' \subseteqq \mathfrak{p}$. Then $A_{\mathfrak{p}'}$ is Cohen-Macaulay if $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \leqq n$, and $\mathrm{depth}A_{\mathfrak{p}'} \geqq n$ otherwise.

In fact, if $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \leqq n$, taking $\mathfrak{p}'' \in$ Spec$A$ with $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) = \mathrm{ht}(\mathfrak{p}''/\mathfrak{p}_1)$, we see that $(A/\mathfrak{p}_1)_{\mathfrak{p}''}$ is Cohen-Macaulay. Replacing A by $\overline{A}$, we may assume that $\mathfrak{p}_1^r = 0$. Then it is easy to see that $A_{\mathfrak{p}''}$ is Cohen-Macaulay, and so is $A_{\mathfrak{p}'}$. If $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \geqq n$, we have $\mathrm{grade}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1, A/\mathfrak{p}_1) \geqq n$. Hence $y_1, \cdots, y_n$ is an $A/\mathfrak{p}_1 (= \overline{A}/\overline{\mathfrak{p}_1})$-sequence for some $y_i \in \mathfrak{p}'$. Then we easily see that this is an $\overline{A}/\overline{\mathfrak{p}_1}^r (= A/(\mathbf{x}))$-sequence. It follows that this is an $A_{\mathfrak{p}'}$-sequence, hence $\mathrm{depth}A_{\mathfrak{p}'} \geqq n$.

Thus, we see that $A_\mathfrak{p}$ satisfies (S_n), contrary to the choice of $\mathfrak{p}$.

3. Outline of proof of Theorem 2

Before the proof, we consider the following condition for a local ring A.

(R'_n) : For every $\mathfrak{p} \in \mathrm{Spec}A$ with $\mathrm{codim}\mathfrak{p} \leqq n$, $A_\mathfrak{p}$ is regular.

Here we put $\mathrm{codim}I = \dim A - \dim A/I$ for an ideal I of A. It is easy to see that A satisfies (R_n) if and only if $A_\mathfrak{p}$ satisfies (R'_n) for all $\mathfrak{p} \in \mathrm{Spec}A$.

Now we start to prove Theorem 2. Discussing similarly to the proof of Theorem 1, we may assume that a ring A satisfies (R_{n-1}), $\overline{\mathrm{R}_n(A)^c} = \mathrm{V}(I)$ for some ideal I of A, $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ with $\mathfrak{p}_i \in \mathrm{Spec}A$, there are no inclusion relations between the $\mathfrak{p}_i$'s, and $\mathrm{ht}\mathfrak{p}_1 \leqq \mathrm{ht}\mathfrak{p}_i$ for all i.

We claim that $\mathrm{ht}I \geqq n$. If this is true, we easily see that $\mathrm{R}_n(A)^c$ is the union of $\mathrm{V}(\mathfrak{p}_i)$ such that $\mathrm{ht}\mathfrak{p}_i = n$ and $A_{\mathfrak{p}_i}$ is not regular.

To prove the claim, it suffices to show that $\mathrm{ht}\mathfrak{p}_1 \geqq n$. Suppose that $l := \mathrm{ht}\mathfrak{p}_1 \leqq n-1$. Then $A_{\mathfrak{p}_1}$ is regular, so replacing A by its localization, we may assume that for some $x_1, \cdots, x_l \in \mathfrak{p}_1$ $\mathbf{x}$ is an A-sequence in $\mathfrak{p}_1$, $(\mathbf{x}) = \mathfrak{p}_1$, $I = \mathfrak{p}_1$, and $A/\mathfrak{p}_1$ satisfies (R_n). Let $\mathfrak{p} \in \mathrm{R}_n(A)^c$ and $\mathfrak{q} \in \mathrm{Spec}A$ with $\mathfrak{q} \subsetneqq \mathfrak{p}$. Then $A_\mathfrak{q}$ satisfies (R'_n).

In fact, let $\mathfrak{p}' \in \mathrm{Spec}A$ such that $\mathfrak{p}' \subsetneqq \mathfrak{p}$ and $\mathrm{codim}\mathfrak{p}'A_\mathfrak{p} \leqq n$. Then, since $\mathrm{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_\mathfrak{p} = \mathrm{codim}(x_1, \cdots, x_l)(A/\mathfrak{p}')_\mathfrak{p} \leqq l$, we obtain $\mathrm{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_\mathfrak{p} \leqq \mathrm{codim}\mathfrak{p}'A_\mathfrak{p} \leqq n$. Taking $\mathfrak{p}'' \in \mathrm{Spec}A$ with $\mathrm{codim}(\mathfrak{p}'+\mathfrak{p}_1/\mathfrak{p}_1)A_\mathfrak{p} = \mathrm{codim}(\mathfrak{p}''/\mathfrak{p}_1)A_\mathfrak{p}$, we see that $(A/\mathfrak{p}_1)_{\mathfrak{p}''}$ is regular, so is $A_{\mathfrak{p}''}$, and so is $A_{\mathfrak{p}'}$. It follows from this that $A_\mathfrak{p}$ satisfies (R'_n).

Thus, we see that $A_\mathfrak{p}$ satisfies (R_n), contrary to the choice of $\mathfrak{p}$.

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A characterization of one dimensional $\mathbf{N}$-graded Gorenstein rings of finite Cohen-Macaulay representation type

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Let $R = \oplus R_n$ be an $\mathbf{N}$-graded Cohen-Macaulay ring where $R_0 = k$ is a field. We denote by mod$R$ the category of finitely generated graded R-modules whose morphisms are graded R-homomorphisms that preserve degrees. We also denote by CMR the full subcategory of mod$R$ consisting of all graded maximal Cohen-Macaulay modules. In the last conference, we proved that if $R = \oplus R_n$ be a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type where $R_0 = k$ is an algebraically closed field of characteristic 0, then there exists the exceptional sequence that generates all graded indecomposable maximal Cohen-Macaulay modules. (we call such an exceptional sequence *MCM generating.* (c.f. [2])) In that work, we had to compute the dimension of $\mathrm{Ext}^n_R(X, Y)$ as k-vector space, for all indecomposable graded maximal Cohen-Macaulay modules X and Y and for all $n \in \mathbf{N}$. Through this computation, we noticed the importance of the invariants $\mathrm{d}(R)$ and $\mathrm{d}_n(R)$ of R that are defined as follows:

Definition 1

$$\mathrm{d}(R) := \sup\{\sum_{n\geq 0} \dim_k \mathrm{Ext}^n_R(X, Y) \mid X, Y \in \mathrm{CM}R \text{ are indecomposable}\},$$

$$\mathrm{d}_n(R) := \sup\{\dim_k \mathrm{Ext}^n_R(X, Y) \mid X, Y \in \mathrm{CM}R \text{ are indecomposable}\}.$$

The main Theorem of this lecture is following;

Theorem 2 [1, Theorem 3.2]

Let k be an algebraically closed field of characteristic 0 and let R be a positively dimensional $\mathbf{N}$-graded Gorenstein ring with isolated singularity where $R_0 = k$. Then the following conditions are equivalent.

(i) *R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type.*

(ii) $\mathrm{d}(R) < \infty$

(ii') $\mathrm{d}_0(R) < \infty$

(iii) $\mathrm{d}(R) \leq 9$

(iii') $\mathrm{d}_0(R) \leq 9$

1 Preliminaries

In this section, we assume $R = \oplus R_n$ is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type and assume that $R_0 = k$ is an algebraically closed field of characteristic 0. In this case, it is known that R is isomorphic to one of the following rings (c.f.[5]).

$$\begin{array}{lll} (A_n) & R = k[x,y]/(y^2 - x^n) & (n \geq 2) \\ (D_n) & R = k[x,y]/(xy^2 - x^n) & (n \geq 3) \\ (E_6) & R = k[x,y]/(x^3 + y^4) & \\ (E_7) & R = k[x,y]/(x^3 + xy^3) & \\ (E_8) & R = k[x,y]/(x^3 + y^5) & \end{array} \quad (1)$$

Moreover the Auslander-Reiten quiver of CMR for each type can be described as they are shown in [Figures (1) – (7)]. We denote by Γ the Auslander-Reiten quiver of CMR.

For indecomposable graded maximal Cohen-Macaulay modules X and Y, we write $X \preceq Y$ if $X \cong Y$ or if there exists a finite path from X to Y in Γ.

Lemma 3 [2, Lemma 3.3.], [1, Lemma 2.1, Lemma 2.2] *The following hold for indecomposable graded maximal Cohen-Macaulay modules X and Y.*

(i) *There are no cyclic paths in Γ.*

(ii) *If $\mathrm{Hom}(X,Y) \neq 0$, then $X \preceq Y$.*

(iii) *If $\mathrm{Ext}^1_R(X,Y) \neq 0$, then $\Omega X \preceq Y \preceq \tau X$. Here, τX denotes the Auslander-Reiten translation of X.*

As a corollary of this Lemma, we get the following Lemma.

Lemma 4 [1, Lemma 2.3]
For any indecomposable graded maximal Cohen-Macaulay modules X and Y, we have $\sharp\{n \in \mathrm{N} \mid \mathrm{Ext}^n_R(X,Y) \neq 0\} \leq 1$.

2 Proof of the main Theorem

In this section, we shall prove the main Theorem. To show the theorem, we need the graded version of Brauer-Thrall 1 theorem for graded maximal Cohen-Macaulay modules, due to [5], [4] and [3].

Theorem 5 (graded version of Brauer-Thrall 1 theorem) *Let R be an $\mathbf{N}$-graded Cohen-Macaulay ring with isolated singularity where $R_0 = k$ is a perfect field. If* $\sup\{\mathrm{e}(X) \mid X \in \mathrm{CM}R \text{ is indecomposable}\} < \infty$, *then R is of finite Cohen-Macaulay representation type. Here* $\mathrm{e}(X)$ *denotes the multiplicity of the irrelevant maximal ideal along X.*

Proof of 2 The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (ii') and (iii) $\Rightarrow$ (iii') $\Rightarrow$ (ii') are trivial. First, we show (ii') $\Rightarrow$ (i). Since $\mathrm{d}_0(R) < \infty$, we see that $\dim_k R_n = \dim_k \mathrm{Hom}(R, R(n)) \leq \mathrm{d}_0(R) < \infty$ for all n. Therefore the Hilbert polynomial of R is constant. Hence $\dim R = 1$. For any indecomposable graded maximal Cohen-Macaulay module X, $\dim_k X_n = \dim_k (R, X(n)) \leq \mathrm{d}_0(R)$ for all n. Therefore the multiplicity $\mathrm{e}(X)$ of X is bounded by $\mathrm{d}_0(R)$. Hence R is of finite Cohen-Macaulay representation type by theorem 5.

To prove (i) $\Rightarrow$ (iii'), it is enough to compute $\sup\{\dim_k \mathrm{Hom}(R, Y), \dim_k \mathrm{Hom}(Y, R), \dim_k \mathrm{Hom}(X_i, Y), \dim_k \mathrm{Hom}(Y_i, Y) \mid Y \in \mathrm{CM}R \text{ is indecomposable}\}$ where X_i and Y_i are in [2, Figures (1) – (7)] . For an indecomposable graded maximal Cohen-Macaulay module X, we denote by X^+ (resp. X^-) the smallest additive full subcategory of $\mathrm{CM}R$ containing all indecomposable graded maximal Cohen-Macaulay modules Y with $X \preceq Y$ (resp. $Y \preceq X$). Then, by induction on the length of the path from X to Y (resp. from Y to X), one can easily check that $\dim_k \mathrm{Hom}(X, Y) = 1$ (resp. $\dim_k \mathrm{Hom}(Y, X) = 1$) for all indecomposable $Y \in X^+$ (resp. $Y \in X^-$) with Y is not free and $\tau Y \notin X^+$ (resp. $\tau^- Y \notin X^-$). Since R is a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type, we may assume that R is one of the rings given in (1) . Thus we are able to compute $\dim_k \mathrm{Hom}(R, R(n)) = \dim_k \mathrm{Hom}(R(-n), R)$ for all n by Hilbert function. Since the functor $\mathrm{Hom}(R, -)$ (resp. $\mathrm{Hom}(-, R)$) is an exact functor on R^+ (resp. R^-), it is possible to compute $\dim_k \mathrm{Hom}(R, Y)$ (resp. $\dim_k \mathrm{Hom}(Y, R)$) for all $Y \in R^+$ (resp. $Y \in R^-$) by using Auslander-Reiten quiver. Since $\mathrm{Hom}(R, Y) = 0$ (resp. $\mathrm{Hom}(Y, R) = 0$) for all $Y \notin R^+$ (resp. $Y \notin R^-$) by lemma 3, it is possible to compute $\dim_k \mathrm{Hom}(R, Y)$ and $\dim_k \mathrm{Hom}(Y, R)$ for all $Y \in \mathrm{CM}R$. For any $X \in \{X_i, Y_i\}_i$, since we have already computed $\dim_k \mathrm{Hom}(X, R(n)) = \dim_k \mathrm{Hom}(X(-n), R)$ and since $\mathrm{Hom}(X, -)$ is an exact functor on X^+, it is also possible to compute $\dim_k \mathrm{Hom}(X, Y)$ for all $Y \in X^+$ by using Auslander-Reiten quiver. In this way we can accomplish the computation of $\dim_k \mathrm{Hom}(X, Y)$ for any indecomposable $X, Y \in \mathrm{CM}R$ and get the invariant $\mathrm{d}_0(R)$. The result is shown in table 1. Looking at this table we have $\mathrm{d}_0(R) \le 9$.

Table 1:

type	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$\mathrm{d}_0(R)$	1	2	3	4	4	6	9

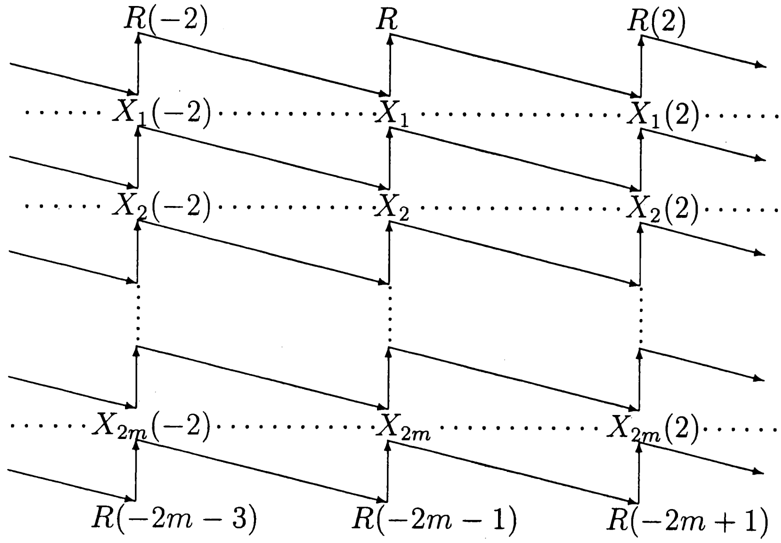
Finally, we prove (iii') $\Rightarrow$ (iii). Because we have already proved (iii') $\Rightarrow$ (i), we may assume that R is one given in (1). Since $\mathrm{Ext}^n_R(X,Y) \cong \mathrm{Ext}^1_R(\Omega^{n-1}X,Y)$ for all $n>0$ and by lemma 4, it is enough to show $\mathrm{d}_0(R) \ge d_1(R)$. For any indecomposable graded maximal Cohen-Macaulay module X, the first syzygy ΩX of X is also an indecomposable graded maximal Cohen-Macaulay module. Since there exists a natural epimorphism $\mathrm{Hom}(\Omega X, Y) \twoheadrightarrow \mathrm{Ext}^1_R(X,Y)$, one can see $\mathrm{d}_0(R) \ge \mathrm{d}_1(R)$ and get $\mathrm{d}(R) \le 9$.□

Remark 6 Let R be a one dimensional $\mathbf{N}$-graded Gorenstein ring of finite Cohen-Macaulay representation type with $R_0 = k$ being algebraically closed field of characteristic 0 (i.e. R is isomorphic to one of the rings given in (1)). In the above proof, we showed how to compute the invariant $\mathrm{d}_0(R)$. Remark that we can also compute the invariant $\mathrm{d}_n(R)$ $(n \ge 1)$ by using Auslander-Reiten quiver in a similar way to this. Since $\mathrm{Ext}^n_R(X,Y) \cong \mathrm{Ext}^1_R(\Omega^{n-1}X,Y)$, we have $\mathrm{d}_n(R) = \mathrm{d}_1(R)$ for $n \ge 1$. We will show how to compute $\mathrm{d}_1(R)$. For an indecomposable graded maximal Cohen-Macaulay module X, we denote by $X^{(1)}$ the smallest additive full subcategory of CMR containing all indecomposable graded non-free maximal Cohen-Macaulay modules Y with $\Omega X \preceq Y \preceq \tau X$. We also denote by $X^{(1)'}$ the smallest additive full subcategory of CMR containing all indecomposable graded non-free maximal Cohen-Macaulay modules Y with $\tau X \prec Y$ and $X \not\preceq Y$. It turns out from lemma 3 that $\mathrm{Ext}^1_R(X,Y) = 0$ for all $Y \notin X^{(1)}$ and $\mathrm{Ext}^n_R(X,Y) = 0$ for all $Y \in X^{(1)'}$ and for all n. And it follows from lemma 4 that $\mathrm{Ext}^1_R(X,-)$ is an exact functor on $X^{(1)} \cup X^{(1)'}$. Hence it is possible to compute $\mathrm{d}_1(R)$ (and therefore $\mathrm{d}_n(R)$ for all $n \ge 1$) by using Auslander-Reiten quiver. The results are given in following table.

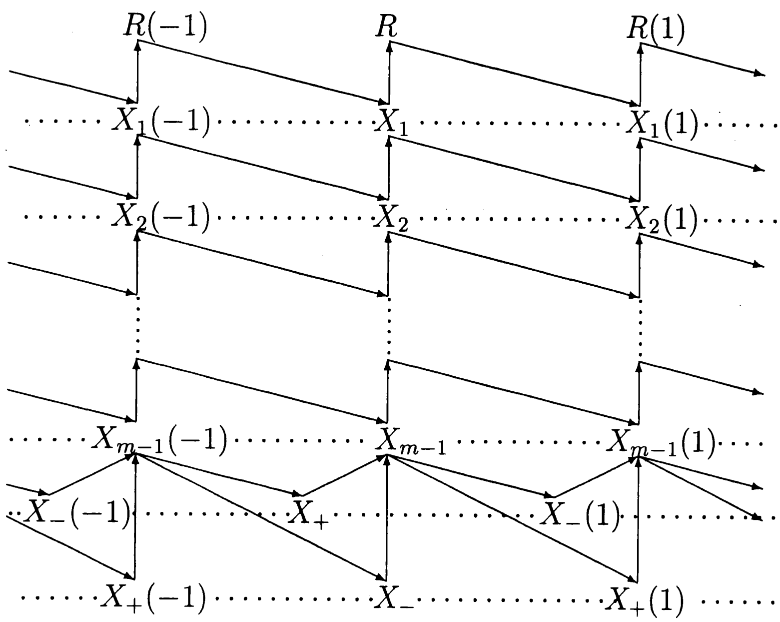
Table 2:

	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$\mathrm{d}(R) = \mathrm{d}_0(R)$	1	2	3	4	4	6	9
$\mathrm{d}_n(R)$ $(n \ge 1)$	1	2	1	2	3	4	6

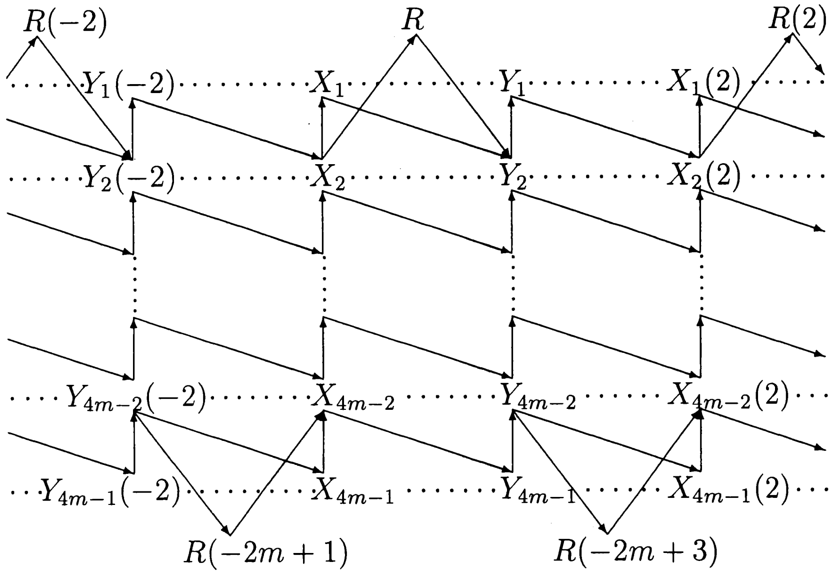
Figure(1) : the type of (A_n) with $n = 2m + 1$.



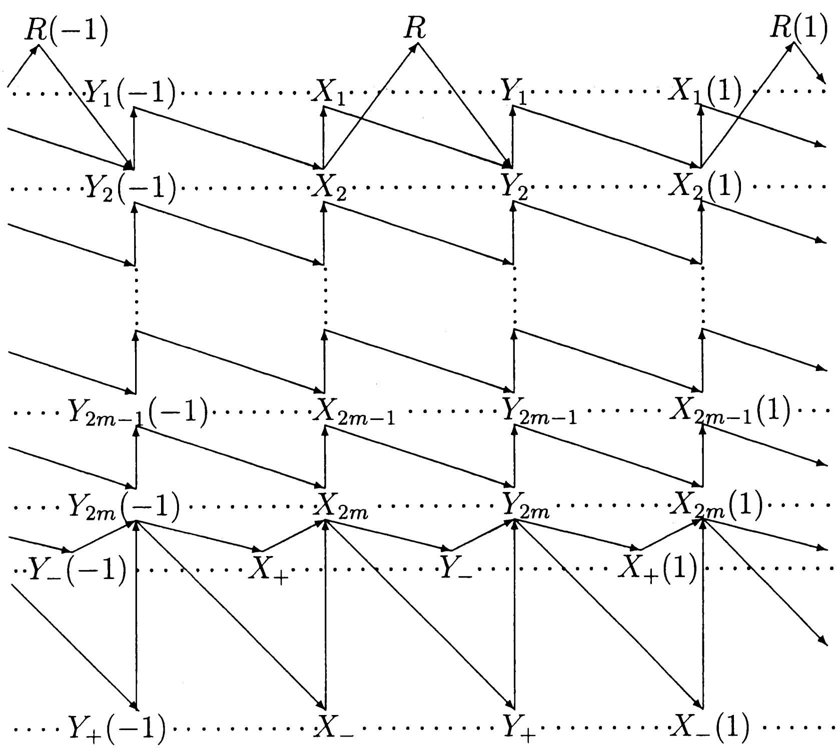
Figure(2) : the type of (A_n) with $n = 2m$.



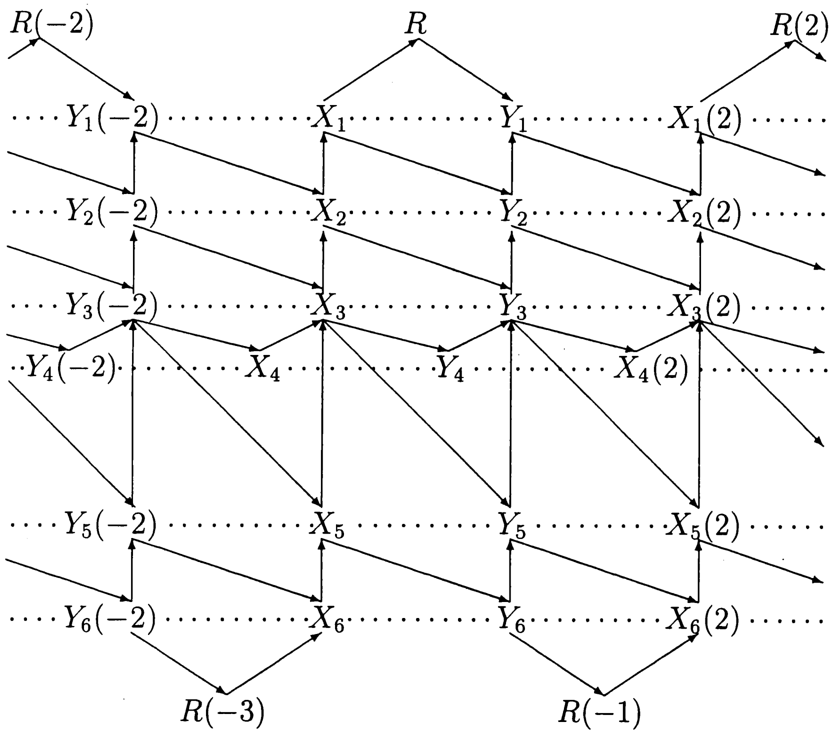
Figure(3) : the type of (D_n) with $n = 2m$.



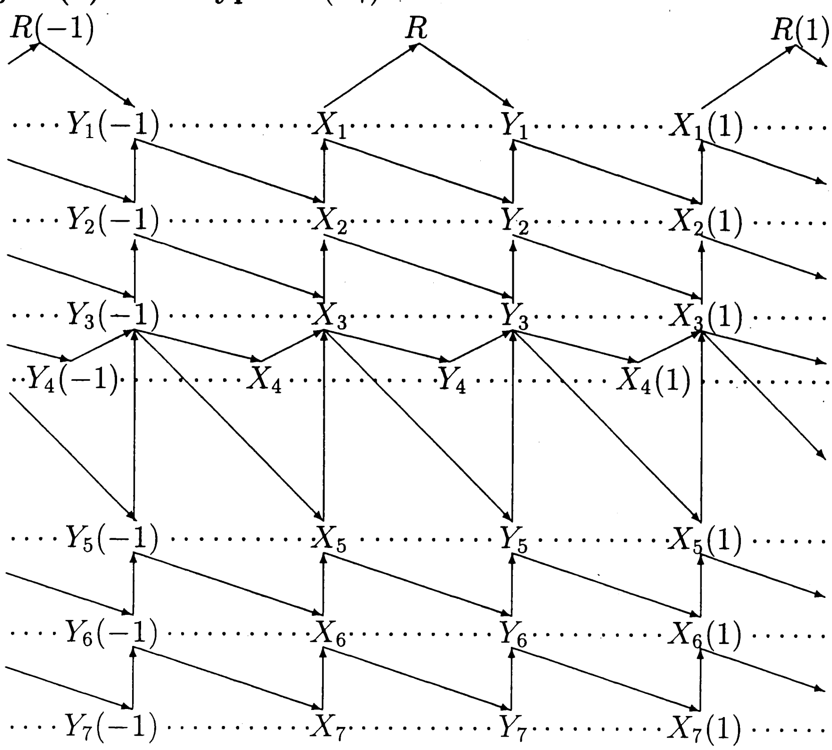
Figure(4) : the type of (D_n) with $n = 2m + 1$.



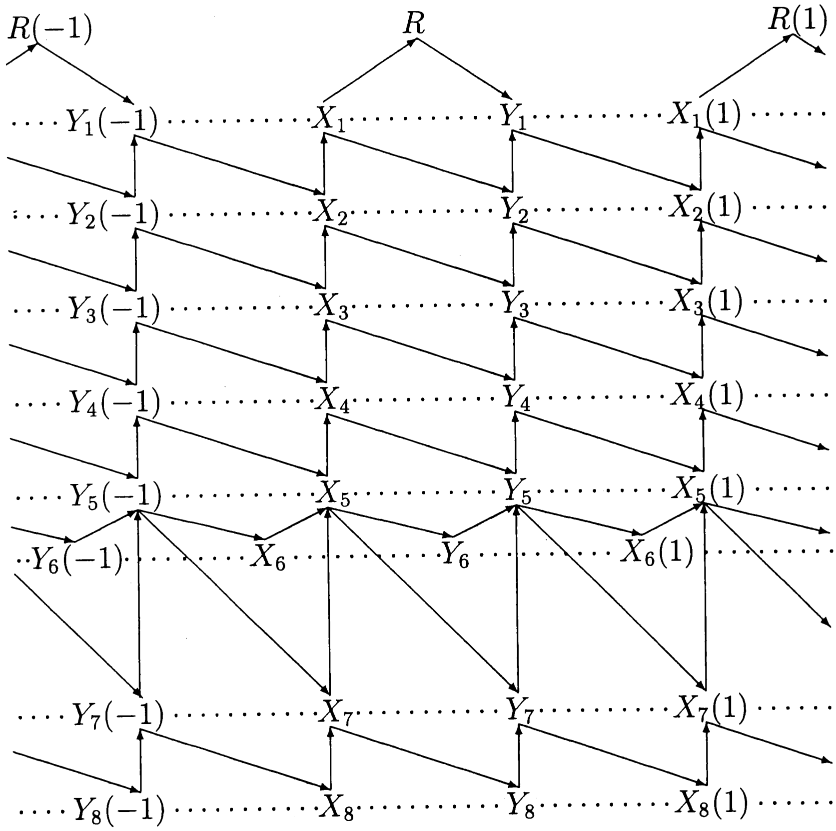
Figure(5) : the type of (E_6).



Figure(6) : the type of (E_7).



Figure(7) : the type of (E_8).



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Rationality, Algebraicity and Transcendency in $\mathbb{F}_q[[z_1, ..., z_m]]$ and Language Theory

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Abstruct: Everyone knows that the formal power series $f = \sum_{0 \le n} z^{p^n} = z + z^p + z^{p^2} + ... \quad \in \mathbb{F}_p[[z]]$ is algebraic, and the series $g = \sum_{0 \le n} z^n = 1 + z + z^2 + ... \in \mathbb{F}_p[[z]]$ is rational, i.e. $g \in \mathbb{F}_p[[\bar{z}]]$. Our results explain this relation as a special case.

Section 1. At first we need the following:

Definition 1. Fixing a base number q, $|n| = |n|_q$ denotes the number of digits in q-adic(base q) expression of non-negative integer n. For an non-negative integer vector $\mathbf{n} = (n_1, ..., n_m)$, $|\mathbf{n}|$ is defined to be equal to $\sup\{|n_1|, |n_2|, ..., |n_m|\}$.

Using this definition we have:

Theorem. If $f = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{F}_q[[z_1, z_2, ..., z_m]]$ is algebraic, then it follows that

$$\sum_{\mathbf{n}} a_{\mathbf{n}} z^{|\mathbf{n}|} \in \mathbb{F}_q(z) \ .$$

More precisely, if $f = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{F}_q[[z_1, z_2, ..., z_m]]$ satisfies an algebraic equation of degree d with coefficients in $\mathbb{F}_q[z_1, z_2, ..., z_m]$ and degrees at most s, then

$$g = \sum_{\mathbf{n}} a_{\mathbf{n}} z^{|\mathbf{n}|} \in \mathbb{F}_q(z)$$

and

$$\deg g \leq d((q^{d-2}+1)s+1)^m.$$

Corollary. If $f = \sum_{n=0}^{\infty} a_n z^n$ is algebraic in $\mathbb{F}_q[[z]]$ then

$$\sum_{l=0}^{\infty}\left(\sum_{q^{l-1} \leq n \leq q^l - 1} a_n\right) z^l \in \mathbb{F}_q(z).$$

Example 1. For each prime p, $f = 1 + x + x^p + x^{p^2} + x^{p^3} + ...$ is algebraic in $\mathbb{F}_p[[x]]$. $\Phi(f) = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$ is rational, and contained in $\mathbb{F}_p(x)$. On the other hand $g = 1 + x^p + x^{p^{p^1}} + x^{p^{p^2}} + ... \in \mathbb{F}_p[[x]]$ is transcendental. Because, $\Phi(g) = 1 + x^2 + x^{p^1+1} + x^{p^2+1} + ...$ is algebraic but not rational.

Example 2. As we have $-1/2 = \frac{1}{1-3} = 1 + 3 + 3^2 + 3^3 + ... \in \mathbb{Q}_3$, it follows that

$$f = (1+x)^{-1/2} = (1+x)^{1+3+3^2+3^3+...}$$

$$= (1+x)(1+x^3)(1+x^{3^2})....$$

Terms of x^m with $|m| = n+1$ are comming from expansion of $(1+x)...(1+x^{3^{n-1}})x^{3^n}$. The number of these terms is 2^n and coefficients are 1. Therefor we have

$$\Phi(f) = 1 + x + \sum_{n=1}^{\infty} 2^n x^{n+1} = \frac{1-x}{1-2x} \in \mathbb{F}_3(x).$$

Section 2. Let k be a perfect field of $char(k) = p > 0$. As usual $k[[z_1, ..., z_m]]$ stands for the formal power series ring over k. A monomial $z_1^{i_1} z_2^{i_2} ... z_m^{i_m}$ in $k[[z_1, ..., z_m]]$ will be abbriviated to $\mathbf{z}^{\mathbf{i}} \in k[[\mathbf{z}]]$. For each $\mathbf{r} = (r_1, r_2, ..., r_m)$, $0 \leq r_j \leq q-1$ $(q = p^e)$, we defined in [2] the operator $A_{\mathbf{r}}$ over $k[[z_1, z_2, ..., z_m]]$:

$$A_{\mathbf{r}}(\sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}) = \sum_{\mathbf{n}} a_{q\mathbf{n}+\mathbf{r}}^{1/q} \mathbf{z}^{\mathbf{n}}.$$

These operators are naturally extended to those over $k((\mathbf{z}))$.

Let X be an alphabet, that is, a finite set of letters(indeterminates). After the notations of [7], $k \ll X \gg$ denotes the non-commutative ring of formal power series of words in X^* over k :

$$k \ll X \gg = \{ \sum_{w \in X^*} \alpha_w w | \alpha_w \in k \}.$$

Here X^* is the free monoid generated by X (or the set of words over X including 1). For each letter x in X the right operator x^{-1} over $k \ll X \gg$ is defined by

$$\left(\sum_{w \in X^*} a_w w\right) x^{-1} = \sum_{w \in X^*} a_{wx} w.$$

For each word $w \in X^*$ operator w^{-1} is defined recursively by $(wx)^{-1} = x^{-1}w^{-1}$. A subset $M \subset k \ll X \gg$ is called stable if and only if $S \in M$ implies $Sx^{-1} \in M$ for each letter $x \in X$.

In the sequel we use special alphabet

$$X_{q^m} = \{x_{r_1 r_2 \ldots r_m} | 0 \le r_1, r_2, \ldots, r_m \le q-1\}$$

with indeterminates $x_{r_1 r_2 \ldots r_m}$.

Definition 2. For $\mathbf{i} = (i_1, \ldots, i_m)$, let $l = |\mathbf{i}|$ and let the base q expression of i_j be $(i_j)_l\ (i_j)_{l-1} \ldots (i_j)_1$ with $(i_j)_h = r_{jh} \in X_{q^m}$. We define a mapping Φ from the monomials in $k[[\mathbf{z}]]$ to $X_{q^m}^*$ by the following: if $l \neq 0$ then

$$\Phi(z_1^{i_1} z_2^{i_2} \ldots z_m^{i_m}) = x_{r_{1l} r_{2l} \ldots r_{ml}} x_{r_{1,l-1} r_{2,l-1} \ldots r_{m,l-1}} \ldots x_{r_{11} r_{21} \ldots r_{m1}},$$

and $\Phi(1) = 1$. Φ can be extended to k-linear and continuous map from $k[[z_1, z_2, \ldots, z_m]]$ to $k \ll X_{q^m} \gg$.

Example 1. For a monomial $x^5y^2z^6 \in \mathbb{F}_2[[x,y,z]]$ we have

$$\Phi(x^5y^2z^6) = \Phi(x^{(101)_2}y^{(10)_2}z^{(110)_2}) = x_{101}x_{011}x_{100}$$

$$\in \mathbb{F}_2 \ll X_{2^3} \gg .$$

By generalizing the results of G. Christol and al [1], we have:

Theorem 1. For every perfect field of positive characteristic k, the following statements are equvalent.

(1) $f \in k((\mathbf{z}))$ is algebraic over $k(\mathbf{z})$

(2) f is contained in an A-stable $k(\mathbf{z})$-finite submodule in $k((\mathbf{z}))$.

(3) f is contained in an A-stable k-finite subspace in $k((\mathbf{z}))$.

Here, by definition, a subset $M \subset k((\mathbf{z}))$ is A-stable if and only if M contains $A_{\mathbf{r}}(f)$ with f for each $\mathbf{r}$.

On the other hand, in the early years of 60'th, Schützenberger and other Language theorists studied the relations between formal power series of words and recognizability of languages. Their results will be summarized as follows [7]:

Theorem 2. For arbitrary semiring k, the following conditions for $f \in k \ll X \gg$ are equivalent.

(1) f is recognizable.

(2) f is contained in stable k-finite submodule in $k \ll X \gg$.

(3) f is rational.

From now, we assume $q = p^e (e > 0)$, $k = \mathbb{F}_q$ and $X = X_{q^m}$. So the operator $A_{\mathbf{r}}$ is reduced to

$$A_{\mathbf{r}}(\sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}) = \sum_{\mathbf{n}} a_{q\mathbf{n}+\mathbf{r}} \mathbf{z}^{\mathbf{n}}$$

for each $\mathbf{r} = (r_1, r_2, ..., r_m)$, $0 \leq r_j \leq q-1$.

It is easy to see the following key lemma of this paper:

Proposition 1. Let $\mathbf{r} = (r_1, r_2, ..., r_m)$ and let $x = x_{r_1 r_2 \ldots r_m} \in X$. Suppose $f = \sum_{\mathbf{n} \neq \mathbf{0}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in k[[\mathbf{z}]]$ have no constant term. Then it follows that

$$\Phi(A_{\mathbf{r}}(f)) = (\Phi(f)) x^{-1}.$$

Remark. It must be noted that $A_{\mathbf{0}}(1) = 1$ in $k[[\mathbf{z}]]$, but $1 x_{0\ldots0}^{-1} = 0$ in $k \ll X \gg$.

By using the Proposition 1, we can prove the following:

Proposition 2. A finite subspace $M \in k[[\mathbf{z}]]$ is A-stable if and only if $\Phi(M) + k$ is stable in $k \ll X_{q^m} \gg$ and finite.

Using this Proposotion, Theorem 1 and 2, we have that $f \in k[[\mathbf{z}]]$ is algebraic if and only if $\Phi(f)$ is rational. Considering the ring homomorphism

$$k[[z_1, ..., z_m]] \to k[[z]] \ (z_i \to z),$$

the theorem can be proved. Qualitative results are obtained by estimating the dimension of $\Phi(M)$. cf. [2] and [3].

Section 3. Comments. We have, in fact, proved that $f = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{F}_q[[z_1, z_2, ..., z_m]]$ is algebraic if and only if $\Phi(f) \in \mathbb{F}_q \ll X_{q^m} \gg$ is rational. This result is known in single variable case.(cf. [1]) As an application of the result, we can prove the transcendency of number theoritical functions such as

$$\sum_{m,n} (\mu(m) + \phi(n)) x^m y^n,$$

where μ (resp. ϕ) is the Moebius (resp. Euler) function.

In the case of positive characteristics, many theorems concerning the algebraicity of formal power seiries are valid for several variables, contrary to the characstic zero.(cf. [1],[2],[3],[6]) The obove equivalence shows the reason of these facts.

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Some recent results on simple extensions of commutative rings

By K. Yoshida (Okayama Univ of Sci)
with S. Oda, J. Sato, K. Baba and K. Yamamoto

R を integral domain, K を R の quotient field とする。α を K 上代数的な元で $\pi : R[X] \longrightarrow R[\alpha]$ を $\pi(X) = \alpha$ によって定まる自然な準同像とする。

$\varphi_\alpha(X)$ で α の K 上での monic な最小多項式とする。

α の拡大次数を t とすれば

$$\varphi_\alpha(X) = X^t + \eta_1 X^{t-1} + \cdots + \eta_t \quad , \quad \eta_i \in K$$

としてあらわされる。

ここで $I_{[\alpha]} \stackrel{def}{=} \bigcap_{i=1}^{t} I_{\eta_i}$, I_{η_i} は η_i の分母イデアル,

と定め α の拡張された分母イデアルと呼ぶ。

$J_{[\alpha]} \stackrel{def}{=} I_{[\alpha]}(1, \eta_1, \cdots, \eta_t)$ は拡大 $R[\alpha]/R$ の flatness の obstruction ideal である。すなわち

$\mathfrak{p} \in \operatorname{Spec} R$ に対して

$$R_{\mathfrak{p}}[\alpha]/R_{\mathfrak{p}} : \text{flat ext} \Leftrightarrow \mathfrak{p} \not\supset J_{[\alpha]}$$

α が anti-integral であるとは $\operatorname{Ker} \pi = I_{[\alpha]} \varphi_\alpha(X) R[X]$ である。

本論文ではアファイン分数変換 $\frac{cx+d}{ax+b}$,

$a, b, c, d \in R$ で $ad-bc \in R^{\times}$ ($R^{\times}$ は R の単元の全体) なるものを考える。このアファイン分数変換はその合成を演算とするとき

$$GL_2(R)$$

と同型である。

さて, 近々 Journal of Algebra に出る論文

Extensions $R[\alpha-a] \cap R[(\alpha-a)^{-1}]$ with an anti-integral element α are unchange for any $a \in R$

というタイトルで次の結果が示されている

Theorem 1 Let R be an integral domain and α be an anti-integral element of degree d over R. Then for any $a \in R$ with $\alpha - a \neq 0$,

$$R[\alpha] \cap R[\alpha^{-1}] = R[\alpha-a] \cap R[(\alpha-a)^{-1}]$$

この結果を利用すると実に興味深い二つの結果が出てくる。

Theorem 2 Assume that α is an anti-integral element of degree t over R. Let $\beta = \dfrac{c\alpha + d}{a\alpha + b}$ with $a, b, c, d \in R$ and $ad - bc \in R^{\times}$. Then β is anti-integral over R and the equality $R[\beta] \cap R[\beta^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$ holds

この証明のためにいくつかの補題を準備しなければなりませんがすべてこれまでに知られている結果なので証明は付けません.

Lemma 3 Assume that α is an anti-integral element of degree t over R: Then the following assertions hold

(1) If $u \in R^{\times}$, then $u\alpha$ is an anti-integral element over R

(2) If $a \in R$, then $\alpha - a$ is anti-integral element over R

(3) α^{-1} is anti-integral element over R.

そして (1), (2), (3) について $J_{[\alpha]}$ は不変, すなわち

$$J_{[\alpha]} = J_{[u\alpha]} = J_{[\alpha - a]} = J_{[u^{-1}]}$$ である。

Lemma 4　If α is anti-integral over R_m for all maximal ideal m of R, then α is anti-integral over R.

さて定理の証明に入りましょう。

Proof of Theorem 2

Lemma 4 より (R, m) local ring の Case で考えればよい。$ad - bc \in R^{\times}$ なので (i) $ad \in R^{\times}$ 又は (ii) $bc \in R^{\times}$

Case (i)

$$\beta = \frac{c\alpha + d}{a\alpha + b} = \frac{c}{a} - \frac{\frac{1}{a^2}(bc - ad)}{\alpha + \frac{b}{a}}$$

$e = \frac{b}{a}$, $f = \frac{c}{a}$, $g = \frac{1}{a^2}(bc - ad)$ とおくと

$e, f \in R$, $g \in R^{\times}$

$$\beta = f + \frac{g}{\alpha + e}$$

$\gamma = \frac{g}{\alpha + e}$ とおくと　$\beta = \gamma + f$

α は R 上 anti-integral なので $\alpha + e$ も又 anti-integral

よって $\frac{1}{\alpha + e}$ も又 anti-integral, よって $\gamma = \frac{g}{\alpha - e}$ も又

同じく anti-integral なので $\beta = \gamma + f$ は anti-integral

J. of Algebra の結果から

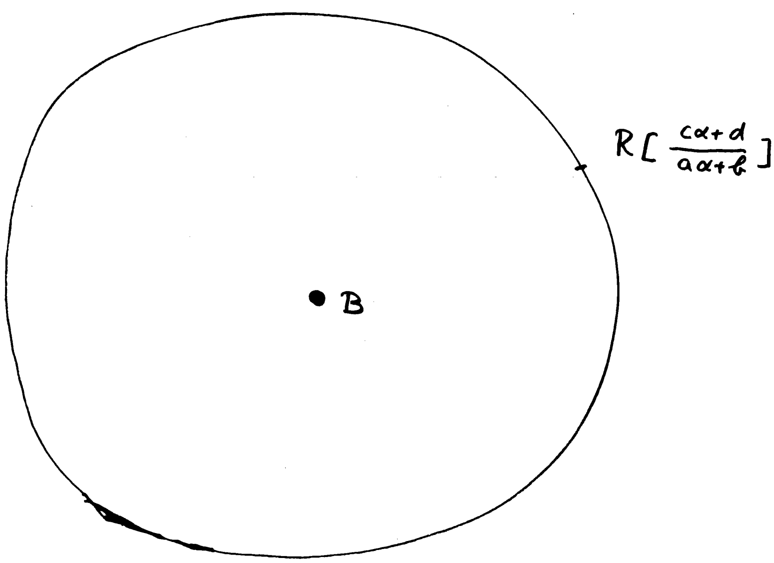
$$R[\gamma] \cap R[\frac{1}{\gamma}] = R[\gamma + f] \cap R[\frac{1}{\gamma + f}] = R[\beta] \cap R[\frac{1}{\beta}].$$

更に

$$R[\gamma] \cap R[\frac{1}{\gamma}] = R[\frac{g}{\alpha + e}] \cap R[\frac{\alpha + e}{g}] = R[\frac{1}{\alpha + e}] \cap R[\alpha + e] = R[\alpha] \cap R[\frac{1}{\alpha}]$$

Case ii も同様である。

今 $B = R[\alpha] \cap R[\frac{1}{\alpha}]$ とおくと



という図が考えられる。

この証明から次の結果は簡単に与えられる

Theorem 5　定理2の仮定のもとで

$$J_{[\alpha]} = J_{[\beta]}$$

可換環 $B = R[\alpha] \cap R[\frac{1}{\alpha}]$ は R 上 integral な環であるが次のようなきれいな R-module としての構造を持つ

$$\zeta_i = \alpha^i + \eta_1 \alpha^{i-1} + \cdots + \eta_i$$ とおくと

$$B = R + I_{[\alpha]}\zeta_1 + \cdots + I_{[\alpha]}\zeta_{d-1} \quad (\text{直和})$$

従って $\varphi_\beta(X) = X^d + \eta_1' X^{d-1} + \cdots + \eta_d'$ とすれば

$$\zeta_i' = \beta^i + \eta_1' \beta^{i-1} + \cdots + \eta_i'$$

$$B = R + I_{[\beta]}\zeta_1' + \cdots + I_{[\alpha]}\zeta_{d-1}'$$

でもある。この Base change matrix はわかっていないが $\beta = \alpha - a$ の場合は富山大修士の山本君によって与えられた。

$$E_{[\alpha]} \overset{\text{def}}{=} \left\{ \frac{c\alpha+d}{a\alpha+b} \;\middle|\; a, b, c, d \in R,\ ad-bc \in R^{\times} \right\}$$

はアフィン分数変換群 ($GL_2(R)$) の orbit であるから

$$E_{[\alpha]} \cap E_{[\beta]} \neq \phi \iff E_{[\alpha]} = E_{[\beta]}$$

従って $E_{[\alpha]} \ni \beta$ であれば $E_{[\alpha]} = E_{[\beta]}$

であることが簡単に分かる。

ここで

$$R\left[\frac{c_1\alpha + d_1}{a_1\alpha + b_1}\right] = R\left[\frac{c_2\alpha + d_2}{a_2\alpha + b_2}\right]$$

となるための条件を与えることが出来るが, localize すれば次の結果が本質的であることがわかる.

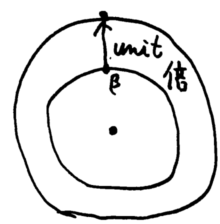
Theorem 6 Assume that R is a Noetherian domain and that α is an anti-integral element of degree t over R. Let a and b be elements of R such that $\alpha - a \neq 0$ and $\alpha - b \neq 0$. Then $R[\frac{1}{\alpha-a}] = R[\frac{1}{\alpha-b}]$ if and only if $a - b \in \sqrt{I_{[\alpha]}\varphi_\alpha(a)} = \sqrt{I_{[\alpha]}\varphi_\alpha(b)}$ and if and only if $R[\alpha, \frac{1}{\alpha-a}] = R[\alpha, \frac{1}{\alpha-b}]$.

最後にもう少し $E_{[\alpha]}$ について考えてみましょう。
これから述べる結果も近々論文として出ますので
結果だけ述べさせていただきます。

Proposition 7

$E_{[\alpha]} \ni {}^{\forall}\beta$ に対して，$\lambda \in K$ について

$$\lambda\beta \in E_{[\alpha]} \Leftrightarrow \lambda \in R^{\times}$$

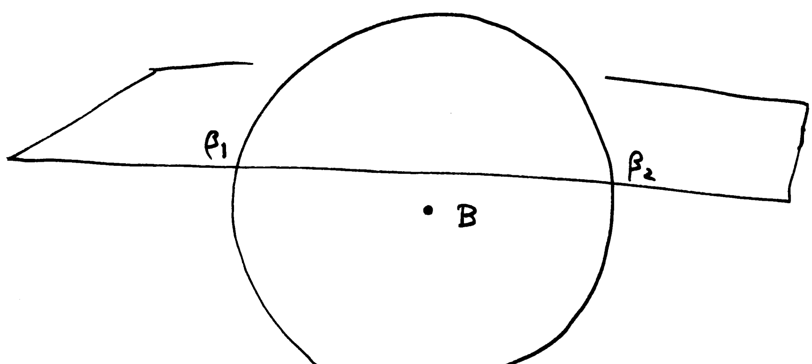


Theorem 8

$E_{[\alpha]} \ni \beta_1, \beta_2$ で $\dfrac{\beta_2}{\beta_1} \notin R^{\times}$ かつ $R^{\times}$ が無限集合であれば $s, t \in R$ $st \neq 0$ であれば

$$s\beta_1 + t\beta_2 \notin E_{[\alpha]}$$

これは



という図が思い浮かぶ。