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第26回可換環論シンポジウム

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序

これは第 26 回可換環論シンポジウムの報告集です．シンポジウムは，2004 年 11 月 24 日 (水) から 27 日 (土) の日程で，倉敷アイビースクエアに於いて開催されました．今回は，Bernd Ulrich 氏 (Purdue Univ., USA) を招待講演者として迎えました．総勢 65 名の参加の下，26 の講演が行われ，大変有意義なものでした．

シンポジウムの開催にあたり，平成１６年度科学研究費補助金 基盤研究 (B)(1)（代表：西田憲司），基盤研究 (B)(2)（代表：吉野雄二），基盤研究 (B)(2)（代表：渡辺敬一）からの援助を頂きました．Ulrich 氏の招聘は，基盤研究 (B)(1)（代表：西田憲司) からの援助により実現しました．ここにあらためてお礼申し上げます．

2005 年 1 月 10 日
青山 陽一（島根大学教育学部）

Preface

This is the Proceedings of the 26-th Symposium on Commutative Ring Theory in Japan, November 24–27, 2004, Kurashiki IVY Square, which was financially supported by Prof. Kenji Nishida (Shinshu Univ.), Prof. Yuuji Yoshino (Okayama Univ.) and Prof. Kei-ichi Watanabe (Nihon Univ.) (Grant-in-Aid for Scientific Research of Japan Society for the Promotion of Science). We had 65 participants, including a guest speaker Prof. Bernd Ulrich (Purdue Univ., USA), and there were 26 lectures. The members of the organizing committee would like to express their hearty thanks for valuable contribution to the conference of all the participants.

January 10, 2005
Yoichi Aoyama (Shimane Univ.)

Contents

DUALIZING COMPLEX OF THE INCIDENCE ALGEBRA OF A FINITE REGULAR CELL COMPLEX ARISING FROM POINCARÉ-VERDIER DUALITY

KOHJI YANAGAWA

This note is an abridged version of my recent paper [13]. The formal version has been submitted for publication elsewhere.

ABSTRACT. Let Σ be a finite regular cell complex with $\emptyset \in \Sigma$, and regard it as a *poset* (i.e., partially ordered set) by inclusion. Let R be the incidence algebra of the poset Σ over a field k. Corresponding to the Verdier duality for constructible sheaves on Σ, we have a dualizing complex $\omega^\bullet \in D^b(\text{mod}_{R\otimes_k R})$ giving a duality functor from $D^b(\text{mod}_R)$ to itself. $\omega^\bullet$ satisfies the Auslander condition. Our duality is somewhat analogous to the Serre duality for projective schemes (the cell $\emptyset \in \Sigma$ plays a similar role to that of "irrelevant ideals"). If $H^i(\omega^\bullet) \neq 0$ for exactly one i, then the underlying topological space of Σ is Cohen-Macaulay (in the sense of the Stanley-Reisner ring theory). The converse also holds when Σ is a simplicial complex. R is always a Koszul ring with $R^! \cong R^{\text{op}}$. The relation between the Koszul duality for R and the Verdier duality is discussed. This result is a variant of a theorem of Vybornov.

1. PREPARATION

A *finite regular cell complex* (c.f. [2, §6.2] and [3]) is a non-empty topological space X together with a finite set Σ of subsets of X such that:

(i) $\emptyset \in \Sigma$ and $X = \bigcup_{\sigma\in\Sigma} \sigma$;
(ii) the subsets $\sigma \in \Sigma$ are pairwise disjoint;
(iii) for each $\sigma \in \Sigma$, $\sigma \neq \emptyset$, there exists a homeomorphism from an i-dimensional disc $B^i = \{x \in \mathbb{R}^i \mid ||x|| \leq 1\}$ onto the closure $\bar\sigma$ of σ which maps the open disc $U^i = \{x \in \mathbb{R}^i \mid ||x|| < 1\}$ onto σ.

An element $\sigma \in \Sigma$ is celled a *cell*. We regard Σ as a poset by $\sigma \geq \tau \overset{\text{def}}{\Longleftrightarrow} \bar\sigma \supset \tau$. If $\sigma \in \Sigma$ is homeomorphic to U^i, set $\dim\sigma = i$. Here $\dim\emptyset = -1$. We also set $\dim X = \max\{\dim\sigma \mid \sigma \in \Sigma\}$.

A finite simplicial complex is a primary example of finite regular cell complexes. A convex polytope P can be regarded as a finite regular cell complex. Here cells are the relative interior of the faces of P. More generally, polyhedral complexes are regular cell complexes.

Let $\sigma, \sigma' \in \Sigma$. If $\dim\sigma = i+1$, $\dim\sigma' = i-1$ and $\sigma' < \sigma$, then there are exactly two cells $\sigma_1, \sigma_2 \in \Sigma$ between σ' and σ. (Here $\dim\sigma_1 = \dim\sigma_2 = i$.) A remarkable property of a regular cell complex is the existence of an *incidence function* ε (c.f. [3, II. Definition 1.8]). The definition of an incidence function is the following.

(i) To each pair (σ, σ') of cells, ε assigns a number $\varepsilon(\sigma, \sigma') \in \{0, \pm1\}$.

(ii) $\varepsilon(\sigma, \sigma') \neq 0$ if and only if $\dim \sigma' = \dim \sigma - 1$ and $\sigma' < \sigma$.
(iii) If $\dim \sigma = 0$, then $\varepsilon(\sigma, \emptyset) = 1$.
(iv) If $\dim \sigma = i+1$, $\dim \sigma' = i-1$ and $\sigma' < \sigma_1, \sigma_2 < \sigma$, $\sigma_1 \neq \sigma_2$, then we have $\varepsilon(\sigma, \sigma_1)\, \varepsilon(\sigma_1, \sigma') + \varepsilon(\sigma, \sigma_2)\, \varepsilon(\sigma_2, \sigma') = 0$.

We can compute the (co)homology groups of X using the cell decomposition Σ and an incidence function ε.

Let P be a finite poset. The incidence algebra R of P over a field k is the k-vector space with a basis $\{e_{x,y} \mid x, y \in P \text{ with } x \geq y\}$. The k-bilinear multiplication defined by $e_{x,y}\, e_{z,w} = \delta_{y,z}\, e_{x,w}$ makes R a finite dimensional associative k-algebra. Set $e_x := e_{x,x}$. Then $1 = \sum_{x \in P} e_x$ and $e_x\, e_y = \delta_{x,y}\, e_x$. We have $R \cong \bigoplus_{x \in P} Re_x$ as a left R-module, and each Re_x is indecomposable.

Denote the category of finitely generated left R-modules by mod_R. If $N \in \mathrm{mod}_R$, we have $N = \bigoplus_{x \in P} N_x$ as a k-vector space, where $N_x := e_x N$. Note that $e_{x,y}\, N_y \subset N_x$ and $e_{x,y}\, N_z = 0$ for $y \neq z$. If $f : N \to N'$ is an R-morphism, then $f(N_x) \subset N'_x$.

For each $x \in P$, we can construct an indecomposable injective module $E_R(x) \in \mathrm{mod}_R$. (When confusion does not occur, we simply denote it by $E(x)$.) Let $E(x)$ be the k-vector space with basis $\{e(x)_y \mid y \leq x\}$. Then we can regard $E(x)$ as a left R-module by

$$e_{z,w}\, e(x)_y = \begin{cases} e(x)_z & \text{if } y = w \text{ and } z \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $E(x)_y = k\, e(x)_y$ if $y \leq x$, and $E(x)_y = 0$ otherwise. An indecomposable injective in mod_R is of the form $E(x)$ for some $x \in P$. Since $\dim_k R < \infty$, mod_R has enough projectives and injectives. Note that R has a finite global dimension.

Let Σ be a finite regular cell complex, and X its underlying topological space. We make Σ a poset as above. In the rest of this paper, R is the incidence algebra of Σ over k. For $M \in \mathrm{mod}_R$, we have $M = \bigoplus_{\sigma \in \Sigma} M_\sigma$ as a k-vector space, where $M_\sigma := e_\sigma M$.

Let $\mathrm{Sh}(X)$ be the category of sheaves of finite dimensional k-vector spaces on X. We say $\mathcal{F} \in \mathrm{Sh}(X)$ is a *constructible sheaf* with respect to the cell decomposition Σ, if $\mathcal{F}|\sigma$ is a constant sheaf for all $\emptyset \neq \sigma \in \Sigma$. Here, $\mathcal{F}|\sigma$ denotes the inverse image $j^*\mathcal{F}$ of $\mathcal{F}$ by the embedding map $j : \sigma \to X$. Let $\mathrm{Sh}_c(X)$ be the full subcategory of $\mathrm{Sh}(X)$ consisting of constructible sheaves with respect to Σ. It is well-known that $D^b(\mathrm{Sh}_c(X)) \cong D^b_{\mathrm{Sh}_c(X)}(\mathrm{Sh}(X))$. (See [5, Theorem 8.1.11]. There, it is assumed that Σ is a simplicial complex. But this assumption is irrelevant. In fact, the key lemma [5, Corollay 8.1.5] also holds for regular cell complexes. See also [8, Lemma 5.2.1].) So we will freely identify these categories.

There is a functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$ which is well-known to specialists (see for example [9, Theorem A]). But we give a precise construction here for the reader's convenience. See [9, 11] for detail.

For $M \in \mathrm{mod}_R$, consider the set

$$\mathrm{Sp\acute{e}}(M) := \coprod_{\emptyset \neq \sigma \in \Sigma} \sigma \times M_\sigma.$$

Let $\pi : \text{Spé}(M) \to X$ be the projection map which sends $(p, m) \in \sigma \times M_\sigma \subset \text{Spé}(M)$ to $p \in \sigma \subset X$. For an open subset $U \subset X$ and a map $s : U \to \text{Spé}(M)$, we will consider the following conditions:

(∗) $\pi \circ s = \text{Id}_U$ and $s_q = e_{\tau,\sigma} \cdot s_p$ for all $p \in \sigma$, $q \in \tau$ with $\tau \geq \sigma$. Here s_p (resp. s_q) is the element of M_σ (resp. M_τ) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).

(∗∗) There is an open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ such that the restriction of s to U_λ satisfies (∗) for all $\lambda \in \Lambda$.

Now we define a sheaf $M^\dagger \in \text{Sh}_c(X)$ from M as follows. For an open set $U \subset X$, set

$$M^\dagger(U) := \{\, s \mid s : U \to \text{Spé}(M) \text{ is a map satisfying } (**) \,\}$$

and the restriction map $M^\dagger(U) \to M^\dagger(V)$ is the natural one. It is easy to see that $M^\dagger$ is a constructible sheaf. For $\sigma \in \Sigma$, let $U_\sigma := \bigcup_{\tau \geq \sigma} \tau$ be an open set of X. Then we have $M^\dagger(U_\sigma) \cong M_\sigma$. Moreover, if $\sigma \leq \tau$, then we have $U_\sigma \supset U_\tau$ and the restriction map $M^\dagger(U_\sigma) \to M^\dagger(U_\tau)$ corresponds to the multiplication map $M_\sigma \ni x \mapsto e_{\tau,\sigma} x \in M_\tau$. For a point $p \in \sigma$, the stalk $(M^\dagger)_p$ of $M^\dagger$ at p is isomorphic to M_σ. We have the exact functor $(-)^\dagger : \text{mod}_R \to \text{Sh}_c(X)$.

For example, we have $E(\sigma)^\dagger \cong j_* \underline{k}_{\bar{\sigma}}$, where j is the embedding map from the closure $\bar{\sigma}$ of σ to X and $\underline{k}_{\bar{\sigma}}$ is the constant sheaf on $\bar{\sigma}$. Similarly, we have $(Re_\sigma)^\dagger \cong h_! \underline{k}_{U_\sigma}$, where h is the embedding map from the open subset U_σ to X.

Remark 1.1. Let $\Sigma' := \Sigma \setminus \emptyset$ be a subposet, and T its incidence algebra over k. Then we have a functor $\text{mod}_T \to \text{Sh}_c(X)$ defined by the same way as $(-)^\dagger$, and it gives an equivalence $\text{mod}_T \cong \text{Sh}_c(X)$ (c.f. [9, Theorem A]). On the other hand, by virtue of $\emptyset \in \Sigma$, our $(-)^\dagger : \text{mod}_R \to \text{Sh}_c(X)$ is neither full nor faithful. But we will see that mod_R have several interesting features which mod_T does not possess.

For $M \in \text{mod}_R$, set $\Gamma_\emptyset(M) := \{\, x \in M_\emptyset \mid Rx \subset M_\emptyset \,\}$. It is easy to see that $\Gamma_\emptyset(M) \cong \text{Hom}_R(k, M)$. Here we regard k as a left R-module by $e_{\sigma,\tau}\, k = 0$ for all $e_{\sigma,\tau} \neq e_\emptyset$. Clearly, $\Gamma_\emptyset$ gives a left exact functor from mod_R to itself (or vect_k). By the definition, we have $(M/\Gamma_\emptyset(M))^\dagger \cong M^\dagger$. We denote the ith right derived functor of $\Gamma_\emptyset(-)$ by $H^i_\emptyset(-)$. In other words, $H^i_\emptyset(-) = \text{Ext}^i_R(k, -)$.

Theorem 1.2 (c.f. [11, Theorem 3.3]). *For $M \in \text{mod}_R$, we have an isomorphism*

$$H^i(X, M^\dagger) \cong H^{i+1}_\emptyset(M) \quad \textit{for all } i \geq 1,$$

and an exact sequence

$$0 \to H^0_\emptyset(M) \to M_\emptyset \to H^0(X, M^\dagger) \to H^1_\emptyset(M) \to 0.$$

Here $H^\bullet(X, M^\dagger)$ stands for the cohomology with coefficients in the sheaf $M^\dagger$.

Proof. Let $I^\bullet$ be an injective resolution of M, and consider the exact sequence

$$0 \to \Gamma_\emptyset(I^\bullet) \to I^\bullet \to I^\bullet/\Gamma_\emptyset(I^\bullet) \to 0 \tag{1.1}$$

of cochain complexes. Put $J^\bullet := I^\bullet/\Gamma_\emptyset(I^\bullet)$. Each component of $J^\bullet$ is a direct sum of copies of $E(\sigma)$ for various $\emptyset \neq \sigma \in \Sigma$. Since $E(\sigma)^\dagger$ is the constant sheaf on $\bar{\sigma}$ which is homeomorphic to a closed disc, we have $H^i(X, E(\sigma)^\dagger) = H^i(\bar{\sigma}; k) = 0$ for all $i \geq 1$. Hence $(J^\bullet)^\dagger\, (\cong (I^\bullet)^\dagger\,)$ gives a $\Gamma(X, -)$-acyclic resolution of $M^\dagger$. It is easy to see that $[J^\bullet]_\emptyset \cong \Gamma(X, (J^\bullet)^\dagger)$. So the assertions follow from (1.1), since $H^0(I^\bullet) \cong M$ and $H^i(I^\bullet) = 0$ for all $i \geq 1$. □

Remark 1.3. (1) We regard a polynomial ring $S := k[x_0, \dots, x_n]$ as a graded ring with $\deg(x_i) = 1$ for each i. Let $I \subset S$ be a graded ideal, and set $A := S/I$. For a graded A-module M, we have the algebraic quasi-coherent sheaf $\tilde{M}$ on the projective scheme $Y := \operatorname{Proj} A$. It is well-known that $H^i(Y, \tilde{M}) \cong [H^{i+1}_{\mathfrak{m}}(M)]_0$ for all $i \geq 1$, and

$$0 \to [H^0_{\mathfrak{m}}(M)]_0 \to M_0 \to H^0(Y, \tilde{M}) \to [H^1_{\mathfrak{m}}(M)]_0 \to 0 \qquad \text{(exact)}.$$

Here $H^i_{\mathfrak{m}}(M)$ stands for the local cohomology module with support in the irrelevant ideal $\mathfrak{m} := (x_0, \dots, x_n)$, and $[H^i_{\mathfrak{m}}(M)]_0$ is its degree 0 component ($H^i_{\mathfrak{m}}(M)$ has a natural $\mathbb{Z}$-grading). See also Remark 3.6 (2) below.

(2) Assume that Σ is a simplicial complex with n vertices. The *Stanley-Reisner ring* $k[\Sigma]$ of Σ is the quotient ring of the polynomial ring $k[x_1, \dots, x_n]$ by the squarefree monomial ideal I_Σ corresponding to Σ. In [10], we defined *squarefree* $k[\Sigma]$-modules which are certain $\mathbb{N}^n$-graded $k[\Sigma]$-modules. For example, $k[\Sigma]$ itself is squarefree. The category $\operatorname{Sq}(\Sigma)$ of squarefree $k[\Sigma]$-modules is equivalent to mod_R of the present paper (see [12]). Let $\Psi : \operatorname{mod}_R \to \operatorname{Sq}(\Sigma)$ be the functor giving this equivalence. In [11], we defined a functor $(-)^+ : \operatorname{Sq}(\Sigma) \to \operatorname{Sh}_c(X)$. For example, $k[\Sigma]^+ \cong \underline{k}_X$. It is easy to see that $(-)^\dagger \cong (-)^+ \circ \Psi$. For $M \in \operatorname{mod}_R$, we have $H^i_\emptyset(M) \cong H^i_{\mathfrak{m}}(\Psi(M))_0$. So the above theorem is a variation of [11, Theorem 3.3].

2. Dualizing complexes

Let $D^b(\operatorname{mod}_R)$ be the bounded derived category of mod_R. For $M^\bullet \in D^b(\operatorname{mod}_R)$ and $i \in \mathbb{Z}$, $M^\bullet[i]$ denotes the i^{th} translation of $M^\bullet$, that is, $M^\bullet[i]$ is the complex with $M^\bullet[i]^j = M^{i+j}$. So, if $M \in \operatorname{mod}_R$, $M[i]$ is the cochain complex $\cdots \to 0 \to M \to 0 \to \cdots$, where M sits in the $(-i)^{\text{th}}$ position.

In this section, from Verdier's dualizing complex $\mathcal{D}^\bullet_X \in D^b(\operatorname{Sh}_c(X))$, we construct a cochain complex $\omega^\bullet$ of injective left $(R \otimes_k R)$-modules which gives a duality functor from $D^b(\operatorname{mod}_R)$ to itself. Let M be a left $(R \otimes_k R)$-module. When we regard M as a left R-module via a ring homomorphism $R \ni x \mapsto x \otimes 1 \in R \otimes_k R$ (resp. $R \ni x \mapsto 1 \otimes x \in R \otimes_k R$), we denote it by ${}_RM$ (resp. $M_{R^{\mathrm{op}}}$).

For $i \leq 1$, the i^{th} component ω^i of $\omega^\bullet$ has a k-basis

$$\{ e(\sigma)^\tau_\rho \mid \sigma, \tau, \rho \in \Sigma, \dim \sigma = -i, \sigma \geq \tau, \rho \},$$

and its module structure is defined by

$$(e_{\sigma', \tau'} \otimes 1) \cdot e(\sigma)^\tau_\rho = \begin{cases} e(\sigma)^\tau_{\sigma'} & \text{if } \tau' = \rho \text{ and } \sigma' \leq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(1 \otimes e_{\sigma', \tau'}) \cdot e(\sigma)^\tau_\rho = \begin{cases} e(\sigma)^{\sigma'}_\rho & \text{if } \tau' = \tau \text{ and } \sigma' \leq \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have ${}_R(\omega^i) \cong (\omega^i)_{R^{\mathrm{op}}} \cong \bigoplus_{\dim \sigma = -i} E(\sigma)^{\mu(\sigma)}$, where $\mu(\sigma) := \#\{\tau \in \Sigma \mid \tau \leq \sigma\}$. Note that $R \otimes_k R$ is isomorphic to the incidence algebra of the poset $\Sigma \times \Sigma$. For each $\sigma \in \Sigma$ with $\dim \sigma = -i$, set $I(\sigma)$ to be the subspace $\langle\, e(\sigma)^{\tau}_{\rho} \mid \tau,\, \rho \leq \sigma \,\rangle$ of ω^i. Then, as a left $R \otimes_k R$-module, $I(\sigma)$ is isomorphic to the injective module $E_{R\otimes_k R}(\,(\sigma, \sigma)\,)$, and $\omega^i \cong \bigoplus_{\dim \sigma = -i} I(\sigma)$. The differential of $\omega^\bullet$ is given by

$$\omega^i \ni e(\sigma)^{\tau}_{\rho} \longmapsto \sum_{\sigma' \geq \tau,\, \rho} \varepsilon(\sigma, \sigma') \cdot e(\sigma')^{\tau}_{\rho} \in \omega^{i+1}$$

makes $\omega^\bullet$ a complex of $(R \otimes_k R)$-modules.

Let $M \in \mathrm{mod}_R$. Using the left R-module structure $I(\sigma)_{R^{\mathrm{op}}}$, $\mathrm{Hom}_R(M, {}_RI(\sigma))$ can be regarded as a left R-module again. Moreover, we have the following.

Lemma 2.1. *For $M \in \mathrm{mod}_R$, we have $\mathrm{Hom}_R(M, {}_RI(\sigma)) \cong E(\sigma) \otimes_k (M_\sigma)^\vee$ as left R-modules. Here $(M_\sigma)^\vee$ is the dual vector space $\mathrm{Hom}_k(M_\sigma, k)$ of M_σ.*

Proof. Set $M_{\geq \sigma} = \bigoplus_{\tau \in \Sigma, \tau \geq \sigma} M_\tau$ to be a submodule of M. By the short exact sequence $0 \to M_{\geq\sigma} \to M \to M/M_{\geq\sigma} \to 0$, we have

$$0 \to \mathrm{Hom}_R(M/M_{\geq\sigma}, {}_RI(\sigma)) \to \mathrm{Hom}_R(M, {}_RI(\sigma)) \to \mathrm{Hom}_R(M_{\geq\sigma}, {}_RI(\sigma)) \to 0.$$

Since $(M/M_{\geq\sigma})_\sigma = 0$, we have $\mathrm{Hom}_R(M/M_{\geq\sigma}, {}_RI(\sigma)) = 0$. So we may assume that $M = M_{\geq\sigma}$. Let $\{f_1, \ldots, f_n\}$ be a k-basis of $(M_\sigma)^\vee$. Since $({}_RI(\sigma))_\tau = 0$ for $\tau > \sigma$, $\mathrm{Hom}_R(M_{\geq\sigma}, {}_RI(\sigma))$ has a k-basis $\{\, e(\sigma)^{\tau}_{\sigma} \otimes f_i \mid \tau \leq \sigma,\ 1 \leq i \leq n \,\}$. □

In the sequel, we simply denote $\mathrm{Hom}_R(-, {}_R\omega^i)$ by $\mathrm{Hom}_R(-, \omega^i)$, etc.

For a bounded complex $M^\bullet$ in mod_R, set $\mathbf{D}(M^\bullet) := \mathrm{Hom}^\bullet_R(M^\bullet, \omega^\bullet)$. Since each ${}_R\omega^i$ is injective, we have $\mathbf{D}(M^\bullet) \cong \mathbf{R}\mathrm{Hom}_R(M^\bullet, \omega^\bullet)$. And $\mathbf{D}$ gives a contravariant functor from $D^b(\mathrm{mod}_R)$ to itself.

We can describe $\mathbf{D}(M^\bullet)$ explicitly. Since $\omega^i \cong \bigoplus_{\dim\sigma=-i} I(\sigma)$, we have

$$\mathrm{Hom}_R(M, \omega^i) \cong \bigoplus_{\dim\sigma=-i} \mathrm{Hom}_R(M, I(\sigma)) \cong \bigoplus_{\dim\sigma=-i} E(\sigma) \otimes_k (M_\sigma)^\vee$$

for $M \in \mathrm{mod}_R$ by Lemma 2.1. So we can easily check that $\mathbf{D}(M)$ is of the form

$$\mathbf{D}(M) : 0 \longrightarrow \mathbf{D}^{-d}(M) \longrightarrow \mathbf{D}^{-d+1}(M) \longrightarrow \cdots \longrightarrow \mathbf{D}^{1}(M) \longrightarrow 0,$$

$$\mathbf{D}^i(M) = \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma=-i}} E(\sigma) \otimes_k (M_\sigma)^\vee.$$

Here the differential sends $e(\sigma)_\rho \otimes f \in E(\sigma) \otimes_k (M_\sigma)^\vee$ to

$$\sum_{\tau\in\Sigma, \tau\geq\rho} \varepsilon(\sigma, \tau) \cdot e(\tau)_\rho \otimes f(e_{\sigma,\tau} -) \in \bigoplus_{\substack{\tau\in\Sigma \\ \dim\tau=\dim\sigma-1}} E(\tau) \otimes_k (M_\tau)^\vee.$$

For a bounded cochain complex $M^\bullet$ in mod_R, we have

$$\mathbf{D}^t(M^\bullet) = \bigoplus_{i-j=t} \mathbf{D}^i(M^j) = \bigoplus_{\substack{\sigma\in\Sigma, j\in\mathbb{Z} \\ -\dim\sigma-j=t}} E(\sigma) \otimes_k (M^j_\sigma)^\vee,$$

and the differential is given by

$$\mathbf{D}^t(M^\bullet) \supset E(\sigma) \otimes_k (M_\sigma^j)^\vee \ni x \otimes y \mapsto d(x \otimes y) + (-1)^t (x \otimes \partial^\vee(y)) \in \mathbf{D}^{t+1}(M^\bullet),$$

where $\partial^\vee : (M_\sigma^j)^\vee \to (M_\sigma^{j-1})^\vee$ is the k-dual of the differential ∂ of $M^\bullet$, and d is the differential of $\mathbf{D}(M^j)$.

Since the underlying space X of Σ is locally compact and finite dimensional, it admits Verdier's dualizing complex $\mathcal{D}_X^\bullet \in D^b(\mathrm{Sh}(X))$ with the coefficients in k (see [4, V. §2]).

Theorem 2.2. *For $M^\bullet \in D^b(\mathrm{mod}_R)$, we have*

$$\mathbf{D}(M^\bullet)^\dagger \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}_X^\bullet).$$

Proof. An explicit description of $\mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}_X^\bullet)$ is given in the unpublished thesis [8] of A. Shepard. When Σ is a simplicial complex, this description is treated in [9, §2.4], and also follows from the author's previous paper [11] (and [12]). The general case can be reduced to the simplicial complex case using the barycentric subdivision. Shepard's description of $\mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}_X^\bullet)$ is the same thing as the above mentioned description of $\mathbf{D}(M^\bullet)$ under the functor $(-)^\dagger$. □

Lemma 2.3. *For each $\sigma \in \Sigma$, the natural map $E(\sigma) \to \mathbf{D} \circ \mathbf{D}(E(\sigma))$ is an isomorphism in $D^b(\mathrm{mod}_R)$.*

Proof. We may assume that $\sigma \neq \emptyset$. Put $C^\bullet := \mathbf{D}(E(\sigma))$, and let $\tau_{\leq 0}C^\bullet : \cdots \to C^{-1} \to C^0 \to 0 \to \cdots$ be the truncated complex of $C^\bullet$. Note that $(\tau_{\leq 0}C^\bullet)^\dagger = (C^\bullet)^\dagger$. Since $E(\sigma)^\dagger$ is the constant sheaf over the closure $\overline{\sigma}$ which is homeomorphic to a closed disc, we have $(C^\bullet)^\dagger \cong j_! \underline{k}_\sigma[\dim \sigma]$ in $D^b(\mathrm{Sh}(X))$ by the Poincaré-Verdier duality. Here $j : \sigma \to X$ is the embedding map. Note that $(C^\bullet)^\dagger$ is a $\Gamma(X, -)$-acyclic resolution of $j_!\underline{k}_\sigma[\dim \sigma]$ and $\Gamma(X, (C^\bullet)^\dagger) \cong (\tau_{\leq 0}C^\bullet)_\emptyset$ as cochain complexes of k-vector spaces. Hence we have $H^i(\tau_{\leq 0}C^\bullet)_\emptyset \cong H^{i+\dim\sigma}(X, j_!\underline{k}_\sigma) \cong H_c^{i+\dim\sigma}(\sigma, \underline{k}_\sigma) \cong H_c^{i+\dim\sigma}(\sigma; k)$. Here $H_c^i(-)$ stands for the cohomology with the compact supports. Therefore $H^0(\tau_{\leq 0}C^\bullet)_\emptyset = k$ and $H^i(\tau_{\leq 0}C^\bullet)_\emptyset = 0$ for all $i \neq 0$. Since $(C^1)_\emptyset = k$, $(C^i)_\emptyset = 0$ for all $i > 1$, and the differential map $(C^0)_\emptyset \to (C^1)_\emptyset$ is non-zero, we have $H^i(C^\bullet)_\emptyset = 0$ for all i. Hence $C^\bullet \cong M[\dim \sigma]$ in $D^b(\mathrm{mod}_R)$, where M is a simple module with $M = M_\sigma = k$. So $\mathbf{D} \circ \mathbf{D}(E(\sigma)) \cong \mathbf{D}(M[\dim \sigma]) \cong E(\sigma)$. □

Theorem 2.4. *(1) $\omega^\bullet \in D^b(\mathrm{mod}_{R \otimes_k R})$ is a dualizing complex in the sense of* [14, Definition 1.1].

(2) The dualizing complex $\omega^\bullet$ satisfies the Auslander condition in the sense of [14, Definition 2.1].

Proof. (1) The conditions (i) and (ii) of [14, Definition 1.1] obviously hold in our case. So it remains to prove the condition (iii). To prove this, it suffices to show that the natural morphism $R \to \mathbf{D} \circ \mathbf{D}(R)$ is an isomorphism. But this follows from "Lemma on Way-out Functors" and Lemma 2.3.

(2) For a non-zero module $M \in \mathrm{mod}_R$, set $j_\omega(M) := \min\{\, i \mid \mathrm{Ext}_R^i(M, \omega^\bullet) \neq 0 \,\}$. By the description of $\mathbf{D}(M)$, we have $j_\omega(M) = -\max\{\, \dim \sigma \mid \sigma \in \Sigma, M_\sigma \neq 0 \,\}$, and $\mathrm{Ext}_R^i(M, \omega^\bullet)_\sigma = 0$ for $\sigma \in \Sigma$ with $\dim \sigma > -i$. Hence, for any submodule $N \subset \mathrm{Ext}_R^i(M, \omega^\bullet)$, we have $j_\omega(N) \geq i$. □

Corollary 2.5. *We have* $\mathrm{Ext}^i_R(M^\bullet, \omega^\bullet)_\emptyset \cong H_\emptyset^{-i+1}(M^\bullet)^\vee$ *for all* $i \in \mathbb{Z}$ *and all* $M^\bullet \in D^b(\mathrm{mod}_R)$.

Proof. Since $\mathbf{D}\circ\mathbf{D}(M^\bullet)$ is an injective resolution of $M^\bullet$, we have $\mathbf{R}\Gamma_\emptyset(M^\bullet) = \Gamma_\emptyset(\mathbf{D}\circ\mathbf{D}(M^\bullet))$. By the structure of $\mathbf{D}(-)$, we have $\Gamma_\emptyset(\mathbf{D}\circ\mathbf{D}(M^\bullet)) = (\mathbf{D}(M^\bullet)_\emptyset)^\vee[-1]$. So we are done. □

Proposition 2.6. *For any* $\sigma \in \Sigma$, $\mathbf{D}(Re_\sigma)^\dagger \cong \mathbf{R}j_*\mathcal{D}^\bullet_{U_\sigma}$ *where* $j : U_\sigma := \bigcup_{\tau\geq\sigma}\tau \to X$ *is the embedding map. In particular,* $\mathbf{D}(Re_\emptyset)^\dagger \cong \mathcal{D}^\bullet_X$.

Proof. See [13, Proposition 5.2]. □

3. Miscellaneous Remarks

The former half of this section is closely related to Shepard's thesis [8].

For $M, N \in \mathrm{mod}_R$ and $\sigma \in \Sigma$, set $\underline{\mathrm{Hom}}_R(M,N)_\sigma := \mathrm{Hom}_R(M_{\geq\sigma}, N)$. We make $\underline{\mathrm{Hom}}_R(M,N) := \bigoplus_{\sigma\in\Sigma}\underline{\mathrm{Hom}}_R(M,N)_\sigma$ a left R-module as follows: For $f \in \underline{\mathrm{Hom}}_R(M,N)_\sigma$ and a cell τ with $\tau \geq \sigma$, set $e_{\tau,\sigma}f$ to be the restriction of f into the submodule $M_{\geq\tau}$ of $M_{\geq\sigma}$.

Lemma 3.1. *For* $M \in \mathrm{mod}_R$, *we have* $\underline{\mathrm{Hom}}_R(M, E(\sigma)) \cong E(\sigma)\otimes_k (M_\sigma)^\vee$.

Proof. Similar to Lemma 2.1. □

If a complex $M^\bullet$ in mod_R is exact, then so is $\underline{\mathrm{Hom}}_R(M^\bullet, E(\sigma))$ by Lemma 3.1. Moreover, if $M^\bullet$ is bounded and exact, and $I^\bullet$ is a bounded complex such that I^i is injective for all i, then $\underline{\mathrm{Hom}}^\bullet_R(M^\bullet, I^\bullet)$ is exact.

For cells $\sigma, \tau \in \Sigma$, consider the subset $\{\rho \in \Sigma \mid \rho \geq \sigma, \tau\}$. Assume that this subset has the minimum element (we denote it by $\sigma\vee\tau$) for any $\sigma, \tau \in \Sigma$. (If Σ is a simplicial complex, then this condition is satisfied.) Since $\underline{\mathrm{Hom}}_R(Re_\sigma, N)_\tau \cong N_{\sigma\vee\tau}$ in this case, the complex $\underline{\mathrm{Hom}}^\bullet_R(Re_\sigma, N^\bullet)$ is exact for an exact complex $N^\bullet$. Hence if $N^\bullet$ and $P^\bullet$ are bounded, $N^\bullet$ is exact, and each P^i is projective, then $\mathrm{Hom}^\bullet_R(P^\bullet, N^\bullet)$ is exact. Thus we have the following lemma.

Lemma 3.2. *(1) If* $I^\bullet$ *is an injective resolution of* $N^\bullet \in D^b(\mathrm{mod}_R)$, *then*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet) \cong \underline{\mathrm{Hom}}^\bullet_R(M^\bullet, I^\bullet).$$

(2) Assume that, for each $\sigma, \tau \in \Sigma$, *the subset* $\{\rho \in \Sigma \mid \rho \geq \sigma, \tau\}$ *has the minimum element. Then*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet) \cong \underline{\mathrm{Hom}}^\bullet_R(P^\bullet, N^\bullet)$$

for a projective resolution $P^\bullet$ *of* $M^\bullet \in D^b(\mathrm{mod}_R)$.

Remark 3.3. The additional assumption on $\{\rho \in \Sigma \mid \rho \geq \sigma, \tau\}$ in Lemma 3.2 (2) is really necessary. Without this, there is an easy example with $\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet) \not\cong \underline{\mathrm{Hom}}^\bullet_R(P^\bullet, N^\bullet)$. See [13, Example 4.3].

By Lemmas 3.1 and 3.2, we have the following. Recall that $\mathbf{D}(Re_\emptyset)^\dagger = \mathcal{D}^\bullet_X$.

Proposition 3.4. *If* $M^\bullet \in D^b(\mathrm{Sh}_c(X))$, *then* $\mathbf{D}(M^\bullet) \cong \mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, \mathbf{D}(Re_\emptyset))$.

If $\mathcal{F},\mathcal{G} \in \mathrm{Sh}_c(X)$, then it is easy to see that $\mathcal{H}om(\mathcal{F},\mathcal{G}) \in \mathrm{Sh}_c(X)$. For $M,N \in \mathrm{mod}_R$ and $\emptyset \neq \sigma \in \Sigma$, we have $\mathcal{H}om(M^\dagger, N^\dagger)(U_\sigma) = \mathrm{Hom}_{\mathrm{Sh}(U_\sigma)}(M^\dagger|_{U_\sigma}, N^\dagger|_{U_\sigma}) = \mathrm{Hom}_R(M_{\geq\sigma}, N_{\geq\sigma}) = \mathrm{Hom}_R(M_{\geq\sigma}, N) = \underline{\mathrm{Hom}}_R(M,N)_\sigma$. Hence

$$\underline{\mathrm{Hom}}_R(M,N)^\dagger \cong \mathcal{H}om(M^\dagger, N^\dagger).$$

If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(\mathrm{Sh}_c(X))$, then $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet,\mathcal{G}^\bullet) \in D^b(\mathrm{Sh}_c(X))$ (see [5, Proposition 8.4.10]). Thus we can use an injective resolution of $\mathcal{G}^\bullet$ in $D^b(\mathrm{Sh}_c(X))$ to compute $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet,\mathcal{G}^\bullet)$. If $I^\bullet$ is an injective resolution of $N^\bullet \in D^b(\mathrm{mod}_R)$, then $(I^\bullet)^\dagger$ gives an injective resolution of $(N^\bullet)^\dagger$ in $D^b(\mathrm{Sh}_c(X))$. Hence we have the following.

Proposition 3.5 ([8, Theorem 5.2.5]). *If $M^\bullet, N^\bullet \in D^b(\mathrm{mod}_R)$, then*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet)^\dagger \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, (N^\bullet)^\dagger).$$

By Lemma 3.2 (2), if $\sigma \vee \tau \in \Sigma$ exists for any $\sigma,\tau \in \Sigma$ (e.g., Σ is a simplicial complex), then we can compute $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet,\mathcal{G}^\bullet)$ for $\mathcal{F}^\bullet,\mathcal{G}^\bullet \in D^b(\mathrm{Sh}_c(X))$ using a projective resolution of $\mathcal{F}^\bullet$ in $D^b(\mathrm{Sh}_c(X))$.

Remark 3.6. Let $\mathrm{mod}_\emptyset$ be the full subcategory of mod_R consisting of modules M with $M_\sigma = 0$ for all $\sigma \neq \emptyset$. In other words, $M \in \mathrm{mod}_\emptyset$ if and only if $j_\omega(M) = 1$, where j_ω is the invariant defined in the proof of Theorem 2.4 (2). Then $\mathrm{mod}_\emptyset$ is a *dense subcategory* of mod_R. That is, for a short exact sequence $0 \to M' \to M \to M'' \to 0$ in mod_R, M is in $\mathrm{mod}_\emptyset$ if and only if M' and M'' are in $\mathrm{mod}_\emptyset$. So we have the quotient category $\mathrm{mod}_R / \mathrm{mod}_\emptyset$ by [7, Theorem 4.3.3]. Let $\pi : \mathrm{mod}_R \to \mathrm{mod}_R / \mathrm{mod}_\emptyset$ be the canonical functor. It is easy to see that $\pi(M) \cong \pi(M')$ if and only if $M_{>\emptyset} \cong M'_{>\emptyset}$, where $M_{>\emptyset} = \bigoplus_{\sigma\neq\emptyset} M_\sigma$. Hence $\mathrm{Sh}_c(X) \cong \mathrm{mod}_R / \mathrm{mod}_\emptyset$.

Let $A = \bigoplus_{i\geq 0} A_i$ be a commutative homogeneous k-algebra as in Remark 1.3 (1), Gr_A the category of graded A-modules, and gr_A its full subcategory of all finitely generated modules. We say $M \in \mathrm{Gr}_A$ is a *torsion* module, if for all $x \in M$ there is some $i \in \mathbb{N}$ with $A_{\geq i} \cdot x = 0$. So, for $0 \neq M \in \mathrm{gr}_A$, M is a torsion module if and only if $\operatorname{Krull-dim} M = 0$. Let Tor_A (resp. tor_A) be the full subcategory of Gr_A (resp. gr_A) consisting of torsion modules. These subcategories are dense, and the category $\mathrm{Qco}(Y)$ (resp. $\mathrm{Coh}(Y)$) of quasi-coherent (resp. coherent) sheaves on the projective scheme $Y := \operatorname{Proj} A$ is equivalent to the quotient category $\mathrm{Gr}_A / \mathrm{Tor}_A$ (resp. $\mathrm{gr}_A / \mathrm{tor}_A$). The Krull dimension of $M \in \mathrm{gr}_A$ equals to $-\min\{ i \mid \mathrm{Ext}^i_A(M, \omega_A^\bullet) \neq 0\}$, where $\omega_A^\bullet$ is a normalized dualizing complex of A. In this sense, $\mathrm{Sh}_c(X) \cong \mathrm{mod}_R / \mathrm{mod}_\emptyset$ is an imitation of $\mathrm{Coh}(Y) \cong \mathrm{gr}_A / \mathrm{tor}_A$.

Let J be the left ideal of R generated by $\{ e_{\sigma,\emptyset} \mid \sigma \neq \emptyset\}$. Then we have that $\underline{\mathrm{Hom}}_R(J,M)^\dagger \cong M^\dagger$ and $\underline{\mathrm{Hom}}_R(J,M)_\emptyset \cong \Gamma(X, M^\dagger)$. It is easy to check that $\underline{\mathrm{Hom}}_R(J,-)$ gives a functor $\eta : \mathrm{mod}_R / \mathrm{mod}_\emptyset \to \mathrm{mod}_R$ with $\pi \circ \eta = \mathrm{Id}$. Moreover, η is a *section functor*, in other words, η is a right adjoint to π. See [7, §4.4] for properties of section functors.

We have the section functor $\mathrm{Qco}(Y) \to \mathrm{Gr}_A$ given by $\mathcal{F} \mapsto \bigoplus_{i\in\mathbb{Z}} H^0(Y, \mathcal{F}(i))$. (This idea does not work for $\mathrm{Coh}(Y)$ and gr_A.) Our $\underline{\mathrm{Hom}}_R(J,-)$ is an imitation of this functor, while objects in mod_R are finitely generated modules (a key point is that mod_R has enough injectives).

For a finite poset P, the *order complex* $\Delta(P)$ is the set of chains of P. Recall that a subset C of P is a *chain* if any two elements of C are comparable. Obviously, $\Delta(P)$ is an (abstract) simplicial complex. The geometric realization of the order complex $\Delta(\Sigma')$ of the poset $\Sigma' := \Sigma \backslash \emptyset$ is homeomorphic to the underlying space X of Σ. See, for example, [3, VI. Definition 3.1]. We say a finite regular cell complex Σ is *Cohen-Macaulay* (resp. *Buchsbaum*) if $\Delta(\Sigma')$ is Cohen-Macaulay (resp. Buchsbaum) over k. (If Σ itself is a simplicial complex, we can use Σ directly instead of $\Delta(\Sigma')$.) These are topological properties of the underlying space X.

Proposition 3.7. *Assume that, for each $\sigma, \tau \in \Sigma$, the subset $\{\, \rho \in \Sigma \mid \rho \geq \sigma, \tau \,\}$ has the minimum element (e.g., Σ is a simplicial complex). Set $d := \dim \Sigma$. Then we have the following.*

(1) $H^i(\omega^\bullet) = 0$ for all $i \neq -d$ if and only if Σ is Cohen-Macaulay over k.
(2) $H^i(\omega^\bullet)^\dagger = 0$ for all $i \neq -d$ if and only if Σ is Buchsbaum over k.

For the proof, see [13, Proposition 5.5]. We remark that Lemma 3.2 (2) is crucial for this result. So the assumption on $\{\, \rho \in \Sigma \mid \rho \geq \sigma, \tau \,\}$ is really essential.

4. Relation to Koszul duality

Let $A = \bigoplus_{i \geq 0} A_i$ be an $\mathbb{N}$-graded associative k-algebra with $A_0 \cong k^n$ (as algebras) for some $n \in \mathbb{N}$. Then $\mathfrak{r} := \bigoplus_{i>0} A_i$ is the graded Jacobson radical. We say A is *Koszul*, if a left A-module $A/\mathfrak{r}$ admits a graded projective resolution

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to A/\mathfrak{r} \to 0$$

such that P^{-i} is generated by its degree i component as an A-module (i.e., $P^{-i} = AP^{-i}_i$). If A is Koszul, it is a quadratic ring, and its *quadratic dual ring* $A^!$ (see [1, Definition 2.8.1]) is Koszul again, and isomorphic to the opposite ring of the Yoneda algebra $\mathrm{Ext}^\bullet_A(A/\mathfrak{r}, A/\mathfrak{r})$.

Note that the incidence algebra R of Σ is a graded ring with $\deg(e_{\sigma,\sigma'}) = \dim \sigma - \dim \sigma'$. So we can discuss the Koszulity of R.

Proposition 4.1 (c.f. [12, Lemma 4.5]). *The incidence algebra R of a finite regular cell complex Σ is always Koszul. And we have $R^! \cong R^{\mathsf{op}}$.*

When Σ is a simplicial complex, the above result was proved by Polishchuk [6] in much wider context (but, $\emptyset \notin \Sigma$ in his convention). More precisely, he put a new partial order on the set $\Sigma \setminus \emptyset$ associated with a perversity function p, and construct two rings from this new poset. Then he proved that these two rings are Koszul and quadratic dual rings of each other. Our R and R^{op} correspond to the case when p is a bottom (or top) perversity. In the middle perversity case, Σ has to be a simplicial complex to make his rings Koszul.

Proof. Let Λ be the incidence algebra of a finite poset P over k. It is known that Λ is Koszul if and only if the order complex $\Delta(I)$ is Cohen-Macaulay over k for any open interval I of Σ. The Koszulity of R follows from this. For $R^! \cong R^{\mathsf{op}}$, an incidence function ε of Σ plays a roll. See [13, Proposition 6.1] for detail. $\square$

Since $R^! \cong R^{\mathsf{op}}$, $\operatorname{Hom}_k(-,k)$ gives duality functors $\mathbf{D}_k : \operatorname{mod}_R \to \operatorname{mod}_{R^!}$ and $\mathbf{D}_k^{\mathsf{op}} : \operatorname{mod}_{R^!} \to \operatorname{mod}_R$. These functors are exact, and they can be extended to the duality functors between $D^b(\operatorname{mod}_R)$ and $D^b(\operatorname{mod}_{R^!})$.

Let gr_R (resp. $\operatorname{gr}_{R^!}$) be the category of finitely generated graded left R-modules (resp. R^{op}-modules). Let $DF : \operatorname{mod}_R \to \operatorname{mod}_{R^!}$ and $DG : \operatorname{mod}_{R^!} \to \operatorname{mod}_R$ be the functors defined in [1, Theorem 2.12.1]. Since R and $R^!$ are artinian, DF and DG give an equivalence $D^b(\operatorname{gr}_R) \cong D^b(\operatorname{gr}_{R^!})$ by the Koszul duality ([1, Theorem 2.12.6]).

If $M \in \operatorname{mod}_R$, then we can regard M as a graded module by $\deg M_\sigma = \dim \sigma$. The same thing is true for $\operatorname{mod}_{R^!}$. In this way, we can make DF and DG functors between $D^b(\operatorname{mod}_R)$ and $D^b(\operatorname{mod}_{R^!})$ giving an equivalence $D^b(\operatorname{mod}_R) \cong D^b(\operatorname{mod}_{R^!})$.

Theorem 4.2 (c.f. Vybornov, [9, Corollary 4.3.5]). *Under the above situation, we have $DF \cong \mathbf{D}_k \circ \mathbf{D}$ and $DG \cong \mathbf{D} \circ \mathbf{D}_k^{\mathsf{op}}$.*

The proof follows from the explicit description of the Koszul duality functors. See [13, Theorem 6.2]. When Σ is a simplicial complex (with $\emptyset \notin \Sigma$), the above theorem was given by Vybornov [9]. Independently, I also proved a similar result ([12, Theorem 4.7]).

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Arithmetical rank of squarefree monomial ideals of small arithmetic degree

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1 Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring with n variables over a field k with $\deg x_i = 1$ $(i = 1, 2, \ldots, n)$. In this article we determine the arithmetical rank of squarefree monomial ideals in R with small arithmetic degree. More precisely, we prove the following theorem:

Theorem. *Let I be a squarefree monomial ideal. Then we have:*

(1)
$$\mathrm{arithdeg} I = \mathrm{reg}\, I \Rightarrow \mathrm{ara}\, I = \mathrm{projdim}\ (R/I).$$

(2)
$$\mathrm{arithdeg} I = \mathrm{indeg} I + 1 \Rightarrow \mathrm{ara}\, I = \mathrm{projdim}\ (R/I).$$

First we fix the terminology we use in this article.

Let I be an ideal of R. We define the arithmetical rank $\mathrm{ara} I$ of I by

$$\mathrm{ara} I\ := \min\{r;\ \exists a_1, a_2, \ldots, a_r \in I \text{ such that } \sqrt{(a_1, a_2, \ldots, a_r)} = \sqrt{I}\ \}.$$

In general, $\mathrm{ara} I \geq \mathrm{ht}\, I$. And I is said to be a *set-theoretic complete intersection*, if $\mathrm{ara} I = \mathrm{ht} I$.

Let I be a homogeneous ideal in R and

$$0 \to \bigoplus_j R(-j)^{\beta_{pj}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow I \to 0$$

a graded minimal free resolution of I over R. Here p is called the *projective dimension* of I over R and denote it by $\mathrm{projdim} I$. We have $\mathrm{projdim}\ (R/I) = \mathrm{projdim} I + 1$. Put $\mu(I) := \sum_j \beta_{0j}$, which stands for the minimum number of generators of I. The *initial degree* indeg I of I and the *relation type* $\mathrm{rt}(I)$ of I are defined respectively by

$$\mathrm{indeg} I = \min\{j \,:\, \beta_{0j} \neq 0\},$$
$$\mathrm{rt} I = \max\{j \,:\, \beta_{0j} \neq 0\}.$$

And the *(Castelnuovo-Mumford) regularity* of I is defined by

$$\mathrm{reg} I = \max\{j - i \,:\, \beta_{ij} \neq 0\}.$$

We say that I has *linear resolution* if $\mathrm{reg} I = \mathrm{indeg} I$.

For a simplicial complex Δ on the vertex set $V = \{1, \ldots, n\}$, we mean that Δ is a collection of subsets of V such that

$$F \in \Delta,\ G \subset F \Rightarrow G \in \Delta.$$

We call

$$I_\Delta = (x_{i_1} \cdots x_{i_p};\ i_1 < i_2 < \ldots < i_p,\ \{i_1, \ldots, i_p\} \notin \Delta)$$

the *Stanley–Reisner ideal* of Δ.

Put

$$\Delta^* = \{F \in 2^V \,:\, V \setminus F \notin \Delta\},$$

which is also a simplicial complex, and called the *Alexander dual* of Δ. We call I_{Δ^*} the Alexander dual ideal of I_Δ.

2 Arithmetical rank of squarefree monomial ideals

Let $H_I^i(R)$ be the i-th local cohomology module of R with respect to I. The *cohomological dimension* cd I of I is defined to be $\mathrm{cd}\, I := \max\{i;\ H_I^i(R) \neq 0\}$. It is easy to see $\mathrm{ara} I \geq \mathrm{cd}\, I$.

When I is a squarefree monomial ideal, the following theorem is known :

Theorem 2.1 (Lyubeznik [Ly1] see also [Te2]). *Let I be a squarefree monomial ideal. Then we have*

$$\mathrm{projdim}\,(R/I) = \mathrm{cd}\, I.$$

Corollary 2.2. *Let I be a squarefree monomial ideal. Then we have* ara $I \geq$ projdim (R/I).

In particular, if I is a set-theoretic complete intersection, then R/I is Cohen-Macaulay.

Problem 2.3. Let I be a squarefree monomial ideal. Under what conditions do we have ara $I =$ projdim (R/I)?

We do not always have ara $I =$ projdim (R/I) as the following example shows.

Example 2.4 (Yan [Ya]). Let I be the ideal in $R = k[u, v, w, x, y, z]$ generated by $uvw, uvy, vwx, uwz, uxy, uxz, vxz, vyz, wxy, wyz$. Then I is the Stanley-Reisner ideal of a triangulation of $\mathbf{P}^2(\mathbf{R})$ with six vertices. In this case, ara $I = 4$, which is proved by Yan, using the étale cohomology. On the other hand projdim $(R/I) = 3$ if char $(k) \neq 2$.

We pick up some classes for whose members the equality holds.

Proposition 2.5 ([Te3]). *Let I be a squarefree monomial ideal. If $\mu(I)-$projdim $(R/I) \leq 1$, then we have*

$$\text{ara } I = \text{projdim } (R/I).$$

For an ideal I in R, we define the *deviation* $d(I)$ of I by $d(I) = \mu(I) - \text{ht } I$.

Theorem 2.6 ([Te4]). *Let I be a squarefree monomial ideal of deviation 2. Then we have*

$$\text{ara } I = \text{projdim } (R/I).$$

Proposition 2.7. *Let Δ be a disconnected simplicial complex. I.e., let I_Δ be a squarefree monomial ideal with* depth $R/I_\Delta = 1$. *Then we have*

$$\text{ara } I_\Delta = \text{projdim} R/I_\Delta.$$

(*Proof.*) By [Ei-Ev] we have $n - 1 = \text{projdim} R/I_\Delta \leq \text{ara } I_\Delta \leq n - 1$.

Proposition 2.8. *Let Δ be a non-acyclic simplicial complex such that I_Δ has linear resolution. (E.g., I_Δ is a non-Cohen-Macaulay Buchsbaum squarefree monomial ideal with linear resolution.) Then we have*

$$\text{ara } I_\Delta = \text{projdim} R/I_\Delta.$$

(*Proof.*) By [Gr] we have $n - \text{indeg} I_\Delta + 1 = \text{projdim} R/I_\Delta \leq \text{ara } I_\Delta \leq n - \text{indeg} I_\Delta + 1$.

3 Squarefree monomial ideals of small arithmetic degree

We define the *arithmetic degree* arithdeg I of a squarefree monomial ideal I by

$$\text{arithdeg } I = \sharp(\text{Ass R/I}).$$

For squarefree monomial ideals, we have the following relations:

Theorem 3.1 (Hoa-Trung[Ho-Tr], Stückrad, Frübis-Terai[Fr-Te]). *Let I be a squarefree monomial ideal. Then we have*

$$\text{indeg } I \leq \text{reg} I \leq \text{arithdeg} I.$$

The arithmetical rank is known when the arithmetic degree agrees with the initial degree:

Theorem 3.2 (Schenzel-Vogel[Sche-Vo], Schmitt-Vogel[Schm-Vo]). *If a squarefree monomial ideal I satisfies* arithdeg$I$ = indeg I*, then after a suitable change of variables, I is of the form*

$$I = (x_{11}, x_{12}, \dots, x_{1j_1}) \cap (x_{21}, x_{22}, \dots, x_{2j_2}) \cap \dots \cap (x_{q1}, x_{q2}, \dots, x_{qj_q}),$$

and $\text{projdim}(R/I) = \sum_{i=1}^{q} j_i - q + 1$.

Put $a_\ell = \sum_{\ell_1+\ell_2+\dots+\ell_q=\ell} x_{1\ell_1}x_{2\ell_2}\cdots x_{q\ell_q}$ *for* $\ell = q, q+1, \dots, \sum_{i=1}^{q} j_i$*. Then we have* $\sqrt{(a_\ell;\ \ell = q, q+1, \dots, \sum_{i=1}^{q} j_i)} = I$.

Hence ara I = projdim (R/I).

Now we consider the case that the arithmetic degree is equal to regularity:

Theorem 3.3. *Let I be a squarefree monomial ideal with* arithdeg$I$ = reg I*. Then we have*

$$\text{ara } I = \text{projdim } (R/I).$$

To prove the above theorem we define the *size* of a monomial ideal I, which is introduced by Lyubeznik. Let $I = \cap_{j=1}^{r} Q_j$ be an irredundant primary decomposition of I, where the Q_i are monomial primary ideals. Let h be the height of $\sum_{j=1}^{r} Q_j$, and denote by v the minimum number t such that there exist $j_1, \ldots, j_t$ with $\sqrt{\sum_{i=1}^{t} Q_{j_i}} = \sqrt{\sum_{j=1}^{r} Q_j}$. Then size$I = v + (n - h) - 1$. Then we have:

Lemma 3.4 (Lyubeznik[Ly2]). *Let I be a (squarefree) monomial ideal in R. Then* ara$I \leq n -$ size$I$.

The form is determined for a squarefree monomial ideal I with arithdeg$I$ = reg I as follows:

Lemma 3.5 (Hoa-Trung[Ho-Tr]). *Let I be a squarefree monomial ideal in R such that* arithdeg$I$ = reg I. *Then after a suitable change of variables, I is of the form*

$$I = (y_1, x_{i_{11}}, x_{i_{12}}, \ldots, x_{i_{1j_1}}) \cap (y_2, x_{i_{21}}, x_{i_{22}}, \ldots, x_{i_{2j_2}}) \cap \ldots \cap (y_q, x_{i_{q1}}, x_{i_{q2}}, \ldots, x_{i_{qjq}}),$$

and

$$\text{projdim}(R/I) = \deg \text{lcm}(x_{i_{11}}, x_{i_{12}}, \ldots, x_{i_{1j_1}}, x_{i_{21}}, x_{i_{22}}, \ldots, x_{i_{2j_2}}, \ldots, x_{i_{q1}}, x_{i_{q2}}, \ldots, x_{i_{qjq}}) + 1.$$

Lemma 3.6. *Let I be a squarefree monomial ideal in R such that* arithdeg$I$ = reg I. *Then we have*

$$\text{projdim } (R/I) = n - \text{size}I.$$

(Proof.) We may assume that every variable is zero divisor on R/I. Since size$I + 1$ = arithdeg$I$ = reg I by the above lemma, it is enough to prove to

$$\text{projdim } (R/I) + \text{reg}I = n + 1.$$

Let J be the Alexander daul ideal of I. Then we have

$$J = (y_1 x_{i_{11}} x_{i_{12}} \cdots x_{i_{1j_1}}, y_2 x_{i_{21}} x_{i_{22}} \cdots x_{i_{2j_2}}, \ldots, y_q x_{i_{q1}} x_{i_{q2}} \cdots x_{i_{qjq}}).$$

Since projdim (R/I) = reg$J$ and reg$I$ = projdim (R/J) (see [Te1]), it is enough to prove

$$\text{projdim } (R/J) + \text{reg}J = n + 1.$$

Because of the form of the ideal J, the Taylor resolution of J gives a minimal free resolution of J. Hence the last syzygy determines the regularity. Since every variable is zero divisor on R/J, reg$J = n -$ projdim$J = n -$ projdim$(R/J) + 1$. QED

Now Theorem 3.3 is clear by Lemmas 3.4 and 3.6.

Next we consider a squarefree monomial ideal whose arithmetic degree is one bigger than its initial degree:

Theorem 3.7. *Let I be a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$. *Then we have*

$$\operatorname{ara} I = \operatorname{projdim}\,(R/I).$$

To prove the above theorem we use:

Lemma 3.8. *Let I be a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$. *Then I is one of the following forms after a suitable change of the variables:*
(1)

$$\begin{aligned} I = &(x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \ldots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \\ &\cap (x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $q \geq p \geq 2$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_q \geq 1$.
(2)

$$\begin{aligned} I = &(x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \ldots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \\ &\cap (x_{q+1,1}, x_{q+1,2}, \ldots, x_{q+1,j_{q+1}}, x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $q \geq p \geq 1$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_q, j_{q+1} \geq 1$.
(3)

$$\begin{aligned} I = &(x_{11}, x_{12}, \ldots, x_{1j_1}, y_1, \ldots, y_p) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}, y_1, \ldots, y_p) \cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \\ &\cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \cap (x_{q+1,1}, x_{q+1,2}, \ldots, x_{q+1,j_{q+1}}, x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}), \end{aligned}$$

where $q \geq 2$, $p \geq 1$, $1 \leq i_\ell \leq j_\ell$ $(\ell = 1, 2)$, $j_3, \ldots, j_q \geq 1$, $j_{q+1} \geq 0$.

(*Proof.*) Let I be a squarefree monomial ideal with $\operatorname{arithdeg} I = \operatorname{indeg} I + 1$, and J its Alexander dual ideal. Then J satisfies that $\mu(J) = \operatorname{ht} J + 1$, that is J is an almost complete intersection. Such J are classified in [Te3]. QED

(*Proof of Theorem 3.7.*) We check the equality for all the cases in the above lemma. Let J be the Alexander dual ideal of I.

(1)We may assume that $j_1 - i_1 = \min\{j_\ell - i_\ell;\ \ell = 1, 2, \ldots, p\}$. Then

$$\operatorname{projdim}\,(R/I) = \operatorname{reg} J = i_1 + j_2 + \cdots + j_q - q + 1.$$

Put $a_\ell = \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_q=\ell \\ \ell_1 \le i_1 \text{ or } \ell_2 \le i_2 \text{ or } \ldots \text{ or } \ell_p \le i_p}} x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q}$ for $\ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t$. Then we have $\sqrt{(a_\ell;\ \ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t)} = I$ by [Schm-Vo, Lemma]. Hence ara I = projdim (R/I).

(2)By Theorem 3.3 the equality holds in this case.

(3) (i) The case of $j_{q+1} > 0$. By Theorem 3.3 the equality holds.

(ii) The case of $j_{q+1} = 0$ and $i_\ell < j_\ell$ ($\ell = 1, 2$). We may assume that $j_1 - i_1 \le j_2 - i_2$. Then

$$\text{projdim } (R/I) = \text{reg} J = i_1 + j_2 + \cdots + j_q - q + 1 + p.$$

For simplicity, we mean that $x_{1j_1+i} = y_i$ and $x_{2j_2+i} = y_i$ for $i = 1, 2, \ldots, p$.

Put $a_\ell = \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_q=\ell, \\ \ell_1 \le i_1 \text{ or } \ell_2 \le i_2}} x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q}$ for $\ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t + p$. Then we have $\sqrt{(a_\ell;\ \ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t + p)} = I$ by [Schm-Vo, Lemma]. Hence ara I = projdim (R/I).

(iii) The case of $j_{q+1} = 0$ and ($i_1 = j_1$ or $i_2 = j_2$). We may assume that every variable is a zero divisor on R/I. Then R/J is Cohen-Macaulay with $a(R/J) = 0$. Hence by Proposition 2.8 the equality holds in this case. QED

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2次元単項式イデアルのRatliff-Rush閉包とRees代数のBuchsbaum性について

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1 はじめに

以下，A は可換環とし，I を環 A のイデアルとする。元 $x \in A$ が I 上整であるとは，ある自然数 n と元 $c_i \in I^i$ が存在して，環 A 内で次の等式 $x^n + a_1x^{n-1} + \cdots + a_n = 0$ が満たすことをいう。記号 $\overline{I}$ によって，I 上整であるような A の元全体のなす集合を表し，イデアル I の整閉包と呼ぶ。$\overline{I}$ は A のイデアルであって，I を含み，等式 $\overline{\overline{I}} = \overline{I}$ が成り立つ。また，$\widetilde{I} = \bigcup_{n\geq 0}[I^{n+1} : I^n]$ とおき，イデアル I の Ratliff-Rush 閉包という。任意の整数 $n \geq 0$ について $I^{n+1} : I^n \subseteq I^{n+2} : I^{n+1}$ が成り立つから，集合 $\widetilde{I}$ は環 A のイデアルであって，$I \subseteq \widetilde{I}$ が成り立つ。

命題 1.1. (1) I, J を環 A のイデアルとすれば，$\widetilde{I}\widetilde{J} \subseteq \widetilde{IJ}$ である。特に，任意の自然数 n について，$(\widetilde{I})^n \subseteq \widetilde{I^n}$ である。

(2) I を環 A の有限生成なイデアルであって，環 A の非零因子を少なくとも一つは含むと仮定する。このとき，$\widetilde{I} \subseteq \overline{I}$ が成り立つ。

有限生成でないイデアルや零因子のみからなるイデアルについては，この包含関係が成り立つとは限らない。特に，環 A が Noether 環で，イデアル I が少なくとも一つは環 A の非零因子を含むなら，$I \subseteq \widetilde{I} \subseteq \overline{I}$ であり，十分大きい整数 n を取れば $\widetilde{I} = I^{n+1} : I^n$ となる。本稿では，次の二つの問いを考える。

問題 1.2. (1) いつ $I = \widetilde{I}$ が成り立つか？

(2) いつ $\overline{I} = \widetilde{I}$ が成り立つか？

このような等式が成り立つための実際的な判定法を与えるとともに，豊富な具体例を探し，そのようなイデアルが持つ固有の性質を解明することが目的である。Ratliff-Rush 閉包 $\widetilde{I}$ は非常に面白い性質を持ったイデア

ルであり，この問題を考える中で得られた副産物も多い。本稿では特に2変数多項式環内のやや特殊な形をした単項式イデアルについて，上記の問いに対する解答を探したいと思う。

本稿の構成を説明しておこう。主結果とその証明は第4節で与える。第2節では，この論文を通して必要となるイデアルの整閉包の基本的な事項を纏めておくことにしたい。第3節では，まず Ratliff-Rush 閉包に関する基本的な事実を整理し，次に，基礎環 A が2次元正則局所環のとき，問題1.3が実は何を問うているかを解明し，今回の研究の動機付けを明確にしたい。

2 イデアルの整閉包

本節の目的は，Northcott-Rees [NR] に従いながら，イデアルの整閉包に関し，後に必要となる事柄を纏めておくことにある。以下，A は可換環とし，I は環 A のイデアルとする。J を A のイデアルで I を含むものとする。イデアル J が I の reduction であるとは，ある整数 $r \geq 0$ に対し等式 $I^{r+1} = JI^r$ が成り立つことをいう。t を不定元として，$\mathcal{R}(I) = A[It] \subseteq A[t]$ とおき，イデアル I の Rees 代数と呼ぶ。Rees 代数は，可換環論をはじめ代数幾何など数学の様々な分野において多くの関心を集める重要な研究対象であり，特にイデアルの整閉包との関連で深い研究がなされている。イデアルの整閉包と Rees 代数の関連を示す事実として例えば，環 A の元 x をとり，$xt \in A[t]$ を考えると，x がイデアル I 上整であることと xt が環 $\mathcal{R}(I)$ 上整であることは同値であり。従って，イデアルの整拡大の理論は Rees 代数 $\mathcal{R}(I)$ を通すことによって，環の整拡大の理論に帰着され，整閉包 $\overline{I}$ の基本的な性質をが得られる。主なものを整理し纏めると，下記の通りである。

命題 2.1. I,J を環 A のイデアルとする。このとき，次が成り立つ。

(1) $\overline{I}$ は A のイデアルで，$I \subseteq \overline{I} \subseteq \sqrt{I}$ である。

(2) $J \subseteq I$ とすると，$\overline{J} \subseteq \overline{I}$ である。

(3) $\overline{\overline{I}} = \overline{I}$ である。

(4) 環 A が Noether で $J \subseteq I$ なら，$I \subseteq \overline{J}$ であるための必要十分条件は，J が I の reduction であることである。

(5) $\overline{I} \cdot \overline{J} \subseteq \overline{IJ}$ である。

3 Ratliff-Rush 閉包

Ratliff-Rush 閉包の基本的な性質を述べよう。Ratliff-Rush 閉包は，Ratliff と Rush[RR] により導入された。

命題 3.1 (cf. [M]). A は Noether 環とし，I, J は環 A のイデアルで，イデアル I は環 A の非零因子を少なくとも一つは含むと仮定する。このとき，次が成り立つ。

(1) $I \subseteq \widetilde{I} \subseteq \overline{I}$ である。

(2) 十分大なる任意の整数 $n \gg 0$ について，等式 $\widetilde{I^n} = I^n$ が成り立つ。

(3) $I \subseteq J$ なら，次の 3 条件は互いに同値である。

  (i) $J \subseteq \widetilde{I}$ である。
  (ii) ある整数 $\ell > 0$ に対し等式 $I^\ell = J^\ell$ が成り立つ。
  (iii) 十分大なる任意の整数 $n \gg 0$ に対し等式 $I^n = J^n$ が成り立つ。
  (iv) $\widetilde{I} = \widetilde{J}$ である。

(4) $\widetilde{\widetilde{I}} = \widetilde{I}$ である。

系 3.2. A は極大イデアル $\mathfrak{m}$ を持つ次元正の Noether 局所環，I は環 A の $\mathfrak{m}$-準素イデアルとすると，$\widetilde{I}$ は次の条件を満たす最大のイデアルである：(1) $I \subseteq J \subsetneq A$ であって，(2) イデアル J は I と同じ Hilbert 多項式を持つ。

命題 3.1 の証明の鍵は次の補題である。

補題 3.3. A は局所環とは限らない Noether 環とし I は A のイデアルとする。このとき，次が正しい。

(1) ある整数 $\ell \geq 0$ が存在して，任意の整数 $n \geq \ell$ に対し等式 $(I^{n+1} : I) \cap I^\ell = I^n$ が成り立つ。

(2) イデアル I は少なくとも 1 つの非零因子を含むとする。このとき十分大きな整数 $n \gg 0$ を取ると，任意の整数 $h > 0$ に対して等式 $I^{n+h} : I^h = I^n$ が成り立つ。

命題 3.4. A は可換環，I, J を環 A のイデアルとする。$J = (a_1, a_2, \cdots, a_d)$ が I の reduction なら，等式 $\widetilde{I} = \bigcup_{n>0}[I^{n+1} : (a_1^n, a_2^n, \cdots, a_d^n)]$ が成り立つ。故に，環 A が Noether で $J \subseteq I \subseteq \overline{J}$ なら，$\widetilde{J} \subseteq \widetilde{I}$ である。

この命題により，イデアル $\widetilde{I}$ を計算したければ，reduction の生成元の冪についてだけ見ればよいことがわかり，特に今回設定する状況の下では非常に強力な鍵となる。

次に，2 次元正則局所環内での Ratliff-Rush 閉包の性質を述べておきたい。以下，環 A は極大イデアル $\mathfrak{m}$ を持つ 2 次元の正則局所環とし，I は $\mathfrak{m}$-準素イデアルとする。2 次元正則局所環の議論において最も重要な事実は，次の定理である。証明は [H] を参照されたい。

定理 3.5 (Zariski). I, J を 2 次元正則局所環 A のイデアルとすると，等式 $\overline{I} \cdot \overline{J} = \overline{IJ}$ が成り立つ。故に，任意の整数 $n > 0$ について $\overline{J^n} = (\overline{J})^n$ である。

さて，$R = \mathcal{R}(I)$ を I の Rees 代数とし，$M = \mathfrak{m}R + R_+$ を Rees 代数 R の斉次極大イデアルとする。次の命題は，2 次元正則局所環における等式 $\overline{I} = \widetilde{I}$ を特徴づける。

定理 3.6. 次の条件は同値である。

(1) 等式 $\overline{I} = \widetilde{I}$ が成り立つ。

(2) ある整数 $\ell > 0$ について，$\overline{I^\ell} = I^\ell$ が成り立つ。

(3) 十分大きい任意の整数 $n \gg 0$ について，$\overline{I^n} = I^n$ が成り立つ。

(4) ある整数 $\ell > 0$ について，$\overline{I^\ell} = \widetilde{I^\ell}$ が成り立つ。

(5) 任意の整数 $n > 0$ について，$\overline{I^n} = \widetilde{I^n}$ が成り立つ 。

(6) $\operatorname{Proj} R$ は正規スキームである。

(7) 斉次極大イデアル M とは異なる R の任意の素イデアル P について，局所環 R_P は正規である。

このとき Rees 代数 R は FLC を持つ。

証明. (1) $\Rightarrow$ (3) 定理 3.5 と命題 3.1 (2) により，十分大なる整数 $n \gg 0$ について $\overline{I^n} = \overline{I}^n = (\widetilde{I})^n = I^n$ である。

(2) $\Rightarrow$ (1) $I \subseteq \overline{I}$ で，$I^\ell = \overline{I^\ell} = \overline{I}^\ell$ であるから，命題 3.1 (3) により $\overline{I} \subseteq \widetilde{I}$ が成り立つ。故に $\overline{I} = \widetilde{I}$ である。

(1) $\Rightarrow$ (5) 定理 3.5 と命題 1.1 (1) により，任意の整数 $n > 0$ について $\overline{I^n} = \overline{I}^n = (\widetilde{I})^n \subseteq \widetilde{I^n}$ であって，$\overline{I^n} = \widetilde{I^n}$ が成り立つ。

(4) $\Rightarrow$ (2) 定理 3.5 と命題 1.1 (1) により，任意の整数 $n > 0$ について $\overline{I^{\ell n}} = (\overline{I^\ell})^n = (\widetilde{I^\ell})^n \subseteq \widetilde{I^{\ell n}}$ であり，整数 n を十分大きく取れば $\widetilde{I^{\ell n}} = I^{\ell n}$ が成り立つので等式 $I^{\ell n} = \overline{I^{\ell n}}$ が従う。

(3) $\Rightarrow$ (7) 次数 R-加群の完全列 $0 \to R \to \overline{R} \to C \to 0$ を考える。但し $\overline{R}$ は環 R の商体内での整閉包とする。環 R と $A[t]$ は商体を共有し，$A[t]$ は整閉整域であるから，$\overline{R}$ は R の $A[t]$ 内での整閉包に他ならない。故に $\overline{R} = \mathcal{R}(\overline{I})$ である。今，$\overline{R}$ は有限生成 R-加群であって，条件 (3) より C の (0) でない斉次成分は高々有限個しか存在しない。故に，整数 $\ell > 0$ を $R_+^\ell C = (0)$，$\mathfrak{m}^\ell C = (0)$ が同時に成り立つように取ることができる。故に，$\operatorname{Supp}_R C \subseteq \{M\}$ である。環 R の素イデアル $P \neq M$ を取れば，$C_P = (0)$ であるから，$R_P = \overline{R}_P$ は正規環である。

(6) $\Rightarrow$ (2) 素イデアル $Q \in \mathrm{Supp}_R C$ を任意に取れば，$P \subseteq Q$ を満たす斉次素イデアル $P \in \mathrm{Ass}_R C$ が存在する。$R_+ \not\subseteq P$ ならば，$P \in \mathrm{Proj}\, R$ であって，条件 (6) より局所環 R_P は正規であり，$C_P = (0)$ を得る。しかし $P \in \mathrm{Ass}_R C$ であるのでこれは不可能である。故に $R_+ \subseteq P$ であるから $R_+ \subseteq Q$ であり，$R_+ \subseteq \sqrt{(0) :_R C}$ が成り立つ。よって十分大なる整数 $\ell \gg 0$ に対し $R_+^\ell C = (0)$ であり，次数 R-加群 C は高々有限個しか (0) でない斉次成分を持たない。故に，$\overline{I^n} = I^n$ $(n \gg 0)$ が成り立つ。

完全列 $0 \to R \to \overline{R} \to C \to 0$ を見る。環 $\overline{R} = \mathcal{R}(\overline{I})$ は Cohen-Macaulay(cf. [HS]) であって，$\ell(C) < \infty$ であるから，環 R の極大イデアル M に関する局所コホモロジーを考えることにより $\mathrm{H}^1_M(R) = C$, $\mathrm{H}^p_M(R) = (0)$ $(p \neq 1, 3)$ を得る。故に環 R は FLC を持つ。 □

このように，2 次元正則局所環内では等式 $\overline{I} = \widetilde{I}$ が Rees 代数の環構造を大きく規定しているのである。

系 3.7. 次の条件は同値である。

(1) R は Buchsbaum 環であって，斉次極大イデアル M とは異なる R の任意の素イデアル P について，R_P は正規環である。

(2) R は Buchsbaum 環であって，$\overline{I} = \widetilde{I}$ である。

(3) $\mathfrak{m}\overline{I} \subseteq I$ であって，等式 $I\overline{I} = I^2$ が成り立つ。

$Q = (a, b)$ がイデアル I の reduction ならば，次の条件を付け加えることができる。

(4) $\mathfrak{m}\overline{I} \subseteq I$ であって，$Q\overline{I} \subseteq I^2$ である。

証明. (1) と (2) の同値性は定理 3.6 による。

(2) $\Rightarrow$ (3) $\overline{I} = \widetilde{I}$ であるから，定理 3.6 より $\mathrm{H}^1_M(R) = \overline{R}/R$ であり，R が Buchsbaum 環であるから $M(\overline{R}/R) = (0)$ となる。よって $\mathfrak{m}\overline{I} \subseteq I$ かつ $I\overline{I} = I^2$ が成り立つ。

(3) $\Rightarrow$ (2) $I\overline{I} = I^2$ より $\overline{I} \subseteq I^2 : I \subseteq \widetilde{I}$ である。故に $\overline{I} = \widetilde{I}$ である。n についての帰納法で $\mathfrak{m}\overline{I^n} \subseteq I^n$ と $I\overline{I^n} = I^{n+1}$ を示すことができる。よって，$M\overline{R} \subseteq R$ が従い，$M\,\mathrm{H}^1_M(R) = (0)$ となって，$\mathrm{H}^p_M(R) = (0)$ $(p \neq 1, 3)$ より，R は Buchsbaum 環であることがわかる。

$Q = (a, b)$ がイデアル I の reduction なら，Q は $\overline{I}$ の reduction で，等式 $\overline{I}^2 = Q\overline{I}$ が成り立つ ([HS])。故に，条件 (3)(4) が同値となる。 □

次節において，2 次元正則局所環の典型として，2 変数多項式環を考察したい。

4 単項式イデアル

$A = k[X_1, X_2, \cdots, X_d]$ を体 k 上 d 変数多項式環とする。$a_1, a_2, \cdots, a_d > 0, \ell \geq 0$ を整数で，$f_1, f_2, \cdots, f_\ell \in A$ を単項式とし $I = (X_1^{a_1}, X_2^{a_2}, \cdots, X_d^{a_d}) + (f_1, f_2, \cdots f_\ell)$ とおく。次が正しい。

定理 4.1 (早坂太). $0 \leq \ell < d$ なら，等式 $I = \widetilde{I}$ が成り立つ。

証明. $f_i = X_1^{b_{i1}} X_2^{b_{i2}} \cdots X_d^{b_{id}}$ と書く。$I \neq \widetilde{I}$ と仮定し，単項式 $f \in \widetilde{I} \setminus I$ を取る。$f = X_1^{c_1} X_2^{c_2} \cdots X_d^{c_d}$ $(c_i < a_i)$ とすれば，整数 $1 \leq i \leq \ell$ について，整数 $1 \leq j_i \leq d$ で $c_{j_i} < b_{ij_i}$ となるものが存在する。$\ell < d$ なので，整数 $j \in \{1, 2, \cdots, d\} \setminus \{j_1, j_2, \cdots, j_\ell\}$ を取れる。さて，整数 $n > 0$ を $f \in I^{n+1} : (X_j^{a_j})^n$ が成り立つように取る。故に $fX_j^{a_j n} \in I^{n+1}$ である。任意の整数 $1 \leq i \leq \ell$ について，$c_{j_i} < b_{ij_i}$，$j_i \neq j$ であるから，$f_i \nmid fX_j^{a_j n}$ である。一方で，$i \neq j$ なら，$c_i < a_i$ であるから，$X_i^{a_i} \nmid fX_j^{a_j n}$ である。よって $(X_j^{a_j})^{n+1} \mid fX_j^{a_j n}$ でなければならないので $(n+1)a_j \leq na_j + c_j$ となるが，これは不可能である。故に $I = \widetilde{I}$ である。 □

命題 4.2. $\overline{(X_1^{a_1}, X_2^{a_2}, \cdots, X_d^{a_d})} = (X_1^{b_1} X_2^{b_2} \cdots X_d^{b_d} \mid \sum_{i=1}^{d} \frac{b_i}{a_i} \geq 1)$.

以下，$d = 2$ とする。定理 4.1 により $\ell = 1$ のときは $I = \widetilde{I}$ であるから，$\ell \geq 2$ のときを考える。状況を確認しておく。$A = k[X, Y]$ を体 k 上 2 変数多項式環とし，$a, b, \ell > 0$ を整数とし，さらに $a_1, a_2, \cdots, a_\ell, b_1, b_2, \cdots, b_\ell > 0$ を整数とする。$I = (X^a, Y^b) + (X^{a_i} Y^{b_i} \mid 1 \leq i \leq \ell)$ とおく。イデアル (X^a, Y^b) が I の reduction となるよう，次の条件 $\frac{a_i}{a} + \frac{b_i}{b} \geq 1 \quad (1 \leq i \leq \ell)$ を仮定する (cf. 命題 4.2)。よって，命題 3.4 により $\widetilde{I} = \bigcup_{n>0} I^{n+1} : (X^{na}, Y^{nb})$ である。さらに，次を仮定する。

$$a_0 := 0 < a_1 < a_2 < \cdots < a_\ell < a_{\ell+1} := a,$$
$$b_0 := b > b_1 > b_2 > \cdots > b_\ell > b_{\ell+1} := 0.$$

これは，イデアル I は $\ell + 2$ 個の元 $\{X^{a_i} Y^{b_i} \mid 0 \leq i \leq \ell + 1\}$ で極小的に生成されるという仮定である。

以上の仮定の下に，本節の主結果は次の定理である。

定理 4.3. $\ell = 2$ とせよ。このとき，等式 $I = \widetilde{I}$ が成り立つことと，$2a_1 \geq a_2$ であるかまたは $2b_2 \geq b_1$ であることは同値である。

定理 4.4. $\ell = 2$，$4 \leq a \leq b$ と仮定すると，次の条件は同値である。

(1) 等式 $\overline{I} = \widetilde{I}$ が成り立つ。

(2) 次の 3 条件が満たされる：(i) $a \mid b$ (ii) $a_1 = 1$, $b_1 = b - \frac{b}{a}$ (iii) $a_2 = a - 1$, $b_2 = \frac{b}{a}$

定理の証明を述べる前に，いくつか準備をしておきたい。$L = \{(\alpha,\beta) \in \mathbb{Z}^2 \mid 0 \le \alpha, \beta\}$, $S = \{(\alpha,\beta) \in \mathbb{Z}^2 \mid 0 \le \alpha < a_1, 0 \le \beta < b\}$, $T = \{(\alpha,\beta) \in \mathbb{Z}^2 \mid 0 \le \alpha < a, 0 \le \beta < b_\ell\}$ とおく。J を多項式環 A の単項式イデアルとしたとき，$(\alpha,\beta) \in L$ について，$X^\alpha Y^\beta \in J$ であることを単に $(\alpha,\beta) \in J$ と表すこともある。実数 $\alpha \in \mathbb{R}$ について $\lceil \alpha \rceil = \min\{n \in \mathbb{Z} \mid \alpha \le n\}$ と定め，整数 $i \in \mathbb{Z}$ について $c(i) = \lceil b - \frac{ib}{a} \rceil$ とおく。$c(0) = b$, $c(a) = 0$ である。$a \le b$ ならば，$c(i) > c(i+1)$ であって，命題 4.2 によって $\overline{I} = \overline{(X^a, Y^b)} = (X^i Y^{c(i)} \mid 0 \le i \le a)$ となる。次が正しい。

補題 4.5. $(\alpha,\beta) \in L$ とする。もし $(\alpha,\beta) \in S \cup T$ ならば，$(\alpha,\beta) \notin \widetilde{I}$ である。

証明. $(\alpha,\beta) \in S$ で $(\alpha,\beta) \in \widetilde{I}$ とする。定義より，ある整数 $n > 0$ で $(X^\alpha Y^\beta)Y^{bn} \in I^{n+1}$ である。よって $(\alpha, \beta + nb) = \sum_{i=0}^{\ell+1} m_i(a_i, b_i) + \xi$ と書ける (但し, $m_i \ge 0$, $\sum_{i=0}^{\ell+1} m_i = n+1$, $\xi \in L$)。$\alpha < a_1 < \cdots < a_{\ell+1}$ より $m_1 = \cdots = m_{\ell+1} = 0$ であるから，$m_0 = n+1$ となり，$\beta + nb \ge (n+1)b$ を得るが，$(\alpha,\beta) \in S$ であるので，これは不可能である。$(\alpha,\beta) \in T$ のときは，$(X^\alpha Y^\beta)X^{an} \in I^{n+1}$ を用いて同様に証明される。 □

補題 4.6. $I = \widetilde{I}$ であるための必要十分条件は，$X^{a_2-1}Y^{b_1-1} \notin \widetilde{I}$ である。

証明. $(a_2-1, b_1-1) \notin I$ であるので，必要条件である。逆に $(a_2-1, b_1-1) \notin \widetilde{I}$ と仮定せよ。$I \ne \widetilde{I}$ ならば，$(\alpha,\beta) \in \widetilde{I} \setminus I$ なる $(\alpha,\beta) \in L$ が存在する。補題 4.5 により $(\alpha,\beta) \notin S \cup T$ である。故に，$(\alpha,\beta) \notin I$ であるから $a_1 \le \alpha \le a_2 - 1$ かつ $b_2 \le \beta \le b_1 - 1$ となり，$(a_2-1, b_1-1) \in \widetilde{I}$ が得られるが，これは不可能である。 □

命題 4.7. $a \le b$ とする。このとき，もしも $\overline{I} = \widetilde{I}$ ならば, $a_1 = 1$, $b_1 = c(1)$, $a_\ell = a-1$, $b_\ell = c(a-1)$ である。$a \mid b$ ならば逆も正しい。

証明. $\widetilde{I} = \overline{I} \ni XY^{c(1)}$ より $(1, c(1)) \notin S$ である。故に $a_1 = 1$ である。定義より，ある整数 $n > 0$ で $(XY^{c(1)})Y^{nb} \in I^{n+1}$ であるから，$(1, nb + c(1)) = \sum_{i=0}^{\ell+1} m_i(a_i, b_i) + \xi$ $(\sum_{i=0}^{\ell+1} m_i = n+1,\ \xi \in L)$ と書ける。$1 < a_2 < \cdots < a_{\ell+1}$ より $m_2 = \cdots m_{\ell+1} = 0$ であり，$m_1 \le 1$ である。よって，$nb + c(1) \ge m_1 b_1 + (n+1-m_1)b$ であるから，$c(1) \ge m_1 b_1 - (m_1 - 1)b$ である。もしも $m_1 = 0$ ならば $c(1) \ge b = c(0)$ で，これは不可能である。よって $m_1 = 1$ であり，$c(1) \ge b_1 \ge b - \frac{a_1 b}{a}$ が成り立つ。よって $b_1 = c(1)$ である。$X^{a-1}Y^{c(a-1)} \in \overline{I} = \widetilde{I}$ であるので，補題 4.5 より $(a-1, c(a-1)) \notin T$ となる。故に $c(a-1) \ge b_\ell$ である。$a_\ell \le a-1$ より $b_\ell \ge c(a_\ell) \ge c(a-1)$ であるから，$b_\ell = c(a-1) = c(a_\ell)$ が成り立つ。故に $a_\ell = a-1$ である。 □

それでは定理の証明を与える。

定理 4.3 の証明. $I=\widetilde{I}$ で，$2a_1<a_2$ かつ $2b_2<b_1$ と仮定する。$a_2=ma_1+r$，$b_1=nb_2+s$ $(r,s,m,n\in\mathbb{Z},0<r\le a_1,0<s\le b_2,\ m,n\ge 2)$ とおく。$a_2-1+(n-1)a-na_2\ge 0$ であるから，$(X^{a_2-1}Y^{b_1-1})X^{(n-1)a}\in(X^{a_2}Y^{b_2})^n$ である。同様に，$b_1-1+(m-1)b-mb_1\ge 0$ であるから，$(X^{a_2-1}Y^{b_1-1})Y^{(m-1)b}\in(X^{a_1}Y^{b_1})^m$ である。故に，$X^{a_2-1}Y^{b_1-1}\in\widetilde{I}$ を得る。逆に，$2a_1\ge a_2$ と仮定すると，$X^{a_2-1}Y^{b_1-1}\notin\bigcup_{n>0}I^{n+1}:Y^{nb}$ である。実際，$n>0$ について $(X^{a_2-1}Y^{b_1-1})Y^{nb}\in I^{n+1}$ とすると，$(a_2-1,b_1-1+nb)=\sum_{i=0}^{3}m_i(a_i,b_i)+\xi$ $(\sum_{i=0}^{3}m_i=n+1,\ \xi\in L)$ と書ける。$a_2-1<a_2<a$ より $m_2=m_3=0$ で，$2a_1>a_2-1$ より $m_1\le 1$ である。よって $b_1-1+nb\ge m_1b_1+(n+1-m_1)b$ より $b_1-1\ge m_1b_1-(m_1-1)b$ である。$m_1=0$ なら $b_1-1+nb\ge(n+1)b$ で，$m_1=1$ なら $b_1-1+nb\ge b_1+nb$ である。$b>b_1$ より，これらは不可能である。$2b_2\ge b_1$ の場合も $X^{a_2-1}Y^{b_1-1}\notin\bigcup_{n>0}I^{n+1}:X^{na}$ が同様に確かめられる。 □

定理 4.4 の証明. (2) ⇒ (1) 命題 4.7 に従う。

(1) ⇒ (2) 命題 4.7 より，$(a_1,b_1)=(1,\lceil b-\frac{b}{a}\rceil)$，$(a_2,b_2)=(a-1,\lceil\frac{b}{a}\rceil)$ である。$a\mid b$ を示そう。$\frac{b}{a}=r+\alpha$ $(r\in\mathbb{Z},0\le\alpha<1)$ とおく。

Claim 4.8. $0<\alpha\le\frac{1}{2}$ ならば $X^{a-2}Y^k\notin\bigcup_{n>0}I^{n+1}:X^{na}$ である。但し，$k=c(a-2)=\lceil\frac{2b}{a}\rceil$ とする。

証明. $\alpha\le\frac{1}{2}$ だから $k=2r+1$ であって，$\alpha>0$ だから $b_2=r+1$ である。整数 $n>0$ を取り，$(X^{a-2}Y^k)X^{na}\in I^{n+1}$ と仮定する。$(a-2+na,k)=\sum_{i=0}^{3}m_i(a_i,b_i)+\xi$ $(\sum_{i=0}^{3}m_i=n+1,\ \xi\in L)$ と書ける。$2b_2>k$ なので，$m_2\le 1$ である。一方で，$a\ge 4$ だから $k<b_1<b$ であるので，$m_0=m_1=0$ となる。従って，$a-2+na\ge m_2(a-1)+(n+1-m_2)a$ となり，$m_2\ge 2$ を得るが，これは不可能である。 □

Claim 4.9. $1>\alpha>\frac{1}{2}$ ならば $X^2Y^m\notin\bigcup_{n>0}I^{n+1}:Y^{nb}$ である。但し，$m=c(2)=\lceil b-\frac{2b}{a}\rceil$ とする。

証明. $\alpha>\frac{1}{2}$ より $m=b-2r-1$ で，$\alpha>0$ なので $c(1)=b-r$ である。整数 $n>0$ を取り，$(X^2Y^m)Y^{nb}\in I^{n+1}$ と仮定する。$(2,m+nb)=\sum_{i=0}^{3}m_i(a_i,b_i)+\xi$ $(\sum_{i=0}^{3}m_i=n+1,\ \xi\in L)$ と書ける。$a\ge 4$ より，$2<a_2<a$ であって，$m_2=m_3=0$ を得る。$a_1=1$ なので，$m_1\le 2$ である。よって，$m+nb\ge m_1b_1+(n+1-m_1)b$ となり，$m\ge m_1b_1-(m_1-1)b$ を得るが，$m<b_1<b$ であるので，$m_1=2$ でなければならない。しかしながら，$m-(2b_1-b)=b-2r-1-(2(b-r)-b)=-1<0$ であるので，$m\ge m_1b_1-(m_1-1)b$ は不可能である。 □

よって $a\mid b$ である。 □

定理4.4は，$\ell=2$で$4\le a\le b$という仮定の下で，$\overline{I}=\widetilde{I}$となるような単項式イデアル$I$の形を完全に決めている。特に，$b=aq$とおいたとき，次のような変数変換$f:k[X,Z]\to k[X,Y]\ X\mapsto X\ Z\mapsto Y^q$を考えると，等式$\overline{I}=\widetilde{I}$を満たすイデアル$I$は，多項式環$k[X,Z]$内のイデアル$(X^a,X^{a-1}Z,XZ^{a-1},Z^a)$の拡大として得られるものしかない。つまり，本質的に$(X^a,X^{a-1}Y,XY^{a-1},Y^a)$しかないということがわかる。

系 4.10. $\ell=2$, $4\le a\le b$で$\overline{I}=\widetilde{I}$とせよ。このとき，$I^{a-2}=\overline{I^{a-2}}$であって，$I^{a-3}\ne\overline{I^{a-3}}$である。

証明. 定理4.4より$a\mid b$である。$q=\frac{b}{a}$とおき，環拡大$f:B=k[X,Z]\to A=k[X,Y]\ X\mapsto X\ Z\mapsto Y^q$を考える。$K=(X^a,X^{a-1}Z,XZ^{a-1},Z^a)B$とおく。$\overline{K}=(X,Z)^aB$である。定理4.4より，$I=KA$が成り立つ。さて，$1\le n\le a-2$なら$K^n=(X^{(a-1)p+i}Y^{na-(a-1)p-i}\mid 0\le i\le n, 0\le p\le n)$であるので，$K^{a-2}=(X,Z)^{a(a-2)}=\overline{K^{a-2}}$であって，$K^{a-3}\ne\overline{K^{a-3}}$であることがわかる。また，任意の整数$n>0$について，$I^n=K^nA$であって，$\overline{I^n}=\overline{K^n}A$である。故に，$A$は$B$-freeであるので，$I^{a-2}=\overline{I^{a-2}}$と$I^{a-3}\ne\overline{I^{a-3}}$が従う。 □

定理4.4から次の事実が従う。

系 4.11. $\ell\ge 2$, $4\le a\le b$とする。もしも，次の3条件：(i) $a\mid b$ (ii) $a_1=1$, $b_1=b-\frac{b}{a}$ (iii) $a_\ell=a-1$, $b_\ell=\frac{b}{a}$が満たされるならば，等式$\overline{I}=\widetilde{I}$が成り立つ。

証明. $J=(X^a,XY^{b_1},X^{a-1}Y^{b_2},Y^b)$とおくと，$J\subseteq I$, $\overline{J}=\widetilde{J}$，$\widetilde{J}\subseteq\widetilde{I}$, $\overline{J}=\overline{I}$が成り立つ。故に，$\widetilde{I}\supseteq\widetilde{J}=\overline{J}=\overline{I}$である。 □

任意の整数$\ell\ge 3$に対し，$\overline{I}=\widetilde{I}$であるが$a\nmid b$となるような単項式イデアル$I$が存在する。

例 4.12. 整数$a\ge 5$を任意に取る。$b=a+2$とせよ。このとき$I=(X^iY^{c(i)}\mid 0\le i\le a, i\ne 2)$とおくと$\overline{I}=I+(X^2Y^{c(2)})$であって，$\ell=a-2\ge 3$, $X^2Y^{c(2)}\in\widetilde{I}$となる。よって等式$\overline{I}=\widetilde{I}$が成り立つが，もちろん$a\nmid b$である。

最後に，Rees代数$R=\mathcal{R}(IA_{\mathfrak{M}})$ (但し,$\mathfrak{M}=(X,Y)$である)のBuchsbaum性について述べよう。

定理 4.13. $\ell=2$, $4\le a\le b$とすると，次の条件は同値である。

(1) RはBuchsbaum環であって，Rの任意の素イデアル$P\ne\mathfrak{M}R+R_+$について局所環R_Pは正規である。

(2) RはBuchsbaum環であって，$\overline{I}=\widetilde{I}$が成り立つ。

(3) $I = (X^4, X^3Y, XY^3, Y^4)$ である。

証明. $\sqrt{I} = \mathfrak{M}$ であるので，$\overline{I} = \widetilde{I}$ であることと $\overline{IA_{\mathfrak{M}}} = \widetilde{IA_{\mathfrak{M}}}$ であることは同値である。

(1) と (2) の同値性は，系 3.8 を見よ。

(2) ⇒ (3) $\overline{I} = \widetilde{I}$ なので，定理 4.4 によって $a \mid b$ である。$b = aq$ とする。系 3.8 より $(X, Y)\overline{I} \subseteq I$ であるので，$q = 1$ で $a = 4$ を得る。実際，$c(2) = q(a-2)$ であるから $X^2Y^{q(a-2)} \in \overline{I}$ となり，$X^2Y^{q(a-2)+1} \in I$ である。故に，$q(a-2) + 1 \geq q(a-1) = c(1)$ であって，$q = 1$ を得る。故に $I = (X^a, X^{a-1}Y, XY^{a-1}, Y^a)$ である。$a \geq 5$ ならば，明らかに $X^2Y^{a-2} \cdot X \notin I$ である。

(3) ⇒ (2) $\overline{I} = (X^4, X^3Y, X^2Y^2, XY^3, Y^4) = I + (X^2Y^2)$ である。$\mathfrak{M}\overline{I} = \mathfrak{M}I + \mathfrak{M}{\cdot}X^2Y^2 \subseteq I$ であることと，$(X^4, Y^4)\overline{I} = (X^4, Y^4)I + (X^4, Y^4){\cdot}X^2Y^2 \subseteq I^2$ であることは，明らかである。故に，系 3.8 より主張 (2) が従う。 □

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On j-multiplicity

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This is a joint work with B. Ulrich.

The notion of j-multiplicity was introduced by Achilles and Manaresi in [1] and the theory was developed in [4], [2] and [3]. The j-multiplicity $\mathrm{j}(I)$ is an invariant of an ideal I in a Noetherian local ring $(R, \mathfrak{m})$. If I is $\mathfrak{m}$-primary, then $\mathrm{j}(I)$ coincides with the usual multiplicity $\mathrm{e}(I)$. In this note we give a length formula of j-multiplicity which enables us to compute $\mathrm{j}(I)$ of a given ideal I.

Let us begin with the definition of j-multiplicity. It can be defined for a finitely generated module L over a positively graded Noetherian ring $T = \oplus_{n \geq 0} T_n$ such that $(T_0, \mathfrak{m})$ is local and $T = T_0[T_1]$. We assume that $T_0/\mathfrak{n}$ is an infinite field. Let d be a positive integer with $\dim_T L \leq d$. We denote the Krull dimension of $L/\mathfrak{n}L$ as an T-module by $\ell(T, L)$ and call it the analytic spread of L. Let $W = \mathrm{H}^0_{\mathfrak{n}T}(L)$, which is the 0-th local cohomology module of L with respect to $\mathfrak{n}T$. By the Artin-Rees lemma, we see that $W \cap \mathfrak{n}^k L = 0$ for $k \gg 0$. Then $W = \oplus_{n \geq 0} \mathrm{H}^0_{\mathfrak{n}}(L_n)$ can be embedded in $L/\mathfrak{n}^k L$ as a graded $T/\mathfrak{n}^k T$-module. Because $T/\mathfrak{n}^k T$ is a standard graded ring over an Artinian local ring and $\dim_T L/\mathfrak{n}^k L = \ell(T, L) \leq d$, there exists an integer $\alpha \geq 0$ such that

$$\mathrm{length}_{T_0} \mathrm{H}^0_{\mathfrak{n}}(L_n) = \frac{\alpha}{(d-1)!} n^{d-1} + (\text{terms of lower degree})$$

for $n \gg 0$. This number α is called the j-multiplicity of the T-module L and is denoted by $\mathrm{j}_d(T, L)$.

Lemma 1 (cf. [4]) *$\mathrm{j}_d(T, L) \neq 0$ if and only if $\ell(T, L) = d$.*

Lemma 2 (cf. [4]) *Let $d \geq 2$ and $\dim_{T_0} L_n < \dim_T L$ for any $n \geq 0$. We choose $f \in T_1$ generally so that the following two conditions are satisfied;*

(1) *f is T_+-filter regular for L,*

(2) $\ell(T, L/fL + W) \leq d - 2$.

Then we have $\dim_T L/fL \leq d-1$ *and* $\mathrm{j}_d(T, L) = \mathrm{j}_{d-1}(T, L/fL)$.

Now we consider a Noetherian local ring $(R, \mathfrak{m})$ with $|R/\mathfrak{m}| = \infty$ and a finitely generated R-module M. We take an ideal I of R and a positive integer d with $\dim_R M \leq d$. We set $\mathrm{j}_d(I, M) = \mathrm{j}_d(\mathrm{gr}_I R, \mathrm{gr}_I M)$ and call it the j-multiplicity of I with respect to M. Let us simply denote $\mathrm{j}_{\dim R}(I, R)$ by $\mathrm{j}(I)$. By Lemma 1 and Lemma 2, we have the following assertion.

Lemma 3 $\mathrm{j}(I) \neq 0$ *if and only if* $\ell(I) = \dim R > 0$, *where* $\ell(I)$ *denotes the usual analytic spread of* I.

Lemma 4 *Let* $d \geq 2$ *and* $\dim_R M/IM < \dim_R M$. *Then, for a general element* $a \in I$, *we have* $\dim_R M/aM \leq d-1$ *and* $\mathrm{j}_d(I, M) = \mathrm{j}_{d-1}(I, M/aM)$.

In the case where $\dim_R M/IM = \dim_R M$, we need the following result.

Lemma 5 *Let* $N = M/\mathrm{H}^0_I(M)$. *Then* $N = 0$ *or* $\dim_R N/IN < \dim_R N$. *Moreover we have* $I^n M/I^{n+1}M \cong I^n N/I^{n+1}N$ *for* $n \gg 0$, *and so* $\mathrm{j}_d(I, M) = \mathrm{j}_d(I, N)$.

Applying Lemma 4 and Lemma 5 successively, we get the next result.

Theorem 6 *Let* $a_1, \ldots, a_{d-1}, a_d$ *be sufficiently generic elements of* I. *Then*

$$\begin{aligned} \mathrm{j}(I) &= \mathrm{e}_I(R/(a_1, \ldots, a_{d-1}) : I^\infty) \\ &= \mathrm{length}_R\, R/a_d R + ((a_1, \ldots, a_{d-1}) : I^\infty), \end{aligned}$$

where $(a_1, \ldots, a_{d-1}) : I^\infty = \cup_{n>0} ((a_1, \ldots, a_{d-1}) : I^n)$.

As an application of the theorem above, we get the following assertion.

Example 7 *Let* $R = K[[X, Y, Z]]$ *be the formal power series ring over an infinite field* K. *Let* $\mathfrak{p}$ *be the defining ideal of a space monomial curve:* $X = t^k, Y = t^\ell, Z = t^m$, *where* k, ℓ *and* m *are positive integers with* $GCD\{k, \ell, m\} = 1$. *Then* $\mathfrak{p}$ *is generated by the maximal minors of the matrix*

$$\begin{pmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{pmatrix},$$

where α, β, γ, α', β' and γ' are positive integers. Replacing the variables X, Y and Z, we may assume

$$k\alpha = \min\{k\alpha, \ell\beta', m\gamma', \ell\beta, m\gamma, k\alpha'\}.$$

Then we have $\mathrm{j}(\mathfrak{p}) = \alpha\beta(\gamma+\gamma')$.

We give a sketch of proof for this example. We put $f = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}, g = X^{\alpha+\alpha'} - Y^{\beta}Z^{\gamma'}$ and $h = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma}$. Then $\mathfrak{p} = (f, g, h)$ and the ideal generated by general two elements in $\mathfrak{p}$ can be written in the form $(af - g, bf - h)$ with $0 \neq a, b \in K$. We put $\xi = af - g$ and $\eta = bf - h$. It is easy to see that

$$\begin{aligned}(\xi, \psi) :_R \mathfrak{p}^{\infty} &= (\xi, \eta) :_R f \\ &= (X^{\alpha} + aY^{\beta'} + bZ^{\gamma'}, Y^{\beta} + aZ^{\gamma} + bX^{\alpha'}).\end{aligned}$$

Therefore, by Theorem 6, we get

$$\mathrm{j}(\mathfrak{p}) = \mathrm{length}_R\, R/\mathfrak{p} + (X^{\alpha} + aY^{\beta'} + bZ^{\gamma'}, Y^{\beta} + aZ^{\gamma} + bX^{\alpha'}).$$

Let $A = K[[t^k, t^\ell, t^m]]$. Then ϕ induces an isomorphism $R/\mathfrak{p} \xrightarrow{\sim} A$, which implies

$$\mathrm{j}(\mathfrak{p}) = \mathrm{length}_A\, A/(t^{k\alpha}u, t^{\ell\beta}u)A,$$

where $u = 1 + at^{\ell\beta' - k\alpha} + bt^{m\gamma' - k\alpha} = 1 + at^{m\gamma - \ell\beta} + bt^{k\alpha' - \ell\beta} \in K[[\,t\,]]$. Therefore we get $\mathrm{j}(\mathfrak{p}) = \alpha\beta(\gamma+\gamma')$ since

$$\begin{aligned}&\mathrm{length}_A\, A/(t^{k\alpha}u, t^{\ell\beta}u)A \\ =\ &\mathrm{length}_A\, A/(t^{k\alpha}, t^{\ell\beta})A \\ =\ &\mathrm{length}_R\, R/(X^{\alpha}, Y^{\beta})R + \mathfrak{p} \\ =\ &\mathrm{length}_R\, R/(X^{\alpha}, Y^{\beta}, Z^{\gamma+\gamma'})R.\end{aligned}$$

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On the graded rings whose Veronese subring is a polynomial ring

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Let $R = \oplus_{k \geq 0} R_k$ be a normal graded ring with R_0 an algebraically closed field. Classification of normal two-dimensional graded U.F.D. was first done for hypersurface case by K.-i. Watanabe, and finished by S. Mori[3]. It turn out that normal 2-dimensional graded U.F.D. is written as a graded complete intersection of Brieskorn-Fermat-Pham polynomials with certain conditions. In [3], the essential parts are arguments on Veronese subrings and studies of several ramifications. Further, for graded 2-dimensional U.F.D. R, there is an integer N such that the N-th Veronese subring $R^{(N)}$ is a polynomial.

In [6], we rediscover the Mori's structure theorem as a comparison theorem of cyclic cover of Kummer type and cyclic cover obtained by the Veronese subring.

In this talk, I want to discuss on the graded rings whose Veronese subring is a polynomial ring, that is, graded rings where there is an integer N such that the N-th Veronese subring $R^{(N)}$ is a polynomial.

Our result is the following:

Theorem. *Let R be a normal graded ring. There is an integer N such that the N-th Veronese subring $R^{(N)}$ is a polynomial if and only if the following conditions are satisfied; R is written as a complete intersection over $S = R^{(N)}$*

$$R = S[x_1, \dots, x_s]/(x_1^{m_1} - v_1, \dots\dots, x_s^{m_s} - v_s)$$

where S : polynomial ring and $v_i \in S$ is reduced, $1 \leq i \leq s$, and v_iS, v_jS have no common components in the case $i \neq j$. Further the numerical conditions $(m_i, m_j) = 1$ $(i \neq j)$, and $(m_i, \deg_S(v_i)) = 1$ $(i = 1, \dots, s)$ hold.

The theorem is a corollary of the results of [6].

§1 Some results from [6].

Theorem (3.2) of [6]. *Let R be a normal domain. For non-units $v_1, \dots v_n \in R$ and integers $m_1, \dots, m_n \in \mathbf{Z}$ with $m_i \geq 2$, we define the ring S as $S = R[X_1, \dots, X_n]/(X_1^{m_1} - v_1, \dots, X_n^{m_n} - v_n)$. Then the following hold:*

(i) S is a normal domain if and only if all v_iR are reduced and no pair v_iR and v_jR with $i \neq j$ has a common prime component.

(ii) S is a cyclic cover of R if and only if $(m_i, m_j) = 1$ for any $i \neq j$.

Theorem (3.3) of [6]. *Let $R = R(X, D)$ be a normal domain as described in (1.1), let $v_1, \dots v_n \in R$ be homogeneous reduced non-units, and let $m_1, \dots, m_n \in \mathbf{Z}$ with $m_i \geq 2$. Here we assume that no pair v_iR and v_jR has a common prime component and that $(m_i, m_j) = 1$ for any $i \neq j$. We define the cyclic $m_1 \cdots m_n$-cover S of R by $S = R[X_1, \dots, X_n]/(X_1^{m_1} - v_1, \dots, X_n^{m_n} - v_n)$. Then the following hold:*

(i) $S^{(m_1 \cdots m_n)} \cong R$ if and only if $(m_i, \deg_R(v_i)) = 1$ for $1 \leq i \leq n$.

(ii) Under the conditions stated in of (i), representing $v_iR = R(-\mathcal{D}(E_i))$ by $E_i = \sum_{k=1}^{r_i} \frac{1}{q_{V_{i,k}}} V_{i,k} \in \mathrm{Div}(X, D)$, $1 \leq i \leq n$, and with $S = R(X, \tilde{D})$ as given in Theorem (1.3), $\tilde{D}$ can be written

$$\tilde{D} = \sum_{i=1}^{n} \sum_{k=1}^{r_i} \frac{\tilde{p}_{V_{i,k}}}{m_i q_{V_{i,k}}} V_{i,k} + \sum_{V \neq V_{i,j}} \frac{\tilde{p}_V}{q_V} V,$$

where $(\tilde{p}_{V_{i,k}}, m_i q_{V_{i,k}}) = 1$, and $(\tilde{p}_V, q_V) = 1$ for $V \neq V_{i,k}$.

§2. Remarks.

In the situation of our Main Theorem, the class group of R is a finite group and computed by following the arguments of Theorem (3.6) of [6].

Example Let $R = k[x, y, z]/(x^a + y^b + z^c)$ and put the weight of each variable as $wt(x) = L/a, wt(y) = L/b, wt(z) = L/c$, where

$L = LCM(a, b, c)$. We can represent as $R = k[y, z][x]/(x^a + y^b + z^c)$, and can regard that $S = k[y, z]$ and $v = -y^b - z^c$. Now, $R^{(N)} = k[x, y]$ for some integer N, if and only if, $m = a$ and $\deg(y^b + z^c) = LCM(b, c)$ are coprime. Hence $(a, b) = (a, c) = 1$. In the above, the arguments depend on the representation of R.

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Generic fibrations by affine curves over discrete valuation rings

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1. Introduction

Let $(R, \pi R)$ be a discrete valuation ring (DVR for short) with quotient field K and residue field k, and let A be an integral domain containing R. Then, for a given K-algebra D, we say that A is a generic D-fibration (or a generic fibration by D) over R if the generic fiber $A \otimes_R K$ is K-isomorphic to D. We are interested in a k-algebraic structure of the special fibre $A \otimes_R k$ for a generic D-fibration A. In the case where D is a polynomial ring $K[x]$ or a Laurent polynomial ring $K[x, x^{-1}]$ in one variable x over K, we have already proved the following

Theorem 1. *Suppose that A is a normal and finitely generated generic D-fibration over R.*
(1) *If $D = K[x]$, then*

$$A/\sqrt{\pi A} \cong_k (k_1 \times \cdots \times k_n)[x]$$

for some finite algebraic extensions $k_1, \ldots, k_n$ of k.
(2) *If $D = K[x, x^{-1}]$, then $A/\sqrt{\pi A}$ is k-isomorphic to one of the following:*

(a) $T := (k_1 \times \cdots \times k_m)[x]$,
(b) $L := k[x, y]/(xy - 1)$,
(c) $T \times L$,
(d) $U := k[x, y]/(xy)$,
(e) $T \times U$,

where m is a positive integer, $k_1, \ldots, k_m$ are finite algebraic extensions of k, and $k[x, y]$ is a polynomial ring in two variables over k (cf. [1]).

Let $D := K[x,y]/(f(x,y))$ be a normal domain of dimension one, and let A be a finitely generated normal generic D-fibration over R. Then there exist DVRs $V_1, \ldots, V_n$ of the algebraic function field $K(x,y)$ such that

$$A = D \cap V_1 \cap \cdots \cap V_n.$$

For each i, let P_i be the maximal ideal of V_i. Then we have

$$\mathrm{qt}(A/\sqrt{\pi A}) = V_1/P_1 \times \cdots \times V_n/P_n,$$

where $\mathrm{qt}(*)$ denotes the total quotient ring. From this we know that the residue fields V_i/P_i have important meaning to our purpose. We are thus led to consider the residue field V_1/P_1 of a valuation ring V_1 of $K(x,y)$ dominating R. In this note we take up the case where $f(x,y) = y^2 - f(x)$ with $\deg f(x) = 3$. The main result is as follows.

Theorem 2. *Let (V_1, P_1) be a DVR of an algebraic function field $K(x,y)$, where $y^2 = f(x)$ with $\deg f(x) = 3$, and suppose that V_1 dominates R. If $\mathrm{ch}\, k \neq 2$ and V_1/P_1 is not algebraic over k, then there exist a finite extension field k_1 of k and a polynomial $g(t) \in k_1[t]$ with $\deg g(t) \leq 3$ such that $V_1/P_1 \cong k_1(t,u)$, where $u^2 = g(t)$. In particular, if k is algebraically closed, then V_1/P_1 is a rational function field in one variable or an elliptic function field over k.*

2. Proof of Theorem 2

We keep the same notation and assumption as in Theorem 2. For $w \in V_1$ we denote by $\bar{w}$ the residue class of w in V_1/P_1. Let $V_x = V_1 \cap K(x)$ and $P_x = P_1 \cap V_x$. Then (V_x, P_x) is a DVR of $K(x)$ dominating R. Note that V_x/P_x is transcendental over k, because V_1/P_1 is algebraic over V_x/P_x. Hence, by Nagata's theorem (cf. [1], [2]), we have

$$V_x/P_x = k_1(\bar{z})$$

for some $z \in V_x$, where k_1 is the algebraic closure of k in V_x/P_x. Let $v_1 \colon K(x,y) \to G_1$ (resp. $v_x \colon K(x) \to G_x$) be the valuation corresponding to V_1 (resp. V_x). Then

$$[V_1/P_1 : V_x/P_x] \cdot |G_1/G_x| \leq [K(x,y) : K(x)] = 2,$$

so that $V_1/P_1 = V_x/P_x$ if $|G_1/G_x| \neq 1$. Hence, for the proof of the theorem, we may assume $G_1 = G_x$. In particular, we have $v_x = v_1|_{K(x)}$.

Lemma 3. Let $\hat{R}$ be the completion of R and let $\hat{K} = \mathrm{qt}(\hat{R})$. Then there exists a DVR (W_1, Q_1) of $\hat{K}(x, y)$ such that W_1 dominates V_1 and $W_1/Q_1 = V_1/P_1$ (cf. [1, Lemma 5.3]).

Proof. Let ξ be a uniformzing parameter of V_1 and let $V_1' = V_1 \otimes_R \hat{R}$. Then

$$V_1'/\xi V_1' = V_1/\xi V_1 \otimes_R \hat{R} = V_1/\xi V_1,$$

and hence ξ is a prime element of V_1'. Note that

$$V_1' = V_1 \otimes_R \hat{R} \subset K(x, y) \otimes_R \hat{R} \subset \hat{K}(x, y),$$

and hence we can identify V_1' with $\hat{R}[V_1] \subset \hat{K}(x, y)$. In particular $V_1' \subset \hat{V}_1$, where $\hat{V}_1$ is the completion of V_1, so that

$$\bigcap\nolimits_{r>0} \xi^r V_1' \subset \bigcap\nolimits_{r>0} \xi^r \hat{V}_1 = (0),$$

which implies $\mathrm{ht}(\xi V_1') = 1$. Thus $W_1 := V'_{1(\xi)}$ is a DVR satisfying the required conditions. □

By Lemma 3, in what follows we assume that R is complete. Hence, if S is a domain finitely generated over R, then the dimension formula holds between R and S. Write $v_x(x) = qv_x(\pi) + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < v_x(\pi)$. Then, replacing x by $\pi^{-q}x$, we assume $x \in V_x$ with $v_x(x) < v_x(\pi)$.

Now, starting from $S_0 = R[x]$, we inductively define a finite sequence of rings

$$R \subset S_0 \subset S_1 \subset \cdots \subset S_n \subset V_x$$

such that $S_i = S_{i-1}[\xi_i, \eta_i]$ and $S_i/\xi_i S_i = \kappa_i[\eta_i'] = \kappa_i^{[1]}$, where $\xi_i \in P_x$, $\eta_i \in V_x$, η_i' is the residue class of η_i in $S_i/\xi_i S_i$, and κ_i is a finite algebraic extension of k. We set $\xi_0 = \pi$ and $\eta_0 = x$. Suppose that we have defined S_i. If $P_x \cap S_i = \xi_i S_i$, then $n = i$ and the sequence ends. If not, we define $S_{i+1} = S_i[\xi_{i+1}, \eta_{i+1}]$ as follows. Since $S_i/\xi_i S_i = \kappa_i[\eta_i'] = \kappa_i^{[1]}$, we have $P_x \cap S_i = (\xi_i, \theta_i)S_i$ for some $\theta_i \in S_i$. We choose $\theta_i = \eta_i$ when

$v_x(\eta_i) > 0$. If $v_x(\xi_i) \le v_x(\theta_i)$, then we set $\xi_{i+1} = \xi_i$ and $\eta_{i+1} = \theta_i/\xi_i$. If $v_x(\xi_i) > v_x(\theta_i)$, then write $v_x(\xi_i) = \lambda_i d_i$ and $v_x(\theta_i) = \mu_i d_i$, where $d_i = \gcd(v_x(\xi_i), v_x(\theta_i))$, and let α_i, β_i be nonnegative integers satisfying $\lambda_i\alpha_i - \mu_i\beta_i = 1$. Then we set

$$\xi_{i+1} = \xi_i^{\alpha_i}/\theta_i^{\beta_i}, \quad \eta_{i+1} = \theta_i^{\lambda_i}/\xi_i^{\mu_i}.$$

Note that in this case we have $v_x(\xi_{i+1}) = d_i < v_x(\xi_i)$, because $v_x(\xi_i) > v_x(\theta_i)$ implies $\lambda_i \ge 2$. Moreover we have $v_x(\eta_{i+1}) = 0$ and

$$\xi_i = \xi_{i+1}^{\lambda_i}\eta_{i+1}^{\beta_i}, \quad \theta_i = \xi_{i+1}^{\mu_i}\eta_{i+1}^{\alpha_i}. \tag{1}$$

We will show that $S_{i+1}/\xi_{i+1}S_{i+1} \cong \kappa_{i+1}^{[1]}$, where $\kappa_{i+1} = S_i/(\xi_i, \theta_i)S_i$. If $v_x(\xi_i) \le v_x(\theta_i)$, then we have

$$S_{i+1} = S_i[\theta_i/\xi_i] \cong S_i[X]/(\xi_i X - \theta_i),$$

because ξ_i, θ_i is a regular sequence in S_i. Hence the claim follows. When $v_x(\xi_i) > v_x(\theta_i)$, let $\rho\colon S_i[X, Y] \to S_{i+1} = S_i[\xi_{i+1}, \eta_{i+1}]$ be the natural S_i-algebra homomorphism. Then

$$(X^{\lambda_i}Y^{\beta_i} - \xi_i,\ X^{\mu_i}Y^{\alpha_i} - \theta_i) \subset \ker\rho,$$

and hence

$$(X,\ X^{\lambda_i}Y^{\beta_i} - \xi_i,\ X^{\mu_i}Y^{\alpha_i} - \theta_i) = (X, \xi_i, \theta_i) \subset (X) + \ker\rho.$$

Since $S_{i+1}/\xi_{i+1}S_{i+1} \cong S_i[X, Y]/((X) + \ker\rho)$ and

$$S_i[X, Y]/(X, \xi_i, \theta_i) \cong \kappa_{i+1}[Y] = \kappa_{i+1}^{[1]},$$

we know that ρ induces a surjective S_i-algebra homomorphism

$$\bar\rho\colon \kappa_{i+1}^{[1]} \to S_{i+1}/\xi_{i+1}S_{i+1}.$$

Suppose that $\ker\bar\rho \ne 0$, and let Q be a minimal prime ideal of $\xi_{i+1}S_{i+1}$. Then S_{i+1}/Q is a surjective image of $\kappa_{i+1}^{[1]}/\ker\bar\rho$, so that S_{i+1}/Q is a finite extension field of k. On the other hand note that $\xi_i \in \xi_{i+1}S_{i+1}$ by the equation (1). Hence, inductively, we have $\xi_j \in \xi_{i+1}S_{i+1}$ for every $j < i$, and therefore $\pi = \xi_0 \in \xi_{i+1}S_{i+1}$. Thus $\pi \in Q$, so that, by the dimension formula, we have

$$\mathrm{ht}(Q) = \mathrm{ht}(\pi R) + \mathrm{tr.deg}_R S_{i+1} - \mathrm{tr.deg}_k S_{i+1}/Q = 2,$$

which is a contradiction. Hence $\ker \bar{\rho} = 0$, and $\bar{\rho}$ is an isomorphism, as desired.

Next we will show that the sequence is of finite length, namely, there exists n such that $P_x \cap S_n = \xi_n S_n$. Recall that $v_x(\xi_i) \geq v_x(\xi_{i+1})$, where the equality holds if and only if $\xi_i = \xi_{i+1}$. Hence it suffices to prove that it cannot be the case

$$\xi_i = \xi_{i+1} = \cdots \tag{2}$$

for some i. Suppose that there exists i satisfying (2), and let $B = \cup_{j \geq i} S_j$. Then, for $g \in P_x \cap B$, we have $g \in P_x \cap S_j \subset \xi_{j+1} S_{j+1} \subset \xi_{j+1} B$ for some j with $j \geq i$, which implies $P_x \cap B = \xi_i B$ because $\xi_{j+1} = \xi_i$. On the other hand, we have

$$\bigcap_{r>0} \xi_i^r B \subset \bigcap_{r>0} P_x^r = 0,$$

and hence $\mathrm{ht}(\xi_i B) = 1$. Thus $B_{(\xi_i)}$ is a DVR dominated by V_x, and therefore $B_{(\xi_i)} = V_x$. However,

$$B/\xi_i B = \bigcup_{j \geq i} (S_j/(\xi_j, \theta_j)) = \bigcup_{j \geq i} \kappa_j,$$

which is algebraic over k. This is a contradiction.

For the sequence defined above, let $\xi = \xi_n$ and $z = \eta_n$. Then $V_x = S_{n(\xi)}$. Hence $v_x(\xi) = 1$ and $V_x/P_x = \kappa_{n-1}(\bar{z}) = \kappa_{n-1}^{(1)}$, so that we have $k_1 = \kappa_{n-1}$.

Lemma 4. For each $i \geq 0$, let $l_i = v_x(\xi_i)$ and $m_i = v_x(\theta_i)$. Then

$$\overline{\xi^{-l_i} \xi_i} = a_i \bar{z}^{s_i}, \quad \overline{\xi^{-m_i} \theta_i} = b_i \bar{z}^{t_i}$$

for some $a_i, b_i \in k_1 \setminus \{0\}$ and $s_i, t_i \in \mathbb{Z}_{\geq 0}$.

Proof. It follows from the definition that $\eta_i = \theta_i$ or $\bar{\eta}_i \in k_1 \setminus \{0\}$ for each $1 \leq i < n$. Thus, using the equation (1), we can easily verify the assertion by induction on $n - i$. □

Lemma 5. Let $m = v_x(f(x))$. If $\bar{x}$ is transcendental over k, then

$$\overline{\xi^{-m} f(x)} = a h(\bar{z})$$

for some $a \in k_1 \setminus \{0\}$ and $h(\bar{z}) \in k_1[\bar{z}]$ with $\deg h(\bar{z}) \leq 3$. If $\bar{x}$ is algebraic over k, then

$$\overline{\xi^{-m} f(x)} = a\bar{z}^s h(\bar{z})$$

for some $a \in k_1 \setminus \{0\}, s \in \mathbb{Z}$ and $h(\bar{z}) \in k_1[\bar{z}]$ with $\deg h(\bar{z}) \leq 1$.

Proof. Let $v\colon K(x) \to G$ be the valuation corresponding to the valuation ring $R[x]_{(\pi)}$, so that v is the canonical extension of the valuation of K corresponding to R. Write

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

with $a_i \in K$. If $\bar{x}$ is transcendental over k, then we have $V_x = R[x]_{(\pi)}, \xi = \pi, z = x$ and $v_x(f(x)) = \min\{v(a_i) \mid 0 \leq i \leq 3\}$. Thus the assertion is obvious in this case.

Suppose that $\bar{x}$ is algebraic over k. Then there exists a monic polynomial $\varphi(x) \in R[x]$ such that $P_x \cap V[x] = (\pi, \varphi(x))$, namely $\varphi(x) = \theta_0$. Write

$$v_x(\pi) = \lambda_0 d_0, \quad v_x(\theta_0) = \mu_0 d_0$$

with $\gcd(\lambda_0, \mu_0) = 1$, as in the definition of the sequence of rings given above.

Case 1. $\deg \varphi(x) \geq 4$.

Let $r = \min\{v(a_i) \mid 0 \leq i \leq 3\}$. Then $m = rv_x(\pi)$, and $\overline{\pi^{-r} f(x)} \in k_1 \setminus \{0\}$. Thus the assertion follows from Lemma 4.

Case 2. $\deg \varphi(x) = 3$ or 2.

In this case we have

$$f(x) = b_1(x)\varphi(x) + b_0(x) = b_1(x)\theta_0 + b_0(x),$$

where $b_i(x) \in K[x]$ with $\deg b_i(x) < \deg \varphi(x)$ for $i = 0, 1$. Let $r_i = v(b_i(x))$ and $c_i(x) = \pi^{-r_i} b_i(x)$ for $i = 0, 1$. Then $v_x(b_i(x)) = r_i v_x(\pi)$. If $q_1 := v_x(b_1(x)\theta_0) < q_0 := v_x(b_0(x))$, then

$$f(x) = c_1(x)\pi^{r_1}\theta_0(1 + b_0(x)/(b_1(x)\theta_0)),$$

where $b_0(x)/(b_1(x)\theta_0) \in P_x$ and $\overline{c_1(x)} \in k_1 \setminus \{0\}$. Thus the assertion follows from Lemma 4. The proof is similar in the case $q_1 > q_0$. Suppose that $q_0 = q_1$. Then

$$(\lambda_0 r_1 + \mu_0)d_0 = \lambda_0 r_0 d_0,$$

which implies $\lambda_0 = 1$ and $\mu_0 = r_0 - r_1$, because $\gcd(\lambda_0, \mu_0) = 1$. Thus

$$\xi_{\mu_0} = \xi_0 = \pi, \quad \eta_{\mu_0} = \theta_0/\pi^{\mu_0},$$

and hence

$$f(x) = \xi_{\mu_0}^{\lambda_0 r_0}(c_1(x)\eta_{\mu_0} + c_0(x)).$$

Note that if we set $\tau = c_1(x)\eta_{\mu_0} + c_0(x)$, then $\bar{\tau} \in k_1 \setminus \{0\}$ or $\tau = \theta_{\mu_0}$. Thus the assertion is a consequence of Lemma 4.

Case 3. $\deg \varphi(x) = 1$.

We may assume $\theta_0 = x$. Recall that $v_x(\pi) > v_x(x)$, and hence $\lambda_0 \geq 2$.

Suppose that there exists i such that $v_x(a_i x^i) < v_x(a_j x^j)$ for $j \neq i$. Then we have $f(x) = a_i x^i(1 + g)$ with $g \in P_x$, and the assertion follows from Lemma 4.

Suppose that there exist i and j with $i < j$ such that

$$v_x(a_i x^i) = v_x(a_j x^j) = \min\{v_x(a_l x^l) \mid 0 \leq l \leq 3\}.$$

Then, setting $r_l = v(a_l)$ for each l, we have $(\lambda_0 r_i + \mu_0 i)d_0 = (\lambda_0 r_j + \mu_0 j)d_0$, which implies $\lambda_0(r_i - r_j) = \mu_0(j - i)$, and hence

$$\lambda_0 = j - i, \quad \mu_0 = r_i - r_j, \tag{3}$$

because $\gcd(\lambda_0, \mu_0) = 1, \lambda_0 > 1$ and $1 \leq j - i \leq 3$. Let $b_l = \pi^{-r_l} a_l$ for each l, so that $b_l \in R^\times$. Since

$$\pi = \xi_1^{\lambda_0} \eta_1^{\beta_0}, \quad x = \xi_1^{\mu_0} \eta_1^{\alpha_0}$$

by the equation (1), we have

$$a_l x^l = b_l \xi_1^{\lambda_0 r_l + \mu_0 l} \eta_1^{\beta_0 r_l + \alpha_0 l}$$

for each l. Note that it follows from (3) that

$$(\beta_0 r_j + \alpha_0 j) - (\beta_0 r_i + \alpha_0 i) = \lambda_0 \alpha_0 - \mu_0 \beta_0 = 1.$$

Therefore, setting $N = \lambda_0 r_i + \mu_0 i$ and $L = \beta_0 r_i + \alpha_0 i$, we have

$$f(x) = b_j \xi_1^N \eta_1^L (\eta_1 - c_1 + \xi_1 h_1(\xi_1, \eta_1)),$$

where $c_1 = -b_i/b_j \in R^\times$ and $h_1(X,Y) \in R[X,Y]$. If $v_x(\eta_1 - c_1) = 0$, then we are done by Lemma 4. If $v_x(\eta_1 - c_1) > 0$, then $\theta_1 = \eta_1 - c_1$. In this case if $v_x(\theta_1) < v_x(\xi_1)$, then

$$f(x) = b_j \xi_1^N \eta_1^L \theta_1 (1 + (\xi_1/\theta_1) h_1(\xi_1, \eta_1)),$$

which proves the assertion by Lemma 4. If $v_x(\theta_1) \geq v_x(\xi_1)$, then $\xi_2 = \xi_1$ and $\eta_2 = \theta_1/\xi_1$, so that $\eta_1 = \xi_2\eta_2 + c_1$. From this it follows that

$$f(x) = b_j \xi_2^{N+1} \eta_1^L (\eta_2 - c_2 + \xi_2 h_2(\xi_2, \eta_2)),$$

where $c_2 = -h_1(0, c_1)$ and $h_2(X,Y) = (h_1(X, XY + c_1) + c_2)/X$. We are done if $v_x(\eta_2 - c_2) = 0$. If not, then $\theta_2 = \eta_2 - c_2$, and we can repeat the above argument. Since the sequence $\theta_0, \theta_1, \theta_2, \ldots$ is of finite length, the assertion is verified. □

Now we will prove Theorem 2. Let $m = v_x(f(x))$. Then m is even, because $v_x(f(x)) = v_1(y^2) = 2v_1(y)$. By Lemma 5, we have

$$\overline{\xi^{-m} y^2} = a\bar{z}^s h(\bar{z})$$

for some $a \in k_1 \setminus \{0\}, s \in \mathbb{Z}$ and $h(\bar{z}) \in k_1[\bar{z}]$. Write $s = 2q + r$ with $q \in \mathbb{Z}$ and $r = 0$ or 1. Set $g(\bar{z}) = a\bar{z}^r h(\bar{z})$ and $w = \xi^{-m/2} z^{-q} y \in V_1$. Then $K(x,y) = K(x,w)$ and $\bar{w}^2 = g(\bar{z})$. Note that $\deg g(\bar{z}) = r + \deg h(\bar{z}) \leq 3$ by Lemma 5. First suppose that $H(Y) := Y^2 - g(\bar{z})$ is an irreducible polynomial of $k_1(\bar{z})[Y]$. Then, setting $A = V_x[w]$, we have

$$A \cong V_x[Y]/(Y^2 - w^2),$$

which implies

$$A/\xi A \cong k_1(\bar{z})[Y]/(Y^2 - g(\bar{z})).$$

This show that ξA is a maximal ideal of A, and hence $P_1 \cap A = \xi A$. Thus $V_1 = A_{(\xi)}$, so that

$$V_1/P_1 \cong k_1(t, u)$$

with $u^2 = g(t)$. Next suppose that $H(Y)$ is not an irreducible polynomial. Since $\operatorname{ch} k \neq 2$, it follows that $g(\bar{z}) = \psi^2$ for some $\psi = \psi(\bar{z}) \in k_1[\bar{z}]$.

Choose $\tau \in V_x$ such that $\bar{\tau} = \psi(\bar{z})$. Then $w^2 - \tau^2 = (w - \tau)(w + \tau) \in P_1$, and hence we may assume $w - \tau \in P_1$ and $w + \tau \notin P_1$. Let

$$A = V_x[w,\ (w + \tau)^{-1}] \cong V_x[Y,\ (Y + \tau)^{-1}]/(Y^2 - w^2),$$

which is a subring of A. Then

$$A/\xi A \cong k_1(\bar{z})[Y,\ (Y + \psi)^{-1}]/(Y^2 - \psi^2) = k_1(\bar{z})$$

and hence ξA is a maximal ideal of A. Thus $V_1 = A_{(\xi)}$, and

$$V_1/P_1 = k_1(\bar{z}) = k_1^{(1)}$$

in this case. This completes the proof of Theorem 2.

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AFFINE SURFACES WHICH ARE CLOSE TO THE AFFINE PLANE

M. MIYANISHI

1. Problems and Motivations

This talk is based on the works [2], [4] and [6]. The ground field is assumed to be the complex field $\mathbb{C}$ throughout this talk. A classification of smooth affine varieties is roughly done according to the value of log Kodaira dimension. In the case dim $= 2$, this classification works nicely thanks to the theory of "open algebraic surfaces". One of the results is stated as follows.

Theorem 1.1. *Let X be a smooth affine surface. Then X has log Kodaira dimension $-\infty$ if and only if there is an $\mathbb{A}^1$-fibration $f : X \to B$, where B is a smooth curve.*

Here we mean by an $\mathbb{A}^1$-fibration a surjective morphism whose general fibers are isomorphic to the affine line $\mathbb{A}^1$. If B is affine, there is a G_a-action $\sigma : G_a \times X \to X$ such that f is the associated quotient morphism $X \to X/G_a$. This follows from local triviality of an $\mathbb{A}^1$-fibration over a curve. But if B is complete, there is no such G_a-action giving rise to f.

Makar-Limanov introduced an invariant, which we call the *Makar-Limanov invariant* and denote by $\mathrm{ML}(X)$, for an affine algebraic variety $X = \operatorname{Spec} A$ as $\bigcap_\delta \operatorname{Ker} \delta$, where δ ranges over all locally nilpotent derivations on the coordinate ring A which correspond bijectively to G_a-actions on X. When we try to classify smooth affine surfaces in a more refined fashion, we have to make use of this Makar-Limanov invariant.

Definition **1.2.** A smooth affine surface X is an ML_i surface if tr.deg $Q(\mathrm{ML}(X)) = i$, where $i = 0, 1, 2$, where $Q(\mathrm{ML}(X))$ is the quotient field.

Roughly speaking, an ML_0 surface (resp. ML_1 surface, or ML_2 surface) has two independent (resp. one, or no) G_a-actions. In particular, an ML_0-surface is a rational surface. Meanwhile, an ML_2 surface could have an $\mathbb{A}^1$-fibration over a complete curve.

Definition **1.3.** A smooth affine surface X is *unruled* (resp. *simply ruled, multi-ruled*) if X has no $\mathbb{A}^1$-fibrations (resp. only one $\mathbb{A}^1$-fibration, two independent $\mathbb{A}^1$-fibrations). X is also called 0-*ruled,* 1-*ruled,* 2-*ruled* if X is unruled, simply-ruled, multi-ruled, respectively.

If X is an ML_0 surface then it is 2-ruled. But the converse is not necessarily the case. The base of an $\mathbb{A}^1$-fibration for an ML_0 surface is not necessarily the affine line. If X is an ML_0 surface with Picard number $\rho(X)$ positive, there is always an $\mathbb{A}^1$-fibration with base isomorphic to $\mathbb{P}^1$.

Let $f : X \to B$ be an $\mathbb{A}^1$-fibration as above. A scheme-theoretic fiber $f^*(P)$ over a point $P \in B$ is *smooth* (or *singular*) if it is isomorphic to $\mathbb{A}^1$ (or otherwise). If $f^*(P)$ is singular, write it as $f^*(P) = \sum_{i=1}^n m_i C_i$, where C_i is an irreducible curve. Then every C_i is isomorphic to $\mathbb{A}^1$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. Let $m = \gcd(m_1, \ldots, m_n)$. If $m > 1$, we say that $f^*(P)$ is a multiple fiber with multiplicity m.

Our intention is to define smooth affine surfaces which are close to the affine plane $\mathbb{A}^2$ in various properties and consider whether or not they still enjoy the same results as $\mathbb{A}^2$ does. We have to specify when a smooth affine surface is close to $\mathbb{A}^2$ and what king of results we have in mind.

We consider a smooth affine surface X which has an $\mathbb{A}^1$-fibration $f : X \to B$ such that (1) B is isomorphic to $\mathbb{A}^1$ and (2) every fiber of f is irreducible. In particular, $\rho(X) = 0$. Note that multiple fibers are admitted. More precisely, we say such an X is an *affine pseudo-plane* if there is at most one multiple fiber. An affine pseudo-plane is either an ML_0 surface or an ML_1 surface. The results or problems we have in mind are, for example, those listed below.

1. Let C be a curve on X which is isomorphic to $\mathbb{A}^1$. Is it then a fiber component of a (possibly new) $\mathbb{A}^1$-fibration on X? In other terms, is it true that $\overline{\kappa}(X - C) = -\infty$? (Analogy of theorem of Abhyankar-Moh-Suzuki)
2. Suppose $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$. Does this imply that $X \cong Y$? (Cancellation Problem)
3. Suppose that X is an ML_0 surface. Let $f : X \to Y$ be a proper morphism, where Y is a smooth affine surface. Is Y an ML_0 surface? (Ascent and descent of ML_i property)
4. Let $\varphi : X \to X$ be an étale endomorphism. Is φ an automorphism ? (Generalized Jacobian problem)

The following criteria of ML_0 surface and ML_1 surface turn to be very useful. The first one is essentially due to M.H. Gizatullin and J. E. Bertin.

Lemma 1.4. *Let X be a smooth affine surface and let V be a minimal smooth normal completion of X. Then X is an* ML_0 *surface if and only if* $\operatorname{rank}\Gamma(X, \mathcal{O}_X)/\mathbb{C}^* = 0$ *and the dual graph of the boundary divisor $D := V - X$ is a linear chain.*

Lemma 1.5. *Let X be a smooth affine rational surface and let $X \hookrightarrow V$ be a smooth minimal normal completion. Let $D := V - X$. Then the following conditions are equivalent.*

(1) *X is an* ML_1 *surface which is different from $\mathbb{A}^1 \times \mathbb{C}^*$.*
(2) *The dual weighted graph $\Gamma(D)$ of D has a non-admissible twig and it is not a linear chain.*

2. Analogy of theorem of Abhyankar-Moh-Suzuki

A result is positive here. The following result (Theorem 2.2) is obtained in [2] and independently by Kishimoto-Kojima. Let us begin with some preliminary result on ML_0 surfaces.

Lemma 2.1. *Let X be an* ML_0 *surface. Then the following assertions hold.*

(1) $\operatorname{rank}\Gamma(X, \mathcal{O}_X)^*/\mathbb{C}^* = 0$.
(2) *The torsion part* $\operatorname{Pic}(X)_{\mathrm{tor}}$ *is isomorphic to $\pi_1(X)$ and it is a finite cyclic group, while $H_2(X)$ is the free part of* $\operatorname{Pic}(X)$.
(3) *Any $\mathbb{A}^1$-fibration $f : X \to B$ has at most two (resp. one) multiple fibers if $B \cong \mathbb{P}^1$ (resp. $B \cong \mathbb{A}^1$).*
(4) *Suppose that $\rho(X) = 0$. Let $f : X \to B$ be an $\mathbb{A}^1$-fibration, where B is necessarily isomorphic to $\mathbb{A}^1$. Let $f^*(P_0) = dF_0$ be a unique multiple fiber of f. Let $\widetilde{X}$ be the universal covering space of X. Then $\widetilde{X}$ is the normalization of $X \times_B (B', \sigma)$, where $\sigma : B' \to B$ is a cyclic covering of order d totally ramifying at the point P_0 and the point at infinity P_∞. The surface $\widetilde{X}$ is an affine hypersurface in $\mathbb{A}^3$ defined by $xz = y^d - 1$, where the Galois group G acts as $(x, y, z) \mapsto (\zeta x, \zeta^n y, \zeta^{-1} z)$ when G is identified with the group of d-th roots of unity.* (See [3]).

Theorem 2.2. *Let X be a $\mathbb{Q}$-homology plane. We suppose that X is an* ML_0 *surface. Let C be a curve isomorphic to the affine line on X. Then there exists an $\mathbb{A}^1$-fibration $f : X \to B$ such that $B \cong \mathbb{A}^1$ and C is a fiber component of f.*

Proof is outlined as follows. For a topological manifold Y, $e(Y)$ denotes its Euler-Poincaré characteristic. Since $e(X) = e(C) = 1$, we have $e(X - C) = 0$. Furthermore, $\overline{\kappa}(X - C) \leq 1$. If $\overline{\kappa}(X - C) = -\infty$, the result follows from Theorem 1.1. So, it suffices to show that $\overline{\kappa}(X - C) = 0, 1$ is impossible, and we will show it by making use of the theory of open algebraic surfaces.

An affine line C on a smooth affine surface X is called *anomalous* if $\overline{\kappa}(X - C) \geq 0$. Even if X is an ML_0 surface, there may exist an anomalous affine line unless $\rho(X) = 0$. Here is an example.

EXAMPLE **2.3.** *Let Σ_0 be the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$. We denote any fiber of the vertical (resp. the horizontal) $\mathbb{P}^1$-fibration by ℓ (resp. M) and call it a* fiber *(resp.* a section*). Take two horizontal sections M_0, M_1 and three fibers $\ell_0, \ell_1, \ell_\infty$. Let $P_0 := M_0 \cap \ell_0$ and $P_1 := M_1 \cap \ell_1$. Let A be a smooth irreducible curve such that $A \sim M + 2\ell$ and touches M_0 (resp. M_1) at P_0 (resp. P_1) with order of contact 2. Hence A meets ℓ_∞ at a point other than $M_0 \cap \ell_\infty$ and $M_1 \cap \ell_\infty$. Blow up the point P_0 (resp. P_1) and its infinitely near point of the first order lying on M_0 (resp. M_1) to produce irreducible exceptional curves E_1, E_2 (resp. F_1, F_2), where $({E_1}^2) = ({F_1}^2) = -2$ and $({E_2}^2) = ({F_2}^2) = -1$. Then the proper transform A' of A meets E_2 and F_2. We blow up these two intersection points to obtain the exceptional curves E_3 and F_3. We denote the proper transforms of E_i, F_i $(i = 1, 2)$ by the same letters. Now we have $({E_i}^2) = ({F_i}^2) = -2$ for $i = 1, 2$ and $({E_3}^2) = ({F_3}^2) = -1$. Let A' be anew the proper transform of A'. Let $\sigma : V \to \Sigma_0$ be the composite of these blowing-ups. We have now the following relations*

$$
\begin{aligned}
K_V &= -\sigma^*(2\ell_\infty + M_0 + M_1) + E_1 + 2E_2 + 3E_3 + F_1 + 2F_2 + 3F_3 \\
\sigma^*(M_0) &= M_0' + E_1 + 2E_2 + 2E_3 \\
\sigma^*(M_1) &= M_1' + F_1 + 2F_2 + 2F_3 \\
\sigma^*(\ell_0) &= \ell_0' + E_1 + E_2 + E_3 \\
\sigma^*(\ell_1) &= \ell_1' + F_1 + F_2 + F_3 \\
\sigma^*(A) &= A' + E_1 + 2E_2 + 3E_3 + F_1 + 2F_2 + 3F_3
\end{aligned}
$$

where $M_0', M_1', \ell_0', \ell_1', A'$ signify the proper transforms of $M_0, M_1, \ell_0, \ell_1, A$ on V. We set $X := V - D$, where $D := M_0' + M_1' + \sigma^*(\ell_\infty) + E_1 + E_2 + F_1 + F_2$ and $C := A' \cap X$. Then the following assertions hold.

(1) *X is an ML_0 surface with $\rho(X) = 1$.*
(2) *$\overline{\kappa}(X - C) = 0$ and hence C is an anomalous affine line.*

If X is an ML_1 surface with $\rho(X) = 0$, there exists an anomalous affine line. Namely we have the following examples and result.

EXAMPLE **2.4.** *Let V_0 be a Hirzebruch surface of degree $n = 0, 1$ with the $\mathbb{P}^1$-fibration $p_0 : V_0 \to \mathbb{P}^1$. Let M_0 and ℓ be respectively the minimal* *section and a general fiber. Let* H_0 *be a smooth curve such that* $H_0 \sim 2M_0+\ell$ *(resp.* $H_0 \sim 2(M_0+\ell)$*) if* $n=0$ *(resp.* $n=1$*). Let* P_0, P_∞ *be the points of the base curve of* p_0 *over which* $p_0|_{H_0}\colon H_0 \to \mathbb{P}^1$ *ramifies and let* $\ell_0 = p_0^{-1}(P_0)$ *and* $\ell_\infty = p_0^{-1}(P_\infty)$*. Let* $\sigma : V \to V_0$ *be the blowing-ups of the point* $\ell_\infty \cap H_0$ *and its infinitely near point on* H_0 *which produce a* (-2) *curve* E_1 *and a* (-1) *curve* E_2*. Let* $H = \sigma'(H_0), L = \sigma'(\ell_\infty)$ *and* $\overline{C} = \sigma'(\ell_0)$*. Let* $X = V - (H + E_1 + E_2 + L)$ *and let* $C = \overline{C} \cap X$*. Then the following assertions hold.*

(1) $(H^2) = 2$*. Let* $\tau : V' \to V$ *be the blowing-ups of the point* $H \cap E_2$ *and its infinitely near point on* H *which produce a* (-2) *curve* E_3 *and a* (-1) *curve* E_4*. Write* $\tau'(E_1), \tau'(E_2)$ *by the same letters* E_1, E_2 *and let* $H' = \tau'(H)$*. Then* $(H'^2) = 0$ *and* $|H'|$ *defines a* $\mathbb{P}^1$*-fibration* $\overline{f} : V' \to \mathbb{P}^1$ *such that* $f = \overline{f}|_X\colon X \to \mathbb{A}^1$ *is an* $\mathbb{A}^1$*-fibration. In the fibration* $\overline{f}$*,* E_4 *is a cross-section and* $E_3 + L + 2(E_2 + E_1 + A)$ *(resp.* $E_3 + E_1 + 2(E_2 + L + A)$*) is a fiber of* $\overline{f}$ *if* $n = 0$ *(resp.* $n = 1$*), where* A *is a* (-1) *curve meeting* X*.*

(2) X *is an* ML_1 *surface with* $\rho(X) = 0$ *and one multiple fiber of multiplcity* 2*.*

(3) C *is an affine line lying transversally to* f *and* $\overline{\kappa}(X - C) = 0$*.*

EXAMPLE **2.5.** *Let* V_0 *be a Hirzebruch surface of degree* $n \geq 0$ *with the* $\mathbb{P}^1$*-fibration* $p_0 : V_0 \to \mathbb{P}^1$ *and let* M_0, ℓ *be the same as in Example 2.4. Let* M_1 *be a section disjoint from* M_0 *(hence* $({M_0}^2) = n$*). Choose three fibers* $\ell_0, \ell_1, \ell_\infty$*. Let* $\sigma : V \to V_0$ *be a sequence of blowing-ups which produce the following degenerate fibers* Γ_i *from* ℓ_i *for* $i = 0, 1$*:*

$$\Gamma_0 :\ M_0' \ - \ \underset{\overline{C}}{(-m_1)} \ - \ \underset{E_0}{(-1)} \ - \ \underset{E_1}{(-2)} \ - \ \cdots \ - \ \underset{E_{m_1-1}}{(-2)} \ - \ M_1'$$

$$\Gamma_1 :\ M_0' - (-a_1) - \cdots - (-a_s) \ - \ \underset{F_0}{(-1)} \ - \ (-b_t) - \cdots - (-b_1) - M_1'$$

where $a_i \geq 2\ (1 \leq i \leq s), b_j \geq 2\ (1 \leq j \leq t), \overline{C} = \sigma'(\ell_0)$ *and* $M_k' = \sigma'(M_k)$ *for* $k = 0, 1$*. Let* m_2 *be the multiplicity of the component* F_0 *in the fiber* $\sigma^*(\ell_1)$ *and let* $D = M_0' + M_1' + \ell_\infty + (\sigma^*(\ell_0)_{\mathrm{red}} - (\overline{C} + E_0)) + (\sigma^*(\ell_1)_{\mathrm{red}} - F_0)$ *and let* $X = V - D$*. Let* $C = \overline{C} \cap X$*. Suppose that* $m_1 \geq 2$ *and* $m_2 \geq 2$*.Then the following assertions hold.*

(1) X *is an* ML_1 *surface.*

(2) C *is an affine line, and it lies transversally to a unique* $\mathbb{A}^1$*-fibration* $f : X \to \mathbb{A}^1$*.*

(3) $\overline{\kappa}(X - C) = 0$ *if and only if* $m_1 = m_2 = 1$*.* $\overline{\kappa}(X - C) = 1$ *otherwise.*

(4) *If* $m_1 = m_2 = 2$, X *is isomorphic to the surface constructed in Example 2.4.*

Theorem 2.6. *Let* X *be a* $\mathbb{Q}$*-homology plane. Suppose that* X *is an* ML_1 *surface and not isomorphic to the surface constructed in Examples 2.4 and 2.5. Then any affine line on* X *is a fiber of the unique* $\mathbb{A}^1$*-fibration* $f : X \to \mathbb{A}^1$. *In other words, there are no affine lines which lie transversally to the unique* $\mathbb{A}^1$*-fibration* $f : X \to \mathbb{A}^1$.

We have the following result, which makes some clear distinction in the cases $\rho(X) = 0$ and $\rho(X) > 0$.

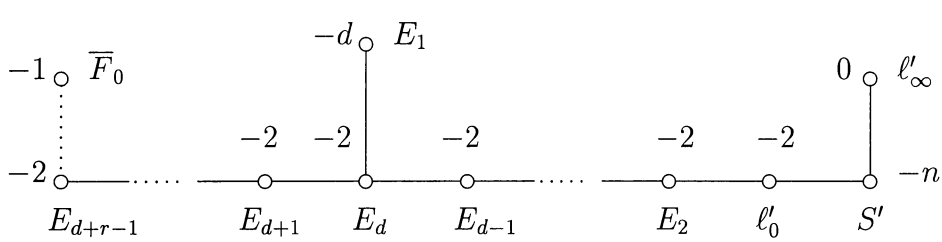
Theorem 2.7. *Let* X *be an* ML_0 *surface with* $\rho(X) > 0$. *Then there exists an* $\mathbb{A}^1$*-fibration* $g : X \to \mathbb{P}^1$.

In the proof, we use effectively Lemma 2.1, (4) and the Derksen invariant $\mathrm{Dk}(X)$ which is defined for an affine variety $X = \mathrm{Spec}\,(R)$ as follows. $\mathrm{Dk}(X)$ is the subalgebra of R over $\mathbb{C}$ generated by $\mathrm{Ker}\,\delta$, where δ ranges over all locally nilpotent derivations on R.

3. Affine pseudo-planes and cancellation problem

The class of affine pseudo-planes is fairly wide, and we have to specify a subclass to state some results meaningful.

Definition **3.1.** *An affine pseudo-plane* X *which has an* $\mathbb{A}^1$*-fibration* $f : X \to B \cong \mathbb{A}^1$ *and a unique multiple fiber* dF_0 *has* type (d, n, r) *if* X *has a smooth normal completion* (V, D) *such that the boundary divisor* $D = V - X$ *has the dual graph as shown below, where* $n \geq 1$ *and* $r \geq 1$ *and* f *is defined by a pencil* $|\ell'_\infty|$. *Furthermore,* $\overline{F}_0$ *is the closure of* F_0 *in* V *and* S' *is the unique cross-section contained in* D.



In the above graph, we can make $(S'^2) = -1$ by applying elementary transformations with centers on ℓ'_∞. We call an affine pseudo-plane of type $(d, 1, r)$ simply an affine pseudo-plane of type (d, r) and denote it by $X(d, r)$. We can construct $X(d, r)$ in more explicit terms.

Lemma 3.2. *The following assertions hold for an affine pseudo-plane $X := X(d, r)$.*

(1) *X is an* ML_0 *surface (resp.* ML_1 *surface) if $r = 1$ (resp. $r \geq 2$).*

(2) *X is, in general, isomorphic to the complement of $M_0 \cup C_d$ if $r < d$, and $M_1 \cup C_d$ if $r \geq d$ in the Hirzebruch surface Σ_n with $n = |r - d|$, where M_0 is the minimal section and where C_d and M_1 are specified as follows. In the case $r < d$, C_d is an irreducible member of the linear system $|M_0 + d\ell_0|$ which meets M_0 in the point $M_0 \cap \ell_0$ with multiplicity r, where ℓ_0 is a fiber of the $\mathbb{P}^1$-fibration of Σ_n. In the case $r \geq d$, M_1 is a section of Σ_n with $({M_1}^2) = n$, and C_d is an irreducible memeber of the linear system $|M_1 + d\ell_0|$ which meets M_1 in the point $M_1 \cap \ell_0$ with multiplicity r. In both cases, $\ell_0 \cap X = \overline{F} \cap X$.*

(3) *X is isomorphic to $\mathbb{P}^2 - C$, where C is a curve on $\mathbb{P}^2$ defined by $X_0 X_1^{d-1} = X_2^d$ with $d \geq 2$ if and only if X has type $(d, d-1)$.*

Proof. The assertions (1) follows from Lemmas 1.4 and 1.5.For the proof of the assertion (2), contract $S', \ell_0', E_2, \ldots, E_d, E_{d+1}, \ldots, E_{d+r-1}$ in this order. Then the resulting surface is the Hirzebruch surface Σ_n with $n = |r - d|$ and the image of ℓ'_∞ provides C_d. The image of E_1 provides M_0 or M_1 according as $r - d < 0$ or $r - d \geq 0$, while the image of $\overline{F}$ is the fiber ℓ_0. The assertion (3) can be proved as in the assertion (2). Q.E.D.

tom Dieck [1] observed some partial cases of affine pseudo-planes of type (d, r) as examples of affine surfaces without cancellation property. The following construction is based on his idea.

Write $\Sigma_n = \mathrm{Proj}\,(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$ as the quotient of $(\mathbb{A}^2 \setminus \{0\}) \times \mathbb{P}^1$ under the relation

$$(z_0, z_1), [w_0, w_1] \sim (\nu z_0, \nu z_1), [\nu^n w_0, w_1]$$

for $\nu \in G_m = \mathbb{C}^*$. The projection $\{(z_0, z_1), [w_0, w_1]\} \mapsto [z_0, z_1]$ induces a $\mathbb{P}^1$-fibration $p_n : \Sigma_n \to \mathbb{P}^1$. In the above definition by quotient and in what follows, the integer n could be negative. If $n \geq 0$, the curve $w_0 = 0$ (resp. $w_1 = 0$) is a section M_1 of p_n with $({M_1}^2) = n$ (resp. the minimal section M_0 with $({M_0}^2) = -n$). Meanwhile, if $n < 0$, then the curve $w_0 = 0$ (resp. $w_1 = 0$) is the minimal section M_0 (resp. a section M_1 with $({M_1}^2) = |n|$) of $\Sigma_{|n|}$. Let $d \geq 2$ and $r = d + n \geq 1$. With the notations of Lemma 3.2, we assume that the fiber ℓ_0 is defined by $z_0 = 0$. Let $w = w_0/w_1$. Then $\{z_0/z_1, w/z_1^n\}$ is a system of local coordinates at the point $M_1 \cap \ell_0$ (resp. $M_0 \cap \ell_0$) if $n \geq 0$ (resp. $n < 0$). Let Λ be a linear subsystem of $|M_1 + d\ell_0|$ if $n \geq 0$ (resp. $|M_0 + d\ell_0|$ if $n < 0$) consisting of members which meet the curve M_1 (resp. M_0) at the point $M_1 \cap \ell_0$ (resp. $M_0 \cap \ell_0$) with multiplicity r if $n \geq 0$ (resp. $n < 0$). Then any member of Λ is defined by an equation

$$\frac{w}{z_1^n}\left\{a_0 + a_1\left(\frac{z_0}{z_1}\right) + \cdots + a_{d-1}\left(\frac{z_0}{z_1}\right)^{d-1} + a_d\left(\frac{z_0}{z_1}\right)^d\right\} + a_{d+1}\left(\frac{z_0}{z_1}\right)^r = 0$$

or equivalently by

$$w_0\left(a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_{d-1} z_0^{d-1} z_1 + a_d z_0^d\right) + a_{d+1} z_0^r w_1 = 0 \qquad (1)$$

for $(a_0, a_1, \ldots, a_{d+1}) \in \mathbb{P}^{d+1}$. In fact, it is readily computed that $\dim \Lambda = d+1$. So, the curve C_d is defined by such an equation with $a_0 \neq 0$ and $a_{d+1} \neq 0$. Hence it follows that

$$X(d,r) = \Sigma_{|n|} \setminus (\{w_0 = 0\} \cup C_d)$$

where $n = r - d$ and C_d is the curve defined by (1) with $a_0 \neq 0$ and $a_{d+1} \neq 0$. tom Dieck [1], in fact, considered the case below where $X(d,r)$ admits a G_m-action.

Lemma 3.3. *Let $r \geq 2$ and let $X = X(d,r)$. Let $\sigma : G_m \times X \to X$ be a non-trivial action of the algebraic torus $G_m = \mathbb{C}^*$. Then the following assertions hold true.*

(1) *The action σ induces an action $\sigma : G_m \times \Sigma_{|n|} \to \Sigma_{|n|}$ such that ${}^{\sigma(\mu)}M_i \subseteq M_i$ for $i = 0, 1$, ${}^{\sigma(\mu)}C_d \subseteq C_d$ and ${}^{\sigma(\mu)}\ell_0 \sim \ell_0$, where ${}^{\sigma(\mu)}M_i$ denotes the image of M_i under the action of $\mu \in \mathbb{C}^*$, etc.*

(2) *The curve C_d is defined by an equation*

$$z_1^d w_0 + a z_0^r w_1 = 0 \quad \textit{for} \quad a \in \mathbb{C}^*.$$

These construction allows us to write down explicitly a defining equation of the universal covering space $\widetilde{X}(d,r)$ of $X(d,r)$.

Lemma 3.4. *The universal covering $\widetilde{X}(d,r)$ is isomorphic to an affine hypersurface in $\mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x,y,z]$ defined by an equation*

$$x^r z + (y^d + a_1 x y^{d-1} + \cdots + a_{d-1} x^{d-1} y + a_d x^d) = 1.$$

The Galois group of the covering $\widetilde{X}(d,r) \to X(d,r)$ is a cyclic group $H(d) := \mathbb{Z}/d\mathbb{Z}$ of order d and acts as

$$\lambda \cdot (x,y,z) = (\lambda x, \lambda y, \lambda^{-r} z)$$

for $\lambda \in H(d)$.

The universal covering spaces of the $X(d,r)$ satisfies the following stability property. They do not have cancellation property as shown by the assertion (2) below.

Theorem 3.5. *Let $d \geq 2$ and $r, s > d$. Let $\widetilde{X}(d,r)$ and $\widetilde{X'}(d,s)$ be the affine hypersurfaces defined by the equations $x^r z + (y^d + a_2 y^{d-2} + \cdots + a_d x^d) = 1$ and $x^s z + (y^d + a'_2 y^{d-2} + \cdots + a'_d x^d) = 1$, respectively. Then the following assertions hold.*

(1) *For any r and s,*

$$\widetilde{X}(d,r) \times \mathbb{A}^1 \cong \widetilde{X'}(d,s) \times \mathbb{A}^1.$$

(2) *The isomorphism $\widetilde{X}(d,r) \cong \widetilde{X'}(d,s)$ holds if and only if $r = s$ and $a'_i = \mu^i a_i$ for $\mu \in \mathbb{C}^*$ and $2 \leq i \leq d$.*

In order to find counterexamples to the cancellation problem in the calss of affine pseudo-planes $X(d,r)$, we have to find an $H(d)$-equivariant isomorphism in the above assertion (1) with possibly non-trivial actions on $\mathbb{A}^1$. This is much more difficult problem. We say that tom Dieck [1] pioneered this construction with the $X(d,r)$ admitting G_m-actions.

Theorem 3.6. *Let $d \geq 2$ and let $r, s > 1$ and $r \equiv s \equiv 1 \pmod{d}$. Let $\widetilde{X}(d,r;f)$ be the affine hypersurface defined by $x^r z + f(x,y) = 1$ where $f(x,y)$ is of the form $y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d$ with $a_j \in \mathbb{C}$. Then the quotient of $\widetilde{X}(d,r;f)$ by the Galois group $H(d)$ is an affine pseudo-plane $X(d,r;f)$ of type (d,r), and the following assertions hold.*

(1) *For any r and s,*

$$X(d,r;f_1) \times \mathbb{A}^1 \cong X(d,s;f_2) \times \mathbb{A}^1,$$

where f_1 and f_2 are monic homogeneous polynomials of the form stated above.

(2) *The isomorphism $X(d,r;f_1) \cong X(d,s;f_2)$ holds if and only if $r = s$ and $f_1(x,y) = f_2(\mu x, y)$ for $\mu \in \mathbb{C}^*$.*

4. Ascent and descent of the ML_i property

A crucial result is the following.

Lemma 4.1. *Let $\varphi : X \to Y$ be an étale finite morphism of smooth affine surfaces. Then any G_a-action on Y lifts up to a G_a-action on X. In particular, if Y is ML_0 then so is X.*

A main ingredient of proof is that an $\mathbb{A}^1$-fibration is locally trivial in the Zariski topology and the inverse image of an affine line decomposes completely under an étale finite morphism.

We ask if the converse holds and obtain the following.

Theorem 4.2. *Let $\varphi : X \to Y$ be a finite morphism of smooth affine surfaces with X an ML_0 surface. Assume that either φ is étale or φ is a Galois (possibly ramified) covering. Then Y is an ML_0 surface.*

Proof. If φ is étale then $\pi_1(X)$ is a subgroup of finite index in $\pi_1(Y)$. Hence by taking a normal subgroup of finite index in $\pi_1(Y)$ contained in $\pi_1(X)$ we can find a smooth affine surface Z and an étale finite morphism $\psi : Z \to X$ such that $\varphi \cdot \psi : Z \to Y$ is an étale Galois covering. Since X is ML_0, it follows by Lemma 4.1 that Z is also an ML_0 surface. By replacing X by Z if necessary, we shall assume that φ is a (possibly ramified) Galois covering with Galois group G.

By the equivariant completion theorem of Sumihiro [7] and G-equivariant resolution of singularities, we can find a smooth normal G-completion $X \hookrightarrow V$, where G acts on the boundary divisor $D := V - X$. If the completion is minimal, then D is a linear chain because X is ML_0 (cf. Lemma 1.4). We shall show that V can be so chosen that D is linear.

Assume that D is not minimal. Then D has an irreducible component D_1 such that D_1 is a (-1) curve and meets at most two other components. Then all the conjugates of D_1 in D have the same property. Let $D_1, D_2, \ldots, D_r$ be all the conjugates of D_1. If $D_i \cap D_j = \emptyset$ for $1 \le i < j \le r$, then we can contract all of them simultaneously and obtain a new normal G-completion. Assume that $D_1 \cap D_2 \neq \emptyset$. Let Γ_1 be the connected subtree of D containing D_1 but not D_2, and let Γ_2 be the connected subtree of D containing D_2 but not D_1. Then Γ_1 and Γ_2 are also conjugate. By the assumption, D_1 meets only one other irreducible component of Γ_1 and similarly for D_2. If Γ_1 contains a branch component of D then so does Γ_2. If we contract D_1 and any subsequent (-1) curves which meet at most two other irreducible components, then we reach a minimal divisor with simple normal crossings which is a tree but still has two branching components. This contradicts the assumption that X is ML_0. Hence we can assume that D is a G-stable linear chain.

Consider the quotient surface V/G which contains Y as an open set. Then V/G is normal and D/G is a simply-connected divisor. Furthermore, V/G has at most quotient singular points on D/G which are the images of intersection points of irreducible components of D.

We shall show that Y has a smooth normal completion W such that $W - Y$ is a linear chain of smooth rational curves. To see this, we let H be a subgroup of G which keeps all the irreducible components stable. Then H has index at most 2 in G. Then V/H has at most cyclic quotient singularities. By taking a minimal G/H-equivariant resolution of singularities, we obtain a normal completion $X/H \hookrightarrow U$ such that U

is smooth along $U - X/H$ and $U - X/H$ is a linear chain of smooth rational curves. We note here that $X//H$ may have singular points if φ is not étale.

Now we can assume that $G \cong \mathbb{Z}/2\mathbb{Z}$ and the generator of G permutes the end components of D which is a linear chain. Then a local analysis at a possible fixed point on D shows that G-action is given by $(x, y) \mapsto (y, x)$ with respect to a suitable system of local coordinates at the fixed point and hence that $U/(\mathbb{Z}/2\mathbb{Z})$ is, in fact, smooth. Let $W := U/(\mathbb{Z}/2\mathbb{Z})$. Then $W - Y$ is a linear chain of smooth rational curves. Hence Y is an ML_0 surface by Lemma 1.4 because $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ implies $\Gamma(Y, \mathcal{O}_Y)^* = \mathbb{C}^*$.
Q.E.D.

In the case of étale finite morphism, we have ascent and descent both ways for ML_i surfaces with $i = 0, 1, 2$.

Theorem 4.3. *Let $\varphi : X \to Y$ be an étale finite morphism. Then Y is* ML_i $(i = 1, 2)$ *if and only if so is X.*

Proof. Consider first the ML_1-property. Suppose that Y is ML_1. Then X is ML_1 or ML_0 by Lemma 4.1. If X is ML_0 then Y is ML_0 by Theorem 4.2. This is a contradiction. Hence X is ML_1. Conversely, suppose that X is ML_1. As in the proof of Theorem 4.2, there exists a Galois étale finite covering $\psi : Z \to X$ such that $\varphi \cdot \psi : Z \to X$ is a Galois étale covering. Since Z is ML_1 by what we have proved above, we may assume that $\varphi : X \to Y$ is a Galois étale finite covering with group G. Let $f : X \to B$ be an $\mathbb{A}^1$-fibration with $B \cong \mathbb{A}^1$. Since this $\mathbb{A}^1$-fibration is unique on X, the G-action preserves f. Namely, the G-translates of a fiber of F are again fibers of f. Hence f induces an $\mathbb{A}^1$-fibration $g : Y \to \mathbb{A}^1$. So, Y is ML_1 or ML_0. If Y is ML_0 then X is ML_0 by Theorem 4.2, a contradiction. Hence Y is ML_1.

The case of ML_2-property follows readily if one uses the ascent and the descent of the ML_i-property for $i = 0, 1$. Q.E.D.

We can prove the following result, which generalizes a result that if $f : \mathbb{A}^2 \to Y$ is a finite morphism to a smooth affine surface Y then Y is isomorphic to $\mathbb{A}^2$.

Theorem 4.4. *Let X, Y be smooth affine surfaces such that there is a proper morphism $f : X \to Y$. Suppose that X is an* ML_0 *surface with $\rho(X) = 0$. Then Y is also an* ML_0 *surface with $\rho(Y) = 0$.*

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微分の核とヒルベルトの第14問題

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1　ヒルベルトの第14問題

体 K 上の n 変数多項式環 $K[X]=K[X_1,\ldots,X_n]$ の商体を $K(X)$ とする．中間体 $K\subset L\subset K(X)$ に対し，$K[X]$ の K 部分代数 $L\cap K[X]$ の有限生成性を問う問題はヒルベルトの第14問題と呼ばれる．1954年にザリスキ [19] は，L の K 上の超越次数が2以下ならばこの問題の答えは肯定的であることを示した．一方，永田 [16] は1958年に初めての反例を $n=32$ の場合に超越次数が4の中間体 L として与えた．

$K[X]$ における微分の核の有限生成性の問題は，ヒルベルトの第14問題の特別な場合にあたり，最近活発に研究されている．ここで，一般に可換な K 代数 A に対し，K 線形写像 $D: A\to A$ は任意の $a,b\in A$ に対して $D(ab)=D(a)b+aD(b)$ が成り立つとき，A における微分という．K 部分ベクトル空間 $V\subset A$ に対し $V^D=\{a\in V\mid D(a)=0\}$ は V の K 部分ベクトル空間となるが，さらに V が K 部分代数ならば V^D は V の K 部分代数になる．D は，任意の $a\in A$ に対し $D^r(a)=0$ となる $r\in\mathbf{N}$ が存在するとき，局所冪零であるという．$K[X]$ における微分 D は $K(X)$ における微分に拡張でき，$K[X]^D=K(X)^D\cap K[X]$ が成り立つ．$K(X)^D$ は K を含む $K(X)$ の部分体なので，D の核 $K[X]^D$ の有限生成性の問題はヒルベルトの第14問題の一部である．ザリスキ [19] より $n\leq 3$ ならば $K[X]^D$ は常に有限生成であることが従う．

1990年にロバーツ [18] は，$n=7$ の場合に超越次数が6の新しい種類の反例を与えた．これを改良し，フロイデンバーグ [5] は $n=6$ で超越次数が5の反例を，さらにデイグルとフロイデンバーグ [2] は $n=5$ で超越次数が4の反例を構成した．ロバーツの反例の高次元化や一般化には，小島と宮西 [6] や筆者 [9] によるものもある．これらの反例は，$K[X]$ における局所冪零微分の核

として実現できる.

ヒルベルトの第14問題は，これまで $n=3,4$ の場合と，L の超越次数が 3 の場合だけが未解決のまま残されていたが，筆者 [11] により否定的に決着した．また $K[X]$ における微分の核の有限生成性の問題は $n=4$ の場合だけ未解決だったが，筆者 [10], [12] によりこれも否定的に決着した．なお，$K[X]$ における局所冪零微分の核の有限生成性の問題は，$n=4$ ではいくつかの肯定的な結果 (例えば [3], [7], [15]) はあるものの，依然未解決である．

ヒルベルトの第14問題の反例を構成するための筆者による理論は，ロバーツ [18] やそれに端を発する [2], [5], [6], [9] などを範疇に収めるだけでなく，様々な種類の新しい反例を自在に与え，また詳細に分析することを可能にする．この理論の概要は [14] でも扱ったが，本稿では特に微分の核として実現可能な反例に焦点を当て，それらを調べるための方法について述べる．理論の応用として，ある種類の局所冪零微分に対し，その核が有限生成でないための十分条件を与える．また，そうした有限生成でない微分の核の生成系を，有限回の手続きで求めるためのアルゴリズムについても説明する．

2 反例の構成

以下，K の標数は零とする．$K[Y]=K[Y_1,\ldots,Y_m]$ と $K[Y^{\pm 1}]=K[Y_1^{\pm 1},\ldots,Y_m^{\pm 1}]$ をそれぞれ K 上の m 変数の多項式環とローラン多項式環とする．$\mathbf{Z}^m$ の元 $b=(b_1,\ldots,b_m)$ に対し $Y^b=Y_1^{b_1}\cdots Y_m^{b_m}$ とおき，ローラン多項式 $g=\sum_{b\in\mathbf{Z}^m}\nu_bY^b$ に対し，$\mathbf{Z}^m$ の部分集合

$$\mathrm{supp}(g)=\{b\in\mathbf{Z}^m \mid \nu_b\neq 0\}$$

を g の台と呼ぶ．これらは $\mathbf{Z}^n$ の元やローラン多項式環 $K[X^{\pm 1}]$ の元に対しても同様とする．$K[Y]$ における微分 D に対し，

$$\mathrm{supp}(D)=\bigcup_{i=1}^{m}\mathrm{supp}(Y_i^{-1}D(Y_i)) \tag{2.1}$$

を D の台と呼ぶ．$\mathrm{supp}(g)$ や $\mathrm{supp}(D)$ の $\mathbf{R}^m$ における凸包 $\mathrm{New}(g)$ や $\mathrm{New}(D)$ を，それぞれ g や D のニュートン多面体と呼ぶ．$\{\delta-\delta' \mid \delta,\delta'\in\mathrm{supp}(D)\}$ が生成する $\mathbf{Z}$ 加群を M_D とおく．各 $\gamma\in\mathbf{Z}^m/M_D$ に対し，$\mathbf{Z}^m/M_D$ における像が γ と等しい $b\in(\mathbf{Z}_{\geq 0})^m$ に対する Y^b 全体が生成する K ベクトル空間を $K[Y]_\gamma$

とする．このとき，$K[Y]^D$ に $\mathbf{Z}^m/M_D$ 次数付け $K[Y]^D = \bigoplus_{\gamma\in\mathbf{Z}^m/M_D}(K[Y]^D_\gamma)$ が定まる．

$K[Y]$ の K 部分代数 B と有理数成分の (n,m) 行列 $\Omega = (\omega_{i,j})_{i,j}$ および $\mathbf{Z}^n + \{\Omega\cdot b \mid b\in\mathbf{Z}^m\}$ の部分群 M からなる三つ組 $\mathcal{D} = (B,\Omega,M)$ を考える．各 $s\in M$ に対し，任意の $b\in\mathrm{supp}(g)$ が $\Omega\cdot b+s\in\mathbf{Z}^m$ を満たす $g\in B$ 全体を B^Ω_s とおく．また，$K[M] = \bigoplus_{s\in M}K\mathbf{e}(s)$ を M の群環とする．ただし，記号 $\mathbf{e}(s)$ $(s\in M)$ に対し，積を $\mathbf{e}(s)\mathbf{e}(s') = \mathbf{e}(s+s')$ $(s,s'\in M)$ により定める．このとき，$B^\Omega_M = \bigoplus_{s\in M}B^\Omega_s\otimes\mathbf{e}(s)$ は $B\otimes_K K[M]$ の M 次数つき K 部分代数となる．準同型 $\Phi_\mathcal{D}: B^\Omega_M\to K(X)$ を $(\sum_b \nu_b Y^b)\otimes\mathbf{e}(s)\mapsto\sum_b\nu_b X^{\Omega\cdot b+s}$ により定義する．そして，$\Phi_\mathcal{D}(B^\Omega_M)$ の商体を $K(\mathcal{D})$ とおく．このとき三つ組 $\mathcal{D}$ をうまく選ぶことで，ヒルベルトの第 14 問題に対する実に様々な反例を $K(\mathcal{D})$ として得ることができる．

一般に，$K[Y]$ の微分 D が条件

(D) 任意の $\delta,\delta'\in\mathrm{supp}(D)$ に対し $\Omega\cdot(\delta-\delta')$ は $\mathbf{Z}^n$ に含まれる．

を満たすとき，任意の $g\in K[Y]_\gamma$ $(\gamma\in\mathbf{Z}^m/M_D)$ に対し

(G) 任意の $b,b'\in\mathrm{supp}(g)$ に対し $\Omega\cdot(b-b')$ は $\mathbf{Z}^n$ に含まれる．

が成り立つ．従って，$b\in\mathrm{supp}(g)$ を任意にとり $s=-\Omega\cdot b$ とおいたとき，g は $(K[Y]^D)_s$ に含まれる．

我々が特に興味のある三つ組は，B が (D) を満たす $K[Y]$ の局所冪零微分の核 $K[Y]^D$ の場合のものである．このとき，$K(\mathcal{D})$ は $K[Y]$ の微分の核を用いて定義されるが，$K(\mathcal{D})\cap K[X]$ 自身を核とする $K[X]$ の微分が存在するとは限らない．そこで，そのような微分が存在するための三つ組 $\mathcal{D}$ の条件を探すことが問題となる．

準同型 $\mathbf{Z}^m\times M\ni(b,s)\mapsto\Omega\cdot b+s\in\mathbf{Q}^n$ が単射のとき，(Ω,M) は単射的であるという．(Ω,M) が単射的ならば準同型 $\Phi_\mathcal{D}$ は単射だが，逆は一般に正しくない．実際，$n=3$ における反例 [11] の構成には，(Ω,M) は単射的でないが $\Phi_\mathcal{D}$ は単射であるような三つ組 $\mathcal{D}$ を用いる．

筆者 [12, Theorem 1.4] の帰結として次が従う．

定理 2.1 $K[Y]$ の局所冪零微分 D は条件 (D) と $D(Y_i)\neq 0$ $(i=1,\ldots,m)$ を満たし，さらに (Ω,M) は単射的であるとする．このとき，$(M\otimes_\mathbf{Z}\mathbf{R})\cap\mathbf{Z}^n$ が M に含まれるならば，$K(\mathcal{D})\cap K[X] = K[X]^E$ を満たす $K[X]$ の微分 E が存在する．

しかし，定理 2.1 において E を局所冪零であるようにとれるとは限らない．実際，$n=4$ における反例 [10] の場合，三つ組 $\mathcal{D}$ は定理 2.1 の条件を満た

すので $K(\mathcal{D}) \cap K[X] = K[X]^E$ となる $K[X]$ の微分 E は存在する．しかし，$K(\mathcal{D}) \cap K[X]$ は $K[X]$ における零でない任意の局所冪零微分によって零に写らない元を含み，$K[X]$ における局所冪零微分の核にはなり得ない．

3 微分とその核の記述

以下，本稿では $m \leq n$ とし，(Ω, M) が次のような特別な形の場合を考える．すなわち Ω は，$n-m+1 \leq i \leq n$ と $1 \leq j \leq m$ に対し，$i+j = n-m+1$ ならば $\omega_{i,j} = 1$, $i+j \neq n-m+1$ ならば $\omega_{i,j} = 0$ という条件を満たし，

$$M = \{(a_1, \ldots, a_n) \in \mathbf{Z}^n + \{\Omega \cdot b \mid b \in \mathbf{Z}^m\} \mid a_{n-m+1} = \cdots a_n = 0\}$$

とする．このとき (Ω, M) は単射的である．記述を簡単にするため n を $m+n$ と取り替え，多項式環 $K[X]$ の代わりに $K[X][Y] = K[X_1, \ldots, X_n][Y_1, \ldots, Y_m]$ を考える．零だけからなる後ろの m 個の成分を無視することで，M は $\mathbf{Q}^n$ の部分群とみなす．この場合，Ω の最初の n 行に対応する (n, m) 行列 Ω_0 と，条件

(D$'$) 任意の $\delta, \delta' \in \mathrm{supp}(D)$ に対し $\Omega_0 \cdot (\delta - \delta')$ は $\mathbf{Z}^n$ に含まれる．

を満たす $K[Y]$ の局所冪零微分 D を与えることで，三つ組 $\mathcal{D}$ は一意的に決まる．

各 i に対し Ω の第 i 行を ω_i とおく．$\mathbf{Z}^m$ の有限部分集合 S に対し，

$$v_{\Omega_0}(S) = (\min\{\omega_1 \cdot b \mid b \in S\}, \ldots, \min\{\omega_n \cdot b \mid b \in S\})$$

と定める．ローラン多項式 $g = \sum_{b \in \mathbf{Z}^m} \nu_b Y^b$ が (G) を満たすとき，全ての $b \in \mathrm{supp}(g)$ に対し $a(b) = \Omega_0 \cdot b - v_{\Omega_0}(\mathrm{supp}(g))$ は $\mathbf{Z}^n$ に含まれる．そこで，$g^{\Omega_0} = \sum_{b \in \mathbf{Z}^m} \nu_b X^{a(b)} Y^b$ とおく．(D$'$) を満たす $K[Y]$ の微分 D に対し，$K[X][Y]$ の微分 D^{Ω_0} を $D^{\Omega_0}(X_i) = 0$ $(i = 1, \ldots, n)$ および

$$D^{\Omega_0}(Y_i) = (Y_i^{-1} D(Y_i))^{\Omega_0} X^{c_i} Y_i \quad (i = 1, \ldots m)$$

から定める．ここで，$c_i = v_{\Omega_0}(\mathrm{supp}(Y_i^{-1} D(Y_i))) - v_{\Omega_0}(\mathrm{supp}(D))$ とする．このとき，(G) を満たす $g \in K[Y]^D$ に対し $D^{\Omega_0}(g^{\Omega_0}) = 0$ が成り立つ．また，D が局所冪零ならば D^{Ω_0} も局所冪零になる．

定理 3.1 $K[Y]$ の局所冪零微分 D が (n, m) 行列 Ω_0 に対して条件 (D$'$) を満たすとする．このとき，$K(\mathcal{D}) \cap K[X][Y] = K[X][Y]^{D^{\Omega_0}}$ が成り立つ．

ヒルベルトの第 14 問題に対する反例のうち，[2], [5], [6], [9], [18] で与えられたものは全て適当な D と Ω_0 に対する D^{Ω_0} の核として実現できる．これらの構造の本質は，D^{Ω_0} を三つ組 $(K[Y]^D, \Omega, M)$ に分解して初めて明らかになるといえる．

微分 D^{Ω_0} の定め方から $K[X][Y]^{D^{\Omega_0}}$ は $K[X]$ を含む．

命題 3.2 $K[Y]$ の微分 D が (n,m) 行列 Ω_0 に対して条件 (D′) を満たすとき，$K[X]$ 加群 $K[X][Y]^{D^{\Omega_0}}$ は $\{g^{\Omega_0} \mid g \in K[Y]^D_\gamma \ (\gamma \in \mathbf{Z}^m/M_D)\}$ によって生成される．

4 斉次三角微分

$K[Y]$ における微分 D は，$D(Y_i) \in K[\{Y_j \mid jr)$ が成り立つとき，半斉次であるという．半斉次な三角微分 D の型を次のように定義する．D の $K[Y_1,\ldots,Y_r]$ への制限 D_r の型を $(\nu_2,\ldots,\nu_r)$ とする．各 $r0$ $(i \in \{1,2,3,(4,l)\} \setminus \{j\})$ を満たす $k \in \{1,\ldots,n\}$ 全体を I_j^l とおく．各 $k \in \bigcup_{j=1}^3 \bigcup_{l=1}^\alpha I_j^l$ に対し，

$$\tilde{\omega}_k = \begin{cases} \min\left\{\omega_k^{2,1}, s^{-1}\omega_k^{3,1}\right\} & k \in \bigcup_{l=1}^\alpha I_1^l \text{ のとき} \\ \min\left\{\omega_k^{1,2}, (\nu_3 - \nu_2 s')^{-1}\omega_k^{3,2}\right\} & k \in \bigcup_{l=1}^\alpha I_2^l \text{ のとき} \\ \min\left\{\omega_k^{1,3}, \nu_2^{-1}\omega_k^{2,3}\right\} & k \in \bigcup_{l=1}^\alpha I_3^l \text{ のとき} \end{cases}$$

と定める．このとき，全ての k に対し $\tilde{\omega}_k$ は正数となる．

各 $1 \le l \le \alpha$ に対し $\nu_{4,l}\tilde{\omega}_k > \nu_2\omega_k^{(4,l),1}$ $(k \in J \cap I_1^l)$ と $\nu_{4,l}\tilde{\omega}_k > \omega_k^{(4,l),j}$ $(k \in J \cap I_j^l, j = 2,3)$ を成り立たせる部分集合 $J \subset \{1,\ldots,n\}$ に対し，記号をいくつか定義する．各 $1 \le j \le 3$ と $1 \le l \le \alpha$ に対し $\xi_{j,J}^l = \max\{\omega_k^{(4,l),j}\tilde{\omega}_k^{-1} \mid k \in J \cap I_j^l\}$ とおき，

$$\xi_J^l = \mathrm{lcm}\{\nu_2,\nu_3\}\xi_{1,J}^l + \frac{\nu_3}{\nu_3 - \nu_2 s'}\xi_{2,J}^l + \xi_{3,J}^l, \tag{4.1}$$

$$\tau_{1,J}^l = \frac{\xi_{1,J}^l}{\nu_{4,l} - \nu_2\xi_{1,J}^l}, \quad \tau_{j,J}^l = \frac{\nu_j\xi_{j,J}^l}{\nu_{4,l} - \xi_{j,J}^l} \ \ (j = 2,3) \tag{4.2}$$

と定める．ただし，$J\cap I_j^l=\emptyset$ のときは $\xi_{j,J}^l=0$ とする．$\alpha>1$ のとき，

$$\begin{aligned}
\eta_{1,J}&=\nu_{4,1}\tau_{1,J}^1-\nu_{4,\alpha}\tau_{1,J}^\alpha+\nu_2\tau_{1,J}^1\tau_{1,J}^\alpha(\nu_{4,1}-\nu_{4,\alpha}),\\
\eta_{2,J}&=\nu_{4,\alpha}-\nu_{4,1}+\nu_2(\nu_{4,\alpha}\tau_{1,J}^1-\nu_{4,1}\tau_{1,J}^\alpha),\\
\eta_{3,J}&=(\nu_3-\nu_2 s)\eta_{1,J}+s\eta_{2,J},\\
\eta_{4,J}&=\nu_{4,1}\nu_{4,\alpha}(\tau_{1,J}^1-\tau_{1,J}^\alpha)
\end{aligned}\tag{4.3}$$

とおく．

$\{(\Psi_2^D)^{c_2}(\Psi_3^D)^{c_3}\mid c_2,c_3\in\mathbf{Z}_{\geq 0},c_2\nu_2+c_3\nu_3=\nu_{4,l},1\leq l\leq\alpha\}$ が生成する K ベクトル空間を V とする．V は有限次元なので $\{\mathrm{New}(\Psi_4^D-Y_4+f)\mid f\in V\}$ は有限集合であり，包含関係に関する有限個の極小元 $P_1,\ldots,P_r$ がとれる．

このとき，次が成り立つ．

定理 4.1 $\gcd\{\nu_2,\nu_3\}$ は $\nu_{4,1},\ldots,\nu_{4,\alpha}$ を割り切り，$c_3^D\neq 0$ であるとする．また，各 $1\leq k\leq n$ と $j\in\{1,2,3,(4,1),\ldots,(4,\alpha)\}$ に対し $\omega_k^{1,j}$ は整数であるとする．このとき，次を満たす $J\subset\{1,\ldots,n\}$ が存在すれば $K[X][Y]^{D^{\Omega_0}}$ は有限生成でない．

(i) 各 $1\leq l\leq\alpha$ に対し，$\nu_{4,l}\tilde{\omega}_k>\nu_2\omega_k^{(4,l),1}$ $(k\in J\cap I_1^l)$ および $\nu_{4,l}\tilde{\omega}_k>\omega_k^{(4,l),j}$ $(k\in J\cap I_j^l, j=2,3)$ が成り立つ．

(ii) $\xi_J^1\leq 1$ および $\xi_J^\alpha\leq 1$ が成り立つ．

(iii) 各 $k\in J\cap\bigcup_{l=1}^\alpha(I_2^l\cup I_3^l)$ と $b\in\mathrm{supp}(\Psi_4^D-Y_4)$ に対し，$\omega_k\cdot b\geq\omega_{k,4}$ が成り立つ．

(iv) 全ての $(b_1,\ldots,b_4)\in\mathrm{supp}(D(Y_4))$ に対して次が成り立つ．

(a) $\alpha=1$ のときは $b_2-\tau_{1,J}^1b_1\geq\tau_{1,J}^1(1-\nu_2)-1$．

(b) $\alpha>1$ のときは $(b_1+1-\nu_2)\eta_{1,J}+(b_2+1)\eta_{2,J}+b_3\eta_{3,J}-\eta_{4,J}\geq 0$．

(v) 各 $1\leq i\leq r$ に対し，$\omega_k\cdot b<\omega_{k,4}$ を満たす $k\in J\cap\bigcup_{j=1}^3\bigcup_{l=1}^\alpha I_j^l$ と $b\in P_i$ が存在する．

ヒルベルトの第 14 問題に対するロバーツ [18] やフロイデンバーグ [5], デイグルとフロイデンバーグ [2], 筆者 [9, Theorem 1.4] などによる反例は，定理 4.1 の帰結として得られる．

5 $K[X][Y]^{D^{\Omega_0}}$ の生成系の求め方

筆者 [9, Section 3] は，ロバーツの反例 [18] を実現する有限生成でない三角微分の核の生成系を決定した．これは特殊な具体例の計算結果に過ぎない

が，その方法自体は一部に改良を施すことで飛躍的に一般化できる．この節では，D と Ω_0 が定理 4.1 の条件の他に，さほど一般性を損ねない若干の補足的条件を満たしさえすれば，$K[X][Y]^{D^{\Omega_0}}$ は有限生成でないにも関わらず，その生成系を有限回の手続きで求められることを説明する．この方法を使うと，[2], [5], [9, Theorem 1.4], [18] で与えられた有限生成でない三角微分の核の生成系も求められる．

まず，生成系の決定に用いる原理を説明する．$K[X][Y]$ の K 部分代数 A と項順序 $\preceq$ に対し，次の主張はよく知られている．

補題 5.1 ([17, Proposition 1.16]) A の部分集合 S に対し，$\{\mathrm{in}_{\preceq}(f) \mid f \in S\}$ が K 代数 $\mathrm{in}_{\preceq}(A)$ を生成するならば，S は K 代数 A を生成する．

$K[X][Y]$ の項順序 $\preceq$ で，任意の $a, a' \in (\mathbf{Z}_{\geq 0})^n$ と $b, b' \in (\mathbf{Z}_{\geq 0})^4$ に対し $b' - b$ の第 4 成分が正ならば $X^aY^b \preceq X^{a'}Y^{b'}$ となるものを考える．このとき，ヒルベルトの第 14 問題の反例を構成するための筆者の理論を使うと，$\mathrm{in}_{\preceq}(K[X][Y]^{D^{\Omega_0}})$ に含まれる各単項式 T に対し，$\mathrm{in}_{\preceq}(F) = T$ を満たす $F \in K[X][Y]^{D^{\Omega_0}}$ を実際に構成できる．従って，イニシャル代数 $\mathrm{in}_{\preceq}(K[X][Y]^{D^{\Omega_0}})$ の生成系が記述できれば，命題 5.1 より $K[X][Y]^{D^{\Omega_0}}$ の生成系も求まったことになる．

以下，$\alpha = 1$ とする．さらに，定理 4.1 の条件 (i) から (v) が $J = \{1, \ldots, n\}$ に対して成り立ち，任意の $k \in J \setminus \bigcup_{j=1}^{3} I_j^1$ が $\omega_k^{i,(4,1)} \geq 0$ $(i = 1, 2, 3)$ を満たすと仮定する．一般に，定理 4.1 の条件が満たされるとき，n を集合 $J' = J \cap \bigcup_{j=1}^{3} \bigcup_{l=1}^{\alpha} I_j^l$ の元の個数と取り替え，Ω_0 を $(\omega_k)_{k \in J'}$ と取り替えても再び定理 4.1 の条件は満たされるが，そのときこの条件も満たされる．

本節の残りの部分ではこの仮定の下で，有限回の手続きにより判定可能な後述の条件 (P) が成り立つ場合に，$X_1 \preceq \cdots \preceq X_n \preceq Y_1 \preceq \cdots \preceq Y_4$ を満たす $K[X][Y]$ の辞書式順序 $\preceq$ に関するイニシャル代数 $R = \mathrm{in}_{\preceq}(K[X][Y]^{D^{\Omega_0}})$ の生成系を，有限回の手続きで記述する方法について説明する．

各 $\lambda \in \mathbf{Z}_{\geq 0}$ と $1 \leq k \leq n$ に対し，$a \leq \lambda \nu_i^{-1} \omega_{k,i}$ $(i = 2, 3)$ を満たす最大の $a \in \lambda\omega_{k,1} - \mathbf{Z}_{\geq 0}$ を $a'_k(\lambda)$ とおく．このとき，任意の $b = (b_2, b_3) \in (\mathbf{Z}_{\geq 0})^2$ に対し

$$
\begin{aligned}
a_k(b) &= \omega_{k,2}b_2 + \omega_{k,3}b_3 - a'_k(b_2\nu_2 + b_3\nu_3) \\
&= b_2\omega_k^{2,1} + b_2\omega_k^{3,1} + (b_2\nu_2 + b_3\nu_3)\omega_{k,1} - a'_k(b_2\nu_2 + b_3\nu_3)
\end{aligned}
$$

は整数である．また，$a_k(b) \geq \nu_j^{-1}(b_2\omega_k^{2,j} + b_2\omega_k^{3,j})$ $(j = 1,2,3)$ が成り立つ．$k \in I_j$ を満たす j に対し $\omega_k^{2,j}$ と $\omega_k^{3,j}$ は非負なので，$a_k(b)$ は非負整数である．よって，$T_b = X_1^{a_1(b)} \cdots X_n^{a_n(b)} Y_2^{b_2} Y_3^{b_3}$ は $K[X][Y]$ の元となる．

簡単のため $\Psi_i = \Psi_i^D$ $(i = 2,3,4)$ とおく．定理 ?? より，$R \cap K[X][Y_1, Y_2, Y_3]$ は $R' = K[X][\{\mathrm{in}_{\preceq}(\Psi_i^{\Omega_0}) \mid i = 1,2,3\}]$ と等しい．

命題 5.2 R に含まれる任意の単項式 TY_4^l $(T \in K[X][Y_1, Y_2, Y_3],\ l \in \mathbf{Z}_{\geq 0})$ に対し，$X^c T_b Y_4^l$ が R に含まれ，$T(X^c T_b)^{-1}$ が R' に含まれるような $c \in (\mathbf{Z}_{\geq 0})^n$ と $b \in (\mathbf{Z}_{\geq 0})^2$ が存在する．

各 $b \in (\mathbf{Z}_{\geq 0})^2$ に対し $\mathcal{T}^b = \{(c, l) \in (\mathbf{Z}_{\geq 0})^n \times \mathbf{N} \mid X^c T_b Y_4^l \in R\}$ とおく．命題 5.2 とその上の注より，

$$R = R'[\{X^c T_b Y_4^l \mid (c, l) \in \mathcal{T}^b, b \in (\mathbf{Z}_{\geq 0})^2\}] \tag{5.1}$$

が成り立つ．$B = \{(b_2, b_3) \in (\mathbf{Z}_{\geq 0})^2 \mid b_2\nu_2 + b_3\nu_3 < \mathrm{lcm}\{\nu_2, \nu_3\}\}$ とおく．

命題 5.3 任意の $b \in (\mathbf{Z}_{\geq 0})^2 \setminus B$ に対して $\mathcal{T}^b = \{0\} \times \mathbf{N}$ が成り立つ．

各 $b \in B$ に対し，部分集合 $\mathcal{S}^b \subset (\mathbf{Z}_{\geq 0})^n$ と $N_c^b \subset \mathbf{N}$ $(c \in \mathcal{S}^b)$ が

$$\mathcal{T}^b = \{(d + c, l) \mid d \in (\mathbf{Z}_{\geq 0})^n, l \in N_c^b, c \in \mathcal{S}^b\} \tag{5.2}$$

を満たすとする．このとき，(5.1) と命題 5.3 より R は

$$\{T_b Y_4^l \mid l \in \mathbf{N}, b \in (\mathbf{Z}_{\geq 0})^2 \setminus B\} \cup \{X^c T_b Y_4^l \mid l \in N_c^b, c \in \mathcal{S}^b, b \in B\}$$

によって R' 上生成される．従って，R の生成系を記述するためには，各 $b \in B$ に対し (5.2) を満たす部分集合 $\mathcal{S}^b \subset (\mathbf{Z}_{\geq 0})^n$ と $N_c^b \subset \mathbf{N}$ $(c \in \mathcal{S}^b)$ を求めればよい．B は有限集合なので，これが $b \in B$ ごとに有限回の手続きで行えればよい．

B の元 $b = (b_2, b_3)$ を任意にとり $\lambda_0 = b_2\nu_2 + b_3\nu_3$ とおく．各 $l \in \mathbf{N}$ に対し，

$$\{\Psi_2^{c_2}\Psi_3^{c_3} \mid c_2, c_3 \in \mathbf{Z}_{\geq 0}, c_2\nu_2 + c_3\nu_3 = \lambda_0 + l\nu_{4,1}\}$$

が生成する K ベクトル空間を V_l とおく．また，$\{\Psi_2^{b_2+i\nu_3'}\Psi_3^{b_3-i\nu_2'} \mid 0\frac{l\nu_{4,1}}{\mathrm{lcm}\{\nu_2,\nu_3\}}+1$

は，直接計算により具体的に記述できる．

以下では，条件

(P) 任意の $P \in \mathcal{N}$ に対して $L(c_P)$ は有限集合である．

が満たされる場合に，(5.2) を満たす部分集合 $\mathcal{S}^b \subset (\mathbf{Z}_{\geq 0})^n$ と $N_c^b \subset \mathbf{N}$ $(c \in \mathcal{S}^b)$ を求めるためのアルゴリズムを与える．$\mathcal{N}$ は有限集合なので (P) の成立は有限回の手続きで判定できる．なお，仮定 (P) はさほど一般性を損ねるものでなく，例えば $\xi_J^1 < 1$ ならば常に成り立つ．また，[2], [5] や [18] の場合，$\xi_J^1 = 1$ でも (P) は成り立つ．実際，これらの例の場合には (P) よりも強い

(P′) 任意の $P \in \mathcal{N}$ に対して $L(c_P) = \emptyset$ である．

が成り立つ．$\mathcal{I}_\infty = \{c \in \mathcal{I}_0 \mid L(c) = \emptyset\}$ とおく．

命題 5.4 $\mathcal{I}_\infty \times \mathbf{N} \subset \mathcal{T}^b \subset \mathcal{I}_0 \times \mathbf{N}$ が成り立つ．

(P′) が満たされるとき $\mathcal{I}_0$ は $\mathcal{I}_\infty$ と等しい．よって，命題 5.4 より $\mathcal{T}^b = \mathcal{I}_0 \times \mathbf{N}$ が従う．この場合は $\mathcal{S}^b = \{c_P \mid P \in \mathcal{N}\}$, $N_c^b = \mathbf{N}$ $(c \in \mathcal{S}^b)$ とすればよい．

次に，一般の場合を考える．有限集合 $\bigcup_{P \in \mathcal{N}} L(c_P)$ の最大元を l_0 とし，

$$H = \left\{(c_1, \ldots, c_4) \in \mathbf{R}^4 \mid c_1, c_2, c_3 \geq 0, c_4 \geq -l_0, c_1 + c_2\nu_2 + c_3\nu_3 + c_4\nu_4 = 0\right\}$$

に含まれる整凸多面体全体を $\mathcal{P}$ とおく．H は有界なので $\mathcal{P}$ は有限集合である．各 $P \in \mathcal{P}$ に対し，

$$d_{P,k} = \max(\{a_k'(\lambda_0) - \omega_k \cdot c' \mid c' \in P\} \cup \{0\}) \quad (k = 1, \ldots, n)$$

とおくと，$d_P = (d_{P,1}, \ldots, d_{P,n})$ は $(\mathbf{Z}_{\geq 0})^n$ の元になる．$\mathcal{S}^b = \{d_P \mid P \in \mathcal{P}\} \cap \mathcal{I}_0$ とおく．各 $c \in \mathcal{S}^b$ に対し，$(\Psi_2^{b_2}\Psi_3^{b_3}\Psi_4^l + \sum_{i=0}^l f_i \Psi_4^{l-i})Y_4^{-l}$ のニュートン多面体と H の交わりが P に含まれる $f_i \in V_i$ $(i = 0, \ldots, \min\{l, l_0\})$ が存在するような $l \in \mathbf{N}$ 全体を N_c^b とする．ただし，P は $c = d_P$ を満たす $\mathcal{P}$ の元とする．

命題 5.5 上のように定めた $\mathcal{S}^b$ と N_c^b $(c \in \mathcal{S}^b)$ に対し，(5.2) が成り立つ．

最後に，各 $c \in \mathcal{S}^b$ に対し N_c^b を有限回の手続きで具体的に記述できることを説明する．各 $l \in \mathbf{N}$ に対し，l が N_c^b に含まれることは，有限個の変数に関する K 上のある連立 1 次方程式が解を持つことと同値になる．l_0 以上の一般の l については $K[l]$ 上の連立 1 次方程式の解の存在の問題に帰着でき，l が N_c^b に含まれることは l が K 上のある代数方程式の解であることと同値になる．従って，N_c^b は実際に計算可能である．

以上の方法により，(P) の仮定の下で $K[X][Y]^{D^{\Omega_0}}$ のイニシャル代数 R の生成系はいつでも記述できる．

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付記 原稿を提出した数日後，筆者はNijmegen大学を訪れ [4] の著者 van den Essen 氏に会った．その際，予想 ?? について話したところ，de Bondt [1] による局所冪零微分 D が反例になることを教えられた．この場合，$m = 6$ だが D のニュートン多面体の次元は 3 になる．

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ヒルベルトの第14問題に対する フロイデンバーグの反例について

谷本 龍二

フロイデンバーグ [2] は，ヒルベルトの第14問題に対する次のような反例を構成した．

k を標数0の体とし，$R := k[x, y, s, t, u, v]$ を k 上の6変数多項式環とする．R における局所べき零微分 D を

$$D := x^3\frac{\partial}{\partial s} + y^3 s\frac{\partial}{\partial t} + y^3 t\frac{\partial}{\partial u} + x^2y^2\frac{\partial}{\partial v} \in \mathrm{Der}_k(R)$$

で定める．このとき，D の核 R^D は k 代数として無限生成である．

ここで，k の加法群 $(k, +)$ を $\mathbb{G}_a$ であらわすことにする．すると，D による $\mathbb{G}_a$ の R への作用 $\mathbb{G}_a \times R \to R$ が

$$t \cdot f := \sum_{n=0}^{\infty} \frac{D^n(f)}{n!}t^n \quad (t \in \mathbb{G}_a,\ f \in R)$$

で定まる．この $\mathbb{G}_a$ 作用による不変式環 $R^{\mathbb{G}_a}$ は R^D と一致することに注意しておく．

本稿の目的は，次の問題に解答を与えることである．

問題. 無限生成 k 代数 R^D の生成系を記述せよ．

解答の方針は，無数の不変式を構成し，それらが生成系になることを示すことである．

はじめに，R^D の元 f_1, f_2, f_3, f_4, f_5 を

$$\begin{aligned}
f_1 &:= x,\\
f_2 &:= y,\\
f_3 &:= 2x^3t - y^3s^2,\\
f_4 &:= 3x^6u - 3x^3y^3st + y^6s^3,\\
f_5 &:= 9x^6u^2 - 18x^3y^3stu + 6y^6s^3u + 8x^3y^3t^3 - 3y^6s^2t^2
\end{aligned}$$

とおく．v 次数が 0 の R^D の元は f_i $(1 \leq i \leq 5)$ の多項式としてあらわせることが簡単な計算からわかる．

v 次数が 1 以上の R^D の生成元を構成する手がかりは次の補題にある．

補題 1． k 代数の間の準同型 $\pi_v : R \to R_{xy}$ を

$$\pi_v(f) := \sum_{n=0}^{\infty} \frac{D^n(f)}{n!} \left(-\frac{v}{x^2y^2} \right)^n$$

で定める．A における列 $\{b_m\}_{m \geq 0}$ が，

$$(*) \quad \begin{cases} b_0 \in A^{\Delta}, \\ \Delta(b_m) = x^2y^2b_{m-1} \quad (^{\forall}m \geq 1) \end{cases}$$

を満たせば，各 $m \geq 0$ に対して $\pi_v(b_m) \in R^D$ が成り立つ．さらに，$\pi_v(b_m)$ を v 次数について展開すると，

$$\pi_v(b_m) = \frac{(-1)^m}{m!} b_0 v^m + (v \text{ 次数について低次の項})$$

がえられる．

証明． π_v の定義から容易に従う． **証明終**

上記の補題をふまえ，フロイデンバーグ [2] は以下の定理を示した．

定理 2． $A := k[x, y, s, t, u]$ とおき，$\Delta := D|_A$ とおく．このとき，ある A における列 $\{b_m^{(1)}\}_{m \geq 0}$ で次の条件 (1), (2), (3) を満たすものが存在する．

(1) $b_0^{(1)} = f_1$.

(2) $\Delta(b_m^{(1)}) = x^2y^2b_{m-1}^{(1)} \quad (^{\forall}m \geq 1)$.

(3) $b_m^{(1)} \in xA \quad (^{\forall}m \geq 2)$.

したがって，各 $m \geq 1$ に対し

$$f_1 v^m + (v \text{ 次数について低次の項})$$

の形の R^D の元が存在する．

A における列 $\{b_m^{(2)}\}_{m\geq 0}$ と $\{b_m^{(3)}\}_{m\geq 0}$ を

$$b_m^{(2)} := \begin{cases} f_2f_3 & m=0\text{のとき} \\ x^2(3x^3u - y^3st) & m=1\text{のとき} \\ \dfrac{1}{x}(b_1^{(2)}b_{m-1}^{(1)} - (m-1)b_0^{(2)}b_m^{(1)}) & m\geq 2\text{のとき}, \end{cases}$$

$$b_m^{(3)} := \begin{cases} f_4 & m=0\text{のとき} \\ x^2y^2(3x^3su + y^3s^2t - 4x^3t^2) & m=1\text{のとき} \\ \dfrac{1}{x}(b_1^{(3)}b_{m-1}^{(1)} - (m-1)b_0^{(3)}b_m^{(1)}) & m\geq 2\text{のとき} \end{cases}$$

とおく．すると，次が成り立つ．

補題 3. 各 $i=2,3$ に対し，$\{b_m^{(i)}\}_{m\geq 0}$ は補題 1 の条件 $(*)$ を満たす．したがって，各 $m\geq 1$ に対し，

$$f_2f_3v^m + (v\text{ 次数について低次の項}),$$
$$f_4v^m + (v\text{ 次数について低次の項})$$

の形の R^D の元が存在する．

証明. 定理 2 から容易に従う． **証明終**

上記で構成した R^D の元 $f_1, f_2, f_3, f_4, f_5, \pi_v(b_m^{(i)})$ $(i=1,2,3,\ m\geq 1)$ の集合を $\mathcal{S}$ とおく．本稿の主定理は次である．

定理 4. $R^D = k[\mathcal{S}]$.

$\mathcal{S}$ が生成系になることの証明において，次のファン・デン・エッセンによる補題 [1, Proposition 1.4.15] を用いる．

補題 5. $\mathcal{A} := k[x_1,\ldots,x_n]$ を k 上の n 変数多項式環とする．d を $\mathcal{A}$ における局所べき零微分で，$\mathcal{A}^d$ は $f_1,\ldots,f_\ell$ で生成されているものとする．α を $d(\alpha)\neq 0$ かつ $d^2(\alpha)=0$ なる $\mathcal{A}$ の元とする．さらに，$\mathcal{A}' := \mathcal{A}[y_1,\ldots,y_\ell]$ を $\mathcal{A}$ 上の ℓ 変数多項式環とし，$\mathcal{A}'$ 上の単項式順序 $\preceq$ を

$$y_1 \prec y_2 \prec \cdots \prec y_\ell \prec x_1 \prec x_2 \prec \cdots \prec x_n$$

なるものとし，B を $\mathcal{A}'$ のイデアル $J := (y_i - f_i)_{1 \leq i \leq \ell}$ の $\preceq$ に関するグレブナ基底とする．このとき，$\mathcal{A}$ の任意の 0 でない元 a について次は同値である．

(1) $a \in \Delta(\mathcal{A})$.

(2) $\mathcal{A}$ の元

$$d(\alpha)^{m+1} \sum_{i=0}^{m} \frac{(-1)^i}{(i+1)!} d^i(a) \left(\frac{\alpha}{d(\alpha)} \right)^{i+1}$$

の B による正規形は $k[y_1, \ldots, y_\ell]$ に属する．ただし，m は，$d^m(a) \neq 0$ かつ $d^{m+1}(a) = 0$ なる整数である．

以下，定理 4 の証明をする．$R^D \subset k[\mathcal{S}]$ を，R^D の元の v 次数についての帰納法を用いて証明すればよい．f を R^D から任意にとる．$\deg_v(f) = 0$ なら，f は f_1, f_2, f_3, f_4, f_5 の k 上の多項式として表示されるので，$f \in k[\mathcal{S}]$ が成立する．$\deg_v(f) \geq 1$ なら，帰納法の仮定により，ある $g \in k[\mathcal{S}]$ が存在し $\deg_v(f-g) < \deg_v(f)$ なることを示せばよい．f を v 次数について展開し

$$f = a_\ell v^\ell + a_{\ell-1} v^{\ell-1} + \cdots + a_0 \in R^D, \qquad a_i \in A \ \ (0 \leq i \leq \ell), \quad a_\ell \neq 0$$

と表示すると，その最高次の係数 a_ℓ について $a_\ell \in A^\Delta = k[f_1, f_2, f_3, f_4, f_5]$ が成り立つ．したがって，定理 2 と補題 3 より，ある $g \in k[\mathcal{S}]$ で，

$$f - g = \left(\sum_{(\alpha,\beta) \in \Lambda_1} a_{\alpha,\beta} f_2^\alpha f_5^\beta + \sum_{(\lambda,\mu) \in \Lambda_2} b_{\lambda,\mu} f_3^\lambda f_5^\mu \right) v^\ell + (v \text{ について低次の項})$$

となるものが存在する．ただし，各 $(\alpha, \beta) \in \Lambda_1$ と各 $(\lambda, \mu) \in \Lambda_2$ に対し $a_{\alpha,\beta}, b_{\lambda,\mu} \in k$ とし，さらに，各 $\beta \geq 0$ に対し $a_{0,\beta} = 0$ とする．ここで，

$$b_\ell := \sum_{(\alpha,\beta) \in \Lambda_1} a_{\alpha,\beta} f_2^\alpha f_5^\beta + \sum_{(\lambda,\mu) \in \Lambda_2} b_{\lambda,\mu} f_3^\lambda f_5^\mu$$

とおくと，$f - g \in R^D$ より $x^2 y^2 b_\ell \in \Delta(A)$ が成立する．したがって，補題 5 にあらわれる $\mathcal{A}, d, \alpha, a$ として，特に $A, \Delta, s, x^2 y^2 b_\ell$ の場合を考えれ

ば，$x^2y^2b_\ell s$ の B による正規形は $k[y_1,y_2,y_3,y_4,y_5]$ の元であることがわかる．ただし，B は，$A[y_1,y_2,y_3,y_4,y_5]$ 上の

$$y_1 \prec y_2 \prec y_3 \prec y_4 \prec y_5 \prec x \prec y \prec s \prec t \prec u.$$

を満たす辞書式順序 $\preceq$ による $J=(y_i-f_i)_{1\le i\le 5}$ のグレブナ基底であり，

$$q_1 := y_1^3, \quad q_2 := -y_2^3y_3^3 - y_4^2, \quad q_3 := -x + y_1,$$
$$q_4 := -y + y_2, \quad q_5 := -sy_2^3y_3 - y_4, \quad q_6 := -y_3^2 + sy_4,$$
$$q_7 := -s^2y_2^3 - y_3, \quad q_8 := -6uy_3^2 + 3t^2y_4 + sy_5, \quad q_9 := 3t^2y_2^3y_4 + 6uy_4 - y_5$$

で与えることができる．実際，$x^2y^2b_\ell s$ の B による正規形を求めれば，

$$y_1^2y_2^2\left(\sum_{(\alpha,\beta)\in\Lambda_1} a_{\alpha,\beta}f_2^\alpha f_5^\beta + \sum_{(\lambda,\mu)\in\Lambda_2} b_{\lambda,\mu}f_3^\lambda f_5^\mu\right)s$$

であることが容易にわかる．以上のことから $b_\ell=0$ が成立する．　**証明終**

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対称群 が作用する 0 次元 GORENSTEIN 環

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1 Introduction

この タイトルで言う「対称群が作用する 0 次元 Gorenstein 環」を考える動機をいくつか挙げてみよう.

1. F. S. Macaulay による (非自明な)Gorenstein 環の最初の例は「対称群が作用する Gorenstien 環」であった. この例は以下の通り. ([1] p.29)
$$K[x,y,z]/(x^2,y^2,z^2,x(y-z),y(z-x)) \tag{1}$$
$K=\mathbb{C}$(または, 代数的閉体でも同じ) とすれば, Hilbert 関数に (1 3 1) を持つ Gorenstein 環はこの形に限ることが証明できる. 3 変数 2 次形式の標準形を考えることとほぼ同じことだ. 従ってこの例は自明ではない Gorenstein 環のもっとも簡単な例になっている.
$K=\mathbb{R}$ とすれば, 上記のほかに, 次のものがある.
$$K[x,y,z]/(xy,yz,zx,x^2-y^2,x^2-z^2). \tag{2}$$
また, 1970 年代前半に佐久間-奥山によって報告された Gorenstein 環の 2 つの例も矢張り, 対称群が作用しているものであった. この例は以下の通り.

$$K[x,y,x]/(x^3-y^3,y^3-z^3,xy,yz,zx)$$

$$K[x,y,x]/(xy-xz,yx-yz,x^3,y^3,z^3)$$

2. 表現論においては久しい以前から,「対称群が作用する Gorenstein 環」が研究されていた. たとえば対称群の余不変式部分環を S_n の左加群と見たとき, それは, S_n の正則表現と同値であることが以前から知られている. ここで言う, S_n の余不変式環とは, 次のものである.

$$K[x_1,\cdots,x_n]/(e_1,\cdots,e_n)$$

ただし, e_i は, i 次基本対称式. Terasoma-Yamad [3] は, その既約分解の基底を求めた.

3. 有限集合の Transversal Theory において, 順序同型群がレベル集合毎に推移的に作用するものを考えると, Sperner 性の証明が簡単になる. (このことは [2] に書いてある.) 一方, Sperner 性を持たない Gorenstein 環の例 (主に, 池田ひでみ氏によって発見されたものが多い) は対称群が作用していない. このようなことを考えると
「対称群が変数の置換として作用する Gorenstein 環は Sperner 性を持つ」
と予想したくなる. (ただし,「レベル集合毎に順序同型群が推移的に作用する」と言う条件に置き換わるある種の条件が必要だろう.)
"This is too good to be true."（虫が良すぎる）と言われそうだが, 少なくとも作業仮説としては意味を持つだろう.「Sperner 性」は,「強い Lefschetz 性」または「弱い Lefschetz 性」に置き換えても, 同様の仮説が成立する.

4. 次のアルチン環を考える.

$$K[x_1,\cdots,x_k]/(x_1^n,\cdots,x_k^n) \tag{3}$$

これは明らかに, 対称群が作用する Gorenstein 環である. これは, n 次元ベクトル空間の k 重テンソル空間と見ることができるので, Weyl の相互法則を可換環論の立場から見直すことが出来る. 一方, Weyl の相互法則から導かれる全ての結果は, 直ちにこの Gorenstein 環に応用できる.

5. 対称群が作用する0次元Gorenstein環は，正整数nと，対称式または交代式$f \in K[x_1, \cdots, x_k]$を用いて，

$$K[x_1, \cdots, x_k]/(x_1^n, \cdots, x_k^n) \colon f \tag{4}$$

と表すことができる．従って，前項目4で述べたWeylの相互法則が何らかの形で新たな応用を持つと期待できる．ついでに言うと，この表記法で，nとfが一意的に決まるわけではない．しかし，nを最小にとり，また，fに現れる変数のべきの次数は，高々$(n-1)$であると言う条件を付ければ，一意的に決まる．Gorensteinイデアルをコロンで表す表し方はあまり知られていないようだ．いくつか例を挙げてみよう．次の例では，Δは差積，p_iは変数のi次のべき和，また，h_iはi次の完全対称式，すなわち，i次の単項式の総和である．

(a) $(x^2, y^2, z^2) \colon x + y + z = (x^2, y^2, z^2, xy - xz, xy - yz)$

(b) $(x^3, y^3, z^3) \colon x^3y^3 + y^3z^3 + z^3x^3 = (xy, yz, zx, x^3 - y^3, y^3 - x^3)$

(c) $(x^{r+2}, y^{r+2}, z^{r+2}) \colon \Delta = (h_r, h_{r+1}, h_{r+2}), \quad r = 1, 2, \cdots$

(d) $(x^{3r}, y^{3r}, z^{3r}) \colon \Delta(h_{2(r-1)})^3 = (p_r, p_{r+1}, p_{r+2}), \quad r = 1, 2, \cdots$

2 同次単項式完全交差環とWeylの相互法則

以下，体Kの標数は零とする．$R = K[x_1, \cdots, x_k]$を多項式環とする．$f \in R$の偏次数とは，各変数に関する次数の中の最大値を意味するものとする．$A(n, k)$で，偏次数が高々$(n-1)$の多項式からなるRの部分ベクトル空間を表すことにする．RのイデアルIを$I = (x_1^n, \cdots, x_k^n)$とおくと，ベクトル空間として，次の通り直和に分解する．

$$R = A(n, k) \oplus I$$

$A(n, k)$はn次元ベクトル空間のk重テンソル空間と見ることができる．$A(n, k) = R/I$だらから，$A(n, k)$は，可換環の構造を持つ．今まで，テンソル空間$A(n, k)$を可換環として扱うことは，あまり無かったようだ．同じく，R/Iをテンソル空間と見ることも少なかったように思える．もう少し詳しく，テンソル空間としての$R/I = A(n, k)$の扱い方をはっきりさせておこう．$A = A(n, k)$と置く．Aの基底として，偏次数が高々

$(n-1)$ の単項式を取る:

$$\{x_1^{i_1}\cdots x_k^{i_k}|0\le i_1,\cdots,i_k\le n-1\}$$

A の元は,

$$\sum F(i_1,i_2,\cdots,i_k)x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$$

と書ける. $A(n,k)$ を n 次元ベクトル空間 K^n の k 重テンソル空間と見ているのだから, 一般線形群 $GL(n)$ のテンソル表現 $\phi\colon GL(n)\to GL(A)$ が自然に考えられる. これを具体的に書くと次の様になる. $g=(g_{\alpha\beta})\in GL(n)$ とするとき, $g(x_1^{i_1}\cdots x_k^{i_k})$ は,

$$g(x_1^{i_1}\cdots x_k^{i_k})=(\sum_{\beta=0}^{n-1}g_{i_1\beta}x_1^{\beta})\cdots(\sum_{\beta=0}^{n-1}g_{i_k\beta}x_k^{\beta})$$

である. ただし, 行列 $g=(g_{\alpha\beta})$ の成分の番号 α,β は, 0 から $(n-1)$ までを動くものとする. 同時に, 対称群 S_k が, 変数の置換として A に作用する.

Weyl の相互法則によれば, A の $GL(n)$ 加群としての既約分解を求めることは, S_k 加群としての既約分解とほぼ同等である. だから, A を, S_k 加群として既約分解できれば, 自動的に $GL(n)$ 加群としての分解が求まることになる. この意味で S_k 加群としての既約分解は興味深い. 以下では, (い) n は任意で, $k=2$, (ろ) k は任意で, $n=2$, の 2 つの場合について, この分解を求めてみる. そして, それが Introduction の項目 1 で述べた, Macaulay や佐久間-奥山の例とどのように結びつくのか紹介する.

その前に, べきゼロ行列について, 一言注意をしたい.

3 べきゼロ行列の Jordan 基底

"Jordan 基底" と言う言い方は一般的ではない. ここで言う意味は, 行列を Jordan 標準形として表すための基底のことである. $J\in\mathrm{End}(V)$ をべきゼロ写像としよう. 今更言うまでも無いのだろうが, J の Jordan 基底

を構成する手順は次の通り．まず，次のような部分空間の真減少列を考える．

$$V = \mathrm{Im}J^0 \supset \mathrm{Im}J \supset \mathrm{Im}J^2 \supset \cdots \supset \mathrm{Im}J^p = 0 \tag{5}$$

これを，Ker$J$ に制限して，Ker$J$ の部分空間の減少列を作る．

$$\mathrm{Ker}J = \mathrm{Ker}J \cap \mathrm{Im}J^0 \supset \mathrm{Ker}J \cap \mathrm{Im}J \supset \mathrm{Ker}J \cap \mathrm{Im}J^2 \supset \cdots \supset \mathrm{Ker}J \cap \mathrm{Im}J^p = 0 \tag{6}$$

この部分空間のうち繰り返しのあるものは除き，名前をつけなおして，次の様に書く．

$$\mathrm{Ker}J = W_s \supset W_{s-1} \supset \cdots \supset W_1 \supset 0 \tag{7}$$

すなわち，$W_i = \mathrm{Ker}J \cap \mathrm{Im}J^j \ \exists j$ で，$W_{i+1} \neq W_i$ であり，s は，この性質を持つもののうちの最大である．(7) で得た旗を尊重しながら，Ker$J$ の基底を取る．それが，J の Jordan 基底の一部分になる．すなわち，集合 B を求むべき J の Jordan 基底とすれば，$B_0 := B \cap \mathrm{Ker}J$ が決定できたことになる．いったん，B_0 が決まれば，各 b $\in B_0$ に対して，$a \in V \setminus \mathrm{Im}J$ を見つけ，$aJ^i = b$ とできる．集合

$$\bigcup_{b \in B_0} \{a, aJ, \cdots .aJ^i\}$$

(i は b による) が求める基底である．基底元の番号は，必要に応じてつければよい．

4 $A = A(n, 2)$ すなわち K^n の 2 重テンソル

$k = 2$ とすると，我々の Gorenstein 環 $A = K[x_1, \cdots, x_k]/(x_1^n, \cdots, x_k^n)$ は次のようになる．

$$A = K[x, y]/(x^n, y^n)$$

A の正則表現を $\times: A \to \mathrm{End}(A)$ で表す．すなわち，$a \in A$ に対して，$\times a$ は，$\times a(x) = ax$ で定義される線形写像 $A \to A$ を表すものとする．$l = x + y$ と置き，$\times l \in \mathrm{End}(A)$ の Jordan 基底を前節の方法で求めてみよう．このとき，A の "強 Lefschetz 性" が決定的に重要な役割を果たす．"強 Lefschetz 性" を復習すると次のようになる．

Theorem 1 $A = A(n,k)$ とし，$l = x_1 + x_2 + \cdots + x_k$ とする．更に，$A = \bigoplus_{i=0}^{c} A_i$ を A の次数分解とする．ただし，c は，$A_c \neq 0$ とする最大値である．(実際は，$c = nk - k$.) このとき，$\times l^{c-2i}$: $A_i \to A_{c-i}$ が，全ての i，ただし，$0 \leq i \leq [c/2]$ について全単射となる．

元通り $k = 2$ とし，変数は，x, y を使う．前節で紹介した方法で，$\times l$ の斉次 Jordan 基底を求めてみよう．そのためには，まず，$\mathrm{Ker}(\times l)$ の基底を求めるのであった．

A の Hilbert 関数は，長さ $(2n-1)$ の数列

0	1	2	⋯	n-2	n-1	n	⋯	2n-3	2n-2
1	2	3	⋯	n-1	n	n-1	⋯	2	1

である．次数 $(n-1)$ でピークに達し，その後一つづつ落ちてくる．$W = \mathrm{Ker}(\times l)$ と置くと，強 Lefschetz 性から，各々の $i = n-1, n, n+1, \cdots, 2n-2$ について，$\dim W_i = 1$，で，それ以外の次数では $W_i = 0$ となることが分かる．W は各次数毎に 1 次元だから，$\mathrm{Ker}(\times l)$ の基底元は定数倍を除くと一意的に決まる．(もちろん斉次という条件は必要だ.)

$W_{2n-2} = < b >$ とすると，強 Lefschetz 条件から，A_0 の元 a が存在して，$\times l^{2n-2}(a) = b$ となるはずだが，A_0 は定数しかないのだから，$a = 1, b = l^{2n-2}$ と考えてよい．同様に考えると，各 $i = 0, 1, 2, \cdots, n-1$ に付き，$(a_i, b_i) \in A_i \times A_{2n-2-i}$ で次の性質をものものが存在する．

1. $b_i \in \mathrm{Ker}(\times l)$
2. $\times l^{2n-2-2i}(a_i) = b_i$

少し考えると，$b_1, b_3, b_5, \cdots$ は対称式，また，$b_2, b_4, b_6, \cdots$ は交代式であることが分かる．したがって，奇数番号 i については，

$$a_i, \times l(a_i), \times l^2(a_i), \times l^3(a_i), \cdots$$

が対称式，偶数番号 i については，

$$a_i, \times l(a_i), \times l^2(a_i), \times l^3(a_i), \cdots$$

が交代式であることが分かる．以上により $\times l$ の Jordan 基底を求めることができた．それらは，必然的に，対称式，または，交代式になることも分かった．このうち，対称式の個数は

$$(2n-1)+(2n-5)+(2n-9)+\cdots=\frac{n(n+1)}{2} \tag{8}$$

であり，交代式の個数は

$$(2n-3)+(2n-7)+(2n-11)+\cdots=\frac{n(n-1)}{2} \tag{9}$$

である．

$A(n,2)$ は，n 次元ベクトル空間 $V:=K^n$ の 2 重テンソル空間 $W:=(K^n)^{\otimes 2}$ であり，対称式は対称テンソル，また，交代式は交代テンソルに他ならない．もちろん，上記の $n(n+1)/2$ と $n(n-1)/2$ は対称テンソルと交代テンソルの個数を表しているのだが，しからば，その分割 (8) と (9) は，何を表すのだろう．これは，次のように説明される．$\phi\colon SL(2)\to GL(n)=GL(V)$ を $SL(2)$ の n 次既約表現としよう．(各 n についてただ一つある．) また，$W=V\otimes V$ とし，$W=W_s\oplus W_a$ を対称テンソルと交代テンソルへの分解とする．このとき，W_s と W_a は，$GL(n)$ 加群だから，ϕ によって，$SL(2)$ 加群と考えることができ，$SL(2)$ 加群として既約分解する．その分解の次元が，丁度，上記の分割 (8) と (9) になっている．このことは，$SL(2)$ 加群の分解が Lie 環 $s\ell(2)$ の分解で得られることから分かる．$\times l$ を Jordan 標準形に置くことは，$s\ell(2)$ 加群の分解を求めることと同じだ．そして，$GL(2)$ の表現と $SL(2)$ の表現は，determinant のべきしか違わない．

5 $A(2,k)$ すなわち Boolean algebra

$A(n,k)$ において $n=2$ とすれば，Boolean algebra

$$A=K[x_1\cdots,x_k]/(x_1^2,\cdots,x_k^2)$$

を得る．言うまでもなく，集合 $2^{\{x_1,\cdots,x_k\}}$ が，A の基底となる．一般的には，集合 $2^{\{x_1,\cdots,x_k\}}$ 自身（と 2 つの演算 $\wedge,\vee$ を込めて）を Boolean algebra

と呼ぶのだろう．ここで言う“Boolean algebra”は，一般的な言い方ではないかも知れない．$L = \times(x_1 + x_2 + \cdots + x_k)$ とし，更に，

$$D = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_k}$$

と置く．L と D は多項式環 $R = K[x_1, \cdots, x_k]$ に作用していると考える．D を A に制限したものを $D|_A$ と書くことにする．また，$A(n,k) = R/I$ と考え，L が $A(n,k)$ に引き起こす写像を $L|_A$ と書く．従って，$L|_A, D|_A \in \mathrm{End}(A)$ である．更に，$H = [L|_A, D|_A]$(交換子) と置く．やはり，$H \in \mathrm{End}(A)$ である．

Proposition 2 (a) $\langle L|_A, D|_A, H\rangle$ は，$s\ell_2$-**triple** である．
(b) $L|_A$ の Jordan 基底 B が存在して，$B \setminus \mathrm{Im}L$ が $\mathrm{Ker}(D|_A)$ の基底となる．

Proof. (a) と (b) は，Lie 環 (それも $s\ell_2$ だけ) の表現論から分かる．

次の定理により，$L|_A$ の Jordan 基底を求めることができる．

Theorem 3 $i = 0, 1, 2, \cdots, [n/2]$ について，$(\mathrm{Ker}D) \cap A_i$ は i 次の Specht 多項式で張られる．標準ヤングタブローで定義される Specht 多項式が基底になる．

証明のために，Specht 多項式の定義をしなければならない．ヤング図形は周知のものとする．サイズ k のヤング図形に，1 から k までの整数を配置したものを，ヤングタブローと言う．ヤングタブローのうち，各行，各列とにも単調増加になっているものが標準ヤングタブローである．さて，$\mathcal{T}$ をヤングタブローとしよう．$X = \{x_1, \cdots, x_k\}$ を変数の集合とし，X_j を $\mathcal{T}$ の第 j 列に現れる変数の集合とする．(厳密には，その番号を持つ変数の集合と言うべきだ．) X_j の元に関する差積を $\Delta(X_j)$ で表すとき，列ごとの差積を全て掛けたもの，

$$\prod_j \Delta(X_j) \tag{10}$$

を，**Specht 多項式**と呼ぶ．

Theorem 3 の証明の概略を述べる．まず，次の補題に注意しよう．

Lemma 4 Ker $D = K[\{x_i - x_j | 1 \leq i, j \leq k\}]$

Proof. $R = K[x_1, \cdots, x_k]$ と置く．言うまでも無く，Ker D は，$\{f \in R | Df = 0\}$ の意味である．これは，algebra になるので，右辺が左辺に含まれていることはすぐ分かる．また，右辺は $(k-1)$ 次元多項式環であるから，左辺の Hilbert 関数を計算すれば，両者は一致することが分かる．(証明終)

Lemma 5 (Ker D) $\cap A$ の Hilbert 関数は，$\binom{k}{i} - \binom{k}{i-1}$ である．

Proof. (Ker D) $\cap A$ は Ker $D|_A$: $A \to A$ のことである．まず，A の Hilbert 関数は，2 項係数 $\binom{k}{i}$ で与えられる．Proposition 2 を使うと線形写像 $D|_A$: $A \to A$ は各次数毎に単射または全射である．よって主張が証明された．(証明終)

Lemma 6 （変数の数を k としている．これが，ヤング図形等のサイズである．）$\mathcal{T}$ をヤングタブローとし，T をその台とし，S を $\mathcal{T}$ から構成される Specht 多項式とする．このとき，

1. $S = 1 \iff T$ の行数は 1.
2. $S \in A$ and $S \neq 1 \iff T$ の行数は 2.
3. $S \in A$ とするとき，S の次数は，T の第 2 行の長さに等しい．

Proof. いずれも S の定義から明らかである．

Proposition 7 k, i を非負整数とし，$k - i \geq i$ とする．$T = T(k-i, i)$ を行の長さが $k-i, i$ の 2 行からなるヤング図形とする．

(a) T を台として持つ標準ヤングタブローの個数は $\binom{k}{i} - \binom{k}{i-1}$ である．
(a) k, i を固定するとき，台を T とする標準ヤングタブローから構成される Specht 多項式の集合は一次独立である．

Proof. (a) ヤングタブロー$\mathcal{T}$の台は，Tであるとする．このときkの位置は, 必ず行の右端である．しかも，1行目の最後にkが位置するなら，その真下は空である．よってkを除くことで，ヤングタブローのサイズを小さくできる．これに注意して，kに関する帰納法が働く．(後は簡単．)

(b) 番号の大きい変数を上位とする正辞書式モノミアル順序を考える．このときSpecht多項式の頭項は，$\mathcal{T}$の第2行に現れる（番号を持つ）変数の積である. (理由:変数の差分の積としてSが表されている．それを展開する際に上位の単項式を追及していくと，差分ごとに大きい番号を選ぶことになる．あとは下段の数の真上にはそれより小さい数が入っていることに注意すれば良い.) また，下段の数列により，$\mathcal{T}$が決定される．(理由:下段の数が決まれば，残りの数が，大きさの順に上段に並ぶことになる．) つまり，異なるSpect多項式は異なる頭項を持つ．よって，いろいろの$\mathcal{T}$から決まるSpecht多項式達は，1次独立である．

Theorem 3の証明. Lemma 5とLemma 6とProposition 7からしたがう．(証明終)

3節で紹介した方法で，LのJordan基底を求めることができる．また，$GL(2)$の表現の既約分解は．LのJordan分解と一致する．

Example 8 $k=4$として，LのJordan基底を書いてみよう．まず，Hilbert関数は，(1, 4, 6, 4, 1)である．その階差数列は, (1, 3, 2).
0次のSpecht多項式は1,
1次のSpecht多項式は，$x_1-x_2, x_1-x_3, x_1-x_4$（これを順に$a, b, c$と置く）
2次のSpecht多項式は，$(x_1-x_2)(x_3-x_4), (x_1-x_3)(x_2-x_4)$（これを順に$f, g$と置く).
この時，$A=A(2,4)$のGL(2)加群としての既約分解は,

次元5のものが1個:　$\langle 1, L, L^2, L^3, L^4\rangle$

次元3のものが3個:　$\langle a, aL, aL^2\rangle$ と $\langle b, bL, bL^2\rangle$ と $\langle c, cL, cL^2\rangle$

次元1のものが2個:　$\langle f\rangle$ と $\langle g\rangle$

である.

言うまでもなく，$2^4 = 5 \times 1 + 3 \times 3 + 1 \times 2$ となる．注意をすると，L を掛ける時に形の上であらわれる“変数の2次以上のべき”は0とする．従って，L^i は実際には，基本対称式（の定数倍）である．

Remark 9 以上，あえて表現論の一般論を極力使わない方針で証明を構成した．実際に使ったのは，$s\ell_2$ の表現論だけ (それもほとんどべきゼロ行列の Jordan 基底だけ) だ．仮に，表現論の結果を全面的に使うなら，別の書き方があったに違いない．

6 Gorenstein 環への応用

前節どおり，$R = K[x_1, \cdots, x_k]$，$L = x_1 + \cdots + x_k$，$A = R/(x_1^2, \cdots, x_k^2)$，$l = x_1 + x_2 + \cdots + x_k \in A$ とする．(L と l は，同じ形になってしまった．l は，モッド $(x_1^2, \cdots, x_k^2)$ の意味を込めて使う．前後関係からどちらか判定できるか，またはどちらでも良い場合が多いので混乱は生じないはずだ．) (A, l) は強 Lefschetz 性を持つから，$A/0{:}\, l^j$ の Hilbert 関数が分かる．それは，A の Hilbert 関数の中央の部分を j 個取り除いてできる数列である．$\times l$ の Jordan 基底が分かったのだから，$0{:}\, l^j$ のベクトル空間としての基底が分かったことになる．ただし，その中からイデアルとしての極小生成を選ぶとなると必ずしも簡単なことではない．ここでは，$j = 1$ の場合すなわち，$0{:}\, l$ のイデアル (極小) 基底を書き出してみよう．

このため，$(\ )^\star \colon A \to A$ で A の“双対写像”を表す．すなわち，$M \in A$ を単項式とするとき，$M^\star = (x_1 \cdots x_k)/M$ と定めた線形写像である．Specht 多項式 $S \in A$ に対して，$S^\star$ を S の双対 Specht 多項式と言うことにする．2節で定めたとおり，$A \subset R$ なので，$S^\star \in R$ と考えることができる．

次の定理では，$T = T(r, s)$ で行の長さが r, s からなるヤング図形を表す．$\mathcal{T}$ をヤングタブローとするとき，$|\mathcal{T}|$ でその台なるヤング図形を表す．また，$\mathcal{T}$ が定義する Specht 多項式を $S(\mathcal{T})$ で表す．

Theorem 10 R において，$J = (x_1^2, \cdots, x_k^2) : L$ とする．J の極小生成系の個数を $\mu(J)$ とすると，

$$\mu(J) = k + \binom{k}{h} - \binom{k}{h+1}$$

である．ただし，h は，k の偶奇に応じて，$h = k/2$ または，$h = (k+1)/2$ と定める．実際には，k が偶数のとき，

$$J = (x_1^2, \cdots, x_k^2) + \{S(\mathcal{T})^\star | \ |\mathcal{T}| = T(h,h)\}R$$

であり，k が奇数のとき，

$$J = (x_1^2, \cdots, x_k^2) + \{S(\mathcal{T})^\star | \ |\mathcal{T}| = T(h,h-1)\}R$$

である．(偶数の場合は $\star$ はあってもなくても同じだ．)

証明の前に一つ補題が必要である．

Lemma 11 $\mathcal{T}$ を，標準的とは限らないヤングタブローで，その台を $T = T(r,s)$ とする．また，S を $\mathcal{T}$ が定義する Specht 多項式とする．更に，(i,j) が $\mathcal{T}$ の一つの列の成分であるとする．このとき，$(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})S$ は，Specht 多項式である．(定数倍は無視する．)

更に，次がなりたつ．

$$((\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})S)^\star = (x_i - x_j)S^\star$$

Proof. 仮定より，$(x_i - x_j)$ は S の因子であり，それ以外には，この 2 個の変数は表れない．よって，$(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})S = \pm 2S/(x_i - x_j)$ である．後半の主張も簡単に証明できる．

Proof of Theorem 10. 後半を証明すればよい．すでに，$\mathrm{Ker}(\times l)\colon A \to A$ は，ベクトル空間としては，双対 Specht 多項式で張られることが分かっている．A のイデアル $\mathfrak{a}$ を最小次数の双対 Specht 多項式が生成する A のイデアルとする．このとき，$\mathfrak{a}$ は，Lemma 11 よりすべての双対 Specht 多項式を含むことが分かる．(証明終)

Remark 12 Introductionで紹介したMacaulayの例に，双対Specht多項式が現れている．

Corollary 13 $A = A(2,k)$において，

$$\mu(0\colon l) = \binom{k}{h} - \binom{k}{h+1}$$

ただし，hはTheorem 10と同様とする．

Proof. Theorem 10からすぐ分かる．

Theorem 14 前と同様$A = R/(x_1^2, \cdots, x_k^2)$，また，$l = x_1 + \cdots + x_k$とする．$A/(l)$の*Macaulay type*は$h$番目のCatalan数（$\frac{1}{h+1}\binom{2h}{h}$）である．ただし，$h$はTheorem 10の通り．

言い換えれば，$J = (x_1^2, \cdots, x_k^2, L)$と置き，$R/J$の極小自由分解を

$$0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to R \to R/J \to 0$$

とするとき，rank $F_k = \frac{1}{h+1}\binom{2h}{h}$となる．

Proof. $A/(l)$のMacaulay typeは，$0\colon l$の生成元の数である．またそれは，極小自由分解の最後のランクとして現れる．(証明終)

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CASTELNUOVO-MUMFORD REGULARITY FOR DIVISORS ON RATIONAL NORMAL SCROLLS

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Let X be a projective scheme of $\mathbb{P}_K^N$ over a field K. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \cdots, x_N)$ be the irrelevant ideal. Then we put $\mathbb{P}_K^N = \mathrm{Proj}(S)$. We denote by $\mathcal{I}_X$ the ideal sheaf of X. Let m be an integer. Then X is said to be m-regular if $\mathrm{H}^i(\mathbb{P}_K^N, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_K^N$ is the least such integer m and is denoted by $\mathrm{reg}(X)$. The interest in this concept stems partly from the well-known fact that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the p-th syzygy module of the defining ideal I of $X \subseteq \mathbb{P}_K^N$ occur in degree $\leq m + p$, see, e.g., [5].

In what follows, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to ℓ, and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to ℓ.

The starting point of our research on the Castelnuovo-Mumford regularity is an inequality $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$ for the generic hyperplane section of nondegenerate projective curve $X \subseteq \mathbb{P}_K^N$, which is a consequence of the Uniform Posision Lemma [1] for the characteristic zero case and Ballico's corresponding result [2] for the positive characteristic case. Then the corresponding bound works for ACM (arithmetically Cohen-Macaulay) nondegenerate projective variety. Further we have the extremal examples for this bound.

Theorem 1 (See [10, 11]). Let $X \subseteq \mathbb{P}_K^N$ be a generic hyperplane section of nondegenerate projective curve over an algebraically closed field K. Then we have $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$. Further, assume that $\deg(X) \geq N^2 + 2N + 2$. If the equality $\mathrm{reg}(X) = \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$ holds, then X is contained in a rational normal curve in $\mathbb{P}_k^N$.

Next let us consider the regularity bounds for non-ACM projective varieties. From now on, we always assume that X is locally Cohen-Macaulay and equi-dimensional. In order to evaluate the intermediate cohomologies, we introduce the notion of the k-Buchsbaum property. Let k be a nonnegative integer. Then X is called k-Buchsbaum if the graded S-module $\mathrm{M}^i(X) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(\ell))$, called the deficiency module of X, is annihilated by $\mathfrak{m}^k$ for $1 \leq i \leq \dim(X)$. The minimal nonnegative integer n, if it exists, such that X is n-Buchsbaum, it denoted by $k(X)$, see, e.g., [8, 9].

Theorem 2 (See [3, 10, 13, 14]). Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K. If X is not ACM, then we have

$$\mathrm{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\mathrm{codim}(X)} \right\rceil + k(X) \dim(X).$$

Further, assume that $\deg(X) \geq 2\,\mathrm{codim}(X)^2 + \mathrm{codim}(X) + 2$. Then the equality holds only if X is a curve on a rational ruled surface.

The proof for the bounds consists in the study of spectral sequences. This powerful tool is applied for several interesting results on the regularity problem, see, e.g., [4]. The extremal varieties are obtained by dimensional induction, where the characteristic zero case is easier than the positive characteristic case thanks to the Socle Lemma [7]. This observation implies that the above bound is sharp for the curve case, but no extremal examples are found for higher dimensional case. In order to describe a conjecture towards better bounds, we define $\tilde{k}(X)$ as the minimal integer k such that all successive hyperplane sections of X, that is, $X \cap L$ with $\mathrm{codim}(X \cap L) = \mathrm{codim}(X) + \mathrm{codim}(L)$ for any linear space L of $\mathbb{P}^N_K$, have the k-Buchsbaum property, see, e.g., [6]. Now we state a variation of the Hoa's conjecture.

Conjecture 3 (See [10]). Let X be a nondegenerate projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K. Then we have

$$\mathrm{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\mathrm{codim}(X)} \right\rceil + \max\{\tilde{k}(X), 1\}.$$

Furthermore, assume that $\deg(X)$ is large enough. Then the equality holds only if X is a divisor on a rational normal scroll.

The following result shows that his conjecture should be best possible.

Theorem 4 (See [12]). Let X be a nondegenerate projective variety in $\mathbb{P}^N_K$ of dimension r over an algebraically closed field K. Put $k = \tilde{k}(X)$

Assume that X is a divisor on a rational normal scroll. Then we have

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X)-1}{\operatorname{codim}(X)} \right\rceil + \max\{k, 1\}.$$

Furthermore, there exist extremal varieties for all r and k.

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ON SYZYGIES OF THE RESIDUE FIELD

RYO TAKAHASHI

1. Introduction

Throughout the present paper, we assume that all rings are commutative noetherian local rings and all modules are finitely generated modules.

Dutta [3] proved the following theorem in his research into the homological conjectures:

Theorem 1.1 (Dutta). *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that the nth syzygy module of k has a non-zero direct summand of finite projective dimension for some $n \geq 0$. Then R is regular.*

Since G-dimension is similar to projective dimension, this theorem naturally leads us to the following question:

Question 1.2. Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that the nth syzygy module of k has a non-zero direct summand of finite G-dimension for some $n \geq 0$. Then is R Gorenstein?

It is obviously seen from the indecomposability of k that this question is true if $n = 0$. Hence this question is worth considering just in the case where $n \geq 1$.

We are able to answer in this paper that the above question is true if $n \leq 2$. Furthermore, as the theorems below say, we can even determine the structure of a ring satisfying the assumption of the above question for $n = 1, 2$.

2. Basic definitions

Throughout this section, let $(R, \mathfrak{m}, k)$ be a local ring. In this section, we introduce several notions to explain and prove in the next section the main theorems of this paper.

We begin by recalling the notions of a (pre)cover and a (pre)envelope of a module. Let $\mathrm{mod}\, R$ denote the category of finitely generated R-modules.

Definition 2.1. Let $\mathcal{C}$ be a full subcategory of $\mathrm{mod}\, R$.

The detailed version of this paper has been submitted for publication elsewhere.

(1) Let $\phi : X \to M$ be a homomorphism from $X \in \mathcal{C}$ to $M \in \operatorname{mod} R$.
  (i) We call ϕ or X a *$\mathcal{C}$-precover* of M if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{C}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.
  (ii) Assume that ϕ is a $\mathcal{C}$-precover of M. We call ϕ or X a *$\mathcal{C}$-cover* of M if any endomorphism f of X with $\phi = \phi f$ is an automorphism.

(2) Let $\phi : M \to X$ be a homomorphism from $M \in \operatorname{mod} R$ to $X \in \mathcal{C}$.
  (i) We call ϕ or X a *$\mathcal{C}$-preenvelope* of M if for any homomorphism $\phi' : M \to X'$ with $X' \in \mathcal{C}$ there exists a homomorphism $f : X \to X'$ such that $\phi' = f\phi$.
  (ii) Assume that ϕ is a $\mathcal{C}$-preenvelope of M. We call ϕ or X a *$\mathcal{C}$-envelope* of M if any endomorphism f of X with $\phi = f\phi$ is an automorphism.

A $\mathcal{C}$-precover (resp. $\mathcal{C}$-cover, $\mathcal{C}$-preenvelope, $\mathcal{C}$-envelope) is also called a *right $\mathcal{C}$-approximation* (resp. *right minimal $\mathcal{C}$-approximation*, *left $\mathcal{C}$-approximation*, *left minimal $\mathcal{C}$-approximation*).

We denote by $\mathcal{F}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all free R-modules. Recall that a homomorphism $f : M \to N$ of R-modules is said to be *minimal* if the induced homomorphism $f \otimes_R k : M \otimes_R k \to N \otimes_R k$ is an isomorphism. Let $\nu_R(M)$ denote the minimal number of generators of an R-module M, i.e., $\nu_R(M) = \dim_k(M \otimes_R k)$. An $\mathcal{F}(R)$-cover of an R-module M is nothing but a minimal homomorphism from a free module to M.

Let M be an R-module. Take its $\mathcal{F}(R)$-cover $\pi : F \to M$. The *first syzygy module* $\Omega_R M = \Omega_R^1 M$ of M is defined to be the kernel of the homomorphism π, and the *nth syzygy module* $\Omega_R^n M$ of M is defined inductively: $\Omega_R^n M = \Omega_R(\Omega_R^{n-1} M)$ for $n \geq 2$. Dually to this, we can define the cosyzygy modules of any module.

Definition 2.2. Let M be an R-module.

(1) Take the $\mathcal{F}(R)$-envelope $\theta : M \to F$ of M. We set $\Omega_R^{-1} M = \operatorname{Coker} \theta$, and call it the *first cosyzygy module* of M.

(2) Let $n \geq 2$. Assume that the $(n-1)$th cosyzygy module $\Omega_R^{-(n-1)} M$ is defined. Then we set $\Omega_R^{-n} M = \Omega_R^{-1}(\Omega_R^{-(n-1)} M)$ and call it the *nth cosyzygy module* of M.

A module is said to be *stable* if it has no non-zero free summand.

Now, we recall the definition of G-dimension.

Definition 2.3. (1) We denote by $\mathcal{G}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all R-modules M satisfying the following three conditions:

(i) M is reflexive,
(ii) $\mathrm{Ext}^i_R(M,R)=0$ for every $i>0$,
(iii) $\mathrm{Ext}^i_R(M^*,R)=0$ for every $i>0$.

(2) Let M be an R-module. If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $G_i \in \mathcal{G}(R)$ for every i, then we say that M has *G-dimension at most n*, and write $\text{G-dim}_R M \le n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

If an R-module M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has *G-dimension n*, and write $\text{G-dim}_R M = n$. Note that for an R-module M we have $\text{G-dim}_R M = 0$ if and only if $M \in \mathcal{G}(R)$, and that all free R-modules belong to $\mathcal{G}(R)$.

We denote by $\underline{\mathcal{G}}(R)$ the full subcategory of $\mathcal{G}(R)$ consisting of all stable modules in $\mathcal{G}(R)$. The dual functor $(-)^*$ and the syzygy functor $\Omega(-)$ make good correspondences between the category $\underline{\mathcal{G}}(R)$ and itself.

3. Main results

In this section, using the results given in the previous section, we shall state and prove our main theorems.

First of all, we consider an idealization possessing a non-free reflexive module.

Proposition 3.1. *Let $(S,\mathfrak{n},k)$ be a local ring, $V \neq 0$ a finite-dimensional k-vector space, and $R = S \ltimes V$ the idealization of V over S. Let M be a non-free indecomposable reflexive R-module. Then*

(1) $M \cong \mathrm{Soc}\, R \cong V \cong k$,
(2) *If* $\mathrm{depth}\, S = 0$, *then* $S = k$, *hence* $R \cong k[[X]]/(X^2)$.

PROOF. (1) Denote by $\mathfrak{m}$ the unique maximal ideal of R, and set

$$\begin{cases} I = \mathfrak{n} \ltimes 0 = \{(s,v) \in R \mid s \in \mathfrak{n}, v = 0\}, \\ J = 0 \ltimes V = \{(s,v) \in R \mid s = 0\}. \end{cases}$$

These are ideals of R, and it is easy to see that $\mathfrak{m} = I \oplus J$. We have isomorphisms

$$\begin{aligned} M^* &\cong \mathrm{Hom}_R(M,\mathfrak{m}) \\ &\cong \mathrm{Hom}_R(M, I \oplus J) \\ &\cong \mathrm{Hom}_R(M,I) \oplus \mathrm{Hom}_R(M,J). \end{aligned}$$

Since M^* is also indecomposable, we have either $\mathrm{Hom}_R(M, I) = 0$ or $\mathrm{Hom}_R(M, J) = 0$. However J is isomorphic to k^e as an R-module where $e = \dim_k V$, hence $\mathrm{Hom}_R(M, J) \cong k^{ne} \neq 0$ where $n = \nu_R(M)$. It follows that $\mathrm{Hom}_R(M, I) = 0$. Now the assertions of the proposition immediately follow from this. $\square$

Corollary 3.2. *Let $(S, \mathfrak{n}, k)$ be a local ring, $V \neq 0$ a finite-dimensional k-vector space, and $R = S \ltimes V$ the idealization of V over S. Then the following conditions are equivalent:*

1. *There is a non-free R-module in $\mathcal{G}(R)$;*
2. *R is Gorenstein;*
3. *$R \cong k[[X]]/(X^2)$.*

The decomposability of the maximal ideal and the existence of a non-free module of G-dimension zero played essential roles in the achievement of Corollary 3.2. From now on, we consider a local ring satisfying these conditions in more general settings.

Proposition 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that there is a direct sum decomposition $\mathfrak{m} = I \oplus J$ where I, J are non-zero ideals of R. Let M be a non-free indecomposable module in $\mathcal{G}(R)$. Then there exist elements $x, y \in \mathfrak{m}$ such that*

1. *$I = (x)$ and $J = (y)$,*
2. *$(0 : x) = (y)$ and $(0 : y) = (x)$,*
3. *M is isomorphic to either (x) or (y).*

Hence the minimal free resolution of k is as follows:

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x\ y)} R \longrightarrow k \longrightarrow 0.$$

PROOF. We see that M^* and ΩM are also non-free indecomposable modules in $\mathcal{G}(R)$. There are isomorphisms

$$\begin{aligned} M^* &\cong \mathrm{Hom}_R(M, \mathfrak{m}) \\ &= \mathrm{Hom}_R(M, I \oplus J) \\ &\cong \mathrm{Hom}_R(M, I) \oplus \mathrm{Hom}_R(M, J). \end{aligned}$$

The indecomposability of M^* implies that either $\mathrm{Hom}_R(M, I) = 0$ or $\mathrm{Hom}_R(M, J) = 0$. We may assume that

$$\mathrm{Hom}_R(M, J) = 0. \tag{3.3.1}$$

There is an exact sequence

$$0 \to \Omega M \to R^n \to M \to 0. \tag{3.3.2}$$

Dualizing this by J, we obtain another exact sequence

$$\mathrm{Hom}_R(M, J) \to J^n \to \mathrm{Hom}_R(\Omega M, J).$$

We have $\mathrm{Hom}_R(\Omega M, J) \neq 0$ by (3.3.1). Applying the above argument to the module ΩM yields

$$\mathrm{Hom}_R(\Omega M, I) = 0. \tag{3.3.3}$$

Also, dualizing (3.3.2) by I, we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, I) \to I^n \to \mathrm{Hom}_R(\Omega M, I),$$

and hence $M^* \cong \mathrm{Hom}_R(M, I) \cong I^n$. The indecomposability of M^* implies that $n = 1$ (i.e. M is cyclic), and $M^* \cong I$.

We also have

$$\begin{aligned} M &\cong M^{**} \\ &\cong \mathrm{Hom}_R(M^*, \mathfrak{m}) \\ &\cong \mathrm{Hom}_R(M^*, I) \oplus \mathrm{Hom}_R(M^*, J). \end{aligned}$$

Note that $\mathrm{Hom}_R(M^*, I)$ is isomorphic to $\mathrm{Hom}_R(I, I)$, which contains the identity map of I. Hence $\mathrm{Hom}_R(M^*, I) \neq 0$ and therefore

$$\mathrm{Hom}_R(M^*, J) = 0.$$

Applying the above argument to the module M^*, we see that M^* is also cyclic and $M \cong M^{**} \cong I$. Thus, we have shown that $M \cong M^* \cong I$ and these modules are cyclic. Noting (3.3.3) and applying the above argument to the module ΩM, we see that $\Omega M \cong (\Omega M)^* \cong J$ and these modules are cyclic.

Write $I = (x)$ and $J = (y)$. We easily see $(0 : (0 : x)) = (x)$. Similarly, we also have $(0 : (0 : y)) = (y)$. Since $(0 : x) = \Omega(x) \cong \Omega M \cong (y)$, we have $(x) = (0 : y)$, and therefore $(0 : x) = (y)$. Thus we obtain the minimal free resolutions of (x) and (y):

$$\begin{cases} \cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \to (x) \to 0, \\ \cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to (y) \to 0. \end{cases}$$

Taking the direct sum of these exact sequence, we get

$$\cdots \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Joining this to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ constructs the minimal free resolution of k in the assertion. □

From the above proposition we can get the following theorem.

Theorem 3.4. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the following conditions are equivalent:*

(1) *The maximal ideal of R is decomposable;*

(2) $\dim S = 2$ *and* $I = (xy)$ *for some regular system of parameter* x, y *of* S.

Using Theorem 3.4 and Cohen's structure theorem, we obtain the following corollary.

Corollary 3.5. *Let* $(R, \mathfrak{m})$ *be a complete local ring. The following conditions are equivalent:*

(1) *There is a non-free module in* $\mathcal{G}(R)$*, and* $\mathfrak{m}$ *is decomposable;*
(2) R *is Gorenstein, and* $\mathfrak{m}$ *is decomposable;*
(3) *There are a complete regular local ring* S *of dimension two and a regular system of parameters* x, y *of* S *such that* $R \cong S/(xy)$.

Note that the finiteness of G-dimension is independent of completion. Thus, Corollary 3.5 not only gives birth to a generalization of [4, Proposition 2.3] but also guarantees that Question 1.2 is true if $n = 1$.

As far as here, we have observed a local ring whose maximal ideal is decomposable. From here to the end of this paper, we will observe a local ring such that the second syzygy module of the residue class field is decomposable. We begin with the following theorem, which implies that Question 1.2 is true if $n = 2$.

Theorem 3.6. *Let* $(R, \mathfrak{m}, k)$ *be a local ring. Suppose that* $\mathfrak{m}$ *is indecomposable and that* $\Omega_R^2 k$ *has a non-zero proper direct summand of finite G-dimension. Then* R *is a Gorenstein ring of dimension two.*

Proof. Replacing R with its $\mathfrak{m}$-adic completion, we may assume that R is a complete local ring. In particular, note that R is Henselian.

We have $\Omega_R^2 k = M \oplus N$ for some non-zero R-modules M and N with $\text{G-dim}_R M < \infty$. There is an exact sequence

$$0 \longrightarrow M \oplus N \xrightarrow{(f,g)} R^e \longrightarrow \mathfrak{m} \longrightarrow 0$$

of R-modules, where $e = \operatorname{edim} R$. Setting $A = \operatorname{Coker} f$ and $B = \operatorname{Coker} g$, we get exact sequences

$$(3.6.1) \qquad \begin{cases} 0 \to M \xrightarrow{f} R^e \xrightarrow{\alpha} A \to 0, \\ 0 \to N \xrightarrow{g} R^e \xrightarrow{\beta} B \to 0. \end{cases}$$

It is easily observed that there are exact sequences

$$(3.6.2) \qquad 0 \longrightarrow R^e \xrightarrow{\binom{\alpha}{\beta}} A \oplus B \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Here we can prove that $\operatorname{Ext}_R^2(k, R) \neq 0$.

Fix a non-free indecomposable module $X \in \mathcal{G}(R)$. Applying the functor $\operatorname{Hom}_R(X, -)$ to (3.6.2) gives an exact sequence

$$0 \to (X^*)^e \to \operatorname{Hom}_R(X, A) \oplus \operatorname{Hom}_R(X, B) \to \operatorname{Hom}_R(X, \mathfrak{m}) \to 0$$

and an isomorphism

$$\text{(3.6.3)} \qquad \mathrm{Ext}_R^1(X,A) \oplus \mathrm{Ext}_R^1(X,B) \cong \mathrm{Ext}_R^1(X,\mathfrak{m}).$$

We have $(X^*)^e \in \mathcal{G}(R)$ and $\mathrm{Hom}_R(X,\mathfrak{m}) \in \mathcal{G}(R)$, hence

$$\mathrm{Hom}_R(X,A) \in \mathcal{G}(R).$$

Take the first syzygy module of X; we have an exact sequence

$$0 \to \Omega X \to R^n \to X \to 0.$$

Dualizing this sequence by A, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(X,A) \to A^n \to \mathrm{Hom}_R(\Omega X, A) \to \mathrm{Ext}_R^1(X,A) \to 0.$$

Divide this into two short exact sequences

$$\text{(3.6.4)} \qquad \begin{cases} 0 \to \mathrm{Hom}_R(X,A) \to A^n \to C \to 0, \\ 0 \to C \to \mathrm{Hom}_R(\Omega X, A) \to \mathrm{Ext}_R^1(X,A) \to 0 \end{cases}$$

of R-modules. Since ΩX is also a non-free indecomposable module in $\mathcal{G}(R)$, applying the above argument to ΩX instead of X shows that the module $\mathrm{Hom}_R(\Omega X, A)$ also belongs to $\mathcal{G}(R)$. We have $\text{G-dim}_R(A^n) < \infty$ by the first sequence in (3.6.1). Hence it follows from (3.6.4) that $\text{G-dim}_R C < \infty$, and

$$\text{(3.6.5)} \qquad \text{G-dim}_R(\mathrm{Ext}_R^1(X,A)) < \infty.$$

On the other hand, applying the functor $\mathrm{Hom}_R(X,-)$ to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_R(X,\mathfrak{m}) \to X^* \to \mathrm{Hom}_R(X,k) \to \mathrm{Ext}_R^1(X,\mathfrak{m}) \to 0.$$

We have $\mathrm{Hom}_R(X,k) \cong \mathrm{Ext}_R^1(X,\mathfrak{m})$, hence $\mathrm{Ext}_R^1(X,\mathfrak{m})$ is a k-vector space. Since $\mathrm{Ext}_R^1(X,A)$ is contained in $\mathrm{Ext}_R^1(X,\mathfrak{m})$ by (3.6.3),

$$\text{(3.6.6)} \qquad \mathrm{Ext}_R^1(X,A) \text{ is a } k\text{-vector space.}$$

It follows from (3.6.4) and (3.6.6) that R is Gorenstein. Since the only number i such that $\mathrm{Ext}_R^i(k,R) \neq 0$ is the Krull dimension of R if R is Gorenstein, it follows from the above two claims that R is a Gorenstein local ring of dimension two, which completes the proof of the theorem. □

The above theorem interests us in the observation of a Gorenstein local ring of dimension two such that the second syzygy module of the residue class field is decomposable.

Theorem 3.7. *Let $(S,\mathfrak{n},k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that R is a Henselian Gorenstein ring of dimension two. Then the following conditions are equivalent:*

(1) $\Omega_R^2 k$ *is decomposable;*
(2) $\dim S = 3$ *and* $I = (xy - zf)$ *for some regular system of parameters* x, y, z *of* S *and* $f \in \mathfrak{n}$.

PROOF. (2) $\Rightarrow$ (1): We can show this implication by easy calculations.

(1) $\Rightarrow$ (2): First of all, note that the local ring R is not regular. We denote by $\mathfrak{m}$ the maximal ideal $\mathfrak{n}/I$ of R. It suffices to show the existence of an R-regular element $w \in \mathfrak{m} - \mathfrak{m}^2$ such that $\mathfrak{m}/(w)$ is decomposable. Let E denote the fundamental module of R. We can write $E = M \oplus N$ for some non-zero R-modules M and N. Hence the fundamental sequence of R is as follows:

$$\text{(a)} \qquad 0 \longrightarrow R \xrightarrow{\binom{\sigma}{\tau}} M \oplus N \xrightarrow{(f,g)} \mathfrak{m} \longrightarrow 0.$$

Take an R-regular element $w \in \mathfrak{m} - \mathfrak{m}^2$, and set $\overline{(-)} = (-) \otimes_R R/(w)$. If $\mathfrak{m}\overline{R}$ is decomposable, then our aim is attained. Hence let $\mathfrak{m}\overline{R}$ be indecomposable. The sequence (a) induces another exact sequence

$$0 \longrightarrow \overline{R} \xrightarrow{\binom{\overline{\sigma}}{\overline{\tau}}} \overline{M} \oplus \overline{N} \xrightarrow{(\overline{f},\overline{g})} \overline{\mathfrak{m}} \longrightarrow 0.$$

The natural surjection

$$\pi : \overline{\mathfrak{m}} \to \mathfrak{m}\overline{R}$$

is a split-epimorphism with kernel isomorphic to k. Hence there exists a split-monomorphism

$$\rho : \mathfrak{m}\overline{R} \to \overline{\mathfrak{m}}$$

such that $\pi\rho = 1$. Then note that the cokernel of ρ is isomorphic to k. On the other hand, the homomorphism $(\overline{f}, \overline{g})$ is a $\mathcal{G}(\overline{R})$-precover of $\overline{\mathfrak{m}}$. Therefore there exists a homomorphism

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \mathfrak{m}\overline{R} \to \overline{M} \oplus \overline{N}$$

such that $\rho = (\overline{f}, \overline{g})\binom{\alpha}{\beta} = \overline{f}\alpha + \overline{g}\beta$. Set $e = \operatorname{edim} R$, $m = \nu_R(M)$, and $n = \nu_R(N)$.

Claim 1. *We have either* $\left\{ \begin{array}{lcl} m & = & e-1 \\ n & = & 2 \end{array} \right\}$ *or* $\left\{ \begin{array}{lcl} m & = & 2 \\ n & = & e-1 \end{array} \right\}$.

On the other hand, we have

$$1 = \pi\rho = \pi\overline{f}\alpha + \pi\overline{g}\beta$$

in $\operatorname{End}_{\overline{R}}(\mathfrak{m}\overline{R})$. Since $\mathfrak{m}\overline{R}$ is indecomposable, the endomorphism ring $\operatorname{End}_{\overline{R}}(\mathfrak{m}\overline{R})$ is a local ring, and hence either $\pi\overline{f}\alpha$ or $\pi\overline{g}\beta$ is a unit of this ring, in other words, is an automorphism. Put $\mathfrak{a} = \operatorname{Im} f$ and $\mathfrak{b} = \operatorname{Im} g$.

Claim 2. *If $\pi \overline{f} \alpha$ (resp. $\pi \overline{g} \beta$) is an automorphism, then* $\mathfrak{m} = \mathfrak{a} + (w)$ *(resp.* $\mathfrak{m} = \mathfrak{b} + (w)$*) and there is an R-regular element in* $\mathfrak{a} - \mathfrak{m}^2$ *(resp.* $\mathfrak{b} - \mathfrak{m}^2$*).*

Claim 3. *We have both* $\operatorname{grade} \mathfrak{a} > 0$ *and* $\operatorname{grade} \mathfrak{b} > 0$.

Put $x = \sigma(1)$ and $y = \tau(1)$. Then $f(x) + g(y) = (f, g)\binom{\sigma}{\tau}(1) = 0$. Set $v = f(x) = -g(y) \in \mathfrak{a} \cap \mathfrak{b}$. Take an element $a \in \mathfrak{a} \cap \mathfrak{b}$. Then we have $a = f(p) = g(q)$ for some $p \in M$ and $q \in N$. Hence $\binom{p}{-q} \in \operatorname{Ker}(f, g) = \operatorname{Im}\binom{\sigma}{\tau}$, and therefore $\binom{p}{-q} = b\binom{x}{y}$ for some $b \in R$. Thus $p = bx$, and we get $a = f(p) = f(bx) = bv \in (v)$. It follows that

$$\mathfrak{a} \cap \mathfrak{b} = (v).$$

Since $\operatorname{grade}(v) = \operatorname{grade}(\mathfrak{a} \cap \mathfrak{b}) = \inf\{\operatorname{grade} \mathfrak{a}, \operatorname{grade} \mathfrak{b}\} > 0$ by Claim 3, the element v is an R-regular element.

Set $\overline{(-)} = (-) \otimes_R R/(v)$. Since $\mathfrak{a} + \mathfrak{b} = \mathfrak{m}$ and $\mathfrak{a} \cap \mathfrak{b} = (v)$, there is a natural exact sequence

$$\omega : 0 \to \overline{R} \to R/\mathfrak{a} \oplus R/\mathfrak{b} \to k \to 0$$

of $\overline{R}$-modules. Suppose that this exact sequence splits. Then we have an isomorphism

$$R/\mathfrak{a} \oplus R/\mathfrak{b} \cong \overline{R} \oplus k,$$

and it is seen from the Krull-Schmidt theorem that k is isomorphic to either $R/\mathfrak{a}$ or $R/\mathfrak{b}$. Hence we have either $\mathfrak{m} = \mathfrak{a}$ or $\mathfrak{m} = \mathfrak{b}$, and the same argument as the end of the proof of Claim 3 yields a contradiction. Thus the exact sequence ω does not split.

Dualizing the sequence ω, we obtain an exact sequence

$$\begin{array}{llll} & \operatorname{Hom}_{\overline{R}}(k, \overline{R}) \longrightarrow & \operatorname{Hom}_{\overline{R}}(R/\mathfrak{a} \oplus R/\mathfrak{b}, \overline{R}) \longrightarrow & \overline{R} \\ \xrightarrow{\ \zeta\ } & \operatorname{Ext}^1_{\overline{R}}(k, \overline{R}) \longrightarrow & \operatorname{Ext}^1_{\overline{R}}(R/\mathfrak{a} \oplus R/\mathfrak{b}, \overline{R}) \longrightarrow & 0, \end{array}$$

where the map ζ sends the identity of $\overline{R}$ to the non-zero element of $\operatorname{Ext}^1_{\overline{R}}(k, \overline{R})$ corresponding to the non-split exact sequence ω. Since $\operatorname{Hom}_{\overline{R}}(k, \overline{R}) = 0$ and $\operatorname{Ext}^1_{\overline{R}}(k, \overline{R}) \cong k$, it is observed from the above exact sequence that

$$\begin{aligned} \mathfrak{m}\overline{R} &= \operatorname{Ker} \zeta \\ &\cong \operatorname{Hom}_{\overline{R}}(R/\mathfrak{a} \oplus R/\mathfrak{b}, \overline{R}) \\ &\cong \operatorname{Hom}_{\overline{R}}(R/\mathfrak{a}, \overline{R}) \oplus \operatorname{Hom}_{\overline{R}}(R/\mathfrak{b}, \overline{R}) \end{aligned}$$

and $\operatorname{Ext}^1_{\overline{R}}(R/\mathfrak{a}, \overline{R}) \oplus \operatorname{Ext}^1_{\overline{R}}(R/\mathfrak{b}, \overline{R}) \cong \operatorname{Ext}^1_{\overline{R}}(R/\mathfrak{a} \oplus R/\mathfrak{b}, \overline{R}) = 0$. Hence $\operatorname{Ext}^1_{\overline{R}}(R/\mathfrak{a}, \overline{R}) = \operatorname{Ext}^1_{\overline{R}}(R/\mathfrak{b}, \overline{R}) = 0$, and thus $\operatorname{Ext}^i_{\overline{R}}(R/\mathfrak{a}, \overline{R}) = \operatorname{Ext}^i_{\overline{R}}(R/\mathfrak{b}, \overline{R}) = 0$ for every $i > 0$ because the self injective dimension of $\overline{R}$ is equal to one. It follows that both $R/\mathfrak{a}$ and $R/\mathfrak{b}$ belong to

$\mathcal{G}(\overline{R})$, hence they are reflexive over $\overline{R}$. Therefore the $\overline{R}$-dual modules $\mathrm{Hom}_{\overline{R}}(R/\mathfrak{a}, \overline{R})$ and $\mathrm{Hom}_{\overline{R}}(R/\mathfrak{b}, \overline{R})$ are non-zero, which proves that $\mathfrak{m}\overline{R}$ is decomposable. This completes the proof of our theorem. $\square$

Combining Theorem 3.6 with Theorem 3.7 gives birth to the following corollary. Compare it with Corollary 3.5.

Corollary 3.8. *Let $(R, \mathfrak{m}, k)$ be a complete local ring. Suppose that $\mathfrak{m}$ is indecomposable. Then the following conditions are equivalent:*

(1) *$\Omega_R^2 k$ has a non-zero proper direct summand of finite G-dimension;*
(2) *R is Gorenstein, and $\Omega_R^2 k$ is decomposable;*
(3) *There are a complete regular local ring $(S, \mathfrak{n})$ of dimension three, a regular system of parameters x, y, z of S, and $f \in \mathfrak{n}$ such that $R \cong S/(xy - zf)$.*

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A GENERALIZED HOCHSTER'S FORMULA FOR LOCAL COHOMOLOGIES OF MONOMIAL IDEALS

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ABSTRACT. The Hilbert series of local cohomologies for monomial ideals, which are not necessarily square-free, is established. As applications, we give a sharp lower bound of the non-vanishing degree of local cohomologies and also a sharp lower bound of the positive integer k of k-Buchsbaumness for generalized Cohen-Macaulay monomial ideals.

INTRODUCTION

Let K be a field and let $S = K[X_1, \ldots, X_n]$ be a polynomial ring with the standard grading. For a graded ideal $I \subset S$ we set $R = S/I$. We denote by x_i the image of X_i in R for $i = 1, \ldots, n$ and set $\mathfrak{m} = (x_1, \ldots, x_n)$, the unique graded maximal ideal. Also $H^i_{\mathfrak{m}}(R)$ denotes the local cohomology module of R with regard $\mathfrak{m}$.

The aim of this paper is to show a generalization of Hochster's formula on local cohomologies for square-free monomial ideals (Stanley-Reisner ideals) [4] to monomial ideals that are not necessarily square-free. The obtained formula, although its topological meaning is not clear as compared to the original formula, tells much about the non-vanishing degrees of the local cohomologies $H^i_{\mathfrak{m}}(R)$. In particular, we consider generalized Cohen-Macaulay monomial ideals.

A residue class ring R is called *generalized Cohen-Macaulay* ring, or simply *generalized CM*, if $H^i_{\mathfrak{m}}(R)$ has finite length for $i \neq \dim R$. In this case, we will call the ideal $I \subset S$ a *generalized CM ideal.* For a generalized CM ring R, there exists an integer $k \in \mathbb{Z}$, $k \geq 1$, such that $\mathfrak{m}^k H^i_{\mathfrak{m}}(R) = 0$ for $i \neq \dim R$. If this condition holds, we will also call R, or $I \subset S$, *k-Buchsbaum.* An ideal I is generelized CM if and only if it is k-Buchsbaum for some k. If I is k-Buchsbaum but not $(k-1)$-Buchsbaum, then we will call I *strictly k-Buchsbaum.* As a main application of our formula, we will give a sharp bound of k for the k-Buchsbaumness for a generalized CM monomial ideals.

For a finite set S we denote by $\mid S \mid$ the cardinarity of S, and, for sets A and B, $A \subset B$ means that A is a subset of B, which may be equal to A.

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1. Generalized Hochster's Formula

We first consider an extension of Hochster's formula on local cohomologies of Stanley-Reisner ideals.

Let $I \subset S$ be a monomial ideal, which is not necessarily square-free. Then we have

$$H^i_{\mathfrak{m}}(R) \cong H^i(C^\bullet)$$

where $C^\bullet$ is the Čech complex defined as follow:

$$C^\bullet : 0 \longrightarrow C^0 - \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0, \qquad C^t = \bigoplus_{1 \le i_1 < \cdots < i_t \le n} R_{x_{i_1} \cdots x_{i_t}}.$$

and the differenctial $C^t \longrightarrow C^{t+1}$ of this complex is induced by

$$(-1)^s nat : R_{x_{i_1} \cdots x_{i_t}} \longrightarrow R_{x_{j_1} \cdots x_{j_{t+1}}} \qquad \text{with } \{i_1, \ldots, i_t\} = \{j_1, \ldots, \hat{h}_s, \ldots, j_{t+1}\}$$

where nat is the natural homomorphism to localized rings and $R_{x_{i_1} \cdots x_{i_t}}$, for example, denotes localization by $x_{i_1}, \ldots, x_{i_t}$.

We can consider a $\mathbb{Z}^n$-grading to $H^i_{\mathfrak{m}}(R)$, $C^\bullet$ and $R_{x_{i_1} \cdots x_{i_t}}$ induced by the multi grading of S. See for example [3] for more detailed information about this complex.

Now we will consider the degree a subcomplex $C^\bullet_a$ of $C^\bullet$ for any $a \in \mathbb{Z}^n$. Before that we will prepare the notation. For a monomial ideal $I \subset S$, we denote by $G(I)$ the minimal set of monomial generators. Let $u = X_1^{a_1} \cdots X_n^{a_n}$ be a monimial with $a_i \ge 0$ for all i, then we define $\nu_j(u) = a_j$ for all $j = 1, \ldots, n$ and $\operatorname{supp}(u) = \{i \mid a_i \ne 0\}$. Now for $a \in \mathbb{Z}^n$, we set $G_a = \{i \mid a_i < 0\}$ and $H_a = \{i \mid a_i > 0\}$.

Lemma 1.1. *Let $x = x_{i_1} \cdots x_{i_r}$ with $i_1 < \cdots < i_r$ and set $F = \operatorname{supp}(x)$. For all $a \in \mathbb{Z}^n$ we have $\dim_K(R_x)_a \le 1$ and the following are equivalent*

(*i*) $(R_x)_a \cong K$
(*ii*) *$F \supset G_a$ and for all $u \in G(I)$ there exists $j \notin F$ such that $\nu_j(u) > a_j \ge 0$.*

Notice that the condition $a_i \ge 0$ in (*ii*) is redundant because it follows from the condition $F \supset G_a$. But it is written for the readers convenience.

Proof. The proof of $\dim_K(R_x)_a \le 1$ is verbatim the same as that of Lemma 5.3.6 (a) in [3]. Now we assume (*i*), i.e., $(R_x)_a \ne 0$. This is equivalent to the condition that there exists a monomial $\sigma \in R$ and $\ell \in \mathbb{N}$ such that

(*a*) $x^m \sigma \ne 0$ for all $m \in \mathbb{N}$, and
(*b*) $\deg \dfrac{\sigma}{x^\ell} = a$,

where deg denotes the multidegree. We know from (*b*) that we have $F \supset G_a$ because a negative degree $a_i (< 0)$ in a must come from the denominator of the fraction σ / x^ℓ and $F = \operatorname{supp}(x^\ell)$. Now we know that (*a*) is equivalent to the following condition: for all $u \in G(I)$ and for all $m \in \mathbb{N}$ we have $u \not| (X_{i_1}^m \cdots X_{i_r}^m)(X_1^{b_1} \cdots X_n^{b_n})$ where we set $\sigma = x_1^{b_1} \cdots x_n^{b_n}$ with some integers $b_j \ge 0$, $j = 1, \ldots, n$. This is equivalent to the following: for all $u \in G(I)$ there exists $i \notin F$ such that $\nu_i(u) > b_i$. Furthermore, we know from the condition $F \supset G_a$ that we have $a_i = b_i$ for $i \notin F$ since by (b) non-negative degrees in a must come from σ. Consequently we obtain (ii).

Now we show the converse. Assume that we have (ii). Set $\tau = \prod_{i \in H_a} x_i^{a_i}$ and $\rho = \prod_{i \in G_a} x_i^{-a_i}$. Then since $F \supset G_a$ there exists $\ell \in \mathbb{N}$ and a monomial σ (in R) such that

$$(1) \qquad x^\ell = \rho\sigma$$

We show that $\frac{\sigma\tau}{x^\ell} \neq 0$ in R_x. $\frac{\sigma\tau}{x^\ell} \neq 0$ is equivalent to the condition that $x^m(\sigma\tau) \neq 0$ for all $m \in \mathbb{N}$, as in the above discussion. This is equivalent to the condition

$$(2) \qquad \text{for all } u \in G(I) \text{ there exists } i \notin F \text{ such that } \nu_i(u) > b_i$$

where we set $\sigma\tau = x_1^{b_1} \cdots x_n^{b_n}$ for some integers $b_j \geq 0$, $j = 1, \ldots, n$. But by (1) we have $i \notin \mathrm{supp}(\sigma)$ for $i \notin F$, so that $b_i = \nu_i(\tau) = a_i (> 0)$ (i.e., $i \in H_a$) or $a_i = b_i = 0$ (i.e., $i \notin H_a \cup G_a$). Hence (2) is exactly the condition that for all $u \in G(I)$ there exists $j \notin F$ such that $\nu_j(u) > a_j \geq 0$, which is assured by the assumption. Thus we have $\frac{\sigma\tau}{x^\ell} \neq 0$ in R_x. Therefore

$$\deg \frac{\sigma\tau}{x^\ell} = \deg \frac{\sigma\tau}{\rho\sigma} = \deg \prod_{i \in H_a \cup G_a} x_i^{a_i} = \deg x^a = a$$

as required. □

Let $a \in \mathbb{Z}^n$. By Lemma 1.1 we see that $(C^i)_a$ has a K-linear basis

$$\{b_F : F \supset G_a, \text{ and for all } u \in G(I) \text{ there exists } j \notin F \text{ such that } \nu_j(u) > a_j \geq 0\}.$$

Restricting the differentation of $C^\bullet$ to the ath graded piece we obtain a complex $(C^\bullet)_a$ of finite dimensional K-vector spaces with differentation $\partial : (C^i)_a \longrightarrow (C^{i+1})_a$ given by $\partial(b_F) = \sum (-1)^{\sigma(F,F')} b_{F'}$ where the sum is taken over all F' such that $F' \supset F$ with $|F'| = i+1$ and for all $u \in G(I)$ there exists $j \notin F'$ such that $\nu_j(u) > a_j \geq 0$. Also we define $\sigma(F, F') = s$ if $F' = \{j_0, \ldots, j_i\}$ and $F = \{j_0, \ldots, \hat{j_s}, \ldots, j_i\}$. Then we describe the ath component of the local cohomology in terms of this subcomplex: $H^i_{\mathfrak{m}}(R)_a \cong H^i(C^\bullet)_a = H^i(C^\bullet_a)$.

Now we fix our notation on simplicial complex. A simplicial complex Δ on a finite set $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ such that $F \in \Delta$ whenever $F \subset G$ for some $G \in \Delta$. Notice that we do not assume the condition that $\{i\} \in \Delta$ for $i = 1, \ldots, n$. We define $\dim F = i$ if $| F |= i+1$ and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which will be called the dimension of Δ. If we assume a linear order on $[n]$, say $1 < 2 < \cdots < n$, then we will call Δ *oriented*, and in this case we always denote an element $F = \{i_1, \ldots, i_k\} \in \Delta$ with the orderd sequence $i_1 < \ldots < i_k$. For a given oriented simplicial complex of dimension $d-1$, we denote by $\mathcal{C}(\Delta)$ the augumented oriented chain complex of Δ:

$$\mathcal{C}(\Delta) : 0 \longrightarrow \mathcal{C}_{d-1} \xrightarrow{\partial} \mathcal{C}_{d-2} \longrightarrow \cdots \longrightarrow \mathcal{C}_0 \xrightarrow{\partial} \mathcal{C}_{-1} \longrightarrow 0$$

where

$$\mathcal{C}_i = \bigoplus_{F \in \Delta, \dim F = i} \mathbb{Z}F \qquad \text{and} \qquad \partial F = \sum_{j=0}^{i} (-1)^j F_j$$

for all $F \in \Delta$. Here we define $F_j = \{i_0, \ldots, \hat{i}_j, \ldots, i_k\}$ for $F = \{i_0, \ldots, i_k\}$. Now for an abelian group G, we define the ith reduced simplicial homology $\tilde{H}_i(\Delta; G)$ of Δ to be the ith homology of the complex $\mathcal{C}(\Delta) \otimes G$ for all i. Also we define the ith reduced simplicial cohomology of Δ $\tilde{H}^i(\Delta; G)$ to be the ith cohomology of the dual chain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta), G)$ for all i. Notice that we have

$$\tilde{H}_{-1}(\Delta; G) = \begin{cases} G & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{otherwise} \end{cases},$$

and if $\Delta = \emptyset$ then $\dim \Delta = -1$ and $\tilde{H}_i(\Delta; G) = 0$ for all i.

Now we will establish an isomorphism between the complex $(C^\bullet)_a$, $a \in \mathbb{Z}^n$, and a dual chain complex. For any $a \in \mathbb{Z}^n$, we define a simplicial complex

$$\Delta_a = \left\{ F - G_a \;\middle|\; \begin{array}{l} F \supset G_a \text{ and} \\ \text{for all } u \in G(I) \text{ there exists } j \notin F \text{ such that } \nu_j(u) > a_j \geq 0 \end{array} \right\}.$$

Notice that we may have $\Delta_a = \emptyset$ for some $a \in \mathbb{Z}^n$.

Lemma 1.2. *For all $a \in \mathbb{Z}^n$ there exists an isomorphism of complexes*

$$\alpha^\bullet : (C^\bullet)_a \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_a)[-j-1], K) \qquad j = |G_a|$$

where $\mathcal{C}(\Delta_a)[-j-1]$ means shifting the homological degree of $\mathcal{C}(\Delta_a)$ by $-j-1$.

Proof. The assignment $F \mapsto F - G_a$ induces an isomorphism $\alpha^\bullet : (C^\bullet)_a \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_a)[-j-1], K)$ of K-vector spaces such that $b_F \mapsto \varphi_{F-G_a}$, where

$$\varphi_{F'}(F") = \begin{cases} 1 & \text{if } F' = F" \\ 0 & \text{otherwise.} \end{cases}$$

That this is a homomorphism of complexes can be checked in a straightforward way. □

Now we come to our main theorem.

Theorem 1.1. *Let $I \subset S = K[X_1, \ldots, X_n]$ be a monomial ideal. Then the multigraded Hilbert series of the local cohomology modules of $R = S/I$ with respect to the $\mathbb{Z}^n$-grading is given by*

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) = \sum_{F \in \Delta} \sum \dim_K \tilde{H}_{i-|F|-1}(\Delta_a; K)\mathbf{t}^a$$

where $\mathbf{t} = t_1 \cdots t_n$, the second sum runs over $a \in \mathbb{Z}^n$ such that $G_a = F$ and $a_j \leq \rho_j - 1$, $j = 1, \ldots, n$, with $\rho_j = \max\{\nu_j(u) \mid u \in G(I)\}$ for $j = 1, \ldots, n$, and Δ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$.

Proof. By Lemma 1.2 and universal coefficient theorem for simplicial (co)homology, we have

$$\begin{aligned}\mathrm{Hilb}(H^i_{\mathfrak{m}}(R),\mathbf{t}) &= \sum_{a\in\mathbb{Z}^n} \dim_K H^i_{\mathfrak{m}}(R)_a \mathbf{t}^a \\ &= \sum_{a\in\mathbb{Z}^n} \dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a;K)\mathbf{t}^a.\end{aligned}$$

It is clear from the definition that $\Delta_a=\emptyset$ if for all $j\notin G_a$ we have $a_j\geq\rho_j$. In this case, we have $\dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a;K)=0$. Thus we obtain

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(R),\mathbf{t}) = \sum_{\substack{a\in\mathbb{Z}^n\\ a_j\leq\rho_j-1\\ j=1,\dots,n}} \dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a;K)\mathbf{t}^a.$$

Now if $\Delta_a\neq\emptyset$, we must have $(G_a-G_a=)\ \emptyset\in\Delta_a$, i.e., G_a must be a subset of $\{1,\dots,n\}$ such that for all $u\in G(I)$ there exists $j\notin G_a$ such that $\nu_j(u)>a_j\geq 0$, and this condition is equivalent to "$G_a\not\supset \mathrm{supp}(u)$ for all $u\in G(I)$", which can further be refined as "G_a is not a non-face of Δ, i.e., $G_a\in\Delta$". Thus we finally obtain the required formula. □

The original Hochster's formula is a special case of Theorem 1.1. For a simplicial complex Γ and $F\in\Gamma$, we define $\mathrm{lk}_\Gamma F=\{G|F\cup G\in\Gamma, F\cap G=\emptyset\}$ and $\mathrm{st}_\Gamma F=\{G|F\cup G\in\Gamma\}$.

Corollary 1.1 (Hochster). *Let Δ be a simplicial complex and let $K[\Delta]$ be the Stanley-Reisner ring corresponding to Δ. Then we have*

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(K[\Delta]),\mathbf{t}) = \sum_{F\in\Delta} \dim_K \tilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;K)\prod_{j\in F}\frac{t_j^{-1}}{1-t_j^{-1}}.$$

Proof. By Theorem 1.1 we have

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(R),\mathbf{t}) = \sum_{F\in\Delta}\sum_{\substack{a\in\mathbb{Z}^n_-\\ G_a=F}} \dim_K \tilde{H}_{i-|F|-1}(\Delta_a;K)\mathbf{t}^a$$

where $\mathbb{Z}^n_-=\{a\in\mathbb{Z}^n|a_j\leq 0 \text{ for } j=1,\dots,n\}$ and

$$\begin{aligned}\Delta_a &= \left\{F-G_a \;\middle|\; \begin{array}{l}F\supset G_a, \text{ and for all } u\in G(I) \text{ there exists } j\notin F\\ \text{such that } j\in\mathrm{supp}(u) \text{ and } j\notin H_a\cup G_a\end{array}\right\}.\\ &= \{F-G_a \mid F\supset G_a, \text{ and for all } u\in G(I) \text{ we have } H_a\cup F\not\subset \mathrm{supp}(u)\}.\\ &= \{L\mid L\cap G_a=\emptyset, L\cup G_a\cup H_a\in\Delta\} = \mathrm{lk}_{\mathrm{st}_\Delta H_a} G_a.\end{aligned}$$

Then the rest of the proof is exactly as in Theorem 5.3.8 [3]. □

2. Applications

In this section, we give some application of Theorem 1.1. We define $a_i(R) = \max\{j | H^i_{\mathfrak{m}}(R)_j \neq 0\}$ if $H^i_{\mathfrak{m}}(R) \neq 0$ and $a_i(R) = -\infty$ if $H^i_{\mathfrak{m}}(R) = 0$. Similarly, we define and $b_i(R) = \inf\{j | H^i_{\mathfrak{m}}(R)_j \neq 0\}$ if $H^i_{\mathfrak{m}}(R) \neq 0$ and $b_i(R) = +\infty$ if $H^i_{\mathfrak{m}}(R) = 0$.

Recall that $\rho_j = \max\{\nu_j(u) \mid u \in G(I)\}$ for $j = 1, \ldots, n$.

Corollary 2.1. *Let $I \subset S = K[X_1, \ldots, X_n]$ be a monomial ideal. Then $a_i(R) \leq \sum_{j=1}^{n} \rho_j - n$ for all i.*

Proof. By Theorem 1.1, the terms in $\mathrm{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t})$ with the highest total degree are at most $\dim_K \tilde{H}_{i-|F|-1}(\Delta_a; K)\mathbf{t}^a$ with $a_j = \rho_j - 1$ for all j. Thus the total degree is at most $\sum_j \rho_j - n$. □

From Corollary 2.1, we can recover the following well known result.

Corollary 2.2. *Let $I \subset S$ be a Stanley-Reisner ideal. Then $a_i(R) \leq 0$ for all i.*

Proof. If I is square-free, then $\rho_j \leq 1$ for $j = 1, \ldots, n$. □

For a Stanley-Reisner generalized Cohen-Macaulay ideal $I \subset S$ with $\dim R = d$, it is well known that it is Buchsbaum and $b_i(R) \geq 0$ for all $i (\neq d)$. The following theorem extends this result to monomial ideals in general.

Theorem 2.1. *Let $I \subset S = K[X_1, \ldots, X_n]$ be a generalized CM monomial ideal. Then $b_i(R) \geq 0$ for all $i < \dim R$.*

Proof. Let $d = \dim R$. Assume that there exists i and j with $0 \leq i < d$ and $j < 0$ such that $H^i_{\mathfrak{m}}(R)_j \neq 0$. Then by Theorem 1.1 there exists $a \in \mathbb{Z}^n$ such that

(*i*) $\sum_{k=1}^{n} a_k = j < 0$, in particular $G_a \neq \emptyset$, and

(*ii*) $\dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a; K) \neq 0$, in particular $\Delta_a \neq \emptyset$.

Now observe that by the definition of Δ_a, the coditions (*ii*) is independent of the values of a_j for $j \in G_a$. This means that the total degree $j = \sum_{k=1}^{n} a_k$ can be any negative integer so that $H^i_{\mathfrak{m}}(R)$ is not of finite length, which contradicts the assumption. □

Remark 1. By Kodaira Vanishing Theorem (Corollary 2.4 [5]), we have $b_i(R) \geq 0$ for $i \neq \dim R$ if $\mathrm{char}(K) = 0$ and R is a normal domain and has an isolated singularity at $\mathfrak{m}$. Theorem 2.1 is a case that is not covered by Kodaira Vanishing Theorem.

For a generalized Cohen-Macaulay ideal $I \subset S$ with $d = \dim R$, there exists some positive integer k such that $\mathfrak{m}^k H^i_{\mathfrak{m}}(R) = 0$ for all $i \neq d$. Then we refer R as a k-Buchsbaum ring. Now we consider the question: what is the lower bound of k?

Theorem 2.2. *Let $I \subset S = K[X_1, \ldots, X_n]$ be a generalized CM monomial ideal. Then $R = S/I$ is $\left(\sum_{j=1}^{n} \rho_j - n + 1\right)$-Buchsbaum.*

Proof. R is $\max_{i \neq d}(a_i(R) - b_i(R) + 1)$-Buchsbaum. The required result follows immediately from Corollary 2.1 and Theorem 2.1. □

We can immediately recover the following well known result.

Corollary 2.3. *Let $I \subset S$ be a Stanley-Reisner ideal. If R is generalized Cohen-Macaulay, then R is 1-Buchsbaum.*

In fact, it is well-known that a generalized Cohen-Macaulay Stanley-Reisner ideal is Buchsbaum, which is stronger than 1-Buchsbaumness.

The bound of k-Buchsbaumness given in Theorem 2.2 is best possible. In fact, we can construct strictly $(\sum_{j=1}^{n} \rho_j - n + 1)$-Buchsbaum ideals as in the following example.

Example 2.1. Let $I \subset S$ be a Stanley-Reisner Buchsbaum ideal. Notice that such ideals can be constructed with the method presented in [1] and $H^i_{\mathfrak{m}}(S/I)$ $(i \neq \dim R)$ is a K-vector space for $i \neq \dim R$.

Now consider a K-homomorphism

$$\varphi : S \longrightarrow S, \qquad X_i \longmapsto X_i^{a_i} \ (i = 1, \dots, n)$$

where $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $a_i \geq 1$ for $i = 1, \dots, n$. We define $\varphi(M) = M \otimes_S {}^{\varphi}S$ for a S-module M, where a left-right S-module ${}^{\varphi}S$ is equal to S as a set, it is a right S-module in the ordinary sense and its left S-module structure is determined by φ. Then we have

1. $\varphi(S/I) = S/\varphi(I)S$,
2. φ is an exact functor.

Thus, for $i \neq \dim R$, we have $H^i_{\mathfrak{m}}(S/\varphi(I)S) \cong \varphi(H^i_{\mathfrak{m}}(S/I))$ and since $H^i_{\mathfrak{m}}(S/I)$ is a direct sum of $S/\mathfrak{m}$, $H^i_{\mathfrak{m}}(S/\varphi(I)S)$ is a direct sum of $S/(X_1^{a_1}, \dots, X_n^{a_n})$. Then we know that $\mathfrak{m}^k H^i_{\mathfrak{m}}(S/I) = 0$ but $\mathfrak{m}^{k-1} H^i_{\mathfrak{m}}(S/I) \neq 0$ with $k = \sum_{j=1}^{n} \rho_j - n + 1 = \sum_{j=1}^{n} a_j - n + 1$.

Remark 2. Bresinsky and Hoa gave a bound for k-Buchsbaumness for ideals generated by monomials and binomials (Theorem 4.5 [2]). For monomial ideals our bound is stronger than that of Bresinsky and Hoa.

Finally, we consider vanishing cohomological dimensions of generalized CM monomial ideals. Recall that Castelnuovo-Mumford regularity of the ring R is defined by

$$\operatorname{reg}(R) = \max\{i + j \mid H^i_{\mathfrak{m}}(R)_j \neq 0\}.$$

Let $r = \operatorname{reg}(R)$. Then we have $H^i_{\mathfrak{m}}(R)_j = 0$ for $j > r - i$. Then we have

Corollary 2.4. *Let $I \subset S$ be a generalized CM monomial ideal with $d = \dim R$ and $r = \operatorname{reg}(R)$. Then $H^i_{\mathfrak{m}}(R) = 0$ for $r + 1 \leq i < d$. In particular, if I has a q-linear resolution, we have $H^i_{\mathfrak{m}}(R) = 0$ for $q \leq i < d$.*

Proof. First part is clear from Theorem 2.1. If R has a q-linear resolution, we have $\operatorname{reg}(R) = q - 1$. Thus the second statement also follows immediately. $\square$

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LOCAL COHOMOLOGIES OF THE CANONICAL MODULE

KAWASAKI, TAKESI

1. Introduction

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module of dimension d. A finitely generated A-module K_M is called the canonical module of M if $K_M \otimes \hat{A} \cong \operatorname{Hom}(H^d_{\mathfrak{m}}(M), E)$ where E is the injective envelope of the residue field $A/\mathfrak{m}$. Although the canonical module may not exist if A is not complete, it is unique up to isomorphism if it exists.

Local cohomologies of the canonical module are closely related to the Cousin complex of M [3]. We would like to compute them. It is known that K_M satisfies Serre's (S_2)-condition and that $\dim K_M = d$. Therefore K_M is Cohen-Macaulay if $d \leq 2$ and $H^0_{\mathfrak{m}}(K_M) = H^1_{\mathfrak{m}}(K_M) = 0$ if $d > 2$. If M has finite local cohomologies, that is, $H^p_{\mathfrak{m}}(M)$ is of finite length for $0 \leq p < d$, then Schenzel [10] showed that

$$H^p_{\mathfrak{m}}(K_M) \cong \operatorname{Hom}(H^{d-p+1}_{\mathfrak{m}}(M), E)$$

for $2 \leq p < d$. In 1986, Suzuki [11] gave a generalization of isomorphisms above by using an unconditioned strong d-sequence. If M has a system of parameters which is an uncoditioned strong d-sequence, then Suzuki's result gives another proof of Schenzel's isomorphisms. However, a finitely generated module M has finite local cohomologies if it has a system of parameters for M which is an unconditioned strong d-sequence on M. Therefore we need a generalized notion of unconditioned strong d-sequences. We give a notion of p-standard sequence in Section 2. By using it, we can describe the main theorem.

Theorem 1.1. *If A is complete, $d \geq 2$ and $x_1, \ldots, x_d$ a system of parameters for M which is a p-standard sequence on M of some type, then*

$$H^p_{\mathfrak{m}}(K_M) \cong \operatorname{Hom}(H^0_{x_{p-1}} H^{d-p+1}_{(x_p,\ldots,x_d)}(M), E)$$

for $2 \leq p < d$.

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This is not in final form. The detailed version of this paper will be submitted for publication elsewhere.

This theorem will be proven in Section 3.

2. P-STANDARD SEQUENCES

Let A be a commutative ring and M an A-module.

Definition 2.1. A sequence $x_1, \dots, x_d$ in A is said to be a d-sequence [6] on M if

$$(x_1, \dots, x_{i-1})M : x_i x_j = (x_1, \dots, x_{i-1})M : x_j$$

whenever $1 \le i \le j \le d$. The sequence $x_1, \dots, x_d$ is said to be a strong d-sequence on M if $x_1^{n_1}, \dots, x_d^{n_d}$ is a d-sequence on M for any integers $n_1, \dots, n_d > 0$. The sequence $x_1, \dots, x_d$ is said to be an unconditioned strong d-sequence [5] on M if it is a strong d-sequence on M in any order.

We note that the sequence $x_1, \dots, x_d$ is an unconditioned strong d-sequence on M if and only if

$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_i^{n_i} x_j^{n_j} = (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_j^{n_j}.$$

for any integers $n_1, \dots, n_d > 0$, any subset $\Lambda \subsetneq \{1, \dots, d\}$ and i, $j \in \{1, \dots, d\} \setminus \Lambda$.

We give a weaker notion.

Definition 2.2. A sequence $x_1, \dots, x_d$ in A is said to be an unconditioned p-sequence [1, Definition 2.1] on M if

$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_i^{n_i} = (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_i$$

for any integers $n_1, \dots, n_d > 0$, any subset $\Lambda \subsetneq \{1, \dots, d\}$ and $i \in \{1, \dots, d\} \setminus \Lambda$. Let $0 \le s < d$ be an integer. The sequence $x_1, \dots, x_d$ is said to be a p-standard sequence [7] of type s on M if

$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_i^{n_i} x_j^{n_j} = (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_j^{n_j}$$

for any integes $n_1, \dots, n_d > 0$, any subset $\Lambda \subsetneq \{1, \dots, d\}$ and i, $j \in \{1, \dots, d\} \setminus \Lambda$ such that $i \le j$ or $j > s$.

Such sequences fulfilled many good properties. Next two propositions are firstly proved by Goto and Yamagishi [5, Propositions 2.3 and 2.4] for unconditioned strong d-sequences. However their proofs still work under weaker assumption.

Proposition 2.3. *Let $x_1, \dots, x_d$ be an unconditioed p-sequence on M, $n_1, \dots, n_d > 1$ integers, $\Lambda \subsetneq \{1, \dots, d\}$ and $i \in \{1, \dots, d\} \setminus \Lambda$. Then*

$$(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M : x_i = \sum_{\Lambda' \subset \Lambda} \left(\prod_{\lambda \in \Lambda'} x_\lambda^{n_\lambda - 1} \right) [(x_\lambda \mid \lambda \in \Lambda')M : x_i].$$

Here we set $(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda') = (0)$ and $\prod_{\lambda \in \Lambda'} x_\lambda^{n_\lambda - 1} = 1$ if $\Lambda' = \emptyset$.

Proposition 2.4. *Let $x_1, \ldots, x_d$ be a p-standard sequence on M of any type and $n_1, \ldots, n_d, m_1, \ldots, m_d > 0$ integers. Then*

$$(x_1^{n_1+m_1}, \ldots, x_d^{n_d+m_d})M : x_1^{m_1} \cdots x_d^{m_d}$$
$$= \sum_{i=1}^{d} (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{i+1}^{n_{i+1}}, \ldots, x_d^{n_d})M : x_i + (x_1^{n_1}, \ldots, x_d^{n_d})M.$$

Let $\boldsymbol{x} = x_1, \ldots, x_d$ be a sequence on A. The local cohomology of M with respect to $\boldsymbol{x}$ is the direct limit of Koszul cohomologies

$$H^p_{\boldsymbol{x}}(M) = \varinjlim{}_n H^p(x_1^n, \ldots, x_d^n; M).$$

If A is Noetherian, then $H^p_{\boldsymbol{x}}(M)$ is equal to the local cohomology of M with respect to the ideal $(x_1, \ldots, x_d)$.

The next proposition is a generalization of [5, Theorem 3.9].

Proposition 2.5. *Let $x_1, \ldots, x_d$ be a strong d-sequence on M. Then*

$$H^p_{\boldsymbol{x}}(M) = \operatorname{inj}\lim_{n} \frac{(x_1^n, \ldots, x_p^n)M : x_{p+1}}{(x_1^n, \ldots, x_p^n)M}$$

for $p < d$.

Next we consider the existence of p-standard sequences. We assume that A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module of dimension $d > 0$. For any A-module N, let

$$\mathfrak{a}(N) = \prod_{0 \leq p < \dim N} \operatorname{ann} H^p_{\mathfrak{m}}(N).$$

Definition 2.6. Let $0 \leq s < d$ be an integer. A system of parameters $x_1, \ldots, x_d$ for M is called a p-standard system of parameters [2, 8] of type s if

$$x_{s+1}, \ldots, x_d \in \mathfrak{a}(M)$$

and

$$x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$$

for $i \leq s$.

It is known that there is a p-standard system of parameters for M of type $d-1$ if A has a dualizing complex [1, p. 482] or if A is universally catenary, all the formal fibers of A are Cohen-Macaulay and M is equidimensional [9, Theorem 2.5].

Theorem 2.7. *A p-standard system of parameters for M of type s is a p-standard sequence on M of type s.*

Proof. See [7]. □

Thus, many finitely generated modules have a system of parameters which is a p-standard sequence on M.

3. THE PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. Let A be a Noetherian complete local ring with maximal ideal $\mathfrak{m}$, E the injective envelope of $A/\mathfrak{m}$ and M a finitely generated A-module of dimension $d > 0$. Let $\boldsymbol{x} = x_1$, ..., x_d be a system of parameters for M which is p-standard sequence on M of some type. First we prove the following proposition. It is the dual of well-known result. See the proof of [4, Lemma 2.2].

Proposition 3.1. *$H_0(\boldsymbol{x}; H^d_{\boldsymbol{x}}(M)) = 0$. If $d > 1$, then $H_1(\boldsymbol{x}; H^d_{\boldsymbol{x}}(M)) = 0$.*

Proof. Since $x_1[0:_M x_1] = 0$ and $x_1[M/x_1M] = 0$, we obtain exact sequences

$$H^{d-1}_{\boldsymbol{x}}(M/x_1M) \longrightarrow H^d_{\boldsymbol{x}}(M/0:_M x_1) \longrightarrow H^d_{\boldsymbol{x}}(M) \longrightarrow 0$$

and

$$0 \longrightarrow H^d_{\boldsymbol{x}}(M) \longrightarrow H^d_{\boldsymbol{x}}(M/0:_M x_1) \longrightarrow 0.$$

Therefore

$$H^{d-1}_{\boldsymbol{x}}(M/x_1M) \longrightarrow H^d_{\boldsymbol{x}}(M) \xrightarrow{x_1} H^d_{\boldsymbol{x}}(M) \longrightarrow 0$$

is exact and hence $x_1[H^d_{\boldsymbol{x}}(M)] = H^d_{\boldsymbol{x}}(M)$. Thus $H_0(\boldsymbol{x}; H^d_{\boldsymbol{x}}(M)) = 0$.

If $d > 1$, then we can also show that

$$x_2[H^{d-1}_{x_2,\dots,x_d}(M/x_1M)] = H^{d-1}_{x_2,\dots,x_d}(M/x_1M).$$

Because of the spectral sequence

$$E_2^{pq} = H^p_{x_1} H^q_{x_2,\dots,x_d}(M/x_1M) \Rightarrow H^{p+q}_{\boldsymbol{x}}(M/x_1M),$$

we obtain

$$x_2[H^{d-1}_{\boldsymbol{x}}(M/x_1M)] = H^{d-1}_{\boldsymbol{x}}(M/x_1M).$$

Therefore $x_2[0:_{H^d_{\boldsymbol{x}}(M)} x_1] = 0:_{H^d_{\boldsymbol{x}}(M)} x_1$. Thus we obtain

$$H_p(x_1, x_2; H^d_{\boldsymbol{x}}(M)) = 0 \quad \text{for } p = 0, 1.$$

The spectral sequence

$$E^2_{pq} = H_p(x_3, \dots, x_d; H_q(x_1, x_2; H^d_{\boldsymbol{x}}(M))) \Rightarrow H_{p+q}(\boldsymbol{x}; H^d_{\boldsymbol{x}}(M))$$

gives $H_1(\boldsymbol{x}; H^d_{\boldsymbol{x}}(M)) = 0$. □

Since x_1, ..., x_d is a p-standard sequence on M of some type, we obtain

Proposition 3.2. *$H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M)$ is finitely generated if $1 \leq k < d$.*

Proof. Since

$$\begin{aligned} H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M) &= H^0_{x_k}(\operatorname{inj}\lim_n M/(x^n_{k+1},\dots,x^n_d)M) \\ &= \operatorname{inj}\lim_n H^0_{x_k}(M/(x^n_{k+1},\dots,x^n_d)M) \\ &= \operatorname{inj}\lim_n (x^n_{k+1},\dots,x^n_d)M : x_k/(x^n_{k+1},\dots,x^n_d)M, \end{aligned}$$

there are homomorphisms

$$\varphi_n\colon (x^n_{k+1},\dots,x^n_d)M : x_k \to H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M)$$

such that $H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M) = \bigcup_n \operatorname{Im}\varphi_n$, $\varphi_n((x^n_{k+1},\dots,x^n_d)M) = 0$ and

$$\varphi_n(a) = \varphi_m((x_{k+1}\cdots x_d)^{m-n}a)$$

if $m > n$ and $a \in (x^n_{k+1},\dots,x^n_d)M : x_k$.

Let $a \in (x^n_{k+1},\dots,x^n_d)M : x_k$. By using Proposition 2.3, we can put

$$a = \sum_{\Lambda\subset\{k+1,\dots,d\}} \left(\prod_{\lambda\in\Lambda} x^{n-1}_\lambda\right) a_\Lambda$$

where $a_\Lambda \in (x_\lambda \mid \lambda \in \Lambda)M : x_k$. If $\Lambda \neq \{k+1,\dots,d\}$ and $j \in \{k+1,\dots,d\} \setminus \Lambda$, then

$$a_\lambda \in (x_\lambda \mid \lambda \in \Lambda)M : x_k \subset (x_\lambda \mid \lambda \in \Lambda)M : x_k x_j = (x_\lambda \mid \lambda \in \Lambda)M : x_j$$

and hence $x_{k+1}\cdots x_d a_\Lambda \in (x^2_\lambda \mid \lambda \in \Lambda)M$. Therefore

$$\begin{aligned} \varphi_n(a) &= \varphi_1(a_{\{k+1,\dots,d\}}) + \sum_{\Lambda'\subsetneq\{k+1,\dots,d\}} \varphi_{n+1}((x_{k+1}\cdots x_d)\left(\prod_{\lambda\in\Lambda} x^{n-1}_\lambda\right) a_\lambda) \\ &= \varphi_1(a_{\{k+1,\dots,d\}}). \end{aligned}$$

Thus $H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M) = \operatorname{Im}\varphi_1$ is finitely generated. □

We prove the following poposition by descending induction on k. The case that $k = 1$ is Theorem 1.1.

Proposition 3.3. *If* $1 \le k < d$, *then*

$$H^p_{\mathfrak{m}}(\operatorname{Hom}(H^{d-k+1}_{x_k,\dots,x_d}(M), E)) \cong \operatorname{Hom}(H^0_{x_{k+p-2}} H^{d-k-p+2}_{x_{k+p-1},\dots,x_d}(M), E)$$

for $2 \le p < d-k+1$.

Proof. If $k = d-1$, then there is no p such that $2 \le p < d-k+1$. Assume that $k < d-1$ and the theorem is true for larger values of k. There is an exact sequence

$$\begin{aligned} 0 \to H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M) \to H^{d-k}_{x_{k+1},\dots,x_d}(M) \to \\ [H^{d-k}_{x_{k+1},\dots,x_d}(M)]_{x_k} \to H^1_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M) \to 0. \end{aligned}$$

Because of the spectral sequence

$$E_2^{pq} = H^p_{x_k} H^q_{x_{k+1},\dots,x_d}(M) \Rightarrow H^{p+q}_{x_k,\dots,x_d}(M),$$

we obtain an isomorphism

$$H^1_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M) \cong H^{d-k+1}_{x_k,\dots,x_d}(M).$$

Therefore we have two exact sequences

$$0 \to \operatorname{Hom}(H^{d-k+1}_{x_k,\dots,x_d}(M), E) \to \operatorname{Hom}([H^{d-k}_{x_k,\dots,x_d}(M)]_{x_k}, E) \to L \to 0$$

$$0 \to L \to \operatorname{Hom}(H^{d-k}_{x_{k+1},\dots,x_d}(M), E) \to \operatorname{Hom}(H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M), E) \to 0.$$

Since $H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M)$ is finitely generated, $\operatorname{Hom}(H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M), E)$ is Artinian. Therefore

$$H^p_{\mathfrak{m}}(\operatorname{Hom}(H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M), E)) = 0$$

for all $p > 0$. By using induction hypothesis, we obtain

$$H^p_{\mathfrak{m}}(L) = \begin{cases} 0, & p = 0; \\ \operatorname{Hom}(H^0_{x_k} H^{d-k}_{x_{k+1},\dots,x_d}(M), E), & p = 1; \\ \operatorname{Hom}(H^0_{x_{k+p-1}} H^{d-k-p+1}_{x_{k+p},\dots,x_d}(M), E), & 2 \le p < d-k. \end{cases}$$

Since the multiplication of x_k on $\operatorname{Hom}([H^{d-k}_{x_{k+1},\dots,x_d}(M)]_{x_k}, E)$ is an isomorphism, $H^p_{\mathfrak{m}}(\operatorname{Hom}([H^{d-k}_{x_{k+1},\dots,x_d}(M)]_{x_k}, E)) = 0$ for all p. Therefore

$$H^p_{\mathfrak{m}}(\operatorname{Hom}(H^{d-k+1}_{x_k,\dots,x_d}(M), E))$$
$$= \begin{cases} 0, & p = 0, 1; \\ \operatorname{Hom}(H^0_{x_{k+p-2}} H^{d-k-p+2}_{x_{k+p-1},\dots,x_d}(M), E), & 2 \le p < d-k+1. \end{cases}$$

□

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1

Remarks on transitivity of exceptional sequences

荒谷　督司

この講演を通して k を標数 0 の代数的閉体とする。$\mathcal{C}$ を enough projectives (もしくは enough injectives) なアーベリアン k-圏とし、 $\mathcal{D}^b(\mathcal{C})$ を $\mathcal{C}$ の有界導来圏とする。

鎖複体 $E^\bullet \in \mathcal{D}^b$ が *exceptional* であるとは、$\mathrm{RHom}(E^\bullet, E^\bullet) \cong k$ をみたすことであり、exceptional の列 $\epsilon = (\cdots, E_i^\bullet, E_{i+1}^\bullet, \cdots)$ が *exceptional sequence* であるとは、$\mathrm{RHom}(E_i^\bullet, E_j^\bullet) = 0\ (\forall i > j)$ をみたすことである。

exceptional sequence の概念は、A. L. Gorodentsev と A. N. Rudakov によって導入され ([5])、A. I. Bondal によって一般化されている ([3])。そして、$\mathcal{C}$ が hereditary k-algebra A 上の有限生成左 A-加群のなす圏 mod$A$ または、weighted projective line $\mathbf{X}$ 上の coherent sheaf のなす圏 coh($\mathbf{X}$) のときには、以下のことが知られている。

任意の exceptional sequence ϵ に対し、ϵ の長さは $\mathcal{C}$ の Grothendieck group のランク n 以下である。そして、ϵ の長さが n のとき、exceptional sequence ϵ を *complete* とよぶ。

complete exceptional sequence は $D(\mathcal{C})$ を三角圏として生成する。

complete exceptional sequence 全体の集合にはブレイド群が推移的に作用している。

本公演では、$R = \bigoplus_{n \in \mathbb{N}} R_n$ を 1-次元次数付き Gorenstein 環で、有限 Cohen-Macaulay 表現型であるもの。$\mathcal{C} = \mathrm{mod}R$ を射が次数を保つもののみの次数付き有限生成 R-加群のなす圏の場合を考える。

1 Quivers

この章では、主結果を述べるための準備をする。Q を頂点の数が n の Dynkin quiver とし、$\Gamma = \mathbb{Z}Q$ を translation quiver とする。また、Γ_0 を Γ の頂点集合、τ を translation とする。

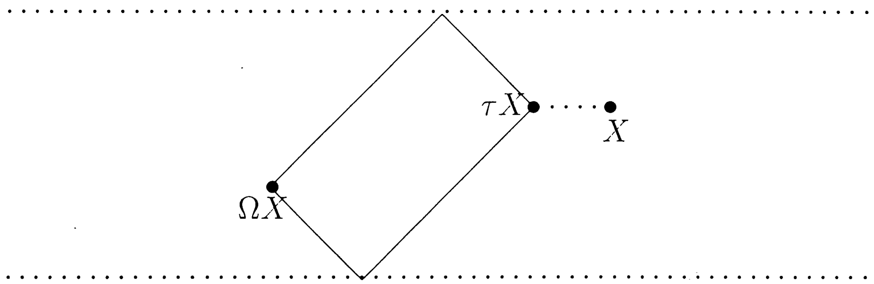
[1]The detailed version of this paper will be submitted for publication elsewhere.

定義 1.1 $X, Y \in \Gamma_0$ とする。

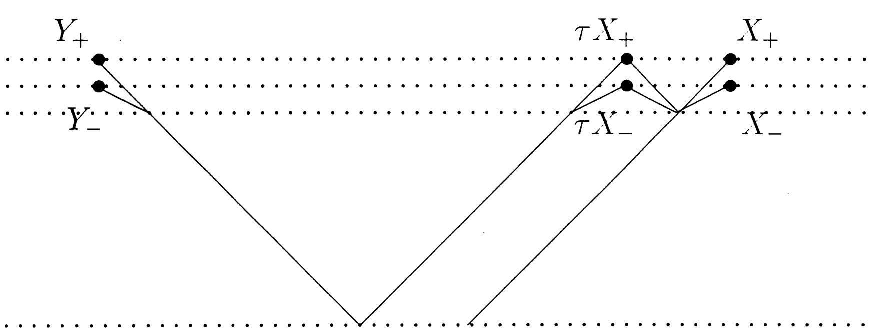
1. X から Y へ arrow があるとき、$X \lessdot Y$ と表す。
2. X から Y へ path があるとき、$X < Y$ と表す。
3. X と Y の間に path がないとき、$X \nsim Y$ と表す。

定義 1.2 各 $\Gamma = \mathbb{Z}Q$ に対し、syzygy functor $\Omega : \Gamma_0 \to \Gamma_0$ を以下のように定義する。

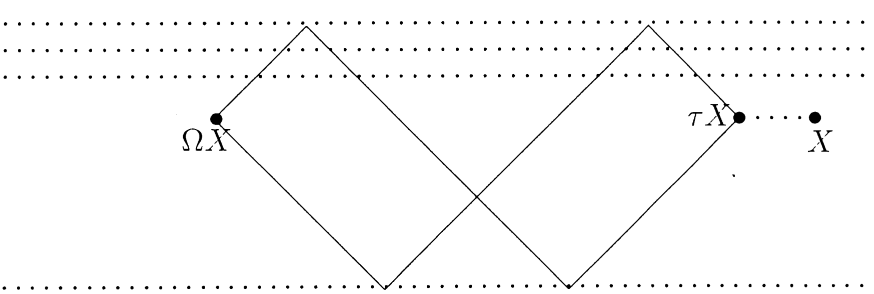
$\Gamma = \mathbb{Z}\mathbf{A}_n$ のとき、



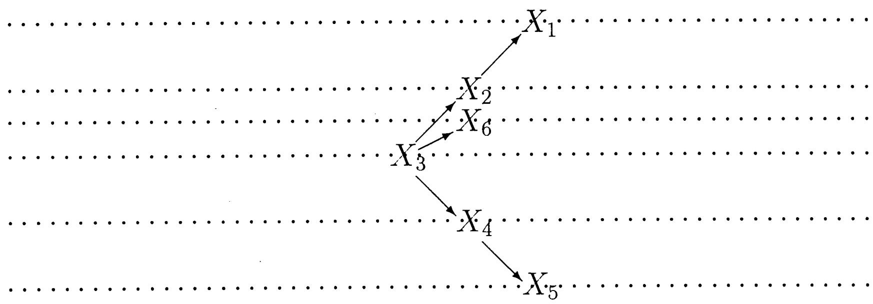
$\Gamma = \mathbb{Z}\mathbf{D}_n$ のとき、



n が奇数のとき、$\Omega X_\pm = Y_\mp$ とし、n が偶数のとき、$\Omega X_\pm = Y_\pm$ とする。一般の X に対しては以下のように定義する。



$\Gamma = \mathbb{Z}\mathbf{E}_6$ のとき、



各 X_i に対し、$\Omega X_1 = \tau^6 X_5$, $\Omega X_2 = \tau^6 X_4$, $\Omega X_3 = \tau^6 X_3$, $\Omega X_4 = \tau^6 X_2$, $\Omega X_5 = \tau^6 X_1$, $\Omega X_6 = \tau^6 X_6$ とする。

$\Gamma = \mathbb{Z}\mathbf{E}_7$ のとき、各 X に対し $\Omega X = \tau^9 X$ とする。

$\Gamma = \mathbb{Z}\mathbf{E}_8$ のとき、各 X に対し $\Omega X = \tau^{15} X$ とする。

$R = \bigoplus_{n\in\mathbb{N}} R_n$ を 1-次元次数付き Gorenstein 環で、$R_0 = k$ とする。さらに有限 Cohen-Macaulay 表現型であるとする。mod$R$ を有限生成次数付き R-加群のなす圏で射は次数を保つものとする。さらに CMR を極大 CM 加群全体のなす充満部分圏とする。このとき、CMR の Auslander-Reiten quiver の射影加群でない極大 CM 加群全体から得られる full subquiver を Γ とおくと、$\Gamma = \mathbb{Z}Q$ である (c.f. [1])。上で定義している Ω は、この状況での syzygy 加群に対応している。

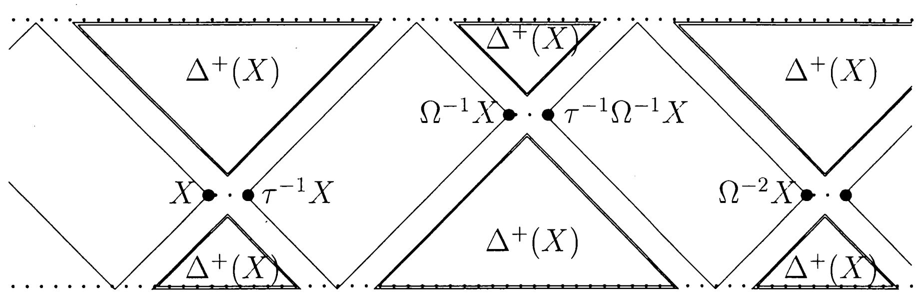
定義 1.3 $S \subset \Gamma_0$ が *slice* であるとは、以下の二条件をみたすことである。

1. 任意の $X \in \Gamma_0$ に対し、$\tau^\ell X \in S$ をみたす整数 ℓ がただ一つ存在する。
2. 任意の $X \in S$ と任意の $X \lessdot Y$ なる $Y \in \Gamma_0$ に対し、$Y \in S$ または $\tau Y \in S$ である。

定義 1.4 各頂点 $X \in \Gamma_0$ に対し、$\mathrm{S}^+(X)$, $\mathrm{S}^-(X)$, $\mathrm{S}'^+(X)$, $\mathrm{S}'^-(X)$, $\Delta(X)$, $\Delta^+(X)$, $\Delta^-(X)$ を以下のように定義する。

1. $\mathrm{S}^+(X)$ は slice であり、$Y \in \mathrm{S}^+(X)$ ならば $X \leq Y$ である。
2. $\mathrm{S}^-(X)$ は slice であり、$Y \in \mathrm{S}^-(X)$ ならば $Y \leq X$ である。
3. $\mathrm{S}'^*(X) = \mathrm{S}^* \setminus \{X\}$ とする。（ここで、$* = +, -$ である）
4. $\Delta(X) = \{\ Y \in \Gamma_0 \mid X \nsim Y\ \}$ とする。
5. $\Delta^+(X) = \bigcup_{\ell \geq 0} (\Delta(\Omega^{-\ell}X) \cup \mathrm{S}'^+(\Omega^{-\ell}X))$ とする。
6. $\Delta^-(X) = \bigcup_{\ell \leq 0} (\Delta(\Omega^{-\ell}X) \cup \mathrm{S}'^-(\Omega^{-\ell}X))$ とする。

$\Gamma = \mathbb{Z}\mathbf{A}_n$ のとき、各 $X \in \Gamma_0$ に対し、$\Delta^+(X)$ は以下のような位置関係にあることに注意する。



定義 1.5 頂点 $E_1, E_2, \cdots, E_r \in \Gamma_0$ に対し、列 $\epsilon = (E_1, E_2, \cdots, E_r)$ が *strongly exceptional sequence* であるとは、次の条件をみたすことである。

$$E_i \in \bigcap_{j<i} \Delta^+(E_j)\ (1 < i \leq r)$$

この条件は次の条件と同値である。

$$E_i \in \bigcap_{j>i} \Delta^-(E_j)\ (1 \leq i < r)$$

この章の主結果を述べる前にもう少し準備をする。

定義 1.6 $\epsilon = (E_1, E_2, \cdots, E_r)$ を strongly exceptional sequence とする。このとき、$E_i \lessdot_\epsilon E_j$ であるとは、以下の二条件をみたすことである。

1. $E_i \in S'^-(E_j)$ である。
2. もし $E_i \leq E_\ell \leq E_j$ ならば $\ell = i$ または j である。

補題 1.7 $\epsilon = (E_1, E_2, \cdots, E_r)$ を strongly exceptional sequence とする。このとき、次のどの列も strongly exceptional sequence になる。

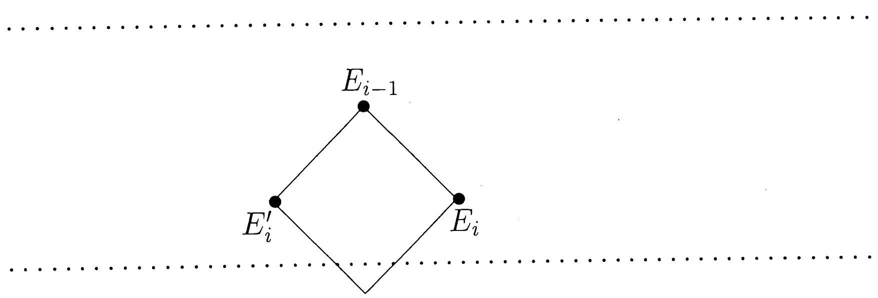
1. $E_{i-1} \nsim E_i$ のとき、$(E_1, E_2, \cdots, E_{i-2}, E_i, E_{i-1}, E_{i+1}, \cdots, E_r)$
2. $E_j \notin \mathrm{S}'^{-}(E_i)$ $(\forall j)$ のとき、$(E_1, E_2, \cdots, E_{i-1}, \Omega E_i, E_{i+1}, \cdots, E_r)$
3. $E_j \notin \mathrm{S}'^{+}(E_i)$ $(\forall j)$ のとき、$(E_1, E_2, \cdots, E_{i-1}, \Omega^{-1} E_i, E_{i+1}, \cdots, E_r)$

さらに $\Gamma = \mathbb{Z}\mathbf{A}_n$ のときは、以下の列も strongly exceptional sequence になる。

4. $E_j \lessdot_\epsilon E_i$ ならば $j = i-1$ のとき、

$$(E_1, E_2, \cdots, E_{i-2}, E'_i, E_{i-1}, E_{i+1}, \cdots, E_r)$$

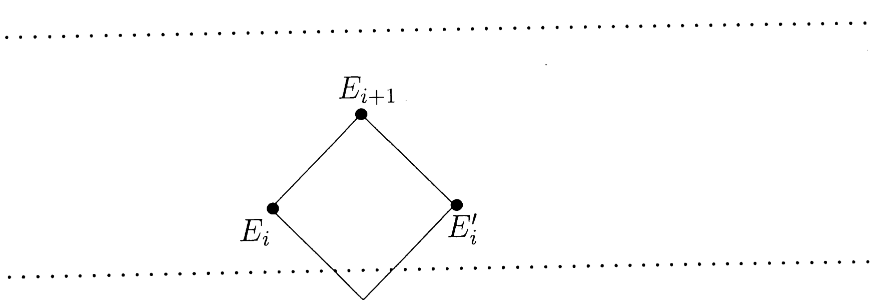
但し、E'_i は以下のように定義されたものである。



5. $E_i \lessdot_\epsilon E_j$ ならば $j = i+1$ のとき、

$$(E_1, E_2, \cdots, E_{i-2}, E'_i, E_{i-1}, E_{i+1}, \cdots, E_r)$$

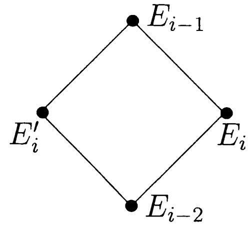
但し、E'_i は以下のように定義されたものである。



6. $E_{i-2} \lessdot_\epsilon E_i$, $E_{i-1} \lessdot_\epsilon E_i$ のとき、

$$(E_1, E_2, \cdots, E_{i-3}, E'_i, E_{i-2}, E_{i-1}, E_{i+1}, \cdots, E_r)$$

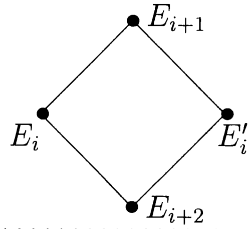
但し、E'_i は以下のように定義されたものである。



7. $E_i \lessdot_\epsilon E_{i+1}$, $E_i \lessdot_\epsilon E_{i+2}$ のとき、

$$(E_1, E_2, \cdots, E_{i-1}, E_{i+1}, E_{i+2}, E'_i, E_{i+3}, \cdots, E_r)$$

但し、E'_i は以下のように定義されたものである。



定義 1.8 ϵ, ϵ' を strongly exceptional sequence とする。ϵ' が ϵ に補題 1.7 の変形を有限回繰り返すことで得られるとき、$\epsilon \sim \epsilon'$ と表す。

次の定理がこの章の主結果である。

定理 1.9 ϵ を長さ r の strongly exceptional sequence とする。このとき、長さ r の strongly exceptional sequence ϵ' と slice S で、次の条件をみたすものが存在する。

1. $\epsilon \sim \epsilon'$ である。
2. ϵ' は S に埋め込むことができる。

この定理より、次の系が導かれる。

系 1.10 ϵ を長さ r の strongly exceptional sequence とする。このとき、

以下のことが成立している。

1. $r \leq n$ である。
2. ϵ を長さ n の strongly exceptional sequence に拡張することができる。

系 1.11 $\Gamma = \mathbb{Z}\mathbf{A}_n$ とする。このとき、長さ n の任意の strongly exceptional sequence ϵ, ϵ' に対し $\epsilon \sim \epsilon'$ である。

2 可換環への応用

この章を通して $R = \bigoplus_{i \geq 0} R_i$ を次数付き環で、$R_0 = k$ を標数 0 の代数的閉体とする。また、R は有限 Cohen-Macaulay 表現型であるとする。mod$R$ で射が次数を保つものである有限生成次数付き R-加群のなす圏とし、CMR で極大 Cohen-Macaulay 加群のなす充満部分圏とする。Γ' を CMR の Auslander-Reiten quiver とする。Γ_0 を射影加群でない直既約極大 Cohen-Macaulay 加群全体からなる頂点集合とし、Γ を Γ_0 から得られる full subquiver とする。このとき、ある n 個の頂点集合の Dynkin quiver Q が存在し、$\Gamma = \mathbb{Z}Q$ であることに注意する。

定義 2.1 有限生成 R-加群 E が *exceptional* であるとは以下の条件をみたすことである。

1. $\mathrm{Hom}(E, E) \cong k$ である。
2. $\mathrm{Ext}^\ell(E, E) = 0$ $(\ell > 0)$ である。

また、exceptional 加群の列 $\epsilon = (\cdots, E_i, E_{i+1}, \cdots)$ が *exceptional sequence* であるとは $\mathrm{Ext}^\ell(E_i, E_j) = 0$ $(i > j, \ell \geq 0)$ をみたすことである。

exceptional 加群に関して次のことがわかっている (c.f.[1],[2])。

補題 2.2

1. 任意の直既約極大 CM 加群は exceptional である。
2. 直既約極大 CM 加群 X, Y に対し、$\mathrm{Hom}(X, Y) \neq 0$ ならば $X \leq Y$ (in Γ) である。

3. 直既約極大 CM 加群 X, Y と正の整数 ℓ に対し、$\mathrm{Ext}^\ell(X,Y) \neq 0$ ならば $\tau^{-1}\Omega^{-\ell+1}Y \leq X \leq \Omega^{-\ell}Y$ (in Γ) である。

直既約極大 CM 加群 X, Y に対し、 $X \in \Delta^+(Y)$ と $Y \in \Delta^-(X)$ は同値である。さらにこのとき、補題 2.2 より、すべての整数 ℓ に対し、$\mathrm{Ext}^\ell(X,Y) = 0$ である。したがって、ϵ が 1 章の意味で strongly exceptional sequence であるならば、加群の意味でも exceptional sequence になっていることに注意する。

定義 2.3 $\epsilon = (\cdots, E_i, E_{i+1}, \cdots)$ を CMR での exceptional sequence とする。このとき、$\underline{\epsilon}$ を ϵ の中で Γ_0 に属するもの全体から得られる full subsequence とする。

1. ϵ が形式的長さが r の *strongly exceptional sequence* であるとは、$\underline{\epsilon}$ が 1 章の意味で長さ r の strongly exceptional sequence であることである。
2. ϵ が *complete* であるとは、以下の二条件をみたすことである。
   (a) $\underline{\epsilon}$ は長さ n の exceptional sequence である。
   (b) 任意の直既約射影加群は ϵ に属している。

定理 1.9 および系 1.10 より次の定理が得られる。

定理 2.4 ϵ を形式的長さが r の strongly exceptional sequence とする。このとき、以下のことが成立する。

1. $r \leq n$ である。
2. ϵ を complete strongly exceptional sequence に拡張することができる。

さらに以下のことも導かれる。

系 2.5 $\epsilon = (\cdots, E_i, E_{i+1}, \cdots)$ を strongly exceptional sequence とすると、ϵ は三角圏として $\mathcal{D}^b(\mathrm{mod}R)$ を生成する。すなわち、すべての E_i を含む $\mathcal{D}^b(\mathrm{mod}R)$ の最小の三角圏が $\mathcal{D}^b(\mathrm{mod}R)$ である。

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ASYMPTOTIC STABILITY OF PRIMES ASSOCIATED TO HOMOGENEOUS COMPONENTS OF MULTIGRADED MODULES

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1. Introduction

Let R be a commutative Noetherian ring and let I be an ideal in R. In 1979, M. Brodmann showed that for any finitely generated R-module M, $\mathrm{Ass}_R(M/I^nM)$ is stable for all large $n \gg 0$ ([1]). In this report, I would like to extend this result and give a few applications. Our main result is the following.

Theorem 1.1. *Let $A \subseteq B$ be a homogeneous inclusion of Noetherian standard $\mathbb{N}^r$-graded rings with $A_0 = B_0 = R$ and N a finitely generated $\mathbb{N}^r$-graded A-module and M a finitely generated $\mathbb{N}^r$-graded B-module. Assume that N is a graded A-submodule of M. Then $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}})$ is stable for all large $\mathbf{n} \gg \mathbf{0}$. That is, there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}}) = \mathrm{Ass}_R(M_{\mathbf{k}}/N_{\mathbf{k}})$ for all $\mathbf{n} \geq \mathbf{k}$.*

Here we use the partial order on $\mathbb{N}^r$ as follows: If $\mathbf{n} = (n_1, \ldots, n_r)$ and $\mathbf{k} = (k_1, \ldots, k_r)$, then $\mathbf{n} \geq \mathbf{k}$ if and only if $n_i \geq k_i$ for all $i = 1, \ldots, r$.

This is a natural generalization of Brodmann's result. Let $J_1 \subseteq I_1, \ldots, J_r \subseteq I_r$ be ideals of R. Let E be a finitely generated R-module and F a submodule of E. Then, applying Theorem 1.1 to the case where A (resp. B) to be the multi-Rees ring of ideals $I_1, \ldots, I_r$ (resp. $J_1, \ldots, J_r$) and N (resp. M) to be the multi-Rees module of F (resp. E) with respect to ideals $I_1, \ldots, I_r$ (resp. $J_1, \ldots, J_r$), we have the following Corollary.

Corollary 1.2. $\mathrm{Ass}_R(I_1^{n_1} \cdots I_r^{n_r} E / J_1^{n_1} \cdots J_r^{n_r} F)$ *is stable for all large* $\mathbf{n} = (n_1, \ldots, n_r) \gg \mathbf{0}$.

Brodmann's result is in the case where $r = 1, I_1 = R$ and $F = E$. Corollary 1.2 is also a generalization of results in [4] and [5].

We shall give a proof of Theorem 1.1 in Section 2. In Section 3, we will give a few applications of Theorem 1.1. Let $(R, \mathfrak{m})$ be a local ring and I an ideal in R. In [2], Brodmann showed that $\operatorname{depth}_R(R/I^n)$ is constant for all large $n \gg 0$ by using the stability of $\operatorname{Ass}_R(R/I^n)$, and improved the Burch's inequality for the analytic spread of ideals. Following the method in [2], we can extend these results to the case of homogeneous extension of multigraded rings.

2. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. Let $A \subseteq B$ be a homogeneous inclusion of Noetherian standard $\mathbb{N}^r$-graded rings with $A_0 = B_0 = R$ and N a finitely generated $\mathbb{N}^r$-graded A-module and M a finitely generated $\mathbb{N}^r$-graded B-module. Assume that N is a graded A-submodule of M. We begin with the following.

Proposition 2.1. *With the above situation, there exists a vector* $\mathbf{k}$ *such that* $\operatorname{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}}) \subseteq \operatorname{Ass}_R(M_{\mathbf{n}+\mathbf{e}_j}/N_{\mathbf{n}+\mathbf{e}_j})$ *for all* $\mathbf{n} \geq \mathbf{k}$ *and all* $j = 1, \ldots, r$, *where* $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^r$ *is the standard basis element of* $\mathbb{N}^r$.

Proof. First, since N is a finitely generated over A, there exits a vector $\mathbf{t} \in \mathbb{N}^r$ such that $N_{\mathbf{n}+\mathbf{t}} = A_{\mathbf{n}} N_{\mathbf{t}}$ for all $\mathbf{n} \geq \mathbf{0}$. Let $F = B \cdot N_{\mathbf{t}} \subseteq M$ be a B-submodule of M generated by $N_{\mathbf{t}}$ and let $J_j = A_{\mathbf{e}_j} B \subseteq B$ be an ideal in B generated by $A_{\mathbf{e}_j}$ $(j = 1, \ldots, r)$. Then we have the following.

Claim There exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that

(1) $[(0) :_M (B_{\mathbf{e}_j})]_{\mathbf{n}} = (0)$,

(2) $J_1^{n_1} \cdots J_j^{n_j+1} \cdots J_r^{n_r} F :_M J_j \subseteq J_1^{n_1} \cdots J_r^{n_r} F + \mathrm{H}^0_{J_j}(M)$,

(3) $J_1^{n_1} J_2^{n_2} \cdots J_r^{n_r} F \cap \mathrm{H}^0_{J_j}(M) = (0)$,

for all $\mathbf{n} \geq \mathbf{k}$ and all $j = 1, 2, \ldots, r$.

We want to show that $\operatorname{Ass}_R(M_{\mathbf{n}+\mathbf{t}}/N_{\mathbf{n}+\mathbf{t}}) \subseteq \operatorname{Ass}_R(M_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}/N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j})$ for all $\mathbf{n} \geq \mathbf{k}$ and all $j = 1, \ldots, r$. Take $\mathfrak{p} \in \operatorname{Ass}_R(M_{\mathbf{n}+\mathbf{t}}/N_{\mathbf{n}+\mathbf{t}})$ and fix an integer $1 \leq j \leq r$. To show that $\mathfrak{p} \in \mathrm{Ass}_R(M_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}/N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j})$, we may assume that $(R, \mathfrak{p})$ is local. Write $\mathfrak{p} = N_{\mathbf{n}+\mathbf{t}} :_R h$ for some $h \in M_{\mathbf{n}+\mathbf{t}}$. We consider the following two cases.

Case 1 $A_{\mathbf{e}_j} \cdot h \not\subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$.
Let a be an element in $A_{\mathbf{e}_j}$ such that $ah \notin N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$. Then

$$\mathfrak{p}(ah) = a(\mathfrak{p}h) \subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j},$$

and so we have $\mathfrak{p} \subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j} :_R ah \neq R$. Hence we get the equality $\mathfrak{p} = N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j} :_R ah$.

Case 2 $A_{\mathbf{e}_j} \cdot h \subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$.
Suppose $A_{\mathbf{e}_j} \cdot h \subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$. Then $J_j \cdot h \subseteq J_1^{n_1} \cdots J_j^{n_j+1} \cdots J_r^{n_r} F$. Hence we have

$$\begin{aligned} h &\in J_1^{n_1} \cdots J_j^{n_j+1} \cdots J_r^{n_r} F :_M J_j \\ &\subseteq J_1^{n_1} \cdots J_r^{n_r} F + \mathrm{H}^0_{J_j}(M) \quad \text{by Claim (2).} \end{aligned}$$

We write $h = h_1 + h_2$ with $h_1 \in J_1^{n_1} \cdots J_r^{n_r} F$ and $h_2 \in \mathrm{H}^0_{J_j}(M)$. Since h is a homogeneous element of degree $\mathbf{n}+\mathbf{t}$, we may assume that $h_1 \in [J_1^{n_1} \cdots J_r^{n_r} F]_{\mathbf{n}+\mathbf{t}} = N_{\mathbf{n}+\mathbf{t}}$. So, we may assume that $h \in \mathrm{H}^0_{J_j}$. Assume that $B_{\mathbf{e}_j} \cdot h \subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$. Then we have

$$B_{\mathbf{e}_j} \cdot h \subseteq J_1^{n_1} \cdots J^{n_j+1} \cdots J_r^{n_r} F \cap \mathrm{H}^0_{J_j}(M) = (0),$$

by Claim (3). Hence $h \in [(0) :_M (B_{\mathbf{e}_j})]_{\mathbf{n}+\mathbf{t}} = (0)$ by Claim (1) and so we have $h = 0$. This is a contradiction. Therefore $B_{\mathbf{e}_j} \cdot h \not\subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$. Let $b \in B_{\mathbf{e}_j}$ with $bh \notin N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j}$. Since

$$\mathfrak{p}(bh) = b(\mathfrak{p}h) \subseteq J_1^{n_1} J_2^{n_2} \cdots J_r^{n_r} F,$$

we have $\mathfrak{p}(bh) \subseteq J_1^{n_1} J_2^{n_2} \cdots J_r^{n_r} F \cap \mathrm{H}^0_{J_j}(M) = (0)$ by Claim (3). Hence

$$\mathfrak{p} \subseteq (0) :_R bh \subseteq N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j} :_R bh \neq R,$$

and so we have the equality $\mathfrak{p} = N_{\mathbf{n}+\mathbf{t}+\mathbf{e}_j} :_R bh$. □

Proof of Theorem 1.1. By Proposition 2.1, there exists $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ such that $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}}) \subseteq \mathrm{Ass}_R(M_{\mathbf{n}+\mathbf{e}_j}/N_{\mathbf{n}+\mathbf{e}_j})$ for all $\mathbf{n} \geq \mathbf{k}$ and all

$j = 1, \dots, r$. Let $k_0 = \max\{k_1, \dots, k_r\}$ and set $\mathbf{k_0} = (k_0, \dots, k_0)$. Then we have an increasing sequence

$$\mathrm{Ass}_R(M_{\mathbf{k_0}}/N_{\mathbf{k_0}}) \subseteq \mathrm{Ass}_R(M_{\mathbf{k_0+1}}/N_{\mathbf{k_0+1}}) \subseteq \dots,$$

where $\mathbf{1} = (1, \dots, 1)$. Therefore it is enough to show that

Claim the set $\bigcup_{m>0} \mathrm{Ass}_R(M_{(m,\dots,m)}/N_{(m,\dots,m)})$ is finite.

Let $\mathcal{F}(m) = \mathrm{Ass}_R(M_{(m,\dots,m)}/N_{(m,\dots,m)})$ for any $m > 0$. Since N is finitely generated over A, there exists $\ell > 0$ such that

$$\begin{aligned} N_{(m,\dots,m)} &= A_{(m-\ell,\dots,m-\ell)} N_{(\ell,\dots,\ell)} \\ &= [J^{m-\ell} L]_{(m,\dots,m)} \end{aligned}$$

for any $m \geq \ell$, where $J = J_1 \cdots J_r$ and $L = B \cdot N_{(\ell,\dots,\ell)}$. So, for any large $m \gg 0$, we have $M_{(m,\dots,m)}/N_{(m,\dots,m)} = [M/J^{m-\ell}L]_{(m,\dots,m)}$. Hence we have

$$\begin{aligned} \mathcal{F}(m) &\subseteq \mathrm{Ass}_R(M/J^{m-\ell}L) \\ &= \{P \cap R \mid P \in \mathrm{Ass}_B(M/J^{m-\ell}L)\} \end{aligned}$$

for $m \gg 0$. It is easy to see that the set $\bigcup_{n>0} \mathrm{Ass}_B(M/J^n L)$ is finite, and so we have Claim. □

3. Asymptotic depth and spread

In this section, we will give a few consequences of Theorem 1.1. First of all, by using the stability of $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}})$, we can prove the following.

Theorem 3.1. *Let A, B, N, M and R be the same as in Theorem 1.1. Then for any ideal $\mathfrak{a}$ in R,* $\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}})$ *is constant for all large* $\mathbf{n} \gg \mathbf{0}$. *That is, there exists a vector* $\mathbf{k} \in \mathbb{N}^r$ *such that* $\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}}) = \mathrm{grade}(\mathfrak{a}, M_{\mathbf{k}}/N_{\mathbf{k}})$ *for all* $\mathbf{n} \geq \mathbf{k}$.

Proof. We may assume that $\mathfrak{a} \neq R$. For any $\mathbf{k} \in \mathbb{N}^r$, we put

$$c_{\mathbf{k}} = \inf\{\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}}) \mid \mathbf{n} \geq \mathbf{k}\}.$$

Fix any increasing sequence $\mathbf{k}^{(1)} < \mathbf{k}^{(2)} < \dots < \mathbf{k}^{(m)} < \dots$ of $\mathbb{N}^r$ and put $c = \lim_{m\to\infty} c_{\mathbf{k}^{(m)}}$. If $c = \infty$, then $\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}}) = \infty$ for all large

$\mathbf{n} \gg \mathbf{0}$. Assume $c < \infty$. We proceed by induction on c. By Theorem 1.1, there exists $\mathbf{k}^{(j)}$ such that

$$\mathrm{Ass}_R(M_\mathbf{n}/N_\mathbf{n}) = \mathrm{Ass}_R(M_{\mathbf{k}^{(j)}}/N_{\mathbf{k}^{(j)}}) \quad \text{for all } \mathbf{n} \geq \mathbf{k}^{(j)}.$$

Let $\mathcal{F} = \mathrm{Ass}_R(M_{\mathbf{k}^{(j)}}/N_{\mathbf{k}^{(j)}})$. When $c = 0$. Then there exists $\mathbf{m} \geq \mathbf{k}^{(j)}$ such that grade$(\mathfrak{a}, M_\mathbf{m}/N_\mathbf{m})$=0, because $c_{\mathbf{k}^{(j)}} = 0$. Hence

$$\mathfrak{a} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M_\mathbf{m}/N_\mathbf{m})} \mathfrak{p} = \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}.$$

This implies grade$(\mathfrak{a}, M_\mathbf{n}/N_\mathbf{n}) = 0$ for all $\mathbf{n} \geq \mathbf{k}^{(j)}$. Suppose $c > 0$. Then grade$(\mathfrak{a}, M_\mathbf{n}/N_\mathbf{n}) > 0$ for all $\mathbf{n} \geq \mathbf{k}^{(j)}$. Hence

$$\mathfrak{a} \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M_\mathbf{n}/N_\mathbf{n})} \mathfrak{p} \quad \text{for all } \mathbf{n} \geq \mathbf{k}^{(j)}.$$

So, we can choose $x \in \mathfrak{a}$, which is a non-zerodivisor on $M_\mathbf{n}/N_\mathbf{n}$ for all $\mathbf{n} \geq \mathbf{k}^{(j)}$. Set $A' = (A + xB)/xB \subseteq B' = B/xB$ and $N' = (N + xM)/xM \subseteq M' = M/xM$. We put $c'_\mathbf{k} = \inf\{\mathrm{grade}(\mathfrak{a}, M'_\mathbf{n}/N'_\mathbf{n}) \mid \mathbf{n} \geq \mathbf{k}\}$ and let $c' = \lim_{m\to\infty} c'_{\mathbf{k}(m)}$. Then $c' = c-1$, because x is a non-zerodivisor on $M_\mathbf{n}/N_\mathbf{n}$ for all large $\mathbf{n} \gg \mathbf{0}$. By induction hypothesis, replacing by large $\mathbf{k} \gg \mathbf{0}$ if necessary, there exists $\mathbf{k} \geq \mathbf{k}^{(j)}$ such that

$$\mathrm{grade}(\mathfrak{a}/xR, M'_\mathbf{n}/N'_\mathbf{n}) = \mathrm{grade}(\mathfrak{a}/xR, M'_\mathbf{k}/N'_\mathbf{k}) \quad \text{for all } \mathbf{n} \geq \mathbf{k}.$$

Hence we have grade$(\mathfrak{a}, M_\mathbf{n}/N_\mathbf{n}) =$ grade$(\mathfrak{a}, M_\mathbf{k}/N_\mathbf{k})$ for all $\mathbf{n} \geq \mathbf{k}$. □

By Theorem 3.1, when R is local, depth$_R(B_\mathbf{n}/A_\mathbf{n})$ is constant for all large $\mathbf{n} \gg \mathbf{0}$. We denote by depth$(A, B)$ this constant value and call it *the asymptotic depth* of A and B. The asymptotic depth is related to the spread of A and B. The spread of a multigraded ring is introduced by Kirby and Rees [6]. Let G be a Noetherian standard $\mathbb{N}^r$-graded ring with $G_\mathbf{0} = (R, \mathfrak{m})$ local. Then the spread $s(G)$ of G is defined to be

$$\max\left\{\mathrm{ht}_{G/\mathfrak{m}G}\, P \mid P \text{ is a graded prime of } G/\mathfrak{m}G,\ [G/\mathfrak{m}G]_+ \not\subseteq P\right\} + 1,$$

where $[G/\mathfrak{m}G]_+ = \bigoplus_{n_1,\ldots,n_r>0}[G/\mathfrak{m}G]_{(n_1,\ldots,n_r)}$ is the irrelevant ideal of $G/\mathfrak{m}G$. When $[G/\mathfrak{m}G]_+ \subseteq \sqrt{(0)}$, we define $s(G) = 0$. When $r = 1$, one can check that $s(G) = \dim G/\mathfrak{m}G$.

With this notation, we have the following.

Theorem 3.2. *Let $A \subseteq B$ be a homogeneous inclusion of Noetherian standard $\mathbb{N}^r$-graded rings with $A_0 = B_0 = R$ local. Let d be the dimension of R. Then*

$$s(A) \leq s(B) + d - \operatorname{depth}(A, B),$$

if $\operatorname{depth}(A, B)$ *is finite.*

Proof. By [6, Lemma 1.7], the spread of a multigraded ring is equal to the spread of its diagonal subring. Also, the asymptotic depth is equal to the asymptotic depth of homogeneous extension of its diagonal subrings. So we may assume that $r = 1$. Let $c = \operatorname{depth}(A, B)$. We want to show that

$$\dim A/\mathfrak{m}A \leq \dim B/\mathfrak{m}B + d - c.$$

We proceed by induction on c. Since

$$\begin{aligned} \dim A/\mathfrak{m}A &\leq \dim A \\ &\leq \dim B \\ &\leq d + \dim B/\mathfrak{m}B, \end{aligned}$$

it is true when $c = 0$. Suppose $c > 0$. Then

$$\mathfrak{m} \not\subseteq \left(\bigcup_{Q \in \operatorname{Assh} R} Q \right) \cup \left(\bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p} \right),$$

where $\mathcal{F}$ is the stable value of $\operatorname{Ass}_R(B_n/A_n)$ for sufficiently large $n \gg 0$. Let $a \in \mathfrak{m}$ such that a is a non-zerodivisor on B_n/A_n for all large $n \gg 0$ and $a \notin Q$ for every $Q \in \operatorname{Assh} R$. Set $A' = A/(aB \cap A) \subseteq B' = B/aB$. Then $\operatorname{depth}(A', B') = c - 1$. By induction hypothesis, we have

$$\begin{aligned} \dim A'/\mathfrak{m}A' &\leq \dim B'/\mathfrak{m}B' + (d-1) - (c-1) \\ &\leq \dim B/\mathfrak{m}B + d - c. \end{aligned}$$

We then claim that $\dim A/\mathfrak{m}A = \dim A'/\mathfrak{m}A'$. To see this, it is enough to show that $\sqrt{\mathfrak{m}A} = \sqrt{\mathfrak{m}A + (aB \cap A)}$. Suppose $x \in \sqrt{\mathfrak{m}A + (aB \cap A)}$. We may assume that x is homogeneous. Then $x^\ell \in \mathfrak{m}A + (aB \cap A)$ for some $\ell > 0$. When $\deg x = 0$. Since $x^\ell \in [\mathfrak{m}A + (aB \cap A)]_0 = \mathfrak{m}$, we get $x \in \sqrt{\mathfrak{m}A}$. When $t = \deg x > 0$. Replacing by large $\ell \gg 0$ if necessary, we may assume that a is a non-zerodivisor on $B_{t\ell}/A_{t\ell}$. Hence we have

$$\begin{aligned} x^{\ell} &\in [\mathfrak{m}A + (aB + A)]_{t\ell} \\ &= \mathfrak{m}A_{t\ell} + (aB_{t\ell} \cap A_{t\ell}) \\ &= \mathfrak{m}A_{t\ell} + aA_{t\ell} \\ &= \mathfrak{m}A_{t\ell}. \end{aligned}$$

Therefore $x \in \sqrt{\mathfrak{m}A}$. This completes the proof. □

This bound is sharp in general. In fact, there is the following example.

Example 3.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local domain and $T = R[X, Y]$ a polynomial ring over R. Let $B = T/(XY)$ and denote by x, y the reduction of $X, Y \bmod (XY)$ respectively. Let $A = R[x + y]$ be a subring of B. Then $\operatorname{depth}(A, B) = d = \dim R$ and $s(A) = s(B) = 1$. So, we have the equality $s(A) = s(B) + d - \operatorname{depth}(A, B)$.*

But, in more special case, we can obtain the following better bound by one more.

Theorem 3.4. *Assume that B is a polynomial ring over R and $r = 1$. Then*

$$s(A) \leq s(B) + d - \operatorname{depth}(A, B) - 1,$$

if $\operatorname{depth}(A, B)$ is finite.

Proof. We first prove that $\dim A/\mathfrak{m}A \leq \dim B - 1$. Since A is a subring of the polynomial ring $B = R[t_1, \ldots, t_n]$,

$$\operatorname{Min}(A) = \{\mathfrak{p}B \cap A \mid \mathfrak{p} \in \operatorname{Min}(R)\}.$$

Consequently, if $d > 0$, then we see that $\operatorname{ht}_A(\mathfrak{m}A) > 0$. Hence we have

$$\begin{aligned} \dim A/\mathfrak{m}A &\leq \dim A - 1 \\ &\leq \dim B - 1. \end{aligned}$$

When $d = 0$. Note that $\dim A \leq \dim B$. Assume $\dim A/\mathfrak{m}A = \dim B$. Then $\dim A/(\mathfrak{m}B \cap A) = \dim A = \dim B$. This implies $[A/(\mathfrak{m}B \cap A)]_1 = [B/\mathfrak{m}B]_1$ and so we have $B_1 = \mathfrak{m}B_1 + A_1$. Hence $B_1 = A_1$

by Nakayama's lemma. This contradicts depth(A, B) is finite. Hence we have $\dim A/\mathfrak{m}A \leq \dim B - 1$. Let $c = \text{depth}(A, B)$. Since

$$\begin{aligned} s(A) = \dim A/\mathfrak{m}A &\leq \dim B - 1 \\ &= s(B) + d - 1, \end{aligned}$$

it is true when $c = 0$. Suppose $c > 0$. Then, by the same argument as in a proof of Theorem 3.2, it follows that

$$\dim A/\mathfrak{m}A \leq \dim B/\mathfrak{m}B + d - c - 1.$$

□

Theorem 3.4 is a generalization of Burch's inequality for the analytic spread of ideals. If you take $A = \mathcal{R}(I)$ to be the Rees ring of an ideal I of R and $B = R[t]$ to be the polynomial ring over R in Theorem 3.4, you have the following inequality which is a well-known result of Burch ([3]).

Corollary 3.5 (L. Burch). *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and I an ideal of R. Then*

$$\lambda(I) \leq \dim R - \inf_{n>0}\{\text{depth}_R(R/I^n)\},$$

where $\lambda(I) = \dim(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I))$ the analytic spread of I.

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Buchsbaumness of Rees algebras and associated graded rings with respect to socle ideals of subsystems of parameters in Buchsbaum local rings

Shiro Goto and Hideto Sakurai

1 Introduction.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $0 < r < d$ be an integer. Let $a_1, \ldots, a_r, b \in \mathfrak{m}$ be a subsystem of parameters for A. Let $Q = (a_1, \ldots, a_r)$ and let $I = Q :_A \mathfrak{m}$. Let R be the Rees algebra $\mathrm{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ with respect to I, where t is an indeterminateness. Let $M = \mathfrak{m}R + R_+$ and let $f_i = a_i t$ $(1 \leq i \leq r)$. Let G be the associated graded ring $\mathrm{G}(I) = \mathrm{R}(I)/I\mathrm{R}(I)$ with respec to I.

The purpose of this report is to explore the following questions.

Question 1.1. (1) *When is Q a reduction of I?*

(2) *When does the equality $I^2 = QI$ hold true?*

(3) *Determine the structures of* R *and* G.

When $r = d$, that is, when Q is a parameter ideal, the following results are known, in which $\mathrm{e}^0_{\mathfrak{m}}(A)$ denotes the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$.

Fact 1.2. *The following statements hold true.*

(1) [*GS1*] *Let $\mathfrak{q}$ be a parameter ideal in A. Assume that $\mathrm{e}^0_{\mathfrak{m}}(A) > 1$. Then $\mathfrak{q} :_A \mathfrak{m} \subseteq \overline{\mathfrak{q}}$, where $\overline{\mathfrak{q}}$ denotes the integral closure of $\mathfrak{q}$.*

(2) [*GS1, GS3*] *Assume that* A *is Buchsbaum and* $\mathrm{e}^0_{\mathfrak{m}}(A) > 1$. *Let* $\mathfrak{q}$ *be a parameter ideal. Assume that some assumptions (e.g.,* $\ell_A([\mathfrak{q} :_A \mathfrak{m}]/\mathfrak{q}) = \mathrm{r}(A)$*). Then the equality* $J^2 = \mathfrak{q}J$ *holds true, where* $J = \mathfrak{q} :_A \mathfrak{m}$.

(3) [*Y1, Y2, GN*] *Assume that* A *is Buchsbaum. Let* $\mathfrak{q}$ *be a parameter ideal, and assume that* $J^2 = \mathfrak{q}J$ *,where* $J = \mathfrak{q} :_A \mathfrak{m}$. *Then* $R(J)$ *and* $G(J)$ *are Buchsbaum.*

So we have satisfactory results for parameter ideals. Then in this report we suppose that $0 < r < d$.

In Section 2 we will discuss subsystems of parameters in an arbitrarily Noetherian local ring and give the answer to Question 1.1 (1) (See, Proposition 2.1). Proposition 2.1 shows that Q is always a reduction of I.

In Section 3 and 4 we will discuss subsystems of parameters in a Buchsbaum local ring.

In Section 3 we will give the answer to Question 1.1 (2) in the case where A is a Buchsbaum local ring. In fact the equality $I^2 = QI$ always holds true in Buchsbaum local rings (See, Theorem 3.1).

In Section 4 we will discuss Buchsbaumness of the Rees algebra R and the associated graded ring G in a Buchsbaum local ring. When $r = d-1$, the Rees algebra R and the associated graded ring G are Buchsbaum rings with specific local cohomology modules (See, Corollary 3.6, Theorem 4.1 and Theorem 4.2). We will give the outline of proof of Theorem 4.1 in Section 5. When $0 < r < d-1$, the situation is slightly different from the case where $r = d-1$ (See, Theorem 4.3). In fact G is not necessarily a Buchsbaum ring in the case where $0 < r < d-1$. Theorem 4.3 gives a necessary and sufficient condition that G ia a Buchsbaum ring. Moreover it shows that R is a Buchsbaum ring with specific local cohomology modules if G is a Buchsbaum ring.

2 Subsystems of parameters for a Noetherian local ring.

Throughout this section let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $0 < r < d$ be an integer. Let $a_1, \ldots, a_r \in \mathfrak{m}$ be a subsystem of parameters for A. Let $Q = (a_1, \ldots, a_r)$ and let $I = Q :_A \mathfrak{m}$.

Then we have the following.

Proposition 2.1. *The following two statements hold true.*

(1) $\mathfrak{m}I = \mathfrak{m}Q$.

(2) *I is integral over Q.*

Proof. (1) Suppose that $\mathfrak{m}I \not\subseteq \mathfrak{m}Q$, and choose r as small as possible around counter-examples. Here we can choose an element $f \in \mathfrak{m}I \backslash \mathfrak{m}Q$. Then we may assume that $f = a_r$. If $r = 1$, then $Q = (a_1) = \mathfrak{m}I$. Let $\mathfrak{p}$ be an associated prime of A and suppose that $\dim A/\mathfrak{p} = d$. Since a_1 ia a parameter for A then $a_1 \notin \mathfrak{p}$. Hence the equality $(a_1) = \mathfrak{m}I$ shows that the maximal ideal $\mathfrak{m}/\mathfrak{p}$ of a Northerian local domain $A/\mathfrak{p}$ is invertible. Then $A/\mathfrak{p}$ is a DVR. Therefore $d = 1 = r$. This is impossible. Hence $r > 1$. Let $B = A/(a_1)$. Then $\mathfrak{m}B \cdot IB \neq \mathfrak{m}B \cdot QB$, because $\overline{a_r} \in (\mathfrak{m}B \cdot IB) \backslash (\mathfrak{m}B \cdot QB)$ since $\overline{a_r}$ is a minimal generator of QB. But this is contradicts the minimality of r. Hence the equality $\mathfrak{m}I = \mathfrak{m}Q$ holds true.

(2) Let $C = A/W$, where $W = \mathrm{H}^0_\mathfrak{m}(A)$ the 0-th local cohomology module of A with respect to $\mathfrak{m}$. Thanks to the assertion (1), $\mathfrak{m}C \cdot IC = \mathfrak{m}C \cdot QC$. Hence IC is integral over QC in C because an ideal $\mathfrak{m}C$ contains a non-zero divisor on C. Then I is integral over Q in A since W is nilpotent. $\square$

Therefore Q is always a reduction of I. This statement holds true for any Noetherian local ring. This is the answer to Question 1.1 (1).

Then we have the following Corollary, gives an answer to Question 1.1 (2) in a special case.

Corollary 2.2. *The equality $I^2 = QI$ holds true if $a_1, \ldots, a_r$ forms a regular sequence on A.*

Proof. Let $x \in I^2$. Since $I^2 \subseteq \mathfrak{m}I = \mathfrak{m}Q \subseteq Q = (a_1, \ldots, a_r)$ we can write $x = \sum_{i=1}^r a_i x_i$, where $x_i \in A$ $(1 \leq i \leq r)$. Let $\alpha \in \mathfrak{m}$. Then $\sum_{i=1}^r a_i(\alpha x_i) = \alpha x \in \mathfrak{m}I^2 = \mathfrak{m}Q^2 \subseteq Q^2$. Because $a_1, \ldots, a_r$ forms a regular sequence on A, $\alpha x_i \in Q$ for all $1 \leq i \leq r$. Then $x_i \in Q :_A \mathfrak{m} = I$ for all $1 \leq i \leq r$ and therefore $x = \sum_{i=1}^r a_i x_i \in QI$. Hence the equality $I^2 = QI$ holds true. □

3 Subsystems of parameters for a Buchsbaum local ring.

Throughout this section let A be a Buchsbaum local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $0 < r < d$ be an integer. Let $a_1, \ldots, a_r, b \in \mathfrak{m}$ be a subsystem of parameters for A. Let $Q = (a_1, \ldots, a_r)$ and let $I = Q :_A \mathfrak{m}$. Let R be the Rees algebra $\mathrm{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ with respect to I, where t is an indeterminateness. Let $M = \mathfrak{m}R + R_+$ and let $f_i = a_i t$ $(1 \leq i \leq r)$. Let G be the associated graded ring $\mathrm{G}(I) = \mathrm{R}(I)/I\mathrm{R}(I)$ with respec to I.

Then the first main theorem of this report is the following.

Theorem 3.1. *The equality $I^2 = QI$ holds true.*

When Q is a parameter ideal, the equality $I^2 = QI$ dose not necessarily hold true even in Buchsbaum local rings. But Q is generated by a subsystem of parameters then the equality $I^2 = QI$ always holds true in Buchsbaum local rings. This is the answer to Question 1.1 (2) in Buchsbaum local rings.

Then we have the follwing.

Proposition 3.2. *The following statements hold true for all positive integers $n_1, \ldots, n_r > 0$, $\ell > 0$, and $n \in \mathbb{Z}$, and all $\Lambda \subseteq [1, r] = \{1, 2, \ldots, r\}$.*

(1) $(a_i^{n_i} \mid i \in \Lambda) \cap I^n = \sum_{i \in \Lambda} a_i^{n_i} I^{n - n_i}$.

(2) $[(a_i^{n_i} \mid i \in \Lambda) + (b^\ell)] \cap I^n = \sum_{i\in\Lambda} a_i^{n_i} I^{n-n_i} + b^\ell I^n$.

We can check Proposition 3.2 by using facts that A is Buchsbaum and the equality $I^2 = QI$ holds true. These equalities are useful to determine the structures of the Rees algebra $\mathrm{R} = \mathrm{R}(I)$ and the associated graded ring $\mathrm{G} = \mathrm{G}(I)$ with respect to I.

Corollary 3.3. *$f_1, \ldots, f_r, b$ forms a strong M-sequence on G, that is the following equalities hold true for all positive integers $n_1, \ldots, n_r > 0$, $\ell > 0$, and all $1 \le i \le r$.*

$$\begin{aligned}(f_1^{n_1}, \ldots, f_{i-1}^{n_{i-1}})G : f_i^{n_i} &= (f_1^{n_1}, \ldots, f_{i-1}^{n_{i-1}})G : M.\\ (f_1^{n_1}, \ldots, f_r^{n_r})G : b^\ell &= (f_1^{n_1}, \ldots, f_r^{n_r})G : M.\end{aligned}$$

Corollary 3.3 follows directly from Proposition 3.2. Then we have the following, in which $\mathrm{H}_M^i(*)$ denotes the local cohomology functor with respect to M.

Theorem 3.4. *$M\mathrm{H}_M^i(G) = (0)$ for all $0 \le i \le r$.*

This theorem follows from the following generalization of strong M-sequences.

Proposition 3.5. *Let S be a Noetherian ring and E an S-module. Let $x_1, \ldots, x_t \in S$ $(t > 0)$ and put $\mathfrak{a} = (x_1, \ldots, x_t)$. Let $\mathfrak{b}$ be an ideal of S and suppose $\mathfrak{a} \subseteq \mathfrak{b}$. Assume that $x_1, \ldots, x_t$ forms a strong $\mathfrak{b}$-sequence on E, that is the following equalities hold true for all positive integers $m_1, \ldots, m_t > 0$ and all $1 \le i \le t$.*

$$(x_1^{m_1} \ldots, x_{i-1}^{m_{i-1}})E :_E x_i^{m_i} = (x_1^{m_1} \ldots, x_{i-1}^{m_{i-1}})E :_E \mathfrak{b}.$$

Then $\mathfrak{b}\mathrm{H}_\mathfrak{c}^i(E) = (0)$ for all $0 \le i \le t-1$ and all ideals $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathfrak{b}$.

Theorem 3.4 is a key of our reseaches. In fact Theorem 3.4 detemines the structure of local cohomology modules of the associated graded ring.

Corollary 3.6. $\mathrm{H}_M^i(G) = [\mathrm{H}_M^i(G)]_{1-i} \cong \mathrm{H}_\mathfrak{m}^i(A)$ *for all $0 \le i \le r$.*

Now we are in the position to determine the structures of the Rees algebra and the associated graded ring. Then let me start it in the next section.

4 Buchsbaumness of Rees algebras and associated graded rings.

Throughout this section let A be a Buchsbaum local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $0 < r < d$ be an integer. Let $a_1, \ldots, a_r \in \mathfrak{m}$ be a subsystem of parameters for A. Let $Q = (a_1, \ldots, a_r)$ and let $I = Q :_A \mathfrak{m}$. Let R be the Rees algebra $\mathrm{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ with respect to I, where t is an indeterminateness. Let $M = \mathfrak{m}R + R_+$ and let $f_i = a_i t$ $(1 \leq i \leq r)$. Let G be the associated graded ring $\mathrm{G}(I) = \mathrm{R}(I)/I\mathrm{R}(I)$ with respec to I.

In this section we will discuss Buchsbaumness of the Rees algebra R and the associated graded ring G. Firstly let me explore the case where $r = d - 1$.

4.1 The case where $r = d - 1$.

Assume that $r = d - 1$. In this case, we have the following.

Theorem 4.1. *G is a Buchsbaum ring.*

Let us give the outline of proof of Theorem 4.1 in Section 5. Then we can determine the structure of the Rees algebra.

Theorem 4.2. *R is a Buchsbaum ring with specific local cohomology modules, that is,*

$$\begin{aligned} &\mathrm{H}^0_M(R) = [\mathrm{H}^0_M(R)]_0 \oplus [\mathrm{H}^0_M(R)]_1 \cong \mathrm{H}^0_{\mathfrak{m}}(A) \oplus \mathrm{H}^0_{\mathfrak{m}}(A), \\ &\mathrm{H}^1_M(R) = [\mathrm{H}^1_M(R)]_0 \cong \mathrm{H}^1_{\mathfrak{m}}(A), \\ \textit{and} \quad &\mathrm{H}^i_M(R) = \bigoplus_{n \in [3-i, -1]} [\mathrm{H}^i_M(R)]_n \cong \bigoplus_{n \in [3-i, -1]} \mathrm{H}^{i-1}_{\mathfrak{m}}(A) \quad (2 \leq i \leq d). \end{aligned}$$

4.2 The case where $0 < r < d - 1$.

Assume that $0 < r < d - 1$.

In this case, the situation is slightly different from the case where $r = d - 1$. In fact we have the following.

Theorem 4.3. *The following conditions are equivalent.*

(1) *G has FLC.*

(2) $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ $(2 \leq i \leq d-1)$.

(3) $\mathrm{H}^i_M(G) = (0)$ $(2 \leq i \leq d-1)$.

(4) *G is Buchsbaum.*

When this is the case, R is a Buchsbaum ring with specific local cohomology modules, that is,

$$\begin{aligned} & \mathrm{H}^0_M(R) = [\mathrm{H}^0_M(R)]_0 \oplus [\mathrm{H}^0_M(R)]_1 \cong \mathrm{H}^0_{\mathfrak{m}}(A) \oplus \mathrm{H}^0_{\mathfrak{m}}(A), \\ & \mathrm{H}^1_M(R) = [\mathrm{H}^1_M(R)]_0 \cong \mathrm{H}^1_{\mathfrak{m}}(A), \\ \textit{and} \quad & \mathrm{H}^i_M(R) = (0) \quad (2 \leq i \leq d). \end{aligned}$$

5 The outline of proof of Theorem 4.1.

In this section, we give the outline of proof of Theorem 4.1.

We shall prove Theorem 4.1 by induction on d. Now assume that $d > 1$ and our assertion holds true for $d-1$.

Firstly, suppose that depth $A > 0$. Then f_1 is a regular element on G and hence we get the exact sequence

$$0 \longrightarrow G(-1) \xrightarrow{\widehat{f_1}} G \longrightarrow G_{A/(a_1)}(I/(a_1)) \longrightarrow 0.$$

This exact sequence implies the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^p(M; G_{A/(a_1)}(I/(a_1))) & \longrightarrow & \mathrm{H}^{p+1}(M; G)(-1) \\ \varphi \downarrow & & \psi \downarrow \\ \mathrm{H}^p_M(G_{A/(a_1)}(I/(a_1))) & \xrightarrow{\varepsilon} & \mathrm{H}^{p+1}_M(G)(-1) \end{array}$$

for all $0 \leq p \leq d-2$, where $\mathrm{H}^p(M; *)$ denotes the Koszul cohomology module with respect to M. Here ε is surjective by Theorem 3.4. Then by the hypothesis of induction on d, φ is surjective because $G_{A/(a_1)}(I/(a_1))$ is Buchsbaum if and only if the natural map φ is surjective. Then thanks to the above commutative diagram, ψ is surjective and hence G is Buchsbaum.

Now assume that depth $A = 0$. There exists the exact sequence

$$0 \to \mathrm{H}^0_M(G) \to G \to G_B(IB) \to 0,$$

where $B = A/W$ and $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then thanks to this exact sequence, we get the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}^p(M;G) & \xrightarrow{\theta} & \mathrm{H}^p(M;G(IB)) & \to & \mathrm{H}^{p+1}(M;\mathrm{H}^0_M(G)) & \xrightarrow{\eta} & \mathrm{H}^{p+1}(M;G) \\ \psi\downarrow & & \psi'\downarrow & & & & \\ \mathrm{H}^p_M(G) & \overset{\cong}{\to} & \mathrm{H}^p_M(G(IB)) & & & & \end{array}$$

for all $1 \leq p \leq d-1$. The map ψ' is surjective by the case where local ring has positive depth. Here we want to show that ψ is surjective. Because the module $\mathrm{H}^p_M(G)$ is concentrated in the homogeneous component of degree $1-p$ by Corollary 3.6, we only consider the homogeneous component of degree $1-p$. Then we have the following.

Claim 5.1. *$[\eta]_{1-p} : [\mathrm{H}^{p+1}(M;\mathrm{H}^0_M(G))]_{1-p} \to [\mathrm{H}^{p+1}(M;G)]_{1-p}$ is injective.*

To prove this claim, we need complicated careful computations and I don't want to give it. But any way, this claim holds true.

Then $[\theta]_{1-p} : [\mathrm{H}^p(M;G)]_{1-p} \to [\mathrm{H}^p(M;G(IB))]_{1-p}$ is surjective. Hence thanks to the above commutative diagram, ψ is surjective. Therefore G is Buchsbaum.

This is the outline of proof of Theorem 4.1.

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LINEARLY PRESENTED IDEALS AND A SUBADDITIVITY FORMULA FOR THE DEGREES OF SYZYGIES

BERND ULRICH

1. Introduction

This is a report on joint work with **David Eisenbud** and **Craig Huneke**, that will be published in [8].

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field, $\mathfrak{m}$ its homogeneous maximal ideal, and M, N finitely generated graded R-modules. We give bounds for the highest degrees of the local cohomology modules $\mathrm{H}^j_{\mathfrak{m}}(\mathrm{Tor}^R_k(M, N))$ in terms of individual graded Betti numbers of M and N, under the assumption that $\mathrm{Tor}^R_1(M, N)$ has Krull dimension at most one. We apply these results to syzygies, Gröbner bases, products and powers of ideals, and to the relationship of symmetric and Rees algebras. A guiding principle in some of our applications is the observation that $\mathfrak{m}$-primary ideals with linear presentation 'try to be' powers of $\mathfrak{m}$, generalizing the well-known fact that an $\mathfrak{m}$-primary ideal is a power of $\mathfrak{m}$ if its entire resolution is linear.

2. Regularity of Local Cohomology Modules

In this section we state our main technical result, a general estimate on the highest degrees of the local cohomology of Tor modules. We begin by introducing some notation. When T is an Artinian graded R-module we set $\operatorname{reg} T = \max\{i \,|\, T_i \neq 0\}$. Thus we write $\mathrm{t}_p(M) = \mathrm{t}^R_p(M) = \operatorname{reg} \mathrm{Tor}^R_p(M, K)$ for the top generator degree of the minimal p^{th} syzygy of M, and $\operatorname{reg} M = \max_p\{\mathrm{t}_p(M) - p\} = \max_j\{\operatorname{reg} \mathrm{H}^j_{\mathfrak{m}}(M) + j\}$ for the *Castelnuovo-Mumford regularity* of M.

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Theorem 2.1. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field and let M, N be finitely generated graded R-modules. Assume that* $\dim \operatorname{Tor}_1^R(M, N) \leq 1$, *and let k, j, p, q be non-negative integers satisfying $p + q = n - j + k$.*

(a) *If $p \leq \operatorname{codim} A$ and $q \leq \operatorname{codim} B$ then*

$$\operatorname{reg} \mathrm{H}^j_{\mathfrak{m}}(\operatorname{Tor}_k^R(M, N)) \leq \mathrm{t}_p(M) + \mathrm{t}_q(N) - n.$$

(b) *If $n - j + k \geq \operatorname{codim} A + \operatorname{codim} B$ then*

$$\operatorname{reg} \mathrm{H}^j_{\mathfrak{m}}(\operatorname{Tor}_k^R(M, N)) \leq \max_{\substack{p \geq \operatorname{codim} A \\ q \geq \operatorname{codim} B}} \{\mathrm{t}_p(M) + \mathrm{t}_q(N) - n\}.$$

The assumption on the dimension of Tor is automatically satisfied if $\dim M \otimes_R N \leq 1$; we will see in the next section that our assertions fail to hold without any such condition. In the remainder of this article we will draw various conclusions from Theorem 2.1. We assume throughout that $R = K[x_1, \ldots, x_n]$ is a polynomial ring over an arbitrary field, M, N are finitely generated graded R-modules, and I, J are homogeneous R-ideals.

3. Castelnuovo-Mumford Regularity

In this section we apply our main result to obtain estimates for the Castelnuovo-Mumford regularity of Tor modules and of products and powers of ideals. These bounds recover and extend earlier work by Geramita-Gimigliano-Pitteloud, Chandler, Sidman, Conca-Herzog, and Caviglia ([9], [3], [13], [5], [2]; see also [7], [12], [10]).

Corollary 3.1.

(a) (See [2] for the case $k = 0$.) *If* $\dim \operatorname{Tor}_1^R(M, N) \leq 1$ *then*

$$\operatorname{reg} \operatorname{Tor}_k^R(M, N) \leq \operatorname{reg} M + \operatorname{reg} N + k.$$

(b) (See [13].) *If* $\dim R/(I + J) \leq 1$ *then*

$$\operatorname{reg} IJ \leq \operatorname{reg} I + \operatorname{reg} J.$$

Proof. Part (a) is an immediate consequence of Theorem 2.1, whereas part (b) follows from (a) applied to $\operatorname{Tor}_0^R(I, R/J)$. □

The linear bound on the regularity of powers that follows from Corollary 3.1(b) can be improved if one uses the full strength of Theorem 2.1:

Proposition 3.2. *Assume that* $\dim R/I \leq 1$.

(a) $\operatorname{reg} I^s \leq \operatorname{reg} I + (s-1)(\mathrm{t}_1(I) - 1)$ *for every* $s \geq 1$.

(b) *If* I *is linearly presented and* I^ℓ *has a linear resolution for some* $\ell \geq 1$, *then* I^s *has a linear resolution for every* $s \geq \ell$.

Proof. We may assume that $\operatorname{ht} I \geq 2$. For part (a) one uses induction on $s \geq 1$ and applies Theorem 2.1(a) to the local cohomology modules $\mathrm{H}^j_{\mathfrak{m}}(\operatorname{Tor}_1^R(R/I, R/I^{s-1})) \cong \mathrm{H}^j_{\mathfrak{m}}(I^{s-1}/I^s)$, taking $p = 2$ and $q = n - j - 1$. To prove (b) one argues similarly, starting the induction with $s = \ell$ instead. □

The assertion of Proposition 3.2(b) fails dramatically even for two-dimensional ideals, as demonstrated by examples due to Sturmfels ([14]) and Conca ([4]):

Example 3.3. ([4]) For any $s > 1$ let $I = (x_1x_2^s, x_1x_3^s, x_2^{s-1}x_3x_4) + x_2x_3(x_2, x_3)^{s-1} \subset K[x_1, \ldots, x_4]$. Conca has shown that I^ℓ has a linear resolution for every $\ell < s$, whereas I^s is not even linearly presented! Notice that $\dim R/I = 2$.

4. Convexity

Theorem 2.1(a) also yields relations among individual shifts in the resolution. Most notably it leads to convexity results for the function $p \mapsto \mathrm{t}_p(M)$:

Proposition 4.1. *Assume that* $\dim M \leq 1$.

(a) *If* $J \subset \operatorname{ann}(M)$ *and* $0 \leq q \leq \operatorname{ht} J$, *then*

$$\mathrm{t}_n(M) \leq \mathrm{t}_{n-q}(M) + \mathrm{t}_q(R/J).$$

(b) (Convexity) *If* $M = R/I$ *and* $0 \leq q \leq n$, *then*

$$\mathrm{t}_n(M) \leq \mathrm{t}_{n-q}(M) + \mathrm{t}_q(M)$$

and

$$\mathrm{t}_n(M) \leq \mathrm{t}_{n-q}(M) + q \cdot \mathrm{t}_1(M).$$

Proof. For part (a) one uses Theorem 2.1(a) to estimate the regularity $\operatorname{reg} \mathrm{H}^0_{\mathfrak{m}}(\operatorname{Tor}^R_0(M, R/J)) = \operatorname{reg} \mathrm{H}^0_{\mathfrak{m}}(M) = \mathrm{t}_n(M) - n$. Part (b) follows from (a): For the first inequality we choose $J = I$, and for the second one we may assume that K is infinite and take J to be any complete intersection generated by a regular sequence of $n-1$ forms in I of degree $\mathrm{t}_0(I) = \mathrm{t}_1(M)$. □

The two estimates of Proposition 4.1(b) are independent; in fact there is no convexity result if we replace $\mathrm{t}_n(M)$ by $\mathrm{t}_q(M)$ for $1 < q < n$. This is well known, but a particularly simple example has been given by Caviglia:

Example 4.2. ([1]) For $I = (x_1^3, \ldots, x_4^3, (x_1 + \ldots + x_4)^3) \subset K[x_1, \ldots, x_4]$ and $M = R/I$ one has $\mathrm{t}_1(M) = 3$, whereas $\mathrm{t}_2(M) = 7$! Notice that $\dim M = 0$.

5. Specializing Modulo Linear Forms

In this section we are going to describe the behavior of resolutions under specialization modulo elements that are not necessarily regular. This in turn has strong consequences for initial ideals and, as we will see in the next section, for powers of ideals. Thus let L be a homogeneous R-ideal so that $\overline{R} = R/L$ is a polynomial ring over K in p variables, write $\overline{I}$, $\overline{\mathfrak{m}}$ for the images of I, $\mathfrak{m}$ in $\overline{R}$, and assume that $\overline{I}$ is $\overline{\mathfrak{m}}$-primary. Notice that L is generated by $n-p$ linearly independent linear forms in R and that $p \leq \operatorname{ht} I$.

Proposition 5.1 (Specialization). *In the above setting one has*

$$\mathrm{t}_p^{\overline{R}}(\overline{R}/\overline{I}) \leq \mathrm{t}_p^R(R/I).$$

Proof. We apply Theorem 2.1(a) to bound $\operatorname{reg} \mathrm{H}^0_{\mathfrak{m}}(\operatorname{Tor}^R_0(R/I, R/L)) = \operatorname{reg} \mathrm{H}^0_{\overline{\mathfrak{m}}}(\overline{R}/\overline{I}) = \mathrm{t}_p^{\overline{R}}(\overline{R}/\overline{I}) - p$. □

For the next two corollaries to this proposition we will assume that the homogeneous minimal free resolution of I is linear for the first $p-1$ steps, i. e., that it has the form

$$\ldots \longrightarrow \oplus R(-d-p+1) \longrightarrow \ldots \longrightarrow \oplus R(-d-1) \longrightarrow \oplus R(-d) \longrightarrow I \longrightarrow 0.$$

Corollary 5.2. *In the above setting one has* $\overline{I} = \overline{\mathfrak{m}}^d$.

Proof. According to Proposition 5.1, the $\overline{\mathfrak{m}}$-primary ideal $\overline{I}$ has linear resolution. □

Corollary 5.3. *In addition to the assumptions of Corollary* 5.2 *suppose that* I *is* $\mathfrak{m}$*-primary.*

(a) $\operatorname{in}_{\mathrm{revlex}} \supset (x_1, \dots, x_p)^d$, *where* $\operatorname{in}_{\mathrm{revlex}}$ *denotes the initial ideal in the reverse lexicographic term order.*

(b) *If* $\operatorname{char} K = 0$ *and* $p = n - 1$, *then* $\mu(\operatorname{Gin}(I)) = \mu(\mathfrak{m}^d)$, *where* Gin *denotes the generic initial ideal in the reverse lexicographic term order and* μ *is the minimal number of generators.*

Proof. To see part (a) we apply Corollary 5.2 with $L = (x_{p+1}, \dots, x_n)$. By the corollary $\mathfrak{m}^d \subset \operatorname{in}_{\mathrm{revlex}}(I + L) = (\operatorname{in}_{\mathrm{revlex}}(I)) + L$, where the last equality follows from a standard property of the reverse lexicographic order. Thus $(x_1, \dots, x_p)^d \subset \operatorname{in}_{\mathrm{revlex}}(I)$.

As to part (b), we know from Corollary 5.2 that $I + (x) = \mathfrak{m}^d + (x)$ for every linear form x of R. Now the assertion follows from the work of Conca-Herzog-Hibi ([6]). □

6. Regularity of Powers

In this section we revisit the issue of the Castelnuovo-Mumford regularity of powers, pursuing the idea that linearly presented ideals 'try' to be powers of maximal ideals. The first statement, which is asymptotic in nature, relies on the specialization result Corollary 5.2.

Proposition 6.1. *Assume that* I *is a linearly presented* $\mathfrak{m}$*-primary ideal generated in degree* d. *If* $s \gg 0$ *then* $I^s = \mathfrak{m}^{ds}$.

Proof. We may assume that K is algebraically closed. We need to show that the cokernel of the embedding

$$K[I_d] \hookrightarrow K[R_d]$$

has finite length, or equivalently, that the induced map φ of projective varieties

$$\operatorname{Proj}(K[I_d]) \twoheadleftarrow \operatorname{Proj}(K[R_d]) = \mathbb{P}_K^{n-1}$$

is an isomorphism. However, Corollary 5.2 shows that for every R-ideal L generated by $n-2$ linearly independent linear forms, the embedding

$$K[\overline{I}_d] \hookrightarrow K[\overline{R}_d]$$

is an equality. In other words, φ is an isomorphism when restricted to any line $V(L) \subset \mathbb{P}_K^{n-1}$. But then φ has to be an isomorphism. □

Experimental evidence supports the following conjectural bound for the integers s in Proposition 6.1:

Conjecture 6.2. The equality of Proposition 6.1 holds for every $s \geq n-1$.

This conjecture is obviously true for $n \leq 2$ and follows from the next theorem if $n = 3$:

Theorem 6.3. *Assume that* $\dim R/I \leq 1$ *and that the first* $\lceil \frac{n-1}{2} \rceil$ *steps in the resolution of* I *are linear. If* $s \geq 2$ *then* I^s *has a linear resolution.*

Proof. Write $d = t_0(I)$. According to Proposition 3.2(b) it suffices to show that $\operatorname{reg} I^2 \leq 2d$. Our assumption and Theorem 2.1(a) imply

$$\operatorname{reg} I/I^2 = \operatorname{reg} \operatorname{Tor}_1^R(R/I, R/I) \leq 2d-1,$$

and likewise

$$\operatorname{reg} R/I = \operatorname{reg} \operatorname{Tor}_0^R(R/I, R/I) \leq 2d-2.$$

Now the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow R/I^2 \longrightarrow R/I \longrightarrow 0$$

shows that indeed $\operatorname{reg} R/I^2 \leq 2d-1$. □

7. Rees algebras

We apply our results to the problem of finding the equations that define Rees algebras of linearly presented ideals. Thus assume I is linearly presented, let $f_1, \ldots, f_m$ be forms of degree d minimally generating I, and let φ be an m by ℓ matrix of linear forms presenting I. Mapping the symmetric algebra $\mathcal{S} = \operatorname{Sym}(I)$ onto the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ of I one obtains an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S} \longrightarrow \mathcal{R} \longrightarrow 0,$$

where $\mathcal{A}$ is the homogeneous $\mathcal{S}$-ideal consisting of the R-torsion elements in $\mathcal{S}$. To find the defining equations of the algebra $\mathcal{R}$ it suffices to describe the ideal $\mathcal{A}$, since a presentation of $\mathcal{S}$ can be easily obtained. In fact, mapping new variables T_i to $f_i \in [\mathcal{S}]_1$ one has

$$\mathcal{S} \cong K[x_1, \dots, x_n, T_1, \dots, T_m]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by the entries of the row vector $[T_1, \dots, T_m] \cdot \varphi$. Notice that $\mathcal{S}$ is not only an R-algebra, but also an algebra over the polynomial ring $S = K[T_1, \dots, T_m]$. Exploiting this symmetry further one writes

$$[T_1, \dots, T_m] \cdot \varphi = [x_1, \dots, x_n] \cdot B(\varphi)$$

for some n by ℓ matrix $B(\varphi)$ with linear entries in the polynomial ring S. The matrix $B(\varphi)$ can be explicitly computed as a Jacobian matrix of the defining equations of $\mathcal{S}$ and hence has been dubbed the *Jacobian dual* of φ. Consider the S-module

$$E = \operatorname{coker}_S(\varphi).$$

It can also be described as $\Omega_S(\mathcal{S}) \otimes_{\mathcal{S}} S$ or as the S-submodule of $\mathcal{S}$ generated by $x_1, \dots, x_n$. By the very definition of E one has

$$\operatorname{Sym}_R(I) = \operatorname{Sym}_S(E),$$

establishing a symmetry between I and its *Jacobian dual module* E. The advantage is that whereas I is faithful, E will in general have a non-trivial annihilator

$$\mathfrak{a} = \operatorname{ann}_S(E).$$

Since $x_1, \dots, x_n$ generate an ideal of positive grade in $\mathcal{R}$, this annihilator is necessarily contained in $\mathcal{A}$,

$$\mathcal{S} \cdot \mathfrak{a} \subset \mathcal{A}.$$

Naturally one would like this containment to be an equality. If equality holds then $\mathfrak{a}$ is the defining ideal of the special fiber ring $K[I_d] \cong \mathcal{R} \otimes_R K$, solving the elimination problem of determining the defining equations of the image of the rational map

$$\mathbb{P}_K^{n-1} \dashrightarrow \mathbb{P}_K^{m-1}$$

given by $[f_1 : \dots : f_m]$.

One readily sees that the desired equality $\mathcal{A} = \mathcal{S} \cdot \mathfrak{a}$ holds if the R-torsion of every symmetric power $S_s(I)$ is concentrated in degree ds. The next theorem shows that the latter condition often holds in the $\mathfrak{m}$-primary case:

Theorem 7.1. *Assume that I is a linearly presented $\mathfrak{m}$-primary ideal generated in degree d. The R-torsion of every symmetric power $S_s(I)$ is concentrated in degree ds if one of these conditions are satisfied:*

(i) *The first $\lceil \frac{n}{2} \rceil$ steps in the resolution of I are linear.*

(ii) *The ring R/I is Gorenstein and $n = 3$.*

Proof. We use induction on s to prove that the R-torsion $\tau(S_s(I))$ of $S_s(I)$ satisfies $\operatorname{reg} \tau(S_s(I)) \le ds$. We may assume that $s \ge 2$ since $S_1(I)$ is torsion free, and in the setting of (ii) we may even suppose that $s \ge 3$ since $S_2(I)$ is torsion free in this case according to work of Huneke ([11]). Furthermore Theorem 6.3 shows that either $s-1 = 1$ or I^{s-1} has a linear resolution. Thus Theorem 2.1(a) gives

$$\operatorname{reg} \operatorname{Tor}_2^R(R/I, R/I^{s-1}) \le ds.$$

On the other hand $\operatorname{Tor}_2^R(R/I, R/I^{s-1}) \cong \tau(I \otimes_R I^{s-1})$, and there is an exact sequence of Artinian modules

$$\oplus \tau(S_{s-1}(I))(-d) \longrightarrow \tau(S_s(I)) \longrightarrow C \longrightarrow 0$$

with C an epimorphic image of $\tau(I \otimes_R I^{s-1})$. This completes the induction step. □

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THE STRUCTURE THEOREM OF COMPLETE LOCAL RINGS AND ITS APPLICATION

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In this talk, we first review the proof of the structure theorem of complete local rings, where the characteristic of the residue field is p. Then, we apply its idea to prove the existence of a big Cohen-Macaulay module on a mixed characteristic local ring.

1. The structure theorem of complete local rings

Theorem 1.1 (The structure theorem of complete local rings). *Let* $(A, \mathfrak{m}, \kappa)$ *be a complete local ring with its maximal ideal* $\mathfrak{m} = (x_1, \ldots, x_n)$. *Then, there exist a coefficient ring* A_0 *and indeterminates* $X_1, \ldots, X_n$, *which satisfy the following:*

$$A = A_0[[x_1, \ldots, x_n]] \cong A_0[[X_1, \ldots, X_n]]/I.$$

Where

(1) *in the case of* $\operatorname{char} \kappa = 0$, $A_0 \cong \kappa$.
(2) *in the case of* $\operatorname{char} \kappa = p > 0$, *by a Witt ring* (W, pW, κ)*:*
  (a) *when* $\operatorname{char} A = p^n$, $A_0 \cong W/p^nW$.
  (b) *when* $\operatorname{char} A = 0$, $A_0 \cong W$.

The case when $\operatorname{char} \kappa = 0$ follows from Hensel's lemma. We shall only consider the case when κ is a perfect field of characteristic $p > 0$. The following two lemmas give the proof.

Lemma 1.2. *Let* $(A, \mathfrak{m}, \kappa)$ *be a local ring with* $p \in \mathfrak{m}$. *Then, for* $a, b \in A$, *we have*

$$a \equiv b \pmod{\mathfrak{m}} \Longrightarrow a^{p^k} \equiv b^{p^k} \pmod{\mathfrak{m}^{k+1}}.$$

This lemma defines the map $\varphi_k \colon A/\mathfrak{m} \to A/\mathfrak{m}^{k+1}$ $(a \mapsto a^{p^k})$. Denoting $\operatorname{Im} \varphi_k$ by C_k, we see that each element of the subset

$$E_n = C_n + pC_{n-1} + \cdots + p^{n-1}C_1 + p^nC_0 \subset A/\mathfrak{m}^{n+1}$$

has Witt expression: $\alpha = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^{n-1}a_{n-1}^p + p^na_n$.

Lemma 1.3 ([7, p.175]). *Let* w_n *be the n-th Witt polynomial, that is,*

$$w_n(X_0, X_1, \ldots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^nX_n.$$

Let $\Phi(W, W')$ designate any polynomial in two variables with integral coefficients. There will exist polynomials $\varphi_0(X_0; X'_0), \ldots, \varphi_n(X_0, \ldots, X_n; X'_0, \ldots, X'_n)$ such that

$$\Phi(w_n(X_0, \ldots, X_n), w_n(X'_0, \ldots, X'_n)) = w_n(\varphi_0(X_0; X'_0), \ldots, \\ \ldots, \varphi_n(X_0, \ldots, X_n; X'_0, \ldots, X'_n)).$$

And the coefficients of the φ_n are integral.

The existence of a Witt ring (W, pW, κ) is well-known. Because we are supposing κ perfect, the set of Witt expressions E_n consists a complete set of representatives of $W/p^{n+1}W$. Hence, the isomorphism $\phi_0\colon W/pW \to A/\mathfrak{m}$ gives the ring homomorphism (Lemma1.3)

$$\phi_n\colon W/p^{n+1}W \to A/\mathfrak{m}^{n+1},$$

$$a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^{n-1}a_{n-1}^p + p^n a_n \mapsto \\ \phi_0(a_0)^{p^n} + p\phi_0(a_1)^{p^{n-1}} + \cdots + p^{n-1}\phi_0(a_{n-1})^p + p^n\phi_0(a_n)$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} W/p^{n+1}W & \xrightarrow{\phi_n} & A/\mathfrak{m}^{n+1} \\ \downarrow & & \downarrow \\ W/p^nW & \xrightarrow[\phi_{n-1}]{} & A/\mathfrak{m}^n. \end{array}$$

Thus, we get a homomorphism $\phi\colon W \to A$ which induces the isomorphism $\bar{\phi}\colon W/pW \to A/\mathfrak{m}$.

2. Big Cohen-Macaulay Modules

Let $(A, \mathfrak{m}, \kappa)$ be a d-dimensional local ring. Take a system of parameters $x_1, \ldots, x_d$ for A.

Definition 2.1. An A-module M is called a big Cohen-Macaulay module over A with respect to $x_1, \ldots, x_d$ if the following conditions are satisfied:

(1) x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 1, \ldots, d$,
(2) $M/\mathfrak{m}M \neq 0$.

To show the existence of Big Cohen-Macaulay Modules over a local ring, M. Hochster considered the following:

2.2. **Modifications.** Let $(A, \mathfrak{m}, \kappa)$ be a d-dimensional local ring. Take a system of parameters $x_1, \ldots, x_d$ for A. Let M be an A-module with $\sigma \in M$. Suppose that

$$x_1 m_1 + \cdots + x_{r+1} m_{r+1} = 0.$$

We then refer to $\rho = (m_1, \ldots, m_{r+1}) \in M^{r+1}$ as a type r relation with respect to $x_1, \ldots, x_d$ on M. If we let

$$\boldsymbol{v} = m_{r+1} \oplus (x_1, \ldots, x_r) \in M \oplus A^r,$$

we have canonical maps

$$M \to M \oplus A^r \to (M \oplus A^r)/A\boldsymbol{v}$$

and hence

$$M \to (M \oplus A^r)/A\boldsymbol{v} = M'.$$

Let σ' be the image of σ in M'. Then we have a map

$$(M, \sigma) \to (M', \sigma').$$

That is, a map $M \to M'$ which takes σ to σ'.

We call (M', σ') a first modification of (M, σ) with respect to a type r relation ρ for $x_1, \ldots, x_d$.

In general, we may have a sequence

$$(M, \sigma) = (M_0, \sigma_0) \to (M_1, \sigma_1) \to \cdots \to (M_s, \sigma_s) = (N, \tau)$$

in which (M_{k+1}, σ_{k+1}) is a modification of (M_k, σ_k) with respect to a relation ρ_{k+1} on M_k of type r_{k+1} for $x_1, \ldots, x_d$.

We then say that (N, τ) is an sth modification of (M, σ) of type $\mathbf{r} = (r_1, \ldots, r_s)$.

Further, when $\tau \notin (x_1, \ldots, x_d)N$, (N, τ) is called a *non-degenerate* modification of (M, σ) with respect to the system of parameters $x_1, \ldots, x_d$.

Proposition 2.3. *Let $(A, \mathfrak{m}, \kappa)$ be a d-dimensional local ring with a system of parameters $x_1, \ldots, x_d$ for A. Then the following are equivalent:*

1. *There exists a big Cohen-Macaulay module over A with respect to $x_1, \ldots, x_d$.*
2. *Every modification (N, τ) of $(A, 1)$ with respect to $x_1, \ldots, x_d$ is non-degenerate.*

When A is a local ring of characteristic $p > 0$, by using the proposition above and the Frobenius functor, Hochster showed that every modification (N, τ) of $(A, 1)$ with respect to $x_1, \ldots, x_d$ is non-degenerate.

Theorem 2.4. *Let A be a local ring of characteristic $p > 0$ with a system of parameters $x_1, \ldots, x_d$. Then there exists a big Cohen-Macaulay module over A with respect to $x_1, \ldots, x_d$.*

Now we recall Hochster's Observation [4, p.22] on an equational description for degeneracy of modifications, especially, over a mixed characteristic complete local ring.

2.5. **Hochster's Observation.** Let $(A, \mathfrak{m}, \kappa)$ be a mixed characteristic d-dimensional complete local ring with κ a perfect field of characteristic p.

Fix a system of parameters $p, x_2, \ldots, x_d$ for A. Then, by the structure theorem of complete local rings, we get a complete regular local ring

$$S = W[[x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+h}]]$$

with a Witt ring (W, pW, κ) and an ideal $I = (f_1, \ldots, f_r)S$, which expresses $A = S/I$.

We want to describe in an explicit way what an sth modification of (A, a) with $a \in A$ of type $\mathbf{r} = (r_1, \ldots, r_s)$ with respect to a system of parameters p, $x_2, \ldots, x_d$ looks like and what it means if such a modification degenerates.

In working with vectors, it will be convenient to identify a vector of length ℓ with the vector of length $\ell + \ell'$ whose last ℓ' entries are 0. It will also be convenient to define $p = x_1$ and $r_0 = 1$. Let

$$(A, a) = (M_0, \sigma_0) \to (M_1, \sigma_1) \to \cdots \to (M_s, \sigma_s) \tag{2.5.1}$$

be the sequence of modifications considered.

For each m, write $M_m = S/I \otimes_S L_m$ and we may identify

$$L_m = \bigoplus_{l=0}^{m} S^{r_l} \Big/ \sum_{l=1}^{m} S\boldsymbol{V}_l .$$

Here $\boldsymbol{V}_l = \boldsymbol{W}_l \oplus (x_1, \ldots, x_{r_l})$ may be regarded as a vector in $\bigoplus_{k=0}^{l} S^{r_k}$ and $\boldsymbol{W}_l \in \bigoplus_{k=0}^{l-1} S^{r_k}$ satisfies

$$\sum_{i=1}^{r_l} x_i \boldsymbol{U}_{il} + x_{r_l+1}\boldsymbol{W}_l + \sum_{k=1}^{l-1} t_{lk}\boldsymbol{V}_k = \sum_{j=1}^{r} f_j \boldsymbol{Y}_{jl} \tag{2.5.2}$$

for suitable choices of the vectors $\boldsymbol{U}_{il}$, $\boldsymbol{Y}_{jl} \in \bigoplus_{k=0}^{l-1} S^{r_k}$ and elements $t_{lk} \in S$.

The condition $\sigma_s \in (x_1, \ldots, x_d)M_s$ is then expressed by the existence of vectors $\boldsymbol{U}_i$, $\boldsymbol{Y}_j \in \bigoplus_{l=0}^{s} S^{r_l}$ and elements $t_m \in S$ such that

$$(a, 0, \ldots, 0) + \sum_{i=1}^{d} x_i \boldsymbol{U}_i + \sum_{m=1}^{s} t_m \boldsymbol{V}_m = \sum_{j=1}^{r} f_j \boldsymbol{Y}_j. \tag{2.5.3}$$

3. Bertini Theorem or Jacobian Criterion

3.1. **Notation and Assumptions.** Suppose, further A above is *normal*. Then, we get a d-dimensional complete regular local ring

$$R = W[[x_2, \dots, x_d]],$$

which makes A a finite extension.

In the following, q denotes p^e for a fixed sufficiently large $e \in \mathbf{N}$.

Take a qth root ξ_i of x_i for $i = 2, \dots, d$ and a qth root Ξ_j of X_j for $j = d+1, \dots, d+h$. Let

$$\tilde{S} = W[[\xi_2, \dots, \xi_d, \Xi_{d+1}, \dots, \Xi_{d+h}]]$$

and $\tilde{R} = W[[\xi_2, \dots, \xi_d]]$, and let $\tilde{A} = \tilde{S}/I\tilde{S}$.

Proposition 3.2. *With notation and assumptions above, take* any $\nu \in \mathbf{N}$. *There exists a prime ideal $\tilde{I}$ of $\tilde{S}$ which makes the local domain $B = \tilde{S}/\tilde{I}$ satisfy the following:*

(3.2.1) *There exists an $\tilde{R}$-algebra homomorphism $\bar{\varphi}\colon \tilde{A}/x_2^\nu \tilde{A} \to B/x_2^\nu B$.*

(3.2.2) *There exists $\delta \in R \setminus pR$ which makes B_δ an etale $\tilde{R}_\delta$-algebra.*

Proof. Let $pA = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be the primary decomposition of pA with $\mathfrak{p}_l = \sqrt{\mathfrak{q}_l}$. Let P_l be the prime ideal of S such that $P_l/I = \mathfrak{p}_l$. Because $A_{\mathfrak{p}_l}$ is a DVR, we may assume that $J = (f_1, \dots, f_h)S$ is an ideal of height h and that there exists $t \in S \setminus \bigcup_{l=1}^m P_l$ which makes $tI \subset J$, that is,

$$tf_k = \sum_{j=1}^h s_{kj} f_j \text{ with } s_{kj} \in S \text{ for } k = h+1, \dots, r. \tag{3.2.3}$$

Flenner's Proof of Bertini Theorem [3, Theorem 2.1] shows that, by choosing a sequence of elements:

$$\tilde{s}_1, \dots, \tilde{s}_h \in \tilde{S}^{[p]} = W[[\xi_2, \dots, \xi_d, \Xi_{d+1}^p, \dots, \Xi_{d+h}^p]]$$

and sequences of natural numbers $\epsilon_1, \dots, \epsilon_h$ and $\nu_1, \dots, \nu_h$, we get

$$\psi_j = f_j + t^q x_2^{\nu q} \left(\tilde{s}_j \Xi_{d+j} + t^{p^{\nu_j}} x_2^{\nu p^{\nu_j}} \right) \tag{3.2.4}$$

for $j = 1, \dots, h$, which generate an ideal $\tilde{J} = (\psi_1, \dots, \psi_h)\tilde{S}$ of height h such that $(\tilde{S}/(p, \tilde{J}))_{tx_2}$ is a separable extension of $\tilde{R}/p\tilde{R}$.

Further, put $\theta_j = \tilde{s}_j \Xi_{d+j} + t^{p^{\nu_j}} x_2^{\nu p^{\nu_j}}$ for $j = 1, \dots, h$ and let

$$\psi_k = f_k + t^{q-1} x_2^{\nu q} \sum_{j=1}^h s_{kj} \theta_j \tag{3.2.5}$$

for $k = h+1, \ldots, r$. Because $\psi_j = f_j + t^q x_2^{\nu q}\theta_j$ and by (3.2.3), we have

$$t\psi_k = \sum_{j=1}^{h} s_{kj}\psi_j. \tag{3.2.6}$$

Hence, we get

$$\psi_j \in \tilde{I} = \tilde{J}\tilde{S}_{tx_2} \cap \tilde{S} \text{ for } j = 1, \ldots, r. \tag{3.2.7}$$

Consequently, $(I\tilde{S}, x_2^{\nu}) \subset (\tilde{I}, x_2^{\nu})$. □

4. Musical Chairs and Frobenius Maps

In this section, we gather a few basic facts on the p-adic representation of elements in a ring and those on Frobenius maps. For the details, we refer the reader to [7] and [10].

Lemma 4.1. *Let B be a ring and $a, b \in B$. Suppose that $a \equiv b \pmod{p^k B}$ for $k > 0$. Then*

$$a^{p^n} \equiv b^{p^n} \pmod{p^{k+n}B}.$$

Hence, by sending any element b of each residue class $\bar{b} \in B/pB$ to its p^n-th power b^{p^n}, we get a canonical map

$$\phi_{B,n}\colon B/pB \to B/p^{n+1}B.$$

Consequently, the canonical maps $\phi_{B,n-i}$ above induce

$$w_{B,n}\colon (B/pB)^{n+1} \to B/p^{n+1}B$$

by mapping $(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_n) \mapsto b_0^{p^n} + pb_1^{p^{n-1}} + \cdots + p^n b_n$, which is injective when p is not a zero-divisor and when B/pB is reduced.

4.2. **Frobenius Maps.** With notation and assumptions as in the previous sections, let

$$F_W\colon W \to W$$

be the Frobenius map of a Witt ring (W, pW, κ) which lifts the Frobenius map $F_\kappa\colon \kappa \to \kappa$ of a field of characteristic p.

Because $\tilde{R} = W[[\xi_2, \ldots, \xi_d]]$ is a formal power series ring with indeterminates $\xi_2, \ldots, \xi_d$, we get an extension of F_W,

$$F_{\tilde{R},\xi}\colon \tilde{R} \to \tilde{R}$$

by mapping ξ_i to ${\xi_i}^p$ for $i = 2, \ldots, d$, which is a lifting of the Frobenius map $F_{\tilde{R}/p\tilde{R}}\colon \tilde{R}/p\tilde{R} \to \tilde{R}/p\tilde{R}$.

Take B and δ in Proposition 3.2 and put

$$\Delta = \{F^{e}_{\tilde{R},\xi}(\delta) \mid e \in \mathbf{N}_0\}$$

an $F_{\tilde{R},\xi}$-stable multiplicative set of $\tilde{R}$. Then we have a canonical extension $F_{\tilde{R}_\Delta,\xi}\colon \tilde{R}_\Delta \to \tilde{R}_\Delta$, which induces

$$F_{\tilde{R}_\Delta,\xi,n}\colon \tilde{R}_\Delta/p^n\tilde{R}_\Delta \to \tilde{R}_\Delta/p^n\tilde{R}_\Delta$$

for any $n \in \mathbf{N}$. Because B_δ is an etale $\tilde{R}_\delta$-algebra, we get the extension of $F_{\tilde{R}_\Delta,\xi,n}$,

$$F_{B_\Delta,\xi,n}\colon B_\Delta/p^nB_\Delta \to B_\Delta/p^nB_\Delta$$

which satisfy the following commutative diagram

$$\begin{array}{ccc} B_\Delta/p^{n+1}B_\Delta & \xrightarrow{F_{B_\Delta,\xi,n+1}} & B_\Delta/p^{n+1}B_\Delta \\ \downarrow & & \downarrow \\ B_\Delta/p^nB_\Delta & \xrightarrow[F_{B_\Delta,\xi,n}]{} & B_\Delta/p^nB_\Delta. \end{array}$$

Thus $F_{\tilde{R},\xi}$ is extended to the endomorphism of B^*_Δ, the pB_Δ-adic Henselization of B_Δ

$$F_{B^*_\Delta,\xi}\colon B^*_\Delta \to B^*_\Delta.$$

Let C be the integral closure of $B[F^\epsilon_{B^*_\Delta,\xi}(B)]$ in B^*_Δ for some large $\epsilon \in \mathbf{N}$. Then, C is a finite B-algebra via $\varphi\colon B \to C$, the canonical homomorphism induced by $F^\epsilon_{B^*_\Delta,\xi}$.

Further, let $pC = \sqrt{pC} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_u$ be the primary decomposition of pC. Because pB^*_Δ is a prime ideal, we may assume that $pB^*_\Delta \cap C = \mathfrak{p}_1$. Hence, by letting $\mathfrak{p}_1 = \mathfrak{p}$, we have

(4.2.1) $$p^\nu B^*_\Delta \cap C = \mathfrak{p}^{(\nu)} \text{ and } \mathfrak{p}^{(\nu)} \cap \tilde{B} = p^\nu\tilde{B} \text{ for any } \nu \in \mathbf{N}.$$

Lemma 4.3. *With notation and assumptions above, take $\gamma \in B^*_\Delta$. Suppose that*

(4.3.1) $$\gamma \equiv \gamma_0^{p^n} + p\gamma_1^{p^{n-1}} + \cdots + p^n\gamma_n \pmod{p^{n+1}B^*_\Delta}$$

with $\gamma_0, \gamma_1, \ldots, \gamma_n \in B^*_\Delta$. Then

(4.3.2) $$F^\epsilon_{B^*_\Delta,\xi}(\gamma) \equiv \gamma_0^{p^{n+\epsilon}} + p\gamma_1^{p^{n+\epsilon-1}} + \cdots + p^n\gamma_n^{p^\epsilon} \pmod{p^{n+1}B^*_\Delta}.$$

Thus, when $b \in B$ satisfies

(4.3.3) $$b \equiv b_0^{p^n} + pb_1^{p^{n-1}} + \cdots + p^nb_n \pmod{p^{n+1}B}$$

with $b_0, b_1, \ldots, b_n \in B$, we have

(4.3.4) $$\varphi(b) \equiv b_0^{p^{n+\epsilon}} + pb_1^{p^{n+\epsilon-1}} + \cdots + p^nb_n^{p^\epsilon} \pmod{\mathfrak{p}_1^{(n+1)}}$$

(to be continued)

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Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories[1]

Osamu Iyama

ABSTRACT. Auslander-Reiten theory, especially the concept of almost split sequences and their existence theorem, is fundamental to study categories which appear in representation theory, for example, modules over artin algebras [ARS][GR][R], their functorially finite subcategories [AS][S], their derived categories [H], Cohen-Macaulay modules over Cohen-Macaulay rings [Y], lattices over orders [A2,3][RS], and coherent sheaves on projective curves [AR][GL]. In these Auslander-Reiten theory, the number '2' is quite symbolic. For one thing, almost split sequences give minimal projective resolutions of simple objects of projective dimension '2' in functor categories. For another, Cohen-Macaulay rings and orders of Krull-dimension '2' have fundamental sequences and provide us one of the most beautiful situation in representation theory [A4][E][RV][Y], which is closely related to McKay's observation on simple singularities [M]. In this sense, usual Auslander-Reiten theory should be '2-dimensional' theory, and it would have natural importance to search a domain of higher Auslander-Reiten theory from the viewpoint of representation theory and non-commutative algebraic geometry (e.g. [V1,2][Ar][GL]). In this paper, we introduce $(n-1)$-orthogonal subcategories as a natural domain of '$(n+1)$-dimensional' Auslander-Reiten theory. We show that higher Auslander-Reiten translation and higher Auslander-Reiten duality can be defined quite naturally for such categories. Using them, we show that our categories have *n-almost split sequences*, which are completely new generalization of usual almost split sequences and give minimal projective resolutions of simple objects of projective dimension '$n+1$' in functor categories. We also show the existence of higher dimensional analogy of fundamental sequences for Cohen-Macaulay rings and orders of Krull-dimension '$n+1$'. We show that an invariant subring (of Krull-dimension '$n+1$') corresponding to a finite subgroup G of $\mathrm{GL}_{n+1}(\mathbb{C})$ has a natural maximal $(n-1)$-orthogonal subcategory.

Recently, in representation theory and non-commutative algebraic geometry, it seems that the study of 'nice' subcategories becomes more and more important. Especially, Van den Bergh introduced the concept of non-commutative crepant resolutions to study Bondal-Orlov conjecture on derived categories of resolutions of a Gorenstein

[1]The detailed version [I4,5] of this paper have been submitted for publication elsewhere.

singularity. We show that non-commutative crepant resolutions of Krull dimension d are almost same concept with our maximal $(d-2)$-orthogonal subcategories. Moreover, we show that all maximal 1-orthogonal subcategories are derived equivalent, which supports Van den Bergh's generalization of Bondal-Orlov conjecture. The concept of non-commutative crepant resolutions is also closely related to the concept of Auslander's representation dimension, which measures how far an algebra is from being of representation-finite. A lot of recent results on it show that it is really interesting and useful concept. Although the representation dimension is always finite if Krull dimension is at most 1, we show that it is not valid if Krull dimension is more than 2. Finally, we show that a boundedness conjecture of 1-orthogonal subcategories is valid for algebras with the representation dimension at most three.

1 From Auslander-Reiten theory

1.1 Let us recall M. Auslander's classical theorem [A1] below, which introduced a completely new insight to representation theory of algebras (see 2.3 for $\operatorname{dom.dim}\Gamma$).

Theorem A (Auslander correspondence) *There exists a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras* Λ *and that of finite-dimensional algebras* Γ *with* $\operatorname{gl.dim}\Gamma \leq 2$ *and* $\operatorname{dom.dim}\Gamma \geq 2$. *It is given by* $\Lambda \mapsto \Gamma := \operatorname{End}_\Lambda(M)$ *for an additive generator* M *of* $\operatorname{mod}\Lambda$.

In this really surprising theorem, the representation theory of Λ is encoded in the structure of the homologically nice algebra Γ called an *Auslander algebra*. Since the category $\operatorname{mod}\Gamma$ is equivalent to the functor category on $\operatorname{mod}\Lambda$, Auslander correspondence gave us a prototype of the use of functor categories in representation theory. In this sense, Auslander correspondence was a starting point of later Auslander-Reiten theory [ARS] historically. Theoretically, Auslander correspondence gives a direct connection between two completely different concepts, i.e. a representation theoretic property 'representation-finiteness' and a homological property '$\operatorname{gl.dim}\Gamma \leq 2$ and $\operatorname{dom.dim}\Gamma \geq 2$'. It is a quite interesting project to find correspondence between representation theoretic properties and homological properties (e.g. [I2]).

1.2 Let R be a complete regular local ring of dimension d and Λ a module-finite R-algebra. We call Λ an *isolated singularity* [A3] if $\operatorname{gl.dim}\Lambda \otimes_R R_\mathfrak{p} = \operatorname{ht}\mathfrak{p}$ holds for any non-maximal prime ideal $\mathfrak{p}$ of R. We call a left Λ-module M *Cohen-Macaulay* if it is a projective

R-module. We denote by $\mathrm{CM}\,\Lambda$ the category of Cohen-Macaulay Λ-modules. Then $D_d := \mathrm{Hom}_R(\ ,R)$ gives a duality $\mathrm{CM}\,\Lambda \leftrightarrow \mathrm{CM}\,\Lambda^{op}$. We call Λ an *R-order* (or *Cohen-Macaulay R-algebra*) if $\Lambda \in \mathrm{CM}\,\Lambda$ [A2,3]. In this case, let $\underline{\mathrm{CM}}\Lambda := (\mathrm{CM}\,\Lambda)/[\Lambda]$ be the *stable category* and $\overline{\mathrm{CM}}\Lambda := (\mathrm{CM}\,\Lambda)/[D_d\Lambda]$ the *costable category.* A typical example of an order is a commutative complete local Cohen-Macaulay ring Λ containing a field since such Λ contains a complete regular local subring R [Ma]. Let $\mathbf{E} : 0 \to R \to E_0 \to \cdots \to E_d \to 0$ be a minimal injective resolution of the R-module R. We denote by $D := \mathrm{Hom}_R(\ ,E_d)$ the Matlis dual. Let us recall the fundamental theorems [A2,3][Y] below, where we will give the definition of (pseudo) almost split sequences in 2.5 (put $n := 1$ there).

Theorem B *(1)* (Auslander-Reiten translation) *There exists an equivalence $\tau : \underline{\mathrm{CM}}\Lambda \to \overline{\mathrm{CM}}\Lambda$.*

(2) (Auslander-Reiten duality) *There exist functorial isomorphisms* $\overline{\mathrm{Hom}}_\Lambda(Y,\tau X) \simeq D\,\mathrm{Ext}^1_\Lambda(X,Y) \simeq \underline{\mathrm{Hom}}_\Lambda(\tau^{-1}Y,X)$ *for any* $X,Y \in \mathrm{CM}\,\Lambda$.

Theorem C *(1)* $\mathrm{CM}\,\Lambda$ *has almost split sequences.*

(2) If $d = 2$, then $\mathrm{CM}\,\Lambda$ *has pseudo almost split sequeces.*

As an immediate consequence, almost all simple objects in the functor category $\mathrm{mod}(\mathrm{CM}\,\Lambda)$ have projective dimension 2. If $d = 2$, then all simple objects in the functor category $\mathrm{mod}(\mathrm{CM}\,\Lambda)$ have projective dimension 2. In this sense, we can say that *Auslander-Reiten theory for the case $d = 2$ is very nice.* Using (pseudo) almost split sequences, we can define the *Auslander-Reiten quiver* $\mathfrak{A}(\Lambda)$ of Λ (see 2.6 and put $n := 1$ there).

1.3 Let us recall Auslander's contribution [A4][Y] to McKay correspondence [M]. Let k be a field of characteristic zero and G a finite subgroup of $\mathrm{GL}_d(k)$ with $d \geq 2$. Recall that the *McKay quiver* $\mathfrak{M}(G)$ of G [M] is defined as follows: The set of vertices is the set $\mathrm{irr}\,G$ of isoclasses of irreducible representations of G. Let V be the representation of G acting on k^d through $\mathrm{GL}_d(k)$. For $X,Y \in \mathrm{irr}\,G$, we denote by d_{XY} the multiplicity of X in $V \otimes_k Y$, and draw d_{XY} arrows from X to Y.

Theorem D *Let G be a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$, $\Omega := \mathbb{C}[[x,y]]$ and $\Lambda := \Omega^G$ the invariant subring. Assume that G does not contain pseudo-reflection except the identity. Then Λ is representation-finite with $\mathrm{CM}\,\Lambda = \mathrm{add}_\Lambda\,\Omega$, and the Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ of Λ coincides with the McKay quiver $\mathfrak{M}(G)$ of G.*

1.4 Aim We observed that Auslander-Reiten theory is 2-dimensional-like. Now we can state the aim of this paper. *For each* $n \geq 1$, *find a domain of* $(n+1)$*-dimensional Auslander-Reiten theory.* Namely, *find natural categories* $\mathcal{C}$ *such that Theorems above replaced '2' and* CM Λ *by '*$n+1$*' and* $\mathcal{C}$ *respectively hold.*

2 Main results

2.1 Definition Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full subcategory of $\mathcal{A}$ and $n \geq 0$. For a functorially finite [AS] full subcategory $\mathcal{C}$ of $\mathcal{B}$, we put

$$\begin{aligned} \mathcal{C}^{\perp_n} &:= \{X \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{C}, X) = 0 \text{ for any } i\ (0 < i \leq n)\}, \\ {}^{\perp_n}\mathcal{C} &:= \{X \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{C}) = 0 \text{ for any } i\ (0 < i \leq n)\}. \end{aligned}$$

We call $\mathcal{C}$ a *maximal n-orthogonal subcategory* of $\mathcal{B}$ if

$$\mathcal{C} = \mathcal{C}^{\perp_n} = {}^{\perp_n}\mathcal{C}$$

holds. By definition, $\mathcal{B}$ is a unique maximal 0-orthogonal subcategory of $\mathcal{B}$.

2.2 Example Let Λ be a simple singularity of type Δ and dimension $d = 2$, $\mathcal{A} := \mathrm{mod}^{\mathbb{Z}}\,\Lambda$ the category of graded Λ-modules and $\mathcal{B} := \mathrm{CM}^{\mathbb{Z}}\,\Lambda$ the category of graded Cohen-Macaulay Λ-modules. Then the number of maximal 1-orthogonal subcategories of $\mathcal{B}$ is given as follows:

Δ	A_m	B_m, C_m	D_m	E_6	E_7	E_8	F_4	G_2
number	$\frac{1}{m+2}\binom{2m+2}{m+1}$	$\binom{2m}{m}$	$\frac{3m-2}{m}\binom{2m-2}{m-1}$	833	4160	25080	105	8

This is obtained by showing that maximal 1-orthogonal subcategories of $\mathcal{B}$ correspond bijectively to clusters of the cluster algebra of type Δ [I4,5]. See Fomin-Zelevinsky [FZ1,2] and Buan-Marsh-Reineke-Reiten-Todorov [BMRRT]. See also Geiss-Leclerc-Schröer [GLS].

2.3 For a finite-dimensional algebra Γ, we denote by $0 \to \Gamma \to I_0 \to I_1 \to \cdots$ a minimal injective resolution of the Γ-module Γ. Put $\mathrm{dom.dim}\,\Gamma := \inf\{i \geq 0 \mid I_i \text{ is not projective}\}$ [T]. The following theorem gives a higher dimensional version of Theorem A.

Theorem A′ ($(n+1)$-dimensional Auslander correspondence) *For any* $n \geq 1$, *there exists a bijection between the set of equivalence classes of maximal* $(n-1)$*-orthogonal subcategories* $\mathcal{C}$ *of* mod Λ *with additive* *generators M and finite-dimensional algebras Λ, and the set of Morita-equivalence classes of finite-dimensional algebras Γ with* $\mathrm{gl.dim}\,\Gamma \leq n+1$ *and* $\mathrm{dom.dim}\,\Gamma \geq n+1$. *It is given by $\mathcal{C} \mapsto \Gamma := \mathrm{End}_\Lambda(M)$.*

2.4 In the rest of this section, let R be a complete regular local ring of dimension d, Λ an R-order which is an isolated singularity, $\mathcal{A} := \mathrm{mod}\,\Lambda$ and $\mathcal{B} := \mathrm{CM}\,\Lambda$. For $n \geq 1$, we define functors τ_n and τ_n^- by

$$\tau_n := \tau \circ \Omega^{n-1} : \underline{\mathrm{CM}}\Lambda \to \overline{\mathrm{CM}}\Lambda \quad \text{and} \quad \tau_n^- := \tau^- \circ \Omega^{-(n-1)} : \overline{\mathrm{CM}}\Lambda \to \underline{\mathrm{CM}}\Lambda,$$

where $\Omega : \underline{\mathrm{CM}}\Lambda \to \underline{\mathrm{CM}}\Lambda$ is the syzygy functor and $\Omega^- : \overline{\mathrm{CM}}\Lambda \to \overline{\mathrm{CM}}\Lambda$ is the cosyzygy functor. For a subcategory $\mathcal{C}$ of $\mathrm{CM}\,\Lambda$, we denote by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ the corresponding subcategories of $\underline{\mathrm{CM}}\Lambda$ and $\overline{\mathrm{CM}}\Lambda$ respectively.

Theorem B′ *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$ *($n \geq 1$).*

(1) (n-Auslander-Reiten translation) *For any $X \in \mathcal{C}$, $\tau_n X \in \mathcal{C}$ and $\tau_n^- X \in \mathcal{C}$ hold. Thus $\tau_n : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ and $\tau_n^- : \overline{\mathcal{C}} \to \underline{\mathcal{C}}$ are mutually quasi-inverse equivalences.*

(2) (n-Auslander-Reiten duality) *There exist functorial isomorphisms* $\overline{\mathcal{C}}(Y, \tau_n X) \simeq D\,\mathrm{Ext}^n_\Lambda(X, Y) \simeq \underline{\mathcal{C}}(\tau_n^- Y, X)$ *for any $X, Y \in \mathcal{C}$.*

2.5 Definition Let $\mathcal{C}$ be a full subcategory of $\mathrm{CM}\,\Lambda$ and $J_\mathcal{C}$ the Jacobson radical of $\mathcal{C}$. We call an exact sequence

$$0 \to Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0$$
$$(\text{resp. } 0 \to Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X)$$

with terms in $\mathcal{C}$ an *n-almost split sequence* (resp. *pseudo n-almost split sequence*) if $f_i \in J_\mathcal{C}$ holds for any i and the following sequences are exact.

$$0 \to \mathcal{C}(\ ,Y) \xrightarrow{\cdot f_n} \mathcal{C}(\ ,C_{n-1}) \xrightarrow{\cdot f_{n-1}} \cdots \xrightarrow{\cdot f_1} \mathcal{C}(\ ,C_0) \xrightarrow{\cdot f_0} J_\mathcal{C}(\ ,X) \to 0$$
$$0 \to \mathcal{C}(X,\ ) \xrightarrow{f_0 \cdot} \mathcal{C}(C_0,\ ) \xrightarrow{f_1 \cdot} \cdots \xrightarrow{f_{n-1} \cdot} \mathcal{C}(C_{n-1},\ ) \xrightarrow{f_n \cdot} J_\mathcal{C}(Y,\ ) \to 0$$

We call $f_0 : C_0 \to X$ a *sink map* and $f_n : Y \to C_{n-1}$ a *source map.* We say that $\mathcal{C}$ *has n-almost split sequences* if, for any non-projective $X \in \mathrm{ind}\,\mathcal{C}$ (resp. non-injective $Y \in \mathrm{ind}\,\mathcal{C}$), there exists an n-almost split sequence $0 \to Y \to C_{n-1} \to \cdots \to C_0 \to X \to 0$. Similarly, we say that $\mathcal{C}$ *has pseudo n-almost split sequences* if, for any projective $X \in \mathcal{C}$ (resp. injective $Y \in \mathcal{C}$), there exists a pseudo n-almost split sequence $0 \to Y \to C_{n-1} \to \cdots \to C_0 \to X$.

Theorem C′ *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$ *$(n \geq 1)$.*

(1) $\mathcal{C}$ has n-almost split sequences.

(2) If $d = n+1$, then $\mathcal{C}$ has pseudo n-almost split sequeces.

Consequently, almost all simple objects in the functor category $\mathrm{mod}\,\mathcal{C}$ have projective dimension $n+1$. If $d = n+1$, then all simple objects in the functor category $\mathrm{mod}\,\mathcal{C}$ have projective dimension $n+1$. In this sense, we can say that *$(n+1)$-dimensional Auslander-Reiten theory for the case $d = n+1$ is very nice.*

2.6 We will define the *Auslander-Reiten quiver* $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$. For simplicity, we assume that the residue field k of R is algebraically closed. The set of vertices of $\mathfrak{A}(\mathcal{C})$ is $\mathrm{ind}\,\mathcal{C}$. For $X, Y \in \mathrm{ind}\,\mathcal{C}$, we denote by d_{XY} be the multiplicity of X in C for the sink map $C \to Y$, which equals to the multiplicity of Y in C' for the source map $X \to C'$. Draw d_{XY} arrows from X to Y.

Theorem D′ *Let G be a finite subgroup of $\mathrm{GL}_d(\mathbb{C})$, $\Omega := \mathbb{C}[[x_1, \cdots, x_d]]$ and $\Lambda := \Omega^G$ the invariant subring. Assume that G does not contain pseudo-reflection except the identity, and that Λ is an isolated singularity. Then $\mathcal{C} := \mathrm{add}_\Lambda\,\Omega$ is a maximal $(d-2)$-orthogonal subcategory of $\mathrm{CM}\,\Lambda$. Moreover, the Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ of $\mathcal{C}$ coincides with the McKay quiver $\mathfrak{M}(G)$ of G, i.e. there exists a bijection $\mathbb{H} : \mathrm{irr}\,G \to \mathrm{ind}\,\mathcal{C}$ such that $d_{XY} = d_{\mathbb{H}(X),\mathbb{H}(Y)}$ for any $X, Y \in \mathrm{irr}\,G$.*

3 Non-commutative crepant resolution and representation dimension

3.1 Let us generalize the concept of Van den Bergh's non-commutative crepant resolution [V1,2] of commutative normal Gorenstein domains to our situation.

Again let Λ be an R-order which is an isolated singularity. We call $M \in \mathrm{CM}\,\Lambda$ a *NCC resolution* of Λ if $\Lambda \oplus D_d\Lambda \in \mathrm{add}\,M$ and $\Gamma := \mathrm{End}_\Lambda(M)$ is an R-order with $\mathrm{gl.dim}\,\Gamma = d$. Our definition is slightly stronger than original non-commutative crepant resolutions in [V2] where M is assumed to be reflexive (not Cohen-Macaulay) and $\Lambda \oplus D_d\Lambda \in \mathrm{add}\,M$ is not assumed. But all examples of non-commutative crepant resolutions in [V1,2] satisfy our condition. For the case $d \geq 2$, we have the remarkable relationship below between NCC resolutions and maximal $(d-2)$-orthogonal subcategories.

Theorem *Let $d \geq 2$. Then $M \in \mathrm{CM}\,\Lambda$ is a NCC resolution of Λ if and only if* $\mathrm{add}\,M$ *is maximal $(d-2)$-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$.

3.2 Conjecture It is interesting to study relationship among all maximal $(n-1)$-orthogonal subcategories of $\mathrm{CM}\,\Lambda$. Especially, we conjecture that *their endomorphism rings are derived equivalent.* It is suggestive to relate this conjecture to Van den Bergh's generalization [V2] of Bondal-Orlov conjecture [BO], which asserts that *all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category.* Since maximal $(n-1)$-orthogonal subcategories are analogy of non-commutative crepant resolutions from the viewpoint of 3.1, our conjecture is an analogy of Bondal-Orlov-Van den Bergh conjecture. We have the following partial solution.

Theorem *(1) Let $\mathcal{C}_i = \mathrm{add}\,M_i$ be a maximal 1-orthogonal subcategory of $\mathrm{CM}\,\Lambda$ and $\Gamma_i := \mathrm{End}_\Lambda(M_i)$ $(i = 1, 2)$. Then Γ_1 and Γ_2 are derived equivalent. In particular, $\#\,\mathrm{ind}\,\mathcal{C}_1 = \#\,\mathrm{ind}\,\mathcal{C}_2$ holds.*

(2) If $d \leq 3$, then all NCC resolutions of Λ have the same derived category.

3.3 Let us generalize the concept of Auslander's representation dimension [A1] to relate it to non-commutative crepant resolutios. For $n \geq 1$, define the *n-th representation dimension* $\mathrm{rep.dim}_n\,\Lambda$ of an R-order Λ which is an isolated singularity by

$$\mathrm{rep.dim}_n\,\Lambda := \inf\{\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \mid M \in \mathrm{CM}\,\Lambda,\ \Lambda \oplus D_d\Lambda \in \mathrm{add}\,M,\ M \perp_{n-1} M\}.$$

Obviously $d \leq \mathrm{rep.dim}_n\,\Lambda \leq \mathrm{rep.dim}_{n'}\,\Lambda$ holds for any $n \leq n'$. For the case $d = 0$, $\mathrm{rep.dim}_1\,\Lambda$ coincides with the representation dimension defined in [A1]. We call Λ *representation-finite* if $\#\,\mathrm{ind}(\mathrm{CM}\,\Lambda) < \infty$. In the sense of (2) below, $\mathrm{rep.dim}_1\,\Lambda$ measures how far Λ is from being representation-finite.

Theorem *(1) Assume $d \leq n+1$. Then $\mathrm{CM}\,\Lambda$ has a maximal $(n-1)$-orthogonal subcategory $\mathcal{C}$ with $\#\,\mathrm{ind}\,\mathcal{C} < \infty$ if and only if $\mathrm{rep.dim}_n\,\Lambda \leq n+1$.*

(2) Assume $d \leq 2$. Then Λ is representation-finite if and only if $\mathrm{rep.dim}_1\,\Lambda \leq 2$.

(3) Λ has a NCC resolution if and only if $\mathrm{rep.dim}_{\max\{1,d-1\}}\,\Lambda = d$.

3.4 It is an interesting problem raised by Auslander [A1] to calculate the value of $\mathrm{rep.dim}_1\,\Lambda$. In particular, when $\mathrm{rep.dim}_1\,\Lambda$ is finite? For the case $d \leq 1$, we have the finiteness result below (1). This is not true if $d \geq 2$ (2).

Theorem *(1)[I1,3] If $d \leq 1$, then* $\text{rep.dim}_1 \Lambda < \infty$.

(2) Assume $d = 2$ and that Λ is commutative Gorenstein. Then $\text{rep.dim}_1 \Lambda < \infty$ *if and only if Λ is representation-finite.*

3.5 Conjecture It seems that no example of a maximal $(n-1)$-orthogonal subcategory $\mathcal{C}$ of $\text{CM}\,\Lambda$ with $\#\,\text{ind}\,\mathcal{C} = \infty$ is known. This suggests us to study

$$o(\text{CM}\,\Lambda) := \sup_{\mathcal{C} \subseteq \text{CM}\,\Lambda,\ \mathcal{C} \perp_1 \mathcal{C}} \#\,\text{ind}\,\mathcal{C}\,.$$

We conjecture that $o(\text{CM}\,\Lambda)$ *is always finite.* If Λ is a preprojective algebra of Dynkin type Δ, then Geiss-Schröer [GS] proved that $o(\text{mod}\,\Lambda)$ equals to the number of positive roots of Δ. It would be interesting to find a geometric interpretation of $o(\text{CM}\,\Lambda)$ for more general $\text{CM}\,\Lambda$. For some classes of $\text{CM}\,\Lambda$, one can calculate $o(\text{CM}\,\Lambda)$ by using the theorem below. Especially, (1) seems to be interesting in the connection with known results for algebras with representation dimension at most 3 [IT][EHIS].

Theorem *(1)* $\text{rep.dim}_1 \Lambda \leq 3$ *implies* $o(\text{CM}\,\Lambda) < \infty$.

(2) If $\text{CM}\,\Lambda$ has a maximal 1-orthogonal subcategory $\mathcal{C}$, then $o(\text{CM}\,\Lambda) = \#\,\text{ind}\,\mathcal{C}$.

3.6 Concerning our conjecture, let us recall the well-known proposition below which follows by a geometric argument due to Voigt's lemma ([P;4.2]). It is interesting to ask whether it is true without the restriction on R. If it is true, then any 1-orthogonal subcategory of $\text{CM}\,\Lambda$ is 'discrete', and our conjecture asserts that it is finite. It is interesting to study the discrete structure of 1-orthogonal objects in $\text{CM}\,\Lambda$ and the relationship to whole structure of $\text{CM}\,\Lambda$.

Proposition *Assume that R is an algebraically closed field. For any $n > 0$, there are only finitely many isoclasses of 1-orthogonal Λ-modules X with* $\dim_R X = n$.

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A new proof of global F-regularity of Schubert varieties*

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1. Introduction

Let p be a prime number, k an algebraically closed field of characteristic p, and G a simply connected semisimple affine algebraic group over k. Let T be a maximal torus of G. We choose a base of the root system of G. Let B be the negative Borel subgroup of G. Let P be a parabolic subgroup of G containing B. The closure of a B-orbit on G/P is called a Schubert variety.

Recently, Lauritzen, Raben-Pedersen and Thomsen proved that Schubert varieties are globally F-regular [8] utilizing Bott-Samelson resolution. The objective of these notes is to give another proof. Our proof depends on a simple inductive argument utilizing the familiar technique of fibering the Schubert variety as a $\mathbb{P}^1$-bundle over a smaller Schubert variety.

Global F-regularity was first defined by Smith [15]. A projective variety over k is said to be globally F-regular if it admits a strongly F-regular homogeneous coordinate ring. As a corollary, we have that the all local rings of a Schubert variety is F-regular, in particular, F-rational, Cohen-Macaulay and normal.

A globally F-regular variety is Frobenius split. It has long been known that Schubert varieties are Frobenius split [10]. Given an ample line bundle over G/P, the associated projective embedding of a Schubert variety of G/P is projectively normal [12] and arithmetically Cohen-Macaulay [13]. We can prove that the coordinate ring is strongly F-regular in fact.

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As is pointed out in [8], we have that determinantal rings are strongly F-regular as an immediate corollary to the main theorem.

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2. Preliminaries

Let p be a prime number, and k an algebraically closed field of characteristic p. For a ring A of characteristic p, the Frobenius map $A \to A$ ($a \mapsto a^p$) is denoted by F or F_A. So F_A^e maps a to a^{p^e} for $a \in A$ and $e \geq 0$.

Let A be a k-algebra. The ring A with the k-algebra structure given by

$$k \xrightarrow{F_k^{-r}} k \to A$$

is denoted by $A^{(r)}$ for $r \in \mathbb{Z}$. Note that $F_A^e \colon A^{(r+e)} \to A^{(r)}$ is a k-algebra map for $e \geq 0$ and $r \in \mathbb{Z}$. For $a \in A$ and $r \in \mathbb{Z}$, the element a viewed as an element in $A^{(r)}$ is sometimes denoted by $a^{(r)}$. So $F_A^e(a^{(r+e)}) = (a^{(r)})^{p^e}$ for $a \in A$, $r \in \mathbb{Z}$ and $e \geq 0$.

Similarly, for a k-scheme X and $r \in \mathbb{Z}$, the k-scheme $X^{(r)}$ is defined. The Frobenius morphism $F_X^e \colon X^{(r)} \to X^{(r+e)}$ is a k-morphism.

A k-algebra A is said to be F-finite if the Frobenius map $F_A \colon A^{(1)} \to A$ is finite. A k-scheme X is said to be F-finite if the Frobenius morphism $F_X \colon X \to X^{(1)}$ is finite. Let A be an F-finite Noetherian k-algebra. We say that A is strongly F-regular if for any non-zerodivisor $c \in A$, there exists some $e \geq 0$ such that $cF_A^e \colon A^{(e)} \to A$ ($a^{(e)} \mapsto ca^{p^e}$) is a split monomorphism as an $A^{(e)}$-linear map [5]. A strongly F-regular F-finite ring is F-rational in the sense of Fedder–Watanabe [2], and is Cohen–Macaulay normal.

Let X be a quasi-projective k-variety. We say that X is globally F-regular if for any invertible sheaf $\mathcal{L}$ over X and any $a \in \Gamma(X, \mathcal{L}) \setminus 0$, the composite

$$\mathcal{O}_{X^{(e)}} \to F_*^e \mathcal{O}_X \xrightarrow{F_*^e a} F_*^e \mathcal{L}$$

has an $\mathcal{O}_{X^{(e)}}$-linear splitting [15], [4]. X is said to be F-regular if $\mathcal{O}_{X,x}$ is strongly F-regular for any closed point x of X.

Smith [15, (3.10)] proved the following fundamental theorem on global F-regularity. See also [16, (3.4)] and [4, (2.6)].

Theorem 1. *Let X be a projective variety over k. Then the following are equivalent.*

1 *There exists some ample Cartier divisor D on X such that the section ring $\bigoplus_{n\in\mathbb{Z}}\Gamma(X,\mathcal{O}(nD))$ is strongly F-regular.*

2 *The section ring of X with respect to every ample Cartier divisor is strongly F-regular.*

3 *There exists some ample effective Cartier divisor D on X such that there exists some $e \geq 0$ and an $\mathcal{O}_{X^{(e)}}$-linear splitting of $\mathcal{O}_{X^{(e)}} \to F_*^e\mathcal{O}_X \to F_*^e\mathcal{O}(D)$ and that the open set $X - D$ is F-regular.*

4 *X is globally F-regular.*

A globally F-regular variety is F-regular.

An affine k-variety $\operatorname{Spec} A$ is globally F-regular if and only if A is strongly F-regular if and only if $\operatorname{Spec} A$ is F-regular.

A globally F-regular variety is Frobenius split in the sense of Mehta-Ramanathan [10]. As the theorem above shows, if X is a globally F-regular projective variety, then the section ring of X with respect to every ample divisor is Cohen–Macaulay normal.

The following is a useful lemma.

Lemma 2 ([3, Proposition 1.2]). *Let $f\colon X \to Y$ be a k-morphism between projective k-varieties. If X is globally F-regular and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism, then Y is globally F-regular.*

Let G be a simply connected semisimple algebraic group over k, and T a maximal torus of G. We fix a base of the set of roots of G. Let B be the negative Borel subgroup. Let P be a parabolic subgroup of G containing B. Then B acts on G/P from the left. The closure of a B-orbit of G/P is called a Schubert variety. Any B-invariant closed subvariety of G/P is a Schubert variety. The set of Schubert varieties in G/B and the Weyl group $W(G)$ of G are in one-to-one correspondence. For a Schubert variety X in G/B, there is a unique $w \in W(G)$ such that $X = \overline{BwB/B}$, where the overline denotes the closure operation. We need the following theorem later.

Theorem 3. *A Schubert variety in G/P is a normal variety.*

For the proof, see [12, Theorem 3], [1], [14], and [11].

Let X be a Schubert variety in G/P. Then $\tilde{X} = \pi^{-1}(X)$ is a B-invariant reduced subscheme of G/B, where $\pi\colon G/B \to G/P$ is the canonical projection. It has a dense B-orbit, and actually $\tilde{X}$ is a Schubert variety in G/B.

As the projection π is a locally trivial P/B-fibration, so is the restriction $\pi\colon \tilde{X} \to X$. Since P/B is a k-complete variety, we have the following lemma.

Lemma 4. *$\pi_*\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$. In particular, if $\tilde{X}$ is globally F-regular, then so is X.*

Let $w \in W(G)$, and $X = X_w$ be the corresponding Schubert variety $\overline{BwB/B}$ in G/B. Assume that w is nontrivial. Then there exists some simple root α such that $l(ws_\alpha) = l(w) - 1$, where s_α is the reflection corresponding to α, and l denotes the length. Set $X' = X_{w'}$ be the Schubert variety $\overline{Bw'B/B}$, where $w' = ws_\alpha$. Let P_α be the minimal parabolic subgroup $Bs_\alpha B \cup B$. Let Y be the Schubert variety $\overline{BwP_\alpha/P_\alpha}$.

The following is due to Kempf [7, Lemma 1].

Lemma 5. *Let $\pi_\alpha\colon G/B \to G/P_\alpha$ be the canonical projection. Then X' is birationally mapped onto Y. In particular, $(\pi_\alpha)_*\mathcal{O}_{X'} = \mathcal{O}_Y$ (by Theorem 3). We have $(\pi_\alpha)^{-1}(Y) = X$, and $\pi|_X\colon X \to Y$ is a $\mathbb{P}^1$-fibration, hence is smooth.*

Let X be a Schubert variety in G/B. Let ρ be the half-sum of positive roots, and set $\mathcal{L} = \mathcal{L}((p-1)\rho)|_X$, where $\mathcal{L}((p-1)\rho)$ is the invertible sheaf on G/B corresponding to the weight $(p-1)\rho$. The following was proved by Ramanan–Ramanathan [12]. See also Kaneda [6].

Theorem 6. *There is a section $s \in H^0(X, \mathcal{L}) \setminus 0$ such that the composite*

$$\mathcal{O}_{X^{(1)}} \to F_*\mathcal{O}_X \xrightarrow{F_*s} F_*\mathcal{L}$$

splits.

Since $\mathcal{L}$ is ample, we immediately have the following.

Corollary 7. *X is globally F-regular if and only if X is F-regular.*

Proof. The 'only if' part is obvious. The 'if' part follows from the theorem and Theorem 1, **3⇒4**. □

3. Main theorem

Let k be an algebraically closed field, G a semisimple simply connected algebraic group over k, T a maximal torus of G. We fix a basis of the set of roots of G, and let B be the negative Borel subgroup of G.

In this section we prove the following theorem.

Theorem 8. *Let P be a parabolic subgroup of G containing B, and let X be a Schubert variety in G/P. Then X is globally F-regular.*

Proof. Let $\pi\colon G/B \to G/P$ be the canonical projection, and set $\tilde{X} = \pi^{-1}(X)$. Then $\tilde{X}$ is a Schubert variety in G/B. By Lemma 4, it suffices to show that $\tilde{X}$ is globally F-regular. So in the proof, we may and shall assume that $P = B$.

So let $X = \overline{BwB/B}$. We proceed by induction on the dimension of X, in other words, $l(w)$. If $l(w) = 0$, then X is a point and X is globally F-regular. Let $l(w) > 0$. Then there exists some simple root α such that $l(ws_\alpha) = l(w) - 1$. Set $w' = ws_\alpha$, $X' = \overline{Bw'B/B}$, $P_\alpha = Bs_\alpha B \cup B$, and $Y = \overline{BwP_\alpha/P_\alpha}$.

By induction assumption, X' is globally F-regular. By Lemma 5 and Lemma 2, Y is also globally F-regular. In particular, Y is F-regular. By Lemma 5, $X \to Y$ is smooth. By [9, (4.1)], X is F-regular. By Corollary 7, X is globally F-regular. □

Let k be a field, (x_{ij}) an $m \times n$ matrix with variable entries, and $S\colon = k[x_{ij}]$ the polynomial algebra with mn variables. Let $1 \le u_1 < \cdots < u_p \le m$, $0 \le r_1 < \cdots < r_p < m$, $1 \le v_1 < \cdots < v_q \le n$, and $0 \le s_1 < \cdots < s_q < n$ be sequences of integers. Let I be the ideal of S generated by the all $(r_i + 1)$-minors of the first u_i-rows of (x_{ij}) and the all $(s_j + 1)$-minors of the first v_j-columns of (x_{ij}). The ring S/I is called a *determinantal ring.*

As is pointed out in [8], we have the following.

Corollary 9. *Determinantal rings are strongly F-regular.*

Proof. Since a determinantal variety (that is, the prime spectrum of a determinantal ring) is an affine open subscheme of a Schubert variety in a Grassmann variety, the assertion follows from Theorem 8. □

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LOG RESOLUTIONS OF 3-DIMENSIONAL TORIC IDEALS

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1. Introduction

In regular local rings of dimension 2, we have the unique factorization of integrally closed ideals. We want to investidate how far the related results are true in dimension 3. As an experiment, we investigate integrally closed $\mathfrak{m}$ primary ideals in $k[x, y, z]$ generated by monomials. We call such ideals "toric" ideals.

Some ring theoretic properties of ideals are simply expressed by "geometric" language. For this purpose, the "log resolution" of an ideal is important. We explain the procedure to resolve the ideal. Also, we give a "Riemann-Roch" formula to compute the colength an $\mathfrak{m}$ primary ideal in terms of a log resolution.

Note that since we treat only ideals generated by monomials, we have only to consider the blowing-ups centered in the variety defined by monomials. Thus we can use the terminlogy of toric geometry.

(1) Log resolution of toric ideals by composition of blowing-ups.
(2) Cone of anti-nef divisors and "unique" factorization.
(3) Riemann-Roch formula.

2. Log resolution of toric ideals by composition of blowing-ups.

Notation 2.1. *We use the following notation from toric geometry.* $M \cong \mathbb{Z}^3$ *and* $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ *with* $k[M] = k[x, x^{-1}, y, y^{-1}, z, z^{-1}]$. *We put* $\underline{x}^m = x^a y^b z^c$ *for* $m = (a, b, c) \in M$. N *is the lattice of valuations on* $k[M]$ *with* $\langle m, n \rangle = as + bt + cu$ *for* $n = (s, t, u) \in N$ *and* $\underline{x}^m = x^a y^b z^c$.

We describe the blowing-up of $k[x, y, z]$ by triangulations of the triangle with $(1,0,0), (0,1,0), (0,0,1)$ as vertices. This shows the first quadrant of $N_{\mathbb{R}}$ projected to some plain, say, $x + y + z = 1$.

The blowing-up of a point (resp. $\mathbb{P}^1$) is represented by adding a new vertex $n = n_1 + n_2 + n_3$ (resp. $n = n_1 + n_2$) inside the triangle (n_1, n_2, n_3) and take the subdivision making 3 new triangles and 3 new edges (resp. on the edge (n_1, n_2) and divide the triangles on both sides of the edge). Note that by one blowing-up the number of points (resp. edges, resp. triangles) increase by 1 (resp. 3, resp. 2).

If Σ is a triangulation of the triangle with $(1,0,0),(0,1,0),(0,0,1)$ as vertices and if $\sigma = (n_1, n_2, n_3)$ is a triangle, then we define

$$U_\sigma = \mathrm{Spec}(k[\sigma^\vee \cap M]),$$

where $\sigma^\vee = \{m \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle m, n_i \rangle \geq 0 \ (i = 1,2,3)\}$ and $k[\sigma^\vee \cap M] = k[\underline{x}^m \mid m \in \sigma^\vee \cap M] \subset k[M]$. Also, we denote by X_Σ

$$X_\Sigma = \bigcup_\sigma U_\sigma,$$

where σ runs all the triangles of Σ. For example, in the triangulation of Figure , let σ be the triangle generated by $n_1 = (1,1,1), n_4 = (3,2,2), n_3 = (2,2,1)$, then since the dual base of (n_1, n_4, n_3) is $(-2,1,2)$, $(1,-1,0), (0,1,-1)$, $k[\sigma^\vee \cap M] = k[x^{-2}yz^2, xy^{-1}, yz^{-1}]$.

Notation 2.2. *We define the triangulation Σ_s and the space $X = X_{\Sigma_s}$ by giving $n_1 = (1,1,1), n_2, \ldots, n_s$ and understand that n_i is the center of i-th blowing-up by n_i. Namely, n_i is the sum of 2 (resp. 3) vectors from $n_x = (1,0,0), n_y = (0,1,0), n_z = (0,0,1), n_1, \ldots, n_{i-1}$ which are connected (resp. which make a triangle) in Σ_{i-1}. We denote the edge connecting n_i and n_j (resp. n_i and n_x) by l_{ij} (resp. l_{xi}).*

Now, let us explain the algorithm to make a log resolution of an ideal. The key is the folwoing obvious lemma.

Lemma 2.3. *Let $I = (\underline{x}^{m_1}, \ldots, \underline{x}^{m_s})$ and assume σ is a cone generated by n_1, n_2, n_3. Then $I.k[M \cap \sigma^\vee]$ is generated by $\underline{x}^{m_i}$ iff $\forall j, \forall l \ (1 \leq l \leq 3), \langle m_i, n_l \rangle \leq \langle m_j, n_l \rangle$.*

Algorithm 2.4. *We explain an algorithm to construct a log resolution of $I = (\underline{x}^{m_1}, \ldots, \underline{x}^{m_s})$. Assume we get Σ_i blowing up $n_1, \ldots, n_i$. Attach to n_k the subset W_k of $\{m_1, \ldots, m_s\}$ for which $\langle n_k, m_j \rangle$ take the mininal value. Take a triangle σ of Σ_i. If we can take a common m_i taking minimal value at each verteices of σ, then $I.k[M \cap \sigma^\vee]$ is generated by $\underline{x}^{m_i}$. For some $\sigma = (n_k, n_l, n_p)$, if any 2 intersection is empty ($W_k \cap W_l = \phi$ etc.), blow up this triangle putting $n_{i+1} = n_k + n_l + n_p$. If for some edge $n_k n_l$, $W_k \cap W_l = \phi$, then we put $n_{i+1} = n_k + n_l$. Note that by this algorithm, the choice of n_{i+1} is in no ways unique. But since the difference of $\langle n_k, m_j \rangle, m_j \in W_k$ and $\langle n_l, m_{j'} \rangle, m_{i'} \in W_l$ becomes less, this algorithm terminates at some point.*

For hand calculation, this procedure is usually too much to complete. But it will not be too difficult to let computers to do the job.

Example 2.5. *Let $I = (x^2, y^3, z^4)$. After first blowing up at $n_1 = (1,1,1)$, $W_x = \{y^3, z^4\}, W_y = \{x^2, z^4\}, W_z = \{x^2, y^3\}$ and $W_1 = \{x^2\}$. Then the job is finished for the triangle (n_1, n_y, n_z) and not yet for other 2 triangles. (Figure 1). In this manner, take $n_2 = n_x + n_1 = (2,1,1)$, $n_3 = n_2 + n_y = (2,2,1)$ and $n_4 = n_1 + n_2 = (3,2,2)$, getting Figure 2. We denote by the shaded region generated by $x^2.y^3$ or z^4.*

We can complete the process by taking $n_5 = n_2 + n_3 = (4,3,2)$ *and* $n_6 = n_5 + n_2 = (6,4,3)$. *Note that the line segments* $n_x n_6$, $n_y n_6$, $n_z n_6$ *are the boundaries of regions which have* x^2, y^3, z^4 *as generator, respectively (Figure 3).*

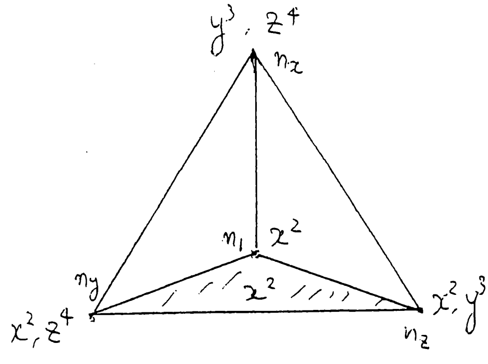

Fig 1

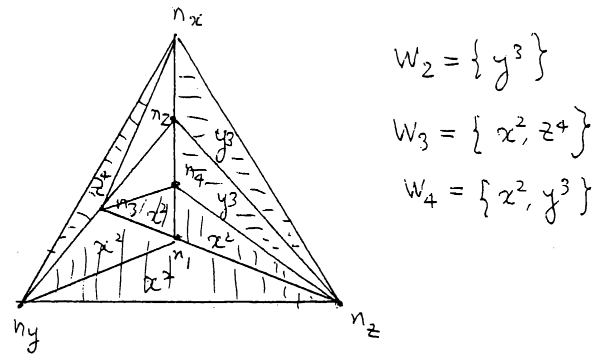

Fig. 2

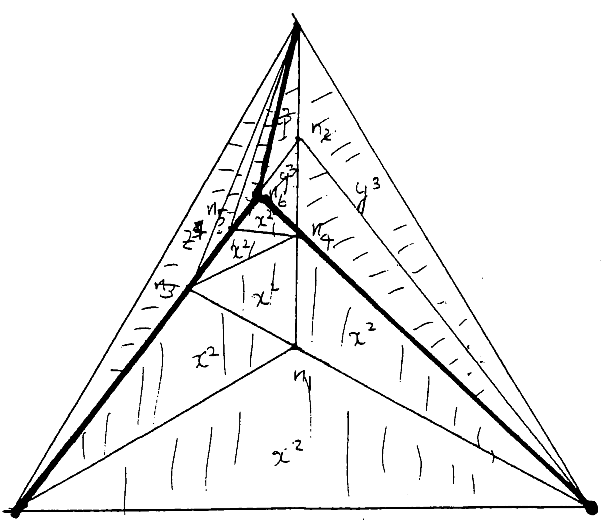

Fig. 3

3. **Cone of anti-nef divisors and "unique" factorization.**

Take $f : X \to \mathrm{Spec}(k[x,y,z]), I\mathcal{O}_X = \mathcal{O}_X(-Z)$, then since $\mathcal{O}_X(-Z)$ is generated by I, we have $(-Z).l \geq 0$ or $Z.l \leq 0$ for every line l. Here, we say a "line" connecting two vertices of the triangulation Σ. In this case we say that $-Z$ is a *nef* divisor and Z is *anti-nef.* In the toric case, if Z is anti-nef, then $\mathcal{O}_X(-Z)$ is generated by $I = H^0(X, \mathcal{O}_X(-Z))$. Hence we have the following.

Proposition 3.1. *The semigroup of integrally closed* $\mathfrak{m}$ *primary toric ideals with the property that* $I\mathcal{O}_X$ *is invertible under* $I * J = \overline{IJ}$ *is isomorphic to anti-nef cones in* $Cl(X)$.

Example 3.2. *(Huneke-Lipman) Take* $n_1 = (1,1,1), n_2 = (2,2,1), n_3 = (2,1,2), n_4 = (1,2,2)$. *Then the anti-nef cone of* X *needs 5 generators* $Z_0 = 2D_1+3D_2+3D_3+3D_4, Z_1 = D_1+D_2+D_3+D_4, Z_2 = D_1+D_2+D_3+2D_4, Z_3 = D_1+D_2+2D_3+D_4, Z_4 = D_1+2D_2+D_3+D_4$ *with the relation* $Z_0+Z_1 = Z_2+Z_3+Z_4$. *This relation induces the relation of ideals* $(x^3,y^3,z^3,xy,yz,zx)(x,y,z) = (x^2,y,z)(x,y^2,z)(x,y,z^2)$.

If X is represented by a certain triangulation, what element of I generates $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ gives a partition of the triangulation. This partition is dual to the Newton polygon P_I of I. (If $I = (\underline{x}^{m_1}, \ldots, \underline{x}^{m_s})$, the Newton polygon is the convex hull of $\bigcup_i (m_i + \mathbb{R}^3_{\leq 0})$. The lines forming the boundary are the lines with the property $Zl < 0$.

Lemma 3.3. *Assume that the line* l_{ij} *is the boundary of two triangles* $\sigma = [n, n_i, n_j]$ *and* $\sigma' = [n', n_i, n_j]$. *If* $I.k[M \cap \sigma^\vee]$ *(resp.* $k[M \cap \sigma^\vee]$*) is generated by* $\underline{x}^{m_i}$ *(resp.* $\underline{x}^{m_j}$*), then* $Z.l_{ij} = \langle m_i - m_j, n\rangle = \langle m_j - m_i, n'\rangle$.

Theorem 3.4. *If we fix the sequence of blowing-ups* $X = X_n \to X_{n-1} \to \ldots \to X_1 \to \mathrm{Spec}(k[x,y,z])$, *we can construct a set of anti-nef divisors* $Z_n, Z_{n-1}, \ldots, Z_1$ *and lines* $L_n, L_{n-1}, \ldots, L_1$ *on* X *with the property* $Z_iL_i = -1$ *and* $Z_il_j = 0$ *if* $i < j$. *Moreover,* $Z_1, \ldots, Z_i$ *is the total transform of ones which are assigned on* X_i *and* L_j *are the same lines or a part of it if some* L_j *is blown-up.*

If Z as an anti-nef divisor on X_n, then Z is written in the unique mannaer as $\mathbb{Z}$-linear combination of $Z_1, \ldots, Z_n$. But some coefficients may be negative. In the above example of Huneke-Lipman, Z_0 is anti-nef but $Z_0 = -Z_1+Z_2+Z_3+Z_4$ and negative coefficient appears. This is written symbolically as $I_0 = (x^3,y^3,z^3,xy,yz,zx) = I_1^{-1}I_2I_3I_4$, which actually means $I_0I_1 = I_2I_3I_4$. So, if we allow "negative" power, then every ideal resolved by X_n is decomposed into a product of $I_1, \ldots, I_n$ in a unique manner.

Sometimes, we can take $Z_1, \ldots, Z_n$ and $L_1, \ldots, L_n$ so that $Z_iL_i = -1$ and $Z_iL_j = 0$ if $i \neq j$. In this case, we need no negative coefficient.

Namely, if $I = H^0(X, \mathcal{O}_X(-Z))$ with Z anti-nef and if $ZL_i = -a_i$, then I is the integral closure of $I_1^{a_1} \cdots I_n^{a_n}$. ([CGL])

It is more natural to think that we assign one ideal to the new blowing-up.

Example 3.5. *If we make a log resolution of $I = \overline{(x^2, y^3, z^4)}$ by $n_1 = (1,1,1), n_2 = n_1 + n_x = (2,1,1), n_3 = n_2 + n_y = (2,2,1)$, $n_4 = n_1 + n_2 = (3,2,2), n_5 = n_2 + n_3 = (4,3,2)$, $n_6 = n_5 + n_2 = (6,4,3)$, then the sequence of ideals are $I_1 = \mathfrak{m}$, $I_2 = \overline{(x, y^2, z^2)}$, $I_3 = (x, y, z^2)$, $I_4 = \overline{(x^2, y^3, yz^2, z^4)}$, $I_5 = \overline{(x^2, y^3, y^2z, z^4)}$, $I_6 = \overline{(x^2, y^3, z^4)}$ and the corresponding divisors generate the anti-nef cone. If we define $L_1 = l_{14}, L_2 = l_{26}, L_3 = l_{13}, L_4 = l_{34}, L_5 = l_{45}, L_6 = l_{46}$ we have $Z_iL_i = -1$ for $i = 1, \ldots, 6$ and $Z_iL_j = 0$ for every $i \neq j$. Thus we have unique factorization for comlete ideals resolved by X in usual sense.*

4. Riemann-Roch Formula

I was informed the following formula from Kimio Watanabe of Tsukuba University.

Theorem 4.1. *If $f : X \to \operatorname{Spec}(k[x, y, z])$ is a log resolution of an $\mathfrak{m}$ primary ideal I in $R = k[x, y, z]$ with $I\mathcal{O}_X = \mathcal{O}_X(-Z)$, then we have*

$$l_R(R/I) = \frac{Z^3}{6} + \frac{Z^2K_X}{4} + \frac{Z(K_X^2 + c_2(X))}{12}$$

Putting nZ in place of Z, we see that Z^3 is the multiplicity of the ideal I.

It is expected that this formula will give us many nice results but until now, we can't handle this well. The proof is by induction on the components of Z and uses Riemann-Roch theorem for surface.

Here, we omit the proof and give some examples.

Example 4.2. *If Z is anti-nef on X and $I = H^0(X, \mathcal{O}_X(-Z))$, then $l_R(I/\overline{\mathfrak{m}I}) = \dfrac{ZZ_0(Z + Z_0 + K_X)}{2} + 1 \leq \mu(I)$, where $\mathfrak{m} = (x, y, z)$ and $\mathfrak{m}\mathcal{O}_X = \mathcal{O}_X(-Z_0)$.*

Example 4.3. *It will be nice if $\mathfrak{m}I$ is integrally closed for all integrally closed toric ideal I. But unfortunately, this is not the case. For example, let $I = \overline{(x^5, y^3z^4, y^{20}, z^{20})}$ and take $u = x^3y^2z^2$. Then $u^2 = xy(x^5)(y^3z^4) \in \mathfrak{m}^2I^2$ and hence $u \in \overline{\mathfrak{m}I}$. But it is easy to see that $u \notin \mathfrak{m}I$.*

Example 4.4. *In the example 3.5, we can calculate $Z_0^3 = 11, Z_0^2K_X = 14$, $Z_0K_X^2 = 28$ and $Z_0c_2(X) = 0$, having $l_R(R/\overline{I^n}) = \dfrac{11n^3 + 21n^2 + 14n}{6}$ by Riemann-Roch formula.*

Remark 4.5. *I stated in my talk the question. "For every "toric" integrally closed ideals, do we have $l_R(R/I) > e(I)/6$ always (where $e(I)$ denotes the multiplicity of I) ?" We can prove that if Z is anti-nef, then $Z^2K_X > 0$ always as is shown in the following Lemma. After my talk, B. Ulrich kindly informed me that it is proved by Lech ([Le]) that $e(I)/l_R(R/I) \leq (d!)e(\mathfrak{m})$ for every $\mathfrak{m}$-primary ideal I in a d dimensional local ring R.*

Lemma 4.6. *Let Z be an anti-nef cycle on X. Then $-ZD_i$ is linearly equivalent to "effective" 1-dimensional cycle, that is, $\sum_l n_l l$, where l is a line and $n_l \geq 0$. Consequently, $ZZ'D_i \geq 0$ for every anti-nef cycle Z'.*

(Proof) Put $Z = \sum a_i D_i$. Then since $\mathcal{O}_X(-Z) = I\mathcal{O}_X$, where $I = H^0(X, \mathcal{O}_X(-Z))$, there exists $x^m \in I$ such that $\langle m, n_i \rangle = a_i$ and $\langle m, n_j \rangle \geq a_j$ for every j. Note that $a_i D_i + \sum_j \langle m, n_j \rangle D_j$ is linearly equivalent to 0.

Then $ZD_i = a_i D_i^2 + \sum_j a_j l_{ij}$, where j runs the indices where n_j and n_i are connected. Now, since $a_i D_i^2$ is linearly equivalent to $-\sum_j \langle m, n_j \rangle D_j$, ZD_i is linearly equivalent to $\sum_{j \neq i} (-\sum_j \langle m, n_j \rangle + a_j) l_{ij}$ and each coefficient is non-positive as noted above.

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An application of the singular Riemann-Roch formula to the theory of Hilbert-Kunz function

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1 Introduction

Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic p, where p is a prime integer. For an $\mathfrak{m}$-primary ideal I and a positive integer e, we set

$$I^{[p^e]} = (a^{p^e} \mid a \in I).$$

It is easy to see that $I^{[p^e]}$ is an $\mathfrak{m}$-primary ideal of A. For a finitely generated A-module M, the function $\ell_A(M/I^{[p^e]}M)$ on e is called the *Hilbert-Kunz funtion* of M with respect to I. It is known that

$$\lim_{e \to \infty} \frac{\ell_A(M/I^{[p^e]}M)}{p^{de}}$$

exists [6], and the real number is called the *Hilbert-Kunz multiplicity*, that is denoted by $e_{HK}(I, M)$. The properties of $e_{HK}(I, M)$ are studied by many authors (Monsky, Watanabe, Yoshida, Huneke, Enescue, e.t.c.).

Recently Huneke, McDermott and Monsky proved the following theorem:

Theorem 1.1 (Huneke, McDermott and Monsky [4]) *Let $(A, \mathfrak{m}, k)$ be a d-dimensional excellent normal local ring of characteristic p, where p is a prime integer. Assume that A is F-finite[1] and the residue class field k is perfect.*

[1] We say that A is F-finite if the Frobenius map $F : A \to A = {}^1A$ is module-finite. We sometimes denote the e-th iteration of F by $F^e : A \to A = {}^eA$.

1. *For an $\mathfrak{m}$-primary ideal I of A and a finitely generated A-module M, there exists a real number $\beta(I, M)$ that satisfies the following equation*[2]*:*

$$\ell_A(M/I^{[p^e]}M) = e_{HK}(I, M) \cdot p^{de} + \beta(I, M) \cdot p^{(d-1)e} + O(p^{(d-2)e})$$

2. *Let I be an $\mathfrak{m}$-primary ideal of A. Then, there exists a $\mathbb{Q}$-homomorphism $\tau_I : \mathrm{Cl}(A)_{\mathbb{Q}} \longrightarrow \mathbb{R}$ that satisfies*

$$\beta(I, M) = \tau_I \left(\mathrm{cl}(M) - \frac{\mathrm{rank}_A M}{p^d - p^{d-1}} \mathrm{cl}({}^1A) \right),$$

for any finitely generated A-module M. In particular,

$$\beta(I, A) = -\frac{1}{p^d - p^{d-1}} \tau_I \left(\mathrm{cl}({}^1A) \right)$$

is satisfied.

For an abelian group N, $N_{\mathbb{Q}}$ stands for $N \otimes_{\mathbb{Z}} \mathbb{Q}$.
It is natural to ask the following question:

Question 1.2 1. *When does* $\mathrm{cl}({}^1A)$ *vanish?*

2. *How does* $\mathrm{cl}({}^eA)$ *behave?*

In the next section, we give a partial answer to this question.

Remark 1.3 The map cl in the theorem as above is called the *determinant map* [1]. Here we recall basic properties on cl.

Let R be a Noetherian normal domain. The group of isomorphism classes of reflexive R-modules of rank 1 is called the *divisor class group* of R, and denoted by $\mathrm{Cl}(R)$. Let $\mathrm{G}_0(R)$ be the Grothendieck group of finitely generated R-modules. Then, there exists the map

$$\mathrm{cl} : \mathrm{G}_0(R) \longrightarrow \mathrm{Cl}(R)$$

that satisfies the following two conditions:

---

[2]Let $f(e)$ and $g(e)$ be functions on e. We denote $f(e) = O(g(e))$ if there exists a real number K that satisfies $|f(e)| < Kg(e)$ for any e.

(i) If M is a reflexive module of rank 1, then $\mathrm{cl}(M)$ is the isomorphism class that contains M.

(ii) Let M be a finitely generated R-module. If the height of the annihilator of M is greater than or equal to 2, then $\mathrm{cl}(M) = 0$.

Example 1.4 1. This example is due to Han-Monsky [3]. Set $A = \mathbb{F}_5[[x_1, \dots, x_4]]/(x_1^4 + \cdots + x_4^4)$ and $m = (x_1, \dots, x_4)A$. Then,

$$\ell_A(A/m^{[p^e]}) = \frac{168}{61}5^{3e} - \frac{107}{61}3^e$$

is satisfied. Therefore, in this case, we have $e_{HK}(m, A) = \frac{168}{61}$ and $\beta(m, A) = 0$. We know that there is no hope to extend Theorem 1.1 under the same assumption.

2. Set

$$A = k[[x_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n]]/I_2(x_{ij}),$$

where k is a perfect field of characteristic $p > 0$.

Suppose $m = 2$ and $n = 3$. Then, K.-i. Watanabe proved

$$\ell_A(A/m^{[p^e]}) = (13p^{4e} - 2p^{3e} - p^{2e} - 2p^e)/8.$$

Therefore, we have $e_{HK}(m, A) = \frac{13}{8}$ and $\beta(m, A) = -\frac{1}{4} \neq 0$.

One can prove that, if $m \neq n$, then there exists an maximal primary ideal I (of finite projective dimension) such that $\beta(I, A) \neq 0$.

In Corollary 2.2, we will see that $\beta(I, A) = 0$ if A is a Gorenstein ring.

2 Main Theorem

Here, we state the main theorem.

Let $F^e : A \to A = {}^eA$ be the e-th iteration of the Frobenius map F.

Theorem 2.1 *Let (A, m, k) be a d-dimensional Noetherian excellent normal local ring of characteristic p, where p is a prime integer, and assume that A is a homomorphic image of a regular local ring. Assume that k is a perfect field and A is F-finite.*

Then, for each integer $e > 0$, we have

$$\mathrm{cl}({}^eA) = \frac{p^{de} - p^{(d-1)e}}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

The following one is an immediate consequence of the above theorem:

Corollary 2.2 *Under the same assumption as in the above theorem, if* $\mathrm{cl}(\omega_A)$ *is a torsion in* $\mathrm{Cl}(A)$*, then* $\beta(I, A) = 0$ *for any maximal primary ideal* I.

The following is an analogue of Theorem 2.1 for normal algebraic varieties.

Theorem 2.3 *Let k be a perfect field of characteristic p, where p is a prime integer. Let X be a normal algebraic variety over k of dimension d. Let $F : X \to X$ be the Frobenius map*[3].

Then, we have

$$c_1(F_*^e\mathcal{O}_X) = \frac{p^{de} - p^{(d-1)e}}{2}[K_X]$$

in $\mathrm{Cl}(X)_{\mathbb{Q}} = A_{d-1}(X)_{\mathbb{Q}}$*, where* $c_1(\ )$ *is the first local Chern character*[4].

We give an outline of a proof of Theorem 2.1 in the next section.

Example 2.4 1. Set

$$A = k[[x_1, x_2, x_3, y_1, y_2, y_3]] \Big/ I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

$\mathfrak{p} = (x_1, x_2, x_3)A$ and $\mathfrak{q} = (x_1, y_1)A$, where $I_2(\ )$ is the ideal generated by all the 2 by 2 minors of the given matrix. Here, assume that k is a perfect field of characteristic 2. Then, using Hirano's formula, we know that

$${}^1A \simeq A^{\oplus 10} \oplus \mathfrak{p} \oplus \mathfrak{q}^{\oplus 5}.$$

Here, recall that $\mathrm{rank}_A {}^1A = p^{\dim A} = 2^4 = 16$.

[3]Remark that, under the assumption, F is a finite morphism.

[4]Set $U = X \setminus \mathrm{Sing}(A)$. Since $\mathrm{codim}_X \mathrm{Sing}(A) \geq 2$, the restriction $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism. Here, remark that $F|_U : U \to U$ is flat. Therefore, $(F_*^e\mathcal{O}_X)|_U = (F|_U)_*^e\mathcal{O}_U$ is a vector bundle on U. Here, $c_1(F_*^e\mathcal{O}_X)$ is defined to be the first Chern class $c_1((F_*^e\mathcal{O}_X)|_U) \in \mathrm{Cl}(U) = \mathrm{Cl}(X)$.

Then, we have

$$\mathrm{cl}({}^1A) = 10\mathrm{cl}(A) + \mathrm{cl}(\mathfrak{p}) + 5\mathrm{cl}(\mathfrak{q}) = 4\mathrm{cl}(\mathfrak{q})$$

since $\mathrm{cl}(A) = 0$ and $\mathrm{cl}(\mathfrak{p}) + \mathrm{cl}(\mathfrak{q}) = 0$.

On the other hand, it is well known that $\omega_A \simeq \mathfrak{q}$. By Theorem 2.1, we have

$$\mathrm{cl}({}^1A) = \frac{2^4 - 2^3}{2}\mathrm{cl}(\omega_A) = 4\mathrm{cl}(\mathfrak{q}).$$

2. Let k be a perfect field of characteristic p, where p is a prime integer. Put $X = \mathbb{P}^1_k$. Let $F : X \to X$ be the Frobenius map. Then, we have $F_*\mathcal{O}_X \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{\oplus(p-1)}$, and

$$c_1(F_*\mathcal{O}_X) = c_1(\wedge^p F_*\mathcal{O}_X) = c_1(\mathcal{O}_X(1-p)) = 1-p.$$

Remark that the natural map $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is an isomorphism in this case.

On the other hand, it is well known that $\omega_X \simeq \mathcal{O}_X(-2)$. Therefore, we have $[K_X] = -2$. By Theorem 2.3, we have

$$c_1(F_*\mathcal{O}_X) = \frac{p-1}{2}[K_X] = 1-p.$$

3 Proof of Theorem 2.1

Now we start to prove Theorem 2.1.

Let $(A, \mathfrak{m})$ be a Noetherian local ring that satisfies the assumption in Theorem 2.1.

Since $(A, \mathfrak{m})$ is a homomorphic image of a regular local ring, we have an isomorphism

$$\tau_A : \mathrm{G}_0(A)_\mathbb{Q} \longrightarrow \mathrm{A}_*(A)_\mathbb{Q}$$

of $\mathbb{Q}$-vector spaces by the singular Riemann-Roch theorem (Chapter 18 in [2]), where $\mathrm{A}_*(A) = \oplus_{i=0}^d \mathrm{A}_i(A)$ is the Chow group of the affine scheme $\mathrm{Spec}(A)$. Let

$$p : \mathrm{A}_*(A)_\mathbb{Q} \longrightarrow \mathrm{A}_{d-1}(A)_\mathbb{Q} = \mathrm{Cl}(A)_\mathbb{Q}$$

be the projection. We set

$$\tau_{d-1} = p\tau_A : \mathrm{G}_0(A)_\mathbb{Q} \longrightarrow \mathrm{Cl}(A)_\mathbb{Q}.$$

Here, we summarize basic facts on the map τ_{d-1}.

(i) Let $\mathfrak{p}$ be a prime ideal of height 1. There exists a natural identification $\mathrm{A}_{d-1}(A) = \mathrm{Cl}(A)$ by $[\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{cl}(\mathfrak{p})$. By the exact sequence

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0,$$

we have

$$\mathrm{cl}(\mathfrak{p}) = \mathrm{cl}(A) - \mathrm{cl}(A/\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

On the other hand, by the top-term property (Theorem 18.3 (5) in [2]), we have $\tau_{d-1}(A/\mathfrak{p}) = [\mathrm{Spec}(A/\mathfrak{p})]$. Therefore we have

$$\tau_{d-1}(A/\mathfrak{p}) = [\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{cl}(\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

Let $\mathfrak{q}$ be a prime ideal of height at least 2. By the top-term property, we have $\tau_{d-1}(A/\mathfrak{q}) = 0$.

(ii) By the covariance with proper maps (Theorem 18.3 (1) in [2]), we have

$$\tau_{d-1}({}^{e}A) = p^{(d-1)e}\tau_{d-1}(A)$$

for each $e > 0$.

(iii) We have

$$\tau_{d-1}(A) = \frac{1}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$ by Lemma 3.5 of [5].

Next we prove the following lemma:

Lemma 3.1 *Let $(A, \mathfrak{m})$ be a local ring that satisfies the assumption in Theorem 2.1. Then, for a finitely generated A-module M, we have*

$$\tau_{d-1}(M) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

Proof. Set $r = \mathrm{rank}_A M$. Then we have an exact sequence

$$0 \to A^r \to M \to T \to 0,$$

where T is a torsion module. By this exact sequence, we obtain

$$\mathrm{cl}(M) = r \cdot \mathrm{cl}(A) + \mathrm{cl}(T) = \mathrm{cl}(T).$$

On the other hand, by the basic fact (iii) as above, we obtain

$$\tau_{d-1}(M) = r \cdot \tau_{d-1}(A) + \tau_{d-1}(T) = \frac{r}{2}\mathrm{cl}(\omega_A) + \tau_{d-1}(T).$$

We have only to prove $\tau_{d-1}(T) = -\mathrm{cl}(T)$.

We may assume that $T = A/\mathfrak{p}$, where $\mathfrak{p} \neq 0$ is a prime ideal of A. If $\mathrm{ht}\,\mathfrak{p} \geq 2$, then we have

$$\tau_{d-1}(A/\mathfrak{p}) = 0 = -\mathrm{cl}(A/\mathfrak{p})$$

by Remark 1.3 and the basic fact (i) as above. If $\mathrm{ht}\,\mathfrak{p} = 1$, then we have

$$\tau_{d-1}(A/\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p})$$

by (i) as above. **q.e.d.**

Now we start to prove Theorem 2.1.

By the basic facts (ii) and (iii), we obtain

$$\tau_{d-1}({}^eA) = p^{(d-1)e}\tau_{d-1}(A) = \frac{p^{(d-1)e}}{2}\mathrm{cl}(\omega_A).$$

By Lemma 3.1, we have

$$\tau_{d-1}({}^eA) = -\mathrm{cl}({}^eA) + \frac{\mathrm{rank}_A\, {}^eA}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. It is easy to see that $\mathrm{rank}_A\, {}^eA = p^{de}$. We have obtained

$$\mathrm{cl}({}^eA) = \frac{p^{de} - p^{(d-1)e}}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. **q.e.d.**

Remark 3.2 By Theorem 2.1 and Lemma 3.1, we have

$$\tau_{d-1}(M) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2}\mathrm{cl}(\omega_A) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{p^d - p^{d-1}}\mathrm{cl}({}^1A).$$

Therefore, we have

$$\beta(I, M) = -\tau_I(\tau_{d-1}(M)).$$

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Schubert cycles are level

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1 Introduction

Grassmannians and their Schubert subvarieties are important and fascinating objects of algebraic geometry and commutative algebra. And their homogeneous coordinate rings are extensively studied. For example, it is known that they are normal Cohen-Macaulay domains and the homogeneous coordinate rings of Grassmannians are Gorenstein. The characterization of the Gorenstein property of the homogeneous coordinate ring of a Schubert variety (Schubert cycle for short) is also known.

On the other hand, Stanley [Sta1] defined the notion called level for standard graded algebras. This is a notion between Cohen-Macaulay property and Gorenstein property for standard graded algebras.

In this note, we prove that every Schubert cycle is level, by using the theory of sagbi basis and Hibi ring.

2 Preliminaries

In this note, all rings and algebras are commutative with identity element.

We first recall the results and notation of Hibi [Hib], with a few modication.

We denote by $\boldsymbol{N}$ the set of all non-negative integers.

Let P be a finite partially ordered set (poset for short).

The length of a chain (totally ordered subset) X of P is $\#X - 1$, where $\#X$ is the cardinality of X.

The rank of P, denoted by rank$P$, is the maximum of the lengths of chains in P.

A poset is said to be pure if its all maximal chains have the same length.

The height (resp. coheight) of an element $x \in P$, denoted by $\mathrm{ht}_P x$ or simply $\mathrm{ht} x$ (resp. $\mathrm{coht}_P\, x$ or $\mathrm{coht}\, x$), is the rank of $\{y \in P \mid y \leq x\}$ (resp. $\{y \in P \mid y \geq x\}$).

A poset ideal (resp. filter) of P is a subset I of P such that $x \in I$, $y \in P$ and $y \leq x$ (resp. $y \geq x$) imply $y \in I$.

For x, $y \in P$, y covers x, denoted by $x \lessdot y$, means $x < y$ and there is no $z \in P$ such that $x < z < y$.

We denote by $\widehat{P}$ the extended poset $P \cup \{\infty, -\infty\}$ where ∞ and $-\infty$ are new elements and $-\infty < x < \infty$ for any $x \in P$.

A map $\nu \colon \widehat{P} \to \boldsymbol{N}$ is order reversing if $x \leq y$ in $\widehat{P}$ implies $\nu(x) \geq \nu(y)$ and strictly order reversing if $x < y$ in $\widehat{P}$ implies $\nu(x) > \nu(y)$.

The set of all order reversing maps (resp. strictly order reversing maps) from $\widehat{P}$ to $\boldsymbol{N}$ which map ∞ to 0 is denoted by $\overline{\mathcal{T}}(P)$ (resp. $\mathcal{T}(P)$).

Now let D be a finite distributive lattice and K a field. A join-irreducible element in D is an element in D which covers exactly one element in D. Recall the result of Birkhoff [Bir]. Let P be the set of all join-irreducible elements in D. Then D is isomorphic to $J(P)$ ordered by inclusion, where $J(P)$ is the set of all poset ideals of P. The isomorphisms $\Phi \colon D \to J(P)$ and $\Psi \colon J(P) \to D$ are given by

$$\begin{aligned} \Phi(\alpha) &:= \{x \in P \mid x \leq \alpha \text{ in } D\} && \text{for } \alpha \in D \text{ and} \\ \Psi(I) &:= \bigvee_{x \in I} x && \text{for } I \in J(P), \end{aligned}$$

where empty join is defined to be the minimal element of D.

Let $\{T_x\}_{x \in \widehat{P}}$ and $\{X_\alpha\}_{\alpha \in D}$ be families of indeterminates. Hibi [Hib] defined the ring $\mathcal{R}_K(D)$, which is now called the Hibi ring, by

$$\mathcal{R}_K(D) := K[\textstyle\prod_{x \in I \cup \{-\infty\}} T_x \mid I \in J(P)]$$

as a subring of the polynomial ring $K[T_x \mid x \in \widehat{P}]$. He also showed that $\mathcal{R}_K(D)$ is isomorphic to

$$K[X_\alpha \mid \alpha \in D]/(X_\alpha X_\beta - X_{\alpha \wedge \beta} X_{\alpha \vee \beta} \mid \alpha, \beta \in D).$$

The isomorphism is induced by the K-algebra homomorphism $K[X_\alpha \mid \alpha \in D] \to K[T_x \mid x \in \widehat{P}]$ sending X_α to $T_{-\infty} \prod_{x \leq \alpha} T_x$. And $\mathcal{R}_K(D)$ is a graded algebra with straightening law (ASL for short) over K generated by D.

It is easily verified that $\mathcal{R}_K(D)$ is an affine semigroup ring such that

$$\mathcal{R}_K(D) = \bigoplus_{\nu \in \overline{\mathcal{T}}(P)} K(\prod_{x \in \widehat{P}} T_x^{\nu(x)}).$$

So by the result of Hocster [Hoc], $\mathcal{R}_K(D)$ is a normal Cohen-Macaulay domain. And by the result of Stanley [Sta2],

$$\bigoplus_{\nu \in \mathcal{T}(P)} K(\prod_{x \in \widehat{P}} T_x^{\nu(x)})$$

is the canonical module of $\mathcal{R}_K(D)$.

3 A sufficient condtion for $\mathcal{R}_K(D)$ to be level

Let D be a finite distributive lattice and let P be the set of all join-irreducible elements in D. We introduce the homogeneous grading on $\mathcal{R}_K(D) \simeq K[X_\alpha \mid \alpha \in D]/(X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta} \mid \alpha, \beta \in D)$ by setting $\deg X_\alpha = 1$ for any $\alpha \in D$. Then $\mathcal{R}_K(D)$ is a standard graded algebra by the terminology of [Sta1].

Stanley [Sta1] defined the level property for standard graded algebra.

Definition 3.1 Let $A = \bigoplus_{n\geq 0} A_n$ be a Cohen-Macaulay standard graded K-algebra, and let $(h_0, \ldots, h_s)$, $h_s \neq 0$ be the h-vector of A. Then we say that A is level if $h_s = \text{type} A$.

Suppose that $\dim_K A_1 = n$ and $\dim A = d$ in the notation above, and $S \to A$ is a natural K-algebra epimorphism, where S is a polynomial ring over K with n variables. Then the following fact is easily verified.

Fact 3.2 *The following conditions are equivalent.*

(1) A is level.

(2) The degree of generators of the canonical module of A is constant.

(3) If $F_\bullet$ is the minimal S-free resolution of A, then the degree of generators of F_{n-d} is constant.

Now we give a sufficient condition for $\mathcal{R}_K(D)$ to be level.

Theorem 3.3 *Assume that $\{y \in P \mid y \geq x\}$ is pure for any $x \in P$, then $\mathcal{R}_K(D)$ is level.*

Note that the condition on P above is equivalent to the following: for any x, $y \in P$ with $x \lessdot y$, $\operatorname{coht}_P x = \operatorname{coht}_P y + 1$.

Theorem 3.3 is a direct consequence of the following lemma and the Stanley's discription of the canonical module of a normal affine semigroup ring.

Lemma 3.4 *In the situation of Theorem 3.3, for any $\nu \in \mathcal{T}(P)$ there is $\nu_0 \in \mathcal{T}(P)$ such that $\nu_0(-\infty) = \operatorname{rank}\widehat{P}$ and $\nu - \nu_0 \in \overline{\mathcal{T}}(P)$, where we set $(\nu - \nu_0)(x) := \nu(x) - \nu_0(x)$ for any $x \in \widehat{P}$.*

proof We set $r = \operatorname{rank}\widehat{P}$ and

$$\nu_0(x) := \max\{\operatorname{coht}_{\widehat{P}} x, r - \nu(-\infty) + \nu(x)\}$$

for any $x \in \widehat{P}$. Then we

Claim *For any elements x, $y \in \widehat{P}$ with $x \lessdot y$, $\nu_0(x) > \nu_0(y)$ and $\nu(x) - \nu(y) \geq \nu_0(x) - \nu_0(y)$.*

We postpone the proof of the claim and finish the proof of the lemma first.

Since $\nu_0(x) > \nu_0(y)$ and $\nu(x) - \nu(y) \geq \nu_0(x) - \nu_0(y)$ for any x, $y \in \widehat{P}$ with $x \lessdot y$, it is easily verified that, $\nu_0(x) > \nu_0(y)$ and $\nu(x) - \nu(y) \geq \nu_0(x) - \nu_0(y)$ for any x, $y \in \widehat{P}$ with $x < y$. And $\nu_0(\infty) = 0$ by definition. So $\nu_0 \in \mathcal{T}(P)$. And $\nu(x) - \nu_0(x) \geq \nu(y) - \nu_0(y) \geq \nu(\infty) - \nu_0(\infty) = 0$ for any x, $y \in P$ with $x < y$. Therefore $\nu - \nu_0 \in \overline{\mathcal{T}}(P)$. Note $\nu_0(-\infty) = r$ by definition.

Now we prove the claim. Consider the case where $\operatorname{coht}_{\widehat{P}} x \leq r - \nu(-\infty) + \nu(x)$ first. Since $\operatorname{coht}_{\widehat{P}} y \leq \operatorname{coht}_{\widehat{P}} x - 1$ and $\nu(y) < \nu(x)$, we see that

$$\operatorname{coht}_{\widehat{P}} y < \operatorname{coht}_{\widehat{P}} x \leq r - \nu(-\infty) + \nu(x)$$

and

$$r - \nu(-\infty) + \nu(y) < r - \nu(-\infty) + \nu(x).$$

Therefore

$$\begin{aligned} \nu_0(x) &= r - \nu(-\infty) + \nu(x) \\ &> \max\{\operatorname{coht}_{\widehat{P}} y, r - \nu(-\infty) + \nu(y)\} \\ &= \nu_0(y) \end{aligned}$$

and

$$\begin{aligned}\nu(x)-\nu(y) &= (r-\nu(-\infty)+\nu(x))-(r-\nu(-\infty)+\nu(y)) \\ &\geq \nu_0(x)-\nu_0(y).\end{aligned}$$

Next consider the case where $\operatorname{coht}_{\widehat{P}} x > r-\nu(-\infty)+\nu(x)$. This happens only when $x \in P$. Therefore $\operatorname{coht}_{\widehat{P}} y = \operatorname{coht}_{\widehat{P}} x - 1$ by assumption. Since

$$r-\nu(-\infty)+\nu(y) < r-\nu(-\infty)+\nu(x) \leq \operatorname{coht}_{\widehat{P}} x - 1,$$

we see that

$$\operatorname{coht}_{\widehat{P}} y > r-\nu(-\infty)+\nu(y).$$

Therefore

$$\nu_0(y) = \operatorname{coht}_{\widehat{P}} y = \operatorname{coht}_{\widehat{P}} x - 1 = \nu_0(x) - 1 < \nu_0(x)$$

and

$$\nu_0(x)-\nu_0(y) = 1 \leq \nu(x)-\nu(y).$$

So we see the claim. ▮

4 Levelness of Schubert cycles

Now we fix a field K and integers m and n with $1 \leq m \leq n$.

For an $m \times n$ matrix M with entries in a K-algebra S, we denote by $K[M]$ the K-subalgebra of S generated by all the entries of M and by $G(M)$ the K-subalgebra of S generated by all maximal minors of M. We also denote by $\Gamma(M)$ the set of all maximal minors of M.

It is known that the homogeneous coordinate ring of the Grassmann variety $G_m(V)$ of m-dimensional subspaces of an n-dimensional K-vector space V is $G(X)$, where X is an $m \times n$ matrix of indeterminates. It is known that $G(X)$ is an ASL over K generated by $\Gamma(X)$, where we identify $\Gamma(X)$ with a combinatorial object $\{[c_1, \ldots, c_m] \mid 1 \leq c_1 < \cdots < c_m \leq n\}$ and define the order of $\Gamma(X)$ by

$$[c_1, \ldots, c_m] \leq [d_1, \ldots, d_m] \overset{\text{def}}{\Longleftrightarrow} c_i \leq d_i \text{ for } i = 1, \ldots, m.$$

Let $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ be a complete flag of subspaces of V and let $a_1, \ldots, a_m$ be integers such that $1 \leq a_1 < \cdots < a_m \leq n$. Then the Schubert subvariety $\Omega(a_1, \ldots, a_m)$ of $G_m(V)$ is defined by

$$\Omega(a_1, \ldots, a_m) := \{W \in G_m(V) \mid \dim(W \cap V_{a_i}) \geq i \text{ for } i = 1, \ldots, m\}.$$

If we put $b_i = n + 1 - a_{m+1-i}$ for $i = 1, \ldots, m$, $\gamma = [b_1, \ldots, b_m]$ and $\Gamma(X;\gamma) = \{\delta \in \Gamma(X) \mid \delta \geq \gamma\}$, then the homogeneous coordinate ring of the Schubert variety $\Omega(a_1, \ldots, a_m)$ (Schubert cycle for short) is

$$G(X;\gamma) := G(X)/(\Gamma(X) \setminus \Gamma(X;\gamma))G(X).$$

This ring is a graded ASL over K generated by $\Gamma(X;\gamma)$ ([DEP], [BV]). It is also known that $G(X;\gamma) \simeq G(U_\gamma)$, where U_γ is the following $m \times n$ matrix with independent indeterminates U_{ij} [BV].

$$\begin{pmatrix} 0 & \cdots & 0 & U_{1b_1} & \cdots & U_{1b_2-1} & U_{1b_2} & \cdots & \cdots & \cdots & \cdots & U_{1n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & U_{2b_2} & \cdots & \cdots & \cdots & \cdots & U_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U_{mb_m} & \cdots & U_{mn} \end{pmatrix}$$

We denote the isomorphism $G(X;\gamma) \xrightarrow{\sim} G(U_\gamma)$ by Φ.

Now we introduce a diagonal term order on the polynomial ring $K[U_\gamma]$. That is, a monomial order on $K[U_\gamma]$ such that the leading monomial of any minor of U_γ is the product of the main diagonal of it. For example, the degree lexicographic order induced by $U_{1b_1} > U_{1b_1+1} > \cdots > U_{1n} > U_{2b_2} > U_{2b_2+1} > \cdots > U_{mn}$.

Then it is easy to see that for standard monomials (of an ASL) μ and μ' on $\Gamma(X;\gamma)$ with $\mu \neq \mu'$, $\mathrm{lm}(\Phi(\mu)) \neq \mathrm{lm}(\Phi(\mu'))$. So $\Phi(\Gamma(X;\gamma))$ is a sagbi basis of $G(U_\gamma)$, since $\{\Phi(\mu) \mid \mu$ is a standard monomial on $\Gamma(X;\gamma)\}$ is a K-vector space basis of $G(U_\gamma)$. And $\{\mathrm{lm}(\Phi(\mu)) \mid \mu$ is a standard monomial on $\Gamma(X;\gamma)\}$ is a K-vector space basis of the initial subalgebra $\mathrm{in}\, G(U_\gamma)$ of $K[U_\gamma]$. Note also that $\mathrm{lm}(\Phi(\alpha))\mathrm{lm}(\Phi(\beta)) = \mathrm{lm}(\Phi(\alpha \wedge \beta))\mathrm{lm}(\Phi(\alpha \vee \beta))$ for any α, $\beta \in \Gamma(X;\gamma)$. So $\mathrm{in}\, G(U_\gamma)$ is the Hibi ring $\mathcal{R}_K(\Gamma(X;\gamma))$.

Since the poset of all join-irreducible elements of $\Gamma(X;\gamma)$ is anti-isomorphic to a finite ideal of $\boldsymbol{N} \times \boldsymbol{N}$ with the componentwise order [Miy], we see by Theorem 3.3 that $\mathrm{in}\, G(U_\gamma)$ is a level ring.

Now by a standard deformation argument, we see the following

Proposition 4.1 *Let A be a K-subalgebra of a polynomial ring $K[Y_1, \ldots, Y_s]$ with finite homogeneous sagbi basis $f_1, \ldots, f_r$. Let $S = K[X_1, \ldots, X_r]$ be the polynomial ring with r variables over K. We make A and $\mathrm{in}\, A$ S-algebras by K-algebra homomorphisms $X_i \mapsto f_i$ and $X_i \mapsto \mathrm{lt}(f_i)$. Then*

$$\dim_K \mathrm{Tor}_i^S(A, K)_j \leq \dim_K \mathrm{Tor}_i^S(\mathrm{in}\, A, K)_j$$

for any i and j.

Since A and in A have the same Hilbert function, A is Cohen-Macaulay, level or Gorenstein if so is in A. Therefore we see the following

Theorem 4.2 *Schubert cycles are level.*

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COHEN–MACAULAY STANLEY–REISNER RINGS AND MULTIPLICITIES

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Dedicated to Professor Kei-ichi Watanabe on the occasion of his sixtieth birthday.

1. Introduction

Throughout this article, let $S = k[X_1, \dots, X_n]$ be a homogeneous polynomial ring with n variables over a field k with $\deg X_i = 1$. For a simplicial complex Δ on the vertex set $V = [n] = \{1, \dots, n\}$, $k[\Delta] = k[X_1, \dots, X_n]/I_\Delta$ is called the *Stanley–Reisner ring* of Δ, where I_Δ is an ideal generated by all square-free monomials $X_{i_1} \cdots X_{i_p}$ such that $\{i_1, \dots, i_p\} \notin \Delta$. Note that $A = k[\Delta]$ is a homogeneous reduced ring with the unique homogeneous maximal ideal $\mathfrak{m} = (X_1, \dots, X_n)k[\Delta]$ and the Krull dimension $d = \dim \Delta + 1$. Let $e(A)$ denote the *multiplicity* of A, which is equal to the number of facets (i.e., maximal faces) F of Δ with $\dim F = d - 1$. Δ is called *pure* if all facets of Δ have the same dimension, that is, A is equidimensional.

Now let $A = S/I$ be any homogeneous k-algebra and

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{p,j}(A)} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}(A)} \xrightarrow{\varphi_1} S \to A \to 0$$

a graded minimal free resolution of A over S. Then the *initial degree* $\operatorname{indeg} A$ (resp. *the relation type* $\operatorname{rt}(A)$) of A is defined by

$$\begin{aligned} \operatorname{indeg} A &= \min\{j \in \mathbb{Z} : \beta_{1,j}(A) \neq 0\}, \\ \operatorname{rt}(A) &= \max\{j \in \mathbb{Z} : \beta_{1,j}(A) \neq 0\}. \end{aligned}$$

Also, the *Castelnuovo–Mumford regularity* of A is defined by

$$\operatorname{reg} A = \max\{j - i \in \mathbb{Z} : \beta_{i,j}(A) \neq 0\}.$$

Then $\operatorname{reg} A \geq \operatorname{indeg} A - 1$, and A (or I) has *q-linear resolution* if equality holds and $\operatorname{indeg} A = q$.

The main purpose of this article is to prove the following theorem:

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Theorem 1.1 (Terai–Yoshida [7]). *Let $A = k[\Delta]$ be a Stanley–Reisner ring with $d = \dim A \geq 2$. Then:*

(1) *If $e(A) \geq \binom{n}{d} - (n-d)$, then A is Cohen–Macaulay.*

(2) *Suppose that Δ is pure.*
If $e(A) \geq \binom{n}{d} - 2(n-d) + 1$, then A is Cohen–Macaulay.

Now let us consider the case of $d = 2$ in the theorem. Then Δ is considered to be a graph having n vertices and e edges. A gragh Δ is *connected* if and only if $k[\Delta]$ is Cohen–Macaulay.

If Δ is disconnected, then $e \leq \binom{n-1}{2}$. Thus we have

(1) $e(k[\Delta]) \geq \binom{n-1}{2} + 1 = \binom{n}{2} - (n-2) \Longrightarrow \Delta$ is connected.

Also, suppose that Δ is pure, that is, Δ has no isolated points. If such a graph Δ is disconnected, then $e \leq \binom{n-2}{2} + 1$. Hence we have

(2) $e(k[\Delta]) \geq \binom{n}{2} - 2(n-2) + 1 = \left[\binom{n-2}{2} + 1\right] + 1 \Longrightarrow \Delta$ is connected.

Therefore Theorem 1.1 can be regarded as a generalization of these facts in graph theory.

2. Proof of the main theorem

Throughout this section, let Δ be a simplicial complex on $V = [n]$, and let $A = k[\Delta] = S/I_\Delta$ be the Stanley–Reisner ring of Δ. Put $d = \dim A$, $c = \operatorname{codim} A = n - d$ and $e = e(A)$. Also $\binom{[n]}{p}$ denotes the family of all distinct p-subsets of $[n]$. Note that $n = \operatorname{embdim} A$.

Let us begin the proof of Theorem 1.1 with giving the following lemma.

Lemma 2.1. *Suppose that $n = d + 1$. If $e(A) \geq d$, then A is a hypersurface.*

Proof. Suppose that A is not a hypersurface. Then we can write $I_\Delta = X_{i_1} \cdots X_{i_p} J$ for some monomial ideal J with height $J \geq 2$ since height $I_\Delta = 1$. In particular, A is not Cohen–Macaulay. Thus we may assume that $\operatorname{indeg} A \leq d$; see e.g. [6, Proposition 1.2]. Then we have $e(A) = p \leq d - 1$. This contradicts the assumption. □

By virtue of the above lemma, we may assume that $c \geq 2$.

Also, **we assume that** $\operatorname{indeg} A = d$ **for simplicity** in this article.

Remark 2.2. We can prove $\operatorname{indeg} A \geq d$ under (1) in the theorem. But the condition (2) does not imply it.

When $c \geq 2$, let Δ^* be the *Alexander dual* of Δ:

$$\Delta^* = \{F \in 2^V : V \setminus F \notin \Delta\}.$$

Then Δ^* is also a simplicial complex on V since $c \geq 2$, and $\Gamma^* = \Delta$. The following theorem is fundamental.

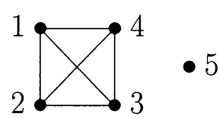
Theorem 2.3 (Eagon–Reiner [2]). *$k[\Delta]$ is Cohen–Macaulay if and only if $k[\Delta^*]$ has linear resolution.*

Also, the following proposition is a "dictionary" between Δ and Δ^*.

Proposition 2.4 (see [2]). *Let $\Gamma = \Delta^*$ be the Alexander dual complex of Δ. Then*

(1) $\operatorname{indeg} k[\Delta] + \dim k[\Gamma] = n$.
(2) $e(k[\Delta]) = \beta_{0,q^*}(I_\Gamma)$, *where* $q^* = \operatorname{indeg} k[\Gamma]$.
(3) Δ *is pure if and only if* $\operatorname{indeg} k[\Gamma] = \operatorname{rt}(k[\Gamma])$.

Example 2.5. $n = 5$, $d = 2$, $c = 3$ and $e = 6$.

$\Delta =$ 

(1) $\operatorname{indeg} k[\Delta] + \dim k[\Gamma] = 2 + 3 = 5$.
(2) $e(k[\Delta]) = \beta_{0,q^*}(I_\Gamma) = 6$, where $q^* = c = 3$.
(3) Δ is *not* pure $\Longrightarrow \operatorname{rt}(k[\Gamma]) = 3 + 1 = 4$.

Δ	I_Δ		Γ	I_Γ
12	X_4X_5	$\longleftrightarrow$	123	$X_3X_4X_5$
13	X_3X_5	$\longleftrightarrow$	124	$X_2X_4X_5$
14	X_2X_5	$\longleftrightarrow$	134	$X_2X_3X_5$
23	X_1X_5	$\longleftrightarrow$	234	$X_1X_4X_5$
24	$X_2X_3X_4$	$\longleftrightarrow$	15	$X_1X_3X_5$
34	$X_1X_3X_4$	$\longleftrightarrow$	25	$X_1X_2X_5$
5	$X_1X_2X_4$	$\longleftrightarrow$	35	$X_1X_2X_3X_4$
	$X_1X_2X_3$	$\longleftrightarrow$	45	

Using Alexander duality, we want to reduce Theorem 1.1 to another one. Let $\Gamma = \Delta^*$ be the Alexander dual of Δ as above. Then

- $\dim k[\Gamma] = n - \operatorname{indeg} k[\Delta] = n - d = c$.
- $q^* := \operatorname{indeg} k[\Gamma] = n - \dim k[\Gamma^*] = n - d = c$.
- $e(k[\Delta]) = \beta_{0,c}(I_\Gamma) = \binom{n}{c} - e(k[\Gamma])$ since $c = \dim k[\Gamma]$.

In particular,

$$e(k[\Delta]) \geq \binom{n}{c} - c \iff e(k[\Gamma]) \leq c,$$

$$e(k[\Delta]) \geq \binom{n}{c} - 2c+1 \iff e(k[\Gamma]) \leq 2c-1.$$

First we consider the Alexander dual version of Theorem 1.1(1).

Theorem 2.6 (Alexander dual version of (1)).
Put $A = k[\Delta]$. Suppose that $\operatorname{indeg} A = \dim A = d \geq 2$. *If $e(A) \leq d$, then A has d-linear resolution.*

Proof. (1) Put $a(A) = \sup\{p \in \mathbb{Z} : [H^d_{\mathfrak{m}}(A)]_p \neq 0\}$, the *a-invariant* of A. From the assumption we obtain that

$$a(A) + d \leq e(A) - 1 \leq d-1,$$

where the first inequality follows from e.g. [4, Lemma 3.1]. Hence $a(A) < 0$. On the other hand, we have that $[H^i_{\mathfrak{m}}(A)]_j = 0$ for all i and $j \geq 1$ since $A = k[\Delta]$. Then

$$\begin{aligned} \operatorname{reg} A &= \max\{p \in \mathbb{Z} : [H^i_{\mathfrak{m}}(A)]_j = 0 \text{ for all } i+j > p\} \\ &\leq d - 1 = \operatorname{indeg} A - 1. \end{aligned}$$

This means that A has d-linear resolution, as required. □

Suppose that $\operatorname{char} k = p$ is prime. Then the homogeneous k-algebra A is called *F-pure* if the Frobenius map $F\colon A \to A$ ($a \mapsto a^p$) is pure. It is known that a Stanley–Reisner ring over k is F-pure, and that if A is F-pure then $[H^i_{\mathfrak{m}}(A)]_j = 0$ for all $j \geq 1$. Therefore the above argument yields the following proposition.

Proposition 2.7. *Let $A = S/I$ be a homogeneous F-pure k-algebra. Suppose that* $\operatorname{indeg} A = \dim A = d \geq 2$. *If $e(A) \leq d$, then A has d-linear resolution.*

Remark 2.8. *Under the hypothesis of the above theorem or proposition, we get* $\operatorname{rt}(A) = \operatorname{indeg} A = d$.

Next we consider the Alexander dual version of Theorem 1.1(2).

Theorem 2.9 (Alexander dual version of (2)).
Put $A = k[\Delta]$. Suppose that $\operatorname{indeg} A = \dim A = d \geq 2$. *If $e(A) \leq 2d-1$ and* $\operatorname{indeg} A = \operatorname{rt}(A)$, *then A has d-linear resolution. In particular, $a(A) < 0$.*

Since $[H_{\mathfrak{m}}^d(k[\Delta])]_0 \cong \widetilde{H}_{d-1}(\Delta)$, it is enough to show the following lemma.

Lemma 2.10. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geq 2$. Suppose that* $\operatorname{rt}(A) \leq d$. *If $e(A) \leq 2d-1$, then* $\operatorname{reg} A \leq d-1$, *equivalently, $\widetilde{H}_{d-1}(\Delta) = 0$.*

Proof. Put $e = e(A)$. Let Δ' be the subcomplex that is spanned by all facets of dimension $d-1$. Replacing Δ with Δ', we may assume that Δ is pure.

We use induction on $d = \dim A \geq 2$. First suppose $d = 2$. The assumption shows that Δ does not contain the boundary complex of a triangle. Hence $\widetilde{H}_1(\Delta) = 0$ since $e(A) \leq 3$.

Next suppose that $d \geq 3$, and that the assertion holds for any complex the dimension of which is less than $d-1$. Assume that $\widetilde{H}_{d-1}(\Delta) \neq 0$. Take one Δ whose multiplicity is minimal among the multiplicities of those complexes. Then Δ does not contain any free face. That is, every face that is not a facet is contained in at least two facets.

First consider the case of $\operatorname{rt}(A) = d$. Take a generator $X_{i_1} \cdots X_{i_d}$ of I_Δ. For every $j = 1, \ldots, d$, each $G_j = \{i_1, \ldots, \widehat{i_j}, \ldots, i_d\}$ is contained in at least two facets as mentioned above. Then $e(A) \geq 2d$ since those facets are different from each other. This is a contradiction.

Next we consider the case of $\operatorname{rt}(A) < d$. For a face G in Δ and $v \in V$, we put

$$\begin{aligned} \Delta_{V \setminus \{v\}} &= \{F \in \Delta : v \notin F\}, \\ \operatorname{star}_\Delta G &= \{F \in \Delta : F \cup G \in \Delta\}, \\ \operatorname{link}_\Delta G &= \{F \in \Delta : F \cup G \in \Delta,\ F \cap G = \emptyset.\}. \end{aligned}$$

Take a Mayer–Vietoris sequence with respect to $\Delta = \Delta_{V\setminus\{n\}} \cup \operatorname{star}_\Delta\{n\}$ as follows:

$$\widetilde{H}_{d-1}(\Delta_{V\setminus\{n\}}) \oplus \widetilde{H}_{d-1}(\operatorname{star}_\Delta\{n\}) \longrightarrow \widetilde{H}_{d-1}(\Delta) \longrightarrow \widetilde{H}_{d-2}(\operatorname{link}_\Delta\{n\}).$$

The minimality of $e(k[\Delta_{V\setminus\{n\}}])$ yields that $\widetilde{H}_{d-1}(\Delta_{V\setminus\{n\}}) = 0$ since $e(k[\Delta_{V\setminus\{n\}}]) < e(k[\Delta])$. On the other hand, since $\widetilde{H}_i(\operatorname{star}_\Delta\{n\})$ vanishes for all i, we have $\widetilde{H}_{d-1}(\Delta) \hookrightarrow \widetilde{H}_{d-2}(\operatorname{link}_\Delta\{n\})$. In particular, $\widetilde{H}_{d-2}(\operatorname{link}_\Delta\{n\}) \neq 0$.

Set $\Delta' = \operatorname{link}_\Delta\{n\}$. Then Δ' is a complex on $V\setminus\{n\}$ with $\dim k[\Delta'] = d-1$, $\operatorname{rt}(k[\Delta']) \leq \operatorname{rt}(k[\Delta]) \leq d-1$. In order to apply the induction hypothesis to Δ', we want to see that $e(k[\Delta']) \leq 2d-3$. In order to do that, we consider $e(k[\Delta_{V\setminus\{n\}}])$. As $\Delta \neq \operatorname{star}_\Delta\{n\}$, one can take

$F = \{i_1, \cdots, i_p, n\} \notin \Delta$ for some $p \leq d-2$ such that $X_{i_1} \cdots X_{i_p} X_n$ is a generator of I_Δ. Then $G := \{i_1, \ldots, i_p\} \in \Delta$, but it is not a facet of Δ. Thus it is contained in at least two facets of Δ, each of which does not contain n. Hence $e(k[\Delta_{V \setminus \{n\}}]) \geq 2$. Thus we get

$$e(k[\Delta']) = e(k[\mathrm{star}_\Delta\{n\}]) = e(k[\Delta]) - e(k[\Delta_{V \setminus \{n\}}]) \leq 2d - 3.$$

By induction hypothesis, we have $\widetilde{H}_{d-2}(\mathrm{link}_\Delta\{n\}) = 0$. This is a contradiction. $\square$

Now let us discuss a generalization of Theorem 2.9. Let $A = S/I$ be an arbitrary homogeneous reduced k-algebra over a field k. We have no results for "F-pure k-algebras" corresponding to Theorem 2.9, but we have the following.

Proposition 2.11. *Let A be a homogeneous integral domain over an algebraically closed field of* $\mathrm{char}\, k = 0$. *If* $e(A) \leq 2d-1$ *and* $\mathrm{codim}\, A \geq 2$, *then* $a(A) < 0$.

Proof. In fact, it is known that an inequality

$$a(A) + d \leq \left\lceil \frac{e(A) - 1}{\mathrm{codim}\, A} \right\rceil$$

holds [1]; see e.g., the remark after Theorem 3.2 in [4]. The assertion immediately follows from this. $\square$

$A = k[\Delta]$ is called *Buchsbaum* if Δ is pure and $k[\mathrm{link}_\Delta\{i\}]$ is Cohen–Macaulay for every $i \in [n]$. As an application of Theorem 1.1, we can provide a sufficient condition for $A = k[\Delta]$ to be Buchsbaum.

Proposition 2.12. *Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geq 3$. Suppose that Δ is pure and* $\mathrm{indeg}\, A = \mathrm{rt}\,(A) = d$. *If* $e(A) = \binom{n}{d} - 2(n-d)$, *then A is Buchsbaum.*

Proof. It is enough to prove the proposition in the case of $e(A) = \binom{n}{d} - 2(n-d)$. We may assume that $c \geq 2$, and that $\Delta \neq \mathrm{star}_\Delta\{i\}$ for every $i \in [n]$. Put $\Gamma_i = \mathrm{link}_\Delta\{i\}$ for each $i \in [n]$.

We first show the following claim.

Claim: $e(A) \leq \binom{n}{d} - \left\{\binom{n-1}{d-1} - e(k[\Gamma_i])\right\}$ for every $i \in [n]$. Also, equality holds if and only if $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$.

[1]Prof. Chikashi Miyazaki informed us that this assertion will be true even if $\mathrm{char}\, k$ is prime.

Putting $W_i = \left\{F \in \binom{[n]}{d} : i \in F \notin \mathrm{star}_\Delta\{i\}\right\}$, we have

$$\binom{n-1}{d-1} - e(k[\Gamma_j]) \leq \binom{n}{d} - e(A)$$

since $\#(W_i) \leq \#(\bigcup_{i=1}^n W_i)$. Also, equality holds if and only if $W_i = \bigcup_{i=1}^n W_i$, that is, $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$.

Now suppose that $e(k[\Gamma_i]) \leq \binom{n-1}{d-1} - 2c - 1$ for some $i \in [n]$. Then the claim implies that $e(A) \leq \binom{n}{d} - 2c - 1$, which contradicts the assumption. Therefore we proved the claim.

Now suppose that height $[I_\Delta]_d S = 1$. Then one can take $i \in [n]$ for which $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$. We may assume $i = n$. Then $\{1, \ldots, \widehat{i}, \ldots, d+1\} \in \Delta$ for all $i \in [d+1]$ because $n-1 \geq d+1$. This means that $X_1 \cdots X_{d+1}$ is a generator of I_Δ; thus $\mathrm{rt}\,(A) = d+1$. This contradicts the assumption. Hence height $[I_\Delta]_d S \geq 2$.

Then there is no element $i \in [n]$ for which $i \in F$ holds for all $F \in \binom{[n]}{d} \setminus \Delta$. Thus the claim yields that

$$\binom{n}{d} - 2c = e(A) \leq \binom{n}{d} - \left[\binom{n-1}{d-1} - e(k[\Gamma_i])\right] - 1,$$

that is, $e(k[\Gamma_i]) \geq \binom{n-1}{d-1} - 2c + 1$ for every $i \in [n]$. Also, we note that Γ_i is pure and $\mathrm{indeg}\, k[\Gamma_i] = \dim k[\Gamma_i] = d-1$. Thus applying Theorem 1.1 to $k[\Gamma_i]$ we obtain that $k[\Gamma_i]$ is Cohen–Macaulay. Therefore A is Buchsbaum since Δ is pure. □

3. Examples

Let c, d be given integers with $c, d \geq 2$. Set $n = c + d$.

Example 3.1. Put $F_{i,j} = \{1, 2, \ldots, \widehat{i}, \ldots, d, j\} \in 2^{[n]}$ for each $i = 1, \ldots, d; j = d+1, \ldots, n$. For a given integers e with $1 \leq e \leq cd$, we choose e faces (say, $F_1, \ldots, F_e$) from $\{F_{i,j} : 1 \leq i \leq d,\ d+1 \leq j \leq n\}$.

Let Δ be a simplicial complex spanned by $F_1, \ldots, F_e$ and all elements of $\binom{[n]}{d-1}$. Then $k[\Delta]$ is a d-dimensional Stanley–Reisner ring with $\mathrm{indeg}\, k[\Delta] = \mathrm{rt}\,(k[\Delta]) = d$ and $e(k[\Delta]) = e$.

In particular, when $e \leq 2d - 1$, the complex Δ satisfies the condition of Theorem 2.9. Thus $k[\Delta]$ has d-linear resolution by the theorem. Also, the Alexander dual complexes of them provide examples satisfying hypothesis of Theorem 1.1.

The following example shows that the assumption "$e(A) \leq 2d - 1$" is optimal in Theorem 2.9.

Example 3.2. There exists a complex Δ on $V = [n]$ $(n = d + 2)$ for which $k[\Delta]$ does not have d-linear resolution with $\dim k[\Delta] = \operatorname{indeg} k[\Delta] = \operatorname{rt}(k[\Delta]) = d$ and $e(k[\Delta]) = 2d$.

In fact, put $n = d+2$. Let Δ_0 be a complex on $V = [d+2]$ such that $k[\Delta_0]$ is a complete intersection defined by $(X_1 \cdots X_d, X_{d+1}X_{d+2})$. Also, let Δ be a complex on V that is spanned by all facets of Δ_0 and all elements of $\binom{[d+2]}{d-1}$. Namely, I_Δ is generated by $X_1 \cdots X_d$ and $X_{i_1} \cdots X_{i_{d-2}} X_{d+1} X_{d+2}$ for all $1 \le i_1 < \cdots < i_{d-2} \le d$. Then $\widetilde{H}_{d-1}(k[\Delta]) \cong \widetilde{H}_{d-1}(k[\Delta_0]) \ne 0$ since $a(k[\Delta_0]) = 0$. Hence $k[\Delta]$ does *not* have linear resolution.

Remark 3.3. *The above example is obtained by considering the case* $c = 2$, $e = 2d$ *in Example* 3.1.

The next example shows that the assumption "$\operatorname{rt}(A) = d$" is *not* superfluous in Theorem 2.9.

Example 3.4. Suppose that $d + 1 \le e \le \binom{n}{d} - 1$. There exists a simplicial complex Δ on $V = [n]$ such that $\dim k[\Delta] = \operatorname{indeg} k[\Delta] = d$, $\operatorname{rt}(A) = d+1$ and $e(A) = e$. In particular, $k[\Delta]$ does not have d-linear resolution.

In fact, put $\mathcal{F} = \binom{[n]}{d} \setminus \binom{[d+1]}{d}$. Let Δ_0 be a simplicial complex on V such that

$$I_{\Delta_0} = (X_1 \cdots X_d X_{d+1})S + (X_{i_1} \cdots X_{i_d} \,:\, \{i_1, \ldots, i_d\} \in \mathcal{F})S.$$

Then $\dim k[\Delta_0] = \operatorname{indeg} k[\Delta_0] = d$, $\operatorname{rt}(k[\Delta_0]) = d+1$, and $e(k[\Delta_0]) = d+1$.

For a given integer e which satisfies the above condition, one obtains the required simplicial complex by adding to Δ_0 any $(e-d-1)$ distinct d-subsets of $2^{[n]}$ that is not contained in $\binom{[d+1]}{d}$.

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加群のG次元と$\mathrm{mod}\underline{C}$のフロベニウス性

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1 序

本稿は私の最近の仕事 [8] に関するシンポジウムでの講演の記録であり，詳しい内容についてはプレプリント [8] を参照にしてください。

以下では，R で常に可換なネータ環を表し，$\mathrm{mod}R$ で有限生成な R-加群のなすアーベル圏を表す。次で定義される二つの $\mathrm{mod}R$ の部分圏 $\mathcal{G}$ と $\mathcal{H}$ に関心がある。

定義 1.1 $\mathcal{G}$ は次の条件を満足する全ての加群 X を対象としてもつ $\mathrm{mod}R$ の充満部分圏である。

$$\mathrm{Ext}_R^i(X, R) = 0 \quad \text{and} \quad \mathrm{Ext}_R^i(\mathrm{Tr}X, R) = 0 \quad \text{for any} \quad i > 0.$$

また，$\mathcal{H}$ で上の第一の条件のみを満足する加群 X からなる部分圏を表すことにする。すなわち，加群 $X \in \mathrm{mod}R$ が $\mathcal{H}$ の対象である必要十分条件は，

$$\mathrm{Ext}_R^i(X, R) = 0 \quad \text{for any} \quad i > 0.$$

である。

明らかにいつも $\mathcal{G} \subseteq \mathcal{H}$ である。$\mathcal{G}$ はG次元0の加群の部分圏と呼ばれることもある。G次元については [2] を参照してほしい。

最近，D.Jorgensen と L.M.Sega [5] が，$\mathcal{G} \neq \mathcal{H}$ であるようなアルチン環の例を構成したと報告している。しかしながら，そのような例の構成が大変難しいことを考えると，非常に多くの場合に等号 $\mathcal{G} = \mathcal{H}$ が成立すると期待することも出来る。

本稿の目的は，この二つの部分圏を関手論的に考察して，$\mathcal{H}$ の部分圏 $\mathcal{C}$ が $\mathcal{G}$ に含まれる条件について考えることである。

先ず最初に以下で使う記号を整理しておく。この稿を通して $\mathrm{mod}\,R$ の部分圏 $\mathcal{C}$ と言ったときには，いつも $\mathcal{C}$ は次の条件を満たすものと仮定しておく。

- $\mathcal{C}$ は essential である。すなわち，もし $X \cong Y$ in $\mathrm{mod}\,R$ かつ $X \in \mathcal{C}$ ならば $Y \in \mathcal{C}$ となる。
- $\mathcal{C}$ は full である。すなわち， $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_R(X, Y)$.
- $\mathcal{C}$ は additive and additively closed である。すなわち，任意の $X, Y \in \mathrm{mod}\,R$ について，$X \oplus Y \in \mathcal{C} \iff X \in \mathcal{C}$ and $Y \in \mathcal{C}$.
- $\mathcal{C}$ は全ての射影加群を含む。

もちろん，上で定義した $\mathcal{G}$ と $\mathcal{H}$ はこの意味で部分圏である。

$\mathcal{C}$ を $\mathrm{mod}\,R$ の任意の部分圏とする。一般的な記号として， $\underline{\mathcal{C}}$ でそれに付随する安定圏 (stable category) を表すことにする。もちろん自然な関手 $\mathcal{C} \to \underline{\mathcal{C}}$ がある。転置関手 Tr とシジジー関手 Ω はそれぞれ安定圏上の関手を定めることは注意すべきである。

$$\mathrm{Tr} : (\underline{\mathcal{C}})^{op} \to \underline{\mathrm{mod}\,R} \qquad \Omega : \underline{\mathcal{C}} \to \underline{\mathrm{mod}\,R}.$$

定義から， Tr がそれぞれ $\underline{\mathcal{G}}$ 上と $\underline{\mathrm{mod}\,R}$ 上で duality を与えることは明らかであろう。したがって， $\mathrm{Tr}\underline{\mathcal{H}}$ は条件 $\mathrm{Ext}^i_R(\mathrm{Tr}X, R) = 0$ $(i > 0)$ を満たす加群 X 全体から成る圏である。したがってとくに，等式 $\underline{\mathcal{G}} = \underline{\mathcal{H}} \cap \mathrm{Tr}\underline{\mathcal{H}}$ が成立する。

任意の加法圏 $\mathcal{A}$ について， $\mathcal{A}$ からアーベル群の圏 (Ab) への反変加法関手のことをここでは $\mathcal{A}$-module と呼ぶことにする。また，それら反変関手の間の自然変換のことを $\mathcal{A}$-module morphism と呼ぶことにする。そして， $\mathrm{Mod}\,\mathcal{A}$ で全ての $\mathcal{A}$-modules とその間の $\mathcal{A}$-module morphisms から成る圏を表す。明らかに $\mathrm{Mod}\,\mathcal{A}$ はアーベル圏である。$\mathcal{A}$-module F が有限表示型であるとは，つぎの完全列があるときを意味する。

$$\mathrm{Hom}_{\mathcal{A}}(\ \ , X_1) \to \mathrm{Hom}_{\mathcal{A}}(\ \ , X_0) \to F \to 0,$$

ただし， $X_0, X_1 \in \mathcal{A}$ である。そして， $\mathrm{mod}\,\mathcal{A}$ で， $\mathrm{Mod}\,\mathcal{A}$ の充満部分圏で全ての有限表示型の $\mathcal{A}$-modules から成る圏を表すことにする。

米田の補題によって，$\mathrm{mod}\,\mathcal{A}$ の射影的対象は $\mathrm{Hom}_{\mathcal{A}}(\ \ ,X)$ $(X \in \mathcal{A})$ の形のものに限る。また，$\mathcal{A}$ の対象 X を $\mathrm{Hom}_{\mathcal{A}}(\ \ ,X)$ に送ることによって定義される関手 $\mathcal{A} \to \mathrm{mod}\,\mathcal{A}$ は充満的埋め込みである。

さて，$\mathcal{C}$ を $\mathrm{mod}\,R$ の任意の部分圏，$\underline{\mathcal{C}}$ をその安定圏とする。このとき，有限表示型 $\mathcal{C}$-modules の圏 $\mathrm{mod}\mathcal{C}$ や有限表示型 $\underline{\mathcal{C}}$-modules の圏 $\mathrm{mod}\underline{\mathcal{C}}$ が上記のようにして構成される。任意の $F \in \mathrm{mod}\mathcal{C}$ (resp. $G \in \mathrm{mod}\underline{\mathcal{C}}$) と任意の $X \in \mathcal{C}$ (resp. $\underline{X} \in \underline{\mathcal{C}}$) に対して，アーベル群 $F(X)$ (resp. $G(\underline{X})$) は自然に R-module 構造をもつことを注意しよう。したがって，F (resp. G) は実際には，反変加法関手 $\mathcal{C}$ (resp. $\underline{\mathcal{C}}$) $\to \mathrm{mod}\,R$ である。

すでに述べたように自然な関手 $\mathcal{C} \to \underline{\mathcal{C}}$ がある。これから，$F \in \mathrm{mod}\underline{\mathcal{C}}$ を F と $\mathcal{C} \to \underline{\mathcal{C}}$ との合成関手に送ることによって，関手 $\iota : \mathrm{mod}\underline{\mathcal{C}} \to \mathrm{mod}\mathcal{C}$ を定義することが出来る。このとき，ι は圏 $\mathrm{mod}\underline{\mathcal{C}}$ と有限表示型 $\mathcal{C}$-modules F で $F(R) = 0$ を満足するするもの全体から成る $\mathrm{mod}\mathcal{C}$ の充満部分圏との間の同値を与えることは容易に示すことが出来る。

2　mod$\underline{\mathcal{G}}$ のフロベニウス性

以下，$\mathcal{C}$ を $\mathrm{mod}\,R$ の任意の部分圏とする。次の条件 (♯) を満たすとき，$\mathcal{C}$ は closed under kernels of epimorphisms であるという。

(♯) $\mathrm{mod}\,R$ の完全列 $0 \to X \to Y \to Z \to 0$ があるとき $Y, Z \in \mathcal{C}$ なら $X \in \mathcal{C}$ である。

(Quillen の言葉では，"all epimorphisms from $\mathrm{mod}\,R$ in $\mathcal{C}$ are admissible" ということである。)

同様に，次の条件 (♭) を満たすとき，$\mathcal{C}$ は closed under extension または extension-closed であるという。

(♭) $\mathrm{mod}\,R$ の完全列 $0 \to X \to Y \to Z \to 0$ があるとき $X, Z \in \mathcal{C}$ なら $Y \in \mathcal{C}$ である。

部分圏 $\mathcal{C}$ は，extension-closed かつ closed under kernels of epimorphisms であるときに，resolving subcategory と呼ばれる。

また，$\mathcal{C}$ が closed under Ω とは次の条件 (♮) が満たされるときをいう。

(♮) $\mathrm{mod}\,R$ の完全列 $0 \to X \to P \to Z \to 0$ があって P が projective であるとき，$Z \in \mathcal{C}$ なら $X \in \mathcal{C}$ である。

同様に，部分圏 $\mathcal{C}$ が closed under Tr であるとは，$X \in \mathcal{C}$ である限り $\mathrm{Tr}X \in \underline{\mathcal{C}}$ となることを意味する。

我々が興味ある二つの圏 $\mathcal{G}$ と $\mathcal{H}$ は，resolving な部分圏であり，$\mathcal{G}$ については closed under Tr であることは明らかである。

次のことを注意しておこう。部分圏 $\mathcal{C}$ が closed under kernels of epimorphisms ならば，closed under Ω である。また，部分圏 $\mathcal{C}$ が extension-closed かつ closed under Ω ならば，それは resolving である。

次の命題は定義から直接容易に示すことができる。[7, Lemma (4.17)] において，R が Cohen-Macaulay 局所環で $\mathcal{C}$ が極大 Cohen-Macaulay 加群の圏のときに，$\mathrm{mod}\underline{\mathcal{C}}$ がアーベル圏であることが証明されているが，この命題の証明はそれと全く同じである。

命題 2.1 $\mathcal{C}$ を $\mathrm{mod}R$ の部分圏で，closed under kernels of epimorphisms であると仮定する。このとき，$\mathrm{mod}\underline{\mathcal{C}}$ は enough projectives を持つアーベル圏である。

圏 $\mathcal{A}$ が Frobenius 圏であるとは，それが enough projectives と enough injectives を持つアーベル圏であって，$\mathcal{A}$ における射影対象と入射対象が一致するときを言う。同様に，(もう少し弱く) 圏 $\mathcal{A}$ が quasi-Frobenius 圏であるとは，それが enough projectives をもつアーベル圏であって，全ての $\mathcal{A}$ における射影対象は入射的であるときを言う。

次の定理が最初に私が得たものであって，この $\mathrm{mod}\underline{\mathcal{C}}$ の研究の出発点となったものである。

定理 2.2 $\mathcal{C}$ を $\mathrm{mod}R$ の部分圏で closed under kernels of epimorphisms と仮定する。このとき，もし $\mathcal{C} \subseteq \mathcal{H}$ ならば，$\mathrm{mod}\underline{\mathcal{C}}$ は quasi-Frobenius 圏である。

この定理の証明は難しくはない。基本的には,「$\mathrm{mod}\underline{\mathcal{C}}$ における入射対象は，$\mathcal{C}$ 上の関手としてみたときには半完全関手のことである」ということに注意すれば容易である。詳しくは [8] を参照してください。

定理 2.3 $\mathrm{mod}R$ の部分圏 $\mathcal{C}$ に対して次のことを仮定する。

(1) $\mathcal{C}$ は resolving である。

(2) $\mathcal{C} \subseteq \mathcal{H}$.

(3) シジジー関手 $\Omega : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ は $\underline{\mathcal{C}}$ の対象の同型類の集合の全射を与える。

このとき，$\mathrm{mod}\underline{\mathcal{C}}$ は Frobenius 圏である。とくに，$\mathrm{mod}\underline{\mathcal{G}}$ は Frobenius 圏である。

定理の第3の条件は，シジジー関手 Ω が圏 $\underline{\mathcal{C}}$ 上の自己同型を与えることを意味している。つまりコシジジー関手 Ω^{-1} が $\underline{\mathcal{C}}$ 上で定義されることを示している。このことを使って，前定理の結果からこの定理が従う。

さて $\mathrm{mod}\,R$ の resolving な部分圏 $\mathcal{C}$ について，次の4条件を考えよう。

$$\begin{cases} (A): & \mathcal{C} \text{ は } \mathcal{H} \text{ の部分圏である。} \\ (B): & \mathrm{mod}\underline{\mathcal{C}} \text{ は quasi-Frobenius 圏である。} \\ (C): & \mathrm{mod}\underline{\mathcal{C}} \text{ は Frobenius 圏である。} \\ (D): & \mathcal{C} \text{ は } \mathcal{G} \text{ の部分圏である。} \end{cases}$$

このとき，次の関係がすでに示された。

$$(A) \Longrightarrow (B) \Longleftarrow (C) \Longleftarrow (D)$$

第一の矢印は定理 2.2 から，第3の矢印は定理 2.3 からそれぞれ従う。(第3の矢印については適当なシジジー関手に関する条件を付加することが必要であるが。) 第2の矢印についてはいつでも成立する。我々のプログラムは，この矢印の逆を考えようというものである。実際，次の節では以下のことを示すことになる。

- もし R がヘンゼル局所環ならば，$(B) \Rightarrow (A)$ が成立する。
- Auslander-Reiten 予想が正しいという仮定のもとで，$(C) \Rightarrow (D)$ が成立する。
- もし $\mathcal{C}$ が有限型ならば，(中山の定理によって) $(B) \Rightarrow (C)$ が成立する。

3　主結果

この節では，R はいつもヘンゼル局所環で，その極大イデアルを $\mathfrak{m}$，剰余体を $k = R/\mathfrak{m}$ と表すことにする。しかし，以下の議論で必要なことは，この条件の下で成立する次の事実だけである。

「$X \in \mathrm{mod}\,R$ が直既約加群であるときには，$\mathrm{End}_R(X)$ は（非可換な）局所環である。」

実際，これより任意の部分圏 $\mathcal{C} \subseteq \mathrm{mod}\, R$ について，$\mathrm{mod}\underline{\mathcal{C}}$ が Krull-Schmidt 圏となることが分かる。

先ず定理 2.2 の逆を，この条件のもとで証明することが出来る。

定理 3.1 $\mathcal{C}$ を $\mathrm{mod}\, R$ の resolving な部分圏とする。さらに，$\mathrm{mod}\underline{\mathcal{C}}$ が quasi-Frobenius 圏であると仮定すると，$\mathcal{C} \subseteq \mathcal{H}$ となる。

ある意味で $\mathcal{H}$ は $\mathrm{mod}\, R$ の resolving な部分圏 $\mathcal{C}$ で $\mathrm{mod}\underline{\mathcal{C}}$ が quasi-Frobenius 圏となるようなもののうちで最大な部分圏であることをこの定理は主張している。

この定理の証明はそれほど易しいわけではない。証明の本質的部分は，$\mathrm{mod}\underline{\mathcal{C}}$ が quasi-Frobenius 圏であるという仮定の下で，任意の対象 $X \in \mathcal{C}$ について $\mathrm{Ext}^1_R(X, R) = 0$ となることを示すことにある。完全な証明については [8] を参照してください。

定理 3.2 上記のように R をヘンゼル局所環とする。さらに次の条件を仮定する。

(1) $\mathcal{C}$ は $\mathrm{mod}\, R$ の resolving な部分圏である。

(2) $\mathrm{mod}\underline{\mathcal{C}}$ は Frobenius 圏である。

(3) 加群 $X \in \mathcal{C}$ について，もし $\mathrm{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}} = 0$ ならば X 射影加群である。

このとき，$\mathcal{C} \subseteq \mathcal{G}$ が成立する。

注意 3.3 我々は,「$\mathcal{G}$ は，$\mathrm{mod}\underline{\mathcal{C}}$ が Frobenius 圏となるような $\mathrm{mod}\, R$ の resolving な部分圏 $\mathcal{C}$ の内で最大なものである」と予想している。定理 3.2 と定理 2.3 をあわせると，この予想は次の予想（Auslander-Reiten 予想）を仮定すれば正しいことが分かる。

(AR) もし任意の $i > 0$ について $\mathrm{Ext}^i_R(X, X \oplus R) = 0$ が成立するならば，X は射影加群であろう。

実際，予想 (AR) が環 R について正しいとすると，定理の中の第 3 の条件は自動的に成立することが分かり，定理 3.2 が我々の予想が正しいことを示している。

定理 3.2 はそれほど短くはない。次の補題がその証明のキーであることだけを注意しておく。

補題 3.4 上記と同じように R をヘンゼル局所環，$\mathcal{C}$ を extension-closed な $\mathrm{mod}\,R$ の部分圏であるとする。二つの対象 $\underline{X}, \underline{Y} \in \underline{\mathcal{C}}$ について，次のことを仮定する。

(1) 圏 $\mathrm{Mod}\underline{\mathcal{C}}$ における単射 φ が存在する。

$$\varphi : \underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}} \to \mathrm{Ext}^1(\ \ ,X)|_{\underline{\mathcal{C}}}$$

(2) $\underline{X}$ は圏 $\underline{\mathcal{C}}$ において直既約である。

(3) 圏 $\underline{\mathcal{C}}$ において $\underline{Y} \not\cong 0$ である。

このとき，加群 X は ΩY の直和因子と同型である。

さて，任意の加法圏 $\mathcal{A}$ に対して，$\mathrm{Ind}(\mathcal{A})$ で $\mathcal{A}$ の直既約対象の同型類の集合を表すことにする。$\mathrm{Ind}(\mathcal{A})$ が有限集合であるときには，圏 $\mathcal{A}$ は有限型であると呼ばれる。次の定理が論文 [8] の主結果であり，有限型の $\mathcal{H}$ の resolving な部分圏は $\mathcal{G}$ に含まれてしまうことを主張している。

定理 3.5 上記のように R はヘンゼル局所環，$\mathcal{C}$ を $\mathrm{mod}R$ の resolving な部分圏であるとする。さらに次の条件を仮定する。

(1) $\mathcal{C} \subseteq \mathcal{H}$.

(2) $\mathcal{C}$ は有限型である。

このとき，$\mathrm{mod}\underline{\mathcal{C}}$ は Frobenius 圏となり，さらに $\mathcal{C} \subseteq \mathcal{G}$ が成立する。

この定理 3.5 の証明のアウトラインについて説明しておく。定理では，$\mathcal{C}$ が有限型であることを仮定したので，このときには圏 $\mathrm{mod}\underline{\mathcal{C}}$ はあるアルティン環 A 上の有限生成左加群の圏と同型になる。（この A は一般的には $\underline{\mathcal{C}}$ の Auslander algebra と呼ばれるものである。）

$$\mathrm{mod}\underline{\mathcal{C}} \quad \cong \quad \mathrm{mod}\ A$$

このとき，A はある可換なアルティン環上の加群として有限生成な（非可換）代数である。一方で，$\mathcal{C} \subseteq \mathcal{H}$ と仮定したので，$\mathrm{mod}\underline{\mathcal{C}}$，それゆえ $\mathrm{mod}\ A$ も，quasi-Frobenius 圏である (定理 2.2 による)。これはアルティン環 A が左

自己入射的 (left selfinjective) であることを意味している。良く知られた中山の定理 ([6] 参照) によれば，このとき A は右自己入射的でもある。この事実と mod A 上の duality を使えば， mod A が Frobenius 圏となることが証明できるのである。

包含関係 $\mathcal{C} \subseteq \mathcal{G}$ を示すには，定理 3.2 を使う。実際，すでに mod$\underline{\mathcal{C}}$ が Frobenious 圏になることは見たのだから，次のことをチェックすればよい。

(*) もし， $X \in \mathcal{C}$ について，圏 $\underline{\mathcal{C}}$ において $\underline{X} \not\cong \underline{0}$ が成立するなら，$\mathrm{Ext}^1_R(\ \ ,X)|_{\underline{\mathcal{C}}} \neq 0$ である。

山形氏による概説 [6] にある中山の定理の証明と同様の議論をすることによって，このことを示すことが出来る。

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