研究集会

第29回可換環論シンポジウム報告集

(The 29th Symposium on Commutative Algebra in Japan)

2007年11月19日～11月22日

於：愛知厚生年金会館 ウェルシティなごや

序 (Preface)

この報告集に収録されている原稿は、第２９回可換環論シンポジウムの講演の記録です。本研究集会は２００７年１１月１９日（月）から１１月２２日（木）にかけて、愛知県厚生年金会館ウェルシティなごや（愛知県名古屋市）において開催されました。この研究集会には、国内(約６５名)の研究者・大学院生の他、海外からも３名の研究者が参加し、合計２８もの興味深い講演が行われました。特に, Aldo Conca 氏にはお忙しい中, 快く招待講演を引き受けて下さり, 大変感謝致しております。また、限られた時間の中で素晴らしい講演をしていただいた講演者の皆様と、研究集会の運営に協力していただいた大学院生の皆様には、この場を借りて感謝したいと思っております。

シンポジウム開催にあたり、下記の援助を受けました。後藤四郎先生には感謝しております。また、明治大学の早坂太氏を始めとして、事務手続きに携わった方々にもこの場を借りて感謝の気持ちを表したいと思います。

- 2007 年度明治大学科学技術研究所重点研究費 A (研究代表者：後藤四郎)
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- 平成 19 年度科学研究費補助金基盤研究 B (研究代表者：吉田健一)
「乗数イデアルと密着閉包の可換代数及び計算代数の視点からの研究」
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２００８年2月

名古屋大学　橋本光靖
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29th Symposium on Commutative Algebra

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F-thresholds, tight closure, integral closure, and multiplicity bounds

Shunsuke Takagi (Kyushu University)

This is a joint work with Craig Huneke, Mircea Mustaţă and Kei-ichi Watanabe.

Let R be a Noetherian ring of prime characteristic p and denote by R° the set of elements of R that are not contained in any minimal prime ideal. The *tight closure* I^* of an ideal $I \subseteq R$ is defined to be the ideal of R consisting of all elements $x \in R$ for which there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$, where $I^{[q]}$ is the ideal generated by the q^{th} powers of all elements of I. The ring R is called *F-rational* if $J^* = J$ for every ideal $J \subseteq R$ generated by parameters.

Let $\mathfrak{a}$ be a fixed proper ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. To each ideal J of R such that $\mathfrak{a} \subseteq \sqrt{J}$, we associate an F-threshold as follows. For every $q = p^e$, let

$$\nu_{\mathfrak{a}}^J(q) := \max\{r \in \mathbb{N} | \mathfrak{a}^r \not\subseteq J^{[q]}\},$$

where $J^{[q]}$ is the ideal generated by the q^{th} powers of all elements of J. Since $\mathfrak{a} \subseteq \sqrt{J}$, this is a nonnegative integer (if $\mathfrak{a} \subseteq J^{[q]}$, then we put $\nu_{\mathfrak{a}}^J(q) = 0$). We put

$$\mathrm{c}_+^J(\mathfrak{a}) = \limsup_{q \to \infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}, \quad \mathrm{c}_-^J(\mathfrak{a}) = \liminf_{q \to \infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}.$$

When $\mathrm{c}_+^J(\mathfrak{a}) = \mathrm{c}_-^J(\mathfrak{a})$, we call this limit the *F-threshold* of the pair $(R, \mathfrak{a})$ (or simply of $\mathfrak{a}$) with respect to J, and we denote it by $\mathrm{c}^J(\mathfrak{a})$. The reader is referred to [3] and [4] for basic properties of F-thresholds.

Example 1. Let R be a Noetherian local ring of characteristic $p > 0$, and let $J = (x_1, \dots, x_d)$, where $x_1, \dots, x_d$ form a full system of parameters in R. It follows from the Monomial Conjecture that $(x_1 \cdots x_d)^{q-1} \notin J^{[q]}$ for every q. Hence $\nu_J^J(q) \geq d(q-1)$ for every q, and therefore $\mathrm{c}_-^J(J) \geq d$. On the other hand, it is easy to see that $\mathrm{c}_+^J(J) \leq d$, and we conclude that $\mathrm{c}^J(J) = d$.

We can describe the tight closure and the integral closure of parameter ideals in terms of F-thresholds.

Theorem 2. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent analytically irreducible Noetherian local domain of characteristic $p > 0$, and let $J = (x_1, \dots, x_d)$ be an ideal generated by a full system of parameters in R. Given an ideal $I \supseteq J$, we have $I \subseteq J^*$ if* *and only if* $c_+^I(J) = d$ *(and in this case* $c^J(J)$ *exists). In particular, R is F-rational if and only if* $c_+^I(J) < d$ *for every ideal* $I \supsetneq J$.

In order to prove Theorem 2, we start with the following lemma.

Lemma 3. *Let* $(R, \mathfrak{m})$ *be an excellent analytically irreducible Noetherian local domain of positive characteristic p. Set* $d = \dim(R)$*, and let* $J = (x_1, \ldots, x_d)$ *be an ideal generated by a full system of parameters in R, and let* $I \supseteq J$ *be another ideal. Then I is not contained in the tight closure* J^* *of J if and only if there exists* $q_0 = p^{e_0}$ *such that* $x^{q_0-1} \in I^{[q_0]}$*, where* $x = x_1 x_2 \cdots x_d$.

Proof. After passing to completion, we may assume that R is a complete local domain. Suppose first that $x^{q_0-1} \in I^{[q_0]}$, and by way of contradiction suppose also that $I \subseteq J^*$. Let $c \in R^\circ$ be a test element. Then for all $q = p^e$, one has $cx^{q(q_0-1)} \in cI^{[qq_0]} \subset J^{[qq_0]}$, so that $c \in J^{[qq_0]} : x^{q(q_0-1)} \subseteq (J^{[q]})^*$, by colon-capturing [2, Theorem 7.15a]. Therefore c^2 lies in $\bigcap_{q=p^e} J^{[q]} = (0)$, a contradiction.

Conversely, suppose that $I \not\subseteq J^*$, and choose an element $f \in I \setminus J^*$. We choose a coefficient field k, and let $B = k[[x_1, \ldots, x_d, f]]$ be the complete subring of R generated by $x_1, \ldots, x_d, f$. Note that B is a hypersurface singularity, hence Gorenstein. Furthermore, by persistence of tight closure [2, Lemma 4.11a], $f \notin ((x_1, \ldots, x_d)B)^*$. If we prove that there exists $q_0 = p^{e_0}$ such that $x^{q_0-1} \in ((x_1, \ldots, x_d, f)B)^{[q_0]}$, then clearly x^{q_0-1} is also in $I^{[q_0]}$. Hence we can reduce to the case in which R is Gorenstein. Since $I \not\subseteq J^*$, it follows from a result of Aberbach [1] that $J^{[q]} \colon I^{[q]} \subseteq \mathfrak{m}^{n(q)}$, where $n(q)$ is a positive integer with $\lim_{q\to\infty} n(q) = \infty$. In particular, we can find $q_0 = p^{e_0}$ such that $J^{[q_0]} : I^{[q_0]} \subseteq J$. Therefore $x^{q_0-1} \in J^{[q_0]} \colon J \subseteq J^{[q_0]} \colon (J^{[q_0]} \colon I^{[q_0]}) = I^{[q_0]}$, where the last equality follows from the fact that R is Gorenstein. □

Proof of Theorem 2. Note first that for every $I \supseteq J$ we have $c_+^J(I) \le d$. Suppose now that $I \subseteq J^*$. It follows from Lemma 3 that $J^{d(q-1)} \not\subseteq I^{[q]}$ for every $q = p^e$. This gives $\nu_J^I(q) \ge d(q-1)$ for all q, and therefore $c_-^I(J) \ge d$. We conclude that in this case $c_+^I(J) = c_-^I(J) = d$.

Conversely, suppose that $I \not\subseteq J^*$. By Lemma 3, we can find $q_0 = p^{e_0}$ such that

$$\mathfrak{b} := (x_1^{q_0}, \ldots, x_d^{q_0}, (x_1 \cdots x_d)^{q_0-1}) \subseteq I^{[q_0]}.$$

If $(x_1, \ldots, x_d)^r \not\subseteq \mathfrak{b}^{[q]}$, then

$$r \le (qq_0 - 1)(d-1) + q(q_0-1) - 1 = qq_0 d - q - d.$$

Therefore $\nu_J^{\mathfrak{b}}(q) \le qq_0 d - q - d$ for every q, which implies $c^{\mathfrak{b}}(J) \le q_0 d - 1$. Since q_0 is a fixed power of p, we deduce

$$c_+^I(J) = \frac{1}{q_0} c_+^{I^{[q_0]}}(J) \le \frac{1}{q_0} c^{\mathfrak{b}}(J) \le d - \frac{1}{q_0} < d.$$

□

Theorem 4. *Let $(R, \mathfrak{m})$ be a d-dimensional formally equidimensional Noetherian local ring of characteristic $p > 0$. If I and J are ideals in R, with J generated by a full system of parameters, then*

(1) $\mathrm{c}_+^J(I) \leq d$ *if and only if* $I \subseteq \overline{J}$.

(2) *If, in addition,* $J \subseteq I$, *then* $I \subseteq \overline{J}$ *if and only if* $\mathrm{c}_+^J(I) = d$.

Proof. Note that if $J \subseteq I$, then $\mathrm{c}_-^J(I) \geq \mathrm{c}_-^J(J) = \mathrm{c}^J(J) = d$, by Example 1. Hence both assertions in (2) follow from the assertion in (1).

One implication in (1) is easy: if $I \subseteq \overline{J}$, then we have $\mathrm{c}_+^J(I) \leq \mathrm{c}_+^J(\overline{J}) = \mathrm{c}^J(J) = d$. Conversely, suppose that $\mathrm{c}_+^J(I) \leq d$. In order to show that $I \subseteq \overline{J}$, we may assume that R is complete and reduced. Indeed, first note that the inverse image of $\overline{J\widehat{R}_{\mathrm{red}}}$ in R is contained in $\overline{J}$, hence it is enough to show that $I\widehat{R}_{\mathrm{red}} \subseteq \overline{J\widehat{R}_{\mathrm{red}}}$. Since $J\widehat{R}_{\mathrm{red}}$ is again generated by a full system of parameters, and since we trivially have

$$\mathrm{c}^{J\widehat{R}_{\mathrm{red}}}(I\widehat{R}_{\mathrm{red}}) \leq \mathrm{c}^J(I) \leq d,$$

we may replace R by $\widehat{R}_{\mathrm{red}}$.

Since R is complete and reduced, we can find a test element c for R.

Claim. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R such that $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. Then $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a}) \leq \alpha$ if and only if for every power q_0 of p, we have $\mathfrak{a}^{\lceil \alpha q \rceil + q/q_0} \subseteq \mathfrak{b}^{[q]}$ for all $q = p^e \gg q_0$.

Proof of Claim. First, assume that $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a}) \leq \alpha$. By the definition of $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a})$, for any power q_0 of p, there exists q_1 such that $\nu_{\mathfrak{a}}^{\mathfrak{b}}(q)/q < \alpha + 1/q_0$ for all $q = p^e \geq q_1$. Thus, $\nu_{\mathfrak{a}}^{\mathfrak{b}}(q) \leq \lceil \alpha q \rceil + q/q_0$, that is, $\mathfrak{a}^{\lceil \alpha q \rceil + q/q_0} \subseteq \mathfrak{b}^{[q]}$ for all $q = p^e \geq q_1$. For the converse implication, note that by assumption, $\nu_{\mathfrak{a}}^{\mathfrak{b}}(q) \leq \lceil \alpha q \rceil + q/q_0 - 1$ for all large $q = p^e \gg q_0$. Dividing by q and taking the limit gives $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a}) \leq \alpha + 1/q_0$. Since q_0 is any power of p, we can conclude that $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a}) \leq \alpha$. □

By the above claim, the assumption $\mathrm{c}_+^J(I) \leq d$ implies that for all $q_0 = p^{e_0}$ and for all large $q = p^e$, we have

$$I^{q(d+(1/q_0))} \subseteq J^{[q]}.$$

Hence $I^q J^{q(d-1+(1/q_0))} \subseteq J^{[q]}$, and thus

$$I^q \subseteq J^{[q]} : J^{q(d-1+(1/q_0))} \subseteq (J^{q-d+1-(q/q_0)})^*,$$

where the last containment follows from the colon-capturing property of tight closure [2, Theorem 7.15a]. We get $cI^q \subseteq cR \cap J^{q-d+1-(q/q_0)} \subseteq cJ^{q-d+1-(q/q_0)-l}$ for some fixed integer l that is independent of q, by the Artin-Rees lemma. Since c is a non-zero divisor in R, it follows that

$$I^q \subseteq J^{q-d+1-(q/q_0)-l}. \tag{1}$$

If ν is a discrete valuation with center in $\mathfrak{m}$, we may apply ν to (1) to deduce $q\nu(I) \geq \left(q - d + 1 - \frac{q}{q_0} - l\right)\nu(J)$. Dividing by q and letting q go to infinity gives $\nu(I) \geq \left(1 - \frac{1}{q_0}\right)\nu(J)$. We now let q_0 go to infinity to obtain $\nu(I) \geq \nu(J)$. Since this holds for every ν, we have $I \subseteq \overline{J}$. ☐

Two years ago (at the 27th Symposium on Commutative Algebra in Japan), we proposed the following conjecture, generalizing a result in [5].

Conjecture 5 (cf. [6, Conjecture 3.2]). *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic $p > 0$. If $J \subseteq \mathfrak{m}$ is an ideal generated by a full system of parameters, and if $\mathfrak{a} \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal, then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{c_{-}^{J}(\mathfrak{a})}\right)^{d} e(J).$$

Example 6. Let $R = k[\![X, Y, Z]\!]/(X^2 + Y^3 + Z^5)$ be a rational double point of type E_8, with k a field of characteristic $p > 0$. Let $\mathfrak{a} = (x, z)$ and $J = (y, z)$. Then $e(\mathfrak{a}) = 3$ and $e(J) = 2$. It is easy to check that $c^{J}(\mathfrak{a}) = 5/3$ and $c^{\mathfrak{a}}(J) = 5/2$. Thus,

$$e(\mathfrak{a}) = 3 > \frac{72}{25} = \left(\frac{2}{c^{J}(\mathfrak{a})}\right)^{2} e(J),$$
$$e(J) = 2 > \frac{48}{25} = \left(\frac{2}{c^{\mathfrak{a}}(J)}\right)^{2} e(\mathfrak{a}).$$

Two years ago, we reported the following result as an evidence of Conjecture 5.

Theorem 7 ([6, Proposition 3.3]). *If $(R, \mathfrak{m})$ is a regular local ring of characteristic $p > 0$ and $J = (x_1^{a_1}, \ldots, x_d^{a_d})$, with $x_1, \ldots, x_d$ a full regular system of parameters for R, and with $a_1, \ldots, a_d$ positive integers, then the inequality given by Conjecture* 5 *holds.*

We will conclude this article with a result related to the graded version of Conjecture 5.

Theorem 8. *Let $R = \bigoplus_{d \geq 0} R_d$ be an n-dimensional graded Cohen-Macaulay ring with R_0 a field of characteristic $p > 0$. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R, then*

$$e(\mathfrak{a}) \geq \left(\frac{n}{c_{-}^{J}(\mathfrak{a})}\right)^{n} e(J).$$

Proof. Suppose that $\mathfrak{a}$ is generated by a full homogeneous system of parameters $x_1, \ldots, x_n$ of degrees $a_1 \leq \cdots \leq a_n$ and J is generated by another homogeneous system of parameters $f_1, \ldots, f_n$ of degrees $d_1 \leq \cdots \leq d_n$. Fix a power $q = p^e$ of p, and define the nonnegative integers $t_1^{(e)}, \dots, t_{n-1}^{(e)}$ inductively as follows: $t_1^{(e)}$ is the least integer t such that $x_1^t \in J^{[q]}$. If $i \geq 2$, then $t_i^{(e)}$ is the least integer t such that $x_1^{t_1^{(e)}-1} \cdots x_{i-1}^{t_{i-1}^{(e)}-1} x_i^t \in J^{[q]}$. We also define the integer $N^{(e)}$ to be the least integer N such that $I^N \subseteq J^{[q]}$. Note that $N^{(e)}$ is greater than $t_1^{(e)} + \cdots + t_{n-1}^{(e)} - n + 1$. Since the lim sup of the ratios $(N^{(e)} + n - 1)/p^e$ is $c_+^J(\mathfrak{a})$, it suffices to prove that

$$(N^{(e)} + n - 1)^n a_1 \cdots a_n \geq n^n q^n d_1 \cdots d_n.$$

First, we will show the following inequality for every $i = 1, \dots, n-1$:

$$t_1^{(e)} a_1 + \cdots + t_i^{(e)} a_i \geq q(d_1 + \cdots + d_i). \tag{2}$$

Let $I_i^{(e)}$ be the ideal of R generated by $x_1^{t_1^{(e)}}, x_1^{t_1^{(e)}-1} x_2^{t_2^{(e)}}, \dots, x_1^{t_1^{(e)}-1} \cdots x_{i-1}^{t_{i-1}^{(e)}-1} x_i^{t_i^{(e)}}$. By the definition of $t_1^{(e)}, \dots, t_i^{(e)}$, we have that $I_i^{(e)} \subseteq J^{[q]}$. The natural surjection of $R/I_i^{(e)}$ onto $R/J^{[q]}$ induces a comparison map between the minimal free resolutions. Looking at the i^{th} free modules, we have the map

$$R(-t_1^{(e)} a_1 - \cdots - t_i^{(e)} a_i) \to \bigoplus_{1 \leq v_1 \leq \cdots \leq v_i \leq n} R(-q d_{v_1} - \cdots - q d_{v_i}).$$

In particular, unless this map is zero, $t_1^{(e)} a_1 + \cdots + t_i^{(e)} a_i$ must be at least as large as the minimum of the twists, which is $q(d_1 + \cdots + d_i)$. So it remains to see the reason why this map cannot be zero. Assume it is zero: then the map

$$\operatorname{Tor}_i^R(R/I_i^{(e)}, R/\mathfrak{b}_i) \to \operatorname{Tor}_i^R(R/J^{[q]}, R/\mathfrak{b}_i)$$

will be zero, where $\mathfrak{b}_i$ is the ideal generated by $x_1, \dots, x_i$. On the other hand, using the Koszul complex on $x_1, \dots, x_i$, we see that this map can be identified with the natural map

$$(I_i^{(e)} : \mathfrak{b}_i)/I_i^{(e)} \to (J^{[q]} : \mathfrak{b}_i)/J^{[q]}.$$

Since the ideal $I_i^{(e)} : \mathfrak{b}_i$ is generated by $x_1^{t_1^{(e)}-1} \cdots x_i^{t_i^{(e)}-1}$ modulo $I_i^{(e)}$, the map is zero if and only if $x_1^{t_1^{(e)}-1} \cdots x_i^{t_i^{(e)}-1}$ is in $J^{[q]}$. However, this contradicts the definition of $t_i^{(e)}$.

Next, we will prove the following estimate:

$$t_1^{(e)} a_1 + \cdots + t_{n-1}^{(e)} a_{n-1} + (N^{(e)} - t_1^{(e)} - \cdots - t_{n-1}^{(e)} + n - 1) a_n \geq q(d_1 + \cdots + d_n). \tag{3}$$

Since $\mathfrak{a}^{N^{(e)}} \subseteq J^{[q]}$, we have that

$$(x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) : J^{[q]} \subseteq (x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) : \mathfrak{a}^{N^{(e)}} = (x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) + \mathfrak{a}^{(n-1)(N^{(e)}-1)}.$$

The ideal $(x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) : J^{[q]}$ is of the form $(x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}, y^{(e)})$, where the extra generator $y^{(e)}$ has degree $N^{(e)}(a_1 + \cdots + a_n) - q(d_1 + \cdots + d_n)$. We write

$$y^{(e)} = \sum_{m_1 + \cdots + m_n = (n-1)(N^{(e)}-1)} r_{m_1 \dots m_n} x_1^{m_1} \dots x_n^{m_n}$$

modulo $(x_1^{N^{(e)}},\dots,x_n^{N^{(e)}})$. Since $x_1^{t_1^{(e)}-1}\cdots x_{n-1}^{t_{n-1}^{(e)}-1}$ is not in $J^{[q]}$, we see that $y^{(e)}$ is not in $(x_1^{N^{(e)}-t_1^{(e)}+1},\dots,x_{n-1}^{N^{(e)}-t_{n-1}^{(e)}+1},x_n^{N^{(e)}})$. To check this, suppose that $y^{(e)}$ is in $(x_1^{N^{(e)}-t_1^{(e)}+1},\dots,x_{n-1}^{N^{(e)}-t_{n-1}^{(e)}+1},x_n^{N^{(e)}})$. Then $J^{[q]}=(x_1^{N^{(e)}},\dots,x_n^{N^{(e)}}):y^{(e)}$ will contain $(x_1^{N^{(e)}},\dots,x_n^{N^{(e)}}):(x_1^{N^{(e)}-t_1^{(e)}+1},\dots,x_{n-1}^{N^{(e)}-t_{n-1}^{(e)}+1},x_n^{N^{(e)}})\ni x_1^{t_1^{(e)}-1}\cdots x_{n-1}^{t_{n-1}^{(e)}-1}$. Thus, some $r_{m_1\dots m_n}$ must be nonzero, where $m_i\le N^{(e)}-t_i^{(e)}$ for $1\le i\le n-1$ and $m_n\le N^{(e)}-1$. Since the degree of D is greater than or equal to the minimal degree of monomials $x_1^{m_1}\dots x_n^{m_n}$ with $r_{m_1\dots m_n}$ nonzero, we can conclude that

$$\begin{aligned}\deg D &= N^{(e)}(a_1+\cdots+a_n)-q(d_1+\cdots+d_n)\\ &\ge (N^{(e)}-t_1^{(e)})a_1+\cdots+(N^{(e)}-t_{n-1}^{(e)})a_{n-1}+(t_1^{(e)}+\cdots+t_{n-1}^{(e)}-n+1)a_n,\end{aligned}$$

which implies the desired estimate.

To finish the proof, we will use the following claim.

Claim. Let $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_n$ be two n-tuple of real numbers, and let $1=\gamma_1\le\gamma_2\le\cdots\le\gamma_n$ be another one. Assume that $\gamma_1\alpha_1+\cdots+\gamma_i\alpha_i\ge\gamma_1\beta_1+\cdots+\gamma_i\beta_i$ for all $i=1,\dots,n$. Then $\alpha_1+\cdots+\alpha_n\ge\beta_1+\cdots+\beta_n$.

Proof of Claim. Let $\lambda_i=\alpha_i-\beta_i$ for $1\le i\le n$. Then $\gamma_1\lambda_1+\cdots+\gamma_i\lambda_i\ge 0$ for all $i=1,\dots,n$. We will prove that $\lambda_1+\cdots+\lambda_n\ge 0$ by induction on n. We may assume that n is greater than one. The assertion is obvious if every $\lambda_i\ge 0$. Suppose that $\lambda_i<0$ for some i. Clearly $i\ge 2$. Since $\gamma_i\ge\gamma_{i-1}$, it follows from that $\gamma_i\lambda_i\le\gamma_{i-1}\lambda_i$. We then define $\gamma'_j=\gamma_j$ for $1\le j\le i-1$ and $\gamma'_j=\gamma_{j+1}$ for $i\le j\le n-1$. Define also $\lambda'_j=\lambda_j$ for $1\le j\le i-2$, $\lambda'_{i-1}=\lambda_{i-1}+\lambda_i$ and $\lambda'_j=\lambda_{j+1}$ for $i\le j\le n-1$. Since $\gamma'_1\lambda'_1+\cdots+\gamma'_j\lambda'_j\ge 0$ for all $j=1,\dots,n-1$, the induction hypothesis implies that $\lambda_1+\cdots+\lambda_n=\lambda'_1+\cdots+\lambda'_{n-1}\ge 0$. □

Set $\alpha_i=t_i^{(e)}$ for $1\le i\le n-1$ and $\alpha_n=N^{(e)}-t_1^{(e)}-\cdots-t_{n-1}^{(e)}+n-1$. Set $\beta_i=qd_i/a_i$ and $\gamma_i=a_i/a_1$ for $1\le i\le n$. Then $\gamma_1\le\cdots\le\gamma_n$, because $a_1\le\cdots\le a_n$. The inequalities $\gamma_1\alpha_1+\cdots+\gamma_i\alpha_i\ge\gamma_1\beta_1+\cdots+\gamma_i\beta_i$ for $1\le i\le n$ follow from the estimates (1) and (2). Using the above claim, we can conclude that

$$N^{(e)}+n-1=\alpha_1+\cdots+\alpha_n\ge\beta_1+\dots\beta_n=q\left(\frac{d_1}{a_1}+\cdots+\frac{d_n}{a_n}\right).$$

Comparing the arithmetic and geometric means of $\{qd_i/a_i\}_i$, we see that

$$(N^{(e)}+n-1)^n a_1\dots a_n\ge n^n q^n d_1\dots d_n.$$

□

Remark 9. Theorem 8 does not imply the graded (Cohen-Macaulay) version of Conjecture 5, because a minimal reduction of an R_+-primary homogeneous ideal is not necessarily homogeneous.

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Department of Mathematics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan
E-mail address: `stakagi@math.kyushu-u.ac.jp`

Stanley–Reinser rings which are complete intersections locally [1]

Department of Mathematics
Faculty of Culture and Education
Saga University
佐賀大学・文化教育学部
寺井 直樹 (Naoki TERAI)
terai@cc.saga-u.ac.jp

Graduate School of Mathematics
Nagoya University
名古屋大学・大学院多元数理科学研究科
吉田 健一 (Ken-ichi YOSHIDA)
yoshida@math.nagoya-u.ac.jp

1. Introduction

By a simplicial complex Δ on the vertex set $V = [n] = \{1, 2, \ldots, n\}$, we mean that Δ is a family of subsets of V which satisfies the following conditions:

(i) $\{i\} \in \Delta$ for every $i \in V$ (ii) $F \in \Delta$, $G \subseteq F$ imply $G \in \Delta$.

An element of Δ is called a *face* of Δ. The *dimension* of Δ, denoted by $\dim \Delta$, is the maximum of the dimension $\dim F = \sharp(F) - 1$, where F runs through all faces of Δ and $\sharp(F)$ denotes the cardinality of a set F. A simplicial complex Δ is called *pure* if all facets (maximal faces with respect to inclusion) of Δ have the same dimension.

For a face F of Δ,

$$\mathrm{link}_\Delta(F) = \{G \in \Delta \,:\, F \cup G \in \Delta,\, F \cap G = \emptyset\}$$

is called the *link* of F. For a subset W of V,

$$\Delta_W = \{F \in \Delta \,:\, F \subseteq W\}$$

is called the *restriction* to W of Δ.

Throughout this talk, let K be a field, and let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over K, unless otherwise specified. The ring S can be viewed as a standard graded K-algebra (i.e., $S = \bigoplus_{n\in\mathbb{N}}$ is an $\mathbb{N}$-graded ring with $S_0 = K$, $S = K[S_1]$) with the unique homogeneous maximal ideal $\mathfrak{m} = (X_1, \ldots, X_n)$. For a simplicial complex Δ, the *Stanley–Reisner ideal* I_Δ and the *Stanley–Reisner ring* $K[\Delta]$ are defined by

$$\begin{aligned} I_\Delta &= (X_{i_1} \cdots X_{i_p} \,:\, 1 \le i_1 < \cdots < i_p \le n,\, \{i_1, \ldots, i_p\} \notin \Delta)S, \\ K[\Delta] &= S/I_\Delta. \end{aligned}$$

Note that any squarefree monomial ideal $I \subseteq S$ with $\mathrm{indeg}\, I \ge 2$ can be written as $I = I_\Delta$ for some simplicial complex Δ, and that $K[\Delta]$ is a graded reduced K-algebra with $\dim K[\Delta] = \dim \Delta + 1$. See [BH, St] about simplicial complexes and Stanley–Reisner rings.

Let $R = S/I$ be an arbitrary standard graded K-algebra. The ring R is said to be *Buchsbaum* (resp. to have *(FLC)*) if $\mathrm{Ext}^i_S(S/\mathfrak{m}, R) \to H^i_\mathfrak{m}(R))$ is surjective (resp. $H^i_\mathfrak{m}(R)$ has finite length) for every $i < \dim R$. In particular, any Buchsbaum ring has (FLC).

[1]In: The 29th symposium on Commutative Ring Theory; Nov.17–21, 2007; at Nagoya

The Stanley–Reisner ring $K[\Delta]$ has (FLC) if and only if Δ is pure and $K[\mathrm{link}_\Delta\{i\}]$ is Cohen–Macaulay for every $i \in V$. When this is the case, $K[\Delta]$ is Buchsbaum; see e.g., [St].

Let Δ be a simplicial complex, and let $G(I_\Delta) = \{m_1, \ldots, m_\mu\}$ denote the minimal set of monomial generators of I_Δ. Then one can easily check the following fact.

Fact 1.1. *Let $I_\Delta = (m_1, \ldots, m_\mu)$ be as above. Then I_Δ is a complete intersection (i.e., I_Δ is generated by a regular sequence) if and only if $\gcd(m_i, m_j) = 1$ for every i, j with $i \neq j$.*

In general, if $I \subseteq S$ is generated by a regular sequence, then S/I^ℓ is Cohen–Macaulay for every integer $\ell \geq 1$. When I is generically a complete intersection (i.e., I_P is a complete intersection for all minimal prime ideal P over I), the converse is also true; see [CN]. Hence, for example, I_Δ is a complete intersection if and only if S/I_Δ^ℓ is Cohen–Macaulay for every $\ell \geq 1$.

In [GT], Goto and Takayama introduced the notion of generalized complete intersection complexes and characterized those complexes: a simplicial complex Δ is said to be a *generalized complete intersection* complex if Δ is pure and $K[\mathrm{link}_\Delta\{i\}]$ is a complete intersection for any vertex $i \in V$. The following theorem gives a motivation of our study.

Theorem 1.2 (Goto–Takayama (see also [GT])). *Let Δ be a simplicial complex on $V = [n]$. Then the following conditions are equivalent:*

(1) *$K[\Delta]$ is a generalized complete intersection in the sense of* [GT].
(2) *S/I_Δ^ℓ has (FLC) for every $\ell \geq 1$.*

Clearly, a complete intersection is a generalized complete intersection. In [GT], they gave examples which are not complete intersections but generalized complete intersection complexes. However, their complexes Δ are disconnected or $\dim \Delta = 1$. So it is natural to ask the following question:

Question 1.3. Assume that a simplicial complex Δ is connected and $\dim \Delta \geq 2$. If Δ is a generalized complete intersection complex, then is it a complete intersection?

The main aim of this talk is to give a complete answer to this question. Before stating our result, let us define the following notion:

Definition 1.4. A simplicial complex $K[\Delta]$ (or Δ) is called *a locally complete intersection* (resp. Gorenstein, Cohen–Macaulay) if $K[\Delta]_P$ is a complete intersection (resp. Gorenstein, Cohen–Macaulay) for every $P \in \mathrm{Proj}\, K[\Delta]$.

Note that $K[\Delta]$ is a locally complete intersection if and only if $K[\Delta]_{X_i}$ is a complete intersection for every $1 \leq i \leq n$. Moreover, since $k[\Delta]_{X_i} \cong K[\mathrm{link}_\Delta\{i\}][X_i, X_i^{-1}]$ we have:

Lemma 1.5. *Let Δ be a simplicial complex on $V = [n]$. Then the following conditions are equivalent:*

(1) *$K[\Delta]$ is a locally complete intersection.*
(2) *$K[\Delta]_{X_i}$ is a complete intersection for every $i \in V$.*
(3) *$K[\mathrm{link}_\Delta\{i\}]$ is a complete intersection for every $i \in V$.*

In particular, Δ is a generalized complete intersection if and only if Δ is pure and a locally complete intersection.

Corollary 1.6. *Let Δ be a simplicial complex on V. If $K[\Delta]$ is a complete intersection (resp. Gorenstein, Cohen–Macaulay), then so is $K[\mathrm{link}_\Delta(F)]$ for any face F of Δ.*

Proof. It immediately follows from the fact $\mathrm{link}_{\mathrm{link}_\Delta\{i\}}(F \setminus \{i\}) = \mathrm{link}_\Delta(F)$ for $i \in F$. □

Example 1.7. Let Δ be a simplicial complex corresponding to 5-gon. That is, $K[\Delta] = K[X_1, X_2, X_3, X_4, X_5]/(X_1X_3, X_1X_4, X_2X_4, X_2X_5, X_3X_5)$. Then $K[\Delta]$ is a locally complete intersection but *not* a complete intersection.

Indeed, $K[\mathrm{link}_\Delta\{1\}] \cong K[X_2, X_5]/(X_2X_5)$ is a complete intersection. Similarly, $K[\mathrm{link}_\Delta\{i\}]$ is also a complete intersection for other $i \in [5]$.

The following theorem is a main result in this talk; see also Section 2.

Theorem 1.8. *Let Δ be a simplicial complex on $V = [n]$ with $\dim \Delta \geq 2$. Assume that Δ is a locally complete intersection. Then it is a disjoint union of finitely many simplicial complexes whose Stanley–Reisner rings are complete intersections.*

In the case $\dim \Delta = 1$, we can also characterize locally complete intersection complexes. See Section 3.

2. Proof of the main theorem

In this section, we will prove the main theorem. First of all, we remark the following lemma.

Lemma 2.1. *Assume that $V = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$. Let Δ_i be a locally complete intersection complex on V_i for $i = 1, 2$. Then a disjoint union $\Delta_1 \cup \Delta_2$ is also a locally complete intersection complex on V.*

Proof. Put $V_1 = [m]$ and $V_2 = [n]$. If we write

$$K[\Delta_1] = K[X_1, \ldots, X_m]/I_{\Delta_1} \quad \text{and} \quad K[\Delta_2] = K[Y_1, \ldots, Y_n]/I_{\Delta_2},$$

then

$$K[\Delta] \cong K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(I_{\Delta_1}, I_{\Delta_2}, \{X_iY_j\}_{1 \leq i \leq m, 1 \leq j \leq n}).$$

Hence

$$K[\Delta]_{X_i} \cong K[\Delta]_{X_i} \quad \text{and} \quad K[\Delta]_{Y_j} \cong K[\Delta_2]_{Y_j}.$$

are complete intersection rings. Thus Δ is also a locally complete intersection. □

Remark 2.2. In the above lemma, we suppose that both Δ_1 and Δ_2 are generalized complete intersections. Then $\Delta_1 \cup \Delta_2$ is a generalized complete intersection if and only if $\dim \Delta_1 = \dim \Delta_2$.

Example 2.3. Let Δ be the disjoint union of the standard $(m-1)$-simplex and the standard $(n-1)$-simplex. Then Δ is a locally complete intersection complex by Lemma 2.1. Moreover, $K[\Delta]$ is isomorphic to

$$K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(X_iY_j \ : \ 1 \le i \le m, 1 \le j \le n)$$

and it is a generalized complete intersection if and only if $m = n$.

By virtue of Lemma 2.1, it suffices to show the following theorem.

Theorem 2.4. *Let Δ be a simplicial complex on $V = [n]$. Assume that Δ is connected and $\dim \Delta \ge 2$. Then the following conditions are equivalent:*

(1) *$K[\Delta]$ is a complete intersection.*
(2) *$K[\Delta]$ is a locally complete intersection.*
(2)' *$K[\Delta]$ is a generalized complete intersection.*

From now on, assume that Δ is a locally complete intersection, connected complex which is not a complete intersection. Suppose that $\dim \Delta \ge 1$. Note that Δ is pure since Δ is connected and locally complete intersection and hence Δ satisfies Serre condition (S_2). Let $G(I_\Delta) = \{m_1, \ldots, m_\mu\}$ denote the minimal set of monomial generators of I_Δ. Then $\mu \ge 2$ and $\deg m_i \ge 2$ for every $i = 1, 2, \ldots, \mu$, and that there exists i, j $(1 \le i < j \le n)$ such that $\gcd(m_i, m_j) \ne 1$.

In order to prove Theorem 2.4, it is enough to show that $\dim \Delta = 1$. In what follows, X_i, $Y_j, \ldots$ denote corresponding variables to vertices x_i, $y_j, \ldots$.

Lemma 2.5. *We may assume that* $\deg m_i = \deg m_j = 2$.

Proof. Take m_j, m_k $(j \ne k)$ such that $\gcd(m_j, m_k) \ne 1$. If $\deg m_j = \deg m_k = 2$, then there is nothing to prove.

Now suppose that $\deg m_k \ge 3$. By [GT, Lemmas 3.4, 3.5], we may assume that $\deg m_j = 2$ and $\gcd(m_j, m_k) = X_p$. Write $m_k = X_pX_{i_1} \cdots X_{i_r}$ and $m_j = X_pX_q$. Then [GT, Lemma 3.6] implies that $X_{i_1}X_q \in G(I_\Delta)$. Set $m_i = X_{i_1}X_q \in I_\Delta$. Then $\deg m_i = \deg m_j = 2$ and $\gcd(m_i, m_j) = X_q \ne 1$, as required. □

The following lemma is simple but important.

Lemma 2.6. *Let x_1, x_2, y be distinct vertices such that X_1Y, $X_2Y \in I_\Delta$. For any $z \in V \setminus \{x_1, x_2, y\}$, at lease one of monomials X_1Z, X_2Z and YZ belongs to I_Δ.*

Proof. It immediately follows from the fact that $K[\mathrm{link}_\Delta\{z\}]$ is a complete intersection. □

Lemma 2.7. *There exist some integers k, $\ell \ge 2$ such that*

(1) $V = \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$.
(2) $X_1Y_1, \ldots, X_kY_1 \in I_\Delta$.

(3) $\sharp\{i \,:\, 1 \le i \le k,\ X_iY_j \notin I_\Delta\} \le 1$ *holds for each* $j = 2, \dots, \ell$.

Proof. By Lemma 2.5, there exists vertices $x_1, x_2, y_1 \in V$ such that X_1Y_1, $X_2Y_1 \in I_\Delta$. Thus one can write $V = \{x_1, \dots, x_k, y_1, \dots, y_\ell\}$ such that

$$\begin{array}{cccccc} X_1Y_1, & X_2Y_1, & \dots, & X_kY_1 & \in & I_\Delta, \\ Y_1Y_2, & Y_1Y_3, & \dots, & Y_1Y_\ell & \notin & I_\Delta. \end{array}$$

If $\ell = 1$, then $\Delta = \Delta_{\{y_1\}} \cup \Delta_{\{x_1,\dots,x_k\}}$ is a disjoint union since $\{y_1, x_i\} \notin \Delta$ for all i. This contradicts the connectedness of Δ. Hence $\ell \ge 2$. Thus it is enough to show (3) in this notation.

Now suppose that there exists an integer j with $2 \le j \le \ell$ such that

$$\sharp\{i \,:\, 1 \le i \le k,\ X_iY_j \notin I_\Delta\} \ge 2.$$

When $k = 2$, we have X_1Y_j, $X_2Y_j \notin I_\Delta$. On the other hand, as X_1Y_1, $X_2Y_1 \in I_\Delta$ and $Y_j \ne X_1, X_2, Y_1$, we obtain that at least one of X_1Y_j, X_2Y_j, Y_1Y_j belongs to I_Δ. It is impossible. So we may assume that $k \ge 3$ and $X_{k-1}Y_j$, $X_kY_j \notin I_\Delta$. Then $\{x_{k-1}\}$, $\{x_k\}$ and $\{y_1\}$ belong to $\mathrm{link}_\Delta\{y_j\}$, and $X_{k-1}Y_1$, X_kY_1 form part of a minimal system of generators of $I_{\mathrm{link}_\Delta\{y_j\}}$. This contradicts the assumption that $K[\mathrm{link}_\Delta\{y_j\}]$ is a complete intersection. □

In what follows, we fix the notation as in Lemma 2.7. First, we suppose that there exists i_0 with $1 \le i_0 \le k$ such that

$$\sharp\{j \,:\, 1 \le j \le \ell,\ X_{i_0}Y_j \notin I_\Delta\} = 1.$$

In this case, we may assume that $X_1Y_2 \notin I_\Delta$ and $X_1Y_j \in I_\Delta$ for all $3 \le j \le \ell$ without loss of generality. Note that $X_2Y_2, \dots, X_kY_2 \in I_\Delta$ by Lemma 2.7. We claim that $\{x_1, y_2\}$ is a facet of Δ. As $X_iY_2 \in I_\Delta$ for each $i = 2, \dots, k$, we have that $\{x_1, y_2, x_i\} \notin \Delta$. Similarly, $\{x_1, y_2, y_j\} \notin \Delta$ since $X_1Y_j \in I_\Delta$ for $j = 1$ or $3 \le j \le \ell$. Hence $\{x_1, y_2\}$ is a facet of Δ, and $\dim \Delta = 1$ because Δ is pure.

By the observation as above, we may assume that for every i with $1 \le i \le k$,

$$\sharp\{j \,:\, 1 \le j \le \ell,\ X_iY_j \notin I_\Delta\} \ge 2$$

or $X_iY_j \in I_\Delta$ holds for all $j = 1, \dots, \ell$.

Now suppose that there exists j_1, j_2 with $1 \le j_1 < j_2 \le \ell$ such that $X_iY_{j_1}$, $X_iY_{j_2} \notin I_\Delta$. Then $X_rY_{j_1}$, $X_rY_{j_2} \in I_\Delta$ for all $r \ne i$ by Lemma 2.7. It follows that $X_rX_i \in I_\Delta$ from Lemma 2.6. Then we can relabel x_i (say $y_{\ell+1}$). Repeating this procedure, we can get one of the following cases:

Case 1: $V = \{x_1, \dots, x_r, y_1, \dots, y_s\}$ such that $X_iY_j \in I_\Delta$ for all i, j with $1 \le i \le r$, $1 \le j \le s$.

Case 2: $V = \{x_1, x_2, y_1, \dots, y_m, z_1, \dots, z_p, w_1, \dots, w_q\}$ such that

$$\left\{ \begin{array}{lll} X_1Y_j \in I_\Delta, & X_2Y_j \in I_\Delta & (j = 1, \dots, m) \\ X_1Z_j \notin I_\Delta, & X_2Z_j \in I_\Delta & (j = 1, \dots, p) \\ X_1W_j \in I_\Delta, & X_2W_j \notin I_\Delta & (j = 1, \dots, q) \end{array} \right.$$

holds for some $m \ge 1$, $p, q \ge 2$.

If Case 1 occurs, then $\Delta = \Delta_{\{x_1,\dots,x_r\}} \cup \Delta_{\{y_1,\dots,y_s\}}$ is a disjoint union. This contradicts the assumption. Thus Case 2 must occur. If $\{x_1, x_2\} \in \Delta$, then it is a facet and so $\dim \Delta = 1$. Hence we may assume that $\{x_1, x_2\} \notin \Delta$. However, since Δ is connected, there exists a path between x_1 and x_2.

Cases (2-a): the case where $\{z_1, w_k\} \in \Delta$ for some k with $1 \le k \le q$.

We may assume that $\{z_1, w_1\} \in \Delta$. Now suppose that $\dim \Delta \ge 2$. Then since $\{z_1, w_1\}$ is *not* a facet, there exists $u \in V \setminus \{x_1, x_2\}$ such that $\{z_1, w_1, u\} \in \Delta$. If $u = z_j$ $(2 \le j \le p)$ (resp. $u = y_i$ $(1 \le i \le m)$), then $G(I_{\mathrm{link}_\Delta \{w_1\}})$ contains X_2Z_1 and X_2Z_j (resp. X_2Y_i); see figure below. It is impossible since $\mathrm{link}_\Delta\{w_1\}$ is a complete intersection. When $u = w_k$, we can obtain a contradiction by a similar argument as above. Therefore $\dim \Delta = 1$.

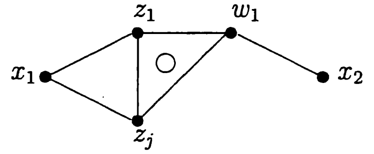

Figure: the case $\{z_1, z_j, w_1\} \in \Delta$ in Case (2-a)

Cases (2-b): the case where $\{z_j, w_k\} \notin \Delta$ for all j, k.

Then we may assume that (i) $\{z_1, y_1\} \in \Delta$ and (ii) $\{y_1, y_2\} \in \Delta$ or $\{y_1, w_1\} \in \Delta$. Now suppose that $\dim \Delta \ge 2$. Then since $\{z_1, y_1\}$ is *not* a facet, we have

$$\{z_1, y_1, y_i\} \in \Delta, \ \{z_1, y_1, w_k\} \in \Delta \text{ or } \{z_1, y_1, z_j\} \in \Delta.$$

When $\{z_1, y_1, y_i\} \in \Delta$, we obtain that $\{X_1Y_1, X_1Y_i\} \in G(I_{\mathrm{link}_\Delta\{z_1\}})$. This is a contradiction. When $\{z_1, y_1, w_k\} \in \Delta$, we can obtain a contradiction by a similar argument as in Case (2-a). Thus it is enough to consider the case $\{z_1, y_1, z_j\} \in \Delta$.

First we suppose that $\{y_1, y_2\} \in \Delta$.

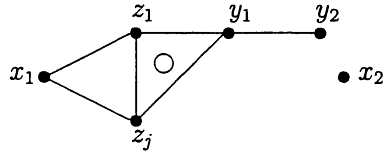

Figure: the case $\{z_1, y_1, z_j\}, \{y_1, y_2\} \in \Delta$ in Case (2-b)

Then $\mathrm{link}_\Delta\{y_1\}$ contains an egde $\{z_1, z_j\}$ and $\{y_2\}$. Since $\mathrm{link}_\Delta\{y_1\}$ is also connected, we can find vertices z_α, y_β such that $\{z_\alpha, y_\beta\} \in \mathrm{link}_\Delta\{y_1\}$. In particular, $\{z_\alpha, y_\beta, y_1\} \in \Delta$. This yields a contradiction because X_1Y_1, X_1Y_β is contained in $G(I_{\mathrm{link}_\Delta\{z_\alpha\}})$.

Next suppose that $\{y_1, w_1\} \in \Delta$.

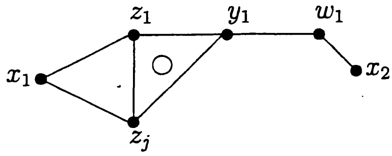

Figure: the case $\{z_1, y_1, z_j\}, \{y_1, w_1\} \in \Delta$ in Case (2-b)

Then $\mathrm{link}_\Delta\{y_1\}$ contains an egde $\{z_1, z_j\}$ and $\{w_1\}$. Since $\mathrm{link}_\Delta\{y_1\}$ is also connected, we can also find vertices z_α, y_β such that $\{z_\alpha, y_\beta\} \in \mathrm{link}_\Delta\{y_1\}$ (notice that $\{z_j, w_k\} \notin \Delta$). Hence we have $\dim \Delta = 1$. We complete the proof of Theorem 2.4.

Let Δ be a simplicial complex with $\dim \Delta \geq 2$. The Stanley–Reisner ring $K[\Delta]$ satisfies the Serre condition (S_2), that is, depth $K[\Delta]_P \geq \min\{2, \mathrm{height}\, P\}$, if and only if Δ is pure and $\mathrm{link}_\Delta(F)$ is connected for every face F with $\dim \mathrm{link}_\Delta(F) \geq 1$.

Corollary 2.8. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Assume that $K[\Delta]$ satisfies (S_2). Then the following conditions are equivalent:*

(1) *$K[\Delta]$ is a complete intersection.*

(2) *For any face F with $\dim \mathrm{link}_\Delta F = 1$, $\mathrm{link}_\Delta F$ is a complete intersection.*

(3) *There exists $W \subseteq V$ such that $\dim \Delta_{V \setminus W} \leq \dim \Delta - 3$ which satisfies the following condition:*

" $\mathrm{link}_\Delta\{x\}$ is a complete intersection for all $x \in W$."

Proof. Note that Δ is pure. Put $d = \dim \Delta + 1$.

(1) $\Longrightarrow$ (3) : It is enough to put $W = V$.

(3) $\Longrightarrow$ (2) : Let W be a subset of V satisfying the condition (3). Let F be a face with $\dim \mathrm{link}_\Delta(F) = 1$. Since Δ is pure, $\sharp(F) = d - 1 - \dim \mathrm{link}_\Delta(F) = d - 2$. As $\dim \Delta_{V \setminus W} \leq d - 4$, F is not contained in $V \setminus W$. Thus there exists $i \in F$ such that $i \in W$. Then since $\mathrm{link}_\Delta\{i\}$ is a complete intersection by the assumption, $\mathrm{link}_\Delta(F)$ is also a complete intersection, as required.

(2) $\Longrightarrow$ (1) : We use an induction on $d \geq 3$. First suppose that $d = 3$. Then for each $i \in V$, we have that $\dim \mathrm{link}_\Delta\{i\} = 1$. Hence $\mathrm{link}_\Delta\{i\}$ is a complete intersection by the assumption (3). Hence by Theorem 2.4, $K[\Delta]$ is a complete intersection.

Next suppose that $d \geq 4$. Let $i \in V$. Since $K[\Delta]$ satisfies (S_2), we have that $\Gamma = \mathrm{link}_\Delta\{i\}$ is connected and $\dim \Gamma = (d-1) - 1 = d - 2 \geq 2$. Moreover, for any face G in Γ with $\dim \mathrm{link}_\Gamma(G) = 1$, $\mathrm{link}_\Gamma(G) = \mathrm{link}_\Delta(G \cup \{i\})$ is a complete intersection by assumption. Hence, by the induction hypothesis, $K[\mathrm{link}_\Delta\{i\}]$ is a complete intersection. Therefore $K[\Delta]$ is a complete intersection by Theorem 2.4 again. □

Combining Theorem 2.4 with Cowsik–Nori's theorem and Goto–Takayama's theorem, we get:

Corollary 2.9. *Let Δ be a simplicial complex with* $\dim \Delta \geq 2$. *Assume that* Δ *is pure and connected. Then the following conditions are equivalent:*

(1) S/I_Δ^ℓ *is Cohen–Macaulay for every* $\ell \geq 1$.
(2) S/I_Δ^ℓ *is Buchsbaum for every* $\ell \geq 1$.
(3) S/I_Δ^ℓ *has (FLC) for every* $\ell \geq 1$.

If S/I_Δ^ℓ is (FLC) (resp. Cohen–Macaulay) for some positive integer ℓ, then S/I_Δ is Buchsbaum (resp. Cohen–Macaulay). In particular, Δ is pure. See [HTT, Theorem 2.6].

If Δ is *not* connected, then (2) and (3) are not necessarily equivalent. See below.

Example 2.10. Let $n \geq 2$ be a positive integer. Let

$$I = I_\Delta = (x_1, \ldots, x_n)(y_1 \ldots, y_n) \subseteq S = K[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Then Δ is the disjoint union of the standard $(n-1)$-simplices. Moreover, S/I^ℓ has (FLC) for every $\ell \geq 1$ by Theorem 1.2. And one can see that S/I^ℓ is *not* Buchsbaum for every $\ell \geq 2$.

3. The case $\dim \Delta = 1$

Proposition 3.1. *Let Δ be a connected simplicial complex of* $\dim \Delta = 1$. *Then the following conditions are equivalent:*

(1) $K[\Delta]$ *is a locally complete intersection.*
(2) $K[\Delta]$ *is a locally Gorenstein.*
(3) Δ *is either one of the following complexes:*
  (a) *n-gon for some* $n \geq 3$;
  (b) *n-pointed path for some* $n \geq 2$.

Proof. Suppose that $\dim \operatorname{link}_\Delta\{i\} = 0$. Then $\operatorname{link}_\Delta\{i\}$ consists of finite points. Hence if it is Gorenstein, then it is either one point or two points. Such a link is also a complete intersection. □

In the case $\dim \Delta = 1$, (1) and (3) in Corollary 2.9 is *not* equivalent in general. But we get the following result.

Proposition 3.2. *Let Δ be a simplicial complex with* $\dim \Delta = 1$. *Assume that Δ is pure and connected. Then the following conditions are equivalent:*

(1) S/I_Δ^ℓ *is Cohen–Macaulay for every* $\ell \geq 1$.
(2) S/I_Δ^ℓ *is Buchsbaum for every* $\ell \geq 1$.

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SEVERAL RESULTS ON FINITENESS PROPERTIES OF LOCAL COHOMOLOGY MODULES OVER COHEN MACAULAY LOCAL RINGS

KEN-ICHIROH KAWASAKI

We assume that all rings are commutative and noetherian with identity throughout this paper.

1. Introduction

In 1993, Huneke and Sharp (cf. [2]) and Lyubeznik (cf. [6]) showed the following results:

Theorem 1 (Huneke, Sharp and Lyubeznik). *Let $(R, \mathfrak{m})$ be a regular local ring containing a field, and I an ideal of R. Then the following assertions hold for all integers $i, j \geq 0$:*

(i) *$H^j_{\mathfrak{m}}(H^i_I(R))$ is an injective module;*
(ii) $\mathrm{inj.dim}_R(H^i_I(R)) \leq \dim H^i_I(R)$;
(iii) *the set of associated prime ideals of $H^i_I(R)$ is a finite set;*
(iv) *all the Bass numbers of $H^i_I(R)$ are finite.*

Our aims in this report are to develop results of Theorem 1 to those over Cohen-Macaulay local rings. We shall introduce the following theorems:

Theorem 2. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat. Let i be a non-negative integer. Further let I be an ideal of A satisfied with the condition that if we set $I \cap R = J$ then $I = JA$.*

(a) *If the set of associated prime ideals of $H^i_J(R)$ is a finite set, then so is the set of associated prime ideals of $H^i_I(A)$;*
(b) *if all the Bass numbers of $H^i_J(R)$ are finite, then so are all the Bass numbers of $H^i_I(A)$.*

Theorem 3. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of regular local rings, which is module-finite and flat, and I an ideal of A. Let i, j be non-negative integers. Set $I \cap R = J$. Suppose that $I = JA$ and R is an unramified regular local ring. Then the following assertions hold:*

(i) $\mathrm{inj.dim}_A H^j_{\mathfrak{n}} H^i_I(A) \leq 1$;
(ii) $\mathrm{inj.dim}_A H^i_I(A) \leq \dim H^i_I(A) + 1$;
(iii) *the set of associated prime ideals of $H^i_I(A)$ is a finite set;*
(iv) *all the Bass numbers of $H^i_I(A)$ are finite.*

Here it is worth while mentioning that S. Takagi and R. Takahashi recently showed finiteness properties of local cohomlogy modules over rings with finite F-representation type (cf. [11]). Mainly we shall prove part (i) and (ii) of Theorem 3 in this report.

2. Proof of Theorem 2: Outline

Definition 1. Let T be a module over a ring A and P a prime ideal of A. We define the j-th Bass number $\mu_j(P,T)$ at P to be

$$\mu_j(P,T) = \dim_{\kappa(P)} \operatorname{Ext}^j_{R_P}(\kappa(P), T_P),$$

where $\kappa(P) = R_P/PR_P$ (cf. [1]).

Remark 1. Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat. Then such properties for an extension are preserved by localization: let P be a prime ideal and set $\mathfrak{p} = P \cap R$. Then the local ring homomorphism of local rings $\phi_P : R_\mathfrak{p} \longrightarrow A_P$ is module-finite and flat (cf. [9, Theorem 7.1, p.46]).

Remark 2. Let $\phi : R \longrightarrow A$ be a ring homomorphism of rings, I an ideal of A and $J = I \cap R$. The condition $I = JA$ of Theorem 2 and 3 is preserved by localization, i.e., for any prime ideal $P \subset A$ we have $JR_\mathfrak{p} = IA_P \cap R_\mathfrak{p}$ where $\mathfrak{p} = P \cap R$.

Proposition 4. *Let $A \longrightarrow B$ be a local ring homomorphism of local rings, which is flat. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals of A. Then $\mathfrak{p}_1 = \mathfrak{p}_2$ if and only if $\operatorname{Ass}_B(B/\mathfrak{p}_1 B) = \operatorname{Ass}_B(B/\mathfrak{p}_2 B)$.*

Proof of Theorem 2. The proof follows from Bourbaki's formula and Propositition 4. □

Thanks to Theorem 1, our theorem proposes the following corollary. One can find the collection of properties for the faithfully flat and flat local ring homomorphisms in [4].

Corollary 5. *Let $(A, \mathfrak{n})$ be a Cohen-Macaulay local ring containing a field k, of dimension d, and $x_1, x_2, \dots, x_d$ a system of parameters. Let I be an ideal of A generated by polynomials over k in $x_1, x_2, \dots, x_d$. Suppose that $A/\mathfrak{n}$ is separable over k (or rather, over the image of k in $A/\mathfrak{n}$ via the natural mapping $A \longrightarrow A/\mathfrak{n}$). The following statements hold for all integers $i, j \geq 0$:*

(a) *the set of associated prime ideals of $H^i_I(A)$ is finite;*
(b) *all the Bass numbers of $H^i_I(A)$ are finite.*

Example 1. Singh [10] and Katzman [3] gave the examples of rings with respect to sets of infinite associated prime ideals of the top local cohomology modules. Especially Katzman's example states that even the second local cohomology module has an infinite set of distinct associated prime ideals. The local ring R and the local cohomology module are as follows:

$$R = k[s,t,x,y,u,v]_\mathfrak{m}/(sx^2v^2 - (s+t)xyuv + ty^2u^2), \quad H^2_{(u,v)}(R),$$

where $\mathfrak{m}$ is the irrelevant maximal ideal (s,t,x,y,u,v).

On the other hand, our result states that $H^j_I(R)$ satisfies finiteness properties (a) and (b) as in Corollary 5, for all $j \geq 0$ and for the ideal I generated by polynomials $f_1 = f_1(s,x,v,t-y,t-u)$, $f_2 = f_2(s,x,v,t-y,t-u)$, $\dots$, $f_r = f_r(s,x,v,t-y,t-u)$ in $s,x,v,t-y,t-u$ over k.

Remark 3. The converse statements of (a) and (b) in Theorem 2 also hold by Proposition 4 and faithfully flatness of ϕ .

3. Several results over regular local rings

In this section, we prove part (i) and (ii) of Theorem 3.

Definition 2. A regular local ring $(R, \mathfrak{m})$ is called unramified if R contains a field or if $p \notin \mathfrak{m}^2$ in the unequal characteristic case, where p is the characteristic of the residue field $R/\mathfrak{m}$. We note that if R contains a field then the characteristic of R and its residue field are equal, and the converse also holds.

We shall introduce several lemmas.

Lemma 6. *Let $(A, \mathfrak{n})$ be a local ring with the maximal ideal $\mathfrak{n}$, M be an (not necessarily finite) A-module with support $V(\mathfrak{n})$. Let l be a non-negative integer. Suppose that M is an A-module of finite injective dimension.*

(i) *If there is an A-module N with finite length such that $\operatorname{Ext}_A^n(N, M) = 0$ for all $n \geq 1$, then M is an injective A-module;*
(ii) *if there is an A-module N with finite length such that $\operatorname{Ext}_A^n(N, M) = 0$ for all $n \geq l+1$, then $\operatorname{inj.dim}_A M \leq l$.*

Lemma 7. *Let A be a ring, and l a positive integer. Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \qquad (\sharp)$$

be an exact sequence of A-modules.

(1) *If M' and M'' are injective A-modules, then M is an injective A-module;*
(2) *if $\operatorname{inj.dim}_A M' \leq l-1$ and $\operatorname{inj.dim}_A M'' \leq l$, then $\operatorname{inj.dim}_A M \leq l$.*

In addition suppose that M is an injective A-module.

(3) *If M' is an injective A-module, then M'' is an injective A-module;*
(4) *if $\operatorname{inj.dim}_A M' \leq l$, then $\operatorname{inj.dim}_A M'' \leq l-1$.*

Lemma 8. *Let A be a ring, $\mathfrak{a}$ an ideal of A, and M an (not necessarily finite) A-module. Let l be a positive integer. Denote by I^* a minimal injective resolution of T:*

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \xrightarrow{\partial^2} \cdots$$
$$\longrightarrow I^{j-1} \xrightarrow{\partial^{j-1}} I^j \xrightarrow{\partial^j} I^{j+1} \xrightarrow{\partial^{j+1}} \cdots$$

Further we denote by d^j the restriction of ∂^j to $\Gamma_{\mathfrak{a}}(I^j)$, after applying the functor $\Gamma_{\mathfrak{a}}(-)$ to I^*.

(1) *If $H_{\mathfrak{a}}^j(M)$ is an injective A-module for all $j \geq 0$, then $\ker d^j$ is an injective A-module for all $j \geq 0$, and $\operatorname{Im} d^j$ is an injective A-module for all $j \geq 0$;*
(2) *if $\operatorname{inj.dim}_A H_{\mathfrak{a}}^j(M) \leq l$ for all $j \geq 0$, then $\operatorname{inj.dim}_A \ker d^j \leq l$ for all $j \geq 0$, and $\operatorname{inj.dim}_A \operatorname{Im} d^j \leq l-1$ for all $j \geq 0$.*

Lemma 9. *Let $(A, \mathfrak{n})$ be a local ring and T an (not necessarily finite) A-module. Let l be a positive integer.*

(1) *If $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $H_{\mathfrak{n}}^j(T)$ is injective for all $j \geq 0$, then T has finite injective dimension and $\operatorname{inj.dim}_A T \leq \dim T$;*
(2) *if $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P + l$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $\operatorname{inj.dim}_A H_{\mathfrak{n}}^j(T) \leq l$ for all $j \geq 0$, then T has finite injective dimension and $\operatorname{inj.dim}_A T \leq \dim T + 2l - 1$.*

Remark 4. More generally, we can slightly improve part (2) of Lemma 9 as follows: if $\text{inj.}\dim_{A_P} T_P \leq \dim T_P + k$ for each prime ideal $P \in \text{Spec}(T)$ with $P \neq \mathfrak{n}$ and $\text{inj.}\dim_A H^j_{\mathfrak{n}}(T) \leq l$ for all $j \geq 0$, then T has finite injective dimension and $\text{inj.}\dim_A T \leq \dim T + k + l - 1$.

Proposition 10. *Let* $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ *be a local ring homomorphism of local rings, which is module-finite and flat, and* T *an* R*-module. Then we have* $\dim_R T = \dim_A T \otimes_R A$.

Proof of Theorem 3. We only prove (i) and (ii). By the results in [7], the assertions (iii) and (iv) in Theorem 3 follow from those of (a) and (b) in Theorem 2.

(i) First we note that $H^j_{\mathfrak{n}} H^i_I(A)$ has a finite injective dimension as an A-module by the regularity condition of A. Since the support of $H^j_{\mathfrak{n}} H^i_I(A)$ is in $V(\mathfrak{n})$, we only prove that

$$\text{Ext}^p_A(A/\mathfrak{n}, H^j_{\mathfrak{n}} H^i_I(A)) = 0$$

for all $p > 1$.

By the results of Zhou [12], $H^j_{\mathfrak{m}} H^i_J(R)$ has injective dimension ≤ 1. Thus we have

$$\text{Ext}^p_R(R/\mathfrak{m}, H^j_{\mathfrak{m}} H^i_J(R)) = 0$$

for all $p > 1$. Since the map $\phi : R \longrightarrow A$ is a module-finite ring homomorphism (hence an integral extension), the radical of $\mathfrak{m}A$ is equal to $\mathfrak{n}$. Then it follows from flatness of ϕ that $H^j_{\mathfrak{n}} H^i_I(A) = H^j_{\mathfrak{m}A} H^i_{JA}(A) = H^j_{\mathfrak{m}} H^i_J(R) \otimes_R A$. Further, we have

$$\text{Ext}^p_A(A/\mathfrak{m}A, H^j_{\mathfrak{m}A} H^i_{JA}(A)) = \text{Ext}^p_R(R/\mathfrak{m}, H^j_{\mathfrak{m}} H^i_J(R)) \otimes A = 0$$

for all $p > 1$ since ϕ is flat. Since $A/\mathfrak{m}A$ is an A-module of finite length and $H^j_{\mathfrak{n}} H^i_I(A)$ has a finite injective dimension as an A-module, it follows from part (ii) of Lemma 6 that $H^j_{\mathfrak{m}A} H^i_{JA}(A)$ has injective dimension ≤ 1. Therefore the injective dimension of $H^j_{\mathfrak{n}} H^i_I(A)$ is not greater than one.

(ii) We shall show the assertion (ii) by induction on $d = \dim H^i_I(A) \geq 0$. Note that $d = \dim H^i_J(R)$ by Proposition 10.

Suppose that $d = 0$. Then the support of $H^i_I(A)$ is contained in $V(\mathfrak{n})$, so the injective dimension of $H^i_I(A) = H^0_{\mathfrak{n}}(H^i_I(A))$ is one by part (i) of the theorem.

Suppose that $d > 0$. Let $P \in \text{Supp}_A(H^i_I(A))$ be a prime ideal such that P is not the maximal ideal. Set $\mathfrak{p} = P \cap R$; $\mathfrak{p}$ is not the maximal ideal of R, since the extension $R \longrightarrow A$ is integral. Then the ring homomorphism $R_{\mathfrak{p}} \longrightarrow A_P$ is a module finite extension and flat between regular local rings by Remark 1. The condition $I = JA$ is preserved by localization (cf. Remark 2), that is $IA_P = (JR_{\mathfrak{p}})A_P$ for a prime ideal P of A. Also, the property of a ring being unramified is preserved by localization. The dimensions of $H^i_I(A)_P$ and $H^i_J(R)_{\mathfrak{p}}$ are less than d and Proposition 10 implies that we can apply the inductive hypothesis for the local cohomology module $H^i_I(A)_P$ over A_P, that is,

$$\text{inj.}\dim_{A_P} H^i_I(A)_P \leq \dim H^i_I(A)_P + 1 \leq d - 1 + 1 = d.$$

Further $H^j_{\mathfrak{n}} H^i_I(A)$ has injective dimension ≤ 1 for all $j \geq 0$ by part (i) of the theorem.

So the assertion follows from Remark 4, that is $\text{inj.}\dim_A H^i_I(A) \leq d + 1$. The proof is completed. □

Corollary 11. *Let* $(A, \mathfrak{n})$ *be a complete ramified regular local ring, of dimension* d*, and* $x_1, x_2, \ldots, x_d$ *a system of parameters, where* $x_1 = p$ *is the characteristic of the residue field* $A/\mathfrak{m}$*. Suppose that* I *is an ideal of* A *generated by polynomials over* $\mathbb{Z}$ *in* $x_2, \ldots, x_d$*. Then we have the following assertions for integers* $i, j \geq 0$*:*

(i) $\text{inj.}\dim_A H^j_{\mathfrak{n}}(H^i_I(A)) \leq 1$;

(ii) inj. $\dim_A H^i_I(A) \leq \dim H^i_I(A) + 1$;
(iii) *the set of associated prime ideals of* $H^i_I(A)$ *is finite*;
(iv) *all the Bass numbers of* $H^i_I(A)$ *are finite.*

Now we propose the following questions:

Question 1. Let i, j be non-negative integers. Let $(A, \mathfrak{n})$ be a regular local ring, I an ideal of A. Is $H^j_{\mathfrak{n}} H^i_I(A)$ injective ?

Question 2. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be an unramified regular local ring, J an ideal of R. Is $H^j_{\mathfrak{m}} H^i_J(R)$ injective ?

If the above questions were answered affirmatively, we could prove that the upper bound of the injective dimension of local cohomology modules is its dimension over an unramified (and also any) regular local ring. Question 2 is suggested in Lyubeznik's paper [7]. We can prove this, modifying that of (iv) of Theorem 3.

Proposition 12. *Let i be a non-negative integer.*

(1) *If Question 1 has an affirmative answer for all $j \geq 0$, then* inj. $\dim_A H^i_I(A) \leq \dim H^i_I(A)$ *holds over a regular local ring $(A, \mathfrak{n})$ for all ideals I of A*;
(2) *if Question 2 has an affirmative answer for all $j \geq 0$, then* inj. $\dim_R H^i_J(R) \leq \dim H^i_J(R)$ *holds over an unramified regular local ring $(R, \mathfrak{m})$ for all ideal J of R.*

Although the following assertions hold not only over a regular local ring but also over other rings, we concentrate rings on regular local rings.

Example 2. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the dimension of I is zero, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Example 3. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the dimension of I is one, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Example 4. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If I is a principal ideal, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Example 5. Let $(R, \mathfrak{m})$ be a regular local ring, Let i, j be non-negative integers, I an ideal of R. If I is generated by a regular sequence, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Proposition 13. *Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R and f is a non-zero and non-unit element of R. If I is a subideal of a principal ideal (f) up to radicals, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

Corollary 14. *Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the height of I is one, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

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DEPARTMENT OF MATHEMATICS, NARA UNIVERSITY OF EDUCATION, TAKABATAKE-CHO, NARA, 630-8528, JAPAN
E-mail address: `kawaken@nara-edu.ac.jp`

29th Symposium on Commutative Algebra
Nagoya November 2007

Cohen-Macaulay property of graded rings associated to contracted ideals in dimension 2

Aldo Conca
Dipartimento di Matematica, Universitá di Genova
Via Dodecaneso 35, I-16146 Genova, Italia

Abstract: We study the Cohen-Macaulay property of the associated graded ring of contracted homogeneous ideals in $K[x,y]$. Surprising, the problem is closely related to the description of the Gröbner fan of the ideal of the rational normal curve. We completely classify the contracted ideals with a Cohen-Macaulay associated graded ring in terms of the numerical invariants arising from Zariski's factorization. These results are contained in "Contracted ideals and the Gröbner fan of the rational normal curve" arXiv0705.3767, joint work with E.De Negri and M.E.Rossi which is going to appear in the first volume of the new journal "Algebra and Number Theory".

Let K be a field, $R = K[x,y]$ and I be a homogeneous ideal of R with $\sqrt{I} = \mathbf{m} = (x,y)$. Denote by $\mathrm{gr}_I(R)$ the associated graded ring of I, that is, $\mathrm{gr}_I(R) = \oplus_k I^k/I^{k+1}$. Denote by $\mu(I)$ the minimal number of generators of I and by $o(I)$ the order of I which is, by definition, the least degree of a non-zero element in I. By the Hilbert-Burch theorem we know that $\mu(I) \leq o(I)+1$. The ideal I is said to be contracted if $\mu(I) = o(I)+1$. Contracted ideals can be characterized also as the ideals which are contracted from a quadratic extension. Explicitly, for a linear form ℓ one considers a quadratic extension $R[x/\ell, y/\ell]$ of R. Then I is contracted if and only if $IR[x/\ell, y/\ell] \cap R = I$ for some ℓ. Contracted ideals have been introduced by Zariski in his studies on the factorization property of integrally closed ideals, see [ZS, App.5]. Every integrally closed ideal I is contracted and has a Cohen-Macaulay associated graded ring $\mathrm{gr}_I(R)$, see [LT]. In general, however, the associated graded ring of a contracted ideal need not be Cohen-Macaulay. So we are led to consider the following:

Problem 0.1. Describe the contracted homogeneous ideals I of $K[x,y]$ such that $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

Zariski proved a factorization theorem for contracted ideals asserting that every contracted ideal I can be written as $I = L_1 \cdots L_s$ where the L_i are themselves contracted but of a very special kind. In the homogeneous case and assuming K is algebraically closed, each L_i is a lex-segment monomial ideal in a specific system of coordinates depending on i.

Recall that a monomial ideal L in R is a lex-segment ideal (lex-ideal for short) if whenever $x^a y^b \in L$ with $b > 0$ then also $x^{a+1}y^{b-1} \in L$. Every lex-ideal L of order d can be written as

$$L = (x^d, x^{d-1}y^{a_1}, \dots, y^{a_d})$$

and hence can be encoded by the vector $a = (a_0, a_1, \dots, a_d)$ with increasing integral coordinates and $a_0 = 0$.

Therefore to every contracted ideal I with factorization $I = L_1 \cdots L_s$ we may associate sequences $a_1, \dots, a_s$, where $a_i = (a_{ij} : j = 0, \dots d_i) \in \mathbf{N}^{d_i+1}$ are increasing and $a_{i0} = 0$. For instance:

Example 0.2. Let

$$X = \begin{pmatrix} y^2 & 0 & 0 & 0 & 0 & 0 \\ -x-3y & y & 0 & 0 & 0 & 0 \\ -9y & -x+3y & y^3 & 0 & 0 & 0 \\ 0 & 0 & -x-y & y^3 & 0 & 0 \\ 0 & 0 & 0 & -x-y & y^2 & 0 \\ 0 & 0 & 0 & 0 & -x-y & y \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$$

and let I be the ideal of 6-minors of X. We have $\mu(I) = 7$ and $o(I) = 6$, so I is contracted. Zariski's factorization of I is

$$I = (x^3, x^2y^2, xy^3, y^9)(x_1^3, x_1^2y_1^4, x_1y_1^7, y_1^9)$$

where $x_1 = x + y$ and $y_1 = y$. Hence we associate to I the sequences $a_1 = (0, 2, 3, 9)$ and $a_2 = (0, 4, 7, 9)$.

With respect to the terminology introduced above, in [CDJR] it is shown that:

Theorem 0.3. *One has*

$$\operatorname{depth} \operatorname{gr}_I(R) = \min\{\operatorname{depth} \operatorname{gr}_{L_i}(R) : i = 1, \dots, s\}.$$

In particular, the Cohen-Macaulayness of $\operatorname{gr}_I(R)$ *is equivalent to the Cohen-Macaulayness of* $\operatorname{gr}_{L_i}(R)$ *for every* $i = 1, \dots, s$.

Therefore, to answer Problem 0.1, one has to characterize the lex-ideals L with Cohen-Macaulay associated graded ring.

Problem 0.4. For every d describe the sequences $a = (a_0, a_1, \dots, a_d) \in \mathbf{N}^{d+1}$ with increasing coordinates and $a_0 = 0$ such that $\operatorname{gr}_L(R)$ Cohen-Macaulay. Here L is the lex-ideal associated to a.

Since R is regular, $\operatorname{gr}_L(R)$ is Cohen-Macaulay iff $\operatorname{Rees}(L)$ is Cohen-Macaulay. As $\operatorname{Rees}(L)$ is an affine semigroup ring, the result of Trung and Hoa [TH] could be applied. But we have not been able to follow this line of investigation.

Denote by P the defining ideal of the Veronese embedding of $\mathbf{P}^1 \to \mathbf{P}^d$ in its standard coordinate system. It is well-known that P is the ideal of $K[t_0, \dots, t_d]$ generated by the 2-minors of the matrix

$$T_d = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & \dots & t_{d-1} \\ t_1 & t_2 & \dots & \dots & t_{d-1} & t_d \end{pmatrix}$$

We show that:

Proposition 0.5. *Let L be a lex-ideal. Denote by a the sequence associated to L. Then*

$$\operatorname{depth} \operatorname{gr}_L(R) = \operatorname{depth} K[t_0, \dots, t_d]/\operatorname{in}_a(P).$$

Here $\operatorname{in}_a(P)$ *denotes the ideal of the initial forms of* P *with respect to the vector* a. *In particular,* $\operatorname{gr}_L(R)$ *is Cohen-Macaulay if and only if* $\operatorname{in}_a(P)$ *is perfect.*

Therefore Problem 0.4 becomes equivalent to:

Problem 0.6. For every d determine the vectors $a \in \mathbf{N}^{d+1}$ such that $\operatorname{in}_a(P)$ is perfect.

To answer 0.6 we first show:

Proposition 0.7. *The initial monomial ideals of P which are perfect are in bijective correspondence with the subsets of $\{1,2,\ldots,d-1\}$. So P has exactly 2^{d-1} perfect initial monomial ideals.*

The fact that P has exactly 2^{d-1} perfect monomial initial ideals can be derived also by combining results of Hosten and Thomas [HT] with results of O'Shea and Thomas [OT].

We explain this bijective correspondence with an example. Suppose $d=6$ and take the sequence and the subset $\{3,4\}$ of $\{1,2,3,4,5\}$, Set $\mathbf{i}=\{0,d\}\cup\{3,4\}=\{0,3,4,6\}$. The corresponding perfect initial ideal I of P is obtained by dividing the matrix T_6 in blocks (from column i_v+1 to i_{v+1})

$$T_6=\begin{pmatrix} t_0 & t_1 & t_2 & | & t_3 & | & t_4 & t_5 \\ t_1 & t_2 & t_3 & | & t_4 & | & t_5 & t_6 \end{pmatrix}$$

and then taking anti-diagonals of minors whose columns belong to the same block,

$$t_1^2, t_1t_2, t_2^2, t_5^2,$$

and main diagonals from minors whose columns belong to different blocks

$$t_0t_4, t_0t_5, t_0t_6, t_1t_4, t_1t_5, t_1t_6, t_2t_4, t_2t_5, t_2t_6, t_3t_5, t_3t_6.$$

The ideal I is the initial ideal of P with respect to every term order τ refining the weight $a=(0,3,5,6,10,16,21)$ obtained from the "permutation" vector $\sigma=(3,2,1|4|6,5)\in S_6$ by setting $a_0=0$ and $a_i=\sum_{j=1}^{i}\sigma_j$. With respect to this term order the 2-minors of T_6 are a Gröbner basis of P but not the reduced Gröbner basis. The corresponding reduced Gröbner basis is

$$\begin{array}{lllll} \underline{t_1^2}-t_0t_2, & \underline{t_1t_2}-t_0t_3, & \underline{t_2^2}-t_1t_3, & \underline{t_0t_4}-t_1t_3 & \underline{t_0t_5}-t_2t_3, \\ \underline{t_1t_4}-t_2t_3, & \underline{t_0t_6}-t_3^2, & \underline{t_1t_5}-t_3^2, & \underline{t_2t_4}-t_3^2, & \underline{t_1t_6}-t_3t_4, \\ \underline{t_2t_5}-t_3t_4, & \underline{t_2t_6}-t_4^2, & \underline{t_3t_5}-t_4^2, & \underline{t_3t_6}-t_4t_5, & \underline{t_5^2}-t_4t_6. \end{array}$$

So for every vector $a=(a_0,a_1,\ldots,a_6)\in\mathbf{Q}^7_{\geq 0}$ satisfying the following system of linear inequalities

$$\begin{array}{llll} 2a_1>a_0+a_2* & a_1+a_2>a_0+a_3 & 2a_2>a_1+a_3* & a_0+a_4>a_1+a_3* \\ a_0+a_5>a_2+a_3 & a_1+a_4>a_2+a_3 & a_0+a_6>2a_3 & a_1+a_5>2a_3 \\ a_2+a_4>2a_3 & a_1+a_6>a_3+a_4 & a_2+a_5>a_3+a_4 & a_2+a_6>2a_4 \\ a_3+a_5>2a_4 & a_3+a_6>a_4+a_5* & 2a_5>a_4+a_6* & \end{array}$$

we have $\mathrm{in}_a(P)=I$. More precisely, if we set

$$C(\mathbf{i})=\{a\in\mathbf{Q}^{d+1}_{\geq 0}:\mathrm{in}_a(P)=I\}$$

then $C(\mathbf{i})$ is the convex cone is defined by above system of inequalities. The $*$ indicates an essential inequalities. One has:

$$\overline{C(\mathbf{i})}=\{a\in\mathbf{Q}^{d+1}_{\geq 0}: I \text{ is an initial ideal of } \mathrm{in}_a(P)\}$$

For a given d we set:

$$CM_d=\{a\in\mathbf{Q}^{d+1}_{\geq 0}:\mathrm{in}_a(P) \text{ is perfect}\}$$

the "Cohen-Macaulay region" of the Gröbner fan of P. Our main theorem is the following:

Theorem 0.8.

$$CM_d = \cup_{\mathbf{i}} \overline{C(\mathbf{i})}$$

where the union is extended to the set of the 2^{d-1} *sequences* $\mathbf{i} = (0 = i_0 < i_1 < \cdots < i_k = d)$.

Combining these results we obtain an explicit characterization, in terms of the numerical invariants arising from the Zariski factorization, of the Cohen-Macaulay property of the associated graded ring to a contracted homogeneous ideal in $K[x,y]$.

Theorem 0.9. *Let* I *be a contracted homogeneous ideal of* $K[x,y]$ *with Zariski factorization* $I = L_1 \cdots L_s$. *Denote by* d_i *the order of* L_i *and by* $a_i \in \mathbb{N}^{d_i+1}$ *the sequence associated to* L_i. *Then* $\mathrm{gr}_I(R)$ *is Cohen-Macaulay iff* $a_i \in CM_{d_i}$ *for all* $i = 1, \ldots, s$.

As the regions CM_{d_i} are the union of cones $\overline{C(\mathbf{i})}$ which are described explicitly in terms of linear inequalities, Theorem 0.9 answers 0.1.

Two of the cones of the Cohen-Macaulay region CM_d are special as they correspond to opposite extreme selections:

(1) (the lex-cone) If $\mathbf{i} = (0,1,2,\ldots,d)$, then the closed cone $\overline{C(\mathbf{i})}$ is described by the inequality system

$$a_i + a_j \geq a_u + a_v$$

with $u = \lfloor (i+j)/2 \rfloor$, $v = \lceil (i+j)/2 \rceil$ for every i,j. Setting

$$b_i = a_i - a_{i-1}$$

the cone $C(\mathbf{i})$ can be described by:

$$b_{i+1} \geq b_i$$

for every $i = 1, \ldots, d-1$. In this case the initial ideal of P is $(t_it_j : j - i > 1)$ and it can be realized by the lex-order with $t_0 < t_1 < \cdots < t_d$ or by the lex-order with $t_0 > t_1 > \cdots > t_d$. This is the only radical monomial initial ideal of P. The lex-ideals "belonging" to $\overline{C(\mathbf{i})}$ are the integrally closed. Indeed, they are the products of d complete intersections of type (x, y^u).

(2) (the revlex-cone) If $\mathbf{i} = (0,d)$ then the closed cone $\overline{C(\mathbf{i})}$ is described by inequality system

$$a_i + a_j \geq a_0 + a_{i+j}$$

if $i + j \leq d$ and

$$a_i + a_j \geq a_d + a_{i+j-d}$$

if $i + j \geq d$. It can be realized by the revlex-order with $t_0 < t_1 < \cdots < t_d$ or by the revlex-order with $t_0 > t_1 > \cdots > t_d$. The corresponding initial ideal of P is $(t_1, \ldots, t_{d-1})^2$. The lex-ideals L "belonging" to the cone are characterized by the fact that $L^2 = (x^d, y^{a_d})L$, that is, they are exactly the lex-ideals with a monomial minimal reduction and reduction number 1. It is not difficult to show that the simple homogeneous integrally closed ideals of $K[x,y]$ are exactly the ideals of the form $\overline{(x^d, y^c)}$ with $\mathrm{GCD}(d,c) = 1$. In other words, $\overline{C(\mathbf{i})}$ contains the exponent vectors of all the simple (i.e. not product of two proper ideals) integrally closed ideals of order d.

Example 0.10 (0.2 continued). For the ideal I the corresponding sequences are $a_1 = (0,2,3,9)$ and $a_2 = (0,4,7,9)$. The region CM_3 is the union of 4 cones: $C_1 = \overline{C(0,3)}$ the revlex-cone, $C_2 = \overline{C(0,1,3)}, C_3 = \overline{C(0,2,3)}$ and $C_4 = \overline{C(0,1,2,3)}$ the lex-cone. The revlex-cone C_1 is described by the inequalities $b_1 \geq b_2 \geq b_3$. The union of the cones C_2, C_3, C_4 form what we call the big cone that is described by the inequality $b_1 \leq b_3$. So we see that

a_1 belongs to the big cone and $a_2 \in C_1$. Hence both a_1 and a_2 belong to CM_3. It follows that $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

For a lex-segment L associated to a vector a there is a closed relationship between the Hilbert series of $\mathrm{gr}_L(R)$ and the multigraded Hilbert series of $\mathrm{in}_a(P)$.

Given a monomial initial ideal I of P (perfect or not) consider the associated closed maximal cone of the Gröbner fan:

$$C_I = \{a \in \mathbf{Q}_{\geq 0}^{d+1} : I \text{ is an initial ideal of } \mathrm{in}_a(P)\}.$$

The key observation is the following:

Lemma 0.11. *Let L be a lex-ideal with associated vector a belonging to C_I. For $k \in \mathbf{N}$ set $M_k(I) = \{\alpha \in \mathbf{N}^{d+1} : t^\alpha \notin I, |\alpha| = k\}$. Denote by $\sum M_k(I)$ the sum of the vectors in $M_k(I)$. By construction $\sum M_k(I) \in \mathbf{N}^{d+1}$ and*

$$\mathrm{length}(R/L^k) = a \cdot \sum M_k(I)$$

for all k.

In terms of Hilbert series Lemma 0.11 can be rewritten as in the following lemma.

Lemma 0.12. *Let L be a monomial ideal with associated sequence a belonging to C_I. Then*

$$H_L^1(z) = a \cdot \nabla H_{S/I}(\underline{t})_{t_i=z}$$

where $\nabla = (\partial/\partial t_0, \ldots, \partial/\partial t_d)$ is the gradient operator.

Where $H_L^1(z)$ is the Hilbert series $\sum \mathrm{length}(R/L^{k+1})z^k$ of L and $H_{S/I}(\underline{t})$ is the $\mathbf{Z}^{d+1}$-graded Hilbert series of S/I.

Combining Lemma 0.11 with Lemma 0.12 we have that Hilbert coefficients, the h-polynomials of L are linear functions in the a_i's whose coefficients just depend on I. The explicit expressions can be computed in terms of the multigraded Betti numbers or in terms of Stanley decompositions of S/I. For example:

```
I:=Ideal(
t[2]^2, t[0]t[6], t[1]t[4], t[0]t[4], t[0]t[3], t[0]t[2],t[3]t[5]^2,
t[0]^2t[5], t[4]t[5]^3, t[5]^5, t[0]t[5]^4, t[4]t[6], t[3]t[6],
t[4]^2, t[2]t[6], t[3]t[4], t[2]t[4], t[2]t[5], t[3]^2, t[2]t[3]
)

Inequalities describing C_I

a[0]+a[5]<a[1]+a[4]     a[1]+a[2]<a[0]+a[3]     2a[1]+a[3]<2a[0]+a[5]
a[1]+4a[6]<5a[5]         a[4]+a[5]<a[3]+a[6]     a[3]+a[5]<a[2]+a[6]

h-vector of L associated to every vector a of C_I

(0)  a[0] + a[1] + a[2] + a[3] + a[4] + a[5] + a[6]
(1)  a[0] + 4a[1] - 2a[2] - a[3] - 2a[4] + 4a[5] + a[6]
(2)  -a[0] + a[1] + a[2] - a[3] + a[4] - 2a[5] + a[6]
(3)  a[3] - a[4] - a[5] + a[6]
(4)  -a[0] + a[1] + 2a[4] - 3a[5] + a[6]
(5)  a[0] - a[1] - a[4] + a[5]
```

We discuss also how the formulas for the Hilbert series and polynomials of $\mathrm{gr}_L(R)$ change by varying the corresponding cones of the Gröbner fan of P. There are two Hilbert polynomials in this setting, the one associated to length(L^k/L^{k+1}) and the one associated to length(R/L^{k+1}). To distinguish one from the other we use an asterisk to denote the second. We have:

Proposition 0.13. *Let I, J be monomial initial ideals of P. Then*

(1) *The formula for the multiplicity that is valid in the cone C_I equals that that is valid in C_J iff $\sqrt{I} = \sqrt{J}$.*
(2) *The formula for the Hilbert series that is valid in the cone C_I equals that that is valid in C_J iff $I = J$.*
(3) *The formula for the Hilbert polynomial* that is valid in the cone C_I equals that that is valid in C_J iff I and J have the same saturation.*

Furthermore there is a conjectural relation with the hypergeometric Gröbner fan introduced by Saito, Sturmfels and Takayama in [SST] and the equality between the formulas giving the Hilbert polynomials. Precisely, we conjecture that the formula for the Hilbert polynomial valid in the cone C_I equals that that is valid in C_J iff I and J have the same minimal components.

The ideals I, J below are non-Cohen-Macaulay initial ideals of P. We display the formulas for the h-vectors and Hilbert coefficients e_0, e_1, e_2 valid in the corresponding cones (computed via Stanley decompositions).

I	$(t_1t_3, t_1t_2, t_0t_2, t_3^3, t_1^2t_4, t_1^3, t_2t_4, t_2t_3, t_2^2)$
(h_0)	$a_0 + a_1 + a_2 + a_3 + a_4$
(h_1)	$2a_0 + a_1 - 3a_2 + a_3 + 2a_4$
(h_2)	$2a_0 - 4a_1 + 3a_2 - 2a_3 + a_4$
(h_3)	$-a_0 + 2a_1 - a_2$
(e_0)	$4a_0 + 4a_4$
(e_1)	$3a_0 - a_1 - 3a_3 + 4a_4$
(e_2)	$-a_0 + 2a_1 - 2a_3 + a_4$

J	$(t_1t_3, t_1t_2, t_1^2, t_3^3, t_2t_4, t_2t_3, t_2^2)$
(h_0)	$a_0 + a_1 + a_2 + a_3 + a_4$
(h_1)	$3a_0 - a_1 - 2a_2 + a_3 + 2a_4$
(h_2)	$a_2 - 2a_3 + a_4$
(h_3)	0
(e_0)	$4a_0 + 4a_4$
(e_1)	$3a_0 - a_1 - 3a_3 + 4a_4$
(e_2)	$a_2 - 2a_3 + a_4$

In this case $I^{top} = J^{top} = (t_1t_3, t_2, t_3^3, t_1^2)$ as conjectured and $J = J^{sat} \neq I^{sat} = (t_2, t_1t_3, t_1^2t_4, t_3^3, t_1^3)$ as we know by Proposition 0.13.

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On codimension-one $\boldsymbol{A}^1$-fibrations over Noetherian normal domains

Nobuharu Onoda (University of Fukui)

1. Introduction

This is a joint work with S. M. Bhatwadekar and A. K. Dutta. Let R be a commutative ring. For a prime ideal P of R, we denote by $k(P)$ the field R_P/PR_P. A polynomial ring in n variables over R is denoted by $R^{[n]}$.

Definition 1.1. We shall call an R-algebra A to be a *codimension-one* $\boldsymbol{A}^1$*-fibration* over R if

$$k(P) \otimes_R A = k(P)^{[1]}$$

for every $P \in \operatorname{Spec} R$ with $\operatorname{ht} P \leq 1$.

Let R be a Noetherian normal domain with field of fractions K. Then the following results were proved in ([2], 3.4) and ([1], 3.10) respectively.

Theorem 1.2. Let A be a flat R-subalgebra of $R^{[m]}$ such that $K \otimes_R A = K^{[1]}$ and $k(P) \otimes_R A$ is an integral domain for every prime ideal P in R of height one. Then $A \cong R[IX]$ for an invertible ideal I of R.

Theorem 1.3. Let A be a faithfully flat finitely generated R-algebra such that $K \otimes_R A = K^{[1]}$ and $k(P) \otimes_R A$ is geometrically integral for every prime ideal P in R of height one. Then $A \cong R[IX]$ for an invertible ideal I of R.

Recently the two results were shown to emanate from the following result.

Theorem 1.4. Let A be a faithfully flat R-algebra such that A is an R-subalgebra of a finitely generated R-algebra B and such that A satisfies the fibre conditions:

(i) $K \otimes_R A = K^{[1]}$.
(ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.

Then $A \cong R[IX]$ for an invertible ideal I of R.

In this note, we explore the structure of a faithfully flat codimension-one $\boldsymbol{A}^1$-fibration over a Krull domain; in particular, over a Noetherian normal domain. As an application we show that all previous results described above can be deduced from this structure theorem.

2. Structure theorem

We begin by noting the following result.

Lemma 2.1. Let R be a Krull domain and A a flat R-algebra. Then A is a codimension-one $\boldsymbol{A}^1$-fibration over R if and only if $A_P = {R_P}^{[1]}$ for every $P \in \operatorname{Spec} R$ with $\operatorname{ht} P = 1$.

Set-Up

Throughout this section we will assume that

R: Krull domain with field of fractions K.

$\Delta = \{P \in \operatorname{Spec} R \mid \operatorname{ht} P = 1\}$.

A: a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every $P \in \Delta$.

x: A fixed element of A such that $T^{-1}A = K[x]$, where $T = R \setminus \{0\}$.

$\Sigma = \{\Gamma \mid \Gamma \text{ is a finite subset of } \Delta\}$.

$\Gamma_a = \{P \in \Delta \mid a \in P\}$, where $0 \neq a \in R$.

Definition 2.2. For $\Gamma \in \Sigma$, we set

$$R_\Gamma = \bigcap_{P \notin \Gamma} R_P$$

and

$$A_\Gamma = S^{-1}A \cap R_\Gamma[x],$$

where $S = R \setminus (\bigcup_{P\in\Gamma} P)$. Note that

$$R_{\Gamma_a} = R[1/a]$$

for $0 \neq a \in R$.

The following result holds for the ring A_Γ defined above.

Lemma 2.3. For $\Gamma \in \Sigma$, we have

$$A_\Gamma = \bigoplus_{n\geq 0}(R \cap d^n R_\Gamma)\left(\frac{x-c}{d}\right)^n$$

for some elements $c, d(\neq 0) \in R$. In particular, for $0 \neq a \in R$,

$$A_{\Gamma_a} = \bigoplus_{n\geq 0}(R \cap d^n R[1/a])\left(\frac{x-c}{d}\right)^n$$

for some $c, d(\neq 0) \in R$. Furthermore, we have
(1) $A_\Gamma \subseteq A$.
(2) $(A_\Gamma)_P = A_P$ for $P \in \Gamma$.
(3) $(A_\Gamma)_P = R_P[x]$ for $P \in \Delta \setminus \Gamma$.

Lemma 2.4. Let I be an ideal of R and suppose that I is R-flat. Then I is an invertible ideal of the form

$$I = R \cap dR[1/a]$$

for some $d \in I, a \in R$. Moreover we have

$$I^n = R \cap d^n R[1/a]$$

for every positive integer n.

From Lemmas 2.3 and 2.4, we have the following

Corollary 2.5. Suppose that A_{Γ_a} is flat over R. Then $A_{\Gamma_a} \cong R[IX]$ for an invertible ideal I of R.

Lemma 2.6. Let Γ_1 and Γ_2 be finite subsets of Δ. If $\Gamma_1 \subseteq \Gamma_2$, then $A_{\Gamma_1} \subseteq A_{\Gamma_2}$.

Lemma 2.6 shows that the rings A_Γ, together with inclusion maps, form a direct system

$$\{A_\Gamma \mid \Gamma \in \Sigma\}$$

indexed by Σ. We now prove the structure theorem:

Theorem 2.7. $A = \varinjlim A_\Gamma \left(= \bigcup_\Gamma A_\Gamma \right)$.

Proof. Set $C = \varinjlim A_\Gamma$. Then $C = \bigcup_\Gamma A_\Gamma$, and hence $C \subset A$. For the converse inclusion $A \subset C$, let w be an arbitrary non-zero element of A. Since $A \subset K[x]$, we can write

$$w = \xi_0 x^n + \xi_1 x^{n-1} + \cdots + \xi_n$$

for some $n \geq 0$ and $\xi_0, \ldots, \xi_n \in K$. Note that $\Delta_\xi := \{P \in \Delta \mid v_P(\xi) < 0\}$ is a finite set for $0 \neq \xi \in K$, because writing $\xi = b/c$ with $b, c (\neq 0) \in R$, we have $v_P(c) > v_P(b) \geq 0$ for $P \in \Delta_\xi$, so that $\Delta_\xi \subset \mathrm{Ass}_R(R/cR)$. Set $\Gamma = \bigcup_{i=0}^n \Delta_{\xi_i}$. Then Γ is a finite subset set of Δ, and $w \in R_P[x]$ for any $P \notin \Gamma$. Therefore

$$w \in \left(\bigcap_{P \in \Gamma} A_P \right) \cap \left(\bigcap_{P \notin \Gamma} R_P[x] \right),$$

which implies $w \in A_\Gamma \subset C$. This completes the proof. □

Lemma 2.8. For $P \in \Delta$, writing $PR_P = pR_P$ with $p \in R$, we have

$$A_P = R_P \left[\frac{x - c}{p^e} \right]$$

for some $c \in R$ and $e \geq 0$. Furthermore, the integer e is uniquely determined for P.

For $P \in \Delta$, we denote by e_P the integer e given in Lemma 2.8 above. Note that

$$e_P > 0 \Longleftrightarrow A_P \neq R_P[x].$$

From Theorem 2.7, we shall now deduce that finite generation of A is equivalent to the finiteness of the set

$$\Delta_0 = \{P \in \Delta \mid e_P > 0\}.$$

Lemma 2.9. Let Γ_1, Γ_2 be elements of Σ such that $\Gamma_1 \subseteq \Gamma_2$. Then $A_{\Gamma_1} \subsetneqq A_{\Gamma_2}$ if and only if there exists $P \in \Gamma_2 \setminus \Gamma_1$ such that $P \in \Delta_0$.

We say that R is a retract of A if there exists an R-algebra map $\varphi\colon A \to R$ such that $\varphi|_R = id_R$.

Theorem 2.10. The following conditions are equivalent:
(1) A is finitely generated over R.
(2) Δ_0 is a finite set.
(3) R is a retract of A and A is a Krull ring.
(4) $A \cong R[IX]$ for an invertible ideal I of R.

Proof. We shall give a proof only for (1)⇒(2) and (2)⇒(4).

(1)⇒(2): Recall that, by Theorem 2.7, we have

$$A = \varinjlim A_\Gamma = \bigcup_\Gamma A_\Gamma. \tag{1}$$

Let $A = R[f_1, \ldots, f_n]$. By (1), for each i there exists $\Gamma_i \in \Sigma$ such that $f_i \in A_{\Gamma_i}$. Then, setting $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$, we have $f_i \in A_\Gamma$ for each i, which implies $A = A_\Gamma$. Now suppose that Δ_0 is an infinite set. Then there exists $P \in \Delta_0 \setminus \Gamma$, because Γ is a finite set. Let $\Gamma' = \Gamma \cup \{P\}$. Then $\Gamma \subseteq \Gamma'$ and $P \in \Gamma' \setminus \Gamma$. Thus $A_\Gamma \neq A_{\Gamma'}$ by Lemma 2.9. On the other hand, by Lemma 2.6, we have

$$A = A_\Gamma \subseteq A_{\Gamma'} \subseteq A,$$

so that $A_\Gamma = A_{\Gamma'}$, a contradiction.

(2)⇒(4): Let $\Delta_0 = \{P_1, \ldots, P_m\}$ and let a be a non-zero element of $P_1 \cap \cdots \cap P_m$. Then $\Delta_0 \subset \Gamma_a$, and hence, by Lemma 2.9, we have $A_{\Gamma_a} = A_\Gamma$ for every $\Gamma \in \Sigma$ such that $\Gamma_a \subseteq \Gamma$. It thus follows from Theorem 2.7 that

$$A = A_{\Gamma_a} = \bigoplus_{n \geq 0} (R \cap d^n R[1/a]) \left(\frac{x - c}{d}\right)^n.$$

Since A is flat over R, we have $A \cong R[IX]$ by Corollary 2.5. □

3. Applications

We now give a few applications of our results.

Theorem 3.1. Let R be a Krull domain with field of fractions K and A a faithfully flat R-algebra such that A is an R-subalgebra of a finitely generated R-algebra B and such that A satisfies the fibre conditions:

(i) $K \otimes_R A = K^{[1]}$.
(ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.

Then $A \cong R[IX]$ for an invertible ideal I of R.

Proof. Let $T = A \backslash \{0\}$ and let $T^{-1}Q$ be a maximal ideal in $T^{-1}B$, where Q is a prime ideal in B. Then $Q \cap A = 0$ and $T^{-1}B/T^{-1}Q$ is algebraic over $T^{-1}A$. Thus, replacing B by B/Q, we may assume that B is an integral domain algebraic over A. Since B is finitely generated over R, there exist elements $f, g_1, \ldots, g_m$ in A such that $B[1/f]$ is integral over $R[g_1, \ldots, g_m, 1/f]$. Let d be a non-zero element in R such that $df \in R[x]$ and $dg_i \in R[x]$ for $1 \leq i \leq m$; such d exists because $A \subset K[x]$. Then we have

$$R[1/d][g_1, \ldots, g_m, 1/f] \subset R[1/d][x, 1/f] \subset A[1/d, 1/f] \subset B[1/d, 1/f],$$

and hence $A[1/d, 1/f]$ is integral over $R[1/d][x, 1/f]$. Note that $R[1/d][x, 1/f]$ is a Krull domain because so is R. Thus $R[1/d][x, 1/f]$ is integrally closed. Note also that both $R[1/d][x, 1/f]$ and $A[1/d, 1/f]$ have the same quotient field $K(x)$. Therefore we have

$$R[1/d][x, 1/f] = A[1/d, 1/f].$$

Let ξ be the coefficient of the highest degree term of f as a polynomial in $K[x]$, and let $b = d\xi$. We will show that $e_P = 0$ for $P \in \Delta$ with $db \notin P$. Indeed, suppose on the contrary that $e := e_P > 0$ for some P with $db \notin P$. Since $d \notin P$, we have $R[1/d] \subset R_P$, so that

$$A_P[1/f] = R_P[x, 1/f].$$

Hence, writing $A_P = R_P[(x-c)/p^e]$ with $c \in R$, we have $f^n(x-c)/p^e \in R_P[x]$ for a sufficiently large integer n. From this it follows that $b^n \in p^e R_P$, and hence $b \in PR_P \cap R = P$, a contradiction. Therefore $e_P = 0$ for P with $db \notin P$, which implies $\Delta_0 \subset \Gamma_{db}$. Thus Δ_0 is a finite set, and hence, by Theorem 2.10, $A \cong R[IX]$ for an invertible ideal I of R. □

Lemma 3.2. Let R be an integral domain and A an R-domain having a retract $\varphi\colon A \to R$. Set $J = \ker \varphi$. Then the following assertions hold.
(1) If A is flat over R, then A is faithfully flat over R.
(2) If f is a non-zero element of J, then f is transcendental over R. In particular R is algebraically closed in A.
(3) Suppose that $\operatorname{tr.deg}_R A > 0$ and J is finitely generated. Let P be a prime ideal in R such that PA remains prime in A. Then $\operatorname{tr.deg}_{R/P} A/PA > 0$.

Theorem 3.3. Let R be a Krull domain and A a flat R-algebra with a retract $\varphi\colon A \to R$. Suppose that A satisfies the following conditions:

(i) $K \otimes_R A = K^{[1]}$.
(ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain.

If $J := \ker \varphi$ is a finitely generated ideal of A, then $A \cong R[IX]$ for an invertible ideal I in R.

Proof. Let P be a prime ideal in R of height one. Then $A/PA \subset A_P/PA_P$ because of flatness of A over R, which implies that PA is a prime ideal in A. Thus, by Theorem 2.10 and Lemma 3.2, it suffices to show that Δ_0 is a finite set. Let $g_1, \ldots, g_m$ be generators of J and let d be a non-zero element of R satisfying $dg_i \in R[x]$ for each $i = 0, \ldots, m$. Let P be an element in $\Delta \setminus \Gamma_d$. We will show that $e := e_P = 0$; if this is the case, then $\Delta_0 \subset \Gamma_d$, and hence Δ_0 is a finite set. Suppose on the contrary that $e > 0$, and write $A_P = R_P[(x-c)/p^e]$ with $c \in R$. For simplicity we set $z = (x-c)/p^e$. Let $\varphi_P\colon A_P \to R_P$ be the retract induced by φ, and let $\varphi_P(z) = c_1$. Since $A_P = R_P[z]$, it then follows that $\ker \varphi_P = (z - c_1)R_P[z]$. Replacing z by $z - c_1$, we may assume that $\ker \varphi_P = zR_P[z]$. Furthermore replacing x by $x - c$, we may assume that $z = x/p^e$. Note that $\ker \varphi_P = J_P$. Note also that $g_i \in R_P[x]$ for each i, because $dg_i \in R[x]$ and $d \notin P$. Hence, for each i, we have $g_i \in R_P[x] \cap zR_P[z] = xR_P[x]$, so that $g_i = xh_i(x)$ where $h_i(x) \in R_P[x]$. Now, since $z \in J_P$ and $J_P = (g_1, \ldots, g_m)R_P[z]$, we can write

$$\frac{x}{p^e} = xh_1(x)u_1(z) + \cdots + xh_m(x)u_m(z)$$

for some $u_1(z), \ldots, u_m(z) \in R_P[z]$. Dividing both sides of the above equation by x, and substituting $x = 0$, we have $1/p^e = h_1(0)u_1(0) + \cdots + h_m(0)u_m(0) \in R_P$. This is a contradiction, as desired. □

Remark 3.4. The condition "J is finitely generated" is necessary. Consider $R = \mathbb{Z}$ and $A = \mathbb{Z}[\frac{X}{p} \mid p \ \ prime \ ]$.

Theorem 3.5. Let R be a locally factorial Krull domain and A a flat codimension-one $\mathbf{A}^1$-fibration over R. Then at each prime ideal $Q \in \operatorname{Spec} R$, either $k(Q) \otimes_R A = k(Q)^{[1]}$ or $k(Q) \otimes_R A = k(Q)$. Suppose in addition that R is a local ring with maximal ideal m and residue field $k (= R/m)$. Then the following conditions are equivalent:

(1) A is finitely generated over R.
(2) $\operatorname{tr.deg}_k A/mA > 0$.
(3) $\dim A/mA > 0$.
(4) $A = R^{[1]}$.

4. Examples

We give below some examples to illustrate the hypotheses in Theorem 3.1.

Example 4.1. The hypothesis on flatness is needed even when A is a finitely generated subalgebra of $R^{[1]}$. For instance, consider $R = k[[t_1, t_2]]$ and

$$A = R[t_1X, t_2X] \cong R[U, V]/(t_2U - t_1V).$$

Example 4.2. The hypothesis on faithful flatness is also necessary. Consider $R = k[[t_1, t_2]]$ and $A = R[U, V]/(t_1U + t_2V - 1)$.

Example 4.3. The condition "$k(P)$ is algebraically closed in $k(P) \otimes_R A$" is necessary. Let $R = \mathbb{R}[[t]]$ and

$$A = R[U, V]/(tU + V^2 + 1).$$

Then A is a finitely generated flat R-algebra, $K \otimes_R A = R^{[1]}$ and $A/tA = \mathbb{C}^{[1]}$. But $A \neq R^{[1]}$.

Example 4.4. Let k be an infinite field and $R = k[[t_1, t_2]]$. Let

$$A = R[\{X/q \mid q \text{ a square-free non-unit in } R\}].$$

Then

$$A_P = R_P[X/p] = R_P{}^{[1]}$$

for every $P \in \operatorname{Spec} R$ with $\operatorname{ht} P = 1$, where $P = pR$. However $A/(t_1, t_2)A = k$. Note that since

$$A = \bigcup_q R[X/q],$$

A is flat over R and hence faithfully flat over R. A is not finitely generated.

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Invariants of the unipotent radical of a Borel subgroup

Mitsuhiro MIYAZAKI
Dept. Math. Kyoto University of Education
Fushimi-ku, Kyoto, 612-8522, Japan
e-mail:g53448@kyokyo-u.ac.jp

1 Introduction

Grassmannians and their Schubert subvarieties are fascinating objects and attract many mathematicians. The homogeneous coordinate ring of the Grassmann variety consisting of m-dimensional subspaces in an n-dimensional vector space over K is the subring of the polynomial ring over K generated by maximal minors of the $m \times n$ matrix of indeterminates. And the homogeneous coordinate ring of a Schubert subvariety is generated by the universal $m \times n$ matrix with the following property for some integers $b_1, b_2, \ldots, b_m$ with $1 \leq b_1 < b_2 < \cdots < b_m \leq n$.

(†) All the i-minors of first $b_i - 1$ columns are zero.

If a matrix M satisfies the property (†), then Mg also satisfies (†) for any upper triangular matrix g, so the Borel subgroup consisting of the upper triangular matrices of the general linear group and its subgroups act on the homogeneous coordinate ring of a Schubert subvariety of a Grassmannian and the algebra generated by the entries of the universal matrix with (†). We study the ring of invariants of the unipotent radical of this Borel subgroup in §3.

It is also known that there is an $m \times n$ universal matrix with conditions on minors related both to rows and columns. The direct product of Borel subgroups, consisting of lower triangular matrices and upper triangular matrices respectively, of the direct product of general linear groups, and its subgroups act on the algebra generated by the entries of the matrix with universal property. We also study the ring of invariants of the unipotent radical of this Borel subgroup.

2 Preliminaries

All rings and algebras in this note are commutative with identity element.

Let K be an infinite field of arbitrary characteristic. For an $s \times t$ matrix $M = (m_{ij})$ with entries in a K-algebra S, we denote by $K[M]$ the K-subalgebra of S generated by the entries of M, by $I_r(M)$ the ideal of S generated by all r-minors of M, by $M_{\leq j}$ the $s \times j$ matrix consisting of the first j columns of M, by $M^{\leq i}$ the $i \times t$ matrix consisting of the first i rows of M and by $\Gamma(M)$ the set of all maximal minors of M.

Let l be a positive integer. We set

$$H(l) := \{[a_1, a_2, \ldots, a_r] \mid 1 \leq a_1 < a_2 < \cdots < a_r \leq l, a_i \in \boldsymbol{Z}\}.$$

For $\alpha = [a_1, a_2, \ldots, a_r] \in H(l)$, we set $\mathrm{size}\alpha = r$. We define the order on $H(l)$ by

$$[a_1, \ldots, a_r] \leq [b_1, \ldots, b_s] \overset{\text{def}}{\Longleftrightarrow} r \geq s, a_i \leq b_i \text{ for } i = 1, 2, \ldots, s.$$

It is easy to verify that $H(l)$ is a distributive lattice.

For positive integers m and n, we set

$$\Delta(m \times n) := \{[\alpha|\beta] \mid \alpha \in H(m), \beta \in H(n), \mathrm{size}\alpha = \mathrm{size}\beta\}$$

and define the order on $\Delta(m \times n)$ by

$$[\alpha|\beta] \leq [\alpha'|\beta'] \overset{\text{def}}{\Longleftrightarrow} \alpha \leq \alpha' \text{ in } H(m) \text{ and } \beta \leq \beta' \text{ in } H(n).$$

For $\delta = [a_1, \ldots, a_r|b_1, \ldots, b_r] \in \Delta(m \times n)$ and an $m \times n$ matrix $M = (m_{ij})$, we set $\delta_M := \det(m_{a_i,b_j})_{i,j}$. We also set $\Delta(m \times n; \delta) := \{\gamma \in \Delta(m \times n) \mid \gamma \geq \delta\}$.

Now we fix integers m and n with $1 \leq m \leq n$. Let X be an $m \times n$ matrix of indeterminates, that is, $X = (X_{ij})$ and $\{X_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n}$ are independent indeterminates. Then

Fact 2.1 ([DEP1]) *$K[X]$ is an algebra with straightening law (ASL for short) over K generated by $\Delta(m \times n)$ with structure map $\delta \mapsto \delta_X$.*

Next we fix $\delta = [a_1, a_2, \ldots, a_r|b_1, b_2, \ldots, b_r] \in \Delta(m \times n)$. Since $\Delta(m \times n) \setminus \Delta(m \times n; \delta)$ is a poset ideal of $\Delta(m \times n)$, we see by [DEP2, Proposition 1.2],

Corollary 2.2

$$R(X;\delta) := K[X]/(\Delta(m \times n) \setminus \Delta(m \times n;\delta))K[X]$$

is an ASL over K generated by $\Delta(m \times n;\delta)$.

The image $\overline{X}$ of X in $R(X;\delta)$ is the universal matrix which satisfies the condition

$$I_i(\overline{X}^{\le a_i - 1}) = I_i(\overline{X}_{\le b_i - 1}) = (0) \quad \text{for } i = 1, 2, \ldots, r+1,$$

where we set $a_{r+1} = m+1$ and $b_{r+1} = n+1$. That is, if M is an $m \times n$ matrix with entries in a K-algebra S and

$$(*) \qquad I_i(M^{\le a_i - 1}) = I_i(M_{\le b_i - 1}) = (0) \quad \text{for } i = 1, 2, \ldots, r+1,$$

then there is a unique K-algebra homomorphism $R(X;\delta) \to S$ mapping $\overline{X}$ to M.

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Now let $G = \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$, B^- the Borel subgroup of $\mathrm{GL}(m,K)$ consisting of lower triangular matrices, B^+ the Borel subgroup of $\mathrm{GL}(n,K)$ consisting of upper triangular matrices and U^- (resp. U^+) the set of all unipotent matrices in B^- (resp. B^+). If $g_1 \in U^-$ and $g_2 \in U^+$, then $g_1^{-1}\overline{X}g_2$ satisfies $(*)$. So there is an automorphism of $R(X;\delta)$ sending $\overline{X}$ to $g_1^{-1}\overline{X}g_2$. Therefore, $U^- \times U^+$ acts on $R(X;\delta)$. We may also consider the action of U^+ on $R(X;\delta)$.

We set

$$Y_\delta := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ Y_{a_1 1} & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ Y_{a_2 1} & Y_{a_2 2} & & 0 \\ \vdots & \vdots & & \vdots \\ Y_{a_r 1} & Y_{a_r 2} & \cdots & Y_{a_r r} \\ \vdots & \vdots & & \vdots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mr} \end{bmatrix}$$

and

$$Z_\delta := \begin{bmatrix} 0 & \cdots & 0 & Z_{1b_1} & \cdots & Z_{1b_2} & \cdots & Z_{1b_r} & \cdots & Z_{1n} \\ 0 & \cdots & 0 & 0 & \cdots & Z_{2b_2} & \cdots & Z_{2b_r} & \cdots & Z_{2n} \\ & \cdots & & & \cdots & & \cdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & Z_{rb_r} & \cdots & Z_{rn} \end{bmatrix},$$

where Y_{ij} and Z_{ij} are independent indeterminates.

Lemma 3.1

$$\begin{array}{l} I_i((Y_\delta Z_\delta)^{\leq a_i-1}) = (0) \\ I_i((Y_\delta Z_\delta)_{\leq b_i-1}) = (0) \end{array} \quad \textit{for } i = 1, \ldots, r, \ r+1.$$

Therefore there is a unique K-algebra homomorphism $R(X;\delta) \to K[Y_\delta, Z_\delta]$ mapping $\overline{X}$ to $Y_\delta Z_\delta$.

We introduce the lexicographic monomial order on $K[Y_\delta, Z_\delta]$ induced by $Y_{a_1 1} > Y_{a_1+1,1} > \cdots > Y_{m1} > Y_{a_2 2} > \cdots > Y_{m2} > Y_{a_3 3} > \cdots > Y_{mr} > Z_{1b_1} > Z_{1b_1+1} > \cdots > Z_{1n} > Z_{2b_2} > \cdots > Z_{2n} > Z_{3b_3} > \cdots > Z_{rn}$.

Lemma 3.2 *If $\gamma = [c_1, \ldots, c_s | d_1, \ldots, d_s]$ is an element of $\Delta(m \times n; \delta)$, then*

$$\mathrm{lm}(\gamma_{Y_\delta Z_\delta}) = Y_{c_1 1} Y_{c_2 2} \cdots Y_{c_s s} Z_{1d_1} Z_{2d_2} \cdots Z_{sd_s}.$$

proof Since

$$\gamma_{Y_\delta Z_\delta} = \sum_{[e_1,\ldots,e_s] \in H(r)} [c_1, \ldots, c_s | e_1, \ldots, e_s]_{Y_\delta} [e_1, \ldots, e_s | d_1, \ldots, d_s]_{Z_\delta}$$

and

$$\begin{aligned} & \mathrm{lm}([c_1, \ldots, c_s | e_1, \ldots, e_s]_{Y_\delta} [e_1, \ldots, e_s | d_1, \ldots, d_s]_{Z_\delta}) \\ = \ & Y_{c_1 e_1} \cdots Y_{c_s e_s} Z_{e_1 d_1} \cdots Z_{e_s d_s}, \end{aligned}$$

the result follows form the definition of monomial order. ∎

If $\mu = \prod_{i=1}^{u} [c_{i1}, \ldots, c_{is(i)} | d_{i1}, \ldots, d_{is(i)}]$ is a standard monomial on $\Delta(m \times n; \delta)$ in the sense of ASL, then

$$\mathrm{lm}(\mu_{Y_\delta Z_\delta}) = \prod_{i=1}^{u} \prod_{j=1}^{s(i)} Y_{c_{ij} j} Z_{j d_{ij}}. \tag{3.1}$$

In particular, we can reconstruct μ form $\mathrm{lm}(\mu_{Y_\delta Z_\delta})$. So

Lemma 3.3 *If μ and μ' are different standard monomials on $\Delta(m \times n; \delta)$, then $\mathrm{lm}(\mu_{Y_\delta Z_\delta}) \neq \mathrm{lm}(\mu'_{Y_\delta Z_\delta})$. In particular, $\{\mu_{Y_\delta Z_\delta} \mid \mu$ is a standard monomial on $\Delta(m \times n; \delta)\}$ is linearly independent over K.*

Therefore

Proposition 3.4 *The K-algebra homomorphism in Lemma 3.1 is injective. In particular, $R(X; \delta) \simeq K[Y_\delta Z_\delta]$.*

For $g \in U^+$, we can define a K-algebra automorphism of $K[Z_\delta]$ which maps Z_δ to $Z_\delta g$. Therefore U^+ acts on $K[Z_\delta]$. As for this action we have

Lemma 3.5 $K[Z_\delta]^{U^+} = K[Z_{1b_1}, Z_{2b_2}, \dots, Z_{rb_r}]$.

proof First we define the row degree on $K[Z_\delta]$ by $\deg Z_{ij} := \boldsymbol{e}_i \in \boldsymbol{N}^r$.

Since the action of U^+ fixes row degree, we may assume, by extending Z_δ, that $[b_1, b_2, \dots, b_r] = [1, 2, \dots, n]$, that is, Z_δ is the $n \times n$ upper triangular matrix of indeterminates.

Let f be an arbitrary element of $K[Z_\delta]^{U^+}$. Since the action of U^+ fixes the row degree, in order to prove that $f \in K[Z_{11}, Z_{22}, \dots, Z_{nn}]$, we may assume that f is homogeneous of row degree $(d_1, d_2, \dots, d_n)$. Write f as

$$\sum_{i_1=0}^{d_1} \cdots \sum_{i_n=0}^{d_n} f_{i_1 i_2 \cdots i_n}(Z_{12}, \dots, Z_{1n}, Z_{23}, \dots, Z_{n-1,n}) Z_{11}^{d_1 - i_1} Z_{22}^{d_2 - i_2} \cdots Z_{nn}^{d_n - i_n}$$

where $f_{i_1 i_2 \cdots i_n}$ is a homogeneous polynomial of Z_{12}, Z_{13}, $\dots$, Z_{1n}, Z_{23}, Z_{24}, $\dots$, Z_{2n}, Z_{34}, $\dots$, $Z_{n-1,n}$ of row degree $(i_1, i_2, \dots, i_n)$.

Let $g = (g_{ij})$ be an element of U^+. Since the image of Z_{ij} by the action of g is

$$\sum_{l=i}^{j} Z_{il} g_{lj} \tag{3.2}$$

for $i \leq j$, we see that the image of f is of the following form.

$$\sum_{i_1=0}^{d_1} \sum_{i_2=0}^{d_2} \cdots \sum_{i_n=0}^{d_n} f_{i_1 i_2 \cdots i_n}(g_{12}, \dots, g_{1n}, g_{23}, \dots, g_{n-1,n}) Z_{11}^{d_1} Z_{22}^{d_2} \cdots Z_{nn}^{d_n}$$
$$+ \quad (\text{terms of lower degree in } Z_{11}, Z_{22}, \dots, Z_{nn})$$

Since $g(f) = f$ for any $g \in U^+$ and K is an infinite field, we see that

$$f_{i_1, i_2, \dots, i_n} = 0 \quad \text{if } (i_1, i_2, \dots, i_n) \neq (d_1, d_2, \dots, d_n),$$

that is, $f \in K[Z_{11}, Z_{22}, \ldots, Z_{nn}]$.

On the contrary, it is clear form (3.2) that $Z_{ii} \in K[Z_\delta]^{U^+}$ for $i = 1$, 2, ..., n. Therefore $K[Z_\delta]^{U^+} = K[Z_{11}, Z_{22}, \ldots, Z_{nn}]$. ∎

By symmetry, we see that U^- acts on $K[Y_\delta]$ and $K[Y_\delta]^{U^-} = K[Y_{a_1 1}, Y_{a_2 2}, \ldots, Y_{a_r r}]$.

Proposition 3.6 *$\{[c_1, \ldots, c_i|b_1, \ldots, b_i]_{Y_\delta Z_\delta} \mid [c_1, \ldots, c_i] \in H(m; [a_1, \ldots, a_r])\}$ is a sagbi basis of*

$$K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta].$$

In particular,

$$K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta] = K[\bigcup_{i=1}^{r} \Gamma((Y_\delta Z_\delta)_{b_1, b_2, \ldots, b_i})],$$

where $M_{b_1, b_2, \ldots, b_i}$ denotes the matrix consisting of b_1, b_2, $\ldots b_{i-1}$ and b_i-th columns of M.

proof It is clear that

$$\begin{aligned} & [c_1, \ldots, c_i|b_1, \ldots, b_i]_{Y_\delta Z_\delta} \\ = \; & [c_1, \ldots, c_i|1, 2, \ldots, i]_{Y_\delta}[1, 2, \ldots, i|b_1, \ldots, b_i]_{Z_\delta} \\ \in \; & K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta]. \end{aligned}$$

Now suppose that $f \in K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta]$ and let

$$f = \sum_\mu r_\mu \mu$$

be the standard representation of f in the ASL $K[Y_\delta Z_\delta] \simeq R(X; \delta)$. Then by Lemma 3.3, we see that there is a unique standard monomial μ such that

$$\mathrm{lm}(f) = \mathrm{lm}(\mu_{Y_\delta Z_\delta}).$$

Since $\mathrm{lm}(\mu_{Y_\delta Z_\delta}) = \mathrm{lm}(f) \in K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}]$, we see, by (3.1), that μ is of the form $\prod_{i=1}^{u}[c_{i1}, \ldots, c_{is(i)}|b_{i1}, \ldots, b_{is(i)}]$. The result follows. ∎

The action of U^+ on $K[Z_\delta]$ induces an action of U^+ on $K[Y_\delta Z_\delta]$. Since

$$K[Y_\delta Z_\delta]^{U^+} = K[Z_\delta]^{U^+}[Y_\delta] \cap K[Y_\delta Z_\delta],$$

we see the following

Theorem 3.7

$$K[Y_\delta Z_\delta]^{U^+} = K[\bigcup_{i=1}^{r} \Gamma((Y_\delta Z_\delta)_{b_1,b_2,\ldots,b_i})].$$

And therefore,

$$R(X;\delta)^{U^+} = K[\bigcup_{i=1}^{r} \Gamma(\overline{X}_{b_1,b_2,\ldots,b_i})].$$

Note 3.8 *If* $[a_1,\ a_2,\ \ldots,\ a_r] = [1,\ 2,\ \ldots,\ m]$, *then* $K[\Gamma(Y_\delta Z_\delta)]$ *is the homogeneous coordinate ring of the Schubert subvariety.*

4 Invariants of the unipotent radical of a Borel subgroup of $\mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$

First we state the following

Proposition 4.1

$$\begin{aligned} & K[Y_\delta Z_\delta] \cap K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Z_\delta, Y_{a_1 1}, Y_{a_2 2}, \ldots, Y_{a_r r}] \\ = \ & K[Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} \mid i = 1, \ldots, r]. \end{aligned}$$

proof It is clear that $Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} = [a_1,\ \ldots,\ a_i | b_1,\ \ldots,\ b_i]_{Y_\delta Z_\delta} \in K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$ for $i = 1,\ 2,\ \ldots,\ r$.

Suppose that $f \in K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$ and let

$$f = \sum_\mu r_\mu \mu$$

be the standard representation of f in the ASL $K[Y_\delta Z_\delta] \simeq R(X;\delta)$. Then there is unique standard monomial μ such that $\mathrm{lm}(f) = \mathrm{lm}(\mu_{Y_\delta Z_\delta})$.

Since $\mathrm{lm}(\mu_{Y_\delta Z_\delta}) = \mathrm{lm}(f) \in K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$, we wee by (3.1) that μ is of the following form.

$$\mu = \prod_{t=1}^{u} [a_1, a_2, \ldots, a_{i(t)} | b_1, b_2, \ldots, b_{i(t)}]$$

So we see that

$$\begin{aligned} & \{Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} \mid i = 1, \ldots, r\} \\ = \ & \{[a_1, \ldots, a_i | b_1, \ldots, b_i]_{Y_\delta Z_\delta} \mid i = 1, \ldots, r\} \end{aligned}$$

is a sagbi basis of $K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},$ $Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$. The result follows. ∎

Since

$$\begin{aligned} & K[Y_\delta Z_\delta]^{U^- \times U^+} \\ = \ & K[Y_\delta Z_\delta]^{U^-} \cap K[Y_\delta Z_\delta]^{U^+} \\ = \ & K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta] \\ & \cap K[Z_\delta, Y_{a_1 1}, Y_{a_2 2}, \ldots, Y_{a_r r}] \cap K[Y_\delta Z_\delta], \end{aligned}$$

We see the following

Theorem 4.2

$$\begin{aligned} & K[Y_\delta Z_\delta]^{U^- \times U^+} \\ = \ & K[Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} \mid i = 1, \ldots, r] \\ = \ & K[[a_1, \ldots, a_i | b_1, \ldots, b_i]_{Y_\delta Z_\delta} \mid i = 1, \ldots, r]. \end{aligned}$$

And therefore,

$$R(X;\delta)^{U^- \times U^+} = K[[a_1, a_2, \ldots, a_i | b_1, b_2, \ldots, b_i]_{\overline{X}} \mid i = 1, 2, \ldots, r].$$

In particular, it is isomorphic to the polynomial ring over K with r variables.

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An upper bound on the reduction number of an ideal

Koji Nishida (Chiba University)

1. Introduction

This is a joint work with Y. Kinoshita, Kensuke Sakata and Ryuta Shinya.

Let Q, I and J be ideals of a commutative ring A such that $Q \subseteq I \subseteq J$. As is noted in [1, 2.6], if J/I is cyclic as an A-module and $J^2 = QJ$, then we have $I^3 = QI^2$. The purpose of this report is to generalize this fact. We will show that if J/I is generated by v elements as an A-module and $J^2 = QJ$, then $I^{v+2} = QI^{v+1}$. We get this result as a corollary of the following theorem, which generalizes Rossi's assertion stated in the proof of [7, 1.3].

Theorem 1.1. *Let A be a commutative ring and $\{F_n\}_{n\geq 0}$ a family of ideals in A such that $F_0 = A$, $IF_n \subseteq F_{n+1}$ for any $n \geq 0$, and $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$ for some $k \geq 0$ and an ideal $\mathfrak{a}$ in A. Suppose that $F_n/(QF_{n-1} + I^n)$ is generated by v_n elements for any $n \geq 0$ and $v_n = 0$ for $n \gg 0$. We put $v = \sum_{n\geq 0} v_n$. Then we have*

$$I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}.$$

If a family $\{F_n\}_{n\geq 0}$ of ideals in A satisfies all of the conditions required in 1.1 in the case where $\mathfrak{a} = (0)$, we have $F_n = QF_{n-1}$ for $n \gg 0$. As a typical example of such $\{F_n\}_{n\geq 0}$, we find $\{\widetilde{I^n}\}_{n\geq 0}$ when I contains a non-zerodivisor, where $\widetilde{I^n}$ denotes the Ratliff-Rush closure of I^n (cf. [9]). If A is an analytically unramified local ring, then $\{\overline{I^n}\}_{n\geq 0}$ is also an important example, where $\overline{I^n}$ denotes the integral closure of I^n. It is obvious that $\{J^n\}_{n\geq 0}$ always satisfies the required condition on $\{F_n\}_{n\geq 0}$ for any ideal J with $I \subseteq J \subseteq \overline{I}$.

We prove 1.1 following Rossi's argument in the proof of [7, 1.3]. However we do not assume that A/I has finite length. And furthermore we can deduce the following corollary which gives an upper bound on the reduction number $\mathrm{r}_Q(I)$ of I with respect to Q using numbers of gerators of certain A-modules.

Corollary 1.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\{F_n\}_{n\geq 0}$ a family of ideals in A such that $F_0 = A$, $IF_n \subseteq F_{n+1}$ for any $n \geq 0$, and $I^{k+1} \subseteq QF_k + \mathfrak{m}F_{k+1}$ for some $k \geq 0$. Then we have*

$$\begin{aligned} \mathrm{r}_Q(I) &\leq k + \sum_{n\geq 1} \mu_A(F_n/(QF_{n-1} + I^n)) \\ &\leq 1 + \mu_A(F_1/I) + \sum_{n\geq 2} \mu_A(F_n/QF_{n-1}). \end{aligned}$$

2. Proof of Theorem 1.1

In order to prove 1.1 we need the following lemma, which generalizes [4, 2.3].

Lemma 2.1. *Let $I_1, I_2, \ldots, I_N$ be finite number of ideals of A. For any $1 \leq n \leq N$, we assume that I_n is generated by v_n elements and*

$$I \cdot I_n \subseteq I^{n+1} + \sum_{\ell=1}^{N} Q^{n+1-\ell} I_\ell .$$

Let $v := v_1 + v_2 + \cdots + v_N > 0$. Then, for any v elements $a_1, a_2, \ldots, a_v$ in I, there exists $\sigma \in QI^{v-1}$ such that

$$a_1 a_2 \cdots a_v - \sigma \in \bigcap_{n=1}^{N} [I^{n+v} : I_n] .$$

Proof of Theorem 1.1. If $v = 0$, then we have $F_n = I^n$ for any $n \geq 0$, and so $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1} = QI^k + \mathfrak{a}I^{k+1} \subseteq I^{k+1}$, which means $I^{k+1} = QI^k + \mathfrak{a}I^{k+1}$. Hence we may assume $v > 0$. For any $n \geq 0$, let us take an ideal I_n generated by v_n elements so that $F_n = QF_{n-1} + I^n + I_n$. We can easily show that

$$(\#) \qquad F_n = I^n + \sum_{\ell=0}^{n} Q^{n-\ell} I_\ell$$

for any $n \geq 0$ by induction on n. Now we choose an integer N so that $N > k$ and $I_n = 0$ for any $n > N$. Then by (#) it follows that

$$I \cdot I_n \subseteq F_{n+1} = I^{n+1} + \sum_{\ell=0}^{N} Q^{n+1-\ell} I_\ell$$

for any $0 \leq n \leq N$. Let $a_1, a_2, \ldots, a_v$ be any elements of I. Then, by 2.1 there exists $\sigma \in QI^{v-1}$ such that

$$a_1 a_2 \cdots a_v - \sigma \in \bigcap_{n=0}^{N} [I^{n+v} : I_n] .$$

We put $\xi = a_1 a_2 \cdots a_v - \sigma$. Then by (#) we get

$$\xi F_n = \xi I^n + \sum_{\ell=0}^{n} Q^{n-\ell} \cdot \xi I_\ell \subseteq I^v \cdot I^n + \sum_{\ell=0}^{n} Q^{n-\ell} \cdot I^{\ell+v} \subseteq I^{v+n}$$

for any $0 \leq n \leq N$. Now the assumption that $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$ implies

$$\xi I^{k+1} \subseteq Q \cdot \xi F_k + \mathfrak{a} \cdot \xi F_{k+1} \subseteq Q \cdot I^{v+k} + \mathfrak{a} \cdot I^{v+k+1} .$$

Therefore we get

$$a_1 a_2 \cdots a_v \cdot I^{k+1} = (\xi + \sigma) I^{k+1} \subseteq QI^{v+k} + \mathfrak{a}I^{v+k+1} .$$

Then, as the elements $a_1, a_2, \ldots, a_v$ are chosen arbitrarily from I, it follows that $I^v \cdot I^{k+1} \subseteq QI^{v+k} + \mathfrak{a}I^{v+k+1} \subseteq I^{v+k+1}$. Thus we get $I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}$.

Proof of Corollary 1.2. We put $v = \sum_{n\geq 1}\mu_A(F_n/(QF_{n-1}+I^n))$. We may assume $v<\infty$. Then, setting $\mathfrak{a}=\mathfrak{m}$ in 1.1, it follows that $I^{v+k+1} = QI^{v+k}+\mathfrak{m}I^{v+k+1}$. Hence we get $I^{v+k+1}=QI^{v+k}$ by Nakayama's lemma, and so $\mathrm{r}_Q(I)\leq v+k$. In order to prove the second inequality, we choose k as small as possible. If $k\leq 1$, we have

$$\mathrm{r}_Q(I)\leq k+v\leq 1+\mu_A(F_1/I)+\sum_{n\geq 2}\mu_A(F_n/QF_{n-1}).$$

So, we assume $k\geq 2$ in the rest of this proof. In this case we have

$$(\natural)\qquad \mathrm{r}_Q(I)\leq k+\mu_A(F_1/I)+\sum_{n=2}^{k}\mu_A(F_n/(QF_{n-1}+I^n))+\sum_{n\geq k+1}\mu_A(F_n/QF_{n-1}).$$

If $2\leq n\leq k$, then $I^n\not\subseteq QF_{n-1}+\mathfrak{m}F_n$, and so the canonical surjection

$$F_n/(QF_{n-1}+\mathfrak{m}F_n)\longrightarrow F_n/(QF_{n-1}+I^n+\mathfrak{m}F_n)$$

is not injective, which means

$$\mu_A(F_n/QF_{n-1}+I^n)\leq \mu_A(F_n/QF_{n-1})-1.$$

Thus we get

$$\sum_{n=2}^{k}\mu_A(F_n/QF_{n-1}+I^n)\leq\{\sum_{n=2}^{k}\mu_A(F_n/QF_{n-1})\}-(k-1).$$

Therefore the required inequality follows from $(\natural)$.

3. Corollaries

In this section we collect some results deduced from 1.1 and 1.2.

Corollary 3.1. *Let J be an ideal of A such that $J\supseteq I$ and $J^2=QJ$. If J/I is finitely generated as an A-module, then $\mathrm{r}_Q(I)\leq\mu_A(J/I)+1$.*

Proof. We apply 1.1 setting $F_n=J^n$ for any $n\geq 0$ and $\mathfrak{a}=(0)$. Because $I^2\subseteq J^2=QJ$, we may put $k=1$, and hence we get $I^{v+2}=QI^{v+1}$, where $v=\mu_A(J/I)$. Then $\mathrm{r}_Q(I)\leq v+1$.

Corollary 3.2. *Let $(A,\mathfrak{m})$ be a two-dimensional regular local ring (or, more generally, a two-dimensional pseudo-rational local ring) such that $A/\mathfrak{m}$ is infinite. If I is an $\mathfrak{m}$-primary ideal with a minimal reduction Q, then $\mathrm{r}_Q(I)\leq\mu_A(\overline{I}/I)+1$.*

Proof. This follows from 3.1 since $(\overline{I})^2=Q\overline{I}$ by [5, 5.1] (or [6, 5.4]).

Corollary 3.3. *Let $\mathfrak{p}$ be a prime ideal of A with $\mathrm{ht}\,\mathfrak{p}=g\geq 2$. Let $Q=(a_1,a_2,\ldots,a_g)$ be an ideal generated by a regular sequence contained in the k-th symbolic power $\mathfrak{p}^{(k)}$ of $\mathfrak{p}$ for some $k\geq 2$. Then we have $\mathrm{r}_Q(I)\leq\mu_A((Q:\mathfrak{p}^{(k)})/Q)+1$ for any ideal I with $Q\subseteq I\subseteq Q:\mathfrak{p}^{(k)}$, if one of the following three conditions holds ; (i) $A_\mathfrak{p}$ is not a regular local ring, (ii) $A_\mathfrak{p}$ is a regular local ring and $g\geq 3$, (iii) $A_\mathfrak{p}$ is a regular local ring, $g=2$, and $a_i\in\mathfrak{p}^{(k+1)}$ for any $1\leq i\leq g$.*

Proof. This follows from 3.1 since $(Q:\mathfrak{p}^{(k)})^2 = Q(Q:\mathfrak{p}^{(k)})$ by [10, 3.1].

Corollary 3.4. *Let $(A,\mathfrak{m})$ be a Buchsbaum local ring. Assume that the multiplicity of A with respect to $\mathfrak{m}$ is 2 and* $\operatorname{depth} A > 0$. *Then, for any parameter ideal Q in A and an ideal I with $Q \subseteq I \subseteq Q:\mathfrak{m}$, we have $\mathrm{r}_Q(I) \le \mu_A((Q:\mathfrak{m})/Q)+1$.*

Proof. This follows from 3.1 since $(Q:\mathfrak{m})^2 = Q(Q:\mathfrak{m})$ by [3, 1.1].

In order to state the last corollary, let us recall the definition of Hilbert coefficients. Let $(A,\mathfrak{m})$ be a d-dimensional Noetherian local ring and I an $\mathfrak{m}$-primary ideal. Then there exists a family $\{\,\mathrm{e}_i(I)\,\}_{0\le i\le d}$ of integers such that

$$\ell_A(A/I^{n+1}) = \sum_{i=0}^{d}(-1)^i\,\mathrm{e}_i(I)\binom{n+d-i}{d-i}$$

for $n \gg 0$. We call $\mathrm{e}_i(I)$ the i-th Hilbert coefficient of I. On the other hand, if A is an analytically unramified local ring, then $\{\,\overline{I^n}\,\}_{n\ge 0}$ is a Hilbert filtration (cf. [2]), and so there exists a family $\{\,\overline{\mathrm{e}}_i(I)\,\}_{0\le i\le d}$ of integers such that

$$\ell_A(A/\overline{I^{n+1}}) = \sum_{i=0}^{d}(-1)^i\,\overline{\mathrm{e}}_i(I)\binom{n+d-i}{d-i}$$

for $n \gg 0$. As is proved in [7, 1.5], if A is a two-dimensional Cohen-Macaulay local ring, then we have

$$\mathrm{r}_Q(I) \le \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) + 1$$

for any minimal reduction Q of I. We can generalize this result as follows.

Corollary 3.5. *Let $(A,\mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field and I an $\mathfrak{m}$-primary ideal with a minimal reduction Q. Then we have the following inequalities.*

(1) $\mathrm{r}_Q(I) \le \mathrm{e}_1(J) - \mathrm{e}_0(J) + \ell_A(A/I) + 1$ *for any ideal J such that $I \subseteq J \subseteq \overline{I}$.*
(2) $\mathrm{r}_Q(I) \le \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_A(A/I) + 1$, *if A is analytically unramified.*

Proof. (1) Setting $F_n = \widetilde{J^n}$ for any $n \ge 0$ in 1.2, we get

$$\begin{aligned}\mathrm{r}_Q(I) &\le 1 + \mu_A(\widetilde{J}/I) + \sum_{n\ge 2}\mu_A(\widetilde{J^n}/Q\widetilde{J^{n-1}})\\ &\le 1 + \ell_A(\widetilde{J}/I) + \sum_{n\ge 2}\ell_A(\widetilde{J^n}/Q\widetilde{J^{n-1}})\\ &= \sum_{n\ge 1}\ell_A(\widetilde{J^n}/Q\widetilde{J^{n-1}}) - \ell_A(I/Q) + 1.\end{aligned}$$

Because $\mathrm{e}_1(J) = \sum_{n\ge 1}\ell_A(\widetilde{J^n}/Q\widetilde{J^{n-1}})$ by [2, 1.10] and

$$\ell_A(I/Q) = \ell_A(A/Q) - \ell_A(A/I) = \mathrm{e}_0(J) - \ell_A(A/I),$$

the required inequality follows.

(2) Similarly as the proof of (1), setting $F_n = \overline{I^n}$ for any $n \geq 0$ in 1.2, we get

$$\mathrm{r}_Q(I) \leq \sum_{n\geq 1} \ell_A(\overline{I^n}/Q\overline{I^{n-1}}) - \ell_A(I/Q) + 1.$$

Because the depth of the associated graded ring of the filtration $\{\overline{I^n}\}_{n\geq 0}$ is positive, we have $\overline{\mathrm{e}}_1(I) = \sum_{n\geq 1} \ell_A(\overline{I^n}/Q\overline{I^{n-1}})$ by [2, 1.9]. Hence we get the required inequality as $\ell_A(I/Q) = \overline{\mathrm{e}}_0(I) - \ell_A(A/I)$.

4. Example

In this section we give an example which shows that the maximum value stated in 3.1 can be reached. It provides an example in the case where $\dim A/I > 0$.

Example 4.1. *Let $n \geq 3$ be an integer and $S = k[X_0, X_1, \dots, X_n]$ be the polynomial ring with $n+1$ variables over a field k. Let $A = S/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal of S generated by the maximal minors of the matrix*

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}.$$

We denote the image of X_i in A by x_i for $0 \leq i \leq n$. It is well known that A is a two-dimensional Cohen-Macaulay graded ring with the graded maximal ideal $\mathfrak{m} = (x_0, x_1, \dots, x_n)$.

1. *Let $I = (x_0, x_1, x_n)$ and $Q = (x_0, x_n)$. Then we have $\mathfrak{m}^2 = Q\mathfrak{m}$, $\mu_A(\mathfrak{m}/I) = n-2$, and $\mathrm{r}_Q(I) = n-1$.*
2. *Let $I = (x_0, x_1, x_{n-1})$, $J = (x_0, x_1, \dots, x_{n-1})$, and $Q = (x_0, x_{n-1})$. Then we have $\dim A/I = 1$, $J^2 = QJ$, $\mu_A(J/I) = n-3$, and $\mathrm{r}_Q(I) = n-2$.*

Proof. (1) Let $0 \leq i \leq j \leq n$. If $i = 0$ or $j = n$, then $x_i x_j \in Q\mathfrak{m}$. On the other hand, if $i > 0$ and $j < n$, then the determinant of the matrix

$$\begin{pmatrix} X_{i-1} & X_j \\ X_i & X_{j+1} \end{pmatrix}$$

is contained in $\mathfrak{a}$, and so $x_i x_j = x_{i-1}x_{j+1}$. Hence we can show that $x_i x_j \in Q\mathfrak{m}$ for any $0 \leq i \leq j \leq n$ by descending induction on $j - i$. Thus we get $\mathfrak{m}^2 = Q\mathfrak{m}$. It is obvious that $\mu_A(\mathfrak{m}/I) = n-2$. Therefore $I^n = QI^{n-1}$ by 3.1 (In fact, we have ${x_1}^n = {x_1}^{n-2} \cdot {x_1}^2 = {x_1}^{n-2} \cdot x_0 x_2 = x_0 {x_1}^{n-3} \cdot x_1 x_2 = x_0 {x_1}^{n-3} \cdot x_0 x_3 = {x_0}^2 {x_1}^{n-4} \cdot x_1 x_3 = \cdots = {x_0}^{n-2} \cdot x_1 x_{n-1} = {x_0}^{n-2} \cdot x_0 x_n = {x_0}^{n-1} x_n \in Q^n \subseteq QI^{n-1}$). In order to prove $\mathrm{r}_Q(I) = n-1$, we show ${x_1}^{n-1} \notin QI^{n-2}$. For that purpose we use the isomorphism

$$\varphi : A \longrightarrow k[\{ s^{n-i}t^i \}_{0\leq i\leq n}]$$

of k-algebras such that $\varphi(x_i) = s^{n-i}t^i$ for $0 \leq i \leq n$, where s and t are indeterminates. We have to show $\varphi(x_1)^{n-1} \notin \varphi(Q)\varphi(I)^{n-2}$. Because $\varphi(I) = (s^n, s^{n-1}t, t^n)$, we get

$$\varphi(I)^\ell \subseteq (\{ s^{\alpha n - \beta} t^{(\ell-\alpha)n+\beta} \mid 0 \leq \alpha \leq \ell, 0 \leq \beta \leq \alpha \})$$

for any $\ell \geq 1$ by induction on ℓ, and so

$\varphi(Q)\varphi(I)^{n-2} \subseteq (\{ s^{(\alpha+1)n-\beta}t^{(n-2-\alpha)n+\beta} , s^{\alpha n-\beta}t^{(n-1-\alpha)n+\beta} \mid 0 \leq \alpha \leq n-2 , 0 \leq \beta \leq \alpha \})$.

Therefore, if $\varphi(x_1)^{n-1} = (s^{n-1}t)^{n-1} = s^{(n-1)^2}t^{n-1} \in \varphi(Q)\varphi(I)^{n-2}$, one of the following two cases

(i) $(\alpha+1)n - \beta \leq (n-1)^2$ and $(n-2-\alpha)n + \beta \leq n-1$, or

(ii) $\alpha n - \beta \leq (n-1)^2$ and $(n-1-\alpha)n + \beta \leq n-1$

must occur for some α and β with $0 \leq \alpha \leq n-2$ and $0 \leq \beta \leq \alpha$. Suppose that the case (i) occured. Then we have

$$(\alpha+1)n - \beta \leq (n-1)n - (n-1) \text{ and } (n-2-\alpha)n \leq n-1-\beta .$$

As the first inequality implies

$$n-1-\beta \leq (n-1)n - (\alpha+1)n = (n-2-\alpha)n ,$$

it follows that

$$n-1-\beta = (n-1)n - (\alpha+1)n ,$$

and so

$$\alpha n - \beta = n^2 - 3n + 1 .$$

Then, as $\alpha n > n^2 - 3n = (n-3)n$, we have $n-3 < \alpha \leq n-2$, which implies $\alpha = n-2$. Thus we get

$$(n-2)n - \beta = n^2 - 3n + 1 ,$$

and so $\beta = n-1$, which contradicts to $\beta \leq \alpha$. Therefore the case (ii) must occur. Then we have

$$\alpha n - \beta \leq (n-1)n - (n-1) \text{ and } (n-1-\alpha)n \leq n-1-\beta .$$

As the first inequality implies

$$n-1-\beta \leq (n-1)n - \alpha n = (n-1-\alpha)n ,$$

it follows that

$$n-1-\beta = (n-1)n - \alpha n ,$$

and so

$$\alpha n - \beta = n^2 - 2n + 1 .$$

Then, as $\alpha n > n^2 - 2n = (n-2)n$, we get $\alpha > n-2$, which contradicts to $\alpha \leq n-2$. Thus we have seen that ${x_1}^{n-1} \notin QI^{n-2}$.

(2) Let $\mathfrak{b} = (X_0, X_1, \ldots, X_{n-1})S$. Then $\mathfrak{a} \subseteq \mathfrak{b}$, and so $\mathfrak{b}$ is the kernel of the canonical surjection $S \longrightarrow A/J$. Hence $A/J \cong k[X_n]$, which implies $\dim A/J = 1$. Let $0 \leq i \leq j \leq n-1$. If $i = 0$ or $j = n-1$, then $x_i x_j \in QJ$. On the other hand, if $i > 0$ and $j < n$, then $x_i x_j = x_{i-1}x_{j+1}$. Hence we can show that $x_i x_j \in QJ$ for any $0 \leq i \leq j \leq n-1$ by descending induction on $j - i$. Thus we get $J^2 = QJ$. It is obvious that $\mu_A(J/I) = n-3$.

Therefore $I^{n-1} = QI^{n-2}$ by 3.1. This means $\dim A/I = \dim A/Q = \dim A/J = 1$. In order to prove $\mathrm{r}_Q(I) = n-2$, we show ${x_1}^{n-2} \notin QI^{n-3}$. For that purpose we use again the isomorphism φ stated in the proof of (1). Although we have to prove $\varphi(x_1)^{n-2} \notin \varphi(Q)\varphi(I)^{n-3}$, it is enough to show

$$(s^{n-1}t)^{n-2} \notin (s^n, st^{n-1})(s^n, s^{n-1}t, st^{n-1})^{n-3}B\,,$$

where $B = k[s,t]$. Because

$$(s^{n-1}t)^{n-2} = s^{n-2} \cdot (s^{n-2}t)^{n-2}$$

in B and

$$(s^n, st^{n-1})(s^n, s^{n-1}t, st^{n-1})^{n-3}B = s^{n-2} \cdot (s^{n-1}, t^{n-1})(s^{n-1}, s^{n-2}t, t^{n-1})^{n-3}B\,,$$

we would like to show

$$(s^{n-2}t)^{n-2} \notin (s^{n-1}, t^{n-1})(s^{n-1}, s^{n-2}t, t^{n-1})^{n-3}B\,.$$

However, it can be done by the same argument as the proof of

$$(s^{n-1}t)^{n-1} \notin (s^n, t^n)(s^n, s^{n-1}t, t^n)^{n-1}\,,$$

and hence we have proved (2).

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SALLY MODULES OF RANK ONE

SHIRO GOTO, KOJI NISHIDA, AND KAZUHO OZEKI

1. Introduction

This paper aims to give a structure theorem of Sally modules of rank one.

Let A be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. Then we have integers $\{e_i = \mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

holds true for all $n \gg 0$. Let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \subseteq A[t]$$

denote, respectively, the Rees algebras of I and Q, where t stands for an indeterminate over A. We put

$$R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \text{and} \quad G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

Let $B = T/\mathfrak{m}T$, which is the polynomial ring with d indeterminates over the field k. Following W. V. Vasconcelos [11], we then define

$$\mathrm{S}_Q(I) = IR/IT$$

and call it the Sally module of I with respect to Q. We notice that the Sally module $S = \mathrm{S}_Q(I)$ is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

The Sally module S was introduced by W. V. Vasconcelos [11], where he gave an elegant review, in terms of his *Sally* module, of the works [8, 9, 10] of J. Sally about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of the graded ring G and the Hilbert coefficients e_i's of I.

As is well-known, we have the inequality ([6])

$$e_1 \ge e_0 - \ell_A(A/I)$$

and C. Huneke [3] showed that $e_1 = e_0 - \ell_A(A/I)$ if and only if $I^2 = QI$ (cf. Corollary 2.3). When this is the case, both the graded rings G and $\mathrm{F}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ are Cohen-Macaulay, and the Rees algebra R of I is also a Cohen-Macaulay ring, provided $d \geq 2$. Thus, the ideals I with $e_1 = e_0 - \ell_A(A/I)$ enjoy very nice properties.

J. Sally firstly investigated the second border, that is the ideals I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$ but $e_2 \neq 0$ (cf. [10, 11]). The present research is a continuation of [10, 11] and aims to give a complete structure theorem of the Sally module of an m-primary ideal I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$.

The main result of this paper is the following Theorem 1.1. Our contribution in Theorem 1.1 is the implication (1) $\Rightarrow$ (3), the proof of which is based on the new result that the equality $I^3 = QI^2$ holds true if $e_1 = e_0 - \ell_A(A/I) + 1$ (cf. Theorem 3.1).

Theorem 1.1. *The following three conditions are equivalent to each other.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) $\mathfrak{m}S = (0)$ *and* $\operatorname{rank}_B S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded T-modules for some* $0 < c \leq d$, *where* $\{X_i\}_{1\leq i\leq c}$ *are linearly independent linear forms of the polynomial ring* B.

When this is the case, $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$, *and the following assertions hold true.*

(i) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_T S = d - c + 1$.

(ii) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.

(iii) *Suppose* $c < d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$$

for all $n \geq 0$. *Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iv) *Suppose* $c = d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$$

for all $n \geq 1$. *Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

Thus Theorem 1.1 settles a long standing problem, although the structure of ideals I with $e_1 = e_0 - \ell_A(A/I) + 2$ or the structure of Sally modules S with $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 2$ remains unknown.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1.1 in Section 3. In Section 2 we will pick up from the paper [1] some auxiliary results on Sally modules, all of which are known, but let us note them for the sake of the reader's convenience. In Section 4 we will construct one example in order to see the ubiquity of ideals I which satisfy condition (3) in Theorem 1.1.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. We assume that the field $k = A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let S be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. We put $R = A[It], T = A[Qt], R' = A[It, t^{-1}]$, and $G = R'/t^{-1}R'$. Let

$$\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} :_A I^n] = \bigcup_{n \geq 1} [I^{n+1} :_A (a_1^n, a_2^n, \cdots, a_d^n)]$$

denote the Ratliff-Rush closure of I, which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = e_i$ for all $0 \leq i \leq d$ (cf. [7]). We denote by $\mu_A(*)$ the number of generators.

2. Auxiliary results

In this section let us firstly summarize some known results on Sally modules, which we need throughout this paper. See [1] and [11] for the detailed proofs.

The first two results are basic facts on Sally modules developed by Vasconcelos [11].

Lemma 2.1. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell S = (0)$ *for integers* $\ell \gg 0$.

(2) *The homogeneous components* $\{S_n\}_{n \in \mathbb{Z}}$ *of the graded* T*-module* S *are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \leq 0, \\ I^{n+1}/IQ^n & \text{if } n \geq 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.

(4) *Suppose that* $S \neq (0)$ *and put* $V = S/MS$, *where* $M = \mathfrak{m}T + T_+$ *is the graded maximal ideal in* T. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* T/M*-space* V *with degree* n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then we have* $\Lambda = \{1, 2, \cdots, q\}$ *and* $\mathrm{r}_Q(I) = q + 1$, *where* $\mathrm{r}_Q(I)$ *stands for the reduction number of* I *with respect to* Q.

(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. See [1, Lemma 2.1]. □

Proposition 2.2. *Let* $\mathfrak{p} = \mathfrak{m}T$. *Then the following assertions hold true.*

(1) $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$. *Hence* $\dim_T S = d$, *if* $S \neq (0)$.

(2) $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n)$ *for all* $n \geq 0$.

(3) *We have* $e_1 = e_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$. *Hence* $e_1 = e_0 - \ell_A(A/I) + 1$ *if and only if* $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B\, S = 1$.

(4) *Suppose that* $S \neq (0)$. *Let* $s = \mathrm{depth}_T\, S$. *Then* $\mathrm{depth}\, G = s - 1$ *if* $s < d$. S *is a Cohen-Macaulay* T*-module if and only if* $\mathrm{depth}\, G \geq d - 1$.

Proof. See [1, Proposition 2.2]. □

Combining Lemma 2.1 (3) and Proposition 2.2, we readily get the following results of Northcott [6] and Huneke [3].

Corollary 2.3 ([3, 6]). *We have* $e_1 \geq e_0 - \ell_A(A/I)$. *The equality* $e_1 = e_0 - \ell_A(A/I)$ *holds true if and only if* $I^2 = QI$. *When this is the case,* $e_i = 0$ *for all* $2 \leq i \leq d$.

The following result is one of the keys for our proof of Theorem 1.1.

Theorem 2.4. *The following conditions are equivalent.*

(1) $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.

(2) $S \cong \mathfrak{a}$ *as graded* T*-modules for some graded ideal* $\mathfrak{a}$ $(\neq B)$ *of* B.

Proof. We have only to show (1) ⇒ (2). Because $S_1 \neq (0)$ and $S = \sum_{n\geq 1} S_n$ by Lemma 2.1, we have $S \cong B(-1)$ as graded B-modules once S is B-free.

Suppose that S is not B-free. The B-module S is torsionfree, since $\mathrm{Ass}_T S = \{\mathfrak{m}T\}$ by Proposition 2.2 (1). Therefore, since $\mathrm{rank}_B\, S = 1$, we see $d \geq 2$ and $S \cong \mathfrak{a}(m)$ as graded B-modules for some integer m and some graded ideal $\mathfrak{a}$ $(\neq B)$ in B, so that we get the exact sequence

$$0 \to S(-m) \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules. We may assume that $\mathrm{ht}_B\, \mathfrak{a} \geq 2$, since $B = k[X_1, X_2, \cdots, X_d]$ is the polynomial ring over the field $k = A/\mathfrak{m}$. We then have $m \geq 0$, since $\mathfrak{a}_{m+1} = [\mathfrak{a}(m)]_1 \cong S_1 \neq (0)$ and $\mathfrak{a}_0 = (0)$. We want to show $m = 0$.

Because $\dim B/\mathfrak{a} \leq d-2$, the Hilbert polynomial of $B/\mathfrak{a}$ has degree at most $d-3$. Hence

$$\begin{aligned}\ell_A(S_n) &= \ell_A(B_{m+n}) - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{m+n+d-1}{d-1} - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{n+d-1}{d-1} + m\binom{n+d-2}{d-2} + \text{(lower terms)}\end{aligned}$$

for $n \gg 0$. Consequently

$$\begin{aligned}\ell_A(A/I^{n+1}) &= e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n)\\ &= e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I)+1)\cdot\binom{n+d-1}{d-1} - m\binom{n+d-2}{d-2}\\ &\quad +(\text{lower terms})\end{aligned}$$

by Proposition 2.2 (2), so that we get $e_2 = -m$. Thus $m = 0$, because $e_2 \geq 0$ by Narita's theorem ([5]). $\square$

The following result will enable us to reduce the proof of Theorem 1.1 to the proof of the fact that $I^3 = QI^2$ if $e_1 = e_0 - \ell_A(A/I) + 1$.

Proposition 2.5. *Suppose $e_1 = e_0 - \ell_A(A/I) + 1$ and $I^3 = QI^2$. Let $c = \ell_A(I^2/QI)$. Then the following assertions hold true.*

(1) $0 < c \leq d$ *and* $\mu_B(S) = c$.

(2) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_B S = d - c + 1$.

(3) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.

(4) *Suppose $c < d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ for all $n \geq 0$. Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1 \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(5) *Suppose $c = d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$ for all $n \geq 1$. Hence $e_i = 0$ for $2 \leq i \leq d$.*

Proof. We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$ by Proposition 2.2 (3), while $S = TS_1$ since $I^3 = QI^2$ (cf. Lemma 2.1 (5)). Therefore by Theorem 2.4 we have $S \cong \mathfrak{a}$ as graded B-modules where $\mathfrak{a} = (X_1, X_2, \cdots, X_c)$ is an ideal in B generated by linear forms $\{X_i\}_{1 \leq i \leq c}$. Hence $0 < c \leq d$, $\mu_B(S) = c$, and $\operatorname{depth}_B S = d - c + 1$, so that assertions (1), (2), and (3) follow (cf. Proposition 2.2 (4)). Considering the exact sequence

$$0 \to S \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules, we get

$$\begin{aligned}\ell_A(S_n) &= \ell_A(B_n) - \ell_A([B/\mathfrak{a}]_n)\\ &= \binom{n+d-1}{d-1} - \binom{n+d-c-1}{d-c-1}\end{aligned}$$

for all $n \geq 0$ (resp. $n \geq 1$), if $c < d$ (resp. $c = d$). Thus assertions (4) and (5) follow (cf. Proposition 2.2 (2)). $\square$

3. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. See Proposition 2.2 (3) for the equivalence of conditions (1) and (2) in Theorem 1.1. The implication (3) $\Rightarrow$ (2) is clear. So, we must show the implication (1) $\Rightarrow$ (3) together with the last assertions in Theorem 1.1. Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then, thanks to Theorem 2.4, we get an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Notice that once we are able to show $I^3 = QI^2$, the last assertions of Theorem 1.1 readily follow from Proposition 2.5. On the other hand, since $\mathfrak{a} \cong S = BS_1$ (cf. Lemma 2.1 (5)), the ideal $\mathfrak{a}$ of B is generated by linearly independent linear forms $\{X_i\}_{1 \leq i \leq c}$ $(0 < c \leq d)$ of B and so, the implication (1) $\Rightarrow$ (3) in Theorem 1.1 follows. We have $c = \ell_A(I^2/QI)$, because $\mathfrak{a}_1 \cong S_1 = I^2/QI$ (cf. Lemma 2.1 (2)). Thus our Theorem 1.1 has been proven modulo the following theorem.

Theorem 3.1. *Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then $I^3 = QI^2$.*

Proof. We proceed by induction on d. Suppose that $d = 1$. Then S is B-free of rank one (recall that the B-module S is torsionfree; cf. Proposition 2.2 (1)) and so, since $S_1 \neq (0)$ (cf. Lemma 2.1 (3)), $S \cong B(-1)$ as graded B-modules. Thus $I^3 = QI^2$ by Lemma 2.1 (5).

Let us assume that $d \geq 2$ and that our assertion holds true for $d - 1$. Since the field $k = A/\mathfrak{m}$ is infinite, without loss of generality we may assume that a_1 is a superficial element of I. Let

$$\overline{A} = A/(a_1), \quad \overline{I} = I/(a_1), \text{ and } \quad \overline{Q} = Q/(a_1).$$

We then have $\mathrm{e}_i(\overline{I}) = e_i$ for all $0 \leq i \leq d-1$, whence

$$\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1.$$

Therefore the hypothesis of induction on d yields $\overline{I}^3 = \overline{Q}\,\overline{I}^2$. Hence, because the element $a_1 t$ is a nonzerodivisor on G if $\operatorname{depth} G > 0$, we have $I^3 = QI^2$ in that case.

Assume that $\operatorname{depth} G = 0$. Then, thanks to Sally's technique ([10]), we also have $\operatorname{depth} \mathrm{G}(\overline{I}) = 0$. Hence $\ell_{\overline{A}}(\overline{I}^2/\overline{Q}\,\overline{I}) = d - 1$ by Proposition 2.5 (2), because $\mathrm{e}_1(\overline{I}) =$

$e_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1$. Consequently, $\ell_A(S_1) = \ell_A(I^2/QI) \geq d-1$, because $\overline{I}^2/\overline{Q}\,\overline{I}$ is a homomorphic image of I^2/QI. Let us take an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Then, since

$$\ell_A(\mathfrak{a}_1) = \ell_A(S_1) \geq d-1,$$

the ideal $\mathfrak{a}$ contains $d-1$ linearly independent linear forms, say $X_1, X_2, \cdots, X_{d-1}$ of B, which we enlarge to a basis $X_1, \cdots, X_{d-1}, X_d$ of B_1. Hence

$$B = k[X_1, X_2, \cdots, X_d],$$

so that the ideal $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B$ in the polynomial ring

$$B/(X_1, X_2, \cdots, X_{d-1})B = k[X_d]$$

is principal. If $\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1})B$, then $I^3 = QI^2$ by Lemma 2.1 (5), since $S = BS_1$. However, because $\ell_A(I^2/QI) = \ell_A(\mathfrak{a}_1) = d-1$, we have $\operatorname{depth} G \geq 1$ by Proposition 2.5 (2), which is impossible. Therefore $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B \neq (0)$, so that we have

$$\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1}, X_d^{\alpha})B$$

for some $\alpha \geq 1$. Notice that $\alpha = 1$ or $\alpha = 2$ by Lemma 2.1 (4). We must show that $\alpha = 1$.

Assume that $\alpha = 2$. Let us write, for each $1 \leq i \leq d$, $X_i = \overline{b_i t}$ with $b_i \in Q$, where $\overline{b_i t}$ denotes the image of $b_i t \in T$ in $B = T/\mathfrak{m}T$. Then $\mathfrak{a} = (\overline{b_1 t}, \overline{b_2 t}, \cdots, \overline{b_{d-1} t}, \overline{(b_d t)^2})$. Notice that

$$Q = (b_1, b_2, \cdots, b_d),$$

because $\{X_i\}_{1 \leq i \leq d}$ is a k-basis of B_1. We now choose elements $f_i \in S_1$ for $1 \leq i \leq d-1$ and $f_d \in S_2$ so that $\varphi(f_i) = X_i$ for $1 \leq i \leq d-1$ and $\varphi(f_d) = X_d^2$. Let $z_i \in I^2$ for $1 \leq i \leq d-1$ and $z_d \in I^3$ such that $\{f_i\}_{1 \leq i \leq d-1}$ and f_d are, respectively, the images of $\{z_i t\}_{1 \leq i \leq d-1}$ and $z_d t^2$ in S. We now consider the relations $X_i f_1 = X_1 f_i$ in S for $1 \leq i \leq d-1$ and $X_d^2 f_1 = X_1 f_d$, that is

$$b_i z_1 - b_1 z_i \in Q^2 I$$

for $1 \leq i \leq d-1$ and

$$b_d^2 z_1 - b_1 z_d \in Q^3 I.$$

Notice that

$$Q^3 = b_1 Q^2 + (b_2, b_3, \cdots, b_{d-1})^2 \cdot (b_2, b_3, \cdots, b_d) + b_d^2 Q$$

and write

$$b_d^2 z_1 - b_1 z_d = b_1\tau_1 + \tau_2 + b_d^2\tau_3$$

with $\tau_1 \in Q^2I$, $\tau_2 \in (b_2, b_3, \cdots, b_{d-1})^2{\cdot}(b_2, b_3, \cdots, b_d)I$, and $\tau_3 \in QI$. Then

$$b_d^2(z_1 - \tau_3) = b_1(\tau_1 + z_d) + \tau_2 \in (b_1) + (b_2, b_3, \cdots, b_{d-1})^2.$$

Hence $z_1 - \tau_3 \in (b_1) + (b_2, b_3, \cdots, b_{d-1})^2$, because the sequence $b_1, b_2, \cdots, b_d$ is A-regular. Let $z_1 - \tau_3 = b_1 h + h'$ with $h \in A$ and $h' \in (b_2, b_3, \cdots, b_{d-1})^2$. Then since

$$b_1[b_d^2 h - (\tau_1 + z_d)] = \tau_2 - b_d^2 h' \in (b_2, b_3, \cdots, b_d)^3,$$

we have $b_d^2 h - (\tau_1 + z_d) \in (b_2, b_3, \cdots, b_d)^3$, whence $b_d^2 h \in I^3$.

We need the following.

Claim. *$h \notin I$ but $h \in \tilde{I}$. Hence $\tilde{I} \neq I$.*

Proof. If $h \in I$, then $b_1 h \in QI$, so that $z_1 = b_1 h + h' + \tau_3 \in QI$, whence $f_1 = 0$ in S (cf. Lemma 2.1 (2)), which is impossible. Let $1 \leq i \leq d-1$. Then

$$b_i z_1 - b_1 z_i = b_i(b_1 h + h' + \tau_3) - b_1 z_i = b_1(b_i h - z_i) + b_i(h' + \tau_3) \in Q^2I.$$

Therefore, because $b_i(h' + \tau_3) \in Q^2I$, we get

$$b_1(b_i h - z_i) \in (b_1) \cap Q^2I.$$

Notice that

$$\begin{aligned} (b_1) \cap Q^2I &= (b_1) \cap [b_1 QI + (b_2, b_3, \cdots, b_d)^2 I] \\ &= b_1 QI + [(b_1) \cap (b_2, b_3, \cdots, b_d)^2 I] \\ &= b_1 QI + b_1(b_2, b_3, \cdots, b_d)^2 \\ &= b_1 QI \end{aligned}$$

and we have $b_i h - z_i \in QI$, whence $b_i h \in I^2$ for $1 \leq i \leq d-1$. Consequently $b_i^2 h \in I^3$ for all $1 \leq i \leq d$, so that $h \in \tilde{I}$, whence $\tilde{I} \neq I$. □

Because $\ell_A(\tilde{I}/I) \geq 1$, we have

$$\begin{aligned} e_1 &= e_0 - \ell_A(A/I) + 1 \\ &= e_0(\tilde{I}) - \ell_A(A/\tilde{I}) + [1 - \ell_A(\tilde{I}/I)] \\ &\leq e_0(\tilde{I}) - \ell_A(A/\tilde{I}) \\ &\leq e_1(\tilde{I}) \\ &= e_1, \end{aligned}$$

where $\mathrm{e}_0(\tilde{I})-\ell_A(A/\tilde{I}) \leq \mathrm{e}_1(\tilde{I})$ is the inequality of Northcott for the ideal $\tilde{I}$ (cf. Corollary 2.3). Hence $\ell_A(\tilde{I}/I)=1$ and $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})$, so that

$$\tilde{I} = I + (h) \quad \text{and} \quad \tilde{I}^2 = Q\tilde{I}$$

by Corollary 2.3 (recall that Q is a reduction of $\tilde{I}$ also). We then have, thanks to [4, Corollary 3.1], that $I^3 = QI^2$, which is a required contradiction. This completes the proof of Theorem 1.1 and that of Theorem 3.1 as well. □

4. An example

Lastly we construct one example which satisfies condition (3) in Theorem 1.1. Our goal is the following. See [2, Section 5] for the detailed proofs.

Theorem 4.1. *Let $0 < c \leq d$ be integers. Then there exists an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1, \quad \text{and} \quad c = \ell_A(I^2/QI)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ of I.

To construct necessary examples we may assume that $c = d$.

Let m, $d > 0$ be integers. Let

$$U = k[\{X_j\}_{1\leq j \leq m}, Y, \{V_i\}_{1\leq i\leq d}, \{Z_i\}_{1\leq i \leq d}]$$

be the polynomial ring with $m+2d+1$ indeterminates over an infinite field k and let

$$\begin{aligned}\mathfrak{a} &= [(X_j \mid 1\leq j\leq m) + (Y)]\cdot[(X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)] \\ &\quad +(V_iV_j \mid 1 \leq i,j\leq d,\ i\neq j) + (V_i^2 - Z_iY \mid 1 \leq i \leq d).\end{aligned}$$

We put $C = U/\mathfrak{a}$ and denote the images of X_j, Y, V_i, and Z_i in C by x_j, y, v_i, and a_i, respectively. Then $\dim C = d$, since $\sqrt{\mathfrak{a}} = (X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)$. Let $M = C_+ := (x_j \mid 1 \leq j \leq m) + (y) + (v_i \mid 1 \leq i \leq d) + (a_i \mid 1 \leq i \leq d)$ be the graded maximal ideal in C. Let Λ be a subset of $\{1, 2, \cdots, m\}$. We put

$$J = (a_i \mid 1 \leq i \leq d) + (x_\alpha \mid \alpha \in \Lambda) + (v_i \mid 1 \leq i \leq d) \quad \text{and} \quad \mathfrak{q} = (a_i \mid 1 \leq i \leq d).$$

Then $M^2 = \mathfrak{q}M$, $J^2 = \mathfrak{q}J + \mathfrak{q}y$, and $J^3 = \mathfrak{q}J^2$, whence $\mathfrak{q}$ is a reduction of both M and J, and $a_1, a_2, \cdots, a_d$ is a homogeneous system of parameters for the graded ring C.

Let $A = C_M$, $I = JA$, and $Q = \mathfrak{q}A$. We are now interested in the Hilbert coefficients e_i's of the ideal I as well as the structure of the associated graded ring and the Sally module of I. We then have the following, which shows that the ideal I is a required example.

Theorem 4.2. *The following assertions hold true.*

(1) *A is a Cohen-Macaulay local ring with* $\dim A = d$.

(2) *$S \cong B_+$ as graded T-modules, whence $\ell_A(I^2/QI) = d$.*

(3) $\mathrm{e}_0(I) = m + d + 2$ *and* $\mathrm{e}_1(I) = \sharp\Lambda + d + 1$.

(4) $\mathrm{e}_i(I) = 0$ *for all* $2 \leq i \leq d$.

(5) *G is a Buchsbaum ring with* $\operatorname{depth} G = 0$ *and* $\mathbb{I}(G) = d$.

Proof. See [2, Theorem 5.2] □

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DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND TECHNOLOGY, MEIJI UNIVERSITY, 1-1-1 HIGASHI-MITA, TAMA-KU, KAWASAKI 214-8571, JAPAN

E-mail address: goto@math.meiji.ac.jp

DEPARTMENT OF MATHEMATICS AND INFORMATICS, GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY, CHIBA UNIVERSITY, 1-33 YATOI-CHO, INAGE-KU, CHIBA-SHI, 263 JAPAN

E-mail address: nishida@math.s.chiba-u.ac.jp

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND TECHNOLOGY, MEIJI UNIVERSITY, 1-1-1 HIGASHI-MITA, TAMA-KU, KAWASAKI 214-8571, JAPAN

E-mail address: kozeki@math.meiji.ac.jp

ANALYTIC SPREAD OF SQUAREFREE MONOMIAL IDEALS

KYOUKO KIMURA (NAGOYA UNIVERSITY)

INTRODUCTION

This is a joint work with Naoki Terai (Saga Univ.) and Ken-ichi Yoshida (Nagoya Univ.).

Let S be a polynomial ring with each variable has degree 1 over an infinite field k, and I a squarefree monomial ideal of S. The *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min\left\{ r : \text{there exist } a_1, \dots, a_r \in I \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I} \right\}.$$

It is known by Lyubeznik [2] that $\operatorname{pd}_S S/I \leq \operatorname{ara} I$, where $\operatorname{pd}_S S/I$ denotes the *projective dimension* of S/I. Let J be a minimal reduction of I. The number of a minimal set of generators of J, which is independent on the choice of J, is called the *analytic spread* of I. We denote it by $l(I)$. Since $\sqrt{J} = \sqrt{I}$ holds, we have

$$\operatorname{pd}_S S/I \leq \operatorname{ara} I \leq l(I).$$

Schmitt–Vogel lemma [4, Lemma, pp. 249] is an important and useful tool in the study of the arithmetical rank. Using this lemma, Schmitt–Vogel proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ for

$$(*) \qquad I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q}),$$

where x_{ij} are variables in S pairwise distinct. Note that this ideal I is the Alexander dual of a complete intersection ideal.

In this report, we refine Schmitt–Vogel lemma for reductions and prove $l(I) = \operatorname{pd}_S S/I$ for the ideal $(*)$ as its application.

1. MAIN THEOREM

In this section, we consider a commutative ring R with unitary. Let I, J be ideals in R with $J \subset I$. We say J is a *reduction* of I if there exists $s \in \mathbb{N}$ such that $I^{s+1} = JI^s$. It is easy to see that if J is a reduction of I, then $\sqrt{J} = \sqrt{I}$. The main theorem of this report is the following:

Theorem 1. *Let R be a commutative ring with unitary. Let $P_0, P_1, \ldots, P_r \subset R$ be finite subsets, and we set*

$$P = \bigcup_{\ell=0}^{r} P_\ell,$$

$$g_\ell = \sum_{a \in P_\ell} a, \quad \ell = 0, 1, \ldots, r.$$

Assume that

(C1) $\sharp P_0 = 1$.

(C2) *For all $\ell > 0$ and $a, a'' \in P_\ell$ ($a \neq a''$), there exist some ℓ' ($0 \leq \ell' < \ell$), $a' \in P_{\ell'}$, and $b \in (P)$ such that $aa'' = a'b$.*

Then we have $(g_0, g_1, \ldots, g_r)$ is a reduction of (P).

The difference between our theorem and Schmitt–Vogel lemma is the assumption of the existence of $b \in (P)$ in (C2). The second condition of Schmitt–Vogel lemma is

(C2)' For all $\ell > 0$ and $a, a'' \in P_\ell$ ($a \neq a''$), there exist some ℓ' ($0 \leq \ell' < \ell$) and $a' \in P_{\ell'}$ such that $aa'' \in (a')$;

and the conclusion is $\sqrt{(g_0, g_1, \ldots, g_r)} = \sqrt{(P)}$.

Remark 2. Schmitt–Vogel lemma allows us to add some exponent $e(a)$ for each $a \in P_\ell$ in the sum g_ℓ, i.e., we may put

$$g_\ell = \sum_{a \in P_\ell} a^{e(a)}.$$

Thus we can take g_ℓ as homogeneous if R is graded. But our theorem is not allowed to add such $e(a)$.

2. Proof of Main Theorem

In this section, we prove Theorem 1.

As first, we fix notation. Put $I = (P)$, $J = (g_0, g_1, \ldots, g_r)$, and

$$I_\ell = \left(\bigcup_{j=0}^{\ell} P_j \right), \qquad \ell = 0, 1, \ldots, r.$$

It is enough to show $I_\ell^{2^\ell} \subset JI^{2^\ell - 1}$ for $\ell = 0, 1, \ldots, r$. We show this by induction on ℓ. In fact, we show

$$I_\ell^{2^\ell} \subset I_{\ell-1}^{2^{\ell-1}} I^{2^\ell - 2^{\ell-1}} + JI^{2^\ell - 1}, \qquad \ell = 0, 1, \ldots, r.$$

If $\ell = 0$, then $I_0 = (P_0) = (g_0) \subset J$ because $\sharp P_0 = 1$. Let us consider the case of $\ell > 0$. Take $a_1, \ldots, a_{2^\ell} \in \bigcup_{j=0}^{\ell} P_j$. We may assume $a_1, \ldots, a_m \in P_\ell$ and $a_{m+1}, \ldots, a_{2^\ell} \in \bigcup_{j=0}^{\ell-1} P_j$.

First, we assume that we can renumbering $a_1, \ldots, a_m$ such that

$$\{a_1, a_1''\}, \ldots, \{a_{\lfloor m/2 \rfloor}, a''_{\lfloor m/2 \rfloor}\},$$

where $a_\lambda \neq a''_\lambda$, $a''_\lambda = a_{\lfloor m/2 \rfloor + \lambda}$ ($\lambda = 1, \dots, \lfloor m/2 \rfloor$), and $\lfloor \alpha \rfloor$ denotes the maximal integer which does not exceed α. Then we can use the condition (C2), that is, there are $a'_\lambda \in \bigcup_{j=0}^{\ell-1} P_j$ and $b_\lambda \in I$ such that $a_\lambda a''_\lambda = a'_\lambda b_\lambda$. Thus

$$\begin{aligned} a_1 \cdots a_{2\ell} &= \left(\prod_{\lambda=1}^{\lfloor m/2 \rfloor} a'_\lambda b_\lambda \right) a_{2\lfloor m/2 \rfloor + 1} \cdots a_{2\ell} \\ &= \left(\prod_{\lambda=1}^{\lfloor m/2 \rfloor} a'_\lambda \right) a_{m+1} \cdots a_{2\ell} \left(\prod_{\lambda=1}^{\lfloor m/2 \rfloor} b_\lambda \right) a_{2\lfloor m/2 \rfloor + 1} \cdots a_m. \end{aligned}$$

Note that $m \leq 2^\ell$ and $\lfloor m/2 \rfloor \geq (m-1)/2$. Then it is easy to see that $\lfloor m/2 \rfloor + 2^\ell - m \geq 2^{\ell-1} - 1/2$. Since $\lfloor m/2 \rfloor + 2^\ell - m \in \mathbb{Z}$, we have $\lfloor m/2 \rfloor + 2^\ell - m \geq 2^{\ell-1}$. Therefore

$$a_1 \cdots a_{2\ell} \in I_{\ell-1}^{2^{\ell-1}} I^{2^\ell - 2^{\ell-1}}.$$

Next, we consider the case that we cannot make $\lfloor m/2 \rfloor$ pairs of distinct elements. This case occurs if and only if there exist $a \in P_\ell$ (uniquely) such that

$$a = a_1 = \cdots = a_{\lfloor (m-1)/2 \rfloor + 2},$$

by renumbering $a_1, \dots, a_m$. Then

$$\begin{aligned} a_1 a_2 \cdots a_{2\ell} &= a a_2 \cdots a_{2\ell} \\ &= \left(g_\ell - \sum_{a'' \in P_\ell,\ a'' \neq a} a'' \right) a_2 \cdots a_{2\ell} \\ &= g_\ell a_2 \cdots a_{2\ell} - \sum_{a'' \in P_\ell,\ a'' \neq a} a'' a_2 \cdots a_{2\ell}. \end{aligned}$$

The first term belongs to $JI^{2^\ell - 1}$. Thus we consider $a'' a_2 \cdots a_{2\ell}$ in the second term only. Since $\max\{\sharp\{i : a_i = a\} : a \in P_\ell\}$ is strictly reduced, the problem can be reduced to the first case.

Q.E.D.

3. An application

In this section, we apply Theorem 1 to some ideals and calculate the analytic spread of them.

Consider the ideal

$$(*) \qquad I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q}),$$

where $x_{11}, \dots, x_{qi_q}$ are all distinct variables. Then one can easily see that

$$\operatorname{pd}_S S/I = \sum_{s=1}^{q} i_s - q + 1.$$

Schmitt–Vogel [4] proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ (see also Schenzel–Vogel [3]). They proved it by applying

$$P_\ell = \{x_{1\ell_1} \cdots x_{q\ell_q} \ : \ \ell_1 + \cdots + \ell_q = \ell + q\}, \qquad \ell = 0, 1, \dots, r$$

to Schmitt–Vogel lemma, where $r = \sum_{s=1}^{q} i_s - q$. Since these $P_0, P_1, \dots, P_r$ also satisfy the assumption of Theorem 1, we have the following corollary:

Corollary 3. *Let* $I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q})$. *Then we have*

$$l(I) = \operatorname{pd}_S S/I.$$

In particular, $(g_0, g_1, \dots, g_r)$ *is a minimal reduction of* I.

Although $l(I) = \operatorname{pd}_S S/I$ is also proven by computing the dimension of fiber cone, we construct a minimal reduction of I explicitly.

By giving an example, we remark that the relation between our theorem and the reduction number.

Let $I = (x_{11}, x_{12}) \cap (x_{21}, x_{22}) \cap (x_{31}, x_{32})$. This is a special case of the ideal (∗) and $\operatorname{pd}_S S/I = 2 + 2 + 2 - 3 + 1 = 4$. The minimal reduction of I which derived from Corollary 3 is generated by the following 4 elements:

$$\begin{aligned}
g_0 &= x_{11}x_{21}x_{31}, \\
g_1 &= x_{12}x_{21}x_{31} + x_{11}x_{22}x_{31} + x_{11}x_{21}x_{32}, \\
g_2 &= x_{12}x_{22}x_{31} + x_{12}x_{21}x_{32} + x_{11}x_{22}x_{32}, \\
g_3 &= x_{12}x_{22}x_{32}.
\end{aligned}$$

Put $J = (g_0, g_1, g_2, g_3)$. Then what is the reduction number $r_J(I)$ of J ? From the our proof of Theorem 1, we can only see $r_J(I) \leq 2^3 - 1 = 7$. But $I^3 = JI^2$ holds. In fact, $r_J(I) = 2$. Thus the upper bound of $r_J(I)$ derived from Theorem 1 is very big in general.

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Graduate School of Mathematics, Nagoya University, Nagoya 464–8602, Japan

E-mail address: m04012w@math.nagoya-u.ac.jp

QUASI-SOCLE IDEALS IN LOCAL RINGS WITH GORENSTEIN TANGENT CONES

SHIRO GOTO, SATORU KIMURA, NAOYUKI MATSUOKA, AND TRAN THI PHUONG

1. Introduction

This talk aims at a study of quasi-socle ideals in a local ring with the Gorenstein tangent cone. Our purpose is to answer Question 1.1 below, of when the graded rings associated to the ideals are Cohen-Macaulay and/or Gorenstein rings, estimating their reduction numbers with respect to minimal reductions.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $Q = (x_1, x_2, \cdots, x_d)$ be a parameter ideal in A and let $q \geq 1$ be an integer. We put $I = Q : \mathfrak{m}^q$ and refer to those ideals as quasi-socle ideals in A. Then one can ask the following questions, which are the main subject of the present research.

Question 1.1.
(1) Find the conditions under which $I \subseteq \overline{Q}$, where $\overline{Q}$ stands for the integral closure of Q.
(2) When $I \subseteq \overline{Q}$, estimate or describe the reduction number $\mathrm{r}_Q(I) = \min \{0 \leq n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$ of I with respect to Q in terms of some invariants of Q or A.
(3) Clarify what kind of ring-theoretic properties of the graded rings associated to the ideal I

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n, \ \mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}, \text{ and } \mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

enjoy.

In this talk we shall focus our attention on a certain special kind of quasi-socle ideals. We now assume that the tangent cone, that is the associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ of $\mathfrak{m}$, is a Gorenstein ring and that the maximal ideal $\mathfrak{m}$ contains a system $x_1, x_2, \cdots, x_d$ of elements such that the ideal $(x_1, x_2, \cdots, x_d)$ is a reduction of $\mathfrak{m}$. Let $a_1, a_2, \cdots, a_d$, and q be positive integers and we put $Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d})$ and $I = Q : \mathfrak{m}^q$. Let $\overline{A} = A/Q$, $\overline{\mathfrak{m}} = \mathfrak{m}/Q$, and $\overline{I} = I/Q$. Let $\rho = \max \{n \in \mathbb{Z} \mid \overline{\mathfrak{m}}^n \neq (0)\}$, that is the index of nilpotency of the ideal $\overline{\mathfrak{m}}$ and put $\ell = \rho + 1 - q$. We then have the following, which are the answers to Question 1.1 in our specific setting.

Key words and phrases: Quasi-socle ideal, regular local ring, Cohen-Macaulay ring, Gorenstein ring, associated graded ring, Rees algebra, Fiber cone, integral closure.
2000 Mathematics Subject Classification: 13H10, 13A30, 13B22, 13H15.

Theorem 1.2. *The following three conditions are equivalent to each other.*

(1) $I \subseteq \overline{Q}$.

(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(3) $\ell \geq a_i$ *for all* $1 \leq i \leq d$.

When this is the case, the following assertions hold true.

(i) $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil := \min \{n \in \mathbb{Z} \mid \frac{q}{\ell} \leq n\}$.

(ii) *The graded rings* $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are Cohen-Macaulay.*

Theorem 1.3. *Suppose that* $\ell \geq a_i$ *for all* $1 \leq i \leq d$. *Then we have the following.*

(i) $\mathrm{G}(I)$ *is a Gorenstein ring if and only if* $\ell \mid q$.

(ii) $\mathcal{R}(I)$ *is a Gorenstein ring if and only if* $q = (d-2)\ell$.

Our setting naturally contains the case where A is a regular local ring with $x_1, x_2, \cdots, x_d$ a regular system of parameters, the case where A is an abstract hypersurface with the infinite residue class field, and the case where $A = R_M$ is the localization of the homogeneous Gorenstein ring $R = k[R_1]$ over an infinite field $k = R_0$ at the irrelevant maximal ideal $M = R_+$. In Section 3 we will explore a few examples, including these three cases, in order to see how Theorems 1.2 and 1.3 work for the analysis of concrete examples. The proofs of Theorems 1.2 and 1.3 themselves shall be given in Section 2.

2. Proof of Theorems 1.2 and 1.3

The purpose of this section is to prove Theorems 1.2 and 1.3. First of all, let us restate our setting, which we shall maintain throughout this talk.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume that the associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ of $\mathfrak{m}$ is Gorenstein and that the maximal ideal $\mathfrak{m}$ contains a system $x_1, x_2, \cdots, x_d$ of elements which generates a reduction of $\mathfrak{m}$ (the latter condition is satisfied if the filed $A/\mathfrak{m}$ is infinite). Hence A is a Gorenstein ring and the initial forms $\{X_i\}_{1 \leq i \leq d}$ of $\{x_i\}_{1 \leq i \leq d}$ with respect to $\mathfrak{m}$ constitute a regular sequence in $\mathrm{G}(\mathfrak{m})$ and we have a canonical isomorphism

$$\mathrm{G}(\mathfrak{m}/(x_1, x_2, \cdots, x_d)) \cong \mathrm{G}(\mathfrak{m})/(X_1, X_2, \cdots, X_d)$$

of graded A-algebras ([VV]). Let $a_1, a_2, \cdots, a_d$, and q be positive integers and we put

$$Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d}) \text{ and } I = Q : \mathfrak{m}^q.$$

Let $\overline{A} = A/Q$, $\overline{\mathfrak{m}} = \mathfrak{m}/Q$, and $\overline{I} = I/Q$. Then

$$\mathrm{G}(\overline{\mathfrak{m}}) \cong \mathrm{G}(\mathfrak{m})/({X_1}^{a_1}, {X_2}^{a_2}, \cdots, {X_d}^{a_d}),$$

whence $G(\overline{\mathfrak{m}})$ is a Gorenstein ring. Let $\rho = \max\{n \in \mathbb{Z} \mid \overline{\mathfrak{m}}^n \neq (0)\}$, that is the index of nilpotency of the ideal $\overline{\mathfrak{m}}$, and we have $\rho = a(G(\overline{\mathfrak{m}})) = a(G(\mathfrak{m})) + \sum_{i=1}^{d} a_i$, where $a(*)$ denotes the a-invariant of the corresponding graded ring ([GW, (3.1.4)]).

Let $\ell = \rho + 1 - q$. By [Wat] (see [O, Theorem 1.6] also) we then have the following.

Proposition 2.1. $(0) : \overline{\mathfrak{m}}^i = \overline{\mathfrak{m}}^{\rho+1-i}$ *for all* $i \in \mathbb{Z}$. *In particular* $\overline{I} = (0) : \overline{\mathfrak{m}}^q = \overline{\mathfrak{m}}^\ell$ *whence* $I = Q + \mathfrak{m}^\ell$.

The key for our proof of Theorem 1.2 is the following.

Lemma 2.2. *Suppose that* $\ell \geq a_i$ *for all* $1 \leq i \leq d$. *Then*

$$Q \cap \mathfrak{m}^{n\ell+m} \subseteq \mathfrak{m}^m Q I^{n-1}$$

for all $m \geq 0$ *and* $n \geq 1$.

Proof. We have

$$Q \cap \mathfrak{m}^{n\ell+m} = \sum_{i=1}^{d} x_i^{a_i} \mathfrak{m}^{n\ell+m-a_i}$$

since $x_1, x_2, \cdots, x_d$ is a super regular sequence with respect to $\mathfrak{m}$. Because

$$n\ell + m - a_i = (n-1)\ell + m + (\ell - a_i) \geq (n-1)\ell + m$$

for each $1 \leq i \leq d$, we get

$$\mathfrak{m}^{n\ell+m-a_i} \subseteq \mathfrak{m}^{(n-1)\ell+m} = \mathfrak{m}^m \cdot (\mathfrak{m}^\ell)^{n-1}.$$

Therefore, since $\mathfrak{m}^\ell \subseteq I$ by Proposition 2.1, we have

$$\begin{aligned} Q \cap \mathfrak{m}^{n\ell+m} &= \sum_{i=1}^{d} x_i^{a_i} \mathfrak{m}^{n\ell+m-a_i} \\ &\subseteq \sum_{i=1}^{d} x_i^{a_i} \mathfrak{m}^m (\mathfrak{m}^\ell)^{n-1} \\ &\subseteq \mathfrak{m}^m Q I^{n-1} \end{aligned}$$

as is claimed. □

Let us now prove Theorem 1.2.

Proof of Theorem 1.2. (2) ⇒ (1) This is well-known. See [NR].

(3) ⇒ (2) By Proposition 2.1 we get $\mathfrak{m}^q I = \mathfrak{m}^q Q + \mathfrak{m}^{q+\ell}$, whence $\mathfrak{m}^{q+\ell} \subseteq Q$, so that $\mathfrak{m}^{q+\ell} = Q \cap \mathfrak{m}^{q+\ell} \subseteq \mathfrak{m}^q Q$ by Lemma 2.2, because $\ell \geq a_i$ for all $1 \leq i \leq d$. Thus $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(1) ⇒ (3) Let $1 \leq i \leq d$ be an integer. Then $x_i^\ell \in \mathfrak{m}^\ell \subseteq I \subseteq \overline{Q}$. Consequently, x_i^ℓ is integral over $Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d})$ so that, thanks to the monomial property of the regular sequence $\underline{x} = x_1, x_2, \cdots, x_d$, we get $\frac{\ell}{a_i} \geq 1$. Hence $\ell \geq a_i$ for all $1 \leq i \leq d$.

Let us now consider assertions (i) and (ii). Let $n \geq 1$ be an integer. Then $I^n = QI^{n-1} + \mathfrak{m}^{n\ell}$ since $I = Q + \mathfrak{m}^\ell$ (Proposition 2.1), so that

$$Q \cap I^n = QI^{n-1} + [Q \cap \mathfrak{m}^{n\ell}] \subseteq QI^{n-1}$$

because $Q \cap \mathfrak{m}^{n\ell} \subseteq QI^{n-1}$ by Lemma 2.2. Therefore $Q \cap I^n = QI^{n-1}$ for all $n \geq 1$, whence $\mathrm{G}(I)$ is a Cohen-Macaulay ring ([VV, Corollary 2.7]).

We will show that $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$. Notice that

$$\mathrm{r}_Q(I) = \min\{n \geq 0 \mid I^{n+1} \subseteq Q\},$$

because $I^{n+1} = QI^n$ if and only if $I^{n+1} \subseteq Q$. Firstly, suppose that $I^{n+1} \subseteq Q$. We then have $\overline{\mathfrak{m}}^{(n+1)\ell} = (0)$ (recall that $\overline{I} = \overline{\mathfrak{m}}^\ell$), whence $(n+1)\ell \geq \rho + 1$. Therefore

$$n + 1 \geq \frac{\rho + 1}{\ell} = \frac{q + \ell}{\ell} = \frac{q}{\ell} + 1,$$

because $\ell = \rho + 1 - q$, so that we have $n \geq \frac{q}{\ell}$.

If $n \geq \frac{q}{\ell}$, then $(n+1)\ell \geq (\frac{q}{\ell} + 1)\ell = q + \ell = \rho + 1$ and so $\overline{I}^{n+1} = \overline{\mathfrak{m}}^{(n+1)\ell} = (0)$, whence $I^{n+1} \subseteq Q$. Thus $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$.

To see that $\mathrm{F}(I)$ is a Cohen-Macaulay ring, it suffices to show that

$$Q \cap \mathfrak{m}I^n = \mathfrak{m}QI^{n-1}$$

for all $n \geq 1$. By Lemma 2.2 we have

$$\begin{aligned} Q \cap \mathfrak{m}I^n &= Q \cap [\mathfrak{m}QI^{n-1} + \mathfrak{m}^{n\ell+1}] \\ &= \mathfrak{m}QI^{n-1} + [Q \cap \mathfrak{m}^{n\ell+1}] \\ &\subseteq \mathfrak{m}QI^{n-1}, \end{aligned}$$

whence $Q \cap \mathfrak{m}I^n = \mathfrak{m}QI^{n-1}$. □

Assume that $\ell \geq a_i$ for all $1 \leq i \leq d$ and let Y_i's be the initial forms of $x_i^{a_i}$'s with respect to I. Then $Y_1, Y_2, \cdots, Y_d$ is a homogeneous system of parameters of $\mathrm{G}(I)$, since Q is a reduction of I (Theorem 1.2). It therefore constitutes a regular sequence in $\mathrm{G}(I)$, because $\mathrm{G}(I)$ is a Cohen-Macaulay ring by Theorem 1.2 (ii), so that we have a canonical isomorphism

$$\mathrm{G}(\overline{I}) \cong \mathrm{G}(I)/(Y_1, Y_2, \cdots, Y_d)$$

of graded A-algebras ([VV]). Hence $\mathrm{a}(\mathrm{G}(\overline{I})) = \mathrm{a}(\mathrm{G}(I)) + d$. Let r be the index of nilpotency of $\overline{I}$, that is $r = \mathrm{a}(\mathrm{G}(\overline{I}))$. Then since $r = \mathrm{r}_Q(I)$ (recall that ${x_1}^{a_1}, {x_2}^{a_2}, \cdots, {x_d}^{a_d}$ is a super regular sequence with respect to I) and $\mathrm{a}(\mathrm{G}(I)) = \mathrm{a}(\mathrm{G}(\overline{I})) - d$ ([GW, (3.1.6)]), by Theorem 1.2 (i) we have the following.

Lemma 2.3. *Suppose that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then $\mathrm{a}(\mathrm{G}(I)) = \lceil \frac{q}{\ell} \rceil - d$.*

Corollary 2.4. *Assume that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then $\mathcal{R}(I)$ is a Cohen-Macaulay ring if and only if $\lceil \frac{q}{\ell} \rceil < d$. When this is the case, $d \geq 2$.*

Proof. Since $\mathrm{G}(I)$ is a Cohen-Macaulay ring by Theorem 1.2 (ii), $\mathcal{R}(I)$ is a Cohen-Macaulay ring if and only if $a(\mathrm{G}(I)) < 0$ ([TI]). By Lemma 2.3 the latter condition is equivalent to saying that $\lceil \frac{q}{\ell} \rceil < d$ (cf. [GSh, Remark (3.10)]). When this is case, $d \geq 2$ because $0 < \lceil \frac{q}{\ell} \rceil$. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. (i) Notice that $\mathrm{G}(I)$ is a Gorenstein ring if and only if so is the graded ring

$$\mathrm{G}(\overline{I}) = \mathrm{G}(I)/(Y_1, Y_2, \cdots, Y_d),$$

where Y_i's stand for the initial forms of $x_i^{a_i}$'s with respect to I. Let r be the index of nilpotency of $\overline{I}$. Then $r = \mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$, and $\mathrm{G}(\overline{I})$ is a Gorenstein ring if and only if the equality

$$(0) : \overline{I}^i = \overline{I}^{r+1-i}$$

holds true for all $i \in \mathbb{Z}$ ([O, Theorem 1.6]). Hence if $\mathrm{G}(I)$ is a Gorenstein ring, we have $(0) : \overline{I} = \overline{I}^r = \overline{\mathfrak{m}}^{r\ell}$. On the other hand, since $\overline{I} = \overline{\mathfrak{m}}^\ell$ and $q = \rho + 1 - \ell$, by Proposition 2.1 we get

$$(0) : \overline{I} = (0) : \overline{\mathfrak{m}}^\ell = \overline{\mathfrak{m}}^q.$$

Therefore $q = r\ell$, since $\overline{\mathfrak{m}}^{r\ell} = \overline{\mathfrak{m}}^q \neq (0)$. Thus $\ell \mid q$ and $r = \frac{q}{\ell}$.

Conversely, suppose that $\ell \mid q$. Hence $r = \frac{q}{\ell}$ by Theorem 1.2 (i). Let $i \in \mathbb{Z}$. Then since $\overline{I} = \overline{\mathfrak{m}}^\ell$, we get $\overline{I}^{r+1-i} = \overline{\mathfrak{m}}^{(r+1-i)\ell}$, while

$$(0) : \overline{I}^i = (0) : \overline{\mathfrak{m}}^{i\ell} = \overline{\mathfrak{m}}^{\rho+1-i\ell}$$

by Proposition 2.1. We then have $(0) : \overline{I}^i = \overline{I}^{r+1-i}$ for all $i \in \mathbb{Z}$, since

$$(r + 1 - i)\ell = q + \ell - i\ell = \rho + 1 - i\ell.$$

Thus $\mathrm{G}(\overline{I})$ is a Gorenstein ring, whence so is $\mathrm{G}(I)$.

(ii) The Rees algebra $\mathcal{R}(I)$ of I is a Gorenstein ring if and only if $\mathrm{G}(I)$ is a Gorenstein ring and $a(\mathrm{G}(I)) = -2$, provided $d \geq 2$ ([I, Corollary (3.7)]). Suppose that $\mathcal{R}(I)$ is a Gorenstein ring. Then $d \geq 2$ by Corollary 2.4. Since $a(\mathrm{G}(I)) = \mathrm{r}_Q(I) - d = -2$, by assertion (i) and Theorem 1.2 (i) we have $\frac{q}{\ell} = \mathrm{r}_Q(I) = d - 2$, whence $q = (d-2)\ell$. Conversely, suppose that $q = (d-2)\ell$. Then $d \geq 3$ since $q \geq 1$. By assertion (i) and Theorem 1.2 (i) $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = \frac{q}{\ell} = d - 2$, whence $a(\mathrm{G}(I)) = (d-2) - d = -2$, so that $\mathcal{R}(I)$ is a Gorenstein ring. □

Example 2.5. Suppose that $\rho \geq 5$ is an odd integer, say $\rho = 2\tau + 1$ with $\tau \geq 2$. Let $q = \rho - 1$. Then $\ell = \rho + 1 - q = 2$. Hence, choosing $a_i \leq 2$ for all $1 \leq i \leq d$, we have $I = Q + \mathfrak{m}^2 \subseteq \overline{Q}$ with $\mathrm{r}_Q(I) = \tau$ by Theorem 1.2. Since $\ell \mid q$, by Theorem 1.3 (i) the ring $\mathrm{G}(I)$ is Gorenstein. The ring $\mathcal{R}(I)$ is by Theorem 1.3 (ii) a Gorenstein ring, if $d = \tau + 2$.

3. Examples and applications

In this section we shall discuss some applications of Theorems 1.2 and 1.3. Let us begin with the case where A is a regular local ring.

3.1. **The case where A is a regular local ring.** Let A be a regular local ring with $x_1, x_2, \cdots, x_d$ a regular system of parameters. Similarly as in the previous sections, let

$$Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d}) \text{ and } I = Q : \mathfrak{m}^q$$

with positive integers $a_1, a_2, \cdots, a_d$, and q. Then $\mathrm{G}(\mathfrak{m}) = k[X_1, X_2, \cdots, X_d]$ is the polynomial ring, where $k = A/\mathfrak{m}$ and X_i's are the initial forms of x_i's, so that we have

$$\rho = \sum_{i=1}^{d} a_i - d \text{ and } \ell = \sum_{i=1}^{d} (a_i - 1) + 1 - q,$$

since $\mathrm{a}(\mathrm{G}(\mathfrak{m})) = -d$. Notice that the condition that

$$\ell \geq \max\,\{a_i \mid 1 \leq i \leq d\}$$

is equivalent to saying that

$$\sum_{j \neq i} a_j \geq q + d - 1$$

for all $1 \leq i \leq d$, because $\ell - a_i = \sum_{j \neq i} a_j - (q + d - 1)$. When this is the case, $d \geq 2$.

Example 3.1. The following assertions hold true.

(1) Let $d = 2$. Then $I \subseteq \overline{Q}$ if and only if $\min\{a_1, a_2\} \geq q + 1$.
(2) Let $d = 3$. Then $I \subseteq \overline{Q}$ if and only if $\min\{a_i + a_j \mid 1 \leq i < j \leq 3\} \geq q + 2$.
(3) Choose integers a and q so that $2 \leq a \leq d$ and $(d-1)(a-1) < q \leq d(a-1)$. Let $a_i = a$ for all $1 \leq i \leq d$. Then $I \subsetneq A$ but $I \not\subseteq \overline{Q}$. For example, let $d = 3, a = 2$, and $q = 3$. Then

$$(x_1^2, x_2^2, x_3^2) : \mathfrak{m}^3 = \mathfrak{m} \not\subseteq \overline{(x_1^2, x_2^2, x_3^2)}.$$

Example 3.2. The following assertions hold true.

(1) Let $d = 2$ and assume that $I \subseteq \overline{Q}$. Then $\mathrm{G}(I)$ is not a Gorenstein ring.
(2) Suppose that $d \geq 3$ and let $n \geq d - 1$ be an integer. Let $a_1 = d - 1$, $a_i = n$ for all $2 \leq i \leq d$, and $q = (d-2)n$. Then $\mathcal{R}(I)$ is a Gorenstein ring.
(3) Suppose that $d = 5$ and let $a_i = 4$ for all $1 \leq i \leq 5$. Let $q = 8$. Then $I \subseteq \overline{Q}$ and $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = 1$, but $\mathcal{R}(I)$ is not a Gorenstein ring.

Since the base ring A is regular, the Cohen-Macaulayness in Rees algebras $\mathcal{R}(I)$ follows from that of associated graded rings $\mathrm{G}(I)$ ([L]). Let us note a brief proof in our context.

Proposition 3.3. *Suppose that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay ring.*

Proof. By Corollary 2.4 we have only to show $\lceil \frac{q}{\ell} \rceil < d$. Let $a_k = \max\{a_i \mid 1 \leq i \leq d\}$. Then because $\ell \geq a_k$, we have

$$\frac{q}{\ell} + 1 = \frac{\rho + 1}{\ell} \leq \frac{\sum_{j=1}^{d}(a_j - 1) + 1}{a_k} = \sum_{j \neq k} \frac{a_j - 1}{a_k} + 1 < d,$$

whence $\lceil \frac{q}{\ell} \rceil < d$ as is wanted. □

Let $L = \{(\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}^d \mid \alpha_i \geq 0 \text{ for all} 1 \leq i \leq d\}$. For each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in L$ we put $x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}$. Let $\mathfrak{a}$ be an ideal in A. Then we say that $\mathfrak{a}$ is a monomial ideal, if $\mathfrak{a}$ is generated by monomials in $\{x_i\}_{1 \leq i \leq d}$, that is $\mathfrak{a} = (x^\alpha \mid \alpha \in \Lambda)$ for some $\Lambda \subseteq L$. Monomial ideals behave very well as if they were monomial ideals in the polynomial ring $k[x_1, x_2, \cdots, x_d]$ over a field k (see [HS] for details). For instance, the integral closure $\overline{Q}$ of our monomial ideal Q is also a monomial ideal and we have the following.

Proposition 3.4 ([HS]). *Let $\Delta = \{\alpha \in L \mid \sum_{i=1}^{d} \frac{\alpha_i}{a_i} \geq 1\}$. Then $\overline{Q} = (x^\alpha \mid \alpha \in \Delta)$.*

Corollary 3.5. *Suppose that $d \geq 2$ and let $n \geq 2$ be an integer. We put $\mathfrak{q} = (x_1^{n-1}, x_2^n, \cdots, x_d^n)$. Then $\overline{\mathfrak{q}} = \mathfrak{q} + \mathfrak{m}^n = (x_1^{n-1}) + \mathfrak{m}^n$ and all the powers $\overline{\mathfrak{q}}^m$ $(m \geq 1)$ are integrally closed.*

Proof. Let $J = \mathfrak{q} + \mathfrak{m}^n$ and $\mathfrak{a} = (x_1^n, x_2^n, \cdots, x_d^n)$. Then $\mathfrak{a} \subseteq \mathfrak{q}$ and $\mathfrak{m}^n \subseteq \overline{\mathfrak{a}}$, so that $J \subseteq \overline{\mathfrak{q}}$. Let $m \geq 1$ be an integer and put $K = (x_1^{m(n-1)}, x_2^{mn}, \cdots, x_d^{mn})$. We will show that $\overline{K} \subseteq J^m$. Let $\alpha \in L$ and assume that $\frac{\alpha_1}{m(n-1)} + \sum_{i=2}^{d} \frac{\alpha_i}{mn} \geq 1$. We want to show that $x^\alpha \in J^m$. We may assume that $\alpha_1 < m(n-1)$. Let $\alpha_1 = (n-1)i + j$ with $i, j \in \mathbb{Z}$ such that $0 \leq j < (n-1)$. Then $0 \leq i < m$. Since $\frac{\alpha_1}{m(n-1)} + \sum_{i=2}^{d} \frac{\alpha_i}{mn} \geq 1$, we get

$$n\alpha_1 + (n-1) \cdot \sum_{i=2}^{d} \alpha_i \geq mn(n-1),$$

so that

$$(n-1) \cdot \sum_{i=2}^{d} \alpha_i \geq mn(n-1) - n\alpha_1 = n[(n-1)(m-i) - j],$$

whence

$$\sum_{i=2}^{d} \alpha_i \geq n(m-i) - \frac{nj}{n-1}.$$

Because $\frac{nj}{n-1} = j + \frac{j}{n-1}$ and $0 \leq j < n-1$, we have $\frac{nj}{n-1} = j + \frac{j}{n-1} < j + 1$ and so

$$\sum_{i=2}^{d} \alpha_i \geq n(m-i) - j.$$

Thus

$$x^\alpha = x_1^{(n-1)i} \cdot x_1^j x_2^{\alpha_2} \cdots x_d^{\alpha_d} \in x_1^{(n-1)i}\mathfrak{m}^{n(m-i)} \subseteq J^m,$$

whence $\overline{K} \subseteq J^m$ by Proposition 3.4.

Because $J^m \subseteq \overline{\mathfrak{q}}^m$ and $\mathfrak{q}^m \subseteq \overline{K}$, we have $J^m \subseteq \overline{\mathfrak{q}}^m \subseteq \overline{\mathfrak{q}^m} \subseteq \overline{K}$, whence $J^m = \overline{\mathfrak{q}}^m = \overline{\mathfrak{q}^m} = K$. Letting $m = 1$, we get $J = \overline{\mathfrak{q}}$. This completes the proof of Corollary 3.5. □

Thanks to Corollary 3.5, we get the following characterization for quasi-socle ideals $I = Q : \mathfrak{m}^q$ to be integrally closed.

Theorem 3.6. *Suppose that $d \geq 2$ and $a_i \geq 2$ for all $1 \leq i \leq d$. Then the following two conditions are equivalent to each other.*

(1) $I = \overline{Q}$.

(2) *Either* (a) $a_i = \ell$ *for all* $1 \leq i \leq d$, *or* (b) *there exists* $1 \leq j \leq d$ *such that* $a_i = \ell$ *if* $i \neq j$ *and* $a_j = \ell - 1$.

When this is the case, $\overline{I^n} = I^n$ *for all* $n \geq 1$, *whence* $\mathcal{R}(I)$ *is a Cohen-Macaulay normal domain.*

Proof. (1) ⇒ (2) Since $I = \overline{Q}$, we get $q \leq \rho$ and $I = Q + \mathfrak{m}^\ell$ (Proposition 2.1). Notice that

$$Q \subseteq I = Q : \mathfrak{m}^q \subsetneq (Q : \mathfrak{m}^q) : \mathfrak{m} = Q : \mathfrak{m}^{q+1},$$

because $I \subsetneq A$. Hence $Q : \mathfrak{m}^{q+1} \not\subseteq \overline{Q}$. Consequently $\ell - 1 = \rho + 1 - (q+1) < a_i$ for some $1 \leq i \leq d$ by Theorem 1.2, so that, thanks to Theorem 1.2 again, we have

$$\ell = a_i \geq a_j$$

for all $1 \leq j \leq d$. Let $\Delta = \{1 \leq j \leq d \mid a_j < \ell\}$. We then have the following.

Claim. (1) $a_j = \ell - 1$, *if* $j \in \Delta$.

(2) $\sharp\Delta \leq 1$.

Proof. Let $j \in \Delta$. Then $a_j < \ell = a_i$ whence $j \neq i$ and $\ell \geq 3$. Let $\alpha = (a_j - 1)\mathbf{e}_j + (a_i - a_j)\mathbf{e}_i$. Then $\alpha \in L$ but, thanks to the monomial property of ideals, $x^\alpha \notin Q + \mathfrak{m}^\ell = I = \overline{Q}$, because $\sum_{k=1}^d \alpha_k = a_i - 1 = \ell - 1$ and $x^\alpha \notin Q$. Consequently, $\sum_{k=1}^d \frac{\alpha_k}{a_k} < 1$ by Proposition 3.4, so that $1 < \frac{1}{a_j} + \frac{a_j}{a_i}$, because

$$\frac{a_j - 1}{a_j} + \frac{a_i - a_j}{a_i} < 1.$$

Let $n = a_i - a_j$. Then $a_j(a_i - a_j) < a_i$ as $1 < \frac{1}{a_j} + \frac{a_j}{a_i}$, whence $a_j n < a_i = a_j + n$ so that $0 \leq (a_j - 1)(n-1) < 1$. Hence $n = 1$ (recall that $a_j \geq 2$) and $a_j = a_i - 1 = \ell - 1$.

Assume $\sharp\Delta \geq 2$ and choose $j, k \in \Delta$ so that $j \neq k$. We put $y = x_j x_k^{\ell-2}$. We then have $y^{\ell-1} = (x_j^{\ell-1})(x_k^{\ell-1})^{\ell-2} = (x_j^{a_j})(x_k^{a_k})^{\ell-2} \in Q^{\ell-1}$, because $a_j = a_k = \ell - 1$ by assertion (1). Hence $y \in \overline{Q} = Q + \mathfrak{m}^\ell$, which is impossible because $y \notin Q$ (recall that $\ell \geq 3$) and $y \notin \mathfrak{m}^\ell$, thanks to the monomial property of ideals. Hence $\sharp\Delta \leq 1$. □

If $\Delta = \emptyset$, we then have $\ell = a_j$ for all $1 \leq j \leq d$. If $\Delta \neq \emptyset$, letting $\Delta = \{j\}$, we get $a_i = \ell$ if $i \neq j$ and $a_j = \ell - 1$. This proves the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Suppose condition (b) is satisfied. Then $I = Q + \mathfrak{m}^\ell = (x_j^{\ell-1}) + \mathfrak{m}^\ell = \overline{Q}$ by Proposition 2.1 and Corollary 3.5. Suppose condition (a) is satisfied. Then $I \subseteq \overline{Q}$ by Theorem 1.2 and $I = Q + \mathfrak{m}^\ell = \mathfrak{m}^\ell$ by Proposition 2.1, whence $I = \overline{Q}$. In each case all the powers of I are integrally closed (see Corollary 3.5 for case (b)), whence the last assertion follows from Proposition 3.3. □

Example 3.7. Suppose that $d \geq 3$ and let $n \geq d - 1$ be an integer. We look at the ideal

$$Q = (x_1^{d-1}, x_2^n, x_3^n, \cdots, x_d^n)$$

and let $q = n(d-2)$. Then $\ell = n$, as $\rho = nd - (n+1)$, whence $I \subseteq \overline{Q}$ and $I = Q + \mathfrak{m}^n = (x_1^{d-1}) + \mathfrak{m}^n$. The ring $\mathcal{R}(I)$ is by Theorem 1.3 (ii) a Gorenstein ring, since $q = (d-2)\ell$. If $n = d$, then $I = (x_1^{d-1}) + \mathfrak{m}^d$ and $\overline{I^m} = I^m$ for all $m \geq 1$ by Corollary 3.5, so that $\mathcal{R}(I)$ is a Gorenstein normal ring.

3.2. **The case where $A = R_M$.** Our setting naturally contains the case where $A = R_M$ is the localization of the homogeneous Gorenstein ring $R = k[R_1]$ over an infinite field $k = R_0$ at the irrelevant maximal ideal $M = R_+$. Let us note one example.

Example 3.8. Let $S = k[X, Y, Z]$ be the polynomial ring over an infinite field k and let $R = S/fS$, where $0 \neq f \in S$ is a form with degree $n \geq 2$. Then R is a homogeneous Gorenstein ring with $\dim R = 2$. Let x_1, x_2 be a linear system of parameters in R and let $M = R_+$. We look at the local ring $A = R_M$. Let $a_1 = 2$, $a_2 = n$, and $q = n$. Let $Q = (x_1^2, x_2^n)A$ and $I = Q : \mathfrak{m}^q$, where $\mathfrak{m} = MA$. Then

$$\rho = \mathrm{a}(R) + (a_1 + a_2) = 2n - 1.$$

Hence $\ell = q = n$, so that $I \subseteq \overline{Q}$, $I = Q + \mathfrak{m}^n = (x_1^2) + \mathfrak{m}^n$, and $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = 1$ (Theorems 1.2 and 1.3). We have $Q \not\subseteq \mathfrak{m}^q$, if $n \geq 3$.

3.3. **The case where $A = k[[t^a, t^b]]$.** Let $1 < a < b$ be integers with $\mathrm{GCD}(a, b) = 1$. We look at the ring $A = k[[t^a, t^b]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal powers series ring over a field k. We put $x = t^a$ and $y = t^b$. Then A is a one-dimensional Gorenstein local ring and $\mathfrak{m} = (x, y)$. Because $A \cong k[[X, Y]]/(X^b - Y^a)$ where $k[[X, Y]]$ denotes the formal powers series ring over the field k, we get

$$\mathrm{G}(\mathfrak{m}) \cong k[X, Y]/(Y^a).$$

Let $n, q \geq 1$ be integers, and put $Q = (x^n)$ and $I = Q : \mathfrak{m}^q$. Then because $a(\mathrm{G}(\mathfrak{m})) = a - 2$, we have $\rho = a + n - 2$ and $\ell = (a + n) - (q + 1)$. Consequently $I \subseteq \overline{Q}$ if and only if $q < a$ (Theorem 1.2), whence the condition that $I \subseteq \overline{Q}$ is independent of the choice of the integer $n \geq 1$. When this is the case, by Theorems 1.2 and 1.3 we have the following.

Theorem 3.9. *The following assertions hold true.*

(1) $\mathrm{r}_Q(I) = \lceil \frac{q}{(a+n)-(q+1)} \rceil$.

(2) *The graded rings* $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are Cohen-Macaulay rings.*

(3) *The ring* $\mathrm{G}(I)$ *is a Gorenstein ring if and only if* $(a+n)-(q+1)$ *divides* q.

Hence, if $q = a-1$, we then have, for each integer $n \geq 1$ such that $n \mid q$, that $\mathrm{G}(I)$ is a Gorenstein ring.

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Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan

E-mail address: goto@math.meiji.ac.jp

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan

E-mail address: skimura@math.meiji.ac.jp

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan

E-mail address: matsuoka@math.meiji.ac.jp

Department of Information Technology and Applied Mathematics, Ton Duc Thang University, 98 Ngo Tat To Street, Ward 19, Binh Thanh District, Ho Chi Minh City, Vietnam

E-mail address: sugarphuong@gmail.com

A FAMILY OF GRADED MODULES ASSOCIATED TO A MODULE

FUTOSHI HAYASAKA

This note is a summary of the paper [9] with E. Hyry (University of Tampere). In this note we introduce a certain family of graded modules associated to a given module. These modules provide a natural extension of the notion of the associated graded ring of an ideal. We will investigate their properties. In particular, we will try to extend the Rees theorem on the associated graded ring of an ideal generated by a regular sequence to this context.

1. Introduction

Let A be a commutative ring and let J be an ideal in A. In 1957, Rees proved in [19] that the associated graded ring

$$G(J) = A/J \oplus J/J^2 \oplus J^2/J^3 \oplus \cdots$$

of J is isomorphic to the polynomial ring over A/J, if the ideal J is generated by a regular sequence on A. In particular, the module $J^\ell/J^{\ell+1}$ is A/J-free for all $\ell \geq 0$. Rees's theorem is a key result for the applications of the associated graded ring in commutative algebra and algebraic geometry.

Recently many authors have investigated graded structures associated to modules, especially in connection with the theory of Buchsbaum-Rim multiplicities. Several results valid in the ideal case have been extended to the module case (for example, see [2, 12, 13, 15, 17, 18, 20, 21]). However, a good notion of an associated graded ring of a module satisfying a suitable version of Rees's theorem seems to be lacking.

Two possible candidates for the associated graded ring of a module appear in the article [12] of Katz and Kodiyalam. Let A be a Noetherian ring and let F be a free A-module of rank $r > 0$. Let M be a submodule of F and let $R = \mathcal{R}(M)$ be the Rees algebra of M, which is the subalgebra of the polynomial ring $S = \mathrm{Sym}_A(F)$ defined as the image of the natural homomorphism $\mathrm{Sym}_A(M) \to \mathrm{Sym}_A(F)$. Let $I(M)$ be the 0-th Fitting ideal $\mathrm{Fitt}_0(F/M)$ of F/M. Katz and Kodiyalam investigated the graded $A/I(M)$-algebra

$$R/I(M)R = A/I(M) \oplus M/I(M)M \oplus M^2/I(M)M^2 \oplus \cdots ,$$

where M^ℓ denotes the homogeneous component R_ℓ of degree ℓ in R. When $r = 1$, the ring $R/I(M)R$ is exactly the associated graded ring of the ideal M in A. In order to study the Buchsbaum-Rim polynomial, they also introduced a graded $R/I(M)R$-module, namely

$$RF/R^+ = F/M \oplus MF/M^2 \oplus M^2F/M^3 \oplus \cdots .$$

When $r = 1$, this module coincides with the ordinary associated graded ring of the ideal M in A. They observe in the proof of [12, Proposition 3.4] that the module $M^\ell F/M^{\ell+1}$ is a direct sum of $\binom{\ell+r}{r}$-copies of F/M, if A is a two dimensional regular local ring and M is a parameter module in F (i.e., the length of F/M is finite and the number of generators of M is just $r+1$). This can be viewed as the module version of Rees' theorem. The goal of this note is to generalize this observation as follows:

Theorem 1.1. *Let A be a Noetherian ring and let F be a free A-module of rank $r > 0$. Let M be a submodule of F such that $\widetilde{M}$ is a perfect matrix of size $r \times (r+1)$. Then the natural surjective homomorphism*

$$(F/M)[Y_1, \ldots, Y_{r+1}] \twoheadrightarrow RF/R^+$$

of $R/I(M)R$-modules is an isomorphism. In particular, the A-module $M^\ell F/M^{\ell+1}$ is a direct sum of $\binom{\ell+r}{r}$-copies of F/M for all $\ell \geq 0$.

Here $\widetilde{M}$ denotes the matrix whose columns correspond to the generators of M with respect to a fixed basis of F. Moreover, we say that the matrix $\widetilde{M}$ *perfect* if $I(M)$ is a proper ideal having the maximal possible grade.

As a corollary, we have the following.

Corollary 1.2 (cf. [12]). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two. Let $F = A^r$ be a free module of rank $r > 0$ and M a parameter module in F. Then the map*

$$\varphi_1 : (F/M)[Y_1, \ldots, Y_{r+1}] \twoheadrightarrow RF/R^+$$

is an isomorphism and hence the module $M^\ell F/M^{\ell+1}$ is a direct sum of $\binom{\ell+r}{r}$-copies of F/M for all $\ell \geq 0$.

This is a direct consequence of Theorem 1.1. Since A is a two-dimensional Cohen-Macaulay local ring, the matrix $\widetilde{M}$ of a parameter module M is perfect of size $r \times (r+1)$. Thus the assertion follows from Theorem 1.1.

In section 2, inspired by the article of Katz and Kodiyalam, we are going to introduce a family of graded $R/I(M)R$-modules $\{G_t(M)\}_{t\geq 0}$, where

$$G_t(M) = S_t/MS_{t-1} \oplus MS_t/M^2S_{t-1} \oplus M^2S_t/M^3S_{t-1} \oplus \cdots .$$

This includes the above two graded modules $R/I(M)R = G_0(M)$ and $RF/R^+ = G_1(M)$. We can then ask when the natural surjective homomorphisms $\varphi_t : (S_t/MS_{t-1})[Y_1, \ldots, Y_n] \to G_t(M)$ are isomorphisms. In section 3, we will discuss the generic case. It turns out that in the generic case the maps φ_t are always isomorphisms (see Proposition 3.1). We can then show that the general case can be reduced to this case provided that a certain condition $\mathcal{P}_t$ holds in the generic case (see Theorem 3.2). Finally, in section 4, we will prove in Theorem 4.1 that the condition $\mathcal{P}_1$ holds true in the case of a generic $r \times (r+1)$ matrix. This will imply our main Theorem 1.1.

2. The associated graded modules of a module

Let A be a Noetherian ring and let F be a free A-module with a basis $\{t_1, \ldots, t_r\}$. Let M be a submodule of F with generators $c_1, \ldots, c_n$ and the matrix $\widetilde{M} = (c_{ij})$. We put $I(M) = I_r(\widetilde{M}) = \mathrm{Fitt}_0(F/M)$. Let $S = A[t_1, \ldots, t_r]$ be the symmetric algebra of F. Let $R = \mathcal{R}(M)$ be the Rees algebra of M, which is now the A-subalgebra of S generated by $c_1, \ldots, c_n$. For each integer $\ell \geq 0$, we denote by M^ℓ the homogeneous component R_ℓ of degree ℓ in R. We always understand products and powers of modules to be taken inside the symmetric algebra S of our fixed free module F. We put $R^+ = \oplus_{\ell>0} M^\ell$.

Let $\mathcal{R}(MS)$ be the Rees algebra of the ideal MS in S, which is the S-subalgebra $S[c_1T, \ldots, c_nT]$ of the polynomial ring $S[T]$. The ring $\mathcal{R}(MS)$ becomes a bi-graded A-algebra by letting $\deg t_i = (0,1)$, $\deg c_jT = (1,0)$. That is,

$$\mathcal{R}(MS) \cong \bigoplus_{p,q\geq 0} M^pS_q.$$

Now consider the bi-graded $A/I(M)$-algebra

$$\mathcal{G} = \mathcal{G}(MS) \otimes_A (A/I(M))$$

where $\mathcal{G}(MS) = \mathcal{R}(MS)/(MS)\mathcal{R}(MS)$ is the associated graded ring of MS. So

$$\mathcal{G} \cong \bigoplus_{p,q\geq 0} M^p S_q / M^{p+1} S_{q-1}.$$

Here we set $MS_{-1} = I(M)$.

Definition 2.1. *For any non-negative integer $t \geq 0$, we set*

$$G_t(M) = \bigoplus_{p\geq 0} \mathcal{G}_{(p,t)} = RS_t/R^+S_{t-1},$$

and call the module $G_t(M)$ as the associated graded module of M of type t

Let $S[Y_1, \ldots, Y_n]$ be a polynomial ring over S with $\deg Y_j = (1,0)$. Consider the bi-graded homomorphism

$$\phi : (S/MS)[Y_1, \ldots, Y_n] \twoheadrightarrow \mathcal{G} \ ; \ \ Y_j \mapsto c_j + (MS)^2 \ .$$

Taking the degree $(*, t)$-part, we obtain the homomorphism

$$\varphi_t : (S_t/MS_{t-1})[Y_1, \ldots, Y_n] \twoheadrightarrow G_t(M),$$

of graded $(A/I(M))\,[Y_1, \ldots, Y_n]$-modules. With this notation, our problem is now the following.

Problem 2.2. *Let M be a submodule of F. Assume that the matrix $\widetilde{M}$ is perfect of size $r \times n$ (i.e., $I(M)$ is a proper ideal and* $\operatorname{grade} I(M) =$ *$n - r + 1$). Is the map φ_t then an isomorphism?*

We note here that modules with a perfect matrix are called complete intersection modules in [21]. This problem can be reduced to the generic case provided that a certain condition $\mathcal{P}_t$ holds in the generic case.

3. The reduction to the generic case

In this section we will reduce Problem 2.2 to the generic case. Let $X = (X_{ij})$ be a generic matrix of size $r \times n$ and let

$$B = \mathbb{Z}[X] = \mathbb{Z}[X_{ij} \mid 1 \leq i \leq r, \ 1 \leq j \leq n]$$

be the corresponding polynomial ring over the ring of integers $\mathbb{Z}$. Let $G = B^r$ be a free module of rank r and let $N \subseteq G$ be a submodule of G such that $\widetilde{N}$ is the generic matrix X. Let $V = \mathrm{Sym}_B(G) = B[t_1, \ldots, t_r]$ be the polynomial ring over B and let $U = \mathcal{R}(N)$ be the Rees algebra of N. Let

$$x_j = X_{1j}t_1 + \cdots + X_{rj}t_r \in V, \quad (j = 1, \ldots, n)$$

be the generators of N. For a generic matrix of an arbitrary size, one can check that the sequence $x_1, \ldots, x_n$ form a d-sequence on V and hence the ideal NV is of linear type ([22]). Furthermore, in the generic case, we have the following.

Proposition 3.1. *For any integer $t \geq 0$, the map*

$$\varphi_t : (V_t/NV_{t-1})[Y_1, \ldots, Y_n] \twoheadrightarrow G_t(N) = UV_t/U^+V_{t-1}$$

is an isomorphism. In particular, the natural surjective homomorphism

$$[\varphi_t]_\ell : (V_t/NV_{t-1})^{\binom{\ell+n-1}{n-1}} \twoheadrightarrow N^\ell V_t/N^{\ell+1}V_{t-1}$$

is an isomorphism of B-modules for all $\ell \geq 0$.

We now consider the following condition

$\mathcal{P}_t$: *The B-module $V_{\ell+t}/N^\ell V_t$ is perfect of grade $n - r + 1$ for all $\ell > 0$.*

We will see in the next Theorem 3.2 that if this condition holds true, then the general case of our Problem 2.2 can be reduced to the generic case. For this, we recall here that A is a Noetherian ring, F is a free A-module of rank $r > 0$ and S is the polynomial ring $A[t_1, \ldots, t_r]$.

Theorem 3.2. *Let M be a submodule of F with a perfect matrix $\widetilde{M}$ of size $r \times n$. Let $0 \leq t$ ($\leq n - r$) be a fixed integer. If condition $\mathcal{P}_t$ holds true, then the map*

$$\varphi_t : (S_t/MS_{t-1})[Y_1, \ldots, Y_n] \twoheadrightarrow G_t(M)$$

is an isomorphism.

Here we give some remarks on condition $\mathcal{P}_t$.

Remark 3.3. We do not know whether condition $\mathcal{P}_t$ holds true or not, except for the following cases.

(1) The free resolution of V_{t+1}/NV_t is given by the generalized Koszul complex (see [4, 14] and also [8, Appendix A2.6]). So the length of this resolution is just $n-r+1$ when $0 \leq t \leq n-r$. However, when $t \geq n-r+1$, the resolution is at least of length $n-r+2$. Hence condition $\mathcal{P}_t$ does not hold when $t \geq n-r+1$.
(2) When $t = 0$, it is known that condition $\mathcal{P}_0$ holds true. This follows from the theorem of Buchsbaum-Eisenbud [5, Corollary 3.2] (see also [13, Proposition 3.3]). Hence φ_0 is an isomorphism if the matrix is perfect (cf. [13, Lemma 3.2]).
(3) When $n = r+1$, condition $\mathcal{P}_1$ holds true. This will be proved in the next section (Theorem 4.1).

4. A sketch of proof of Theorem 1.1

In this final section we will give a sketch of proof of Theorem 1.1. Let $X = (X_{ij})$ be a generic matrix of size $r \times n$ and let

$$B = K[X] = K[X_{ij} \mid 1 \leq i \leq r,\ 1 \leq j \leq n]$$

be the corresponding polynomial ring over an arbitrary commutative Noetherian ring K. Let $G = B^r$ be a free module of rank r and let $N \subseteq G$ be a submodule of G such that $\widetilde{N}$ is the generic matrix X. Identify $\mathrm{Sym}_B(G)$ with the polynomial ring $V = B[t_1, \ldots, t_r]$ and let $U = \mathcal{R}(N)$ be the Rees algebra of N. By Theorem 3.2, it is enough to show that condition $\mathcal{P}_1$ (stated before Theorem 3.2) holds true when $n = r+1$.

Theorem 4.1. *Suppose that $n = r+1$. Then the module $V_{\ell+1}/N^\ell V_1$ is a perfect B-module of grade 2 for all $\ell > 0$. In particular, condition $\mathcal{P}_1$ (stated before Theorem 3.2) holds true.*

Let me give a sketch of proof of Theorem 4.1. We assume in the following that $n = r+1$. We put $x_j = X_{1j}t_1 + \cdots + X_{rj}t_r$ and set $I = (x_1, \ldots, x_{r+1})V = NV$. Let $\mathcal{R} = \mathcal{R}(I)$ be the Rees algebra of I, which is the V-subalgebra $V[x_1T, \ldots, x_{r+1}T]$ of the polynomial ring $V[T]$. We regard the ring $V[T]$ as a bi-graded B-algebra by letting $\deg t_i = (0,1)$, $\deg T = (1,-1)$. Then the Rees algebra $\mathcal{R}$ becomes a bi-graded B-subalgebra of $V[T]$. Note that the B-module $N^\ell V_t$ is now isomorphic to the homogeneous component $\mathcal{R}_{\ell,t}$ of $\mathcal{R}$ for all $\ell, t \geq 0$. We thus want to show the following:

$$\text{(1)} \qquad \operatorname{pd}_B \mathcal{R}_{p,1} \le 1 \quad \text{for all} \quad p > 0.$$

Let $\mathcal{S} = V[Y_1, \dots, Y_{r+1}]$ be a polynomial ring over V. We regard $\mathcal{S}$ as a bi-graded B-algebra by setting $\deg Y_j = (1,0)$ for all $j = 1, \dots, r+1$. We now have the bi-graded presentation $\varepsilon : \mathcal{S} \to \mathcal{R}$; $Y_j \mapsto x_j T$ of $\mathcal{R}$. To show the inequality (1), we will first construct a graded $\mathcal{S}$-free resolution of $\mathcal{R}$. This will be achieved by using the $\mathcal{Z}$-complex, which was introduced by Herzog-Simis-Vasconcelos in [11]. We refer the reader to [10, section 3] for a similar approach.

Let $K_\bullet(\underline{x}; \mathcal{S})$ and $K_\bullet(\underline{Y}; \mathcal{S})$ be the Koszul complexes associated to sequences $\underline{x} = x_1, \dots, x_{r+1}$ and $\underline{Y} = Y_1, \dots, Y_{r+1}$ in $\mathcal{S}$, respectively. We denote by d_x and d_Y the corresponding differentials. These complexes become bi-graded complexes:

$$\begin{array}{llllllll} K_\bullet(\underline{x}; \mathcal{S}): & \cdots & \xrightarrow{d_x} & K_{i+1}(0, -i-1) & \xrightarrow{d_x} & K_i(0, -i) & \xrightarrow{d_x} & \cdots, \\ K_\bullet(\underline{Y}; \mathcal{S}): & \cdots & \xrightarrow{d_Y} & K_{i+1}(-i-1, 0) & \xrightarrow{d_Y} & K_i(-i, 0) & \xrightarrow{d_Y} & \cdots, \end{array}$$

where $K_i = \wedge^i \mathcal{S}^{r+1}$. Let

$$\mathcal{Z}_i = \operatorname{Ker}(K_i(0, -i) \xrightarrow{d_x} K_{i-1}(0, -i+1))$$

be the i-th module of cycles of $K_\bullet(\underline{x}; \mathcal{S})$, which is a graded submodule of $K_i(0, -i)$. Since $d_x \circ d_Y + d_Y \circ d_x = 0$, $\{\mathcal{Z}_\bullet, d_Y\}$ is a subcomplex of $K_\bullet(\underline{Y}; \mathcal{S})$. This complex is called the $\mathcal{Z}$-complex associated to a sequence $\underline{x}$ and denoted by $\mathcal{Z}_\bullet(\underline{x})$ ([11]). Note that the 0-th homology $H_0(\mathcal{Z}_\bullet(\underline{x})) \cong \operatorname{Sym}_V(I)$. Since $\underline{x}$ is a d-sequence on V, $\mathcal{Z}_\bullet(\underline{x})$ is acyclic with the 0-th homology $H_0(\mathcal{Z}_\bullet(\underline{x})) \cong \mathcal{R}$ ([11, Theorem 5.4]). Hence we have the graded exact sequence

$$0 \to \mathcal{Z}_{n-1}(-n+1, n-1) \xrightarrow{d_Y} \cdots \xrightarrow{d_Y} \mathcal{Z}_2(-2,2) \xrightarrow{d_Y} \mathcal{Z}_1(-1,1) \xrightarrow{d_Y} \mathcal{S} \xrightarrow{\varepsilon} \mathcal{R} \to 0.$$

If we now resolve each of the modules $\mathcal{Z}_i(-i, i)$ by a certain (graded) complex $P_{\bullet i}$ and lift the differentials $\mathcal{Z}_i(-i,i) \xrightarrow{d_Y} \mathcal{Z}_{i-1}(-i+1, i-1)$ to maps $P_{\bullet i} \to P_{\bullet i-1}$ of complexes, then the associated double complex will give us an $\mathcal{S}$-free resolution of $\mathcal{R}$.

It is well-known that $x_1, \dots, x_r$ is a regular sequence on V (see [1, Proposition 1]). Hence we have for all $i \ge 2$ the exact sequence

$$0 \to K_{r+1}(-i, i-r-1) \xrightarrow{d_x} \cdots \xrightarrow{d_x} K_{i+1}(-i, -1) \xrightarrow{d_x} \mathcal{Z}_i(-i, i) \to 0.$$

This gives us a graded $\mathcal{S}$-free resolution $P_{\bullet i}$ of $\mathcal{Z}_i(-i,i)$ for $i \geq 2$.

In order to find a resolution of $\mathcal{Z}_1(-1,1)$, we need to introduce some more notation. Let $K_\bullet(\underline{t};\mathcal{S})$ be the Koszul complex of the sequence $\underline{t} = t_1,\dots,t_r$ in S with the differential d_t. We interpret it as bi-graded complex

$$K_\bullet(\underline{t};\mathcal{S}): \ \cdots \ \xrightarrow{d_t} \ L_{i+1}(0,-i-1) \ \xrightarrow{d_t} \ L_i(0,-i) \ \xrightarrow{d_t} \ \cdots,$$

where $L_i = \wedge^i \mathcal{S}^r$. Since $[t_1 \ \cdots \ t_r] \circ X = [x_1 \ \cdots \ x_{r+1}]$, there is a map of complexes $\wedge X : K_\bullet(\underline{x};\mathcal{S}) \longrightarrow K_\bullet(\underline{t};\mathcal{S})$. Taking the $\mathcal{S}$-duals, this gives a map $\wedge^t X : K^\bullet(\underline{t};\mathcal{S}) \longrightarrow K^\bullet(\underline{x};\mathcal{S})$ of complexes, where ${}^tX$ denotes the transpose of X. Identifying the Koszul complexes with their duals, we obtain a map of complexes

$$f_\bullet : K_\bullet(\underline{t};\mathcal{S})(0,-1) \to K_\bullet(\underline{x};\mathcal{S})[1],$$

where each $f_i\colon L_i(0,-i-1) \to K_{i+1}(0,-i-1)$ is a graded homomorphism of degree zero. More explicitly, setting $\Delta_j = (-1)^{j-1}\det X_j$ where X_j is the matrix obtained by deleting j-th column of X, one can check, for example, that $f_0 = {}^t\,[\Delta_1 \ \Delta_2 \ \cdots \ \Delta_{r+1}]$ and f_r is the identity map.

We then have the following.

Proposition 4.2. *With the notation above, there is the following graded resolution of $\mathcal{Z}_1(-1,1)$:*

$$0 \to L_{r-1}(-1,-r+1) \xrightarrow{d_{r-1}} \begin{matrix} K_r(-1,-r+1) \\ \oplus \\ L_{r-2}(-1,-r+2) \end{matrix} \xrightarrow{d_{r-2}} \cdots \xrightarrow{d_1} \begin{matrix} K_2(-1,-1) \\ \oplus \\ L_0(-1,0) \end{matrix} \xrightarrow{d_0} \mathcal{Z}_1(-1,1) \to 0,$$

where $d_0 = (d_x \ f_0)$, $d_{r-1} = \begin{pmatrix} f_{r-1} \\ -d_t \end{pmatrix}$, and $d_i = \begin{pmatrix} d_x & f_i \\ 0 & -d_t \end{pmatrix}$ for all $1 \leq i \leq r-2$.

Let $P_{\bullet 1}$ be the above graded resolution of $\mathcal{Z}_1(-1,1)$. Look at the following commutative diagram.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & 0 & & & \\
 & & \uparrow & & \uparrow & & & \uparrow & & & \\
\cdots & \xrightarrow{d_Y} & \mathcal{Z}_3(-3,3) & \xrightarrow{d_Y} & \mathcal{Z}_2(-2,2) & & \xrightarrow{d_Y} & \mathcal{Z}_1(-1,1) & \xrightarrow{d_Y} & S & \to 0. \\
 & & \uparrow & & \uparrow & & & \uparrow & & & \\
\cdots & \xrightarrow{\pm d_Y} & P_{\bullet 3} & \xrightarrow{\pm d_Y} & P_{\bullet 2} & & \xrightarrow{\begin{pmatrix} \pm d_Y \\ 0 \end{pmatrix}} & P_{\bullet 1} & & &
\end{array}$$

Let $\mathrm{Tot}(P_{\bullet\bullet})$ be the total complex of the resulting double complex $P_{\bullet\bullet}$. Now consider the corresponding spectral sequence. Since $\mathcal{Z}_\bullet(\underline{x})$ is acyclic with $H_0(\mathcal{Z}_\bullet(\underline{x})) \cong \mathcal{R}$, standard arguments yield the following graded $\mathcal{S}$-free resolution of $\mathcal{R}$:

$$\mathrm{Tot}(P_{\bullet\bullet}) \to \mathcal{S} \to \mathcal{R} \to 0.$$

Note that this resolution gives us the defining equations of $\mathcal{R}$:

$$\mathcal{K} = I_2\begin{pmatrix} x_1 & x_2 & \cdots & x_{r+1} \\ Y_1 & Y_2 & \cdots & Y_{r+1} \end{pmatrix} + I_{r+1}\begin{pmatrix} Y_1 \quad \cdots \quad Y_{r+1} \\ \hline \\ X \\ \\ \end{pmatrix}.$$

Hence, by [3, Theorem (3.3)], it suffices to prove the inequality (1) in the case $K = \mathbb{Z}/\mathfrak{p}\mathbb{Z}$ where $\mathfrak{p}$ is a prime. In other words we can assume that $B = K[X]$ where K is a field.

When $r = 1$, the assertion is immediate from the above resolution. In the following we therefore assume that $r \geq 2$. Using the above resolution of $\mathcal{R}$, we can compute $\mathrm{Tor}_i^B(K, \mathcal{R})$ for $i \geq 2$ as follows:

$$\mathrm{Tor}_i^B(K,\mathcal{R}) \cong H_{i-1}(\mathrm{Tot}(\overline{P}_{\bullet\bullet})) \cong \begin{cases} \mathcal{B}_i(i,-i) & (2 \leq i \leq r-1) \\ \overline{L}_r(-1,-r) & (i = r). \end{cases}$$

where $\overline{*} = * \otimes_B K$ and $\mathcal{B}_i = \mathrm{Im}(\overline{K}_{i+1}(-i-1,0) \xrightarrow{d_Y} \overline{K}_i(-i,0)) \subseteq \overline{K}_i(-i,0)$ is the i-th module of boundaries of the complex $K_\bullet(\underline{Y};\overline{S})$. Hence we have $\mathrm{Tor}_i^B(K, \mathcal{R}_{p,1}) = (0)$ for all $i \geq 2$, because

$$\begin{cases} [\mathcal{B}_i]_{p+i,q-i} = (0) & \text{if } q < i \\ \left[\overline{L}_r\right]_{p-1,q-r} = (0) & \text{if } q < r. \end{cases}$$

Consequently, we have $\mathrm{pd}_B\, \mathcal{R}_{p,1} \leq 1$. This completes the proof of Theorem 4.1 and hence we have Theorem 1.1. □

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Department of Mathematics, School of Science and Technology, Meiji University, 214-8571 JAPAN
E-mail address: `fhayasaka@math.meiji.ac.jp`

A modification of Ikeda's theorem

Shin-ichiro Iai

1 Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of A with $\operatorname{grade} I \geq 2$. Assume that A is a homomorphic image of a Gorenstein local ring and that the field $A/\mathfrak{m}$ is infinite. Let t be an indeterminate over A. We define $\mathrm{R}(I) := A[It] \subseteq A[t]$, $\mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}]$, and $\mathrm{G}(I) := \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I)$ and call them respectively the Rees algebra, the extended Rees algebra, and the associated graded ring of I. Let $\mathrm{K}_{\mathrm{R}(I)}$, $\mathrm{K}_{\mathrm{R}'(I)}$, and $\mathrm{K}_{\mathrm{G}(I)}$ denote the graded canonical modules of $\mathrm{R}(I)$, $\mathrm{R}'(I)$, and $\mathrm{G}(I)$, respectively. Let $\mathrm{a}(\mathrm{G}(I))$ stand for the a-invariant of $\mathrm{G}(I)$. We always assume A is a quasi-Gorenstein ring, which means that the canonical module of A is a free A-module of rank 1. The purpose of this paper is to prove the following result, which is a modification of theorem given by Ikeda [I].

Theorem 1.1. *Assume that* $\mathrm{R}(I)$ *is a Cohen-Macaulay ring and* $\mathrm{a}(\mathrm{G}(I)) = -2$. *Then the following two conditions are equivalent.*

(1) $\mathrm{R}(I)$ *is a Gorenstein ring.*

(2) $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(I)(-1)$ *as graded* $\mathrm{R}'(I)$*-modules.*

Let us give some consequences of the theorem above. We define $\widetilde{I} := \cup_{n\geq 0} I^{n+1} : I^n$, which is called the Ratliff-Rush closure of I. We set $\mathcal{F} = \{\widetilde{I^i}\}_{i\in\mathbb{Z}}$ and $\mathrm{R}'(\mathcal{F}) := \sum_{i\in\mathbb{Z}} \widetilde{I^i} t^i \subseteq A[t, t^{-1}]$. Let k be a positive integer. With this notation, the first corollary can be stated as follows.

Corollary 1.2. *Assume that* $\mathrm{R}(I^k)$ *is a Cohen-Macaulay ring and* $\mathrm{a}(\mathrm{G}(I^k)) = -2$. *Then the following two conditions are equivalent.*

(1) $\mathrm{R}(I^k)$ *is a Gorenstein ring.*

(2) $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(\mathcal{F})(-k)$ *as graded* $\mathrm{R}'(I)$*-modules.*

To state the second corollary of the theorem, we set up some notation. We put $d = \dim A$. Let $\mathfrak{a}(A) := \prod_{i=0}^{d-1} (0) :_A \mathrm{H}^i_{\mathfrak{m}}(A)$ and let $\mathrm{NCM}(A) := \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}}$ is not a Cohen-Macaulay ring$\}$. Then $\mathrm{NCM}(A) = \mathrm{V}(\mathfrak{a}(A))$. Put $s = \dim \mathrm{NCM}(A)$

and there is a system of parameters $x_1, x_2, \dots, x_d$ for A such that $x_{s+1}, x_{s+2} \dots, x_d \in \mathfrak{a}(A)$. For each $i \le s$, we put $J_i := (x_{i+1}, x_{i+2}, \dots, x_d)$. Then we have

Corollary 1.3. *Assume that $d \ge 2$. Let $s = 0$ and $I = J_0$. Then the following two conditions are equivalent.*

(1) *$\mathrm{R}(I^k)$ is a Gorenstein ring.*

(2) *A is a Cohen-Macaulay ring and $k = d - 1$.*

The implication (2) ⇒ (1) is already known (see [O], 4.3). The converse implication (1) ⇒ (2) is a result in this paper. The third one is the following

Corollary 1.4. *Assume that $d \ge 3$. Let $s \le 1$ and $I = \bigcup_{\ell \ge 0} J_1 : x_1^\ell$. Then the following two conditions are equivalent.*

(1) *$\mathrm{R}(I^k)$ is a Gorenstein ring.*

(2) *A has finite local cohomology modules and $k = d - 2$.*

The implication (2) ⇒ (1) is already shown in the last symposium. The converse implication (1) ⇒ (2) is a result in this paper.

2 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. We put $R = \mathrm{R}(I)$, $R' = \mathrm{R}'(I)$, and $G = \mathrm{G}(I)$. To begin with we note

Lemma 2.1. *Let a be an integer and let $\kappa = \{\kappa_i\}_{i \ge -a-1}$ be an I-filtration of A such that $\kappa_{-a-1} = A$ and $\kappa_{-a-1} \supsetneq \kappa_{-a}$. Set $\mathrm{gr}_A(\kappa) = \bigoplus_{i \ge -a} \kappa_{i-1}/\kappa_i$ that is a graded G-module. If there is an embedding $G(a) \hookrightarrow \mathrm{gr}_A(\kappa)$ of graded G-modules, then $\kappa_i = I^{i+a+1}$ for all integers $i \ge -a-1$.*

Proof. See the proof of Theorem 3.2 in the paper [GI]. □

Let the ideal I be generated by elements $a_1, a_2, \dots, a_n \in A$. We may assume a_1 is an regular element of A. Let $X_1, X_2, \dots X_n, Y$ are indeterminates over A. We consider the A-algebra homomorphisms

$$\varphi : A[X_1, X_2, \dots X_n] \to R$$

such that $\varphi(X_i) = a_i t$ for all $1 \le i \le n$ and

$$\varphi' : A[X_1, X_2, \dots X_n, Y] \to R'$$

such that $\varphi'(X_i) = a_i t$ for all $1 \le i \le n$ and $\varphi'(Y) = t^{-1}$. Put $P = A[X_1, X_2, \dots X_n]$ and $F_i = X_i Y - a_i$. Then we get the following equality.

Claim 2.2. $\ker\varphi' = (F_1, F_2, \ldots F_n)P[Y] + \ker\varphi P[Y]$.

Proof. Take any element $F \in \ker\varphi'$. Dividing F by $F_1, F_2, \ldots F_n$, we can write $F = \sum_{i=1}^{n} Q_i F_i + H + H'$, where $Q_i \in P[Y]$, $H \in P$, and $H' \in A[Y]$. Then $\varphi'(F) = H(a_1 t, a_2 t, \ldots, a_n t) + H'(t^{-1})$, which is an element of $A[t, t^{-1}]$. Since $\varphi'(F) = 0$, we get $H' \in A$, and hence $H + H' \in \ker\varphi$. □

Set $f_i = a_i tY - a_i$, which is an element of $R[Y]$. We note f_1 is a regular element on $R[Y]$ because so is a_1. Look at the graded R-algebra homomorphism

$$\psi : R[Y] \to R'$$

induced by the injection $R \to R'$ of graded rings such that $\psi(Y) = t^{-1}$. Then the claim above implies the following equality.

Lemma 2.3. $\ker\psi = (f_1, f_2, \ldots f_n)$.

Therefore we get the exact sequence $0 \to \dfrac{\ker\psi}{f_1 R[Y]} \to \dfrac{R[Y]}{f_1 R[Y]} \to R' \to 0$ of graded $R[Y]$-modules. Let us now prove the theorem.

Proof of Theorem 1.1. Assume that R is a Gorenstein ring. Then $\mathrm{K}_{R[Y]} \cong R[Y](m)$ for some $m \in \mathbb{Z}$. Taking the $\mathrm{K}_{R[Y]}$-dual of the graded exact sequence

$$0 \to R[Y](1) \xrightarrow{Y} R[Y] \to R \to 0$$

of graded $R[Y]$-modules, we get the graded exact sequence

$$0 \to R[Y](m) \xrightarrow{Y} R[Y](m-1) \to R(-1) \to 0$$

of graded $R[Y]$-modules because $\mathrm{K}_R \cong R(-1)$ as graded R-modules. Therefore $m = 0$. Put $S = \frac{R[Y]}{f_1 R[Y]}$ and we obtain that S is a Gorenstein graded ring with $\mathrm{K}_S \cong S$ as graded S-modules (recall that $\deg f_1 = 0$). Since Y is a regular element on S, we have $\mathrm{K}_{S/YS} \cong [S/YS](-1)$ as graded S-modules. Put $L = \frac{\ker\psi}{f_1 R[Y]}$. Then $R' \cong S/L$ and $G \cong S/YS + L$ as graded rings, so that $\mathrm{K}_{R'} \cong \mathrm{Hom}_S(S/L, S)$ and $\mathrm{K}_G \cong \mathrm{Hom}_{S/Y}(S/YS + L, [S/YS](-1))$ as graded S-modules. Hence we get $\mathrm{K}_{R'} \cong (0) :_S L$ and $\mathrm{K}_G \cong [YS :_S L/YS](-1)$ as graded S-modules. We can check

Claim 2.4. $YS :_S L = [(0) :_S L] + YS$.

The above claim implies that the natural map $\pi : (0) :_S L \to [YS :_S L]/YS$ is surjective. And we have $\ker\pi = Y[(0) :_S L]$. Therefore $\mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'} \cong \mathrm{K}_G(1)$ as graded G-modules. Thanks to [I], we get $\mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'} \cong G(-1)$ as graded G-modules. Let $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ stand for the canonical I-filtration of A (see [GI], 1.1 and notice that the canonical filtration exists if the base ring A satisfies Serre's condition (S_2)), namely the I-filtration ω fulfills $I^i \subseteq \omega_{i-a-1}$ for all $i \in \mathbb{Z}$ and

$K_{R'(I)} \cong \sum_{i\in\mathbf{Z}} \omega_i t^i \subseteq A[t,t^{-1}]$ as graded $R'(I)$-modules. Therefore we get $\omega_{i+1} = I^i$ by Lemma 2.1, and hence $K_{R'(I)} \cong R'(I)(-1)$ as graded $R'(I)$-modules.

Conversely, assume that $K_{R'(I)} \cong R'(I)(-1)$ as graded $R'(I)$-modules. Hence $K_{R'}/t^{-1}K_{R'} \cong G(-1)$ as graded G-modules. The embedding $K_{R'}/t^{-1}K_{R'}(-1) \hookrightarrow K_G$ follows from the exact sequence $0 \to R'(1) \xrightarrow{t^{-1}} R' \to G \to 0$ of graded R'-modules, so that we can find an embedding $G(-2) \hookrightarrow K_G$ of graded R'-modules. By [TVZ], there is an I-filtration $\kappa = \{\kappa_i\}_{i\geq 0}$ of A such that $\kappa_1 = A$, $\kappa_1 \supsetneq \kappa_2$, $K_R \cong \bigoplus_{i\geq 1} \kappa_i$ as a graded R-module, and $K_G = \bigoplus_{i\geq 2} \kappa_{i-1}/\kappa_i$ as a graded G-module because R is a Cohen-Macaulay ring. Therefore we get $\kappa_i = I^{i-1}$ by Lemma 2.1, and hence $K_{R(I)} \cong R(I)(-1)$ as graded $R(I)$-modules. Then R is a Gorenstein ring. □

We remark that Corollary 1.3 does not hold true without the assumption that the ring A is quasi-Gorenstein. Let us close this paper with the following typical example in [HR], (2.2).

Example 2.5. *Let $k[[s,t]]$ be a formal power series ring over a field k. Let $A = k[[s^2,t,s^3,st]]$ and $I = (s^2,t)$. Then $R(I^2)$ is a Gorenstein ring but A is not a Cohen-Macaulay ring.*

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Mathematics laboratory
Hokkaido University of Education Sapporo
002-8502 JAPAN
e-mail: iai@sap.hokkyodai.ac.jp

CONTRAVARIANTLY FINITE RESOLVING SUBCATEGORIES OVER A GORENSTEIN LOCAL RING

RYO TAKAHASHI

INTRODUCTION

The notion of a contravariantly finite subcategory (of the category of finitely generated modules) was first introduced over artin algebras by Auslander and Smalø [5] in connection with studying the problem of which subcategories admit almost split sequences. The notion of a resolving subcategory was introduced by Auslander and Bridger [2] in the study of modules of Gorenstein dimension zero, which are now also called totally reflexive modules. There is an application of contravariantly finite resolving subcategories to the study of the finitistic dimension conjecture [4].

This paper deals with contravariantly finite resolving subcategories over commutative rings. Let R be a commutative noetherian henselian local ring. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules, by $\mathcal{F}(R)$ the full subcategory of free R-modules, and by $\mathcal{C}(R)$ the full subcategory of maximal Cohen-Macaulay R-modules. The subcategory $\mathcal{F}(R)$ is always contravariantly finite, and so is $\mathcal{C}(R)$ provided that R is Cohen-Macaulay. The latter fact is known as the Cohen-Macaulay approximation theorem, which was shown by Auslander and Buchweitz [3].

In this paper, we shall prove the following theorem; the category of finitely generated modules over a henselian Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Theorem A. *If R is Gorenstein, then all the contravariantly finite resolving subcategories of* $\operatorname{mod} R$ *are* $\mathcal{F}(R)$, $\mathcal{C}(R)$ *and* $\operatorname{mod} R$.

The main theorem of this paper asserts the following: let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that the residue field of R has a right $\mathcal{X}$-approximation. Assume that there exists an R-module $G \in \mathcal{X}$ of infinite projective dimension with $\operatorname{Ext}_R^i(G,R) = 0$ for $i \gg 0$. Let M be an R-module such that each $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X,M) = 0$ for $i \gg 0$. Then M has finite injective dimension. From this result, we will prove the following two theorems. Theorem A will be obtained from Theorem B. The assertion of Theorem C is a main result of [12], which has been a motivation for this paper. (Our way of obtaining Theorem C is quite different from the original proof given in [12].)

Theorem B. *Let $\mathcal{X} \neq \operatorname{mod} R$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that there is an R-module $G \in \mathcal{X}$ of infinite projective dimension such that $\operatorname{Ext}_R^i(G,R) = 0$ for $i \gg 0$. Then R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.*

Theorem C (Christensen-Piepmeyer-Striuli-Takahashi). *Suppose that there is a nonfree R-module in $\mathcal{G}(R)$. If $\mathcal{G}(R)$ is contravariantly finite in* $\operatorname{mod} R$, *then R is Gorenstein.*

Here, $\mathcal{G}(R)$ denotes the full subcategory of totally reflexive R-modules. A totally reflexive module, which is also called a module of Gorenstein dimension (G-dimension) zero, was defined by Auslander [1] as a common generalization of a free module and a maximal Cohen-Macaulay module over a Gorenstein local ring. Auslander and Bridger [2] proved that the full subcategory of totally reflexive modules over a left and right noetherian ring is resolving. The other details of totally reflexive modules are stated in [2] and [11].

If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$, and so $\mathcal{G}(R)$ is contravariantly finite by virtue of the Cohen-Macaulay approximation theorem. Thus, Theorem C can be viewed as the converse of this fact. Theorem C implies the following: let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R-module and are only finitely many nonisomorphic indecomposable totally reflexive R-modules. Then R is an isolated simple hypersurface singularity. For the details, see [12].

CONVENTIONS

In the rest of this paper, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a henselian local ring. The unique maximal ideal of R and the residue field of R are denoted by $\mathfrak{m}$ and k, respectively. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. By a *subcategory* of $\operatorname{mod} R$, we always mean a full subcategory of $\operatorname{mod} R$ which is closed under isomorphisms. Namely, in this paper, a subcategory $\mathcal{X}$ of $\operatorname{mod} R$ means a full subcategory such that every R-module which is isomorphic to some R-module in $\mathcal{X}$ is also in $\mathcal{X}$.

1. CONTRAVARIANT FINITENESS OF TOTALLY REFLEXIVE MODULES

In this section, we will state background materials which motivate the main results of this paper. We start by recalling the definition of a totally reflexive module.

Definition 1.1. We denote by $(-)^*$ the R-dual functor $\operatorname{Hom}_R(-, R)$. An R-module M is called *totally reflexive* (or *of Gorenstein dimension zero*) if

(1) the natural homomorphism $M \to M^{**}$ is an isomorphism, and
(2) $\operatorname{Ext}_R^i(M, R) = \operatorname{Ext}_R^i(M^*, R) = 0$ for any $i > 0$.

We introduce three subcategories of $\operatorname{mod} R$ which will often appear throughout this paper.

We denote by $\mathcal{F}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all free R-modules, by $\mathcal{G}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all totally reflexive R-modules, and by $\mathcal{C}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay R-modules. By definition, $\mathcal{F}(R)$ is contained in $\mathcal{G}(R)$. If R is Cohen-Macaulay, then $\mathcal{G}(R)$ is contained in $\mathcal{C}(R)$. If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$.

Next, we recall the notion of a right approximation over a subcategory of $\operatorname{mod} R$.

Definition 1.2. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$.

(1) Let $\phi : X \to M$ be a homomorphism of R-modules with $X \in \mathcal{X}$. We say that ϕ is a *right $\mathcal{X}$-approximation* (of M) if the induced homomorphism $\operatorname{Hom}_R(X', \phi) : \operatorname{Hom}_R(X', X) \to \operatorname{Hom}_R(X', M)$ is surjective for any $X' \in \mathcal{X}$.

(2) We say that $\mathcal{X}$ is *contravariantly finite* (in $\operatorname{mod} R$) if every R-module has a right $\mathcal{X}$-approximation.

The following result is well-known.

Theorem 1.3 (Auslander-Buchweitz). *Let R be a Cohen-Macaulay local ring. Then $\mathcal{C}(R)$ is contravariantly finite.*

Corollary 1.4. *If R is Gorenstein, then $\mathcal{G}(R)$ is contravariantly finite.*

The converse of this corollary essentially holds:

Theorem 1.5. [12] *Suppose that there is a nonfree totally reflexive R-module. If $\mathcal{G}(R)$ is contravariantly finite in* $\operatorname{mod} R$, *then R is Gorenstein.*

This theorem yields the following corollary, which is a generalization of [21, Theorem 1.3].

Corollary 1.6. *Let R be a non-Gorenstein local ring. If there is a nonfree totally reflexive R-module, then there are infinitely many nonisomorphic indecomposable totally reflexive R-modules.*

Combining this with [25, Theorems (8.15) and (8.10)] (cf. [14, Satz 1.2] and [10, Theorem B]), we obtain the following result.

Corollary 1.7. *Let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R-module but there are only finitely many nonisomorphic indecomposable totally reflexive R-modules. Then R is a simple hypersurface singularity.*

2. Contravariantly finite resolving subcategories

In this section, we will give the main theorem of this paper and several results it yields. One of them implies Theorem 1.5, which is the motive fact of this paper.

First of all, we recall the definition of the syzygies of a given module. Let M be an R-module and n a positive integer. Let

$$F_\bullet = (\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$$

be a minimal free resolution of M. We define the *nth syzygy* $\Omega^n M$ of M as the image of the homomorphism d_n. We set $\Omega^0 M = M$.

We recall the definition of a resolving subcategory.

Definition 2.1. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if it satisfies the following four conditions.

(1) $\mathcal{X}$ contains R.
(2) $\mathcal{X}$ is closed under direct summands: if M is an R-module in $\mathcal{X}$ and N is a direct summand of M, then N is also in $\mathcal{X}$.
(3) $\mathcal{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N are in $\mathcal{X}$, then M is also in $\mathcal{X}$.
(4) $\mathcal{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M and N are in $\mathcal{X}$, then L is also in $\mathcal{X}$.

Now we state the main theorem in this paper.

Theorem 2.2. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *such that the residue field k has a right $\mathcal{X}$-approximation. Assume that there exists an R-module $G \in \mathcal{X}$ of infinite projective dimension such that $\operatorname{Ext}_R^i(G,R) = 0$ for $i \gg 0$. Let M be an R-module such that each $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X,M) = 0$ for $i \gg 0$. Then M has finite injective dimension.*

We shall prove Theorem 2.2 in the next section. In the rest of this section, we will state and prove several results by using Theorem 2.2. We begin with two corollaries which are immediately obtained.

Corollary 2.3. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *which is contained in the subcategory $\{ M \mid \operatorname{Ext}_R^i(M,R) = 0 \text{ for } i \gg 0 \}$ of* $\operatorname{mod} R$. *Suppose that in $\mathcal{X}$ there is an R-module of infinite projective dimension. If k has a right $\mathcal{X}$-approximation, then R is Gorenstein.*

Proof. Each module X in $\mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X,R) = 0$ for $i \gg 0$. Hence Theorem 2.2 implies that R has finite injective dimension as an R-module. □

Corollary 2.4. *Let $\mathcal{X}$ be one of the following.*

(1) $\mathcal{G}(R)$.
(2) The subcategory $\{ M \mid \operatorname{Ext}_R^i(M,R) = 0 \text{ for } i > n \}$ of $\operatorname{mod} R$, *where n is a non-negative integer.*
(3) The subcategory $\{ M \mid \operatorname{Ext}_R^i(M,R) = 0 \text{ for } i \gg 0 \}$ of $\operatorname{mod} R$.

Suppose that in $\mathcal{X}$ there is an R-module of infinite projective dimension. If k has a right $\mathcal{X}$-approximation, then R is Gorenstein.

Proof. The subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is resolving. Since $\mathcal{X}$ is contained in the subcategory $\{ M \mid \operatorname{Ext}_R^i(M,R) = 0 \text{ for } i \gg 0 \}$, the assertion follows from Corollary 2.3. □

Remark 2.5. Corollary 2.4 implies Theorem 1.5. Indeed, any nonfree totally reflexive module has infinite projective dimension by [11, (1.2.10)].

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, let $\mathcal{X}^{\perp}$ (respectively, ${}^{\perp}\mathcal{X}$) denote the subcategory of $\operatorname{mod} R$ consisting of all R-modules M such that $\operatorname{Ext}_R^i(X,M) = 0$ (respectively, $\operatorname{Ext}_R^i(M,X) = 0$) for all $X \in \mathcal{X}$ and $i > 0$. Applying Wakamatsu's lemma to a resolving subcategory, we obtain the following lemma.

Lemma 2.6. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$. *If an R-module M has a right $\mathcal{X}$-approximation, then there is an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$.*

By using this lemma and the theorem which was formerly called "Bass' conjecture", we obtain another corollary of Theorem 2.2.

Corollary 2.7. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *such that k has a right $\mathcal{X}$-approximation and that k is not in $\mathcal{X}$. Assume that there is an R-module $G \in \mathcal{X}$ with $\operatorname{pd}_R G = \infty$ and $\operatorname{Ext}_R^i(G,R) = 0$ for $i \gg 0$. Then R is Cohen-Macaulay and* $\dim R > 0$.

Before giving the next corollary of Theorem 2.2, we establish an easy lemma without proof.

Lemma 2.8. *(1) Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Then, $k \in \mathcal{X}$ if and only if $\mathcal{X} = \operatorname{mod} R$.*

(2) Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. *Suppose that every R-module in ${}^{\perp}(\mathcal{X}^{\perp})$ admits a right $\mathcal{X}$-approximation. Then $\mathcal{X} = {}^{\perp}(\mathcal{X}^{\perp})$.*

(3) Let M and N be nonzero R-modules. Assume either that M has finite projective dimension or that N has finite injective dimension. Then one has an equality

$$\sup\{\, i \mid \operatorname{Ext}^i_R(M,N) \neq 0\,\} = \operatorname{depth} R - \operatorname{depth}_R M.$$

Now we can show the following corollary. There are only two contravariantly finite resolving subcategories possessing such G as in the corollary.

Corollary 2.9. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Assume that there is an R-module $G \in \mathcal{X}$ with $\operatorname{pd}_R G = \infty$ and $\operatorname{Ext}^i_R(G,R) = 0$ for $i \gg 0$. Then either of the following holds.*

(1) $\mathcal{X} = \operatorname{mod} R$,

(2) R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.

Proof. Suppose that $\mathcal{X} \neq \operatorname{mod} R$. Then k is not in $\mathcal{X}$. By Corollary 2.7, R is Cohen-Macaulay.

First, we show that $\mathcal{C}(R)$ is contained in $\mathcal{X}$. For this, let M be a maximal Cohen-Macaulay R-module. We have only to prove that M is in ${}^{\perp}(\mathcal{X}^{\perp})$. Let N be a nonzero R-module in $\mathcal{X}^{\perp}$. Theorem 2.2 implies that N is of finite injective dimension. Since M is maximal Cohen-Macaulay, we have $\sup\{\, i \mid \operatorname{Ext}^i_R(M,N) \neq 0\,\} = 0$. Therefore $\operatorname{Ext}^i_R(M,N) = 0$ for all $N \in \mathcal{X}^{\perp}$ and $i > 0$. It follows that M is in ${}^{\perp}(\mathcal{X}^{\perp})$, as desired.

Next, we show that $\mathcal{X}$ is contained in $\mathcal{C}(R)$. We have an exact sequence $0 \to Y \to X \to k \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 2.6. Since k is not in $\mathcal{X}$, the module Y is nonzero. By Theorem 2.2, Y has finite injective dimension. For a nonzero R-module X' in $\mathcal{X}$, we have equalities $0 \geq \sup\{\, i \mid \operatorname{Ext}^i_R(X',Y) \neq 0\,\} = \operatorname{depth} R - \operatorname{depth}_R X' = \dim R - \operatorname{depth}_R X'$. Therefore X' is a maximal Cohen-Macaulay R-module, as desired. □

Next, we study contravariantly finite resolving subcategories all of whose objects X satisfy $\operatorname{Ext}^{\gg 0}_R(X,R) = 0$. We start by considering special ones among such subcategories.

Proposition 2.10. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that every R-module in $\mathcal{X}$ has finite projective dimension. Then either of the following holds.*

(1) $\mathcal{X} = \mathcal{F}(R)$,

(2) R is regular and $\mathcal{X} = \operatorname{mod} R$.

Proof. If $\mathcal{X} = \operatorname{mod} R$, then our assumption says that all R-modules have finite projective dimension. Hence R is regular. Assume that $\mathcal{X} \neq \operatorname{mod} R$. Then there is an R-module M which is not in $\mathcal{X}$. There is an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 2.6. Note that $Y \neq 0$ as $M \notin \mathcal{X}$. Fix a nonzero R-module $X' \in \mathcal{X}$. We have $\operatorname{Ext}^i_R(X',Y) = 0$ for all $i > 0$, and hence $\operatorname{pd}_R X' = \sup\{\, i \mid \operatorname{Ext}^i_R(X',Y) \neq$

$0\} = 0$ by the Auslander-Buchsbaum formula. Hence X' is free. This means that $\mathcal{X}$ is contained in $\mathcal{F}(R)$. On the other hand, $\mathcal{X}$ contains $\mathcal{F}(R)$ since $\mathcal{X}$ is resolving. Therefore $\mathcal{X} = \mathcal{F}(R)$. □

Combining Proposition 2.10 with Corollary 2.9, we can get the following.

Corollary 2.11. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that every module $X \in \mathcal{X}$ is such that $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Then one of the following holds.*

(1) $\mathcal{X} = \mathcal{F}(R)$,
(2) R is Gorenstein and $\mathcal{X} = \mathcal{C}(R)$,
(3) R is Gorenstein and $\mathcal{X} = \operatorname{mod} R$.

Proof. The corollary follows from Proposition 2.10 in the case where all R-modules in $\mathcal{X}$ are of finite projective dimension. So suppose that in $\mathcal{X}$ there exists an R-module of infinite projective dimension. Then Corollary 2.9 shows that either of the following holds.

(1) $\mathcal{X} = \operatorname{mod} R$,
(2) R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.

By the assumption that every $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$, we have $\operatorname{Ext}_R^i(k, R) = 0$ for $i \gg 0$ in the case (i). In the case (ii), since $\Omega^d k$ is in $\mathcal{X}$ where $d = \dim R$, we have $\operatorname{Ext}_R^{i+d}(k, R) \cong \operatorname{Ext}_R^i(\Omega^d k, R) = 0$ for $i \gg 0$. Thus, in both cases, the ring R is Gorenstein. □

Finally, we obtain the following result from Corollary 2.11 and Theorem 1.3. It says that the category of finitely generated modules over a Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Corollary 2.12. *Let R be a Gorenstein local ring. Then all the contravariantly finite resolving subcategories of* $\operatorname{mod} R$ *are $\mathcal{F}(R)$, $\mathcal{C}(R)$ and* $\operatorname{mod} R$.

3. Proof of the main theorem

Let M be an R-module. Take a minimal free resolution $F_\bullet = (\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$ of M. We define the *transpose* $\operatorname{Tr} M$ of M as the cokernel of the R-dual homomorphism $d_1^* : F_0^* \to F_1^*$ of d_1. The transpose $\operatorname{Tr} M$ has no nonzero free summand.

For an R-module M, let M^*M be the ideal of R generated by the subset

$$\{ f(x) \mid f \in M^*, x \in M \}$$

of R. Note that M has a nonzero free summand if and only if $M^*M = R$.

Proposition 3.1. *Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$ *and $0 \to Y \xrightarrow{f} X \to M \to 0$ an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$. Let $G \in \mathcal{X}$, set $H = \operatorname{Tr} \Omega G$, and suppose that $(H^*H)M = 0$. Let $0 \to K \xrightarrow{g} F \xrightarrow{h} H \to 0$ be an exact sequence of R-modules with F free. Then the induced sequence*

$$0 \longrightarrow K \otimes_R Y \xrightarrow{g \otimes_R Y} F \otimes_R Y \xrightarrow{h \otimes_R Y} H \otimes_R Y \longrightarrow 0$$

is exact, and the map $h \otimes_R Y$ factors through the map $F \otimes_R f : F \otimes_R Y \to F \otimes_R X$.

Proof. We can show that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H\otimes_R Y & \xrightarrow{\delta} & H\otimes_R X & \xrightarrow{\varepsilon} & H\otimes_R M & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(H^*,Y) & \xrightarrow{\zeta} & \mathrm{Hom}_R(H^*,X) & \xrightarrow{\eta} & \mathrm{Hom}_R(H^*,M) & \longrightarrow & 0 \end{array}$$

with exact rows, and see that δ is a split monomorphism. Thus, the homomorphism $h\otimes_R Y$ factors through the homomorphism $F\otimes_R f$. We have isomorphisms $\mathrm{Tor}_1^R(H,Y) = \mathrm{Tor}_1^R(\mathrm{Tr}\,\Omega G, Y) \cong \underline{\mathrm{Hom}}_R(\Omega G, Y) = 0$, which completes the proof of the proposition. □

Now we can prove the following, which will play a key role in the proof of Theorem 2.2.

Proposition 3.2. *Let $\mathcal{X}$ be a subcategory of* $\mathrm{mod}\,R$ *which is closed under syzygies. Let $0\to Y\to X\to M\to 0$ be an exact sequence of R-modules with $X\in\mathcal{X}$ and $Y\in\mathcal{X}^{\perp}$. Suppose that there is an R-module $G\in\mathcal{X}$ with $\mathrm{pd}_R G=\infty$ and $\mathrm{Ext}_R^i(G,R)=0$ for $i\gg 0$. Put $H_i=\mathrm{Tr}\,\Omega(\Omega^i G)$ and assume that $((H_i)^*H_i)M=0$ for $i\gg 0$. Let $D=(D^j)_{j\geq 0}:\mathrm{mod}\,R\to\mathrm{mod}\,R$ be a contravariant cohomological δ-functor. If $D^j(X)=0$ for $j\gg 0$, then $D^j(Y)=D^j(M)=0$ for $j\gg 0$.*

Proof. Replacing G with $\Omega^i G$ for $i\gg 0$, we may assume that $\mathrm{Ext}_R^i(G,R)=0$ for all $i>0$ and that $((H_i)^*H_i)M=0$ for all $i\geq 0$. Let $F_\bullet=(\cdots\xrightarrow{d_{i+1}}F_i\xrightarrow{d_i}F_{i-1}\xrightarrow{d_{i-1}}\cdots\xrightarrow{d_2}F_1\xrightarrow{d_1}F_0\to 0)$ be a minimal free resolution of G. Dualizing this by R, we easily see that $H_i\cong(\Omega^{i+3}G)^*$ and $\Omega H_i\cong(\Omega^{i+2}G)^*$ for $i\geq 0$. By Proposition 3.1, for each integer $i\geq 0$ we have an exact sequence

$$0\to(\Omega^{i+2}G)^*\otimes_R Y\to(F_{i+2})^*\otimes_R Y\xrightarrow{f_i}(\Omega^{i+3}G)^*\otimes_R Y\to 0$$

such that f_i factors through $(F_{i+2})^*\otimes_R X$. The homomorphism $D^j(f_i)$ factors through $D^j((F_{i+2})^*\otimes_R X)$, which vanishes for $j\gg 0$. Hence $D^j(f_i)=0$ for $j\gg 0$, and we obtain an exact sequence

$$0\to D^j((F_{i+2})^*\otimes_R Y)\to D^j((\Omega^{i+2}G)^*\otimes_R Y)\xrightarrow{\varepsilon_{i,j}}D^{j+1}((\Omega^{i+3}G)^*\otimes_R Y)\to 0$$

for $i\geq 0$ and $j\gg 0$. Thus, there is a sequence

$$D^j((\Omega^{i+2}G)^*\otimes_R Y)\xrightarrow{\varepsilon_{i,j}}D^{j+1}((\Omega^{i+3}G)^*\otimes_R Y)\xrightarrow{\varepsilon_{i+1,j+1}}D^{j+2}((\Omega^{i+4}G)^*\otimes_R Y)\xrightarrow{\varepsilon_{i+2,j+2}}\cdots$$

of surjective homomorphisms of R-modules, and $\varepsilon_{i,j}$ is an isomorphism. It follows that $D^j((F_{i+2})^*\otimes_R Y)=0$ for $i\geq 0$ and $j\gg 0$. Thus we have $D^j(Y)=0$ for $j\gg 0$, and $D^j(M)=0$ for $j\gg 0$. □

Now we can prove our main theorem.

Proof of Theorem 2.2. Since k admits a right $\mathcal{X}$-approximation, there exists an exact sequence $0\to Y\to X\to k\to 0$ of R-modules with $X\in\mathcal{X}$ and $Y\in\mathcal{X}^{\perp}$ by Lemma 2.6. For an integer $i\geq 0$, put $H_i=\mathrm{Tr}\,\Omega(\Omega^i G)$. The module H_i has no nonzero free summand. We have $(H_i)^*H_i\neq R$. Hence $((H_i)^*H_i)k=0$ for $i\geq 0$. Applying Proposition 3.2 to the contravariant cohomological δ-functor $D=(\mathrm{Ext}_R^j(\ \ ,M))_{j\geq 0}$, we obtain $D^j(k)=0$ for $j\gg 0$. Namely, we have $\mathrm{Ext}_R^j(k,M)=0$ for $j\gg 0$, which implies that M has finite injective dimension. □

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DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, SHINSHU UNIVERSITY, 3-1-1 ASAHI, MATSUMOTO, NAGANO 390-8621, JAPAN

E-mail address: takahasi@math.shinshu-u.ac.jp

Generic initial ideals, graded Betti numbers and k-Lefschetz properties

Tadahito Harima and Akihito Wachi

Department of Mathematics, Hokkaido University of Education, Kushiro 085-8580, JAPAN

Division of Comprehensive Education, Hokkaido Institute of Technology, Sapporo 006-8585, JAPAN

Abstract

We introduce the k-strong Lefschetz property (k-SLP) and the k-weak Lefschetz property (k-WLP) for graded Artinian K-algebras, which are generalizations of the Lefschetz properties. The main results obtained in this article are as follows:

1. Let I be a graded ideal of $R = K[x_1, x_2, x_3]$ whose quotient ring R/I has the SLP. Then the generic initial ideal of I is the unique almost revlex ideal with the same Hilbert function as R/I.

2. Let I be a graded ideal of $R = K[x_1, x_2, \dots, x_n]$ whose quotient ring R/I has the n-SLP. Suppose that all k-th differences of the Hilbert function of R/I are quasi-symmetric. Then the generic initial ideal of I is the unique almost revlex ideal with the same Hilbert function as R/I.

3. We give a sharp upper bound on the graded Betti numbers of Artinian K-algebras with the k-WLP and a fixed Hilbert function.

1 Introduction

The strong and weak Lefschetz properties for graded Artinian K-algebras (Definition 1. SLP and WLP for short) are often used in studying generic initial ideals and graded Betti numbers ([Wat87], [Iar94], [HW03], [HMNW03], [Cim06], [ACP06], [AS07]). We generalize the Lefschetz properties, and define the k-strong Lefschetz property (k-SLP) and the k-weak Lefschetz property (k-WLP) for graded Artinian K-algebras (Definition 12). This notion was first introduced by A. Iarrobino in a private conversation with J. Watanabe in 1995. The first purpose of the article (Theorem 22) is to determine the generic initial ideals of ideals whose quotient rings have the n-SLP and have Hilbert functions satisfying some condition. The second purpose is to give upper bounds of the graded Betti numbers of graded Artinian K-algebras with the k-WLP (Theorem 31).

Let K be a field of characteristic zero. Suppose that a graded Artinian K-algebra A has the SLP (resp. WLP), and $\ell \in A$ is a Lefschetz element. If $A/(\ell)$ again has the SLP

E-mail addresses: harima@kus.hokkyodai.ac.jp, wachi@hit.ac.jp

(resp. WLP), then we say that A has the 2-SLP (resp. 2-WLP). We recursively define the k-Lefschetz properties (Definition 12): A is said to have the k-SLP (resp. k-WLP), if A has the SLP (resp. WLP), and $A/(l)$ has the $(k-1)$-SLP (resp. $(k-1)$-WLP). We have the characterization of the Hilbert functions of graded Artinian K-algebras with the k-SLP or the k-WLP (Proposition 16). In addition, for a graded Artinian ideal $I \subset R$, we show that R/I has the k-SLP (resp. k-WLP) if and only if $R/\operatorname{gin}(I)$ has the k-SLP (resp. k-WLP), where $\operatorname{gin}(I)$ denotes the generic initial ideal with respect to the graded reverse lexicographic order.

We explain our results on generic initial ideals. A monomial ideal I is called an almost revlex ideal, if the following condition holds: for each minimal generator u of I, every monomial v with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to I. Almost revlex ideals play a key role in the article. The characterization of the Hilbert functions for almost revlex ideals is given in Proposition 17. We start with the uniqueness of generic initial ideals in the case of three variables.

Theorem (see Theorem 10). *Let $I \subset R = K[x_1, x_2, x_3]$ be a graded Artinian ideal whose quotient ring has the SLP. Then $\operatorname{gin}(I)$ is the unique almost revlex ideal for the Hilbert function of R/I. In particular, $\operatorname{gin}(I)$ is uniquely determined only by the Hilbert function.*

For related results of the case of three variables, see Cimpoeaş [Cim06] and Ahn-Cho-Park [ACP06], where the uniqueness of $\operatorname{gin}(I)$ is proved under slightly stronger conditions than in the theorem above. We give some examples of complete intersection of height three whose generic initial ideals are the unique almost revlex ideals (Example 11).

By using the n-SLP, we obtain the following result for the case of n variables. In the following theorem, 'quasi-symmetric' is a notion including 'symmetric' (Definition 18).

Theorem (see Theorem 22). *Let $I \subset K[x_1, x_2, \dots, x_n]$ be a graded Artinian ideal whose quotient ring has the n-SLP, and has the Hilbert function h. Suppose that the k-th difference $\Delta^k h$ is quasi-symmetric for every integer k with $0 \le k \le n-4$. Then $\operatorname{gin}(I)$ is the unique almost revlex ideal for the Hilbert function h. In particular, $\operatorname{gin}(I)$ is uniquely determined only by the Hilbert function h.*

Here the operator Δ is defined by $(\Delta h)_i = \max\{h_i - h_{i-1}, 0\}$, and $\Delta^k h$ is the sequence obtained by applying Δ k-times.

The key to proving this theorem is a uniqueness of Borel-fixed ideals whose quotient rings have the n-SLP (Theorem 19). We give some examples of complete intersection of height n whose generic initial ideals are the unique almost revlex ideals (Example 24).

We next explain our result on the maximality of graded Betti numbers. Let $R = K[x_1, x_2, \dots, x_n]$. The following result on the maximal graded Betti numbers is first proved for $k = 1$ by Harima-Migliore-Nagel-Watanabe [HMNW03].

Theorem (see Theorem 31 and Corollary 32). *Let h be the Hilbert function of some graded Artinian K-algebra with the k-WLP. Then there is a Borel fixed ideal I of R such that R/I has the k-SLP, the Hilbert function of R/I is h, and $\beta_{i,i+j}(A) \le \beta_{i,i+j}(R/I)$ for all graded Artinian K-algebra A having the k-WLP and h as Hilbert function, and for any i and j.*

In particular, when $k = n$, the ideal I for the upper bounds is the unique almost revlex ideal for the Hilbert function h.

Some of the results of this article have been obtained independently and at the same time by Constantinescu (see [Con07]) and Cho-Park (see [CP07]).

2 Generic initial ideals in $K[x_1, x_2, x_3]$ and the SLP

In this section, we first recall the Lefschetz properties (Definition 1) and related facts. The main goal of this section is Theorem 10: for a graded Artinian ideal $I \subset K[x_1, x_2, x_3]$ whose quotient ring has the SLP, the generic initial ideal of I with respect to the the graded reverse lexicographic order is the unique almost revlex ideal for the same Hilbert function as $K[x_1, x_2, x_3]/I$.

2.1 The Lefschetz properties

Definition 1. Let A be a graded Artinian algebra over a field K, and $A = \bigoplus_{i=0}^{c} A_i$ its decomposition into graded components. The graded algebra A is said to have the *strong (resp. weak) Lefschetz property*, if there exists an element $\ell \in A_1$ such that the multiplication map $\times \ell^s : A_i \to A_{i+s}$ ($f \mapsto \ell^s f$) is full-rank for every $i \geq 0$ and $s > 0$ (resp. $s = 1$). In this case, ℓ is called a *Lefschetz element*, and we also say that (A, ℓ) has the strong (resp. weak) Lefschetz property. We abbreviate these properties as the *SLP (resp. WLP)* for short.

It is clear that if (A, ℓ) has the SLP, then (A, ℓ) has the WLP. It is also clear that Hilbert functions of graded algebras with the SLP or the WLP are *unimodal*. Namely there exists a non-negative integer i such that $h_0, h_1, \ldots, h_i$ is an increasing sequence and $h_i, h_{i+1}, \ldots$ is a weakly decreasing sequence, where $h_j = \dim_K A_j$.

Suppose that the Hilbert function of the graded Artinian algebra A is *symmetric*, that is, $A = \bigoplus_{i=0}^{c} A_i$ ($A_c \neq (0)$) and $\dim_K A_i = \dim_K A_{c-i}$ for $i = 0, 1, \ldots, \lfloor c/2 \rfloor$. In this case, it is clear that A has the SLP if and only if there exists $\ell \in A_1$ and $\times \ell^{c-2i} : A_i \to A_{c-i}$ is bijective for every $i = 0, 1, \ldots, \lfloor c/2 \rfloor$.

For a graded algebra A, we denote its Hilbert function by $\mathbf{H}_A$. Namely $\mathbf{H}_A(t)$ denotes the linear dimension of the graded component A_t of degree t. We often identify $\mathbf{H}_A$ with a finite sequence $h = (h_0, h_1, \ldots, h_c)$. A sequence $h = (h_0, h_1, \ldots, h_c)$ is called an *O-sequence* if h is a Hilbert function of some graded K-algebra. There is a classification of Hilbert functions of graded Artinian algebras with the SLP or the WLP.

Proposition 2 ([HMNW03, Corollary 4.6]). *Let $h = (h_0, h_1, \ldots, h_c)$ be a sequence of positive integers. The following three conditions are equivalent.*

(i) h is a Hilbert function of some graded algebra with the SLP,

(ii) h is a Hilbert function of some graded algebra with the WLP,

(iii) h is a unimodal O-sequence, and the sequence Δh is an O-sequence.

2.2 Almost revlex ideals and the SLP

We first recall a result of Wiebe [Wie04].

Lemma 3 (Wiebe, [Wie04, Lemma 2.7]). *If I is an Artinian stable ideal of $R = K[x_1, x_2, \dots, x_n]$, then the following two conditions are equivalent:*

(i) R/I has the SLP (resp. WLP),

(ii) x_n is a strong (resp. weak) Lefschetz element on R/I.

We define the notion of almost revlex ideals (almost revlex-segment ideals). Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field of characteristic zero. Let $>_{\text{revlex}}$ denote the graded reverse lexicographic order.

Definition 4. (i) A monomial ideal I is called a *revlex ideal*, if the following condition holds:

> for each monomial $u \in I$, every monomial v with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to I.

(ii) A monomial ideal I is called an *almost revlex ideal*, if the following condition holds:

> for each monomial u in the minimal generating set of I, every monomial v with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to I.

Remark 5. First it is clear that

(i) revlex ideals are almost revlex ideals.

Second,

(ii) if two almost revlex ideals have the same Hilbert function, then they are equal,

since one can determine the minimal generators from low degrees using a given Hilbert function. Finally,

(iii) almost revlex ideals are Borel-fixed,

since it is easy to see that the generating set of any almost revlex ideal is Borel-fixed.

Using the definition of almost revlex ideals (Definition 4) and combinatrics on monomials, we have the following proposition.

Proposition 6. *Let $I \subset R = K[x_1, x_2, \dots, x_n]$ be an Artinian almost revlex ideal. Then $(R/I, x_n)$ has the SLP.*

2.3 Uniqueness of Borel-fixed ideals and generic initial ideals in $K[x_1, x_2, x_3]$

For a given O-sequence, it is known that a Borel-fixed ideal of $K[x_1, x_2]$, whose quotient ring has the O-sequence as the Hilbert function, is unique. It is the unique lex-segment ideal determined by the Hilbert function. Moreover we have the following theorem, which gives the uniqueness of Borel-fixed ideals for $n = 3$, where the quotient rings have the SLP.

Theorem 7. *Let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field of characteristic zero, and I an Artinian Borel-fixed ideal of R where R/I has the SLP. Then the ideal I is the unique almost revlex ideal for the Hilbert function. In particular, the ideal I is uniquely determined only by the Hilbert function.*

Note that Theorem 7 does not hold, if the number of variables is more than three. See Example 21 for a counterexample in the case of four variables.

The following is an immediate corollary to Proposition 2, Proposition 6 , Theorem 7 and Lemma 9 below.

Corollary 8. *Let $R = K[x_1, x_2, x_3]$ and $h = (1, 3, h_2, h_3, \ldots, h_c)$ an O-sequence. The following three conditions are equivalent:*

(i) h is a Hilbert function of R/I for some almost revlex ideal I of R,

(ii) h is a Hilbert function of some graded algebra with the SLP,

(iii) h is a Hilbert function of some graded algebra with the WLP,

(iv) h is a unimodal O-sequence, and Δh is an O-sequence.

In the rest of this section, we study generic initial ideals in $K[x_1, x_2, x_3]$. We recall the definition of generic initial ideals. Fix any term order σ on the polynomial ring $R = K[x_1, x_2, \ldots, x_n]$ over a field of characteristic zero. For a graded ideal I of R, there exists a Zariski open subset $U \subset GL(n; K)$ such that the initial ideals of $g(I)$ are equal to each other for any $g \in U$. This initial ideal is uniquely determined, called the *generic initial ideal* of I, and denoted by $\operatorname{gin}_\sigma(I)$. It is known that generic initial ideals are Borel-fixed with respect to any term order (see [Eis95, 15.9], e.g.).

Thus we have results on generic initial ideals of ideals whose quotient rings have the SLP, as an easy consequence of Theorem 7. We first recall another result of Wiebe. We write simply by $\operatorname{gin}(I)$ the generic initial ideal of I with respect to the graded reverse lexicographic order from now on.

Lemma 9 (Wiebe, [Wie04, Proposition 2.8]). *Take the graded reverse lexicographic order on the polynomial ring $R = K[x_1, x_2, \ldots, x_n]$ over a field K of characteristic zero. Let I be a graded Artinian ideal of R. Then R/I has the SLP if and only if $R/\operatorname{gin}(I)$ has the SLP.*

We thus have the following theorem by Lemma 3, Theorem 7 and Lemma 9.

Theorem 10. *Let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field of characteristic zero, and consider the graded reverse lexicographic order on R. Let I be a graded Artinian ideal of R, and suppose that R/I has the SLP. Then the generic initial ideal* $\text{gin}(I)$ *is the unique almost revlex ideal for the same Hilbert function as R/I. In particular,* $\text{gin}(I)$ *is uniquely determined by the Hilbert function.*

Cimpoeaş [Cim06] shows that the generic initial ideals of complete intersections of height three whose quotient rings have the SLP are almost revlex ideals. Theorem 10 is an improvement of this result. In addition, Theorem 10 is an improvement of a result of Ahn, Cho and Park [ACP06]. They prove that the generic initial ideals of ideals in $K[x_1, x_2, x_3]$ whose quotient rings have the SLP are determined by their graded Betti numbers.

Example 11. We give four examples of complete intersection in $R = K[x_1, x_2, x_3]$ whose quotient rings have the SLP. The generic initial ideals of these ideals are the unique almost revlex ideals with corresponding Hilbert functions by Theorem 10.

(i) Let $I = (f, g, \ell^r) \subset R$, where f and g are any homogeneous polynomials of R, and ℓ is any homogeneous polynomial of degree one. In this case, if I is a complete intersection, then R/I has the SLP ([HW07a, Example 6.2]).

(ii) Let e_1, e_2 and e_3 be the elementary symmetric functions in three variables, where $\deg(e_i) = i$. Let r and s be positive integers, where r divides s. Then, the quotient ring of the ideal $I = (e_1(x_1^r, x_2^r, x_3^r), e_2(x_1^r, x_2^r, x_3^r), e_3(x_1^s, x_2^s, x_3^s))$ of R has the SLP ([HW07a, Example 6.4]).

(iii) Let p_i be the power sum symmetric function of degree i in three variables, and a be a positive integer. Then, the quotient ring of the ideal $I = (p_a, p_{a+1}, p_{a+2})$ of R has the SLP ([HW07b, Proposition 7.1]).

(vi) Let $I = (e_2, e_3, f) \subset R$, where f is any homogeneous polynomial of R. In this case, if I is a complete intersection, then R/I has the SLP ([HW07b, Proposition 3.1]).

3 Generic initial ideals in $K[x_1, x_2, \dots, x_n]$ and the k-SLP

Suppose that a graded Artinian algebra A has the SLP (resp. WLP), and $\ell \in A$ is a Lefschetz element. If the graded algebra $A/(\ell)$ again has the SLP (resp. WLP), then we say that A has the 2-SLP (resp. 2-WLP). We define the notion of the k-SLP and the k-WLP recursively (Definition 12). A characterization of the Hilbert functions of graded Artinian algebras having the k-SLP or the k-WLP is given in Proposition 16. Moreover, the Hilbert functions of quotient rings by almost revlex ideals are determined in terms of the n-SLP in Proposition 17.

The main goal of this section is Theorem 22: Let $I \subset K[x_1, x_2, \dots, x_n]$ be a graded Artinian ideal whose quotient ring has the n-SLP, and every k-th difference of the Hilbert function is quasi-symmetric. The generic initial ideal of I with respect to the graded reverse lexicographic order is the unique almost revlex ideal for the same Hilbert function as $K[x_1, x_2, \dots, x_n]/I$.

3.1 k-SLP and k-WLP

The first author heard from J. Watanabe that the following notion, the 'k-SLP' and the 'k-WLP', has been introduced by A. Iarrobino in a private conversation with J. Watanabe in 1995.

Definition 12. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra, and k a positive integer. We say that A has the *k-SLP* (resp. *k-WLP*) if there exist linear elements $g_1, g_2, \ldots, g_k \in A_1$ satisfying the following two conditions.

(i) (A, g_1) has the SLP (resp. WLP),

(ii) $(A/(g_1, \ldots, g_{i-1}), g_i)$ has the SLP (resp. WLP) for all $i = 2, 3, \ldots, k$.

In this case, we say that $(A, g_1, \ldots, g_k)$ has the k-SLP (resp. k-WLP). Note that a graded algebra with the k-SLP (resp. k-WLP) has the $(k-1)$-SLP (resp. $(k-1)$-WLP).

Remark 13. From Theorem 4.4 in [HMNW03], one knows that all graded K-algebras $K[x_1]/I$ and $K[x_1, x_2]/I$ have the SLP. Hence the following conditions are equivalent for a graded Artinian algebra $A = K[x_1, x_2, \ldots, x_n]/I$, where $I \subset (x_1, \ldots, x_n)^2$.

(i) A has the n-WLP (resp. the n-SLP),

(ii) A has the $(n-1)$-WLP (resp. the $(n-1)$-SLP),

(iii) A has the $(n-2)$-WLP (resp. the $(n-2)$-SLP).

In particular, graded algebras $K[x_1, x_2, x_3]/I$ with the SLP (resp. WLP) has the 3-SLP (resp. 3-WLP) automatically.

Example 14. For every Artinian almost revlex ideal I of $R = K[x_1, x_2, \ldots, x_n]$ where $I \subset (x_1, \ldots, x_n)^2$, the quotient ring R/I has the n-SLP. In particular, for every revlex ideal I of R where $I \subset (x_1, \ldots, x_n)^2$, the quotient ring R/I has the n-SLP.

Example 14 shows that the class of Hilbert functions for almost revlex ideals is a subset of that for ideals with the n-SLP. In fact, Proposition 17 shows that these two classes coincide. We also determine the class of Hilbert functions for ideals with the k-SLP in Proposition 16.

Let $\text{gin}(I)$ denote the generic initial ideal of I with respect to the graded reverse lexicographic order. The following proposition is an analogue of Wiebe's result (Lemma 9) [Wie04, Proposition 2.8].

Proposition 15. *Let I be a graded Artinian ideal of $R = K[x_1, \ldots, x_n]$, and let $1 \le k \le n$. The following two conditions are equivalent:*

(i) R/I has the k-WLP (resp. the k-SLP),

(ii) $(R/\text{gin}(I), x_n, x_{n-1}, \ldots, x_{n-k+1})$ has the k-WLP (resp. the k-SLP).

3.2 Hilbert functions of graded algebras with the k-SLP

We give a characterization of the Hilbert functions that can occur for graded K-algebras having the k-SLP or the k-WLP. Their characterizations are equal as in the case of the SLP and the WLP (Proposition 2). For a sequence $h = (h_0, h_1, \ldots, h_c)$ of positive integers, define a sequence of the t-th difference $\Delta^t h$ by

$$\Delta^t h = \Delta(\Delta(\cdots \Delta(h)\cdots)) \qquad \text{(apply } t \text{ times)},$$

for a positive integer t.

Proposition 16. *Let $R = K[x_1, x_2, \ldots, x_n]$ and k be an integer with $1 \leq k \leq n$. Let $h = (1, n, h_2, h_3, \ldots, h_c)$ be an O-sequence. The following three conditions are equivalent:*

(i) h is a Hilbert function of some graded algebra with the k-SLP,

(ii) h is a Hilbert function of some graded algebra with the k-WLP,

(iii) h is a unimodal O-sequence, $\Delta^t h$ is a unimodal O-sequence for every integer t with $1 \leq t < k$, and $\Delta^k h$ is an O-sequence.

In addition, we have a characterization of the Hilbert functions of quotient rings R/I for Artinian almost revlex ideals I. The characterization is the same as ideals with the n-WLP. This result is an analogue of the result of Deery [Dee96] or Marinari-Ramella [MR99, Proposition 2.13], which gives the characterization of the Hilbert functions for revlex ideals.

Proposition 17. *Let $R = K[x_1, x_2, \ldots, x_n]$ and $h = (1, n, h_2, h_3, \ldots, h_c)$ an O-sequence. The following four conditions are equivalent:*

(i) h is a Hilbert function of R/I for some almost revlex ideal I of R,

(ii) h is a Hilbert function of some graded algebra with the n-SLP,

(iii) h is a Hilbert function of some graded algebra with the n-WLP,

(iv) h is a unimodal O-sequence, and $\Delta^k h$ is a unimodal O-sequence for every integer k with $1 \leq k \leq n$.

3.3 Uniqueness of Borel-fixed ideals and generic initial ideals in $K[x_1, x_2, \ldots, x_n]$

When $n \leq 3$, we already know that Borel-fixed ideals of $K[x_1, x_2, \ldots, x_n]$ whose quotient rings have the n-SLP are the unique almost revlex ideals for given Hilbert functions. Moreover, we have the following Theorem 19 for any n. For a sequence h, we use a convention that the 0-th difference $\Delta^0 h$ is h itself. We need the following definition to state Theorem 19.

Definition 18. A unimodal sequence $h = (h_0, h_1, \ldots, h_c)$ of positive integers is said to be *quasi-symmetric*, if the following condition holds:

Let h_i be the maximum of $\{h_0, h_1, \ldots, h_c\}$. Then every integer h_j $(j > i)$ is equal to one of $\{h_0, h_1, \ldots, h_i\}$.

In particular, unimodal symmetric sequences are quasi-symmetric.

Theorem 19. *Let $I \subset R = K[x_1, x_2, \ldots, x_n]$ be an Artinian Borel-fixed ideal whose quotient ring R/I has the n-SLP, and let h be the Hilbert function of R/I. Suppose that the k-th difference $\Delta^k h$ is quasi-symmetric for every integer k with $0 \leq k \leq n-4$. Then I is the unique almost revlex ideal for which the Hilbert function of R/I is equal to h. In particular, I is determined only by the Hilbert function.*

In particular, we have the following uniqueness for Borel-fixed ideals in the case of four variables.

Corollary 20. *Let $I \subset K[x_1, x_2, x_3, x_4]$ be a Borel-fixed ideal, for which $K[x_1, x_2, x_3, x_4]/I$ has a quasi-symmetric Hilbert function h , and has the 2-SLP. Then I is the unique almost revlex ideal for the Hilbert function h.*

In Theorem 19, if we drop the condition for $\Delta^k h$ to be quasi-symmetric, then the uniqueness does not necessarily hold as follows.

Example 21. (i) There exist two different Borel-fixed ideals with the 4-SLP in $R = K[x_1, x_2, x_3, x_4]$, and their quotient rings have the same non-quasi-symmetric Hilbert function. Define the following ideals:

$$I = (x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_3^3, x_2^2x_4) + (x_1, x_2, x_3, x_4)^4,$$
$$J = (x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_3^3, x_1x_3x_4) + (x_1, x_2, x_3, x_4)^4.$$

We can easily check that both I and J are Borel-fixed, have the 4-SLP, and R/I and R/J have the same Hilbert function $h = (1, 4, 8, 7)$.

In the rest of this section, we study generic initial ideals in the polynomial ring $R = K[x_1, x_2, \ldots, x_n]$ over a field of characteristic zero. The following theorem, which gives a uniqueness of generic initial ideals, follows from Theorem 19 and Proposition 15.

Theorem 22. *Let $I \subset R = K[x_1, x_2, \ldots, x_n]$ be a graded Artinian ideal whose quotient ring R/I has the n-SLP, and let h be the Hilbert function of R/I. Suppose that the k-th difference $\Delta^k h$ is quasi-symmetric for every integer k with $0 \leq k \leq n-4$. Then the generic initial ideal $\mathrm{gin}(I)$ with respect to the graded reverse lexicographic order is the unique almost revlex ideal for the Hilbert function h. In particular, $\mathrm{gin}(I)$ is determined only by the Hilbert function.*

In particular, we have the following corollary, which corresponds to Corollary 20.

Corollary 23. *Let $I \subset K[x_1, x_2, x_3, x_4]$ be a graded Artinian ideal whose quotient ring has the 2-SLP. Suppose that the Hilbert function h of $K[x_1, x_2, x_3, x_4]/I$ is quasi-symmetric. Then the generic initial ideal $\mathrm{gin}(I)$ with respect to the graded reverse lexicographic order is the unique almost revlex ideal for the Hilbert function h. In particular, $\mathrm{gin}(I)$ is determined only by the Hilbert function.*

Example 24. Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field K of characteristic zero. We consider complete intersections as follows.

(a) Let f_1 and f_2 be homogeneous polynomials of degree d_i $(i = 1, 2)$, and let $g_3, \dots, g_n$ be linear forms. Set $I = (f_1, f_2, f_3 = g_3^{d_3}, \dots, f_n = g_n^{d_n})$. Suppose that $\{f_1, f_2, g_3, \dots, g_n\}$ is a regular sequence. Example 6.2 in [HW07a] shows that R/I has the SLP.

(b) For $i = 1, 2, \dots, n$, $f_i \in K[x_i, \dots, x_n]$ be a homogeneous polynomial of degree d_i which is a monic in x_i, and set $I = (f_1, f_2, \dots, f_n)$. Then R/I is always a complete intersection. Corollary 29 in [HW03] and Corollary 2.1 in [HP05] show that R/I has the SLP.

Now, let k be an integer satisfying $1 \leq k \leq n - 2$ and suppose that

$$d_j \geq d_1 + d_2 + \cdots + d_{j-1} - (j-1) + 1$$

for all $j = n-k+1, n-k+2, \dots, n$. Then we have the following.

(i) $A = R/I$ has the k-SLP.

(ii) In particular, when $k = n - 2$, A has the n-SLP.

(iii) The generic initial ideal of I coincides with the unique almost revlex ideal determined by the Hilbert function of A.

(iv) $\text{gin}(x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n}) = \text{gin}(I)$.

We conclude this section by an additional relation of initial ideals with the k-WLP or the k-SLP. Although this result is not used in the rest of this article, it is an analogue of Wiebe's result [Wie04, Proposition 2.9].

Proposition 25. *Let I be a graded Artinian ideal of $R = K[x_1, \dots, x_n]$, let $\text{in}(I)$ be the initial ideal of I with respect to the graded reverse lexicographic order and let $1 \leq k \leq n$. If $R/\text{in}(I)$ has the k-WLP (resp. the k-SLP), then the same holds for R/I.*

4 An extremal property of graded Betti numbers and the k-WLP

In the rest of this article, we study graded Betti numbers for monomial ideals. The goal is Theorem 31 on the maximality of graded Betti numbers. We give a sharp upper bound on the graded Betti numbers of graded Artinian algebras with the k-WLP and a fixed Hilbert function. The upper bounds are achieved by the quotient rings by Borel-fixed ideals having the k-SLP. In particular, when $k = n$, almost revlex ideals give the upper bounds.

4.1 Graded Betti numbers of stable ideals and the k-WLP

Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field of characteristic zero.

Proposition 26. *Let $I \subset R$ be an Artinian stable ideal, for which R/I has the Hilbert function $h = (h_0, h_1, \dots, h_c)$, and $(R/I, x_n)$ have the WLP. Let $\overline{R} = K[x_1, x_2, \dots, x_{n-1}]$ and $\overline{I} = I \cap \overline{R}$. We have the following.*

(i) The graded Betti numbers $\beta_{i,i+j}(R/I)$ of R/I is given as follows:

$$\beta_{i,i+j}(R/I) = \beta_{i,i+j}(\overline{R}/\overline{I}) + \binom{n-1}{i-1} \times c_{j+1} \qquad (i, j \geq 0),$$

$$c_j = \max\{h_{j-1} - h_j, 0\},$$

where we use the convention that $h_{-1} = 0$.

(ii) By the same c_j, the last graded Betti numbers $\beta_{n,n+j}(R/I)$ is given as follows:

$$\beta_{n,n+j}(R/I) = c_{j+1} \qquad (j \geq 0).$$

In particular, they are determined only by the Hilbert function.

We easily generalize Proposition 26 to the case of the k-WLP.

Notation 27. For a unimodal O-sequence $h = (h_0, h_1, \dots, h_c)$, we define

$$c_j^{(h)} = \max\{h_{j-1} - h_j, 0\}$$

for all $j = 0, 1, \dots, c$, where $h_{-1} = 0$.

Proposition 28. *Let I be an Artinian Borel-fixed ideal of $R = K[x_1, x_2, \dots, x_n]$ and suppose that R/I has the k-WLP.*

(i) Let $k < n$. Set $R' = K[x_1, x_2, \dots, x_{n-k}]$ and $I' = I \cap R'$. Then

$$\beta_{i,i+j}(R/I) = \beta_{i,i+j}(R'/I') + \tbinom{n-k}{i-1} \cdot c_{j+1}^{(\Delta^{k-1}h)} + \dots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

(ii) Let $k = n$. Then we have

$$\beta_{i,i+j}(R/I) = \tbinom{0}{i-1} \cdot c_{j+1}^{(\Delta^{n-1}h)} + \tbinom{1}{i-1} \cdot c_{j+1}^{(\Delta^{n-2}h)} + \dots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

In particular, $\beta_{i,i+j}(R/I)$ is determined only by the Hilbert function.

Corollary 29. *Let I be a graded Artinian ideal of $R = K[x_1, x_2, \dots, x_n]$ and suppose that R/I has the k-WLP.*

(i) Let $k < n$. Set $R' = K[x_1, x_2, \dots, x_{n-k}]$ and $I' = \operatorname{gin}(I) \cap R'$. Then

$$\beta_{i,i+j}(R/\operatorname{gin}(I)) = \beta_{i,i+j}(R'/I') + \tbinom{n-k}{i-1} \cdot c_{j+1}^{(\Delta^{k-1}h)} + \dots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

(ii) Let $k = n$. Then we have

$$\beta_{i,i+j}(R/\operatorname{gin}(I)) = \tbinom{0}{i-1} \cdot c_{j+1}^{(\Delta^{n-1}h)} + \tbinom{1}{i-1} \cdot c_{j+1}^{(\Delta^{n-2}h)} + \dots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

In particular, $\beta_{i,i+j}(R/\operatorname{gin}(I))$ is determined only by the Hilbert function.

4.2 Maximality of graded Betti numbers and the k-WLP

Notation and Remark 30. Let h be the Hilbert function of a graded Artinian K-algebra R/I. Then there is the uniquely determined lex-segment ideal $J \subset R$ such that R/J has h as its Hilbert function. We define

$$\beta_{i,i+j}(h, R) = \beta_{i,i+j}(R/J).$$

The numbers $\beta_{i,i+j}(h, R)$ can be computed numerically without considering lex-segment ideals. Explicit formulas can be found in [EK90].

We give a sharp upper bound on the Betti numbers among graded Artinian K-algebras having the k-WLP. Moreover the upper bound is achieved by a graded Artinian K-algebra with the k-SLP. For $k = 1$, this theorem was first proved by [HMNW03, Theorem 3.20].

Theorem 31. *(i) Let $A = R/I$ be a graded Artinian K-algebra with the k-WLP and put $R' = K[x_1, \ldots, x_{n-k}]$. Then the graded Betti numbers of A satisfy*

$$\begin{aligned}\beta_{i,i+j}(A) \le \beta_{i,i+j}(\Delta^k h, R') + \tbinom{n-k}{i-1} \cdot c_{j+1}^{(\Delta^{k-1}h)} + \cdots \\ + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)} \qquad \textit{if } k < n,\end{aligned} \tag{1}$$

and

$$\begin{aligned}\beta_{i,i+j}(A) \le \tbinom{0}{i-1} \cdot c_{j+1}^{(\Delta^{n-1}h)} + \tbinom{1}{i-1} \cdot c_{j+1}^{(\Delta^{n-2}h)} + \cdots \\ + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)} \qquad \textit{if } k = n.\end{aligned} \tag{2}$$

(ii) Let h be an O-sequence such that there is a graded Artinian K-algebra R/J having the k-WLP and h as Hilbert function. Then there is a Borel-fixed ideal I of R such that R/I has the k-SLP, the Hilbert function of R/I is h and the equality holds in (i) for all integers i, j.

The following are immediate consequences of Theorem 31.

Corollary 32. *Let h be the Hilbert function of a graded Artinian K-algebra R/J having the n-WLP (resp. n-SLP). Let I be the unique almost revlex ideal of R whose quotient ring has the same Hilbert function h. Then R/I has the maximal Betti numbers among graded Artinian K-algebras with the same Hilbert function h and the n-WLP (resp. n-SLP).*

Corollary 33. *Let I be a graded Artinian ideal of $R = K[x_1, x_2, \ldots, x_n]$ and suppose that R/I has the n-WLP (resp. n-SLP). Then $R/\operatorname{gin}(I)$ has the maximal Betti numbers among graded Artinian K-algebras with the same Hilbert function R/I and the n-WLP (resp. n-SLP).*

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The differential module of the polynomial ring with the action of the symmetric group

Nagoya, 20, November, 2007

H. Morita, A. Wachi, J. Watanabe

Let $R = K[x_1, \cdots, x_k]$ be the polynomial ring over K, a field of characteristic 0. Let Ω be the module of differentials:

$$\Omega := Rdx_1 \oplus \cdots \oplus Rdx_k$$

Let $G := S_k$ be the symmetric group in k letters. Let G act on R by permutation of the variables. Extend the action to

$$\wedge^j \Omega$$

for

$$j = 0, 1, \cdots, k.$$

We consider the following problems:

Problem 1. Decompose $\wedge^j \Omega$ into irreducible S_k-modules.

Problem 2. Determine the Hilbert series of the isotypic components of

$$\wedge^j \Omega.$$

Recall that the irreducible modules of S_k are parametrized by the partitions of k. Thus we write W^λ for the irreducible S_k-module corresponding to $\lambda \vdash k$.

Let $Y^\lambda(-)$ be the functor from the category of S_k-modules to itself

"to extract the isotypic component"

belonging to λ. Note that it is an exact functor.

For a graded vector space M,

$$h(M, q)$$

denotes the Hilbert series of M. (This is a power series in q with positive integers as coefficients.)

Example 1 : The case where $k = 2$.

Assume $k = 2$. Then $R = K[x, y]$. There are only two partitions: $\lambda = \begin{cases} 2, \\ 11. \end{cases}$ We want to determine

$$h(Y^{\lambda}(\wedge^{j}\Omega), q)$$

for $j = 0, 1, 2$.

For $j = 0$, it is easy to determine the Hilbert series since

$$\begin{cases} Y^{(2)}(\wedge^{0}\Omega) = R^{G} = K[x + y, xy] \\ Y^{(11)}(\wedge^{0}\Omega) = (x - y)R^{G} \end{cases}$$

and since R is the direct sum:

$$R = R^{G} \oplus (x - y)R^{G}.$$

For $j = 2$, we have

$$\wedge^{2}\Omega \cong R(dx \wedge dy).$$

Thus we have

$$\begin{cases} h(Y^{(2)}(\wedge^{2}\Omega), q) = h(Y^{(11)}(R), q), \\ h(Y^{(11)}(\wedge^{2}\Omega), q) = h(Y^{(2)}(R), q). \end{cases}$$

For $j = 1$, we have to decompose the module

$$\Omega \cong Rdx \oplus Rdy.$$

As is easily seen, symmetric 1-forms are of the form either $sdx + sdy$ with $s \in R^{G}$ or $adx - ady$ with $a \in (x - y)R^{G}$, and alternating 1-forms are either $sdx - sdy$ or $adx + ady$.

Thus we have obtained the following table for $h(Y^{\lambda}(\wedge^{j}\Omega), q)$.

	$j = 0$	$j = 1$	$j = 2$
$\lambda = (2)$	$\frac{1}{(1-q)(1-q^2)}$	$\frac{1}{(1-q)^2}$	$\frac{q}{(1-q)(1-q^2)}$
$\lambda = (11)$	$\frac{q}{(1-q)(1-q^2)}$	$\frac{1}{(1-q)^2}$	$\frac{1}{(1-q)(1-q^2)}$

Example 2: The case where $j = 0$

Fix $j = 0$. So $\wedge^j\Omega = R$. If $\lambda = (k)$, the trivial partition, then $Y^\lambda(R)$ is R^G, the ring of invariants. So we have

$$h(R^G, q) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)},$$

since R^G is generated by the elementary symmetric functions.

Put $\overline{R} := R/(R^G_+)$. Then it is easy to see that

$$\begin{aligned} h(Y^\lambda(R), q) &= h(R^G, q)h(Y^\lambda(\overline{R}), q) \\ &= \frac{h(Y^\lambda(\overline{R}), q)}{(1-q)(1-q^2)\cdots(1-q^k)}. \end{aligned}$$

By a result of Terasoma-Yamada. the numerator (which is a polynomial)

$$h(Y^\lambda(\overline{R}), q)$$

is known to be the q-analog of the hook length formula multiplied by the dimension of W^λ with a certain shift of degree determined by λ.

Main result: The general case

Write

$$\Omega(-n) = Rdx_1 \oplus \cdots \oplus Rdx_n,$$

when we give dx_i degree n. We have been considering

$$\Omega = \Omega(0).$$

Similarly $R(-n)$ denotes the free module of rank 1 generated by a generator of degree n. Thus

$$\begin{aligned} \wedge^j(\Omega(-n)) &\cong R^{\binom{k}{j}}(-jn) \\ &\cong (\wedge^j\Omega)(-jn) \end{aligned}$$

Also put

$$A(n) = R/(x_1^n, \cdots, x_k^n).$$

For simplicity put $F = \Omega(-n)$. We would like to construct a minimal free resolution of $A(n)$ as an R-module such that the boundary maps are compatible with the action of S_k. For this the usual minimal free resolution suffices:

$$\to \wedge^3F \to \wedge^2F \to \wedge^1F \to \wedge^0F \to A(n) \to 0 \qquad (1)$$

We want to know

$$\xi_j := h(Y^\lambda(\wedge^j\Omega), q).$$

Assume that we know

$$h_n := h(Y^\lambda(A(n)), q),$$

for $n = 0, 1, 2, \cdots$. Fix $\lambda \vdash k$ and apply the functor $Y^\lambda(-)$ to the sequence (1) above. Then we have

$$\to \; Y^\lambda(\wedge^3 F) \to \; Y^\lambda(\wedge^2 F) \to \; Y^\lambda(\wedge^1 F) \to \; Y^\lambda(\wedge^0 F) \to \; Y^\lambda(A(n)) \to \; 0.$$

Since the sequence is exact it gives us:

$$h_n = \sum_{j=0}^{k} (-1)^j q^{nj} \xi_j \tag{2}$$

This means that we have an infinite set of linear equations relating $\{\xi_i\}$ and $\{h_i\}$. For example if $k = 3$, we have

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -q & q^2 & -q^3 \\ 1 & -q^2 & q^4 & -q^6 \\ 1 & -q^3 & q^6 & -q^9 \\ 1 & -q^4 & q^8 & -q^{12} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \end{pmatrix}$$

Note that any maximal minor of the matrix $\{(-q^n)^j\}$ is non-zero. Thus we have proved the following theorem.

Theorem With $\lambda \vdash k$ fixed, the set of Hilbert series

$$\xi_0, \xi_1, \xi_2, \cdots, \xi_k.$$

in consideration are determined by any $(k+1)$ terms of

$$h_0, h_1, h_2, \cdots.$$

Consequenctly the infinite sequece

$$h_0, h_1, h_2, \cdots.$$

is determined by any $k+1$ terms.

Actually h_0 and h_1 are known for all $\lambda \vdash k$. (This is trivial.) Hence the above theorem is rephrased as follows:

Theorem′ With $\lambda \vdash k$ fixed, any $k-1$ terms in the infinite series

$$h_2, h_3, \cdots$$

determine

$$\xi_0, \xi_1, \xi_2, \cdots, \xi_k$$

and they determine all

$$h_2, h_3, \cdots$$

As is easily conceived we have the duality

$$Y^\lambda(\wedge^j \Omega) \cong Y^{\overline{\lambda}}(\wedge^{k-j}\Omega)$$

Thus we have the following

Theorem″ Any $[(k+1)/2]$ terms in the infinite series

$$h_0, h_1, h_2, \cdots$$

for all $\lambda \vdash k$ determine

$$\xi_0, \xi_1, \xi_2, \cdots, \xi_k.$$

A result of Morita-Wachi-Watanabe says that

$$h_n$$

is the q-analog of the Weyl dimension formula. It means that we have determined

$$h(Y^\lambda(\wedge^j\Omega), q)$$

for all $\left\{ \begin{array}{l} \lambda \vdash k, \\ j = 0, 1, \cdots, k. \end{array} \right.$

q-analog of the hook length formula

For $\lambda \vdash k$ let W^λ be the irreducibe S_k-module corresponding to λ. Then dim W^λ is given by

$$\dim W^\lambda = \frac{k!}{\prod h_{ij}} \tag{3}$$

where h_{ij} is the hook length at the (i,j)-th position. The following is an example which shows the matrix $\{h_{ij}\}$ for the Young diagram $\lambda = (5, 3, 1)$.

7	5	4	2	1
4	2	1		
1				

In (3) replace integer a by the polynomial

$$\begin{aligned}[a] : &= \frac{1-q^a}{1-q} \\ &= 1+q+\cdots+q^{a-1}\end{aligned}$$

It is the "q-analog of the hook length formula."

Since there are same number of integers in the denominator and enumerator of (3), it is the same if we replace a by

$$1-q^a.$$

q-analog of the Weyl dimension formula

Let $\lambda \vdash k$. Let $n > 0$ be any integer. Let V^λ be the irreducible $GL(n)$-module. Then dim V^λ is given by

$$\dim\ V^\lambda = \frac{\mu_1!\mu_2!\cdots\mu_n!}{(n-1)!(n-2)!\cdots 2!1!\prod h_{ij}}$$

where

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$$

and

$$\mu := (\mu_1, \mu_2, \cdots, \mu_n)$$

is defined by

$$\mu = \lambda + (n-1, n-2, \cdots, 1, 0).$$

If λ has more than n parts, we let dim $V^\lambda = 0$.

The Hilbert series h_n of the module

$$Y^\lambda(A(n))$$

is given by the q-analog of the Weyl dimension formula multiplied by $\dim W^\lambda$ with a certain shift of degrees.

29th Symposium on Commutative Algebra
Nagoya November 2007

Koszul algebras and Gröbner bases of quadrics

Aldo Conca
Dipartimento di Matematica, Universitá di Genova
Via Dodecaneso 35, I-16146 Genova, Italia

Abstract: We present results that appear in the papers [C, CTV, CRV] joint with M.E.Rossi, N.V.Trung and G.Valla and also some new results contained in [C1]. These results concern Koszul and G-quadratic properties of algebras associated with points, curves, cubics and spaces of quadrics of low codimension.

1. Introduction

Let R be a standard graded K-algebra, that is, an algebra of the form $R = K[x_1, \dots, x_n]/I$ where $K[x_1, \dots, x_n]$ is a polynomial ring over the field K and I is a homogeneous ideal with respect to the grading $\deg(x_i) = 1$. Let M be a finitely generated graded R-module. Consider the (essentially unique) minimal graded R-free resolution of M

$$\cdots \to R^{\beta_i} \to R^{\beta_{i-1}} \to \cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

The rank β_i of the i-th module in the minimal free resolution of M is called the i-th *Betti number* of M. One can also keep track of the graded structure of the resolution. It follows that the free modules in the resolution are indeed direct sums of "shifted" copies of R:

$$R^{\beta_i} = \oplus_j R(-j)^{\beta_{ij}}$$

where $R(-a)$ denotes the free module with the generator in degree a and that the matrices representing the maps between free modules have homogeneous entries. The number β_{ij} is called the *(i,j)-th graded Betti number* of M. The resolution is finite if $\beta_i = 0$ for $i \gg 0$.

For which algebras R does every module M has a finite minimal free resolution? The answer is given by (the graded version of) the Auslander-Buchsbaum-Serre theorem:

Theorem 1.1. *(Auslander-Buchsbaum-Serre) Let R be a standard graded K-algebra. The following are equivalent:*

1. *Every finitely generated graded R-module M has a finite minimal free resolution as an R-module.*
2. *The field K, regarded as an R-module via the identification $K = R/\oplus_{i>0} R_i$, has a finite minimal free resolution as an R-module.*
3. *R is regular, i.e. R is (isomorphic to) a polynomial ring.*

If R is not regular then the resolution of K is infinite. The *Poincaré series* $P_R(z)$ of R is the formal power series whose coefficients are the Betti numbers of K, i.e.

$$P_R(z) = \sum_{i \geq 0} \beta_i^R(K) z^i$$

where $\beta_i^R(K)$ is the i-th Betti number of K as an R-module.

Serre asked in [S] whether the Poincaré series $P_R(z)$ is a rational series, that is whether there exist polynomials $a(z), b(z) \in \mathbf{Q}[z]$ such that $P_R(z) = a(z)/b(z)$. The positive answer to Serre's question became well-known under the name of Serre's conjecture. This conjecture has been proved for several classes of algebras. For instance it holds for complete intersections (Tate, Assmus) and for algebras defined by monomials (Backelin). But in 1981 Anick [A] discovered algebras with irrational Poincaré series, such as:

$$\mathbf{Q}[x_1, x_2, \ldots, x_5]/(x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5) + m^3$$

More recently Roos and Sturmfels have shown that irrational Poincaré series arise also in the realm of toric rings, see[RS].

The Poincaré series of R takes into account the rank of the free modules in the minimal free resolution of K. One can also consider the degrees of the generators of the free modules. This leads to the introduction of estimates for the growth of the degrees of the syzygies (like for instance Backelin's rate) and to the definition of Koszul algebras:

Definition 1.2. (Priddy) A standard graded K-algebra R is Koszul if for all i the generators of the i-th free module in the minimal free resolution of K have degree i. Equivalently, R is Koszul if the entries of matrices representing the maps in the minimal free resolution of K are homogeneous of degree 1.

Example 1.3. Let $R = K[x]/(x^n)$ with $n > 1$. Then the resolution of K as an R-module is

$$\ldots \xrightarrow{x^{n-1}} R \xrightarrow{x} R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \to K \to 0$$

Hence R is Koszul iff $n = 2$.

The algebra R is said to be:

(1) quadratic if its defining ideal I is generated by quadrics (i.e. homogeneous elements of degree 2).
(2) G-quadratic if I has a Gröbner basis of quadrics with respect to some system of coordinates and some term order.
(3) LG-quadratic (the L stands for lifting) if there exist a G-quadratic algebra S and a S-regular sequence $y_1, \ldots, y_s$ of elements of degree 1 in S such that $S/(y_1, \ldots, y_s) \simeq R$.

One has:

$$(2) \Rightarrow (3) \Rightarrow \text{ Koszul } \Rightarrow (1)$$

Implications (2) $\Rightarrow$ (3) and Koszul $\Rightarrow$ (1) cannot be reversed in general. We do not know examples of Koszul algebras which are not LG-quadratic.

By a theorem of Tate (see [F]) every quadratic complete intersection is Koszul, but not all of them are G-quadratic. Non-G-quadratic complete intersection of quadrics are given in [ERT]. The easiest example of non-G-quadratic and quadratic complete intersection is given by 3 general quadrics in 3 variables. But every complete intersection of quadrics is LG-quadratic as the following argument of G.Caviglia shows.

Example 1.4. If $R = K[x_1, \ldots, x_n]/(q_1, \ldots, q_m)$ is a complete intersection of quadrics then $R = S/(y_1, \ldots, y_m)$ where

$$S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]/(y_1^2 + q_1, \ldots, y_m^2 + q_m).$$

That $y_1^2 + q_1, \dots, y_m^2 + q_m$ is a Gröbner basis is an easy consequence of Buchberger criterion. That $y_1, \dots, y_m$ form a S-regular sequence follows by an Hilbert function computation.

The Koszul property can be characterized in terms of the Poincaré series. Denote by $H_R(z)$ the Hilbert series of R. Then one has:

$$R \text{ is Koszul } \Leftrightarrow P_R(z)H_R(-z) = 1$$

In particular, Koszul algebras have rational Poincaré series.

2. Filtrations, points and curves

Given an algebra R it can be very difficult to detect whether R is Koszul or not. One can compute the first few matrices in the resolution and check whether they are linear. If they are not, then R is not Koszul. If instead they are linear, one can then compute few more matrices. But the growth of the size of the matrices (i.e. the growth of the Betti numbers) is in general very fast. And it is known that the first non-linear syzygy can appear in arbitrarily high homological degree even for algebras with a given Hilbert function.

A quite efficient method to prove that an algebra is Koszul is that given by filtration arguments of various kinds. These notions have been used by various authors. Inspired by the work of Eisenbud, Reeves and Totaro [ERT], Bruns, Herzog and Vetter [BHV] and of Herzog, Hibi and Restuccia [HHR], we have defined:

Definition 2.1. Let R be a standard graded algebra and let F be a family of ideals of R. Then F is said to be a Koszul filtration of R if the following conditions hold:

(1) Every ideal in F is generated by linear forms,
(2) The ideal (0) and the maximal homogeneous ideal $\oplus_{i>0} R_i$ are in F,
(3) For every non-zero I in F there exists J in F such that $J \subset I$, I/J is cyclic and $J : I$ is also in F.

Definition 2.2. Let R be a standard graded algebra. A Gröbner flag of R is a Koszul filtration F of R which consists of a single complete flag. In other words, a Gröbner flag is a set of ideals $F = \{(0), (V_1), (V_2), \dots, (V_n) = (R_1)\}$ where V_i is a i-dimensional subspace of R_1, $V_i \subset V_{i+1}$ and $(V_i) : (V_{i+1}) = (V_j)$ for some j depending on i.

As the names suggest, we have:

Theorem 2.3. (1) *Let F be a Koszul filtration of R. Then $\operatorname{Tor}_i^R(R/I, K)_j = 0$ for all $i \neq j$ and for all $I \in F$. In particular, R is Koszul.*
(2) *If R has a Gröbner flag then R is G-quadratic.*

Example 2.4. Let $R = K[x_1, \dots, x_n]/I$ with I a quadratic monomial ideal. Then the set F of the ideals of R generated by subsets of $\{x_1, \dots, x_n\}$ is a Koszul filtration of R. To check it, one has only to observe that $I : (x_i)$ is generated by variables mod I.

The property of having a Koszul filtration is stronger than just being Koszul as the following example shows:

Example 2.5. Let R be a complete intersection of 5 generic quadrics in 5. As said already above, R is Koszul. But it does not have a Koszul filtration since its defining ideal does not contain quadrics of rank < 3.

As well, to have a Gröbner flag is more than G-quadratic. For instance the algebra $R = K[x, y, z]/(x^2, y^2, xz, yz)$ is obviously G-quadratic but one can easily see that R does not have a Gröbner flag.

However many classes of algebras which are known to be Koszul have indeed a Koszul filtration or even a Gröbner flag. For instance:

Theorem 2.6. *(Kempf) Let X be a set of s (distinct) points of the projective space $\mathbf{P}^n$ and let $R(X)$ denote the coordinate ring of X. If $s \leq 2n$ and the points are in general linear position then the ring $R(X)$ is Koszul, see* [K].

We have shown that:

Theorem 2.7. *With the assumption of Kempf's theorem, the ring $R(X)$ has a Gröbner flag.*

One may ask whether Kempf's theorem holds also for a larger number of points. This is not the case.

Example 2.8. There exists a set of 9 points in $\mathbf{P}^4$ which are in general linear position and whose coordinate ring is quadratic but non-Koszul. It is obtained via Gröbner-lifting form the ideal number (55) in Roos' list [R].

On the other hand for "generic points" (indeterminates coordinates) we have the following:

Theorem 2.9. *Let X be a set of "generic points" in $\mathbf{P}^n$. Then $R(X)$ is Koszul if and only if $|X| \leq 1 + n + (n^2/4)$.*

Let C be a smooth algebraic curve of genus g over an algebraically closed field of characteristic zero. If C is not hyperelliptic, then the canonical sheaf on C gives a canonical embedding $C \to \mathbf{P}^{g-1}$ and the coordinate ring R_C of C in this embedding is the canonical ring of C. It is known that R_C is quadratic unless C is a trigonal curve of genus $g \geq 5$ or a plane quintic. Another important application of the filtration arguments is the following theorem.

Theorem 2.10. *Let R_C be the coordinate ring of a curve C in its canonical embedding. Assume that R_C is quadratic. Then R_C is Koszul.*

This is due to Vishik and Finkelberg [VF]; other proofs are given by Polishchuk [P], and by Pareschi and Purnaprajna [PP]. We are able to show that:

Theorem 2.11. *Let R_C be as in the Theorem 2.10. Then R_C has a Gröbner flag.*

For integers n, d, s the "pinched Veronese" $\mathrm{PV}(n, d, s)$ is the K-algebra generated by the monomials of degree d in n variables and with at most s non-zero exponents, that is,

$$\mathrm{PV}(n, d, s) = K[x_1^{a_1} \cdots x_n^{a_n} : \quad \sum a_j = d \quad \text{and} \quad \#\{j : a_j > 0\} \leq s].$$

It is an open question whether $\mathrm{PV}(n, d, s)$ is Koszul (it is not even clear whether it is quadratic). G.Caviglia shown in [Ca] that the first not trivial pinched Veronese $\mathrm{PV}(3, 3, 2)$ is Koszul by using a combination of filtrations and ad hoc arguments.

3. Artinian Gorenstein algebras of cubics

The algebras R_C are 2-dimensional Gorenstein domains with h-vector $1 + nz + nz^2 + z^3$ and Theorem 2.10 asserts that they are Koszul as soon as they are quadratic. One might ask:

Question 3.1. Let R be a quadratic Gorenstein algebra with h-vector $1 + nz + nz^2 + z^3$. Is R Koszul?

Without loss of generality, one can assume that the algebra is Artinian. Artinian Gorenstein algebras are described via Macaulay inverse system. Let us recall how. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K of characteristic 0. Let f be a non-zero polynomial of S which is homogeneous of degree, say, s. Let I_f be the ideal of S of the polynomials $g(x_1, \dots, x_n)$ such that

$$g(\partial/\partial x_1, \dots, \partial/\partial x_n)f = 0.$$

Set $R_f = S/I_f$. It is known that R_f is a Gorenstein Artinian algebra with socle in degree s and that every such an algebra arises in this way. In particular, in the case $s = 3$ the Hilbert series of R_f is equal to $1 + nz + nz^2 + z^3$ (provided f is not a cone). So Question 3.1 is about algebras R_f with f a cubic form. We are able to show the following:

Theorem 3.2. (1) *Let f be a cubic in S. Assume there exist linear forms y, z such that $\partial f/\partial yz = 0$ and $\partial f/\partial y$ and $\partial f/\partial z$ are quadrics of rank $n-1$. Then R_f has a Koszul filtration.*
(2) *If f is smooth then R_f is not G-quadratic.*
(3) *For the generic cubic f, the ring R_f is Koszul and not G-quadratic.*

Furthermore:

Theorem 3.3. (1) *Let f be a cubic in S. Assume there exists linear form y such that $\partial f/\partial y^2 = 0$ and $\partial f/\partial y$ is quadric of rank $n-1$. Then R_f has a Gröbner flag.*
(2) *For the generic singular cubic f, the ring R_f is G-quadratic.*

We are not able to answer Question 3.1 in general. But in [CRV] we have shown that Question 3.1 has an affermative answer $n = 3, 4$. In both cases the characterization of the f such that R_f quadratic (or Koszul) is very elegant:

Theorem 3.4. *For $n = 3$ or 4, the following are equivalent:*

(1) *R_f is quadratic.*
(2) *R_f is Koszul.*
(3) *The ideal of 2-minors of the Hessian matrix $(\partial f/\partial x_i x_j)$ of f has codimension n.*

Furthermore for $n = 3$ these conditions are equivalent also to:

(4) *f is not in the closure of the $\mathrm{GL}_3(K)$-orbit of the Fermat cubic $x_1^3 + x_2^3 + x_3^3$.*

G.Caviglia shown in his unpublished master thesis that property (1),(2) and (3) of 3.4 are equivalent also in the case of $n = 5$.

Another interesting question is whether the assumption of 3.2(1) indeed characterize Koszul property for R_f. In this case the answer is no, as the following example shows.

Example 3.5. Let f be the Veronese cubic, that is the determinant of a 3×3 symmetric matrix filled with 6 distinct variables $x_1, \dots, x_6$ and let H be its Hessian matrix. The cubic f has a remarkable property: $\det H$ is f^2 up to scalar and the ideal of 5-minors of H is $(x_1, \dots, x_6)^2 f$. These facts imply that f does not satisfy the assumption of 3.2(1), nevertheless R_f is Koszul (even G-quadratic).

Also, one could also ask whether R_f is LG-quadratic provided it is quadratic. We have reasons to believe that the answer to this question might be positive.

4. Space of quadrics of low codimension

Another point of view we have taken is the following. Let V be a vector space of quadrics of dimension d in n variables. Set $c = \binom{n+1}{2} - d$ the codimension of V in the space of quadrics. Let R_V be the quadratic algebra defined by the ideal generated by V. A theorem of Backelin [B] asserts that if $c \leq 2$ then R_V is Koszul. We have proved in [C] that:

Theorem 4.1. (1) *If $c < n$ then the ring R_V is G-quadratic for a generic V.*
(2) *If $c \leq 2$ then R_V is G-quadratic with, essentially, one exception given by $V = \langle x^2, xy, y^2 - xz, yz\rangle$ in $K[x, y, z]$.*

The "exceptional" algebra $K[x,y,z]/(x^2, xy, y^2 + xz, yz)$ has Hilbert series

$$1 + 3z + 2z^2 + z^3 + z^4 + z^5 + \dots$$

and is LG-quadratic since we may deform it to

$$K[x, y, z, t]/(x^2 + xt, xy + yt, yz + xt, y^2 + xz)$$

which is G-quadratic in the given coordinate system and with respect to revlex $t > x > y > z$.

It follows from 4.1 that every quadratic Artinian algebra R with $\dim R_2 \leq 2$ is G-quadratic.

A recent conjecture of Polishchuk [P2] on Koszul configurations of points, suggests that Artinian quadratic algebras R with $\dim R_2 = 3$ should be Koszul. This is what we have proved in [C1]:

Theorem 4.2. *Let R be an Artinian quadratic algebras with $\dim R_2 = 3$. Then:*
(1) *R is Koszul.*
(2) *R is G-quadratic unless it is (up to trivial extension) a complete intersection of 3 general quadrics in 3 variables.*

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Symbolic Rees rings of space monomial curves in characteristic p and existence of negative curves in characteristic 0

Kazuhiko Kurano

Meiji University

This is a joint work with Naoyuki Matsuoka from Meiji University.

We refer the reader to [7] for detail and proofs.

Our aim is to study finite generation of symbolic Rees rings of the defining ideal of the space monomial curves (t^a, t^b, t^c) for pairwise coprime integers a, b, c such that $(a, b, c) \neq (1, 1, 1)$. If such a ring is not finitely generated over a base field, then it is a counterexample to the Hilbert's fourteenth problem. Finite generation of such rings is deeply related to existence of negative curves on certain normal projective surfaces. We prove that, in the case of $(a + b + c)^2 > abc$, a negative curve exists. Using a computer, we shall show that a negative curve exists if all of a, b, c are at most 300. As a corollary, the symbolic Rees rings of space monomial curves are shown to be finitely generated if a base field is of positive characteristic and all of a, b, c are less than or equal to 300.

1 Symbolic Rees rings of monomial curves and Hilbert's fourteenth problem

Throughout of this note, we assume that rings are commutative with unit.

For a prime ideal P of a ring A, $P^{(r)}$ denotes the r-th symbolic power of P, i.e.,

$$P^{(r)} = P^r A_P \cap A.$$

By definition, it is easily seen that $P^{(r)} P^{(r')} \subset P^{(r+r')}$ for any r, $r' \geq 0$, therefore,

$$\bigoplus_{r \geq 0} P^{(r)} T^r$$

is a subring of the polynomial ring $A[T]$. This subring is called the *symbolic Rees ring* of P, and denoted by $R_s(P)$.

Let k be a field and m be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the k-algebra homomorphism

$$\phi_k : k[x_1, \ldots, x_m] \longrightarrow k[t]$$

given by $\phi_k(x_i) = t^{a_i}$ for $i = 1, \ldots, m$, where $x_1, \ldots, x_m$, t are indeterminates over k. Let $\mathfrak{p}_k(a_1, \ldots, a_m)$ be the kernel of ϕ_k. We sometimes denote $\mathfrak{p}_k(a_1, \ldots, a_m)$ simply by $\mathfrak{p}$ or $\mathfrak{p}_k$ if no confusion is possible.

Theorem 1.1 *Let k be a field and m be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the prime ideal $\mathfrak{p}_k(a_1, \ldots, a_m)$ of the polynomial ring $k[x_1, \ldots, x_m]$.*

Let α_1, α_2, β_1, $\ldots$, β_m, t, T be indeterminates over k. Consider the following injective k-homomorphism

$$\xi : k[x_1, \ldots, x_m, T] \longrightarrow k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t)$$

given by $\xi(T) = \alpha_2/\alpha_1$ and $\xi(x_i) = \alpha_1\beta_i + t^{a_i}$ for $i = 1, \ldots, m$.

Then,

$$k(\alpha_1\beta_1+t^{a_1}, \alpha_1\beta_2+t^{a_2}, \ldots, \alpha_1\beta_m+t^{a_m}, \alpha_2/\alpha_1) \cap k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] = \xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$$

holds true.

Remark 1.2 Let k be a field. Let R be a polynomial ring over k with finitely many variables. For a field L satisfying $k \subset L \subset Q(R)$, Hilbert asked in 1900 whether the ring $L \cap R$ is finitely generated as a k-algebra or not. It is called the *Hilbert's fourteenth problem.*

The first counterexample to this problem was discovered by Nagata [10] in 1958. An easier counterexample was found by Paul C. Roberts [11] in 1990. Further counterexamples were given by Kuroda, Mukai, etc.

On the other hand, Goto, Nishida and Watanabe [2] proved that $R_s(\mathfrak{p}_k(7n-3, (5n-2)n, 8n-3))$ is not finitely generated over k if the characteristic of k is zero, $n \geq 4$ and $n \not\equiv 0$ (3). By Theorem 1.1, we know that they are new counterexamples to the Hilbert's fourteenth problem.

Remark 1.3 With notation as in Theorem 1.1, we set

$$\begin{aligned} D_1 &= \alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2} - \beta_1\frac{\partial}{\partial\beta_1} - \cdots - \beta_m\frac{\partial}{\partial\beta_m} \\ D_2 &= a_1t^{a_1-1}\frac{\partial}{\partial\beta_1} + \cdots + a_mt^{a_m-1}\frac{\partial}{\partial\beta_m} - \alpha_1\frac{\partial}{\partial t}. \end{aligned}$$

Assume that the characteristic of k is zero.

Then, one can prove that $\xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$ is equal to the kernel of the derivations D_1 and D_2, i.e.,

$$\xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m))) = \{f \in k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] \mid D_1(f) = D_2(f) = 0\}.$$

2 Symbolic Rees rings of space monomial curves

In the rest of this paper, we restrict ourselves to the case $m = 3$. For the simplicity of notation, we write x, y, z, a, b, c for x_1, x_2, x_3, a_1, a_2, a_3, respectively. We regard the polynomial ring $k[x, y, z]$ as a $\mathbb{Z}$-graded ring by $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$.

$\mathfrak{p}_k(a, b, c)$ is the kernel of the k-algebra homomorphism

$$\phi_k : k[x, y, z] \longrightarrow k[t]$$

given by $\phi_k(x) = t^a$, $\phi_k(y) = t^b$, $\phi_k(z) = t^c$.

By a result of Herzog [3], we know that $\mathfrak{p}_k(a, b, c)$ is generated by at most three elements.

We are interested in the symbolic powers of $\mathfrak{p}_k(a, b, c)$. If $\mathfrak{p}_k(a, b, c)$ is generated by two elements, then the symbolic powers always coincide with ordinary powers because $\mathfrak{p}_k(a, b, c)$ is a complete intersection. However, it is known that, if $\mathfrak{p}_k(a, b, c)$ is minimally generated by three elements, the second symbolic power is strictly bigger than the second ordinary power.

We are interested in finite generation of the symbolic Rees ring $R_s(\mathfrak{p}_k(a, b, c))$. It is known that this problem is reduced to the case where a, b and c are pairwise coprime, i.e.,

$$(a, b) = (b, c) = (c, a) = 1.$$

In the rest of this paper, we always assume that a, b and c are pairwise coprime.

Let $\mathbb{P}_k(a, b, c)$ be the weighted projective space $\mathrm{Proj}(k[x, y, z])$. Then

$$\mathbb{P}_k(a, b, c) \setminus \{V_+(x, y),\quad V_+(y, z),\quad V_+(z, x)\}$$

is a regular scheme. In particular, $\mathbb{P}_k(a, b, c)$ is smooth at the point $V_+(\mathfrak{p}_k(a, b, c))$. Let $\pi : X_k(a, b, c) \longrightarrow \mathbb{P}_k(a, b, c)$ be the blow-up at $V_+(\mathfrak{p}_k(a, b, c))$. Let E be the exceptional divisor, i.e.,

$$E = \pi^{-1}(V_+(\mathfrak{p}_k(a, b, c))).$$

We sometimes denote $\mathfrak{p}_k(a, b, c)$ (resp. $\mathbb{P}_k(a, b, c)$, $X_k(a, b, c)$) simply by $\mathfrak{p}$ or $\mathfrak{p}_k$ (resp. $\mathbb{P}$ or $\mathbb{P}_k$, X or X_k) if no confusion is possible.

It is easy to see that

$$\mathrm{Cl}(\mathbb{P}) = \mathbb{Z}H \simeq \mathbb{Z},$$

where H is the Weil divisor corresponding to the reflexive sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Set $H = \sum_i m_i D_i$, where D_i's are subvarieties of $\mathbb{P}$ of codimension one. We may choose D_i's such that $D_i \not\ni V_+(\mathfrak{p})$ for any i. Then, set $A = \sum_i m_i \pi^{-1}(D_i)$.

One can prove that

$$\mathrm{Cl}(X) = \mathbb{Z}A + \mathbb{Z}E \simeq \mathbb{Z}^2.$$

Since all Weil divisor on X are $\mathbb{Q}$-Cartier, we have the intersection pairing

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \longrightarrow \mathbb{Q},$$

that satisfies

$$A^2 = \frac{1}{abc}, \quad E^2 = -1, \quad A.E = 0.$$

Here, we have the following natural identification:

$$H^0(X, \mathcal{O}_X(nA - rE)) = \begin{cases} [\mathfrak{p}^{(r)}]_n & (r \geq 0) \\ S_n & (r < 0) \end{cases}$$

Therefore, the *total coordinate ring* (or *Cox ring*)

$$TC(X) = \bigoplus_{n,r \in \mathbf{Z}} H^0(X, \mathcal{O}_X(nA - rE))$$

is isomorphic to the extended symbolic Rees ring

$$R_s(\mathfrak{p})[T^{-1}] = \cdots \oplus ST^{-2} \oplus ST^{-1} \oplus S \oplus \mathfrak{p}T \oplus \mathfrak{p}^{(2)}T^2 \oplus \cdots.$$

It is well-known that $R_s(\mathfrak{p})[T^{-1}]$ is Noetherian if and only if so is $R_s(\mathfrak{p})$.

Remark 2.1 By Huneke's criterion [5] and a result of Cutkosky [1], the following four conditions are equivalent:

(1) $R_s(\mathfrak{p})$ is a Noetherian ring, or equivalently, finitely generated over k.

(2) $TC(X)$ is a Noetherian ring, or equivalently, finitely generated over k.

(3) There exist positive integers r, s, $f \in \mathfrak{p}^{(r)}$, $g \in \mathfrak{p}^{(s)}$, and $h \in (x, y, z) \setminus \mathfrak{p}$ such that

$$\ell_{S_{(x,y,z)}}(S_{(x,y,z)}/(f, g, h)) = rs \cdot \ell_{S_{(x,y,z)}}(S_{(x,y,z)}/(\mathfrak{p}, h)),$$

where $\ell_{S_{(x,y,z)}}$ is the length as an $S_{(x,y,z)}$-module.

(4) There exist curves C and D on X such that

$$C \neq D, \quad C \neq E, \quad D \neq E, \quad C.D = 0.$$

Here, a curve means a closed irreducible reduced subvariety of dimension one.

The condition (4) as above is equivalent to that just one of the following two conditions is satisfied:

(4-1) There exist curves C and D on X such that

$$C \neq E, \quad D \neq E, \quad C^2 < 0, \quad D^2 > 0, \quad C.D = 0.$$

(4-2) There exist curves C and D on X such that

$$C \neq E, \quad D \neq E, \quad C \neq D, \quad C^2 = D^2 = 0.$$

Definition 2.2 A curve C on X is called a *negative curve* if

$$C \neq E \text{ and } C^2 < 0.$$

It is proved that two distinct negative curves never exist.

In the case where the characteristic of k is positive, Cutokosky [1] proved that $R_s(\mathfrak{p})$ is finitely generated over k if there exists a negative curve on X.

We remark that there exists a negative curve on X if and only if there exists positive integers n and r such that

$$\frac{n}{r} < \sqrt{abc} \text{ and } [\mathfrak{p}^{(r)}]_n \neq 0.$$

We are interested in existence of a negative curve. Let a, b and c be pairwise coprime positive integers. By the following lemma, if there exists a negative curve on $X_{k_0}(a,b,c)$ for a field k_0 of characteristic 0, then there exists a negative curve on $X_k(a,b,c)$ for any field k.

Lemma 2.3 *Let a, b and c be pairwise coprime positive integers.*

1. *Let K/k be a field extension. Then, for any integers n and r,*

$$[\mathfrak{p}_k(a,b,c)^{(r)}]_n \otimes_k K = [\mathfrak{p}_K(a,b,c)^{(r)}]_n.$$

2. *For any integers n, r and any prime number p,*

$$\dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{F}_p}(a,b,c)^{(r)}]_n \geq \dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}(a,b,c)^{(r)}]_n$$

holds, where $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{F}_p$ is the prime field of characteristic p. Here, $\dim_{\mathbb{F}_p}$ (resp. $\dim_{\mathbb{Q}}$) denotes the dimension as an $\mathbb{F}_p$-vector space (resp. $\mathbb{Q}$-vector space).

Remark 2.4 Let a, b, c be pairwise coprime positive integers. Assume that there exists a negative curve on $X_{k_0}(a,b,c)$ for a field k_0 of characteristic zero.

By Lemma 2.3, we know that there exists a negative curve on $X_k(a,b,c)$ for any field k. Therefore, if k is a field of characteristic positive, then the symbolic Rees ring $R_s(\mathfrak{p}_k)$ is finitely generated over k by a result of Cutkosky [1]. However, if k is a field of characteristic zero, then $R_s(\mathfrak{p}_k)$ is not necessary Noetherian. In fact, assume that k is of characteristic zero and $(a,b,c) = (7n-3,(5n-2)n, 8n-3)$ with $n \not\equiv 0$ (3) and $n \geq 4$ as in Goto-Nishida-Watanabe [2]. Then there exists a negative curve, but $R_s(\mathfrak{p}_k)$ is not Noetherian.

Definition 2.5 Let a, b, c be pairwise coprime positive integers. Let k be a field.
We define the following three conditions:

(C1) There exists a negative curve on $X_k(a,b,c)$, i.e., $[\mathfrak{p}_k(a,b,c)^{(r)}]_n \neq 0$ for some positive integers n, r satisfying $n/r < \sqrt{abc}$.

(C2) There exist positive integers n, r satisfying $n/r < \sqrt{abc}$ and $\dim_k S_n > r(r+1)/2$.

(C3) There exist positive integers q, r satisfying $abcq/r < \sqrt{abc}$ and $\dim_k S_{abcq} > r(r+1)/2$.

Here, $\dim_k$ denotes the dimension as a k-vector space.

By the following lemma, we know the implications

$$(C3) \Longrightarrow (C2) \Longrightarrow (C1)$$

since $\dim_k[\mathfrak{p}^{(r)}]_n = \dim_k S_n - \dim_k[S/\mathfrak{p}^{(r)}]_n$.

Lemma 2.6 *Let a, b, c be pairwise coprime positive integers. Let r and n be non-negative integers. Then,*

$$\dim_k[S/\mathfrak{p}^{(r)}]_n \leq r(r+1)/2$$

holds true for any field k.

Remark 2.7 It is easy to see that $[\mathfrak{p}_k(a,b,c)]_n \neq 0$ if and only if $\dim_k S_n \geq 2$. Therefore, if we restrict ourselves to $r = 1$, then (C1) and (C2) are equivalent.

On can prove that, if (C1) is satisfied with $r \leq 2$ for a field k of characteristic zero, then (C2) is satisfied.

Assume that k is a field of characteristic zero. Let a, b and c be pairwise coprime integers such that $1 \leq a, b, c \leq 300$. As we shall see in Theorem 5.1, a negative curve exists unless $(a,b,c) = (1,1,1)$. In these cases, calculations by a computer show that (C2) is satisfied if (C1) holds with $r \leq 5$.

We shall discuss the difference between (C1) and (C2) in Section 5.1.

Remark 2.8 Let a, b and c be pairwise coprime positive integers. Assume that $\mathfrak{p}_k(a,b,c)$ is a complete intersection, i.e., generated by two elements.

Permuting a, b and c, we may assume that

$$\mathfrak{p}_k(a,b,c) = (x^b - y^a,\ z - x^\alpha y^\beta)$$

for some $\alpha, \beta \geq 0$ satisfying $\alpha a + \beta b = c$. If $ab < c$, then

$$\deg(x^b - y^a) = ab < \sqrt{abc}.$$

If $ab > c$, then

$$\deg(z - x^\alpha y^\beta) = c < \sqrt{abc}.$$

If $ab = c$, then (a,b,c) must be equal to $(1,1,1)$. Ultimately, there exists a negative curve if $(a,b,c) \neq (1,1,1)$.

3 The case where $(a+b+c)^2 > abc$

In the rest of this paper, we set $\xi = abc$ and $\eta = a+b+c$ for pairwise coprime positive integers a, b and c.

For $v = 0, 1, \ldots, \xi - 1$, we set

$$S^{(\xi,v)} = \oplus_{q \geq 0} S_{\xi q + v}.$$

This is a module over $S^{(\xi)} = \oplus_{q \geq 0} S_{\xi q}$.

Lemma 3.1

$$\dim_k [S^{(\xi,v)}]_q = \dim_k S_{\xi q + v} = \frac{1}{2}\left\{\xi q^2 + (\eta + 2v)q + 2\dim_k S_v\right\}$$

holds for any $q \geq 0$.

Lemma 3.2 Assume that a, b and c are pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$. Then, $\eta - \sqrt{\xi} \neq 0, 1, 2$.

Theorem 3.3 *Let a, b and c be pairwise coprime integers such that $(a,b,c) \neq (1,1,1)$. Then, we have the following:*

1. *Assume that $\sqrt{abc} \notin \mathbb{Z}$. Then, (C3) holds if and only if $(a+b+c)^2 > abc$.*

2. *Assume that $\sqrt{abc} \in \mathbb{Z}$. Then, (C3) holds if and only if $(a+b+c)^2 > 9abc$.*

3. *If $(a+b+c)^2 > abc$, then, (C2) holds. In particular, a negative curve exists in this case.*

Remark 3.4 If $(a+b+c)^2 > abc$, then $R_s(\mathfrak{p})$ is Noetherian by a result of Cutkosky [1].

If $(a+b+c)^2 > abc$ and $\sqrt{abc} \notin \mathbb{Q}$, then the existence of a negative curve follows from Nakai's criterion for ampleness, Kleimann's theorem and the cone theorem (e.g. Theorem 1.2.23 and Theorem 1.4.23 in [8], Theorem 4-2-1 in [6]).

The condition $(a+b+c)^2 > abc$ is equivalent to $(-K_X)^2 > 0$. If $-K_X$ is ample, then the finite generation of the total coordinate ring follows from Proposition 2.9 and Corollary 2.16 in Hu-Keel [4].

If $(a,b,c) = (5,6,7)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $12A - E$. Since $(a+b+c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X.C = (18A - E).(12A - E) = 0.028\cdots > 0,$$

$-K_X$ is ample by Nakai's criterion.

If $(a,b,c) = (7,8,9)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $16A - E$. Since $(a+b+c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X.C = (24A - E).(16A - E) = -0.23\cdots < 0,$$

$-K_X$ is not ample by Nakai's criterion.

4 Degree of a negative curve

Proposition 4.1 *Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Suppose that a negative curve exists, i.e., there exist positive integers n and r satisfying $[\mathfrak{p}_k(a,b,c)^{(r)}]_n \neq 0$ and $n/r < \sqrt{abc}$.*

Set n_0 and r_0 to be

$$\begin{aligned} n_0 &= \min\{n \in \mathbb{N} \mid \exists r > 0 \text{ such that } n/r < \sqrt{\xi} \text{ and } [\mathfrak{p}^{(r)}]_n \neq 0\} \\ r_0 &= \lfloor \frac{n}{\sqrt{\xi}} \rfloor + 1, \end{aligned}$$

where $\lfloor \frac{n}{\sqrt{\xi}} \rfloor$ is the maximum integer which is less than or equal to $\frac{n}{\sqrt{\xi}}$.

Then, the negative curve C is linearly equivalent to $n_0 A - r_0 E$.

Remark 4.2 Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Assume that the negative curve C exists, and C is linearly equivalent to $n_0 A - r_0 E$.

Then, by Proposition 4.1, we obtain

$$\begin{aligned} n_0 &= \min\{n \in \mathbb{N} \mid [\mathfrak{p}^{(\lfloor \frac{n}{\sqrt{\xi}} \rfloor + 1)}]_n \neq 0\} \\ r_0 &= \lfloor \frac{n_0}{\sqrt{\xi}} \rfloor + 1. \end{aligned}$$

Proposition 4.3 *Let a, b and c be pairwise coprime positive integers such that $\sqrt{\xi} > \eta$. Assume that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1+1)/2$. Suppose $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$.*

Then, $q_1 < \frac{2 \dim_k S_{v_1}}{\sqrt{\xi} - \eta}$ holds.

In particular,

$$n_1 = \xi q_1 + v_1 < \frac{2\xi \max\{\dim_k S_t \mid 0 \leq t < \xi\}}{\sqrt{\xi} - \eta} + \xi.$$

5 Calculation by computer

5.1 Examples that do not satisfy (C2)

Suppose that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1+1)/2$. Put $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$. If $\sqrt{\xi} > \eta$, then $q_1 < \frac{2\dim_k S_{v_1}}{\sqrt{\xi}-\eta}$ holds by Proposition 4.3.

By a following programming on MATHEMATICA ([7]), we can check whether (C2) is satisfied or not in the case where $\sqrt{\xi} > \eta$.

Calculations by a computer show that (C2) is not satisfied in some cases, for example, $(a,b,c) = (9,10,13), (13,14,17)$.

The examples due to Goto-Nishida-Watanabe [2] have negative curves with $r = 1$. Therefore, by Remark 2.7, they satisfy the condition (C2).

In the case where $(a,b,c) = (9,10,13), (13,14,17)$, the authors do not know whether $R_s(\mathfrak{p}_k)$ is Noetherian or not in the case where the characteristic of k is zero, however the negative curve do exists as in Theorem 5.1 below.

If we input $(a,b,c) = (5,26,43)$, then the output is $(n,r) = (1196,16)$. Therefore, (C2) is satisfied with $(n,r) = (1196,16)$. However, the negative curve on $X_{\mathbb{C}}(5,26,43)$ is linearly equivalent to $515A - 7E$ by a calculation in the next subsection.

5.2 Existence of a negative curve

Theorem 5.1 *Let a, b and c be pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$. Assume that the characteristic of k is zero.*

If all of a, b and c are at most 300, then there exists a negative curve on X.

Let a, b and c be pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$ and $1 \leq a,b,c \leq 300$. Then, by Theorem 5.1, there exists a negative curve in the case where k is of characteristic zero. Then, by Remark 2.4, $R_s(\mathfrak{p}_k(a,b,c))$ is Noetherian in the case where k is of positive characteristic. Thus, we obtain the following corollary immediately.

Corollary 5.2 *Let a, b and c be pairwise coprime positive integers such that all of a, b and c are at most 300. Assume that the characteristic of k is positive.*

Then the symbolic Rees ring $R_s(\mathfrak{p}_k(a,b,c))$ is Noetherian.

Remark 5.3 Assume that the characteristic of k is zero. Let a, b and c be pairwise coprime positive integers such that $a+b+c < \sqrt{abc}$, $(a,b,c) \neq (1,1,1)$ and $1 \leq a \leq b \leq c \leq 300$.

More than 90% in these cases satisfy (C2).

Using this program, it is possible to know n_0 and r_0 such that the negative curve is linearly equivalent to $n_0A - r_0E$ (cf. Remark 4.2).

Calculations show the following.
The maximal value of r_0 is nine.
In the case where $r_0 \leq 5$, (C2) is satisfied, i.e.,

$$\dim_k S_{n_0} > r_0(r_0+1)/2.$$

Suppose $(a,b,c) = (9,10,13)$. In the case where the characteristic of k is zero, the negative curve is linearly equivalent to $305A - 9E$. We know that the negative curve is also linearly equivalent to $305A - 9E$ if the characteristic of k is sufficiently large. On the other hand, the negative curve is linearly equivalent to $100A - 3E$ if the characteristic of k is three as in Morimoto-Goto [9]. Therefore, the linear equivalent class that contains the negative curve depends on the characteristic of a base field. Assume that the characteristic is a sufficiently large prime number. Let C be the negative curve and D be a curve that satisfies (4-1) in Remark 2.1. Suppose that D is linearly equivalent to $n_1A - r_1E$. Since $C.D = 0$, we know

$$\frac{n_1}{r_1} = \frac{9^2 \cdot 10 \cdot 13}{305}.$$

Therefore, r must be a multiple of 305.

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Kazuhiko Kurano
Department of Mathematics
Faculty of Science and Technology
Meiji University
Higashimita 1-1-1, Tama-ku
Kawasaki 214-8571, Japan
kurano@math.meiji.ac.jp
http://www.math.meiji.ac.jp/~kurano

LINEARITY DEFECT AND REGULARITY OVER A KOSZUL ALGEBRA

KOHJI YANAGAWA

ABSTRACT. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a Koszul algebra over a field $K = A_0$, and $^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. The *linearity defect* $\mathrm{ld}_A(M)$ of $M \in {}^*\mathrm{mod}\, A$ is an invariant defined by Herzog and Iyengar. An exterior algebra E is a Koszul algebra which is the Koszul dual of a polynomial ring. Eisenbud et al. showed that $\mathrm{ld}_E(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, E$. Improving their result, we show that the Koszul dual $A^!$ of a Koszul *commutative* algebra A satisfies the following.

- Let $M \in {}^*\mathrm{mod}\, A^!$. If $\{ \dim_K M_i \mid i \in \mathbb{Z} \}$ is bounded, then $\mathrm{ld}_{A^!}(M) < \infty$.
- If A is complete intersection, then $\mathrm{reg}_{A^!}(M) < \infty$ and $\mathrm{ld}_{A^!}(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A^!$.
- If $E = \bigwedge \langle y_1, \ldots, y_n \rangle$ is an exterior algebra, then $\mathrm{ld}_E(M) \leq c^{n!}2^{(n-1)!}$ for $M \in {}^*\mathrm{mod}\, E$ with $c := \max\{ \dim_K M_i \mid i \in \mathbb{Z} \}$.

1. INTRODUCTION

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a (not necessarily commutative) graded algebra over a field $K := A_0$ with $\dim_K A_i < \infty$ for all $i \in \mathbb{N}$, and $^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. Throughout this paper, we assume that A is *Koszul*, that is, $K = A / \bigoplus_{i \geq 1} A_i$ has a graded free resolution of the form

$$\cdots \longrightarrow A(-i)^{\beta_i(K)} \longrightarrow \cdots \longrightarrow A(-2)^{\beta_2(K)} \longrightarrow A(-1)^{\beta_1(K)} \longrightarrow A \longrightarrow K \longrightarrow 0.$$

Koszul duality is a certain derived equivalence between A and its Koszul dual algebra $A^! := \mathrm{Ext}_A^\bullet(K, K)$.

For $M \in {}^*\mathrm{mod}\, A$, we have its minimal graded free resolution $\cdots \to P_1 \to P_0 \to M \to 0$, and natural numbers $\beta_{i,j}(M)$ such that $P_i \cong \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{i,j}(M)}$. We call

$$\mathrm{reg}_A(M) := \sup\{ j - i \mid i \in \mathbb{N}, j \in \mathbb{Z} \text{ with } \beta_{i,j}(M) \neq 0 \}$$

the *regularity* of M. When A is a polynomial ring, $\mathrm{reg}_A(M)$ has been deeply studied. Even for a general Koszul algebra A, $\mathrm{reg}_A(M)$ is still an interesting invariant closely related to Koszul duality (see Theorem 3.5 below).

Let $P_\bullet$ be a minimal graded free resolution of $M \in {}^*\mathrm{mod}\, A$. The *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\mathrm{lin}(P_\bullet)_i = P_i$ for all i and its differential maps are given by erasing all the entries of degree ≥ 2 from the matrices representing the differentials of $P_\bullet$. According to Herzog-Iyengar [8], we call

$$\mathrm{ld}_A(M) := \sup\{ i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \}$$

the *linearity defect* of M. This invariant is related to the regularity via Koszul duality (see Theorem 3.8 below).

In §4, we mainly treat a Koszul *commutative* algebra A or its dual $A^!$. Even in this case, it can occur that $\mathrm{ld}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$ (c.f. [8]), while Avramov-Eisenbud [1] showed that $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. On the other hand, Herzog-Iyengar [8] proved that if A is complete intersection or Golod then $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. Initiated by these results, we will show the following.

Theorem A. *Let A be a Koszul commutative algebra (more generally, a Koszul algebra with $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$). Then we have;*

(1) *Let $N \in {}^*\mathrm{mod}\, A^!$. If $\mathrm{reg}_{A^!}(N) < \infty$ (e.g. $\dim_K N < \infty$), then $\mathrm{ld}_{A^!}(N) < \infty$.*

(2) *The following conditions are equivalent.*

(a) $\mathrm{ld}_A(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, A$.

(a') $\mathrm{ld}_A(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, A$ *with* $M = \bigoplus_{i=0,1} M_i$.

(b) *If $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation, then* $\mathrm{reg}_{A^!}(N) < \infty$.

In Theorem A (2), the implications $(a) \Rightarrow (a') \Leftrightarrow (b)$ hold for a general Koszul algebra.

If A is a complete intersection, then the Koszul dual $A^!$ is left (and right) noetherian and admits a *balanced dualizing complex*, hence we have $\mathrm{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, A^!$ by [9]. So $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ by Theorem A (2). This is a special case of the above mentioned result of [8], but the proof is different.

Let ${}^*\mathrm{fp}\, A^!$ be the full subcategory of ${}^*\mathrm{mod}\, A^!$ consisting of finitely presented modules.

Theorem B. *If A is a Koszul algebra such that $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$, then $A^!$ is left coherent (in the graded context), and ${}^*\mathrm{fp}\, A^!$ is an abelian category. If further A is commutative, then Koszul duality gives $\mathcal{D}^b({}^*\mathrm{mod}\, A) \cong \mathcal{D}^b({}^*\mathrm{fp}\, A^!)^{\mathrm{op}}$. In particular, if A is a Koszul complete intersection, then we have*

$$\mathcal{D}^b({}^*\mathrm{mod}\, A) \cong \mathcal{D}^b({}^*\mathrm{mod}\, A^!)^{\mathrm{op}}.$$

We remark that the last statement of Theorem B also follows from the existence of a balanced dualizing complex and [10, Proposition 4.5].

Let $E := \bigwedge \langle y_1, \dots, y_n \rangle$ be an exterior algebra. Eisenbud et al. [6] showed that $\mathrm{ld}_E(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, E$ (now this is a special case of Theorem A, since E is the Koszul dual of a polynomial ring). If $n \geq 2$, then $\sup\{\, \mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\, E \,\} = \infty$. On the other hand, we will see that

$$\mathrm{ld}_E(N) \leq c^{n!} 2^{(n-1)!} \quad (c := \max\{\, \dim_K N_i \mid i \in \mathbb{Z} \,\}) \tag{1}$$

for $N \in {}^*\mathrm{mod}\, E$. But a computer experiment suggests that the bound could be very far from sharp. R. Okazaki and the author found a graded ideal $I \subset E$ with $n = 6$ and $\mathrm{ld}_E(E/I) = 9$. This is our "best record", but still much lower than the value given in (1).

2. Koszul Algebras and Koszul Duality

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded algebra over a field $K := A_0$ with $\dim_K A_i < \infty$ for all $i \in \mathbb{N}$, $^*\mathrm{Mod}\, A$ the category of graded left A-modules, and $^*\mathrm{mod}\, A$ the full subcategory of $^*\mathrm{Mod}\, A$ consisting of finitely generated modules. We say $M = \bigoplus_{i \in \mathbb{Z}} M_i \in {}^*\mathrm{Mod}\, A$ is *quasi-finite*, if $\dim_K M_i < \infty$ for all i and $M_i = 0$ for $i \ll 0$. If $M \in {}^*\mathrm{mod}\, A$, then it is clearly quasi-finite. We denote the full subcategory of $^*\mathrm{Mod}\, A$ consisting of quasi-finite modules by $\mathrm{qf}\, A$. (In this paper, we mainly treat a Koszul *commutative* algebra A and its dual $A^! := \mathrm{Ext}^\bullet_A(K, K)$. Even in this case, $A^!$ is *not* left noetherian in general. In fact, it is known that $A^!$ is left noetherian if and only if A is complete intersection. So $^*\mathrm{mod}\, A^!$ is not necessarily abelian, and we have to treat $\mathrm{qf}\, A^!$.) Clearly, $\mathrm{qf}\, A$ is an abelian category with enough projectives. For $M \in {}^*\mathrm{Mod}\, A$ and $j \in \mathbb{Z}$, $M(j)$ denotes the shifted module of M with $M(j)_i = M_{i+j}$. For $M, N \in {}^*\mathrm{Mod}\, A$, set $\underline{\mathrm{Hom}}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{^*\mathrm{Mod}\, A}(M, N(i))$ to be a graded K-vector space with $\underline{\mathrm{Hom}}_A(M, N)_i = \mathrm{Hom}_{^*\mathrm{Mod}\, A}(M, N(i))$. Similarly, we also define $\underline{\mathrm{Ext}}^i_A(M, N)$.

Let $\mathcal{C}(\mathrm{qf}\, A)$ be the homotopy category of cochain complexes in $\mathrm{qf}\, A$, and $\mathcal{C}^-(\mathrm{qf}\, A)$ its full subcategory consisting of complexes which are bounded above (i.e., $X^\bullet \in \mathcal{C}(\mathrm{qf}\, A)$ with $X^i = 0$ for $i \gg 0$). We say $P^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ is a free resolution of $X^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$, if each P^i is a free module and there is a quasi-isomorphism $P^\bullet \to X^\bullet$. We say a free resolution $P^\bullet$ is *minimal*, if $\partial(P^i) \subset \mathfrak{m} P^{i+1}$ for all i. Here ∂ denotes the differential map, and $\mathfrak{m} := \bigoplus_{i>0} A_i$ is the graded maximal ideal. Any $X^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ has a minimal free resolution, which is unique up to isomorphism.

Regard $K = A/\mathfrak{m}$ as a graded left A-module, and set

$$\beta^i_j(X^\bullet) := \dim_K \underline{\mathrm{Ext}}^{-i}_A(X^\bullet, K)_{-j} \quad \text{and} \quad \beta^i(X^\bullet) := \sum_{j \in \mathbb{Z}} \beta^i_j(X^\bullet)$$

for $X^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ and $i, j \in \mathbb{Z}$. In this situation, if $P^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ is a minimal free resolution of $X^\bullet$, then we have $P^i \cong \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta^i_j(X^\bullet)}$ for each $i \in \mathbb{Z}$. It is easy to see that $\beta^i_j(X^\bullet) < \infty$ for each i, j.

Following the usual convention, we often describe (the invariants of) a free resolution of a module $M \in \mathrm{qf}\, A$ in the homological manner. So we have $\beta_{i,j}(M) = \beta^{-i}_j(M)$, and a minimal free resolution of M is of the form

$$P_\bullet : \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{1,j}(M)} \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

We say A is *Koszul*, if $\beta_{i,j}(K) \neq 0$ implies $i = j$, in other words, K has a graded free resolution of the form

$$\cdots \longrightarrow A(-i)^{\beta_i(K)} \longrightarrow \cdots \longrightarrow A(-2)^{\beta_2(K)} \longrightarrow A(-1)^{\beta_1(K)} \longrightarrow A \longrightarrow K \longrightarrow 0.$$

Even if we regard K as a right A-module, we get the equivalent definition.

The polynomial ring $K[x_1, \ldots, x_n]$ and the exterior algebra $\bigwedge \langle y_1, \ldots, y_n \rangle$ are primary examples of Koszul algebras. Of course, there are many other important examples. In the noncommutative case, many of them are not left (or right) noetherian. In the rest of the paper, we assume that A is Koszul.

Koszul duality is a derived equivalence between a Koszul algebra A and its dual $A^!$. A standard reference of this subject is Beilinson et al. [3]. But, in the present paper, we follow the convention of Mori [10].

Recall that Yoneda product makes $A^! := \bigoplus_{i \in \mathbf{N}} \operatorname{Ext}^i_A(K, K)$ a graded K-algebra. (In the convention of [3], $A^!$ denotes the opposite algebra of our $A^!$. So the reader should be careful.) If A is Koszul, then so is $A^!$ and we have $(A^!)^! \cong A$. The Koszul dual of the polynomial ring $S := K[x_1, \ldots, x_n]$ is the exterior algebra $E := \bigwedge \langle y_1, \ldots, y_n \rangle$. In this case, since S is regular and noetherian, Koszul duality is very simple. It gives an equivalence $\mathcal{D}^b(^*\mathrm{mod}\, S) \cong \mathcal{D}^b(^*\mathrm{mod}\, E)$ of the bounded derived categories. In the general case, the description of Koszul duality is slightly technical. For example, if A is not left noetherian, then $^*\mathrm{mod}\, A$ is not an abelian category. So we have to treat $\mathrm{qf}\, A$.

Let $\mathcal{C}^\uparrow(\mathrm{qf}\, A)$ be the full subcategory of $\mathcal{C}(\mathrm{qf}\, A)$ (and $\mathcal{C}^-(\mathrm{qf}\, A)$) consisting of complexes $X^\bullet$ satisfying

$$X^i_j = 0 \quad \text{for } i \gg 0 \text{ or } i + j \ll 0.$$

And let $\mathcal{D}^\uparrow(\mathrm{qf}\, A)$ be the localization of $\mathcal{C}^\uparrow(\mathrm{qf}\, A)$ at quasi-isomorphisms. By the usual argument, we see that $\mathcal{D}^\uparrow(\mathrm{qf}\, A)$ is equivalent to the full subcategory of the derived category $\mathcal{D}(\mathrm{qf}\, A)$ (and $\mathcal{D}^-(\mathrm{qf}\, A)$) consisting of the complex $X^\bullet$ such that

$$H^i(X^\bullet)_j = 0 \quad \text{for } i \gg 0 \text{ or } i + j \ll 0.$$

It is easy to see that $\mathcal{D}^\uparrow(\mathrm{qf}\, A)$ is a triangulated subcategory of $\mathcal{D}(\mathrm{qf}\, A)$.

We write V^* for the dual space of a K-vector space V. Note that if $M \in {}^*\mathrm{Mod}\, A$ then $M^* := \bigoplus_{i \in \mathbf{Z}} (M_{-i})^*$ is a graded right A-module. And we fix a basis $\{x_\lambda\}$ of A_1 and its dual basis $\{y_\lambda\}$ of $(A_1)^*$ $(= (A^!)_1)$. Let $(X^\bullet, \partial) \in \mathcal{C}^\uparrow(\mathrm{qf}\, A)$. In this notation, we define the contravariant functor $F_A : \mathcal{C}^\uparrow(\mathrm{qf}\, A) \to \mathcal{C}^\uparrow(\mathrm{qf}\, A^!)$ as follows.

$$F_A(X^\bullet)^p_q = \bigoplus A^!_{q+j} \otimes_K (X^{j-p}_{-j})^*$$

with the differential $d = d' + d''$ given by

$$d' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto (-1)^p \sum a y_\lambda \otimes m x_\lambda \in A^!_{q+j+1} \otimes_K (X^{j-p}_{-j-1})^*$$

and

$$d'' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto a \otimes \partial^*(m) \in A^!_{q+j} \otimes_K (X^{j-p-1}_{-j})^*.$$

The contravariant functor $F_{A^!} : \mathcal{C}^\uparrow(\mathrm{qf}\, A^!) \to \mathcal{C}^\uparrow(\mathrm{qf}\, A)$ is given by a similar way. (More precisely, the construction is different, but the result is similar. See the remark below.) They induce the contravariant functors $\mathcal{F}_A : \mathcal{D}^\uparrow(\mathrm{qf}\, A) \to \mathcal{D}^\uparrow(\mathrm{qf}\, A^!)$ and $\mathcal{F}_{A^!} : \mathcal{D}^\uparrow(\mathrm{qf}\, A^!) \to \mathcal{D}^\uparrow(\mathrm{qf}\, A)$.

Remark 2.1. In [10], two Koszul duality functors are defined individually. The functor denoted by $\bar{E}_A$ is the same as our $\mathcal{F}_A$. The other one which is denoted by $\tilde{E}_A$ is defined using the operations $\underline{\mathrm{Hom}}_K(A^!, -)$ and $\underline{\mathrm{Hom}}_K(-, K)$. But, in our case, it coincides with F_A except the convention of the sign ± 1. So we do not give the precise definition of $\tilde{E}_A$ here.

Theorem 2.2 (Koszul duality. c.f. [3, 10]). *The contravariant functors $\mathcal{F}_A$ and $\mathcal{F}_{A^!}$ give an equivalence*

$$\mathcal{D}^\uparrow(\operatorname{qf} A) \cong \mathcal{D}^\uparrow(\operatorname{qf} A^!)^{\mathrm{op}}.$$

The next result easily follows from Theorem 2.2 and the fact that $\mathcal{F}_A(K) = A^!$.

Lemma 2.3 (cf. [10, Lemma 2.8]). *For $X^\bullet \in \mathcal{D}^\uparrow(\operatorname{qf} A)$, we have*

$$\beta^i_j(X^\bullet) = \dim H^{-i-j}(\mathcal{F}_A(X^\bullet))_j.$$

3. Regularity and Linearity Defect

Throughout this section, $A = \bigoplus_{i\in\mathbf{N}} A_i$ is a Koszul algebra.

Definition 3.1. For $X^\bullet \in \mathcal{D}^\uparrow(\operatorname{qf} A)$, we call

$$\operatorname{reg}_A(X^\bullet) := \sup\{\, i+j \mid i,j \in \mathbb{Z} \text{ with } \beta^i_j(X^\bullet) \neq 0\,\}$$

the *regularity* of $X^\bullet$. We set the regularity of the 0 module to be $-\infty$.

We say A is *left graded coherent*, if any finitely generated graded left ideal of A has a finite presentation. Let $^*\operatorname{fp} A$ be the full subcategory of $^*\operatorname{mod} A$ consisting of finitely presented modules. As is well-known, A is left graded coherent if and only if $^*\operatorname{fp} A$ is an abelian subcategory of $^*\operatorname{mod} A$.

Lemma 3.2. *If $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$ then A is left noetherian. Similarly, if $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\operatorname{fp} A$ then A is left graded coherent.*

Proof. Assume that A is not left noetherian. Then there is a graded left ideal I which is not finitely generated. Clearly, $A/I \in {}^*\operatorname{mod} A$, but $\beta_{1,j}(A/I) = \beta_{0,j}(I) \neq 0$ for arbitrary large j and $\operatorname{reg}_A(A/I) = \infty$.

Assume that A is not left graded coherent. Then there is a graded left ideal I which is finitely generated but not finitely presented. Clearly, $A/I \in {}^*\operatorname{fp} A$, but $\beta_{2,j}(A/I) = \beta_{1,j}(I) \neq 0$ for arbitrary large j and $\operatorname{reg}_A(A/I) = \infty$. □

Remark 3.3. The author does not know any example of a Koszul algebra A which admits $M \in {}^*\operatorname{mod} A$ with $\operatorname{reg}_A(M) = \infty$ but $\beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(M) < \infty$ for all i. In particular, he does not know a left noetherian (resp. graded coherent) Koszul algebra A such that $\operatorname{reg}_A(M) = \infty$ for some $M \in {}^*\operatorname{mod} A$ (resp. $M \in {}^*\operatorname{fp} A$).

Lemma 3.4. (1) *For $M \in \operatorname{qf} A$, we have*

$$\operatorname{reg}_A(M) < \infty \;\Rightarrow\; \beta_i(M) < \infty \textit{ for all } i \;\Rightarrow M \textit{ has a finite presentation.}$$

(2) *If $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ is a triangle in $\mathcal{D}^\uparrow(\operatorname{qf} A)$, then we have*

$$\operatorname{reg}_A(Y^\bullet) \le \max\{\operatorname{reg}_A(X^\bullet), \operatorname{reg}_A(Z^\bullet)\}.$$

If $\operatorname{reg}_A(X^\bullet) \neq \operatorname{reg}_A(Z^\bullet) + 1$, then equality holds.

(3) *If $M \in {}^*\operatorname{mod} A$ has finite length, then $\operatorname{reg}_A(M) \le \max\{\, i \mid M_i \neq 0\}$.*

(4) *For $X^\bullet \in \mathcal{D}^\uparrow(\operatorname{qf} A)$, we have*

$$\operatorname{reg}_A(X^\bullet) \le \sup\{\operatorname{reg}_A(H^i(X^\bullet)) + i \mid i \in \mathbb{Z}\}.$$

The next result directly follows from Lemma 2.3.

Theorem 3.5 (Eisenbud et al [6], Mori [10]). *For $X^\bullet \in D^\uparrow(\mathrm{qf}\, A)$, we have*

$$\mathrm{reg}_A(X^\bullet) = -\inf\{\, i \mid H^i(\mathcal{F}_A(X^\bullet)) \neq 0 \,\}.$$

We say a complex $X^\bullet \in \mathcal{D}^\uparrow(\mathrm{qf}\, A)$ is *strongly bounded*, if $X^\bullet$ is bounded (i.e., $H^i(X^\bullet) = 0$ for $i \gg 0$ or $i \ll 0$) and $\mathrm{reg}_A(X^\bullet) < \infty$. Let $\mathcal{D}^{sb}(\mathrm{qf}\, A)$ be the full subcategory of $\mathcal{D}^\uparrow(\mathrm{qf}\, A)$ consisting of strongly bounded complexes. By Lemma 3.4 (2), $\mathcal{D}^{sb}(\mathrm{qf}\, A)$ is a triangulated subcategory of $\mathcal{D}(\mathrm{qf}\, A)$.

The next result follows from Theorem 3.5.

Proposition 3.6. *The (restriction of) functors $\mathcal{F}_A$ and $\mathcal{F}_{A^!}$ give an equivalence*

$$\mathcal{D}^{sb}(\mathrm{qf}\, A) \cong \mathcal{D}^{sb}(\mathrm{qf}\, A^!)^{\mathrm{op}}.$$

Let $(P^\bullet, \partial) \in \mathcal{C}^\uparrow(\mathrm{qf}\, A)$ be a complex of *free* A-modules such that $\partial(P^i) \subset \mathfrak{m}P^{i+1}$, in other words, $P^\bullet$ is a minimal free resolution of some $X^\bullet \in \mathcal{C}^\uparrow(\mathrm{qf}\, A)$. According to [6], we define the *linear part* $\mathrm{lin}(P^\bullet)$ of $P^\bullet$ as follows:

(1) $\mathrm{lin}(P^\bullet)$ is a complex with $\mathrm{lin}(P^\bullet)^i = P^i$.

(2) The matrices representing the differentials of $\mathrm{lin}(P^\bullet)$ are given by "erasing" all the entries of degree ≥ 2 (i.e., replacing them by 0) from the matrices representing the differentials of $P^\bullet$.

It is easy to check that $\mathrm{lin}(P^\bullet)$ is actually a complex. But, even if $P_\bullet$ is a minimal free resolution of $M \in \mathrm{qf}\, A$, $\mathrm{lin}(P_\bullet)$ is not acyclic (i.e., $H_i(\mathrm{lin}(P_\bullet)) \neq 0$ for some $i > 0$) in general.

Definition 3.7 (Herzog-Iyengar [8]). Let $M \in \mathrm{qf}\, A$ and $P_\bullet$ its minimal graded free resolution. We call

$$\mathrm{ld}_A(M) := \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M.

We say $M \in {}^*\mathrm{mod}\, A$ has a *linear (free) resolution* if there is some $l \in \mathbb{Z}$ such that $\beta_{i,j}(M) \neq 0$ implies that $j - i = l$. In this case, the minimal free resolution $P_\bullet$ of M coincides with $\mathrm{lin}(P_\bullet)$, and $\mathrm{ld}_A(M) = 0$. As shown in [10, Theorem 5.4], we have

$$\mathrm{reg}_A(M) = \inf\{\, i \mid M_{\geq i} := \bigoplus_{j \geq i} M_j \text{ has a linear resolution}\}.$$

For $i \in \mathbb{Z}$ and $M \in \mathrm{qf}\, A$, $M_{\langle i \rangle}$ denotes the submodule of M generated by the degree i component M_i. We say $M \in \mathrm{qf}\, A$ is *componentwise linear*, if $M_{\langle i \rangle}$ has a linear resolution for all $i \in \mathbb{Z}$.

As shown in [11, 12], for $M \in \mathrm{qf}\, A$, we have

$$\mathrm{ld}_A(M) = \inf\{\, i \mid \Omega_i(M) \text{ is componentwise linear} \,\},$$

where $\Omega_i(M)$ is the i^{th} syzygy of M.

Clearly, we have $\mathrm{ld}_A(M) \leq \mathrm{proj.\,dim}_A(M)$. The inequality is strict quite often. For example, we have $\mathrm{proj.\,dim}_A(M) = \infty$ and $\mathrm{ld}_A(M) < \infty$ for many M. On the other hand, we sometimes have $\mathrm{ld}_A(M) = \infty$.

The next result connects the linearity defect with the regularity via Koszul duality. For a complex $X^\bullet$, $\mathcal{H}(X^\bullet)$ denotes the complex such that $\mathcal{H}(X^\bullet)^i = H^i(X^\bullet)$ for all i and all differentials are 0.

Theorem 3.8 (cf. [6, Theorem 3.1]). *Let $X^\bullet \in \mathcal{D}^\uparrow(\mathrm{qf}\, A)$, and $P^\bullet$ a minimal free resolution of $\mathcal{F}_A(X^\bullet) \in \mathcal{D}^\uparrow(\mathrm{qf}\, A^!)$. Then we have*

$$\mathrm{lin}(P^\bullet) = F_A \circ \mathcal{H}(X^\bullet).$$

Hence, for $M \in \mathrm{qf}\, A$,

$$\mathrm{ld}_A(M) = \sup\{\, \mathrm{reg}_{A^!}(H^i(F_A(M))) + i \mid i \in \mathbb{Z}\}.$$

4. Koszul Commutative Algebras and their Dual

If A is a Koszul commutative algebra and $S := \mathrm{Sym}_K A_1$ is the polynomial ring, then we have $A = S/I$ for a graded ideal I of S. In this situation, A is *Golod* if and only if I has a 2-linear resolution as an S-module (i.e., $\beta_{i,j}(I) \neq 0$ implies $j = i+2$), see [8, Proposition 5.8]. We say A *comes from a complete intersection by a Golod map* (see [2, 8]), if there is an intermediate graded ring R with $S \twoheadrightarrow R \twoheadrightarrow A$ satisfying the following conditions:

(1) R is a complete intersection.
(2) Let J be the graded ideal of R such that $A = R/J$. Then J has a 2-linear resolution as an R-module.

If this is the case, R is automatically Koszul. Clearly, if A itself is complete intersection or Golod, then it comes from a complete intersection by a Golod map.

Example 4.1. Set $S = K[s,t,u,v,w]$ and $A = S/(st, uv, sw)$. Then A is neither Golod nor complete intersection, but comes from a complete intersection by a Golod map (as an intermediate ring, take $S/(st, uv)$).

The next result plays a key role in this section.

Theorem 4.2 (Avramov-Eisenbud [1]). *Let A be a Koszul commutative algebra, and $S := \mathrm{Sym}_K A_1$ the polynomial ring. Then we have $\mathrm{reg}_A(M) \leq \mathrm{reg}_S(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$.*

On the other hand, even if A is Koszul and commutative, $\mathrm{ld}_A(M)$ can be infinite for some $M \in {}^*\mathrm{mod}\, A$, as pointed out in [8]. But we have the following.

Theorem 4.3 (Herzog-Iyengar [8]). *Let A be a Koszul commutative algebra. If A comes from a complete intersection by a Golod map (e.g., A itself is complete intersection or Golod), then $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$.*

Now we are interested in $\mathrm{reg}_{A^!}(N)$ and $\mathrm{ld}_{A^!}(N)$ for a Koszul commutative algebra A. First, we recall that a graded left $A^!$-module has a natural graded right $A^!$-module structure in this case, and vice versa (c.f. [8, §3]). In particular, $A^!$ is left noetherian (resp. graded coherent) if and only if it is right noetherian (resp. graded coherent).

Theorem 4.4. *If A is a Koszul commutative algebra, we have the following.*

(1) *Let* $N \in {}^*\mathrm{mod}\, A^!$. *If* $\mathrm{reg}_{A^!}(N) < \infty$, *then* $\mathrm{ld}_{A^!}(N) < \infty$.
(2) *The following conditions are equivalent.*
  (a) $\mathrm{ld}_A(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, A$.
  (a') $\mathrm{ld}_A(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, A$ *with* $M = \bigoplus_{i=0,1} M_i$.
  (b) $\mathrm{reg}_{A^!}(N) < \infty$ *for all* $N \in {}^*\mathrm{fp}\, A^!$.
(3) *Let* $N \in \mathrm{qf}\, A^!$. *If there is some* $c \in \mathbb{N}$ *such that* $\dim_K N_i \leq c$ *for all* $i \in \mathbb{Z}$, *then* $\mathrm{ld}_{A^!}(N) < \infty$.

Proof. (1) The complex $F_{A^!}(N)$ is always bounded above. Hence if $\mathrm{reg}_{A^!}(N) < \infty$ then $H^i(F_{A^!}(N)) \neq 0$ for only finitely many i by Theorem 3.5. Thus the assertion follows from Theorems 3.8 and 4.2.

(2) The implication $(a) \Rightarrow (a')$ is clear.

$(a') \Rightarrow (b)$: First assume that $N \in {}^*\mathrm{fp}\, A^!$ has a presentation of the form $A^!(-1)^{\oplus \beta_1} \to A^{!\,\oplus \beta_0} \to N \to 0$. Then there is $M \in {}^*\mathrm{mod}\, A$ with $M = \bigoplus_{i=0,1} M_i$ such that $F_A(M)$ gives this presentation. Since $\mathrm{ld}_A(M) < \infty$, we have $\mathrm{reg}_{A^!}(N) < \infty$ by Theorem 3.8.

Next take an arbitrary $N \in {}^*\mathrm{fp}\, A^!$. For a sufficiently large s, $N_{\geq s} := \bigoplus_{i \geq s} N_i$ has a presentation of the form $A^!(-s-1)^{\oplus \beta_1} \to A^!(-s)^{\oplus \beta_0} \to N_{\geq s} \to 0$. (To see this, consider the short exact sequence $0 \to N_{\geq s} \to N \to N/N_{\geq s} \to 0$, and use the fact that $\mathrm{reg}_{A^!}(N/N_{\geq s}) < s$.) We have shown that $\mathrm{reg}_{A^!}(N_{\geq s}) < \infty$. So $\mathrm{reg}_{A^!}(N) < \infty$ by the above short exact sequence.

$(b) \Rightarrow (a)$: By Lemma 3.2, $A^!$ is left graded coherent. So ${}^*\mathrm{fp}\, A^!$ is an abelian category. Each term of $F_A(M)$ is a finite free $A^!$-module, in particular, $F_A(M) \in \mathcal{C}^-({}^*\mathrm{fp}\, A^!)$. Hence we have $H^i(F_A(M)) \in {}^*\mathrm{fp}\, A^!$ for all i. By the assumption, $\mathrm{reg}_{A^!}(H^i(F_A(M))) < \infty$. On the other hand, $H^i(F_A(M)) \neq 0$ for finitely many i by Theorems 3.5 and 4.2. So the assertion follows from Theorem 3.8.

(3) Let $\mathcal{S}$ be the set of all graded submodules of $A^{\oplus c}$ which are generated by elements of degree 1. By Brodmann [4], there is some $C \in \mathbb{N}$ such that $\mathrm{reg}_A(M) \leq \mathrm{reg}_S(M) < C$ for all $M \in \mathcal{S}$. Here S denotes the polynomial ring $\mathrm{Sym}_K A_1$. To prove the assertion, it suffices to show that $\mathrm{reg}_A(H^i(\mathcal{F}_{A^!}(N))) + i \leq C$ for all i. We may assume that $i = 0$. Note that $H^0(\mathcal{F}_{A^!}(N))$ is the cohomology of the sequence

$$A \otimes_K (N_1)^* \xrightarrow{\partial^{-1}} A \otimes_K (N_0)^* \xrightarrow{\partial^0} A \otimes_K (N_{-1})^*.$$

Since $\mathrm{Im}(\partial^0)(-1)$ is a submodule of $A^{\oplus \dim_K N_{-1}}$ generated by elements of degree 1 and $\dim_K N_{-1} \leq c$, we have $\mathrm{reg}_A(\mathrm{Im}(\partial^0)) < C$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\partial^0) \longrightarrow A \otimes_K (N_0)^* \longrightarrow \mathrm{Im}(\partial^0) \longrightarrow 0.$$

Since $\mathrm{reg}_A(A \otimes_K (N_0)^*) = 0$, we have $\mathrm{reg}_A(\mathrm{Ker}(\partial^0)) \leq C$. Similarly, we have $\mathrm{reg}_A(\mathrm{Im}(\partial^{-1})) < C$. By the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\partial^{-1}) \longrightarrow \mathrm{Ker}(\partial^0) \longrightarrow H^0(\mathcal{F}_{A^!}(N)) \longrightarrow 0,$$

we are done. □

Remark 4.5. In Theorem 4.4 (2), the implications $(a) \Rightarrow (a') \Leftrightarrow (b)$ hold for a general Koszul algebra.

If A is a (not necessarily commutative) Koszul algebra satisfying $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$, then Theorem 4.4 (1) and (2) hold for A.

By the above remark and Lemma 3.2, we have the following.

Corollary 4.6. *Let A be a Koszul algebra. If $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$, then $A^!$ is left graded coherent.*

In [2, Corollary 3], Backelin and Roos showed that if A is a Koszul commutative algebra which comes from a complete intersection by a Golod map then $A^!$ is left graded coherent. Moreover, they actually proved that $\operatorname{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{fp}\, A^!$ (see [2, Corollary 2] and [8, Lemma 5.1]). So we have $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ by Theorem 4.4, that is, we get a result of Herzog and Iyengar (Theorem 4.3). Their original proof is essentially based on this line too. While, in the case when A is complete intersection, we have another proof using the notion of *balanced dualizing complex* as stated in the introduction.

Lemma 4.7. *Assume that $\operatorname{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{fp}\, A^!$. Let $X^\bullet \in \mathcal{D}^b(\mathrm{qf}\, A^!)$ be a bounded complex. Then $X^\bullet$ is strongly bounded if and only if $H^i(X^\bullet) \in {}^*\mathrm{fp}\, A^!$ for all i.*

Proof. (Sufficiency): If $H^i(X^\bullet) \in {}^*\mathrm{fp}\, A^!$, then $\operatorname{reg}_{A^!}(H^i(X^\bullet)) < \infty$. Since $X^\bullet$ is bounded, we have $\operatorname{reg}_{A^!}(X^\bullet) < \infty$ by Lemma 3.4 (4).

(Necessity): Assume that $X^\bullet$ is strongly bounded (more generally, $\beta^i(X^\bullet) < \infty$ for all i). Let $P^\bullet$ be a minimal free resolution of $X^\bullet$. Clearly, $P^\bullet \in \mathcal{C}^-({}^*\mathrm{fp}\, A^!)$. By Corollary 4.6, ${}^*\mathrm{fp}\, A^!$ is an abelian category. Hence each $H^i(P^\bullet)\, (\cong H^i(X^\bullet))$ belongs to ${}^*\mathrm{fp}\, A^!$. ☐

Theorem 4.8. *Let A be a Koszul commutative algebra such that $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ (e.g. A comes from a complete intersection by a Golod map). Then Koszul duality gives an equivalence $\mathcal{D}^b({}^*\mathrm{mod}\, A) \cong \mathcal{D}^b({}^*\mathrm{fp}\, A^!)^{\mathrm{op}}$.*

Proof. By Proposition 3.6, it suffices to show that $\mathcal{D}^b({}^*\mathrm{mod}\, A) = \mathcal{D}^{sb}(\mathrm{qf}\, A)$ and $\mathcal{D}^b({}^*\mathrm{fp}\, A^!) = \mathcal{D}^{sb}(\mathrm{qf}\, A^!)$.

Let us consider the first equality (this holds for a general Koszul commutative algebra). If $X^\bullet \in \mathcal{D}^b({}^*\mathrm{mod}\, A)$, then $\operatorname{reg}_A(X^\bullet) < \infty$ by Lemma 3.4 (4) and Theorem 4.2. Hence we have $X^\bullet \in \mathcal{D}^{sb}(\mathrm{qf}\, A)$. Conversely, if $Y^\bullet \in \mathcal{D}^{sb}(\mathrm{qf}\, A)$, then $\beta^i(Y^\bullet) < \infty$ for all i, and the minimal free resolution of $Y^\bullet$ is a complex of finite free modules. So we have $Y^\bullet \in \mathcal{D}^b({}^*\mathrm{mod}\, A)$. Hence $\mathcal{D}^b({}^*\mathrm{mod}\, A) = \mathcal{D}^{sb}(\mathrm{qf}\, A)$.

Next we will show that $\mathcal{D}^b({}^*\mathrm{fp}\, A^!) = \mathcal{D}^{sb}(\mathrm{qf}\, A^!)$. By Corollary 4.6, ${}^*\mathrm{fp}\, A^!$ is an abelian category, and closed under extensions in $\mathrm{qf}\, A^!$. Since a free $A^!$-module of finite rank belongs to ${}^*\mathrm{fp}\, A^!$, this category has enough projectives. So we have $\mathcal{D}^b({}^*\mathrm{fp}\, A^!) = \mathcal{D}^b_{{}^*\mathrm{fp}\, A^!}(\mathrm{qf}\, A^!) = \mathcal{D}^{sb}(\mathrm{qf}\, A^!)$. Here the first equality follows from [7, Exercise III.2.2] and the second one follows from Lemma 4.7. ☐

Corollary 4.9. *If A is a Koszul complete intersection, then Koszul duality gives $\mathcal{D}^b({}^*\mathrm{mod}\, A) \cong \mathcal{D}^b({}^*\mathrm{mod}\, A^!)^{\mathrm{op}}$.*

In the rest of the paper, we study the linearity defect over the exterior algebra $E := \bigwedge \langle y_1, \ldots, y_n \rangle$. Eisenbud et al. [6] showed that $\operatorname{ld}_E(N) < \infty$ for all $N \in$

$^*\mathrm{mod}\, E$. Now this is a special case of Theorem 4.4. But the behavior of $\mathrm{ld}_E(N)$ is still mysterious.

If $n \geq 2$, then we have $\sup\{\, \mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\, E \,\} = \infty$. In fact, $N := E/\operatorname{soc}(E)$ satisfies $\mathrm{ld}_E(N) \geq 1$. And the i^{th} *cosyzygy* $\Omega_{-i}(N)$ of N (since E is selfinjective, we can consider cosyzygies) satisfies $\mathrm{ld}_E(\Omega_{-i}(N)) > i$. But we have an upper bound of $\mathrm{ld}_E(N)$ depending only on $\max\{\, \dim_K N_i \mid i \in \mathbb{Z} \,\}$ and n. Before stating this, we recall a result on $\mathrm{reg}_S(M)$ for $M \in {}^*\mathrm{mod}\, S$.

Theorem 4.10 (Brodmann and Lashgari, [5, Theorem 2.6]). *Let $S = k[x_1, \dots, x_n]$ be the polynomial ring. Assume that a graded submodule $M \subset S^{\oplus c}$ is generated by elements whose degrees are at most d. Then we have $\mathrm{reg}_S(M) \leq c^{n!}(2d)^{(n-1)!}$.*

When $c = 1$ (i.e., when M is an ideal), the above bound is a classical result, and there is a well-known example which shows the bound is rather sharp. For our study on $\mathrm{ld}_E(N)$, the case when $d = 1$ (but c is general) is essential.

Proposition 4.11. *Let $E = \bigwedge \langle y_1, \dots, y_n \rangle$ be an exterior algebra, and $N \in {}^*\mathrm{mod}\, E$. Set $c := \max\{\, \dim_K N_i \mid i \in \mathbb{Z} \,\}$. Then $\mathrm{ld}_E(N) \leq c^{n!}2^{(n-1)!}$.*

Proof. Similar to the proof of Theorem 4.4 (3). □

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Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita 564-8680, Japan

E-mail address: yanagawa@ipcku.kansai-u.ac.jp

GOTZMANN IDEALS OF THE POLYNOMIAL RING

SATOSHI MURAI

ABSTRACT. Let A denote the polynomial ring in n variables over a field. All the Gotzmann ideals of A with at most n generators will be classified. This is a joint work with Takayuki Hibi.

1. INTRODUCTION

Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Let $<_{\text{lex}}$ be the lexicographic order on A induced by the ordering $x_1 > x_2 > \cdots > x_n$. Recall that a *lexsegment* ideal of A is a monomial ideal I of A such that, for monomials u and v of A with $u \in I$, $\deg u = \deg v$ and $u <_{\text{lex}} v$, one has $v \in I$. Let I be a homogeneous ideal of A and I^{lex} the (unique) lexsegment ideal ([2] and [11]) with the same Hilbert function as I. A homogeneous ideal I of A is said to be *Gotzmann* if the number of minimal generators of I is equal to that of I^{lex}. Gotzmann ideals were introduced by Herzog and Hibi [9] in the study of maximal Betti numbers of ideals for a given Hilbert function. Indeed, Herzgo and Hibi proved that a homogeneous ideal I is Gotzmann if and only if the graded Betti numbers of I are equal to those of I^{lex}. Our goal is to classify all the Gotzmann ideals of A generated by at most n homogeneous polynomials.

A homogeneous ideal I of A is said to have a *critical function* if I^{lex} is generated by at most n monomials. Let $1 \leq s \leq n$ and $f_1, \ldots, f_s$ homogeneous polynomials with

$$f_i \in K[x_i, x_{i+1}, \ldots, x_n]$$

for each $1 \leq i \leq s$ and with $\deg f_s > 0$. In [7] the ideal $I_{(f_1,\ldots,f_s)}$ of A defined by

$$(1) \qquad I_{(f_1,\ldots,f_s)} = (f_1x_1, f_1f_2x_2, \ldots, f_1f_2\cdots f_{s-1}x_{s-1}, f_1f_2\cdots f_s)$$

was introduced. A homogeneous ideal I of A is called *canonical critical* if $I = I_{(f_1,\ldots,f_s)}$ for some homogeneous polynomials $f_1, \ldots, f_s$ with $f_i \in K[x_i, x_{i+1}, \ldots, x_n]$ for each $1 \leq i \leq s$ and with $\deg f_s > 0$, where $1 \leq s \leq n$.

Theorem 1.1. *Given a homogeneous ideal I of $A = K[x_1, \ldots, x_n]$, the following conditions are equivalent:*

(i) *I has a critical Hilbert function;*
(ii) *there exists a linear transformation φ on A such that $\varphi(I)$ is a canonical critical ideal;*
(iii) *I is a Gotzmann ideal generated by at most n homogeneous polynomials.*

The author is supported by JSPS Research Fellowships for Young Scientists.

2. Universal lexsegment ideals and canonical critical ideals

In this section we study universal lexsegment ideals and canonical critical ideals. A monomial ideal I of $A = K[x_1, \dots, x_n]$ is said to be *universal lexsegment* if for all integers $m \geq 0$ the ideal $I \cdot K[x_1, \dots, x_{n+m}]$ is a lexsegment ideal of $K[x_1, \dots, x_{n+m}]$. Universal lexsegment ideals were introduced by Babson, Novik and Thomas [1]. We recall the following easy fact.

Lemma 2.1. *Let I be a monomial ideal of A. The following conditions are equivalent:*

(i) *I is universal lexsegment;*
(ii) *I is a lexsegment ideal generated by at most n monomials;*
(iii) *there exist integers $b_1, \dots, b_s \in \mathbb{Z}_{\geq 0}$ with $1 \leq s \leq n$ such that*

$$I = (x_1^{b_1+1}, x_1^{b_1}x_2^{b_2+1}, \dots, x_1^{b_1}x_2^{b_2}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1}). \tag{2}$$

Proof. First, we will show that (ii) implies (iii). Let $I = (u_1, \dots, u_s)$ be a lexsegment ideal with $s \leq n$. Suppose $\deg u_1 \leq \cdots \leq u_s$ and $u_i >_{\text{lex}} u_{i+1}$ if $\deg u_i = \deg u_{i+1}$. Since $u_1 = x_1^{\deg u_1}$, one has $u_1 = x_1^{b_1+1}$. Let $1 < k \leq \min\{n, \delta\}$ and suppose that $u_{k-1} = x_1^{b_1}x_2^{b_2}\cdots x_{k-1}^{b_{k-1}+1}$. Since the monomial ideal $(u_1, \dots, u_{k-1})$ is lexsegment, it follows that the smallest monomial with respect to $<_{\text{lex}}$ of degree $\deg u_k$ belonging to $(u_1, \dots, u_{k-1})$ is $u_{k-1}x_n^{b_k}$. Since u_k is the biggest monomial with respect to $<_{\text{lex}}$ which satisfies $\deg u_k = \deg(u_{k-1}x_n^{b_k})$ and $u_k <_{\text{lex}} u_{k-1}x_n^{b_k}$, we have $u_k = (u_{k-1}/x_{k-1})x_k^{b_k+1}$. Thus $u_k = x_1^{b_1}x_2^{b_2}\cdots x_{k-1}^{b_{k-1}}x_k^{b_k+1}$, as desired.

The implication (iii) $\Rightarrow$ (i) is easy. We will show that (i) implies (ii). Suppose that I is universal lexsegment with $|G(I)| \geq n+1$. What we must prove is I is not universal lexsegment. Since $I' = I \cdot K[x_1, \dots, x_{n+1}]$ is a lexsegment ideal with $|G(I')| \geq n+1$, there exists a lexsegment ideal J of $K[x_1, \dots, x_{n+1}]$ such that $G(J) \subset G(I')$ and $|G(J)| = n+1$. Then, by the implication (ii) $\Rightarrow$ (iii), J must contains a generator which is divisible by x_{n+1}. Since $G(J) \subset G(I') = G(I) \subset A$, this is a contradiction. □

Example 2.2. (a) The lexsegment ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, x_2]$ is universal lexsegment. In fact, the ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, \dots, x_m]$ is lexsegment for all $m \geq 2$.

(b) The lexsegment ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2]$ is not universal lexsegment. Indeed, since $x_1x_2^2 <_{\text{lex}} x_1^2x_3$ in $K[x_1, x_2, x_3]$, the ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2, x_3]$ is not lexsegment.

By using Lemma 2.1, it is easy to characterize critical Hilbert functions. Indeed, it was shown in [12] that the Hilbert function $H(I,t)$ of the universal lexsegment ideal (2) is given by

$$H(I,t) = \binom{t - a_1 + n - 1}{n-1} + \cdots + \binom{t - a_s + n - s}{n-s}, \tag{3}$$

where the sequence $(a_1, a_2, \dots, a_s)$ is defined by setting

$$a_i = \deg x_1^{b_1}\cdots x_{i-1}^{b_{i-1}}x_i^{b_i+1}, \qquad 1 \leq i \leq s.$$

Since a lexsegment ideal with a given Hilbert function is uniquely determined, it follows that a homogeneous ideal I of A is critical if and only if there exists a sequence $(a_1, \ldots, a_s)$ of integers with $0 < a_1 \leq a_2 \leq \cdots \leq a_s$, where $1 \leq s \leq n$, such that the Hilbert function of I is of the form (3).

Definition 2.3. A homogeneous ideal I of A with the Hilbert function (3) will be called a *critical ideal of type* $(a_1, a_2, \ldots, a_s)$.

Next, we study the property of canonical critical ideals. We require the following obvious facts.

Lemma 2.4. *Let* $1 < s \leq n$. *Fix homogeneous polynomials* $f_1, \ldots, f_{s-1}$ *with each* $f_i \in K[x_i, \ldots, x_n]$. *Let* $g \in K[x_s, \ldots, x_n]$ *be a homogeneous polynomial with* $\deg g > 0$. *Then*

$$f_1 f_2 \cdots f_{s-1} g \notin (f_1 x_1, f_1 f_2 x_2, \ldots, f_1 f_2 \cdots f_{s-1} x_{s-1}).$$

Corollary 2.5. *As a vector space over* K *the ideal* (1) *is the direct sum*

$$(4) \quad I_{(f_1,\ldots,f_s)} = \left(\bigoplus_{j=1}^{s-1} (f_1 \cdots f_j x_j) K[x_j, \ldots, x_n] \right) \bigoplus (f_1 \cdots f_s) K[x_s, \ldots, x_n].$$

The above facts implies that canonical critical ideals are critical and Gotzmann.

Proposition 2.6. *Let* $I_{(f_1,\ldots,f_s)}$ *denote the ideal* (1).

(a) $I_{(f_1,\ldots,f_s)}$ *is a critical ideal of type* $(a_1, \ldots, a_s)$, *where* $a_i = \deg f_1 f_2 \cdots f_i x_i$, $i = 1, \ldots, s-1$, *and* $a_s = \deg f_1 f_2 \cdots f_s$.

(b) $I_{(f_1,\ldots,f_s)}$ *is minimally generated by*

$$(5) \qquad \{f_1 x_1, \ldots, f_1 f_2 \cdots f_{s-1} x_{s-1}, f_1 f_2 \cdots f_s\}.$$

(c) $I_{(f_1,\ldots,f_s)}$ *is Gotzmann.*

Proof. The direct sum decomposition (4) says that the Hilbert function of $I_{(f_1,\ldots,f_s)}$ is of the form (3) and, in addition, that $I_{(f_1,\ldots,f_s)}$ is minimally generated by (5). Thus (a) and (b) follow. Since the lexsegment ideal with the Hilbert function (3) is the universal lexsegment ideal (2), one has $|G((I_{(f_1,\ldots,f_s)})^{\text{lex}})| = s$. Hence $I_{(f_1,\ldots,f_s)}$ is Gotzmann, as required. □

3. Proof of Theorem 1.1

In the previous section, we already see that canonical critical ideals are Gotzmann ideals having at most n homogeneous generators. On the other hand, it is clear from the definition that Gotzmann ideals generated by at most n homogeneous generators have a critical Hilbert function. Thus, to complete the proof of Theorem 1.1, what we must prove is any critical ideal can be transformed into canonical critical ideals by a linear transformation of A.

For a monomial u of A, we write $m(u)$ for the largest integer j for which x_j divides u. A monomial ideal I of A is called *stable* if, for each monomial u belonging to $G(I)$ and for each $1 \leq i < m(u)$, one has $(x_i u)/x_{m(u)} \in I$.

Lemma 3.1. *A monomial ideal I of A which is both critical and stable is universal lexsegment.*

Proof. (Sketch.) Suppose $|G(I^{\text{lex}})| = s$. It follows from [12] that the projective dimension of S/I is equal to s. Thus, by the Eliahou–Kervaire resolution [6], it follows that there exists a monomial $u_s \in G(I)$ such that $m(u_s) = s$. Then, by using the definition of stable ideals, a straightforward computation implies that there are monomials $u_1, \ldots, u_{s-1} \in G(I)$ such that $m(u_k) = k$ for $k = 1, 2, \ldots, s-1$ and $\deg u_1 \le \cdots \le \deg u_s$ (e.g., [10, Lemma 1.3]). Since $|G(I)| \le |G(I^{\text{lex}})| = s$, we have $G(I) = \{u_1, \ldots, u_s\}$.

Clearly, $u_1 = x_1^{b_1+1}$ for some $b_1 \ge 0$. Then, by arguing inductively, a routine computation implies that $I = (u_1, \ldots, u_s)$ is an ideal of the form (2). □

Let I be an ideal of A. When K is infinite, given a monomial order σ on A, we write $\text{gin}_\sigma(I)$ for the generic initial ideal ([5] and [8]) of I with respect to σ.

Lemma 3.2. *Let I be a critical ideal of A. Then, for an arbitrary monomial order σ on A induced by the ordering $x_1 > \cdots > x_n$, the generic initial ideal $\text{gin}_\sigma(I)$ is stable. Thus in particular $\text{gin}_\sigma(I)$ is universal lexsegment.*

Proof. Since $\text{gin}_\sigma(I)$ is a critical monomial ideal, it follows from [12, Corollary 1.8] that $\text{gin}_\sigma(I)$ is Gotzmann. Thus in particular $\text{gin}_\sigma(I)$ is componentwise linear [9]. Hence [3, Lemma 1.4] says that $\text{gin}_{<_{\text{rev}}}(\text{gin}_\sigma(I)) = \text{gin}_\sigma(I)$ is stable. Here $<_{\text{rev}}$ is the reverse lexicographic order on A induced by the ordering $x_1 > \cdots > x_n$. Since $\text{gin}_\sigma(I)$ is both critical and stable, it follows from Lemma 3.1 that $\text{gin}_\sigma(I)$ is universal lexsegment. □

Note that the above lemma is obvious in characteristic 0, since generic initial ideals are stable in characteristic 0.

Lemma 3.3. *Suppose that a homogeneous ideal I of A is a critical ideal of type $(a_1, \ldots, a_s)$, where $2 \le s \le n$. Then there exists a homogeneous polynomial f of A with $\deg f = a_1 - 1$ together with a homogeneous ideal J of A such that*

$$I = f \cdot J.$$

Proof. (Sketch.) By considering an extension field, we may assume that K is infinite. Then there is a linear transformation φ with $\text{in}_{<_{\text{lex}}}(\varphi(I)) = \text{gin}_{<_{\text{lex}}}(I)$. Considering $\varphi(I)$ instead of I, one may assume that $\text{in}_{<_{\text{lex}}}(I) = \text{gin}_{<_{\text{lex}}}(I)$. Lemma 3.2 says that $\text{in}_{<_{\text{lex}}}(I)$ is universal lexsegment. Hence

$$\text{in}_{<_{\text{lex}}}(I) = (x_1^{b_1+1}, x_1^{b_1}x_2^{b_2+1}, \ldots, x_1^{b_1}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1}),$$

where $b_i = a_i - a_{i-1}$, $1 \le i \le s$, with $a_0 = 1$.

To simplify the notation, let $u_i = x_1^{b_1} \cdots x_i^{b_i}$ for $i = 1, \ldots, s$. Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis of I, where g_i is a homogeneous polynomial of A with $\text{in}_{<_{\text{lex}}}(g_i) = u_i x_i$ for each $1 \le i \le s$, and $\mathcal{G}' = \{g_2, \ldots, g_s\}$. We show that $\mathcal{G}'$ is a Gröbner basis with respect to $<_{\text{lex}}$. For $2 \le i < j \le s$, consider the S-polynomial

$$S(g_i, g_j) = (u_j/u_i)x_j g_i - x_i g_j.$$

Then, since $\mathrm{in}_{<_{\mathrm{lex}}}(g_1) >_{\mathrm{lex}} \mathrm{in}_{<_{\mathrm{lex}}}(S(g_i, g_j))$ and since $\mathcal{G}$ is a Gröbner basis, a remainder of the S-polynomial of g_i and g_j with respect to $\mathcal{G}'$ can be 0. Hence $\mathcal{G}'$ is a Gröbner basis with respect to $<_{\mathrm{lex}}$, as desired.

Now, we prove Lemma 3.3 by using induction on s. Suppose $s > 2$ (the proof for $s = 2$ is similar). Let J be the ideal of A generated by $\mathcal{G}'$. Since

$$\mathrm{in}_{<_{\mathrm{lex}}}(J) = (u_2x_2, \ldots, u_sx_s) = x_1^{b_1}(x_2^{b_2+1}, x_2^{b_2}x_3^{b_3+1}, \ldots, x_2^{b_2}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1})$$

and since

$$(x_2^{b_2+1}, x_2^{b_2}x_3^{b_3+1}, \ldots, x_2^{b_2}\cdots x_{s-1}^{b_{s-1}}x_s^{b_s+1})$$

is universal lexsegment in $K[x_2, \ldots, x_n, x_1]$, the ideal J is a critical ideal of type $(a_2, \ldots, a_s)$. The induction hypothesis guarantees the existence of a homogeneous polynomial f_0 of A with $\deg(f_0) = a_2 - 1$ which divides each of $g_2, \ldots, g_s$. Since $\mathrm{in}_{<_{\mathrm{lex}}}(f_0)$ divides $\mathrm{in}_{<_{\mathrm{lex}}}(g_i) = u_ix_i$ for each $1 < i \leq s$, one has $\mathrm{in}_{<_{\mathrm{lex}}}(f_0) = u_2$. Let $g_i' = g_i/f_0$ for $i = 2, \ldots, s$. Thus in particular $\mathrm{in}_{<_{\mathrm{lex}}}(g_2') = u_2x_2/u_2 = x_2$.

Now, divide the S-polynomial of g_1 and g_2 by $\mathcal{G}$, say,

$$x_2^{b_2+1}g_1 - x_1(f_0g_2') = q_1g_1 + q_2(f_0g_2') + \cdots + q_s(f_0g_s'),$$

where $q_1, \ldots, q_s$ are homogeneous polynomials of A with

$$\mathrm{in}_{<_{\mathrm{lex}}}(q_1g_1) \leq_{\mathrm{lex}} \mathrm{in}_{<_{\mathrm{lex}}}(x_2^{b_2+1}g_1 - x_1(f_0g_2'))$$

and with

$$\mathrm{in}_{<_{\mathrm{lex}}}(q_k(f_0g_k')) \leq_{\mathrm{lex}} \mathrm{in}_{<_{\mathrm{lex}}}(x_2^{b_2+1}g_1 - x_1(f_0g_2'))$$

for each $2 \leq k \leq s$. Let

$$f_0h = q_2(f_0g_2') + \cdots + q_s(f_0g_s').$$

Thus

$$(x_2^{b_2+1} - q_1)g_1 = f_0(x_1g_2' + h).$$

Since $\mathrm{in}_{<_{\mathrm{lex}}}(x_2^{b_2+1} - q_1) = x_2^{b_2+1}$, $\mathrm{in}_{<_{\mathrm{lex}}}(g_1) = x_1^{b_1+1}$ and $\mathrm{in}_{<_{\mathrm{lex}}}(x_1g_2' + h) = x_1x_2$, it follows that $x_1g_2' + h$ can divide neither $x_2^{b_2+1} - q_1$ nor g_1. Thus $x_1g_2' + h$ is a product $(x_1 + h_1)(x_2 + h_2)$, where h_1 and h_2 are homogeneous polynomials of A with $\deg h_1 = \deg h_2 = 1$, such that $x_1 + h_1$ divides g_1 and $x_2 + h_2$ divides $x_2^{b_2+1} - q_1$. Let $f = g_1/(x_1 + h_1)$. Then $\deg f = a_1 - 1$ and f divides both g_1 and f_0. □

We are now in the position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. What we must prove is (i) implies (ii). This will be achieved by induction on s. Let $I \subset A$ be a critical ideal of type $(a_1, \ldots, a_s)$. If $s = 1$, then the statement is obvious.

Let $s > 1$. Lemma 3.3 guarantees that $I = f \cdot J$, where f is a homogeneous polynomial of A with $\deg f = a_1 - 1$ and where J is a homogeneous ideal of A. The Hilbert function of J is $H(J, t) = H(I, t + a_1 - 1)$. Hence J is a critical ideal of type$(1, a_2 - a_1 + 1, \ldots, a_s - a_1 + 1)$. Since $H(J, 1) \neq 0$, there exists a linear transformation φ on A with $x_1 \in \varphi(J)$. Let J' be the ideal

$$J' = \varphi(J) \cap K[x_2, \ldots, x_n]$$

of $K[x_2, \ldots, x_n]$. Then

$$\varphi(J) = x_1 K[x_1, \ldots, x_n] \bigoplus J'.$$

A straightforward computation implies that the ideal J' of $K[x_2, \ldots, x_n]$ is a critical ideal of type $(a_2 - a_1 + 1, \ldots, a_s - a_1 + 1)$. The induction hypothesis then guarantees the existence of a linear transformation ψ on $K[x_2, \ldots, x_n]$ such that $\psi(J')$ is a canonical critical ideal of $K[x_2, \ldots, x_n]$, say

$$\psi(J') = (f_2 x_2, \ldots, f_2 \cdots f_{s-1} x_{s-1}, f_2 \cdots f_s),$$

where $f_i \in K[x_i, x_{i+1}, \ldots, x_n]$ for each $2 \leq i \leq s$ and where $\deg f_s > 0$. Now, regarding ψ to be a linear transformation on A by setting $\psi(x_1) = x_1$, one has

$$\begin{aligned} (\psi \circ \varphi)(I) &= ((\psi \circ \varphi)(f)) \cdot ((\psi \circ \varphi)(J)) \\ &= ((\psi \circ \varphi)(f)) \cdot (\psi(x_1 A \bigoplus J')) \\ &= (\psi \circ \varphi)(f) \cdot (x_1 A \bigoplus \psi(J')). \end{aligned}$$

Let $f_1 = (\psi \circ \varphi)(f)$. Then it follows that

$$(\psi \circ \varphi)(I) = (f_1 x_1, f_1 f_2 x_2, \ldots, f_1 f_2 \cdots f_{s-1} x_{s-1}, f_1 f_2 \cdots f_s)$$

as desired. □

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Satoshi Murai, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: s-murai@ist.osaka-u.ac.jp

Gröbner bases for the polynomial ring with infinite variables and their applications

Kei-ichiro Iima and Yuji Yoshino
(Okayama University)

1 Introduction

Recall that a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers is called a partition of a non-negative integer n if the equality $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ holds and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. In such a case we denote it by $\lambda \vdash n$.

We are concerned with the following sets of partitions:

$$\begin{aligned}
A(n) &= \{ \ \lambda \vdash n \ \mid \ \lambda_i \equiv \pm 1 \pmod 6 \ \}, \\
B(n) &= \{ \ \lambda \vdash n \ \mid \ \lambda_i \equiv \pm 1 \pmod 3, \quad \lambda_1 > \lambda_2 > \cdots > \lambda_r \ \}, \\
C(n) &= \{ \ \lambda \vdash n \ \mid \ \text{each } \lambda_i \text{ is odd, and} \\
&\qquad \text{any number appears in } \lambda_i\text{'s at most two times } \}.
\end{aligned}$$

It is known by famous Schur's equalities that all these sets $A(n)$, $B(n)$ and $C(n)$ have the same cardinality for all $n \in \mathbb{N}$. It is also known that the one-to-one correspondences among these three sets are realized in some combinatorial way using 2-adic or 3-adic expansions of integers. In this article we reconstruct such one-to-one correspondences by using the theory of Gröbner bases. For this, we need to extend the theory of Gröbner bases to a polynomial ring with infinitely many variables.

2 Gröbner bases

Throughout this article, let k be any field and let $S = k[x_1, x_2, \ldots]$ be a polynomial ring with countably infinite variables. We denote by $\mathbb{Z}_{\geq 0}^{(\infty)}$ the set of all sequences $a = (a_1, a_2, \ldots)$ of integers where $a_i = 0$ for all i but finite number of integers. Also we denote by $\mathrm{Mon}(S)$ the set of all monomials in S. Since any monomial is described uniquely as $x^a = \prod_i x_i^{a_i}$ for some $a = (a_1, a_2, \ldots) \in \mathbb{Z}_{\geq 0}^{(\infty)}$, we can identify these sets:

$$\mathrm{Mon}(S) \;\cong\; \mathbb{Z}_{\geq 0}^{(\infty)}.$$

If we attach degree on S by $\deg x_i = d_i$, then a monomial x^a has degree $\deg x^a = \sum_{i=1}^{\infty} a_i d_i$. In the rest of the paper, we assume that the degrees d_i's are chosen in such a way that there are only a finite number of monomials of degree d for each $d \in \mathbb{N}$. For example, the simplest way of attaching degree is that $\deg x_i = i$ for all $i \in \mathbb{N}$.

Definition 2.1. A total order $>$ on $\mathrm{Mon}(S)$ is called a monomial order if $(\mathrm{Mon}(S), >)$ is a well-ordered set, and it is compatible with the multiplication of monomials, i.e. $x^a > x^b$ implies $x^c x^a > x^c x^b$ for all $x^a, x^b, x^c \in \mathrm{Mon}(S)$.

Note that the ordering $x_1 > x_2 > x_3 > \cdots$ is not acceptable for monomial order, since it violates the well-ordering condition. On the other hand, if we are given any monomial order $>$, then, renumbering the variables, we may assume that $x_1 < x_2 < x_3 < \cdots$.

The following are examples of monomial orders on $\mathrm{Mon}(S)$.

Example 2.2. Let $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ be elements in $\mathbb{Z}_{\geq 0}^{(\infty)}$.

(1) The pure lexicographic order $>_{pl}$ is defined in such a way that $x^a >_{pl} x^b$ if and only if $a_i > b_i$ for the last index i with $a_i \neq b_i$.

(2) The degree (resp. anti-) lexicographic order $>_{dl}$ (resp. $>_{dal}$) is defined in such a way that $x^a >_{dl} x^b$ (resp. $x^a >_{dal} x^b$) if and only if either $\deg x^a > \deg x^b$ or $\deg x^a = \deg x^b$ and $a_i > b_i$ for the last (resp. first) index i with $a_i \neq b_i$.

(3) The degree (resp. anti-) reverse lexicographic order $>_{drl}$ (resp. $>_{darl}$) is defined as follows: $x^a >_{drl} x^b$ (resp. $x^a >_{darl} x^b$) if and only if either $\deg x^a > \deg x^b$ or $\deg x^a = \deg x^b$ and $a_i < b_i$ for the first (resp. last) index i with $a_i \neq b_i$.

These monomial orders are all distinct as shown in the following example in which $\deg x_i = i$ for $i \in \mathbb{N}$:

$$\begin{array}{ccccccccc} x_4 & >_{dl} & x_1x_3 & >_{dl} & x_2^2 & >_{dl} & x_1^2x_2 & >_{dl} & x_1^4, \\ x_1^4 & >_{dal} & x_1^2x_2 & >_{dal} & x_1x_3 & >_{dal} & x_2^2 & >_{dal} & x_4, \\ x_4 & >_{drl} & x_2^2 & >_{drl} & x_1x_3 & >_{drl} & x_1^2x_2 & >_{drl} & x_1^4, \\ x_1^4 & >_{darl} & x_1^2x_2 & >_{darl} & x_2^2 & >_{darl} & x_1x_3 & >_{darl} & x_4. \end{array}$$

Now suppose that a monomial order $>$ on $\mathrm{Mon}(S)$ is given and we fix it. Then, any non-zero polynomial $f \in S$ is expressed as

$$f = c_1 x^{a(1)} + c_2 x^{a(2)} + \cdots + c_r x^{a(r)},$$

where $c_i \neq 0 \in k$ and $x^{a(1)} > x^{a(2)} > \ldots > x^{a(r)}$. In such a case, the leading term, the leading monomial and the leading coefficient of f are given respectively as $\ell t(f) = c_1 x^{a(1)}$, $\ell m(f) = x^{a(1)}$ and $\ell c(f) = c_1$. For an ideal $I (\neq (0)) \subset S$, the initial ideal $in(I)$ of I is defined to be the ideal generated by all the leading terms $\ell t(f)$ of non-zero polynomials $f \in I$. The Gröbner base of I is defined similarly to the ordinary case.

Definition 2.3. A subset $\mathcal{G}$ of an ideal I is called a Gröbner base for I if $\{\ell t(g) \mid g \in \mathcal{G}\}$ generates the initial ideal $in(I)$.

Since S is not a Noetherian ring, one cannot expect that there always exists a finite Gröbner base $\mathcal{G}$ for a given ideal I. But any argument concerning Gröbner bases for an ideal of S can be reduced to the ordinary case for the polynomial rings with finite variables by the following theorem.

Theorem 2.4. *Let I be an ideal of S. For a positive integer n, we set $S^{\langle n \rangle} = k[x_1, x_2, \ldots, x_n]$ which is a polynomial subring of S and set $I^{\langle n \rangle} = I \cap S^{\langle n \rangle}$. Now let $\mathcal{G}$ be a subset of I.*

(1) *Suppose that each $\mathcal{G} \cap S^{\langle n \rangle}$ is a Gröbner base for $I^{\langle n \rangle}$ for all $n \in \mathbb{N}$, then $\mathcal{G}$ is a Gröbner base for I.*

(2) *The converse holds when the monomial order is the pure lexicographic order.*

The following division algorithm is proved using Theorem 2.4.

Theorem 2.5 (Division algorithm). *Let $\mathcal{G}$ be a subset of S. Then any non-zero polynomial $f \in S$ has an expression*

$$f = f_1 g_1 + f_2 g_2 + \cdots + f_s g_s + f',$$

with $g_i \in \mathcal{G}$ and $f_i, f' \in S$ such that the following conditions hold:

(1) *If we write $f' = \sum_{i=1}^{t} c_i x^{a(i)}$ with $c_i \neq 0 \in k$, then $x^{a(i)} \notin \{in(g) \mid g \in \mathcal{G}\}S$ for each $i = 1, 2, \ldots, t$.*

(2) *If $f_i g_i \neq 0$, then $\ell m(f_i g_i) \leq \ell m(f)$.*

Any such f' is called a remainder of f with respect to $\mathcal{G}$. Note that a remainder is in general not necessarily unique. But if $\mathcal{G}$ is a Gröbner base for $I = \mathcal{G}S$, then a remainder of f with respect to $\mathcal{G}$ is uniquely determined.

3 Applications

Let $S = k[x_1, x_2, \ldots]$ be a polynomial ring with countably infinite variables as before. We regard S as a graded k-algebra by defining $\deg(x_i) = i$ for each $i \in \mathbb{N}$, and denote by S_n the part of degree n of S for $n \in \mathbb{N}$. Note that there is a bijective mapping between the set of partitions of n and the set of monomials of degree n. In fact, the correspondence is given by mapping a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ to the monomial $x^\lambda = x_{\lambda_r} \cdots x_{\lambda_2} x_{\lambda_1}$ of degree n.

Let W be any subset of $\mathbb{N}$ satisfying $pW \subset W$ for an integer $p \geq 2$, where $pW = \{pw \mid w \in W\}$. In this case, we consider a polynomial subring $R = k[x_i \mid i \in W]$ of S. We are interested in the following two subsets of partitions of n:

$$\begin{aligned} X(n) &= \{ \lambda \vdash n \mid \lambda_i \in W \setminus pW \text{ for each } i \}, \\ Y(n) &= \{ \lambda \vdash n \mid \lambda_i \in W \text{ for each } i, \text{ and} \\ &\qquad \text{any number appears among } \lambda_i\text{'s at most } p-1 \text{ times} \}. \end{aligned}$$

Theorem 3.1. *Under the circumstances above, consider the set of polynomials $\mathcal{G} = \{x_i^p - x_{pi} \mid i \in W\}$ in R. We adopt the degree anti-reverse lexicographic order on the set of monomials in R. Then $\mathcal{G}$ is a reduced Gröbner base for the ideal $\mathcal{G}S$.*

Furthermore, define a mapping $\varphi : X(n) \to Y(n)$ so that $x^{\varphi(\lambda)}$ is a remainder of x^λ with respect to $\mathcal{G}$ for any $\lambda \in X(n)$. Then φ is a well-defined bijective mapping.

In particular we have that $|X(n)| = |Y(n)|$ in the case above. Therefore, just considering the generating functions of $|X(n)|$ and $|Y(n)|$, we see that the following functional equality holds;

$$\prod_{m \in W \setminus pW} \frac{1}{1 - t^m} = \prod_{m \in W} (1 + t^m + t^{2m} + \cdots + t^{(p-1)m}).$$

Example 3.2. Recall that $A(n)$, $B(n)$ and $C(n)$ are the sets of partitions given in Introduction.

(1) If $W = \{n \in \mathbb{N} \mid n \equiv \pm 1 \pmod 3\}$ and $p = 2$, then $X(n) = A(n)$ and $Y(n) = B(n)$.

(2) If $W = \{n \in \mathbb{N} \mid n \equiv 1 \pmod 2\}$ and $p = 3$, then $X(n) = A(n)$ and $Y(n) = C(n)$.

As a consequence of all the above, we obtain one-to-one correspondences among $A(n)$, $B(n)$ and $C(n)$ by using the theory of Gröbner bases.

For another example, let

$$\begin{aligned} P(n) &= \{ \lambda \vdash n \mid \lambda_i \equiv \pm 1 \pmod 5 \}, \\ Q(n) &= \{ \lambda \vdash n \mid \lambda_i - \lambda_{i+1} \geq 2 \}. \end{aligned}$$

By Rogers-Ramanujan equality, it is known that the sets $P(n)$ and $Q(n)$ have the same cardinality for each $n \in \mathbb{N}$. If we can find an ideal I as in the following question, then we will obtain a one-to-one correspondence between $P(n)$ and $Q(n)$ by using division algorithm.

Question 3.3. *Find an ideal I of S and a monomial order $>$ on* $\mathrm{Mon}(S)$ *satisfying* $S/I \cong k[\{x_i \mid i \equiv \pm 1 \pmod 5\}]$ *and* $in(I) = (x_i^2,\ x_i x_{i+1} \mid i \in \mathbb{N})$.

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Department of Math., Okayama University, 700-8530, Okayama, Japan
iima@math.okayama-u.ac.jp, yoshino@math.okayama-u.ac.jp

Remarks on equivalences of additive subcategories

Naoya Hiramatsu and Yuji Yoshino
(Okayama University)

Abstract

We study category equivalences between some additive subcategories of module categories. As its application, we show that the group of autofunctors of the category of reflexive modules over a normal domain is isomorphic to the divisor class group.

1 A necessary condition for equivalences of additive subcategories

Let R be a commutative ring. We denote the category of all finitely generated R-modules by R-mod, and the full subcategory of R-mod consisting of all reflexive modules by $\mathrm{ref}(R)$. If R is a Cohen-Macaulay local ring, we denote the category of maximal Cohen-Macaulay modules by $\mathrm{CM}(R)$ as a full subcategory of R-mod. By an additive subcategory we always mean a full subcategory which is closed under finite direct sums and direct summands.

Theorem 1. Let A and B be commutative rings. Let $\mathfrak{C}$ (resp. $\mathfrak{D}$) be an additive full subcategory of A-mod (resp. B-mod) which contains a nontrivial free module. If there is a category equivalence between $\mathfrak{C}$ and $\mathfrak{D}$, then $A \cong B$ as a ring.

Moreover, if F and G are the functors which give the equivalences above, then F and G are of the forms $F(X) \cong \mathrm{Hom}_A(G(B), X)$ and $G(Y) \cong \mathrm{Hom}_B(F(A), Y)$ for each $X \in \mathfrak{C}$, $Y \in \mathfrak{D}$.

proof. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{C}$ be functors satisfying $F \cdot G \cong 1_{\mathfrak{D}}$ and $G \cdot F \cong 1_{\mathfrak{C}}$. We denote the B-module $F(A)$ by M and the A-module $G(B)$ by N. Since F and G are fully faithful functors, there exist isomorphisms as rings $\mathrm{End}_B(M) \cong \mathrm{End}_A(A) = A$ and $\mathrm{End}_A(N) \cong \mathrm{End}_B(B) = B$. Thus there are natural maps as follows:

$$\begin{array}{ccccccccc} B & \xrightarrow{\beta} & \mathrm{End}_B(M) & \xrightarrow{\cong} & A & \xrightarrow{\alpha} & \mathrm{End}_A(N) & \xrightarrow{\cong} & B \\ b & \longrightarrow & b_M & \longrightarrow & a & \longrightarrow & a_N & \longrightarrow & b', \end{array} \qquad (1.1)$$

where b_M (resp. a_N) denotes the multiplication map on M (resp. N) by b (resp. a).

The title of the talk had been changed.

First of all, we claim that $b - b' \in \mathrm{Ann}_B M$ for such b and b' as above. Since M is finitely generated B-module, we can take a finite free cover of M and get the following diagram.

$$\begin{array}{ccc} B^{\oplus n} & \longrightarrow & M \\ \Big\downarrow{\scriptstyle b'_B} & & \Big\downarrow{\scriptstyle b_M} \\ B^{\oplus n} & \longrightarrow & M \end{array}$$

Applying the functor G to this diagram, we have a diagram

$$\begin{array}{ccc} N^{\oplus n} & \longrightarrow & A \\ \Big\downarrow{\scriptstyle a_N} & & \Big\downarrow{\scriptstyle a_A} \\ N^{\oplus n} & \longrightarrow & A, \end{array}$$

which is commutative. Since G is an equivalence, this implies that the first diagram is also commutative. Hence we have $b - b' \in \mathrm{Ann}_B M$ as desired.

We denote $\mathrm{Ann}_B M$ by $\mathfrak{b}$ and $\mathrm{Ann}_A N$ by $\mathfrak{a}$. Note that the map $A \xrightarrow{\alpha} \mathrm{End}_A(N) \cong B$ induces an injective mapping $q : A/\mathfrak{a} \to B$. We define the map $p : B \to A/\mathfrak{a}$ as the composition of $B \xrightarrow{\beta} \mathrm{End}_B(M) \cong A$ with the natural projection $A \to A/\mathfrak{a}$.

Secondly, we claim that Ker $p = \mathfrak{b}$. Since $\beta(\mathfrak{b}) = 0$, it is clear that Ker $p \supseteq \mathfrak{b}$. To prove the converse let $b \in$ Ker p. Then, since $\beta(b) \in \mathfrak{a}$, we have $b' = 0$ as in the notation as in (1.1). Since we have shown that $b - b' \in \mathfrak{b}$, we have $b \in \mathfrak{b}$ and the equality Ker $p = \mathfrak{b}$ is proved. Therefore the mapping p induces an isomorphism $A/\mathfrak{a} \cong B/\mathfrak{b}$.

Thirdly we note that any object $Y \in \mathfrak{D}$ has structure of a (B, A)-bimodule. In fact, the category $\mathfrak{D}$ is a full subcategory of B-mod, therefore Y is naturally equipped with left B-module structure. Since F is a dense functor, there exists an object $X \in \mathfrak{C}$ such that $F(X) \cong Y$ and there is a natural ring homomorphism

$$A \to \mathrm{End}_A(X) \cong \mathrm{End}_B(Y),$$

which maps a to $F(a_X)$. Now, for any $a \in A$ and $y \in Y$, we define $y \circ a := F(a_X)(y)$. Since A is a commutative ring, it yields right A-module structure for $Y \in \mathfrak{D}$. Since the equality $(b \circ y) \circ a = F(a_X)(by) = bF(a_X)(y) = b \circ (y \circ a)$ holds for $a \in A$, $b \in B$ and $y \in Y$, we see that Y has structure of a (B, A)-bimodule. Similarly any object $X \in \mathfrak{C}$ has structure of an (A, B)-bimodule.

Since F is an equivalence, there exists an isomorphism as A-modules for any object $X \in \mathfrak{C}$:

$${}_AX = \mathrm{Hom}_A({}_AA, {}_AX) \cong \mathrm{Hom}_B({}_BM_A, {}_BF(X)).$$

The second part of the theorem follows from this isomorphism.

To complete the proof, we need to show $\mathfrak{a} = \mathfrak{b} = (0)$. For this, we note from the definition of bimodule structure that N is isomorphic to $\mathrm{Hom}_B({}_BM_A, B)$ as an (A, B)-bimodule. In particular, there are isomorphisms of B-modules;

$$B \cong \mathrm{End}_A N \cong \mathrm{End}_A(\mathrm{Hom}_B({}_BM_A, B)).$$

Since any element $b \in \mathfrak{b}$ acts as a zero map on $\mathrm{Hom}_B({}_BM_A, B)$, it must be zero as an element of $\mathrm{End}_A(\mathrm{Hom}_B({}_BM_A, B))$. Consequently we have $b = 0$. Thus $\mathfrak{b} = 0$, and $\mathfrak{a} = 0$ as well. □

Corollary 2. Let A and B be Cohen-Macaulay local rings. Then $\mathrm{CM}(A)$ and $\mathrm{CM}(B)$ are equivalent as additive categories if and only if A is isomorphic to B as a ring.

Our theorem is somehow a generalization of Morita equivalence theorem which deals with abelian categories over non-commutative rings. See [3]. The difference is that, assuming rings are commutative, we are concerned with additive subcategories which are not necessarily abelian and our functors are not necessarily exact.

2 Groups of autofunctors over additive subcategories

Let R be a commutative ring and let $\mathfrak{C}$ be an additive subcategory of R-mod. By an autofunctor F on $\mathfrak{C}$, we mean a covariant functor $F : \mathfrak{C} \to \mathfrak{C}$ which gives rise to an equivalence of categories. We denote by $\mathrm{Aut}(\mathfrak{C})$ the group of all isomorphism classes of autofunctors over $\mathfrak{C}$. By an easy observation using Morita equivalence, it is known that $\mathrm{Aut}(R\text{-mod})$ is isomorphic to the Picard group $\mathrm{Pic}(R)$. As an application of Theorem 1 we can show the following theorem.

Theorem 3. Let A be a Noetherian normal domain. Then there is an isomorphism of groups

$$\mathrm{Aut}(\mathrm{ref}(A)) \cong C\ell(A),$$

where $C\ell(A)$ denotes the divisor class group of A.

proof. It follows from Theorem 1 that any $F \in \mathrm{Aut}(\mathrm{ref}(A))$ has a description $F \cong \mathrm{Hom}_A(M, \ )$ for some reflexive A-module M. Since F is an autofunctor, there exists a functor G of the form $\mathrm{Hom}_B(N, \ )$ for some $N \in \mathrm{ref}(A)$ satisfying $F \cdot G \cong G \cdot F \cong 1_{\mathrm{ref}(A)}$. Hence we have a sequence of isomorphisms of A-modules

$$A \cong G \cdot F(A) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, A)) \cong \mathrm{Hom}_A(M \otimes_A N, A),$$

which forces $\mathrm{rank} M = 1$. Thus M defines the divisor class $[M]$ in $C\ell(A)$. We define a homomorphism $\alpha : \mathrm{Aut}(\mathrm{ref}(A)) \to C\ell(A)$ by mapping an autofunctor $F \cong \mathrm{Hom}_R(M, \ )$ to $[M]$.

We should remark that α is a well-defined mapping. But it is clear from Yoneda's lemma which claims that if $\mathrm{Hom}_R(M, \ ) \cong \mathrm{Hom}_R(M', \ )$ as functors on $\mathrm{ref}(A)$ for $M, M' \in \mathrm{ref}(A)$, then $M \cong M'$ as A-modules. It is also not difficult to verify that α is a homomorphism of groups. In fact this follows from the isomorphism of functors on $\mathrm{ref}(A)$;

$$\mathrm{Hom}_A(M, \ ) \cdot \mathrm{Hom}_A(N, \ ) \cong \mathrm{Hom}_A((M \otimes_A N)^{**}, \ ).$$

We only have to show that α is an isomorphism. It is obvious from the definition that α is injective. In the rest we shall show that α is surjective. For this let $[I] \in C\ell(A)$ be an arbitrary element, where I is a divisorial fractional ideal of A. It is enough to see that $\mathrm{Hom}_A(I, \ )$ is a well-defined autofunctor on $\mathrm{ref}(A)$.

First we remark from Bourbaki [2, Chapter VII, §2] that an A-lattice M is reflexive if and only if the equality $M = \bigcap_{\mathfrak{p} \in H(R)} M_\mathfrak{p}$ holds, where $H(A)$ is the set of all prime ideal of height one. Secondly we note that that the equality

$$\mathrm{Hom}_A(X, Y) = \bigcap_{\mathfrak{p} \in H(R)} \mathrm{Hom}_A(X_\mathfrak{p}, Y_\mathfrak{p})$$

holds for $X, Y \in \mathrm{ref}(A)$. In fact, any $f \in \bigcap_{\mathfrak{p} \in H(R)} \mathrm{Hom}_A(X_\mathfrak{p}, Y_\mathfrak{p})$ maps X to $Y_\mathfrak{p}$ for all $\mathfrak{p} \in H(A)$, hence $f(X) \subseteq \bigcap_{\mathfrak{p} \in H(R)} Y_\mathfrak{p} = Y$, and thus $f \in \mathrm{Hom}_A(X, Y)$.

Combining the above two claims we see that $\mathrm{Hom}_A(I, X)$ is a reflexive lattice for any $X \in \mathrm{ref}(A)$. Hence $\mathrm{Hom}_A(I, \ )$ yields a functor from $\mathrm{ref}(A)$ to itself.

Since I is a divisorial ideal, there exists an ideal J with $[J] = -[I]$ in $C\ell(A)$, i.e. $(I \otimes_A J)^{**} \cong A$ where $(\ )^*$ denotes $\mathrm{Hom}_A(\ , A)$.

Therefore there are isomorphisms of functors on $\mathrm{ref}(A)$;

$$\begin{aligned} \mathrm{Hom}_A(J, \mathrm{Hom}_A(I, \ )) &\cong \mathrm{Hom}_A(I \otimes_A J, \ ) \cong \mathrm{Hom}_A((I \otimes_A J)^{**}, \ ) \\ &\cong \mathrm{Hom}_A(A, \ ) = 1_{\mathrm{ref}(A)}. \end{aligned}$$

This shows that $\mathrm{Hom}(I, \ )$ is an autofunctor over $\mathrm{ref}(A)$ as desired, and the proof is completed. □

Corollary 4. Let A be a normal domain of dimension at most two. Then $\mathrm{Aut}(\mathrm{CM}(A)) \cong C\ell(A)$.

proof. In fact, the equality $\mathrm{CM}(A) = \mathrm{ref}(A)$ holds in this case. □

Compared with the corollary, the groups of autofunctors of $\mathrm{CM}(A)$ are expected to be rather small for higher dimensional rings A. In fact we can prove the following theorem.

Theorem 5. Let A be a Cohen-Macaulay local ring. Suppose that A has only an isolated singularity with $\dim A \geq 3$. Then $\mathrm{Aut}(\mathrm{CM}(A))$ is a trivial group.

proof. Let F be an autofunctor over $\mathrm{CM}(A)$. By virtue of Theorem 1, there exists a maximal Cohen-Macaulay module M with $F \cong \mathrm{Hom}_A(M, \ )$. Assume that M is not free, and we shall show a contradiction. For this, take a free cover F of M and we obtain an exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0.$$

Recall that $\Omega(M)$ is also a maximal Cohen-Macaulay module. Apply $\mathrm{Hom}_A(M, \ )$ to the sequence, and we get an exact sequence

$$0 \to \mathrm{Hom}(M, \Omega(M)) \to \mathrm{Hom}(M, F) \to \mathrm{Hom}(M, M) \xrightarrow{f} \mathrm{Ext}^1(M, \Omega(M)) .$$

Note that $f \neq 0$ holds, since M is not free. Because A is an isolated singularity, we see that $M_{\mathfrak{p}}$ is free for any $\mathfrak{p} \in \mathrm{Spec}(R)$ except the maximal ideal of A. This implies that the image $\mathrm{Im}(f)$ is a nontrivial A-module of finite length. On the other hand, we notice that the modules $\mathrm{Hom}(M, M)$ and $\mathrm{Hom}(M, F)$ have depth at least two. (Actually this follows from a general fact that if $\mathrm{depth} Y \geq 2$ and if $\mathrm{Hom}_A(X, Y) \neq 0$, then depth $\mathrm{Hom}_A(X, Y) \geq 2$ for $X, Y \in A$-mod.) Hence we conclude from the depth argument [1, Proposition 1.2.9] that $\mathrm{depth}(\mathrm{Hom}(M, \Omega(M))) = 2$. This is a contradiction, because $\mathrm{Hom}(M, \Omega(M)) \cong F(\Omega(M))$ is a maximal Cohen-Macaulay over A and $\mathrm{depth}(A) \geq 3$. □

Example 6. Let k be a field and set $A = k[[x, y, z]]/(x^2 - yz)$. Let $\mathfrak{p}$ be a prime ideal of A generated by $\{x, y\}$. It is known that A is a normal Gorenstein domain of dimension two and $\mathfrak{p}$ is a unique indecomposable non-free maximal Cohen-Macaulay module over A. The class group $C\ell(A)$ is generated by the class of $\mathfrak{p}$ and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence we have $\mathrm{Aut}(\mathrm{ref}(A)) \cong \mathbb{Z}/2\mathbb{Z}$. In fact, the functor $F = \mathrm{Hom}_A(\mathfrak{p}, \ )$ is a unique nontrivial autofunctor over $\mathrm{ref}(A)$.

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Department of Mathematics, Okayama University, 700-8530, Okayama, Japan
`naoya_h@math.okayama-u.ac.jp, yoshino@math.okayama-u.ac.jp`

RING EXTENSIONS OF AB RINGS

SAEED NASSEH AND YUJI YOSHINO

1. Introduction

This is a report of our recent work. For any detail of this article, see the preprint [6]. Throughout the article, R denotes a commutative Noetherian ring with unity and $\dim(R) < \infty$.

Definition 1.1. We define

$$\text{Ext-index}(R) := \sup\{\, n \mid \operatorname{Ext}^i_R(M,N) = 0 \text{ for } i > n \text{ and } \operatorname{Ext}^n_R(M,N) \neq 0 \text{ for some finitely generated } R\text{-modules } M \text{ and } N \,\}.$$

And the ring R is said to be an AB ring if it satisfies $\text{Ext-index}(R) < \infty$.

In this paper we are interested in the AB property for some ring extensions. Note that any rings of the following types are known to be AB rings.

(1) Complete intersections [3].
(2) Cohen-Macaulay local rings with minimal multiplicity [3] [5].
(3) Gorenstein local rings with codimension at most 4 [7].
(4) Golod rings [5].
(5) Artinian local rings $(R, \mathfrak{m})$ with any of the following conditions:
  (a) $\mathfrak{m}^3 = 0$ and $\mu(\mathfrak{m}) = 3$ [5].
  (b) $\mathfrak{m}^3 = 0$ and $2\mu(\mathfrak{m}) \geq \ell_R(R) + 1$ [4].

2. Trivial extension of a local ring by its residue class field

Let M be an R-module. Then the direct sum $R \oplus M$ is equipped with the product:

$$(r, m).(r', m') = (rr', rm' + r'm).$$

This makes $R \oplus M$ a ring which is called the trivial extension of R by M and denoted by $R(M)$. There is a ring homomorphism $\pi : R(M) \longrightarrow R$ with $\pi(r, m) = r$ and any R-module can be regarded as an $R(M)$-module through π.

This work was done during the visit of the first author to Department of Mathematics of Okayama University. The first author would like to thank Okayama University for their hospitality and facilities.

Lemma 2.1. *Let $(R, \mathfrak{m}, k)$ be an arbitrary local ring. Then for R-modules M and N and for $n \geq 1$, we have an isomorphism*

$$\mathrm{Tor}_n^{R(k)}(M,N) \cong \mathrm{Tor}_n^R(M,N) \oplus \coprod_{i+j=n-1} \mathrm{Tor}_i^{R(k)}(M,k) \otimes_k \mathrm{Tor}_j^R(k,N).$$

Remark 2.2. Let S be a local ring with residue class field ℓ and let M, N be S-modules such that $\ell_S(\mathrm{Tor}_n^S(M,N)) < \infty$ for all n. Then we can consider the generating function $P_{M,N}^S(t)$ defined by the equality

$$P_{M,N}^S(t) = \sum_{n \geq 0} \ell_S(\mathrm{Tor}_n^S(M,N))t^n.$$

Recall that the Poincaré series $P_M^S(t)$ of M is defined to be $P_{\ell,M}^S(t)$ and the Poincaré series $P_\ell^S(t)$ of S is denoted simply by $P_S(t)$. Remark that the following is known.

<u>A theorem of Gulliksen:</u> *Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. Then the equality $P_{R(M)}(t) = P_R(t)(1 - P_M^R(t)\ t)^{-1}$ holds.*

Note that by the previous lemma we can show the equality

$$P_{M,N}^{R(k)}(t) = P_{M,N}^R(t) + P_M^{R(k)}(t)\ P_N^R(t)\ t,$$

for finitely generated modules M and N over an Artinian local ring R. Applying this to $M = N = k$, we have

$$P_{R(k)}(t) = P_R(t)(1 - P_R(t)\ t)^{-1},$$

which is a special case of the above mentioned theorem of Gulliksen.

Theorem 2.3. *Let $(R, \mathfrak{m}, k)$ be an arbitrary local ring and let M and N be nonzero non-free finitely generated $R(k)$-modules. Then $\mathrm{Tor}_n^{R(k)}(M,N) \neq 0$ for all $n \geq 3$.*

Proof. Set $A = R(k)$ and suppose $\mathrm{Tor}_n^A(M,N) = 0$ for some $n \geq 3$. Let $\mathfrak{n}$ be the maximal ideal of A and $x = (0,1) \in A$. Notice that $\mathfrak{n} = (0 :_A x)$ holds and $R \cong A/Ax$ as a ring.

Replacing M and N with their first syzygies, we may assume that $\mathrm{Tor}_n^A(M,N) = 0$ for some $n \geq 1$ and that $xM = 0$ and $xN = 0$. Thus we may assume M and N are modules over R through the identification $R \cong A/Ax$. Then by the previous lemma, the equality $\mathrm{Tor}_{n-1}^A(M,k) \otimes_k (N \otimes_R k) = 0$ holds. Since $N \otimes_R k \neq 0$, we see $\mathrm{Tor}_{n-1}^A(M,k) = 0$. This implies that M has finite projective dimension as an A-module. But $\mathrm{depth}(A) = 0$ and by Auslander-Buchsbaum formula, M is a free A-module. This is a contradiction. □

As applications of the theorem we can prove the following corollaries.

Corollary 2.4. *For an Artinian local ring $(R, \mathfrak{m}, k)$, the trivial extension $R(k)$ is an AB ring with* $\text{Ext-index}(R(k)) = 0$.

Corollary 2.5. *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring. Suppose that M is a finitely generated $R(k)$-module such that $\text{Ext}^i_{R(k)}(M, M) = 0$ for all $i > 0$. Then M is either a free or an injective $R(k)$-module.*

Corollary 2.6. *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring. And let $E = E_{R(k)}(k)$ be the injective envelope of the $R(k)$-module k. Then $\text{Ext}^i_{R(k)}(E, R(k)) \neq 0$ for some $i > 0$.*

Corollary 2.7 (Auslander-Reiten conjecture). *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring and let M be a finitely generated $R(k)$-module such that $\text{Ext}^i_{R(k)}(M, M \oplus R(k)) = 0$ for all $i > 0$. Then M is a free $R(k)$-module.*

3. More ring extensions

Let R be an algebra over a field k. And let M be a module over the polynomial ring $R[x]$. The *specialization of M to an element $\alpha \in k$* is defined by

$$M_\alpha := M \otimes_{k[x]} (k[x]/(x - \alpha)k[x]).$$

Remark that if M is a finitely generated $R[x]$-module, then M_α is a finitely generated R-module.

Lemma 3.1. *Let R be a k-algebra and let $\alpha \in k$. Assume that $x - \alpha$ is a nonzero divisor on $R[x]$-modules M and N. Then we have the exact sequence*

$$0 \longrightarrow \text{Ext}^i_{R[x]}(M, N)_\alpha \longrightarrow \text{Ext}^i_R(M_\alpha, N_\alpha) \longrightarrow \text{Tor}_1^{k[x]}(\text{Ext}^{i+1}_{R[x]}(M, N),\ k[x]/(x - \alpha)) \longrightarrow 0,$$

for each $i \geq 0$.

Theorem 3.2. *Suppose that k is an uncountable field and R is a finite dimensional k-algebra which is AB. Then $R \otimes_k k(x)$ is AB with* $\text{Ext-index}(R \otimes_k k(x)) \leq \text{Ext-index}(R)$.

Proof. Set $b := \text{Ext-index}(R)$ and let M' and N' be finitely generated $R \otimes_k k(x)$-modules with $\text{Ext}^i_{R \otimes_k k(x)}(M', N') = 0$ for $i \gg 0$. We have to show that $\text{Ext}^i_{R \otimes_k k(x)}(M', N') = 0$ for $i > b$.

Note that $R \otimes_k k(x)$ is just a localization of $R[x]$ by a multiplicatively closed subset $k[x] \backslash \{0\}$. Hence we can choose a finitely generated $R[x]$-submodule M of M' (resp. N of N') so that $M \otimes_{k[x]} k(x) \cong M'$ (resp. $N \otimes_{k[x]} k(x) \cong N'$). Notice that $x - \alpha$ acts on M and N as a non-zero divisor.

Since we have an isomorphism $\mathrm{Ext}^i_{R\otimes_k k(x)}(M', N') \cong \mathrm{Ext}^i_{R[x]}(M,N) \otimes_{k[x]} k(x)$, we see that $\mathrm{Ext}^i_{R[x]}(M,N) \otimes_{k[x]} k(x) = 0$ for $i \gg 0$.

On the other hand, since R is a finite dimensional k-algebra, each module $\mathrm{Ext}^i_{R[x]}(M,N)$ ($i \geq 0$) is a finitely generated $k[x]$-module. Hence it has a decomposition as a $k[x]$-module as follows:

$$\mathrm{Ext}^i_{R[x]}(M,N) \cong \bigoplus_{j=1}^{s_i} k[x]/(f_{ij}(x)) \oplus k[x]^{r_i},$$

where $f_{ij}(x) \neq 0 \in k[x]$.

Since $\mathrm{Ext}^i_{R[x]}(M,N) \otimes_{k[x]} k(x)$ are vanishing for $i \gg 0$, we have $r_i = 0$ for $i \gg 0$. Since there are only countably many equations $f_{ij}(x)$, we can find an element $\alpha \in k$ with the property $f_{ij}(\alpha) \neq 0$ for all i, j. Then, since $x - \alpha$ acts bijectively on $k[x]/(f_{ij}(x))$, we see that $\mathrm{Tor}_1^{k[x]}(\mathrm{Ext}^{i+1}_{R[x]}(M,N),\ k[x]/(x-\alpha)) = 0$ for all i. And we see as well that $\mathrm{Ext}^i_{R[x]}(M,N)_\alpha = 0$ for $i \gg 0$. Therefore the previous lemma implies that $\mathrm{Ext}^i_R(M_\alpha, N_\alpha) = 0$ for $i \gg 0$. Thus, by the definition of Ext-index, we have $\mathrm{Ext}^i_R(M_\alpha, N_\alpha) = 0$ for all $i > b$. Since $\mathrm{Ext}^i_{R[x]}(M,N)_\alpha$ is a submodule of $\mathrm{Ext}^i_R(M_\alpha, N_\alpha)$, we have $\mathrm{Ext}^i_{R[x]}(M,N)_\alpha = 0$ for all $i > b$. This implies that $r_i = 0$ for $i > b$, which is equivalent to the vanishing $\mathrm{Ext}^i_{R[x]}(M,N) \otimes_{k[x]} k(x) = 0$ for $i > b$. □

Remark 3.3. Let R be a Gorenstein local ring. Suppose there is an integer $n \geq 0$ such that $\mathrm{Ext}^i_R(M,N) = 0$ for $n+1 \leq i \leq n+t$ and $\mathrm{Ext}^j_R(M,N) \neq 0$ for $j = n, n+t+1$. In such a case we say that $\mathrm{Ext}_R(M,N)$ has a gap of length t. Set

$$\text{Ext-gap}(R) := \sup\{t \in \mathbb{N} \mid \mathrm{Ext}_R(M,N) \text{ has a gap of length } t\},$$

where "sup" is taken over all pairs (M,N) of finitely generated R-modules. R is called Ext-bounded if it has finite Ext-gap. Furthermore we should remark from [3] that it is known that $\text{Ext-gap}(R) < \infty \implies R$ is AB.

Keeping in mind this remark, we can prove the following statement completely in a same way as in the proof of Theorem 3.2:

Let R be a finite dimensional k-algebra where k is an infinite field. If R is Ext-bounded, then so is $R \otimes_k k(x)$.

We can also prove the following theorem in a similar way to the proof of Theorem 3.2.

Theorem 3.4. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay AB local ring with dualizing module. Suppose that R contains an uncountable coefficient field k. Then $R[x]_{\mathfrak{m}R[x]}$ is also a Cohen-Macaulay AB local ring.*

We finish this report by adding the following result.

Theorem 3.5. *Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein AB local ring. Assume that the residue class field k is algebraically closed. Then the polynomial ring $R[x_1, ..., x_n]$ is also AB.*

Proof. It is enough to prove that $R[x_1, ..., x_n]_{\mathfrak{M}}$ is AB for every maximal ideal $\mathfrak{M}$ of $R[x_1, ..., x_n]$. Since R is Artinian, we see that $\mathfrak{M} \cap R = \mathfrak{m}$. Therefore, by Hilbert's Nullstellensatz, there are elements $r_1, \ldots, r_n \in R$ with $\mathfrak{M} = (\mathfrak{m}, x_1 - r_1, \ldots, x_n - r_n)R[x_1, \ldots, x_n]$. Since $R \cong R[x_1, ..., x_n]_{\mathfrak{M}}/(x_1 - r_1, ..., x_n - r_n)R[x_1, ..., x_n]_{\mathfrak{M}}$ is AB and since $\{x_1 - r_1, ..., x_n - r_n\}$ is a regular sequence contained in the radical of $R[x_1, ..., x_n]_{\mathfrak{M}}$, it is easy to see that $R[x_1, ..., x_n]_{\mathfrak{M}}$ is also AB. □

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S. Nasseh; Department of Mathematics, Shahid Beheshti University, -and- Institute for Studies in Theoretical Physics and Mathematics (IPM), 19395-5746, Tehran, Iran

E-mail address: `saeed_naseh@mail.ipm.ir`

Y. Yoshino ; Department of Mathematics, Okayama University, 700-8530, Okayama, Japan

E-mail address: `yoshino@math.okayama-u.ac.jp`

Auslander-Reiten conjecture on Gorenstein rings [1]

Tokuji Araya

Nara University of Education

Email: araya@math.okayama-u.ac.jp

1. Introduction

The generalized Nakayama conjecture which was given by M. Auslander and I. Reiten is as follows [3] : Let Λ be an artin algebra. Any indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ.

They showed that above conjecture holds for all artin algebras if and only if the following conjecture holds for all artin algebras.
Let Λ be an Artin algebra and M be a finitely generated Λ-module. If $\mathrm{Ext}^i_\Lambda(M, M \oplus \Lambda) = 0$ $(\forall i > 0)$, then M is projective.
M. Auslander, S. Ding, and Ø. Solberg widened the context to algebras over commutative local rings [2].

(ARC) Let R be a commutative Noetherian local ring and M be a finitely generated R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $(\forall i > 0)$, then M is free.

They showed in [2] that if R is a complete intersection, then R satisfies (ARC). We shall show the following main theorem.

Theorem 1. *Let R be a Gorenstein ring. If R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$ with* $\mathrm{ht}\, p \leq 1$, *then R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$.*

2. Main Results

Through in this paper, we denote by R the d-dimensional commutative Gorenstein local ring with the unique maximal ideal $\mathfrak{m}$. We also denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules and by $\mathrm{CM}\, R$ the full subcategory of $\mathrm{mod}\, R$ consisting of all maximal Cohen-Macaulay modules.

We give a following condition to consider the Auslander-Reiten conjecture.

(ARC) For $M \in \mathrm{mod}\, R$, suppose $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $(i > 0)$, then M is free.

The main theorem of this paper is following;

Theorem 1. *If R_p satisfies* (ARC) *for all $p \in$* $\mathrm{Spec}R$ *with* $\mathrm{ht}\, p \leq 1$, *then R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$.*

It is difficult to check the freeness of modules in general. We give a following theorem to check the freeness of vector bundles.

Theorem 2. *We assume* $\dim R = d \geq 2$. *Let $M \in \mathrm{CM}\, R$ be a vector bundle. Suppose* $\mathrm{Ext}^{d-1}_R(M, M) = 0$, *then M is free.*

[1] The detailed version of this paper will be submitted for publication elsewhere.

We say M is a *vector bundle* if M_p is a free R_p-module for all prime ideal p which is not maximal ideal $\mathfrak{m}$. We want to omit the assumption M is a vector bundle in Theorem 2. But there is a counterexample if M is not a vector bundle.

Example 3. Let k be a field. We set $R = k[x, y, z]/(xy)$ be a 2-dimensional hypersurface and $M = R/(x)$. In this case, we can check that $\mathrm{Ext}_R^i(M, M) = 0$ if and only if i is odd. In particular, we see that $\mathrm{Ext}_R^{2-1}(M, M) = 0$ even if M is not free.

We prepare a lemma to show Theorem 2.

Lemma 4. [9, Lemma 3.10.] *Let R be a d-dimensional Cohen-Macaulay local ring and ω be a canonical module. We denote by $(-)^\vee$ the canonical dual $\mathrm{Hom}_R(-, \omega)$. For vector bundles M and $N \in \mathrm{CM}\, R$, we have a following isomorphism;*

$$\mathrm{Ext}_R^d(\underline{\mathrm{Hom}}(N, M), \omega) \cong \mathrm{Ext}_R^{d+1}(M, (\mathrm{tr}\, N)^\vee)$$

Here, $\underline{\mathrm{Hom}}(N, M)$ is the set of stable homomorphisms.

Proof of Theorem 2. Let $M \in \mathrm{CM}\, R$ be a vector bundle and we assume $\mathrm{Ext}_R^{d-1}(M, M) = 0$. We take a minimal free resolution of M;

$$F_\bullet : \cdots \to F_1 \to F_0 \to M \to 0.$$

Apply $(-)^* := \mathrm{Hom}_R(-, R)$, we get exact sequence;

$$0 \to M^* \to F_0^* \to F_1^* \to \mathrm{tr}\, M \to 0.$$

Since R is Gorenstein and M is maximal Cohen-Macaulay, we have $\Omega^2 M \cong (\mathrm{tr}\, M)^* (\cong (\mathrm{tr}\, M)^\vee)$. Therefore, we have

$$\begin{aligned} \mathrm{Ext}_R^{d+1}(M, (\mathrm{tr}\, N)^\vee) &\cong \mathrm{Ext}_R^{d+1}(M, (\mathrm{tr}\, N)^*) \\ &\cong \mathrm{Ext}_R^{d+1}(M, \Omega^2 M) \\ &\cong \mathrm{Ext}_R^{d-1}(M, M) = 0. \end{aligned}$$

Since M is vector bundle,

$$\underline{\mathrm{Hom}}_R(M, M)_p \cong \underline{\mathrm{Hom}}_{R_p}(M_p, M_p) = 0 \ (\forall p \neq \mathfrak{m}).$$

Thus we have $\underline{\mathrm{Hom}}_R(M, M)$ has finite length and we have

$$\begin{aligned} \underline{\mathrm{Hom}}_R(M, M) &\cong \mathrm{Ext}_R^d(\mathrm{Ext}_R^d(\underline{\mathrm{Hom}}_R(M, M), R), R) \\ &\cong \mathrm{Ext}_R^d(\mathrm{Ext}_R^{d+1}(M, (\mathrm{tr}\, M)^\vee), R) = 0 \end{aligned}$$

Thus we get M is free. □

Proof of Theorem 1. We put $\mathfrak{P} := \{\, p \in \mathrm{Spec} R \mid R_p$ does not satisfy (ARC) $\}$ and assume $\mathfrak{P} \neq \phi$. Let q be a minimal element in $\mathfrak{P}$ and replace R with R_q. By the minimalty, R is a $d(\geq 2)$-dimrnsional Gorenstein local ring which does not satisfy (ARC) but R_p satisfy (ARC) for all prime $p \neq \mathfrak{m}$. There exists $M \in \mathrm{mod}\, R$ s.t. $\mathrm{Ext}_R^i(M, M \oplus R) = 0$ $(\forall i > 0)$ but M is not free. Since $\mathrm{Ext}_R^i(M, R) = 0$ $(i > 0)$, M is maximal Cohen-Macaulay. For any $p \neq \mathfrak{m}$, $\mathrm{Ext}_{R_p}^i(M_p, M_p \oplus R_p) = 0$ $(\forall i > 0)$ and R_p satisfies (ARC), we have M_p is a free

$R_\mathfrak{p}$-module. Thus we get M is vector bundle. Furthermore, $\mathrm{Ext}_R^{d-1}(M,M)=0$ implies M is free. ($\because$Theorem 2.) Therefore we get contradiction and we have $\mathfrak{P}=\phi$. $\square$

Finally, we remark that normal domain satisfies Serre's (R_1)-condition and regular local ring satisfies (ARC), we get the following as a corollay of Theorem 1.

Corollary 5. *Gorenstein normal domain satisfies (ARC).*

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Quotient categories of homotopy categories

Osamu Iyama, Kiriko Kato and Jun-ichi Miyachi

Abstract

We introduce the homotopy category of unbounded complexes with bounded homologies. We study a recollement of its a quotient by the homotopy category of bounded complexes. This leads to the existence of quotient categories which are equivalent to a homotopy category of acyclic comlpexes, that is a stable derived category. In the case of a coherent ring R of self-injective dimension both sides, we show that the above recollement are triangulated equivalent to a recollement of the stable module category of Cohen-Macaulay R-modules.

1 Introduction

We study two types of triangulated categories in this paper. One is the categories of homotopy classes of chain complexes, equipped with triangles induced by chain maps and mapping cones. The other is stable module categories that are module categories mod projective modules. A stable module category is not triangulated in general. If the module category is Frobenius, then it's projective stabilization is triangulated. This type of triangulates categories are called algebraic triangulated categories. The well-known example is a stable module category of Cohen-Macaulay modules over Gorenstein rings.

Let R be a two-sided noetherian ring. The catogories of right R-modules, of finitely generated right R-modules and of finitely generated projective right R-modules are denoted by Mod$R$ and mod$R$, and *proj* R respectively. Let $\mathsf{K} = \mathsf{K}(\mathrm{proj}R)$ be the category of homotopy classes of complexes of finitely generated R-projective complexes. The following triangulated subcategories of K are of our concern.

$$\mathsf{K}^{\infty,b} = \{C \in \mathsf{K} \mid \mathrm{H}^i(C) = 0 \text{ (except for finite } i\text{'s)}\}$$

$$\mathsf{K}^{-,b} = \{C \in \mathsf{K}^{\infty,b} \mid C^i = 0 \text{ (for sufficiently large } i)\}$$

$$\mathsf{K}^{\infty,\emptyset} = \{C \in \mathsf{K}^{\infty,b} \mid \mathrm{H}^i(C) = 0 \ (i \in \mathsf{Z})\}$$

$$\mathsf{K}^{b} = \{C \in \mathsf{K} \mid C^i = 0 \text{ (except for finite } i\text{'s)}\}$$

Those triangulated categories are all épaisse, so the quotient categories are again triangulated.

Definition 1.1 ([Iw]) A two-sided noetherian ring is called Iwanaga-Gorenstein if $\mathrm{id}_R R < \infty$ and $\mathrm{id}_{R^{op}}\ R < \infty$.

If R is an Iwanaga-Gorenstein ring, we define a subcategory $\mathrm{CM}(R)$ of $\mathrm{mod}R$ as $\mathrm{CM}(R) = \{X \in \mathrm{mod}R \mid \mathrm{Ext}^i_R(X, R) = 0 \quad (i > 0)\}$.

Theorem 1.2 (Buchweitz [Bu]) Assume R is Iwanaga-Gorenstein. The quotient category $\mathsf{K}^{-,b}/\mathsf{K}^b$ is triangle equivalent to the stable module category $\underline{\mathrm{CM}}(R)$.

On the other hand, we observe the following.

Theorem 1.3 If R is Iwanaga-Gorenstein.The quotient category $\mathsf{K}^{\infty,b}/\mathsf{K}^{-,b}$ is equivalent to the stable module category $\underline{\mathrm{CM}}(R)$.

Naturally, the question arises: What is $\mathsf{K}^{\infty,b}/\mathsf{K}^b$? Is it realizable as a stable module category?

2 Operations and functors on $\mathsf{K}^{\infty,b}$

For an object A of $\mathsf{K}^{\infty,b}$, define objects X_A and T_A of $\mathsf{K}^{\infty,\emptyset}$ as follows.

Let l be the smallest integer such that $\mathrm{H}_l(A^*) \neq 0$. Then $Cok\, d_A^{l-1}$ is a maximal Cohen-Macaulay module. Define $X_A \in \mathsf{K}^{\infty,\emptyset}$ as

$$\tau_{\leq l} X_A = \tau_{\leq l} A$$

and

$$\cdots \to X_A^{l+1*} \to X_A^{l+2*} \to (Cok\, d_A^{l-1})^* \to 0$$

is exact. Then X_A is totally acyclic and $\mathrm{id}_{Cok\, d_A^{l-1*}}$ induces a canonical chain map $\xi_A : X_A \to A$ as $\xi_A^i = \mathrm{id} \quad (i \leq l)$.

Similarly, let r be the largest integer such that $\mathrm{H}^r(A) \neq 0$. Then $Ker\, d_A^r$ is a maximal Cohen-Macaulay module. Define $T_A \in \mathsf{K}^{\infty,\emptyset}$ as

$$\tau_{\geq r} X_A = \tau_{\geq r} A$$

and

$$\cdots \to T_A^{r-1} \to T_A^r \to (Ker\, d_A^r) \to 0$$

is exact. Then T_A is totally acyclic and $\mathrm{id}_{Ker\, d_A^r}$ induces a canonical chain map $\zeta_A : A \to T_A$ as $\zeta_A^i = \mathrm{id} \quad (i \geq r)$.

Set a chain maps $l_A : L_A \to A$ and $r_{L_A} : L_A \to R_{L_A}$ as follows:

$$\tau_{\leq 0} L_A = \tau_{\leq 0} X_A, \tau_{\geq 1} L_A = \tau_{\geq 1} A,$$

$$\tau_{\leq 0} l_A = \tau_{\leq 0} \xi_A, \tau_{\geq 1} l_A = \tau_{\geq 1} \mathrm{id}_A,$$

$$\tau_{\leq 0} R_{L_A} = \tau_{\leq 0} L_A, \tau_{\geq 1} R_{L_A} = \tau_{\geq 1} T_{L_A},$$

$$\tau_{\leq 0} r_{L_A} = \tau_{\leq 0} \mathrm{id}_{L_A}, \tau_{\geq 1} r_{L_A} = \tau_{\geq 1} \zeta_A$$

Obviously $C(l_A)$ and $C(r_{L_A})$ belongs to K^b, hence as an object of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, A is isomorphic to the complex

$$R_{L_A} : \cdots \to {X_A}^{-1} \to {X_A}^0 \to {T_A}^1 \to {T_A}^2 \to \cdots$$

We may assume $\lambda_A = \mathrm{H}^0(\tau_{\leq 0}\xi_A\zeta_A) : Cok\, d_{X_A}^{-1} \to Ker\, d_{T_A}^1$ to be surjective by adding some split exact sequence of projective modules if necessary.

3 The category of morphisms

We define category $\mathrm{Mor}(R)$ as follows: objects of $\mathrm{Mor}(R)$ are the morphisms $\alpha : X_\alpha \to T_\alpha$ of $\mathrm{Mod}(R)$. For $\alpha, \beta \in \mathrm{mor}(R)$, we define

$$\mathrm{Mor}(R)(\alpha,\beta) = \{(f_X, f_T) \in \mathrm{Hom}_R(X_\alpha, X_\beta) \times \mathrm{Hom}_R(T_\alpha, T_\beta) \mid f_T\alpha = \beta f_X\}.$$

And the subcategory $\mathrm{mor}_s^{CM}(R)$ of $\mathrm{Mor}(R)$ consists of the objects $\alpha : X_\alpha \to T_\alpha$ of $\mathrm{CM}(R)$ that are surjective. The structure of $\mathrm{mor}_s^{CM}(R)$ is obtained by the next lemma.

Lemma 3.1 Let $T_2(R)$ be the category of 2×2 upper triangular matrices with entries in R. Then $\mathrm{Mod}(T_2(R))$ is equivalent to $\mathrm{Mor}(R)$. And $\mathrm{mor}_s^{CM}(R)$ is equivalent to the category $\mathrm{CM}(T_2(R))$.

proof. An object $f : X_f \to T_f$ of $\mathrm{Mor}(R)$ corresponds to an $T_2(R)$-module $M_f = X_f \times T_f$ where $(x\ t)\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = (xa\ f(x)b + tc)$. This correspondence gives an equivalence between $\mathrm{CM}(T_2(R))$ and $\mathrm{mor}_i^{CM}(R)$ consisting of injective maps $\alpha : X_\alpha \to T_\alpha$ with $X_\alpha, T_\alpha, \mathrm{Cok} f \in \mathrm{CM}(R)$. Obviously $\mathrm{mor}_i^{CM}(R)$ is equivalent to $\mathrm{mor}_s^{CM}(R)$. (q.e.d.)

Thus $\mathrm{mor}_s^{CM}(R)$ is a Frobenius category together with projective-injective objects consisting of $p \in \mathrm{mor}_s^{CM}(R)$ that X_p and T_p are projective modules. Hence the stable category $\underline{\mathrm{mor}_s^{CM}}(R)$ is triangulated. We shall construct a functor between $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ and $\underline{\mathrm{mor}_s^{CM}}(R)$.

Let $\alpha : X_\alpha \to T_\alpha$ be an object of $\mathrm{mor}_s^{CM}(R)$ and let F_{X_α} and F_{T_α} be acyclic projective complexes such that $\mathrm{H}^0(\tau_{\leq 0}F_{X_\alpha}) = X_\alpha$ and $\mathrm{H}^0(\tau_{\leq 0}F_{T_\alpha}) = T_\alpha$. Set natural maps $\rho : F_{X_\alpha}^0 \to X_\alpha$ and $\epsilon : T_\alpha \to F_{T_\alpha}$. Make a projective complex F_α as

$$\tau_{\leq 0}F_\alpha = \tau_{\leq 0}F_{X_\alpha},\ \tau_{\geq 1}F_\alpha = \tau_{\geq 1}F_{T_\alpha},\ d_{F_\alpha} = \epsilon\alpha\rho.$$

Lemma 3.2 1) A morphism $f \in \mathrm{mor}_s^{CM}(R)(\alpha,\beta)$ induces a chain map $F_f : F_\alpha \to F_\beta$.
2) For morphisms $f \in \mathrm{mor}_s^{CM}(R)(\alpha,\beta)$ and $g \in \mathrm{mor}_s^{CM}(R)(\beta,\gamma)$, $F_{gf} = F_gF_f$.
3) An exact sequence $0 \to \alpha \xrightarrow{f} \beta \xrightarrow{g} \gamma \to 0$ in $\mathrm{mor}_s^{CM}(R)$ induces an exact sequence $0 \to F_\alpha \xrightarrow{F_f} F_\beta \xrightarrow{F_g} F_\gamma \to 0$ in $\mathsf{C}^{\infty,b}$.

4) An object p of $\mathrm{mor}_s^{CM}(R)$ is projective if and only if F_p is a bounded complex.

Lemma 3.3 The operation F gives a functor $\mathrm{mor}_s^{CM}(R) \to \mathsf{K}^{\infty,b}$. And F induces a functor $\underline{F} : \underline{\mathrm{mor}}_s^{CM}(R) \to \mathsf{K}^{\infty,b}/\mathsf{K}^b$.

Proposition 3.4 The functor $\underline{F} : \underline{\mathrm{mor}}_s^{CM}(R) \to \mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangulated.

proof Let

$$\underline{\alpha} \xrightarrow{f} \underline{\beta} \xrightarrow{g} \underline{\gamma} \xrightarrow{h} \Sigma\underline{\alpha}$$

be a triangle in $\underline{\mathrm{mor}_s^{CM}(R)}$. That is, the injective hull $\alpha \xrightarrow{\epsilon} q$ of α and f make a push-out diagram which implies a commutative diagram in $\mathrm{CM}(\Lambda^*)$ with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \alpha & \xrightarrow{\epsilon} & q & \xrightarrow{\rho} & \Sigma\alpha & \to & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle w} & & \| & & \\
0 & \to & \beta & \xrightarrow{g} & \gamma & \xrightarrow{h} & \Sigma\alpha & \to & 0.
\end{array}$$

This induces a commutative diagram in $\mathsf{C}^{\infty,b}$ with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & F_\alpha & \xrightarrow{F_\epsilon} & F_q & \xrightarrow{F_\varrho} & F_{\Sigma\alpha} & \to & 0 \\
 & & \downarrow{\scriptstyle F_f} & & \downarrow{\scriptstyle F_w} & & \| & & \\
0 & \to & F_\beta & \xrightarrow{F_g} & F_\gamma & \xrightarrow{F_h} & F_{\Sigma\alpha} & \to & 0
\end{array}$$

It remains to show that there is a functorial isomorphism $F_{\Sigma\alpha} \cong \Sigma F_\alpha$ in $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_\alpha & \xrightarrow{F_\epsilon} & F_q & \xrightarrow{F_\rho} & F_{\Sigma\alpha} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F_f} & & \downarrow{\scriptstyle F_w} & & \| & & \\
0 & \longrightarrow & F_\beta & \longrightarrow & F_\gamma & \longrightarrow & F_{\Sigma\alpha} & \longrightarrow & 0
\end{array}$$

induces a morphism between triangles in $\mathsf{K}^{\infty,b}$:

$$\begin{array}{ccccccc}
F_\alpha & \xrightarrow{F_\epsilon} & F_q & \xrightarrow{F_\rho} & F_{\Sigma\alpha} & \xrightarrow{\pi_\alpha} & \Sigma F_\alpha \\
\downarrow{\scriptstyle F_f} & & \downarrow{\scriptstyle F_w} & & \| & & \downarrow{\scriptstyle \Sigma F_f} \\
F_\beta & \longrightarrow & F_\gamma & \longrightarrow & F_{\Sigma\alpha} & \longrightarrow & \Sigma F_\alpha
\end{array}$$

Since $F_q \in \mathsf{K}^b$, it is easy to see that π_α is a functorial isomorphism in $\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b$, and we have a triangle in $\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b$:

$$F_\alpha \xrightarrow{F_f} F_\beta \xrightarrow{F_g} F_\gamma \xrightarrow{F_\alpha \pi_\alpha} \Sigma F_\alpha$$

(q.e.d.)

Theorem 3.5 *The category* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ *is triangle equivalent to* $\underline{\mathrm{mor}}_s^{CM}(R)$.

We shall show that $\underline{F}$ is a category equivalence. We have already seen that $\underline{F}$ is dense from the previous section. For proving $\underline{F}$ is fully faithful, we use the notion of t-strucrture.

4 Stable t-structures

Definition 4.1 ([Mi1]) *For full subcategories* $\mathcal{U}$ *and* $\mathcal{V}$ *of a triangulate category* $\mathcal{C}$, $(\mathcal{U},\mathcal{V})$ *is called a stable t-structure in* $\mathcal{C}$ *provided that*

- $\mathcal{U}$ *and* $\mathcal{V}$ *are stable for translations.*
- $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U},\mathcal{V})=0$.
- *For every* $X\in\mathcal{C}$, *there exists a triangle* $U\to X\to V\to\Sigma U$ *with* $U\in\mathcal{U}$ *and* $V\in\mathcal{V}$.

Proposition 4.2 ([BBD], [Mi1]) *Let* $\mathcal{C}$ *be a triangulated category. The following hold.*

1 *Let* $(\mathcal{U},\mathcal{V})$ *be a stable t-structure in* $\mathcal{C}$, $i_*:\mathcal{U}\to\mathcal{C}$ *and* $j_*:\mathcal{V}\to\mathcal{C}$ *the canonical embeddings. Then there are a right adjoint* $i^!:\mathcal{C}\to\mathcal{U}$ *of* i_* *and a left adjoint* $j^*:\mathcal{C}\to\mathcal{V}$ *of* j_* *which satisfy the following.*

   (a) $j^*i_*=0,\ i^!j_*=0$.

   (b) *The adjunction arrows* $i_*i^!\to\mathbf{1}_{\mathcal{C}}$ *and* $\mathbf{1}_{\mathcal{C}}\to j_*j^*$ *imply a triangle* $i_*i^!X\to X\to j_*j^*X\to\Sigma i_*i^!X$ *for any* $X\in\mathcal{C}$.

   In this case, j^* *(resp.,* $i^!$*) implies the triangulated equivalence* $\mathcal{C}/\mathcal{U}\simeq\mathcal{V}$ *(resp.,* $\mathcal{C}/\mathcal{V}\simeq\mathcal{U}$*).*

2 *If* $\{\mathcal{C},\mathcal{C}'';j^*,j_*\}$ *(resp.,* $\{\mathcal{C},\mathcal{C}'';j_!,j^*\}$*) is a localization (resp., a colocalization) of* $\mathcal{C}$, *thai is,* j_* *(resp.,* i_**) is a fully faithful right (resp., left) adjoint of* $i^!$, *then* $(\mathrm{Ker}j^*,\mathrm{Im}j_*)$ *(resp.,* $(\mathrm{Im}j_!,\mathrm{Ker}j^*)$*) is a stable t-structure. In this case, the adjunction arrow* $\mathbf{1}_{\mathcal{C}}\to j_*j^*$ *(resp.,* $j_!j^*\to\mathbf{1}_{\mathcal{C}}$*) implies triangles*

$$U\to X\to j_*j^*X\to\Sigma U$$
$$(\textit{resp., } j_!j^*X\to X\to V\to\Sigma j_!j^*X)$$

   with $U\in\mathrm{Ker}j^*$, $j_*j^*X\in\mathrm{Im}j_*$ *(resp.,* $j_!j^*X\in\mathrm{Im}j_!$, $V\in\mathrm{Ker}j^*$*) for all* $X\in\mathcal{C}$.

Proposition 4.3 *Let* R *be a coherent ring. Then we have the following.*

- $(\mathsf{K}^{-,b},\mathsf{K}^{\infty,\emptyset})$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}$. *Hence* $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset})$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.
- $(\mathsf{K}^{+,b}/\mathsf{K}^b,\mathsf{K}^{-,b}/\mathsf{K}^b)$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.

- *If R is Iwanaga-Gorenstein, then* $(\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b, \mathsf{K}^{+,b}/\mathsf{K}^b)$ *is a stable t-structure of* $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.

Let R be an Iwanaga-Gorenstein ring. Let $\underline{\mathcal{CM}}_0$ (resp., $\underline{\mathcal{CM}}_1$, $\underline{\mathcal{CM}}_p$) be the full subcategory of $\underline{\mathrm{mor}}_s^{CM}(R)$ consisting of objects of the form $X \to 0$ (resp., $S \xrightarrow{=} S$, $P \to T$, with P being projective).

Proposition 4.4 *The following are stable t-structures of* $\underline{\mathrm{mor}}_s^{CM}(R)$.

$$(\underline{\mathcal{CM}}_0, \underline{\mathcal{CM}}_1),\ (\underline{\mathcal{CM}}_p, \underline{\mathcal{CM}}_0),\ (\underline{\mathcal{CM}}_1, \underline{\mathcal{CM}}_p).$$

Proposition 4.5 *The triangulated functor F induces equivalences*

$$\begin{aligned} \underline{F}\,|_{\underline{\mathcal{CM}}_0} &: \underline{\mathcal{CM}}_0 \to \mathsf{K}^{-,b}/\mathsf{K}^b, \\ \underline{F}\,|_{\underline{\mathcal{CM}}_1} &: \underline{\mathcal{CM}}_1 \to \mathsf{K}^{\infty,\emptyset}, \\ \textit{and}\ \ \underline{F}\,|_{\underline{\mathcal{CM}}_p} &: \underline{\mathcal{CM}}_p \to \mathsf{K}^{+,b}/\mathsf{K}^b. \end{aligned}$$

Now we focus on the stable t-structures $(\mathsf{K}^{-,b}/\mathsf{K}^b, \mathsf{K}^{\infty,\emptyset})$ of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, and $(\underline{\mathcal{CM}}_0, \underline{\mathcal{CM}}_1)$ of $\underline{\mathrm{mor}}_s^{CM}(R)$. For a given object A of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, there uniquely exists a triangle

$$A_- \to A \to A_{ac} \to \Sigma A_-$$

with $A_- \in \mathsf{K}^{-,b}/\mathsf{K}^b$ and $A_{ac} \in \mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b$. And for each object $\underline{\alpha}$ of $\underline{\mathrm{mor}}_s^{CM}(R)$, there uniquely exists a triangle

$$\underline{\alpha}_0 \to \underline{\alpha} \to \underline{\alpha}_1 \to \Sigma\underline{\alpha}_0$$

with $\underline{\alpha}_0 \in \underline{\mathcal{CM}}_0$ and $\underline{\alpha}_1 \in \underline{\mathcal{CM}}_1$. From Proposition 4.5, we have $(\underline{F}_{\underline{\alpha}})_- \cong \underline{F}_{\underline{\alpha}_0}$ and $(\underline{F}_{\underline{\alpha}})_{ac} \cong \underline{F}_{\underline{\alpha}_1}$.

Lemma 4.6 *For objects $\underline{\alpha}$ and $\underline{\beta}$ of $\underline{\mathrm{mor}}_s^{CM}(R)$, $\underline{F}$ induces an isomorphism*

$$\mathrm{Hom}_{\underline{\mathrm{mor}}_s^{CM}(R)}(\underline{\alpha}_1, \underline{\beta}_0) \cong \mathrm{Hom}_{\mathsf{K}^{\infty,b}/\mathsf{K}^b}((\underline{F}_{\underline{\alpha}})_{ac}, (\underline{F}_{\underline{\beta}})_-).$$

The proof of Theorem 3.5. We have only to show that $\underline{F}$ is fully faithful. Let $\underline{\alpha}$ and $\underline{\beta}$ be objects of $\underline{\mathrm{mor}}_s^{CM}(R)$. The triangles

$$\begin{aligned} \underline{\alpha}_0 \to \underline{\alpha} \to \underline{\alpha}_1 \to \Sigma\underline{\alpha}_0, \\ \underline{\beta}_0 \to \underline{\beta} \to \underline{\beta}_1 \to \Sigma\underline{\beta}_0 \end{aligned}$$

induce a diagram of abelian groups with exact rows and columns

$$\begin{array}{ccccc} \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1, \beta_0) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1, \beta) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1, \beta_1) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{mor}}_s^{CM}(R)(\alpha, \beta_0) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha, \beta) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha, \beta_1) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0, \beta_0) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0, \beta) & \longrightarrow & \underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0, \beta_1) \end{array}$$

From Proposition 4.5, $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta_0)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_-,(\underline{F_\beta})_-)$ and $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta_1)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},(\underline{F_\beta})_{ac})$. By Lemma 4.6, $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta_0)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},(\underline{F_\beta})_-)$. These together give us
$\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},\underline{F_\beta})$ and $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta_0)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b(\underline{F_\alpha},(\underline{F_\beta})_0)$. Since $(\underline{\mathcal{CM}_0},\underline{\mathcal{CM}_1})$ and $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset})$ are stable t-structures of $\underline{\mathrm{mor}}_s^{CM}(R)$ and $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ respectively, both $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta_1)$ and $\mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_-,(\underline{F_\beta})_{ac})$ vanish. Therefore $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha,\beta_1)\cong\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_1,\beta_1)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_{ac},(\underline{F_\beta})_{ac})\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b(\underline{F_\alpha},(\underline{F_\beta})_{ac})$. Similarly $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha_0,\beta)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha})_-,\underline{F_\beta})$. Now $\underline{\mathrm{mor}}_s^{CM}(R)(\alpha,\beta)\cong \mathsf{K}^{\infty,b}/\mathsf{K}^b((\underline{F_\alpha}),\underline{F_\beta})$ comes from Five lemma. (q.e.d.)

Together with Theorem 3.1, we obtain Buchweitz-type theorem:

Theorem 4.7 *If R is Iwanaga-Gorenstein, then $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangle equivalent to $\underline{\mathrm{CM}}(T_2(R))$.*

5 Recollements

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be triangulated subcategories of a triangulated category $\mathcal{C}$. Suppose $(\mathcal{U},\mathcal{V})$ and $(\mathcal{V},\mathcal{W})$ are both stable t-structures of $\mathcal{C}$. From Prop 4.2, the canonical embedding $j_*:\mathcal{V}\to\mathcal{C}$ and the quotient $s^*:\mathcal{C}\to\mathcal{C}/\mathcal{V}$ have right adjoints $j^!:\mathcal{C}\to\mathcal{V}$ and $s^*:\mathcal{C}/\mathcal{V}\to\mathcal{C}$ since $(\mathcal{U},\mathcal{V})$ is a stable t-structure. And a stable t-structure $(\mathcal{V},\mathcal{W})$ produeces left adjoints $j^*:\mathcal{C}\to\mathcal{V}$ of j_* and $s_!:\mathcal{C}/\mathcal{V}\to\mathcal{C}$ of $s^*:\mathcal{C}/\mathcal{V}\to\mathcal{C}$ respectively.

Definition 5.1 ([BBD]) *A nine-tuple $\{\mathcal{C}',\mathcal{C},\mathcal{C}'';j^*,j_*,j^!,s_!,s^*,s_*\}$ consisting of triangulated categories and functors*

$$\mathcal{C}' \underset{\underset{j^!}{\longleftarrow}}{\overset{\overset{j^*}{\longleftarrow}}{\xrightarrow{j_*}}} \mathcal{C} \underset{\underset{s_*}{\longleftarrow}}{\overset{\overset{s_!}{\longleftarrow}}{\xrightarrow{s^*}}} \mathcal{C}''$$

is called a recollement if it satisfies the following:

- *j_*, $s_!$, and s_* are fully faithful.*
- *(j^*,j_*), $(j_*,j^!)$, $(s_!,s^*)$, and (s^*,s_*) are adjoint pairs.*
- *$j^*s_!=0$, $s^*j_*=0$, and $j^!s_*=0$.*
- *For each object C of $\mathcal{C}$ has triangles*

$$j_*j^!C\to C\to s_!s^*C\to\Sigma j_*j^!C,$$
$$s_*s^*C\to C\to j_*j^*C\to\Sigma s_*s^*C.$$

Proposition 5.2 ([BBD], [Mil]) *1) If $(\mathcal{U},\mathcal{V})$ and $(\mathcal{V},\mathcal{W})$ are stable t-structures of $\mathcal{C}$, then the canonical embedding $j_* : \mathcal{V} \to \mathcal{C}$ produces a recollement*

$$\mathcal{V} \underset{\overset{\xleftarrow{j^!}}{}}{\overset{\overset{\xleftarrow{j^*}}{}}{\xrightarrow{j_*}}} \mathcal{C} \underset{\overset{\xleftarrow{s_*}}{}}{\overset{\overset{\xleftarrow{s_!}}{}}{\xrightarrow{s^*}}} \mathcal{C}/\mathcal{V}$$

2) If $\{\mathcal{C}',\mathcal{C},\mathcal{C}'';j^*,j_*,j^!,s_!,s^*,s_*\}$ is a recollement, then $(\mathrm{Im}j_*,\mathrm{Im}s_*)$ and $(\mathrm{Im}s_!,\mathrm{Im}j_*)$ are stable t-structures.

Remember that if R is Iwanaga-Gorenstein, three triangulated subcategories $\mathsf{K}^{-,b}/\mathsf{K}^b$, $\mathsf{K}^{\infty,\emptyset}$, and $\mathsf{K}^{+,b}/\mathsf{K}^b$ form three stable t-structures in $\mathsf{K}^{\infty,b}$: $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset})$, $(\mathsf{K}^{\infty,\emptyset},\mathsf{K}^{+,b}/\mathsf{K}^b)$ and $(\mathsf{K}^{+,b}/\mathsf{K}^b,\mathsf{K}^{-,b}/\mathsf{K}^b)$. This implies there are three recollements with respect to the canonical embeddings of each subcategories to $\mathsf{K}^{\infty,b}$.

Definition 5.3 *Let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ be triangulated subcategories of a triangulated category $\mathcal{C}$. We call $(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3)$ a triangle of recollements in $\mathcal{C}$ if $(\mathcal{U}_1,\mathcal{U}_2$, $(\mathcal{U}_2,\mathcal{U}_3$, and $(\mathcal{U}_2,\mathcal{U}_3$ are stable t-structures of $\mathcal{C}$. In this case, there are recollements*

$$\mathcal{U}_n \underset{\overset{\xleftarrow{i_n^!}}{}}{\overset{\overset{\xleftarrow{i_n^*}}{}}{\xrightarrow{j_{n*}}}} \mathcal{C} \underset{\overset{\xleftarrow{j_{n*}}}{}}{\overset{\overset{\xleftarrow{j_{n!}}}{}}{\xrightarrow{j_n^*}}} \mathcal{C}/\mathcal{U}_n$$

for any $n \mod 3$ such that the essential image $\mathrm{Im}j_{n!}$ is $\mathcal{U}_{n-1}$, and that the essential image $\mathrm{Im}j_{n*}$ is $\mathcal{U}_{n+1}$. Therefore, $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$ are triangulated equivalent.

Theorem 5.4 *If R is Iwanaga-Gorenstein, then $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset},\mathsf{K}^{+,b}/\mathsf{K}^b)$ is a triangle of recollements in $\mathsf{K}^{\infty,b}/\mathsf{K}^b$. There is a triangulated equivalence between $\underline{\mathrm{mor}}_s^{CM}(R) \cong \underline{\mathrm{CM}}(T_2(R))$ and $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ that induces the correspondence between a triangle of recollements $(\underline{\mathcal{CM}}_0,\underline{\mathcal{CM}}_1,\underline{\mathcal{CM}}_p)$ and $(\mathsf{K}^{-,b}/\mathsf{K}^b,\mathsf{K}^{\infty,\emptyset},\mathsf{K}^{+,b}/\mathsf{K}^b)$.*

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Shestakov-Umirbaev reductions and Nagata's conjecture on a polynomial automorphism

Shigeru Kuroda

Abstract

In 2003, Shestakov-Umirbaev solved Nagata's conjecture on an automorphism of a polynomial ring. In the present paper, we reconstruct their theory by using the "generalized Shestakov-Umirbaev inequality", which was recently given by the author. As a consequence, we obtain a more precise tameness criterion for polynomial automorphisms. In particular, we show that no tame automorphism of a polynomial ring admits a reduction of type IV.

1 Introduction

Let k be a field, n a natural number, and $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k. In the present paper, we discuss the structure of the automorphism group $\operatorname{Aut}_k k[\mathbf{x}]$ of $k[\mathbf{x}]$ over k. Let $F : k[\mathbf{x}] \to k[\mathbf{x}]$ be an endomorphism over k. We identify F with the n-tuple $(f_1, \ldots, f_n)$ of elements of $k[\mathbf{x}]$, where $f_i = F(x_i)$ for each i. Then, F is an automorphism if and only if the k-algebra $k[\mathbf{x}]$ is generated by $f_1, \ldots, f_n$. Note that the sum $\deg F := \sum_{i=1}^n \deg f_i$ of the total degrees of $f_1, \ldots, f_n$ is at least n whenever F is an automorphism. An automorphism F is said to be *affine* if $\deg F = n$. If this is the case, then there exist $(a_{i,j})_{i,j} \in GL_n(k)$ and $(b_i)_i \in k^n$ such that $f_i = \sum_{j=1}^n a_{i,j} x_j + b_i$ for each i. We say that F is *elementary* if there exist

2000 *Mathematical Subject Classification.* Primary 14R10; Secondary 13F20.

Partly supported by the Grant-in-Aid for Young Scientists (Start-up) 19840041, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

$l \in \{1, \dots, n\}$ and $\phi \in k[x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n]$ such that $f_l = x_l + \phi$ and $f_i = x_i$ for each $i \neq l$. The subgroup $\mathrm{T}_k\, k[\mathbf{x}]$ of $\mathrm{Aut}_k\, k[\mathbf{x}]$ generated by affine automorphisms and elementary automorphisms is called the *tame subgroup*. An automorphism is said to be *tame* if it belongs to $\mathrm{T}_k\, k[\mathbf{x}]$.

It is a fundamental question in polynomial ring theory whether $\mathrm{T}_k\, k[\mathbf{x}] = \mathrm{Aut}_k\, k[\mathbf{x}]$ holds for each n, which is called the *tame generators problem*. The equality is obvious if $n = 1$. This also holds true if $n = 2$. It was shown by Jung [4] in 1942 when k is of characteristic zero, and by van der Kulk [5] in 1953 when k is an arbitrary field. These results are consequences of the fact that each automorphism of $k[\mathbf{x}]$ but an affine automorphism admits an elementary reduction if $n = 2$. Here, we say that F *admits an elementary reduction* if $\deg(F \circ E) < \deg F$ for some elementary automorphism E, that is, there exist $l \in \{1, \dots, n\}$ and $\phi \in k[f_1, \dots, f_{l-1}, f_{l+1}, \dots, f_n]$ such that $\deg(f_l - \phi) < \deg f_l$. By the Jung-van der Kulk theorem, in case $n = 2$, we may find elementary automorphisms $E_1, \dots, E_r$ for some $r \in \mathbf{N}$ such that

$$\deg F > \deg(F \circ E_1) > \cdots > \deg(F \circ E_1 \circ \cdots \circ E_r) = 2$$

for each $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$ with $\deg F > 2$. This implies that F is tame.

When $n = 3$, the structure of $\mathrm{Aut}_k\, k[\mathbf{x}]$ becomes far more difficult. In 1972, Nagata [9] conjectured that the automorphism

$$F = (x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2x_3, x_2 + (x_1x_3 + x_2^2)x_3, x_3) \quad (1.1)$$

is not tame. This famous conjecture was finally solved in the affirmative by Shestakov-Umirbaev [11] in 2003 for a field k of characteristic zero. Therefore, $\mathrm{T}_k\, k[\mathbf{x}] \neq \mathrm{Aut}_k\, k[\mathbf{x}]$ if $n = 3$. However, the question remains open for $n \geq 4$.

Shestakov-Umirbaev [11] showed that, if F does not admit an elementary reduction for $F \in \mathrm{T}_k\, k[\mathbf{x}]$ with $\deg F > 3$, then there exists a sequence of elementary automorphisms $E_1, \dots, E_r$, where $r \in \{2, 3, 4\}$, with certain conditions such that $\deg(F \circ E_1 \circ \cdots \circ E_r) < \deg F$. If this is the case, then F is said to *admit a reduction of type I, II, III or IV* according to the conditions on F and $E_1, \dots, E_r$. Nagata's automorphism is not affine, and does not admit neither an elementary reduction nor reductions of these four types. Therefore, Nagata's automorphism is not tame. We note that there exist tame automorphisms which admit reductions of type I (see [1], [7] and [11]), but it is not known whether there exist automorphisms admitting reductions of the other types.

Shestakov-Umirbaev [11] used an inequality [10, Theorem 3] concerning the total degrees of polynomials as a crucial tool. This result was recently generalized by the author in [6]. The purpose of this paper is to reconstruct the Shestakov-Umirbaev theory using the generalized inequality. As a consequence, we obtain a more precise tameness criterion for polynomial automorphisms. In particular, we show that no tame automorphism of $k[\mathbf{x}]$ admits a reduction of type IV.

This report consists of the first two sections of [8], which is available at

`http://arxiv.org/PS_cache/arxiv/pdf/0801/0801.0117v1.pdf`

Although the full version of [8] is 48 pages long, the details are carefully explained. It is said that the theory of Shestakov and Umirbaev is difficult and still not widely understood. I hope that our article will be helpful in understanding how the tame generators problem was solved.

2 Main result

In what follows, we assume that the field k is of characteristic zero. Let Γ be a totally ordered $\mathbf{Z}$-module, and $\omega = (\omega_1, \ldots, \omega_n)$ an n-tuple of elements of Γ with $\omega_i > 0$ for $i = 1, \ldots, n$. We define the *ω-weighted grading* $k[\mathbf{x}] = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma$ by setting $k[\mathbf{x}]_\gamma$ to be the k-vector subspace generated by the monomials $x_1^{a_1} \cdots x_n^{a_n}$ of $k[\mathbf{x}]$ with $\sum_{i=1}^n a_i\omega_i = \gamma$ for each $\gamma \in \Gamma$. For $f \in k[\mathbf{x}] \setminus \{0\}$, we define the *$\omega$-weighted degree* $\deg_\omega f$ of f to be the maximum among $\gamma \in \Gamma$ with $f_\gamma \neq 0$, where $f_\gamma \in k[\mathbf{x}]_\gamma$ for each γ such that $f = \sum_{\gamma\in\Gamma} f_\gamma$. We define $f^\omega = f_\delta$, where $\delta = \deg_\omega f$. In case $f = 0$, we set $\deg_\omega f = -\infty$, i.e., a symbol which is less than any element of Γ. For example, if $\Gamma = \mathbf{Z}$ and $\omega_i = 1$ for $i = 1, \ldots, n$, then the ω-weighted degree is the same as the total degree. For each k-vector subspace V of $k[\mathbf{x}]$, we define V^ω to be the k-vector subspace of $k[\mathbf{x}]$ generated by $\{f^\omega \mid f \in V\setminus\{0\}\}$. For each l-tuple $F = (f_1, \ldots, f_l)$ of elements of $k[\mathbf{x}]$ for $l \in \mathbf{N}$, we define $\deg_\omega F = \sum_{i=1}^l \deg_\omega f_i$.

For each $\sigma \in \mathfrak{S}_l$, we define $F_\sigma = (f_{\sigma(1)}, \dots, f_{\sigma(l)})$, where $\mathfrak{S}_l$ is the symmetric group of $\{1, \dots, l\}$ for each $l \in \mathbf{N}$.

The degree of a differential form defined in [6] is important in our theory. Let $\Omega_{k[\mathbf{x}]/k}$ be the module of differentials of $k[\mathbf{x}]$ over k, and $\bigwedge^l \Omega_{k[\mathbf{x}]/k}$ the l-th exterior power of the $k[\mathbf{x}]$-module $\Omega_{k[\mathbf{x}]/k}$ for $l \in \mathbf{N}$. Then, we may uniquely express each $\theta \in \bigwedge^l \Omega_{k[\mathbf{x}]/k}$ as

$$\theta = \sum_{1 \le i_1 < \cdots < i_l \le n} f_{i_1,\dots,i_l} dx_{i_1} \wedge \cdots \wedge dx_{i_l},$$

where $f_{i_1,\dots,i_l} \in k[\mathbf{x}]$ for each $i_1, \dots, i_l$. Here, df denotes the differential of f for each $f \in k[\mathbf{x}]$. We define

$$\deg_\omega \theta = \max\{\deg_\omega(f_{i_1,\dots,i_l} x_{i_1} \cdots x_{i_l}) \mid 1 \le i_1 < \cdots < i_l \le n\}.$$

If $\theta \neq 0$, then it follows that

$$\deg_\omega \theta \ge \min\{\omega_{i_1} + \cdots + \omega_{i_l} \mid 1 \le i_1 < \cdots < i_l \le n\} > 0. \tag{2.1}$$

We remark that $f_1, \dots, f_l$ are algebraically independent over k if and only if $df_1 \wedge \cdots \wedge df_l \neq 0$ for $f_1, \dots, f_l \in k[\mathbf{x}]$. Actually, this condition is equivalent to the condition that the rank of the l by n matrix $((f_i)_{x_j})_{i,j}$ is equal to l (cf. [3, Proposition 1.2.9]). Here, f_{x_i} denotes the partial derivative of f in x_i for each $f \in k[\mathbf{x}]$ and $i \in \{1, \dots, n\}$. By definition, it follows that

$$\sum_{i=1}^{l} \deg_\omega df_i \ge \deg_\omega(df_1 \wedge \cdots \wedge df_l). \tag{2.2}$$

In (2.2), the equality holds if and only if $f_1^\omega, \dots, f_l^\omega$ are algebraically independent over k. Actually, we may write $df_1 \wedge \cdots \wedge df_l = df_1^\omega \wedge \cdots \wedge df_l^\omega + \eta$, where $\eta \in \bigwedge^l \Omega_{k[\mathbf{x}]/k}$ with $\deg_\omega \eta < \sum_{i=1}^{l} \deg_\omega f_i$. For each $f \in k[\mathbf{x}] \setminus k$, we have

$$\deg_\omega df = \max\{\deg_\omega(f_{x_i} x_i) \mid i = 1, \dots, n\} = \deg_\omega f, \tag{2.3}$$

since $df = \sum_{i=1}^{n} f_{x_i} dx_i$. If $f_1, \dots, f_n \in k[\mathbf{x}]$ are algebraically independent over k, then

$$\sum_{i=1}^{n} \deg_\omega f_i = \sum_{i=1}^{n} \deg_\omega df_i \ge \deg_\omega(df_1 \wedge \cdots \wedge df_n) \ge \sum_{i=1}^{n} \omega_i =: |\omega| \tag{2.4}$$

by (2.1), (2.3) and (2.4). As will be shown in Lemma 6.1(i), if $\deg_\omega F = |\omega|$ for $F \in \mathrm{Aut}_k k[\mathbf{x}]$, then F is tame.

Now, consider the set $\mathcal{T}$ of triples $F = (f_1, f_2, f_3)$ of elements of $k[\mathbf{x}]$ such that f_1, f_2 and f_3 are algebraically independent over k. We identify each $F \in \mathcal{T}$ with the injective homomorphism $F : k[\mathbf{y}] \to k[\mathbf{x}]$ defined by $F(y_i) = f_i$ for $i = 1, 2, 3$, where $k[\mathbf{y}] = k[y_1, y_2, y_3]$ is the polynomial ring in three variables over k. Let $\mathcal{E}_i$ denote the set of elementary automorphisms E of $k[\mathbf{y}]$ such that $E(y_j) = y_j$ for each $j \neq i$ for $i \in \{1, 2, 3\}$, and $\mathcal{E} = \bigcup_{i=1}^{3} \mathcal{E}_i$. We say that $F = (f_1, f_2, f_3)$ *admits an elementary reduction* for the weight ω if $\deg_\omega(F \circ E) < \deg_\omega F$ for some $E \in \mathcal{E}$, and call $F \circ E$ an *elementary reduction* of F for the weight ω.

Let $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ be elements of $\mathcal{T}$. We say that the pair (F, G) satisfies the *Shestakov-Umirbaev condition* for the weight ω if the following conditions hold:

(SU1) $g_1 = f_1 + af_3^2 + cf_3$ and $g_2 = f_2 + bf_3$ for some $a, b, c \in k$, and $g_3 - f_3$ belongs to $k[g_1, g_2]$;

(SU2) $\deg_\omega f_1 \leq \deg_\omega g_1$ and $\deg_\omega f_2 = \deg_\omega g_2$;

(SU3) $(g_1^\omega)^2 \approx (g_2^\omega)^s$ for some odd number $s \geq 3$;

(SU4) $\deg_\omega f_3 \leq \deg_\omega g_1$, and f_3^ω does not belong to $k[g_1^\omega, g_2^\omega]$;

(SU5) $\deg_\omega g_3 < \deg_\omega f_3$;

(SU6) $\deg_\omega g_3 < \deg_\omega g_1 - \deg_\omega g_2 + \deg_\omega(dg_1 \wedge dg_2)$.

Here, $h_1 \approx h_2$ (resp. $h_1 \not\approx h_2$) denotes that h_1 and h_2 are linearly dependent (resp. linearly independent) over k for each $h_1, h_2 \in k[\mathbf{x}] \setminus \{0\}$. We say that $F \in \mathcal{T}$ *admits a Shestakov-Umirbaev reduction* for the weight ω if there exist $G \in \mathcal{T}$ and $\sigma \in \mathfrak{S}_3$ such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition, and call this G a *Shestakov-Umirbaev reduction* of F for the weight ω. As will be shown in Theorem 4.1(P6), $\deg_\omega G < \deg_\omega F$ if G is a Shestakov-Umirbaev reduction of F.

Note that (SU1) implies that there exist $E_i \in \mathcal{E}_i$ for $i = 1, 2, 3$ such that $F \circ E_1 = (f_1, g_2, f_3)$, $F \circ E_1 \circ E_2 = (g_1, g_2, f_3)$ and $F \circ E_1 \circ E_2 \circ E_3 = G$. Furthermore, $\delta := (1/2) \deg_\omega g_2$ belongs to Γ by (SU3).

Here is our main result.

Theorem 2.1 *Assume that* $n = 3$, *and* $\omega = (\omega_1, \omega_2, \omega_3)$ *is an element of* Γ^3 *such that* $\omega_i > 0$ *for each* i. *Then, each* $F \in \mathrm{T}_k\, k[\mathbf{x}]$ *with* $\deg_\omega F > |\omega|$ *admits an elementary reduction or a Shestakov-Umirbaev reduction for the weight* ω.

Note that F admits an elementary reduction for the weight ω if and only if f_i^ω belongs to $k[f_j, f_l]^\omega$ for some $i \in \{1,2,3\}$, where $j, l \in \mathbf{N} \setminus \{i\}$ with $1 \leq j < l \leq 3$. In case $\deg_\omega f_1$, $\deg_\omega f_2$ and $\deg_\omega f_3$ are pairwise linearly independent, this condition is equivalent to the condition that $\deg_\omega f_i$ belongs to the subsemigroup of Γ generated by $\deg_\omega f_j$ and $\deg_\omega f_l$ for some $i \in \{1,2,3\}$. Indeed, for each $\phi \in k[f_j, f_l] \setminus \{0\}$, there exist $p, q \in \mathbf{Z}_{\geq 0}$ such that $\deg_\omega \phi = \deg_\omega f_j^p f_l^q$, since ϕ is a linear combination of $f_j^p f_l^q$ for $(p,q) \in (\mathbf{Z}_{\geq 0})^2$ over k, in which $\deg_\omega f_j^p f_l^q \neq \deg_\omega f_j^{p'} f_l^{q'}$ whenever $(p,q) \neq (p',q')$. Here, $\mathbf{Z}_{\geq 0}$ denotes the set of nonnegative integers.

Using Theorem 2.1, we can verify that Nagata's automorphism is not tame. Let $\Gamma = \mathbf{Z}^3$ equipped with the lexicographic order, i.e., $a \leq b$ if the first nonzero component of $b - a$ is positive for $a, b \in \mathbf{Z}^3$, and let $\omega = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{e}_i$ is the i-th standard unit vector of $\mathbf{R}^3$ for each i. Then, we have

$$\deg_\omega f_1 = (2,0,3),\ \deg_\omega f_2 = (1,0,2),\ \deg_\omega f_3 = (0,0,1).$$

Hence, $\deg_\omega F = (3,0,6) > (1,1,1) = |\omega|$. On the other hand, the three vectors above are pairwise linearly independent, while any one of them is not contained in the subsemigroup of $\mathbf{Z}^3$ generated by the other two vectors. Hence, F does not admit an elementary reduction for the weight ω. Since $(1/2)\deg_\omega f_i$ does not belong to $\Gamma = \mathbf{Z}^3$ for each $i \in \{1,2,3\}$, we know that F does not admit a Shestakov-Umirbaev reduction for the weight ω.

Therefore, we have the following corollary to Theorem 2.1.

Corollary 2.2 *Nagata's automorphism is not tame.*

We may also check that Nagata's automorphism does not admit a Shestakov-Umirbaev reduction in a different way as follows. By Theorem 4.1(P7), we know that $0 < \delta < \deg_\omega f_i \leq s\delta$ holds each $i \in \{1,2,3\}$ if F admits a Shestakov-Umirbaev reduction for the weight ω. Hence, $s \deg_\omega f_i > \deg_\omega f_j$

for each $i, j \in \{1, 2, 3\}$. On the other hand, in the case of Nagata's automorphism, $l \deg_\omega f_3 = (0, 0, l)$ is less than $\deg_\omega f_i$ for $i = 1, 2$ for any $l \in \mathbf{N}$ by the definition of the lexicographic order. Therefore, F does not admit a Shestakov-Umirbaev reduction for the weight ω.

We define the *rank* of ω as the rank of the $\mathbf{Z}$-submodule of Γ generated by $\omega_1, \ldots, \omega_n$. If ω has maximal rank n, then the k-vector space $k[\mathbf{x}]_\gamma$ is of dimension at most one for each γ. Consequently, it follows that $\deg_\omega f = \deg_\omega g$ if and only if $f^\omega \approx g^\omega$ for each $f, g \in k[\mathbf{x}] \setminus \{0\}$. In such a case, the assertion of Theorem 2.1 can be proved more easily than the general case. Actually, we may omit a few lemmas and propositions needed to prove Theorem 2.1. We note that $\omega = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has maximal rank three, and so it suffices to show the assertion of Theorem 2.1 in this special case to verify that Nagata's automorphism is not tame.

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Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachioji
Tokyo 192-0397, Japan

An algorithm for computing generators of $\mathbb{G}_a$-invariant rings

Ryuji Tanimoto

1. Introduction

Let k be an infinite field of arbitrary characteristic $p \geq 0$ and let $\mathbb{G}_a$ be its additive group. For a $\mathbb{G}_a$-action on an affine variety Spec$A$, we have the corresponding k-algebra homomorphism $\varphi : A \to A \otimes_k k[t]$, where t is an indeterminate over the field k. Write

$$\varphi(f) = \sum_{n \geq 0} D_n(f) t^n,$$

where $D_n(f) \in A$ for all $n \geq 0$. Clearly, each D_n is a k-linear endomorphism of A and the set $\{D_n\}_{n \geq 0}$ satisfies the following conditions (1), (2), (3) and (4):

(1) D_0 is the identity map of A;

(2) $D_n(ab) = \sum_{i+j=n} D_i(a) D_j(b)$ for all $n \geq 0$ and for all $a, b \in A$;

(3) For all $a \in A$, there exists $n \geq 0$ such that $D_m(a) = 0$ for all $m \geq n$;

(4) $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ for all $i, j \geq 0$.

Let $\mathbb{G}_a$ act on A by

$$t \cdot f = \sum_{n \geq 0} D_n(f) t^n$$

for all $t \in k$ and $f \in A$. Denote by $A^{\mathbb{G}_a}$ the invariant ring for the $\mathbb{G}_a$-action on A. So, we have the equality

$$A^{\mathbb{G}_a} = \{a \in A \mid D_n(a) = 0 \text{ for all } n \geq 1\}.$$

An element σ of A is said to be a *local slice* of the $\mathbb{G}_a$-action on A if σ satisfies the following conditions (1) and (2):

(1) $\sigma \notin A^{\mathbb{G}_a}$;

(2) $\deg_t(t \cdot \sigma) = \min\{\deg_t(t \cdot f) \mid f \in A \backslash A^{\mathbb{G}_a}\}$.

An element s of A is said to be a *slice* of the $\mathbb{G}_a$-action if s is a local slice of the $\mathbb{G}_a$-action and the leading coefficient of $t \cdot s \in A[t]$ is 1.

In this report, we solve the following problem:

Problem. Assume that the finitely generated k-algebra A is a domain and the $\mathbb{G}_a$-action on A has a slice s. Give an algorithm for computing generators of the $\mathbb{G}_a$-invariant ring $A^{\mathbb{G}_a}$.

2. Dixmier operator and answer

We know from Miyanishi's theorem [2, 1.5.] that

$$A = A^{\mathbb{G}_a}[s]$$

and s can be considered as an indeterminate over $A^{\mathbb{G}_a}$. So, we have a natural surjective k-algebra homomorphism $\varepsilon_s : A \to A^{\mathbb{G}_a}$ by substituting 0 for s. We call this surjection the *Dixmier operator*.

The following theorem gives another description of the Dixmier operator. Let $P_s(t) := t \cdot s \in A[t]$.

Theorem. *For any $f \in A$, express $t \cdot f$ as*

$$t \cdot f = P_s(t) \cdot Q_f(t) + R_f(t),$$

where $Q_f(t)$, $R_f(t) \in A[t]$ and $\deg_t(R_f(t)) < \deg_t(P_s(t))$. Then $R_f(0) \in A^{\mathbb{G}_a}$ and $\varepsilon_s(f) = R_f(0)$.

Now, we give an answer to the Problem. Let $A = k[\alpha_1, \ldots, \alpha_n]$. Calculate $\varepsilon_s(\alpha_i)$ for all $1 \leq i \leq n$ by using the above Theorem. Thus we have an answer.

3. Proof of the Theorem

Let $D := \{D_n\}_{n \geq 0}$. We denote $t \cdot f$ by $\varphi_{D,t}(f)$, where $t \in \mathbb{G}_a$ and $f \in A$. We define the *D-degree* of $f \in A$ by $\deg_D(f) := \deg_t(\varphi_{D,t}(f))$. By substituting $t + t'$ for t of the equality $\varphi_{D,t}(f) = \varphi_{D,t}(s) \cdot Q_f(t) + R_f(t)$, we have

$$\varphi_{D,t+t'}(f) = \varphi_{D,t+t'}(s) \cdot Q_f(t+t') + R_f(t+t').$$

The left hand side $\varphi_{D,t+t'}(f)$ can be written as

$$\begin{aligned} \varphi_{D,t+t'}(f) &= (\varphi_{D,t} \circ \varphi_{D,t'})(f) \\ &= \varphi_{D,t}(\varphi_{D,t'}(s) \cdot Q_f(t') + R_f(t')) \\ &= \varphi_{D,t+t'}(s) \cdot \varphi_{D,t}(Q_f(t')) + \varphi_{D,t}(R_f(t')). \end{aligned}$$

Thus we have

$$\varphi_{D,t+t'}(s) \cdot (Q_f(t+t') - \varphi_{D,t}(Q_f(t'))) = \varphi_{D,t}(R_f(t')) - R_f(t+t').$$

Let $d := \deg_{t'}(\varphi_{D,t'}(s))$. If $Q_f(t+t') - \varphi_{D,t}(Q_f(t')) \neq 0$, the left hand side is of degree $\geq d$ in t' (note that $\varphi_{D,t+t'}(s)$ is a monic polynomial of degree d in t' over $A[t]$). On the other hand, the right hand side is of degree $< d$ in t'. Hence, we know

$$Q_f(t+t') = \varphi_{D,t}(Q_f(t')) \qquad \text{and} \qquad \varphi_{D,t}(R_f(t')) = R_f(t+t').$$

Substituting 0 for t' in the above two equalities, we have

$$Q_f(t) = \varphi_{D,t}(Q_f(0)) \qquad \text{and} \qquad \varphi_{D,t}(R_f(0)) = R_f(t).$$

We know from the latter of the above two equalities that $\deg_D(R_f(0)) < \deg_D(s)$. Since s is a slice of the $\mathbb{G}_a$-action, we have $R_f(0) \in A^{\mathbb{G}_a}$. Evaluating the equality

$$\varphi_{D,t}(f) = \varphi_{D,t}(s) \cdot Q_f(t) + R_f(t)$$

at $t = 0$, we have the equality

$$f = s \cdot Q_f(0) + R_f(0)$$

in A. Evaluating this equality at $s = 0$, we have $\varepsilon_s(f) = R_f(0)$. This completes the proof of the Theorem.

4. Dixmier operator in characteristic 0

In the following, assume that the characteristic of k is zero. The Dixmier operator $\varepsilon_s : A \longrightarrow A^{\mathbb{G}_a}$ has the following simple form. This means that ε_s coincides with φ_{-s} defined in [1, Page 26].

Corollary. *The Dixmier operator ε_s can be written as*

$$\varepsilon_s(f) = \sum_{n\geq 0} \frac{D_1^n(f)}{n!}(-s)^n \quad \textit{for all } f \in A.$$

Proof. Since k is a field of characteristic zero, we have

$$\varphi_{D,t}(f) = \sum_{n\geq 0} \frac{D_1^n(f)}{n!}t^n \quad \text{for all } f \in A$$

and $\varphi_{D,t}(s) = s + t$. Express $\varphi_{D,t}(f)$ as

$$\varphi_{D,t}(f) = (t+s) \cdot Q_f(t) + R_f,$$

where $R_f \in A$. By substituting $-s$ for t of the above equality, we have

$$\sum_{n\geq 0} \frac{D_1^n(f)}{n!}(-s)^n = R_f.$$

We know from the above Theorem that $\varepsilon_s(f) = R_f$. Hence, we have the desired expression of ε_s. Q.E.D.

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On G-local G-schemes

MASAHIRO OHTANI

Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464–8602 JAPAN
m05011w@math.nagoya-u.ac.jp

This is a joint work with Mitsuyasu Hashimoto.

1 Diagrams of schemes and modules over them

Let I be a small category, $\underline{Sch}$ denote the category of schemes. We think a contravariant functor $X_\bullet : I \to \underline{Sch}$. It can be thought as a diagram of schemes and morphisms. For each $i \in I$, denote the scheme $X_\bullet(i)$ by X_i. And for a morphism ϕ in I, denote the morphism $X_\bullet(\phi)$ by X_ϕ. We can define a category $\mathrm{Zar}(X_\bullet)$ as follows :

$$\mathrm{ob}\big(\mathrm{Zar}(X_\bullet)\big) := \{(i, U) \mid i \in \mathrm{ob}(I),\ U \in \mathrm{Zar}(X_i)\},$$
$$\mathrm{Hom}\big((i,U),(j,V)\big) := \{(\phi, h) \mid \phi : i \leftarrow j \text{ is a morphism in } I,\ h : U \to V$$
$$\text{is a morphism such that it is the restriction of } X_\phi : X_i \to X_j\}$$

In the definiton, for a scheme S, $\mathrm{Zar}(S)$ denote the category consisting of open subschemes of S and inclusion morphisms.

And we can define a Grothendieck topology on $\mathrm{Zar}(X_\bullet)$. A class of morphisms $\{(h_\lambda, \phi_\lambda) : (i_\lambda, U_\lambda) \to (i, U)\}$ is a covering of (i, U) if the following hold :

$$(1)\ i_\lambda = i \text{ and } \phi_\lambda = \mathrm{id} \text{ for any } \lambda, \qquad (2)\ U = \bigcup h_\lambda U_\lambda.$$

So we can think sheaves over $\mathrm{Zar}(X_\bullet)$.

Moreover, we define the sheaf of commutative rings $\mathcal{O}_{X_\bullet}$ on $\mathrm{Zar}(X_\bullet)$ by

$$\Gamma\big((i,U), \mathcal{O}_{X_\bullet}\big) := \Gamma(U, \mathcal{O}_{X_i}),$$

where $\mathcal{O}_{X_i}$ is the structure sheaf of X_i. So $\mathrm{Zar}(X_\bullet)$ is a ringed site, and we can think $\mathcal{O}_{X_\bullet}$-module sheaves. Denote the category of $\mathcal{O}_{X_\bullet}$-modules $\mathrm{Mod}(\mathrm{Zar}(X_\bullet))$ by $\mathrm{Mod}(X_\bullet)$, simply.

For $i \in I$, we can define a functor $[-]_i : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(X_i)$ by

$$\Gamma(U, \mathcal{M}_i) := \Gamma\big((i, U), \mathcal{M}\big).$$

This functor $[-]_i$ is called the restriction functor. The restriction functor $[-]_i$ has both a left adjoint and a right adjoint, so $[-]_i$ preserves limits and colimits, and it is exact (Hashimoto [3], (4.4)).

Let $\phi : i \to j$ be a morphism in I. For $(i, U) \in \mathrm{Zar}(X_\bullet)$ and an $\mathcal{O}_{X_\bullet}$-module $\mathcal{M}$, a morphism $\beta_\phi(\mathcal{M}) : \mathcal{M}_i \to (X_\phi)_*\mathcal{M}_j$ is defined by the following diagram of the sets of sections over U :

$$\begin{array}{ccccc} \Gamma(U, \mathcal{M}_i) & \longrightarrow & \Gamma(X_\phi^{-1}U, \mathcal{M}_j) & = & \Gamma(U, (X_\phi)_*\mathcal{M}_i) \\ \| & & \| & & \\ \Gamma((i, U), \mathcal{M}) & \xrightarrow{f} & \Gamma((j, X_\phi^{-1}U), \mathcal{M}) & & \end{array}$$

where f is the restrction with respect to the morphism $\left(\phi, X_\phi|_{X_\phi^{-1}U}\right)$.

And we can define a morphism $\alpha_\phi : X_\phi^*[-]_i \to [-]_j$ to be the composite

$$X_\phi^*[-]_i \xrightarrow{\beta_\phi} X_\phi^*(X_\phi)_*[-]_j \xrightarrow{\epsilon} [-]_j$$

where ϵ is the counit of the adjoint pair $(X_\phi^*, (X_\phi)_*)$.

Definition 1. Let $\mathcal{M}$ be an $\mathcal{O}_{X_\bullet}$-module.
(1) $\mathcal{M}$ is **equivariant** if α_ϕ is an isomorphism for each morphism ϕ in I.
(2) $\mathcal{M}$ is **locally coherent** (resp. **locally quasi-coherent**) if each $\mathcal{M}_i$ is a coherent (resp. quasi-coherent) $\mathcal{O}_{X_i}$-module for any $i \in I$.
(3) $\mathcal{M}$ is **coherent** (resp. **quasi-coherent**) if $\mathcal{M}$ is locally coherent (resp. locally quasi-coherent) and equivariant.

2 The diagram $B_G^M(X)$ and G-local G-scheme

Denote the set of natural numbers $\{0, 1, \cdots, n\}$ by $[n]$. Let Δ be the category defined as follows :

$$\begin{aligned} \mathrm{ob}(\Delta) &= \{[0], [1], [2]\}, \\ \mathrm{Hom}([i], [j]) &= \text{the set of order-preserving injective maps } [i] \to [j]. \end{aligned}$$

Δ is represented by the following diagram (without identity maps) :

$$\Delta = \left([2] \;\underset{\underset{i_2}{\longleftarrow}}{\overset{\overset{i_0}{\longleftarrow}}{\xleftarrow{i_1}}}\; [1] \;\underset{i_1}{\overset{i_0}{\leftleftarrows}}\; [0] \right)$$

where i_s is the order-preserving injection whose image does not contain s.

From now on, let S be a Noetherian scheme, G be an S-group scheme flat of finite type and X be a Noetherian G-scheme. G-scheme is an S-scheme with G-action. We define a diagram of schemes $B_G^M(X) \in \mathrm{Func}(\Delta^{\mathrm{op}}, \underline{Sch})$ by

$$B_G^M(X) := \left(G \times_S G \times_S X \;\underset{\underset{p_{23}}{\longrightarrow}}{\overset{\overset{\mathrm{id}\times a}{\longrightarrow}}{\xrightarrow{\mu\times\mathrm{id}}}}\; G \times_S X \;\underset{p_2}{\overset{a}{\rightrightarrows}}\; X \right)$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ is the product, and p_{23} and p_2 are projections.

We call a module over this diagram $B_G^M(X)$ **a $(G, \mathcal{O}_X)$-module**, and denote the category of $(G, \mathcal{O}_X)$-modules $\mathrm{Mod}(B_G^M(X))$ by $\mathrm{Mod}(G, X)$. And denote the fullsubcategory of locally quasi-coherent $(G, \mathcal{O}_X)$-modules, of quasi-coherent $(G, \mathcal{O}_X)$-modules and of coherent $(G, \mathcal{O}_X)$-modules by $\mathrm{Lqc}(G, X)$, $\mathrm{Qch}(G, X)$ and $\mathrm{Coh}(G, X)$, respectively.

Let Z be a closed subscheme of X. Denote the scheme theoritic image of the action $a : G \times Z \to X$ by Z^*. This subscheme Z^* has the following properties :

1. Z^* is the smallest G-stable (i.e. the action $a : G \times Z^* \to X$ factors through the inclusion $Z^* \hookrightarrow X$) closed subscheme which contains Z. So if Z is G-stable, then $Z^* = Z$.

2. Assume that G is an S-smooth group scheme with connected geometric fibers. If Z is irreducible (resp. reduced), then so is Z^*. So if Z is integral, then Z^* is integral, too.

Definition 2. A quasi-compact G-scheme X is **G-local** if X is has a unique minimal non-empty G-stable closed subscheme Y of X. In this case, we say that (X, Y) is G-local.

There are some examples of G-local G-schemes.

Example 3. (1) If G is trivial, a G-local G-scheme X is of the form $\operatorname{Spec} A$ where A is a local ring.
(2) Let $S = \operatorname{Spec} \mathbb{Z}$, $G = \mathbb{G}_m$ (multiplicative group) and A be a G-algebra. Let ω be the coaction $A \to A \otimes \mathbb{Z}[G]$ and $X(G)$ the charactor group of G. Now it holds $X(G) \simeq \mathbb{Z}$ as groups. For a character $\lambda \in X(G)$, set $A_\lambda = \{a \in A \mid \omega(a) = a \otimes \lambda\}$. Then $A = \bigoplus_{\lambda \in X(G)} A_\lambda$ hold. And for λ, $\mu \in X(G)$, $A_\mu A_\lambda = \{a_\lambda a_\mu \mid a_\lambda \in A_\lambda, a_\mu \in A_\mu\} \subset A_{\lambda+\mu}$. So the equation $A = \bigoplus A_\lambda$ means that $\mathbb{G}_m$-algebras are $\mathbb{Z}$-graded algebras and that an ideal I of $\mathbb{G}_m$-algebra A is $\mathbb{G}_m$-stable if and only if it is homogeneous.

So affine $\mathbb{G}_m$-scheme $X = \operatorname{Spec} A$ is $\mathbb{G}_m$-local if and only if A is an H-local $\mathbb{Z}$-graded ring in the sense of Goto and Watanabe [1].
(3) If $S = \operatorname{Spec} k$ with k an algebraically closed field, G is an linear algebraic group and B is a Borel subgroup of G, then $(G/B, G/B)$ is G-local and $(G/B, B/B)$ is B-local. But it is not affine unless $G = B$ So a G-local G-scheme is not neccesarily affine even if S and G are affine.
(4) Let k be a field, G a reductive group, C a k-algebra of finite type with G-action, $A := C^G$ and $P \in \operatorname{Spec} A$. Then $X = \operatorname{Spec} C_P$ is a G-local G-scheme.

Until the end of this article, let G be an S-smooth group scheme with connected geometric fibers. For example, a connected algebraic group over an algebraically closed field k has this property. And let (X, Y) be a Noetherian G-local G-scheme.

Under the assumption, the unique minimal non-empty G-stable closed subscheme Y of X is integral. In fact, each irreducible component of Y and the reduction Y_{red} of Y is G-stable, so Y is irreducible and reduced because of minimality of Y. So Y has the generic point. Let η be the generic point of Y, $\mathcal{I}$ the defining ideal of Y and $f : Y \to X$ the inclusion.

The localization at η is very important and useful.

Lemma 4. *The localization functor* $[-]_\eta : \operatorname{Qch}(G, X) \to \operatorname{Mod} \mathcal{O}_{X,\eta}$ *is faithful and exact.*

Proof. A localization functor is exact in general, so it is enough to prove that $[-]_\eta$ is faithful, i.e. $\mathcal{M}_\eta \neq 0$ for any quasi-coherent $(G, \mathcal{O}_X)$-module $\mathcal{M} \neq 0$. A quasi-coherent $(G, \mathcal{O}_X)$-module is represented as an inductive limit of coherent $(G, \mathcal{O}_X)$-modules, so we may assume that $\mathcal{M} \neq 0$ is coherent. Then $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}))$ is coherent, and $\underline{\operatorname{Ann}}\, \mathcal{M} := \ker(\mathcal{O}_X \to$

$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{M}))$ is a coherent G-ideal, so $\mathrm{Supp}\,\mathcal{M}$ is a non-empty G-stable closed subscheme. Since Y is minimal, $\eta \in Y \subset \mathrm{Supp}\,\mathcal{M}$. Then $\mathcal{M}_\eta \neq 0$. ∎

By the lemma, we can prove a G-analogue of Nakayama's Lemma.

Theorem 5 (G-Nakayama's lemma). *For a coherent $(G, \mathcal{O}_X)$-module $\mathcal{M}$, if $f^*\mathcal{M} = 0$ then $M = 0$.*

Proof. $\kappa(\eta)\otimes_{\mathcal{O}_{X,\eta}}\mathcal{M}_\eta = (f^*\mathcal{M})_\eta = 0$, so $M_\eta = 0$ by the usual Nakayama's lemma for the local ring $\mathcal{O}_{X,\eta}$. And $[-]_\eta$ is faithful, so $M = 0$. ∎

By localization at η, we also have criteria for coherentness and length-finiteness of quasi-coherent $(G, \mathcal{O}_X)$-modules.

Proposition 6. (1) *For $\mathcal{M} \in \mathrm{Qch}(G, X)$, the following are equivalent :*

(a) *$\mathcal{M}$ is a Noetherian object of $\mathrm{Qch}(G, X)$.*
(b) *$\mathcal{M}_{[0]}$ is a coherent $\mathcal{O}_X$-module.*
(c) *$\mathcal{M}$ is a coherent $(G, \mathcal{O}_X)$-module.*
(d) *$\mathcal{M}_\eta$ is a Noetherian $\mathcal{O}_{X,\eta}$-module.*

(2) *For $\mathcal{M} \in \mathrm{Qch}(G, X)$, the following are equivalent :*

(a) *$\mathcal{M}$ is of finite length in $\mathrm{Qch}(G, X)$.*
(b) *$\mathcal{M}$ is a coherent $(G, \mathcal{O}_X)$-module, and $\mathcal{I}^n\mathcal{M} = 0$ for some n.*
(c) *$\mathcal{M}_\eta$ is $\mathcal{O}_{X,\eta}$-module of finite length.*

Proof. (1) (a)⇔(b). Hashimoto [3], Lemma 12.8. (b)⇒(c)⇒(d) are trivial. (d)⇒(a). Since $[-]_\eta$ is faithful and exact, then an ascending chain $\mathcal{N}_0 \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots$ of $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$ is stable if and only if an ascending chain $[\mathcal{N}_0]_\eta \subset [\mathcal{N}_1]_\eta \subset [\mathcal{N}_2]_\eta \cdots$ of $\mathcal{O}_{X,\eta}$-submodules of $\mathcal{M}_\eta$ is stable.
(2) (a)⇒(b). $\mathcal{M}$ is a coherent by (1). A descending chain $\mathcal{M} \supset \mathcal{I}^1\mathcal{M} \supset \mathcal{I}^2\mathcal{M} \supset \cdots$ is stable by (a). If $\mathcal{I}^n\mathcal{M} = \mathcal{I}^{n+1}\mathcal{M}$, then $\mathcal{I}_\eta^n\mathcal{M}_\eta = \mathcal{I}_\eta^{n+1}\mathcal{M}_\eta$. So $\mathcal{I}_\eta^n\mathcal{M}_\eta = 0$ by usual Nakayama's lemma, and then $\mathcal{I}^n\mathcal{M} = 0$ by faithfulness of $[-]_\eta$. (b)⇒(c) is trivial. (c)⇒(a) is similar to (1) (d)⇒(a) for a descending chain of $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. ∎

3 G-dualizing complex

For a Noetherian G-scheme Z, a complex $\mathbb{F} \in D(\mathrm{Mod}(G, Z))$ is G-**dualizing** if $\mathbb{F}$ has equivariant cohomology sheaves and if $\mathbb{F}_{[0]} \in D(\mathrm{Mod}\, Z)$

is a dualizing complex of Z. Since Δ is a finite ordered category, $\mathbb{F}$ is G-dualizing if and only if $\mathbb{F}$ has finite injective dimension, has coherent cohomology sheaves, and the natural map $\mathcal{O}_{B_G^M(Z)} \to R\,\underline{\mathrm{Hom}}^\bullet(\mathbb{F},\mathbb{F})$ is a quasi-isomorphism, see [3] Lemma 31.6.

For example, if Z is Gorenstein of finite Krull dimension, then $\mathcal{O}_Z$ itself is a G-dualizing complex of Z.

From now on, assume that X has a fixed G-dualizing complex $\mathbb{I}$.

4 The local cohomology

Let $g : X \setminus Y \hookrightarrow X$ be the open immersion. $u : \mathrm{Id} \to g_*g^*$ denote the unit of the adjoint pair (g_*, g^*). Then we think a functor $\underline{\Gamma}_Y = \ker u : \mathrm{Mod}(G,X) \to \mathrm{Mod}(G,X)$.

The functor $\underline{\Gamma}_Y$ is a left exact functor preserving $\mathrm{Lqc}(G,X)$ and $\mathrm{Qch}(G,X)$, see [4] Lemma 3.2. For $\mathcal{M} \in \mathrm{Lqc}(G,X)$, $\underline{\Gamma}_Y(\mathcal{M})$ is computed as follows :

$$\underline{\Gamma}_Y(\mathcal{M}) = \varinjlim_n \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}),$$

see [4] Lemma 3.21.

And the derived functor $R\,\underline{\Gamma}_Y : D(\mathrm{Mod}(G,X)) \to D(\mathrm{Mod}(G,X))$ preserves $D_{\mathrm{Qch}}(\mathrm{Mod}(G,X))$, see [4] Lemma 4.11. For $\mathbb{M} \in D(\mathrm{Mod}(G,X))$, $R^i\,\underline{\Gamma}_Y(\mathbb{M})$ is denoted by $\underline{\mathrm{H}}^i_Y(\mathbb{M})$.

Lemma 7. *For a G-dualizing complex $\mathbb{F}$ of X, the local cohomology sheaves $\underline{\mathrm{H}}^i_Y(\mathbb{F})$ vanish except for only one i.*

Proof. Over a Noetherian scheme S, $A \in \mathrm{Qch}\,S$ is an injective object of $\mathrm{Mod}\,S$ if and only if it is an injective object of $\mathrm{Qch}\,S$. So we can assume that each term of a dualizing complex $\mathbb{F}_S$ of S is quasi-coherent and injective. As this, we can assume that $\mathbb{F}$ is a K-injective complex whose terms are locally quasi-coherent.

Then the following diagram commutes :

$$\begin{array}{ccc} X \setminus Z & \xrightarrow{g} & X \\ {\scriptstyle f'}\uparrow & & {\scriptstyle f}\uparrow \\ \mathrm{Spec}\,\mathcal{O}_{X,\eta} \setminus \{\eta\} & \xrightarrow{g'} & \mathrm{Spec}\,\mathcal{O}_{X,\eta} \end{array}.$$

We calculate the functor $f^*\,\underline{\Gamma}_Y = f^*\ker(Id \xrightarrow{u} g_*g^*)$ by the commutative diagram :

$$f^*\,\underline{\Gamma}_Z = f^*\ker(\mathrm{Id} \xrightarrow{u} g_*g^*) \simeq \ker(f^* \longrightarrow f^*g_*g^*)$$
$$\xrightarrow{\phi} \ker(f^* \longrightarrow g'_*g'^*f^*) \simeq \ker(\mathrm{Id} \longrightarrow g'_*g'^*)f^* = \Gamma_{\mathcal{I}_\eta} f^*.$$

Each term of $\mathbb{F}$ is locally quasi-coherent, so ϕ is isomorphic. So it holds $[\underline{\Gamma}_Z(\mathbb{F})]_\eta \simeq \Gamma_{\mathcal{I}_\eta}(\mathbb{F}_\eta)$. By definition, $\mathbb{F}_\eta$ is a dualizing complex of $\mathcal{O}_{X,\eta}$.

In general, for a local ring $(A, \mathfrak{m})$, local cohomology groups $\mathrm{H}^i_{\mathfrak{m}}(\mathbb{F})$ of a dualizing complex $\mathbb{F}$ of A with support $\{\mathfrak{m}\}$ vanish except for only one i, see Hartshorne [2] V.6. The functor $[-]_\eta$ is faithful and exact, so cohomology $\underline{\mathrm{H}}^i_Y(\mathbb{F})$ vanish except for only one i. ■

Let $\mathbb{F}$ be a G-dualizing complex of X. If it holds $\underline{\mathrm{H}}^0_Y(\mathbb{F}) \neq 0$, a G-dualizing complex $\mathbb{F}$ is called G-normalized. Assume that our G-dualizing complex $\mathbb{I}$ is G-normalized.

Definition 8. For a G-normalized G-dualizing complex $\mathbb{I}$, the non-vanishing local cohomology $\underline{\mathrm{H}}^0_Y(\mathbb{I})$ with support Y is denoted by $\mathcal{E}_X$, and we call it a **G-sheaf of Matlis**.

For a local ring $(A, \mathfrak{m})$, the non-vanishing local cohomology group $\mathrm{H}^i_{\mathfrak{m}}(\mathbb{F})$ of a dualizing complex $\mathbb{F}$ of A with support $\{\mathfrak{m}\}$ is the injective envelope $E_A(A/\mathfrak{m})$ of the residue field $A/\mathfrak{m}$. So we get an isomprphism $[\mathcal{E}_X]_\eta \simeq E_{\mathcal{O}_{X,\eta}}(\kappa(\eta))$ where $\kappa(\eta)$ is the residue field of the local ring $\mathcal{O}_{X,\eta}$.

A G-sheaf of Matlis $\mathcal{E}_X$ corresponds to the injective envelope $E_A(A/\mathfrak{m})$ of the residue field $A/\mathfrak{m}$ for a local ring $(A, \mathfrak{m})$. But it is not necesarily an injective $(G, \mathcal{O}_X)$-module.

Example 9. Let k be a field of characteristic 2, $V = k^2$ and $G = \mathbb{GL}(V)$. Let $X = \operatorname{Spec} A$ where $A = \operatorname{Sym} V^*$. Then $\mathcal{E}_X$ is a $(G, \mathcal{O}_X)$-module which is defined by $A^\dagger$ ($A^\dagger$ denote the graded dual module of A). It is not injective as a G-module, so $\mathcal{E}_X$ is not injective in $\mathrm{Qch}(G, X)$.

Moreover, G-sheaf of Matlis $\mathcal{E}_X = \underline{\mathrm{H}}^0_Y(\mathbb{I})$ depends on G-normalized G-dualizing complex $\mathbb{I}$, so it is not necesarily unique.

5 Main theorems

Theorem 10 (**G-Matlis duality**). *Let T be the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(-, \mathcal{E}_X) : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X)$, $\mathcal{F}$ denote the category of $(G, \mathcal{O}_X)$-modules of finite length. Then the followings hold :*
(1) T is an exact functor on $\mathrm{Coh}(G, X)$.
(2) If $\mathcal{M} \in \mathcal{F}$, then $T\mathcal{M} \in \mathcal{F}$ and the canonical map $\mathcal{M} \to TT\mathcal{M}$ is an isomorphism.

So the functor $T : \mathcal{F} \to \mathcal{F}$ is an anti-equivalence.

Proof. (1) If $\mathcal{N} \in \mathrm{Coh}(G, X)$ then $\mathcal{N}_\eta$ is a finitely generated $\mathcal{O}_{X,\eta}$-module, see Lemma 6. So it holds

$$[\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}_X)]_\eta \simeq \mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{N}_\eta, [\mathcal{E}_X]_\eta). \qquad (\sharp)$$

$[\mathcal{E}_X]_\eta$ is an injective $\mathcal{O}_{X,\eta}$-module, so the functor $\mathrm{Hom}_{\mathcal{O}_{X,\eta}}([-]_\eta, [\mathcal{E}_{X,\eta}])$ is exact. Then $T = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(-, \mathcal{E}_X)$ is exact because $[-]_\eta$ is faithful and exact.

(2) By Lemma 6, $\mathcal{M}_\eta$ is an $\mathcal{O}_{X,\eta}$-module of finite length for $\mathcal{M} \in \mathcal{F}$. Because of the isomorphism ($\sharp$) and usual Matlis duality for the local ring $\mathcal{O}_{X,\eta}$, $[T\mathcal{M}]_\eta$ is an $\mathcal{O}_{X,\eta}$-module of finite length. By Lemma 6 again, $T\mathcal{M}$ is of finite length.

$\mathcal{M}$ and $T\mathcal{M}$ are both coherent, then

$$[TT\mathcal{M}]_\eta \simeq \mathrm{Hom}_{\mathcal{O}_{X,\eta}}\big(\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{M}_\eta, [\mathcal{E}_X]_\eta), [\mathcal{E}_X]_\eta\big).$$

By usual Matlis duality, it is isomorphic to $\mathcal{M}_\eta$. So it holds $TT\mathcal{M} \simeq \mathcal{M}$ because of faithfulness of $[-]_\eta$. ■

Finally, we state a G-analogue of local duality theorem.

Theorem 11 (G-local duality). *Let $\mathbb{E}$ be a bounded below complex in* $\mathrm{Mod}(G, X)$ *with coherent cohomology. Then there is an isomorphism in* $\mathrm{Qch}(G, X)$ *:*

$$\underline{\mathrm{H}}^i_Y(\mathbb{E}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}\big(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_X}(\mathbb{E}, \mathbb{I}), \mathcal{E}_X\big).$$

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CLASSIFICATION OF 2-DIMENSIONAL GRADED NORMAL HYPERSURFACES WITH $a(R) = 1$.

KEI-ICHI WATANABE (日本大学・文理学部・数学科)

Introduction

Inspired by the talk of Kyoji Saito at the Toyama Conference, Aug. 2007, I tried the classification of 2-dimensional graded normal hypersurfaces with $a(R) = 1$ using Demazure's construction of normal graded rings. Since the classificaion is so simple and nearly automatic, I want to introduce it.

Although this classification is "known" in the literature (cf. [S], [P1]), it seems that the systematic method of classification is not known. So, I think this is worthwhile to be published in some form.

Also, I present here the classification of normal two-dimensional hypersurfaces with $a(R) = 2$ and normal graded complete intersections with $a(R) = 1$ and Proj $(R) \cong \mathbb{P}^1$.

1. Preliminaries

Let $R = k[u, v, w]/(f)$ be a 2-dimensional graded normal hypersurface, where k is an algebraically closed field of any characteristic. We put $X =$ Proj (R). Since $\dim R = 2$ and R is normal, X is a smooth curve. Then by the construction of Zariski and Demazure ([1], [5]), there is an ample $\mathbb{Q}$-Cartier divisor D (that is, ND is an ample divisor on X for some positive integer N), such that

$$R = R(X, D) = \oplus_{n\geq 0} H^0(X, O_X(nD)).T^n \subset k(X)[T],$$

where T is a variable over $k(X)$ and

$$H^0(X, O_X(nD)) = \{f \in k(X) \mid \mathrm{div}_X(f) + nD \geq 0\} \cup \{0\}.$$

Now, let us begin the classification. In the following, X is a smooth curve of genus g and D is a fractional divisor on X such that ND is an ample integral (Cartier) divisor for some $N > 0$.

We denote

$$D = D_0 + \sum_{i=1}^{r} \frac{p_i}{q_i} P_i \quad (\forall i,\ (p_i, q_i) = 1),$$

where D_0 is an integral divisor; a divisor with integer coefficients. In this case, we denote

$$D' = \sum_{i=1}^{r} \frac{q_i - 1}{q_i} P_i.$$

At the same time, by our assumption $R \cong k[u, v, w]/(f)$. If $\deg(u, v, w; f) = (a, b, c; h)$, then by [2],

$$a(R) = h - (a + b + c).$$

We always assume $\deg(u,v,w;f) = (a,b,c;h)$ and also that $a \le b \le c$.

Proposition 1.1. *(**Fundamental formulas**) Assume that* $R = R(X, D) \cong k[u,v,w]/(f)$ *with* $\deg(u,v,w;f) = (a,b,c;h)$ *and* $a(R) = h - (a+b+c) = 1$. *Then we have the following equalities.*

(1) [W] Since R is Gorenstein with $a(R) = 1$, *we have*

$$D \sim K_X + D' = K_X + \sum_{i=1}^{r} \frac{q_i - 1}{q_i} P_i,$$

where, in general, $D_1 \sim D_2$ *means that* $D_1 - D_2 = \mathrm{div}_X(f)$ *for some* $f \in k(X)$.

(2) [Tomari's formula] If $P(R,t) = \sum_{n \ge 0} \dim R_n t^n$,

$$\lim_{t \to 1} (1-t)^2 P(R,t) = \deg D.$$

(3) Since $P(R,t) = \dfrac{1-t^h}{(1-t^a)(1-t^b)(1-t^c)}$, *we have*

$$\deg D = \frac{h}{abc} = \frac{1}{abc} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}.$$

Note that the latter expression is a decreasing function of a, b, c.

2. The classification of the hypersurfaces with $a(R) = 1$.

Henceforce, we put $D = K_X + \sum_{i=1}^{r} \frac{q_i - 1}{q_i} P_i$. **We always use the letter** r **in this meaning.** From 1.1 (3), the maximal value of $\deg D$ is taken when $a = b = c = 1$ and $\deg D = 4$ in that case.

Case A. The case $g > 0$.

Assume that $g \ge 1$. Since $\deg(D) \le 4$, and $\deg D \ge \deg K_X = 2g - 2$, $g \le 3$ and if $g = 3$, $D = K_X$. We list the cases by giving the form of D and $(a,b,c;h)$. We can easily deduce the general form of the equation f from this data. Also, if f with the given weight has an isolated singularity, then $k[u,v,w]/(f) \cong R(X,D)$, where D is a divisor of given form.

(A-1) $g = 3, D = K_X$; $(1,1,1;4)$.

Next, consider the case $g = 2$. Note that $\dim R_1 = \dim H^0(K_X) = g = 2$, we have $a = b = 1$ and $\deg D = 1 + \frac{3}{c} \le \frac{5}{2}$ $(c \ge 2)$. Since, either $\deg D = 2, D = K_X$ or $\deg D \ge \frac{5}{2}$, we have 2 cases.

(A-2) $g = 2, D = K_X$; $(1,1,3;6)$.

(A-3) $g = 2, D = K_X + \frac{1}{2}P$; $(1,1,2;5)$.

Next, assume $g = 1$. In this case, $a = 1$, $2 \le b \le c$ and the maximal value of $\deg D$ is $\frac{3}{2}$. Since on the other hand, $\deg D \ge \frac{r}{2}$ and thus $r \le 3$ and if $r = 3$, $D = \frac{1}{2}(P_1 + P_2 + P_3)$.

(A-4) $\quad g = 1, D = \frac{1}{2}(P_1 + P_2 + P_3); \quad (1, 2, 2; 6).$

Also, since $\dim R_2 = r$, if $r = 2$, then $a = 1, b = 2, c \ge 3$, $\deg(D) \le \frac{7}{6}$.

(A-5) $\quad g = 1, D = \frac{1}{2}(P_1 + P_2); \quad (1, 2, 4; 8).$

(A-6) $\quad g = 1, D = \frac{1}{2}P_1 + \frac{2}{3}P_2; \quad (1, 2, 3; 7).$

If $g = 1$ and $D = \frac{q-1}{q}P$, we have $q - 1$ new generators in degrees $1, 3, \ldots, q$. Hence $q \le 4$.

$a = 1, 3 \le b, c$ and $\deg D \le \frac{8}{9}$.

(A-7) $\quad g = 1, D = \frac{1}{2}P; \quad (1, 4, 6; 12).$

(A-8) $\quad g = 1, D = \frac{2}{3}P; \quad (1, 3, 5; 10).$

(A-9) $\quad g = 1, D = \frac{3}{4}P; \quad (1, 3, 4; 9).$

We have 9 types when $g \ge 1$.

Case B. The case $g = 0$ and $r \ge 4$.

In the following, we always assume $g = 0$. Since $\deg(K_X) = -2$ and $\deg D > 0$, we have $r \ge 3$. On the other hand, since $R_1 = H^0(K_X) = 0$, $a \ge 2$ and $\deg D \le \frac{7}{8} < 1$. Since $\deg D \ge -2 + r/2$, we have $r \le 5$.

In this subsection, we treat the cases where $r = 4, 5$.

Now, since $\deg[2D] = r - 4$, $\dim R_2 = 2, 1$, respectively, if $r = 5, 4$.

Thus if $r = 5$, then $a = b = 2$ and $c \ge 3$. Hence $\deg D \le \frac{2}{3}$. Since $3.\frac{1}{2} + 2.\frac{2}{3} - 2 = \frac{5}{6} > \frac{2}{3}$, the only possible cases for $(q_1, \ldots, q_5)$ are $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 2, 3)$.

(B-1) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + \ldots + P_5); \quad (2, 2, 5; 10).$

(B-2) $\quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3 + P_4) + \frac{2}{3}P_5; \quad (2, 2, 3; 8).$

Henceforce we assume $r = 4$ and express D by (q_1, q_2, q_3, q_4) and **we always assume** $q_1 \leq q_2 \leq q_3 \leq q_4$. In this case, $a = 2$ and $3 \leq b \leq c$. Hence $\deg D \leq \frac{1}{2}$. Since $4.\frac{2}{3} - 2 > \frac{1}{2}$, $q_1 = 2$ and $q_4 \geq 3$.

Let s be the number of $q_i > 2$ $(1 \leq s \leq 3)$. Then since $\deg[3D] = -6 + 8 - s$, $\dim R_3 = 0, 1, 2$ when $s = 1, 2, 3$, respectively.

If $s = 3$, $\dim R_2 + \dim R_3 = 3$ and we must have $(a, b, c; h) = (2, 3, 3; 9)$.

(B-3) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3 + P_4)$; $(2, 3, 3; 9)$.

If $s = 2$, $a = 2, b = 3$ and $c \geq 4$ and $\deg D = \frac{1}{6} + \frac{1}{c} \leq \frac{5}{12}$. Also, since $-2 + (\frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{5}{12}$, we have 2 types.

(B-4) $D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}(P_3 + P_4)$; $(2, 3, 6; 12)$.

(B-5) $D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}P_3 + \frac{3}{4}P_4$; $(2, 3, 4; 10)$.

Now we trat the case $(2, 2, 2, q)$, $q \geq 3$. In this case, $R_3 = 0$ and $\dim R_4 = 1$ or 2 according to $q = 3$ or $q \geq 4$. In the latter case, $\dim R_5 = 0$ or 1 according to $q = 4$ or $q \geq 5$. Hence, if $q \geq 5$, we have already 3 generators of R.

(B-6) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{4}{5}P_4$; $(2, 4, 5; 12)$.

(B-7) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{3}{4}P_4$; $(2, 4, 7; 14)$.

(B-8) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{2}{3}P_4$; $(2, 6, 9; 18)$.

We have 8 types in this case.

Case C. The case $g = 0$ and $r = 3$.

We have to determine (q_1, q_2, q_3). In this case, $R_1 = R_2 = 0$ and $\dim R_3 = 1$ or 0 according to $q_1 = 2$ or $q_1 \geq 3$.

Case 1. $q_1 \geq 3$.

In this case, $a = 3$ and $4 \leq b \leq c$. Hence $\deg D \leq \frac{1}{4}$. Hence either $q_1 = 3$ or $q_1 = q_2 = q_3 = 4$.

(C-1) $D = K_X + \frac{3}{4}(P_1 + P_2 + P_3)$; $(3, 4, 4; 12)$.

Henceforce we assume $q_1 = 3$.

$R_4 \neq 0$ if and only if $q_2 \geq 4$. In this case, $a = 3, b = 4$ and $c \geq 5$. Hence $\deg D \leq \frac{13}{60} = \frac{2}{3} + \frac{3}{4} + \frac{4}{5} - 2$. Hence we have only 2 possibilities;

(C-2) $D = K_X + \frac{2}{3}P_1 + \frac{3}{4}P_2 + \frac{4}{5}P_3$; $(3, 4, 5; 13)$.

(C-3) $D = K_X + \frac{2}{3}P_1 + \frac{3}{4}(P_2 + P_3)$; $(3, 4, 8; 16)$.

Next, assume $q_1 = q_2 = 3$. Hence $\deg D = \frac{q_3 - 1}{q_3} - \frac{2}{3}$. On the other hand, since $R_4 = 0$, $a = 3$, $b \geq 5$ and $c \geq 6$ and $\deg D \leq \frac{1}{6}$. This implies $q_3 \leq 6$.

(C-4) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{5}{6}P_3$; $(3, 5, 6; 15)$.

(C-5) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{4}{5}P_3$; $(3, 5, 9; 18)$.

(C-6) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{3}{4}P_3$; $(3, 8, 12; 24)$.

This completes the case $q_1 = 3$.

Case 2. $q_1 = 2$.

In this case, $a \geq 4$ and $R_4 \neq 0$ if and only if $q_2 \geq 4$.

First, we consider the case $q_1 = 2$ and $q_2 = 3$ $(q_3 \geq 7)$.

In this case, $\deg[4D] = -1 = \deg[5D] = \deg[7D], \deg[6D] = 0$. Hence $a = 6$ and $b \geq 8$. Hence $\deg D \leq \frac{1}{18} = \frac{8}{9} - \frac{5}{6}$. This shows that $7 \leq q_3 \leq 9$ and actually these cases gives the hypersurfaces.

(C-7) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{6}{7}P_3$; $(6, 14, 21; 42)$.

(C-8) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{7}{8}P_3$; $(6, 8, 15; 30)$.

(C-9) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{8}{9}P_3$; $(6, 8, 9; 24)$.

Next, we consider the case $q_1 = 2$ and $q_2 \geq 4$.

In this case, $\deg[4D] = 0$ and $a = 4, b \geq 5, c \geq 6$. Hence $\deg D \leq \frac{2}{15} = (\frac{1}{2} + \frac{4}{5} + \frac{5}{6}) - 2$. Hence $q_2 \leq 5$ and if $q_2 = 5$, the possibility is the following 2 cases.

(C-10) $D = K_X + \frac{1}{2}P_1 + \frac{4}{5}P_2 + \frac{5}{6}P_3$; $(4, 5, 6; 16)$.

(C-11) $D = K_X + \frac{1}{2}P_1 + \frac{4}{5}(P_2 + P_3)$; $(4, 5, 10; 20)$.

The remaining case is $q_1 = 2, q_2 = 4$ $(q_3 \geq 5)$.

Since $\dim R_4 = 1$ and $R_5 = 0$ and hence $a = 4, b \geq 6, c \geq 7$ and $\deg D = \frac{q_3 - 1}{q_3} - \frac{3}{4} \leq \frac{3}{28}$. Hence $5 \leq q_3 \leq 7$ and actually these cases give hypersurfaces.

This finishes the classification !

(C-12) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{4}{5}P_3$; $(4, 10, 15; 30)$.

(C-13) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{5}{6}P_3$; $(4, 6, 11; 22)$.

(C-14) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{6}{7}P_3$; $(4, 6, 7; 18)$.

3. The classification of hypersurfaces with $a(R) = 2$.

In this section, we classify normal graded hypersurfaces of dimension 2 with $a(R) = 2$.

We may assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with

$$\deg(u, v, w; f) = (a, b, c; h); \quad h = a + b + c + 2.$$

We always assume $(a, b, c) = 1$. Since R is Gorenstein with $a(R) = 2$, $2D$ is linearly equivalent to $K_X + D'$. Hence we may assume that

$$D = E + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i,$$

where $2E \sim K_X$ and every q_i is odd.

Since $\deg D = \dfrac{h}{abc} = \dfrac{a+b+c+2}{abc} \leq 5$, $2g - 2 \leq 2\deg D \leq 10$ and we have $g \leq 6$.

First, we divide the cases according to (1) $a \geq 2$, (2) $a = 1, b \geq 2$, or (3) $a = b = 1$.

Case 1. $a \geq 2$.

This is equivalent to say that $R_1 = H^0(X, \mathcal{O}_X(D)) = 0$. If this is the case, we have

$$\deg D \leq \frac{9}{2 \cdot 2 \cdot 3} = \frac{3}{4} < 1.$$

Since $\deg D \geq g - 1$, $g = 0$ or 1 in this case.

For a while, we assume that $g = 0$.

Now, we can write

$$D = -Q + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i.$$

Hence $R_1 = R_2 = 0$ and $\dim R_3 = r - 2$ since $\dfrac{q_i - 1}{2q_i} \geq \dfrac{1}{3}$ for every q_i. Hence $a = 3$ and $\deg D \leq \dfrac{12}{3 \cdot 3 \cdot 4} = \dfrac{1}{3}$. This implies that $r \leq 4$ and if $r = 4$, then $D = -Q + \sum_{i=1}^{4} \frac{1}{3} P_i$.

(2-1) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2 + P_3 + P_4), \quad (3, 3, 4; 12).$

Now, we assume $r = 3$ and $q_1 \leq q_2 \leq q_3$. Then since $\dim R_3 = 1$, we have $a = 3$ and $b \geq 5$. Also, $\dim R_4 = 0$ and $\dim R_5 = 2$ (resp. 1, resp. 0) if $q_3 \geq 5$ (resp. $q_1 = 3, q_2 \geq 5$, resp. $q_2 = 3$).

If $q_1 \geq 5$, $\deg D \leq \dfrac{15}{3 \cdot 5 \cdot 5} = \dfrac{1}{5}$. Hence we must have $D = -Q + \frac{2}{5}(P_1 + P_2 + P_3)$.

(2-2) $\quad g = 0, D = -Q + \frac{2}{5}(P_1 + P_2 + P_3), \quad (3, 5, 5; 15).$

Next, consider the case $q_1 = 3$ and $q_2 \geq 5$. In this case, $a = 3, b = 5$ and $c \geq 7$ and then $\deg D \leq \dfrac{17}{3 \cdot 5 \cdot 7} = \dfrac{1}{3} + \dfrac{2}{5} + \dfrac{3}{7} - 1$. Hence we are restricted to the following 2 cases.

(2-3) $\quad g = 0, D = -Q + \frac{1}{3}P_1 + \frac{2}{5}P_2 + \frac{3}{7}P_3, \quad (3, 5, 7; 17).$

Actually, if we put $D = -\frac{2}{3}(\infty) + \frac{2}{5}(0) + \frac{3}{7}(-1)$, then $R = k[F, G, H]$ with $F = \dfrac{1}{x(x+1)}T^3, G = \dfrac{1}{x^2(x+1)^2}T^5, H = \dfrac{1}{x^2(x+1)^3}T^7$ with the relation

$$F^4G = FH^2 + G^2H.$$

Hence $R \cong k[X, Y, Z]/(XZ^2 + Y^2Z - X^4Y)$.

(2-4) $\quad g = 0, D = -Q + \frac{1}{3}P_1 + \frac{2}{5}(P_2 + P_3), \quad (3, 5, 10; 20).$

If $q_2 = 3$, then $a = 3$ and $b \geq 7$. Hence $\deg D \leq \dfrac{21}{3 \cdot 7 \cdot 9} = 2\dfrac{1}{3} + \dfrac{4}{9} - 1$. Hence in this case, $q_3 = 5, 7$ or 9.

(2-5) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2) + \frac{4}{9}P_3, \quad (3, 7, 9; 21).$

(2-6) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2) + \frac{3}{7}P_3, \quad (3, 7, 15; 30).$

(2-7) $\quad g = 0, D = -Q + \frac{1}{3}(P_1 + P_2) + \frac{2}{5}P_3, \quad (3, 10, 15; 30).$

Now we have finished the case $a \geq 2$ and $g = 0$. Next, we treat the case $a \geq 2$ and $g = 1$. In this case, we put

$$D = E + \sum_{i=1}^{r} \frac{q_i - 1}{2q_i} P_i,$$

where $E \in \mathrm{Div}(X)$ with $E \neq 0$ and $2E \sim 0$. Since $[2D] = 0$ and $\deg[3D] = r > 0$, we have $a = 2$ and $b = 3$. Hence $\deg D \leq \dfrac{10}{2 \cdot 3 \cdot 3} < 1$ and actually, we have $r = 1$ or 2.

If $r = 2$, then $\deg[4D] = 2$ and we must have $(a, b, c) = (2, 3, 4)$ and $\deg D = \dfrac{11}{24}$. But since $\dfrac{q_1 - 1}{2q_1} + \dfrac{q_2 - 1}{2q_2} = \dfrac{11}{24}$ is impossible, this case does not occur. Hence we must have $r = 1$.

Since $a = 2, b = 3, c = 4$ is impossible as we have seen before, we must have $D = E + \dfrac{q-1}{2q}P$ with $E + P \geq 0$ and $\deg D \leq \dfrac{12}{2 \cdot 3 \cdot 5}$. We have 2 possibilities; $D = E + \dfrac{2}{5}P$ and $D = E + \dfrac{1}{3}P$. But the in latter case, we must have $a = 2, b = 3, c = 9$, which contradicts the fact $\deg D = \dfrac{1}{3}$.

Hence we are reduced to the case.

(2-8) $g=1, D=E+\frac{2}{5}P$ with $2E \sim 0$ and $E \neq 0$, $(2,3,5;12)$.

This finishes the case $a \geq 2$.

Case 2. $a=1$ **and** $b \geq 2$.

In this case, $\deg D \leq \frac{7}{1\cdot 2\cdot 2} < 2$. Hence we have $g=1$ or 2 in this case. Moreover, if $g=1$, since $[2D]=0$, we have $b \geq 3$ and $\deg D \leq \frac{9}{1\cdot 3\cdot 3} \leq 1$.

First, we assume $g=1$ and $D=\sum_{i=1}^{r}\frac{q_i-1}{2q_i}P_i$. Since $[2D]=0$ in this case, $b \geq 3$ and we have $\deg D \leq \frac{9}{1\cdot 3\cdot 3}=1$. Hence $r \leq 3$ in this case and if $r=3$, $D=\frac{1}{3}(P_1+P_2+P_3)$.

(2-9) $g=1, D=\frac{1}{3}(P_1+P_2+P_3)$, $(1,3,3;9)$.

Next, we assume $r=2$. Then $b=3$ and $c \geq 5$. We have $\deg D \leq \frac{11}{1\cdot 3\cdot 5}=\frac{1}{3}+\frac{2}{5}$. Hence we have 2 possibilities;

(2-10) $g=1, D=\frac{1}{3}P_1+\frac{2}{5}P_2$, $(1,3,5;11)$.

There is a linear relation between $T^{11}, GT^8, HT^6, G^2T^5, GHT^3, G^3T^2, H^2T, G^2H$, where $\deg T=1, \deg G=3$ and $\deg H=5$.

(2-11) $g=1, D=\frac{1}{3}(P_1+P_2)$, $(1,3,6;12)$.

Next, we assume $D=\frac{q-1}{2q}P$. In this case, $b \geq 5$ and $\deg D \leq \frac{15}{1\cdot 5\cdot 7}=\frac{3}{7}$. Hence we have 3 possibilities; $q=3,5,7$.

(2-12) $g=1, D=\frac{3}{7}P$, $(1,5,7;15)$.

(2-13) $g=1, D=\frac{2}{5}P$, $(1,5,8;16)$.

(2-14) $g=1, D=\frac{1}{3}P$, $(1,6,9;18)$.

Next, we treat the case $g=2$, $a=1, b \geq 2$ and $D=E+\sum_{i=1}^{r}\frac{q_i-1}{2q_i}P_i$, with $2E \sim K_X$. Since $[2D] \sim K_X$ in this case, we have $a=1, b=2$ and $c \geq 3$. Thus we have $\deg D \leq \frac{8}{1\cdot 2\cdot 3}=1+\frac{1}{3}$. Hence $r \leq 1$ and if $r=1$, then $D=D=E+\frac{1}{3}P$.

(2-15) $g=2, D=E$ with $2E \sim K_X$, $(1,2,5;10)$.

(2-16) $g = 2, D = \frac{1}{3}P, \quad (1,2,3;8)$.

This finishes the case $a = 1, b \geq 2$.

Case 3. $a = b = 1$.

In this case, $\deg D = \dfrac{c+4}{c}$. Since $g \geq 3$ in this case, $\deg D \geq 2$ and we have $c \leq 4$.

(2-17) $g = 6, D = E$ with $2E \sim K_X, \quad (1,1,1;5)$.

(2-18) $g = 5, D = E$ with $2E \sim K_X, \quad (1,1,2;6)$.

(2-19) $g = 3, D = E + \frac{1}{3}P$ with $2E \sim K_X, \quad (1,1,3;7)$.

(2-20) $g = 3, D = E$ with $2E \sim K_X, \quad (1,1,4;8)$.

4. Complete intersections with $a(R) = 1$.

In my talk at the conference, I talked about classification of normal graded complete intersections of dimension 2 with $a(R) = 1$. Until now, I can not find a satisfactory way to classify them. Here, I will show the results when the genus of the curve is 0.

Proposition 4.1. *Let $R = \oplus_{n\geq 0} R_n$ be a normal graded complete intersection Of dimension 2 with $R_0 = k$, a field, $a(R) = 1$ and $R_1 = 0$. Then the embedded dimension of R is at most 4.*

This follows from the fact $\mathfrak{m}H^1(X, \mathcal{O}_X) = 0$, where $\mathfrak{m}$ is the graded maximal ideal of R and $X \to$ Spec (R) is a resolution of singularities of R. By the Briançon-Skọda type argument, we can assert that $\mathfrak{m}^3 \subset J$, where J is a minimal reduction of $\mathfrak{m}$. Then by the argument as in [NW], §2, we can deduce that the embedded dimension of R is at most 4. Conversely, if R is Gorenstein with the embedded dimension 4, then R is a complete intersection by the famous result of J.-P. Serre.

Until now, I can not find a satisfactory method of classification for this case. Actually, what I do is only to restrict the embedding dimension. So, I list only the results in this case. We list the divisor D on $X = \mathbb{P}^1$ with $R(X, D) \cong k[u, v, w, z]/(f, g)$. We also put $\deg(u, v, w, z; f, g) = (a, b, c, d; g, h)$ with $g + h = a + b + c + d + 1$ with $a \leq b \leq c \leq d$ and $g \leq h$ in the following table.

(3-1) $D = K_X + \frac{1}{2}(P_1 + \ldots + P_6), \quad (2,2,2,3;4,6)$.

(3-2) $D = K_X + \frac{1}{2}(P_1 + \ldots + P_4) + \frac{3}{4}P_5, \quad (2,2,3,4;6,6)$.

(3-3) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{2}{3}(P_4 + P_5), \quad (2,2,3,3;5,6)$.

(3-4) $D = K_X + \frac{2}{3}(P_1 + \ldots + P_4), \quad (2,3,3,3;6,6)$.

(3-4) $D = K_X + \frac{2}{3}(P_1 + \ldots + P_4), \quad (2,3,3,3;6,6).$

(3-5) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3) + \frac{3}{4}P_4, \quad (2,3,3,4;6,7).$

(3-5) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3) + \frac{3}{4}P_4, \quad (2,3,3,4;6,7).$

(3-6) $D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{3}{4}(P_3 + P_4), \quad (2,3,4,4;6,8).$

(3-7) $D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}P_3 + \frac{4}{5}P_4, \quad (2,3,4,5;7,8).$

(3-8) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{5}{6}P_4, \quad (2,4,5,6;8,10).$

(3-9) $D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{5}{6}P_4, \quad (2,4,5,6;8,10).$

(3-10) $D = K_X + \frac{3}{4}(P_1 + P_2) + \frac{4}{5}P_3, \quad (3,4,4,5;8,9).$

(3-11) $D = K_X + \frac{2}{3}P_1 + \frac{4}{5}(P_2 + P_3), \quad (3,4,5,5;8,10).$

(3-12) $D = K_X + \frac{2}{3}P_1 + \frac{3}{4}P_2 + \frac{5}{6}P_3, \quad (3,4,5,6;9,10).$

(3-13) $D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{6}{7}P_3, \quad (3,5,6,7;10,12).$

(3-14) $D = K_X + \frac{1}{2}P_1 + \frac{5}{6}(P_2 + P_3), \quad (4,5,6,6;10,12).$

(3-15) $D = K_X + \frac{1}{2}P_1 + \frac{4}{5}P_2 + \frac{6}{7}P_3, \quad (4,5,6,7;11,12).$

(3-16) $D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{7}{8}P_3, \quad (4,6,7,8;12,14).$

(3-17) $D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{9}{10}P_3, \quad (6,8,9,10;16,18).$

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DEPARTMENT OF MATHEMATIS, COLLEGE OF HUMANITIES AND SCIENCES, NIHON UNIVERSITY SETAGAYA, TOKYO 156-8550, JAPAN

E-mail address: watanabe@math.chs.nihon-u.ac.jp