研究集会

第31回可換環論シンポジウム

The 31st Symposium on Commutative Algebra Japan

2009年11月24日—27日

於 ホテルアウィーナ大阪

平成21年度科学研究費補助金
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序

これは第31回可換環論シンポジウムの報告集です。このシンポジウムは2009年11月24日(火)から11月27日(金)にかけて、ホテルアウィーナ大阪 に於いて開催されました。 Igor Burban 氏(University of Bonn, Germany) と Tim Römer 氏(University of Osnabrück, Germany)を、海外からの招待講演者として迎え、70名を超える参加者のもと、合計29件の興味深い講演が行われました。

なお、シンポジウム開催にあたり、以下の平成21年度科学研究費補助金からの援助を受けました。

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吉野雄二先生、伊山修先生には、ここに改めてお礼申し上げます。また、研究集会の運営に協力して頂いた、岡山大学、明治大学の大学院生の皆様にも、この場を借りて感謝したいと思います。

2010年1月

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CONTENTS

A generalization of Huckaba-Marley's theorem and its application to the theory of initial ideals

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1 Introduction

Let K be a field and $S = K[X_1, \ldots, X_r]$ a polynomial ring with monomial order. Suppose that S is graded by some positive weight vector and J is a graded ideal. It is known that there is a flat family whose general fiber is S/J and a special fiber is $S/\operatorname{in}(J)$. In particular, $\operatorname{depth}_{S_\mathfrak{m}}(S/\operatorname{in}(J))_\mathfrak{m} \leq \operatorname{depth}_{S_\mathfrak{m}}(S/J)_\mathfrak{m}$, where $\mathfrak{m} = (X_1, \ldots, X_r)$.

The validity of equality is hopeless and there are many examples with strict inequality. Indeed, there are examples such that S/J are Cohen-Macaulay of dimension n and $\operatorname{depth} S/\operatorname{in}(J) = 0$ for any integer $n > 0$.

But in the case where $\operatorname{in}(J)$ is a radical ideal, there may be some hope for handling this phenomenon and the present author have proved the following

Theorem 1.1 ([Miy]) *Let A be a graded Hodge algebra over a field K generated by H and governed by Σ. Suppose that A is Cohen-Macaulay, Σ is square-free and the discrete counterpart A_{dis} of A is Buchsbaum. Then A_{dis} is Cohen-Macaulay.*

The result of Huckaba-Marley [HM, Theorem 3.10] played an important role in the proof of Theorem 1.1.

In this paper, we recall the generalized form of Huckaba-Marley's theorem and as an application, prove the generalization of Theorem 1.1.

2 Generalized Huckaba-Marley's theorem

First we settle the following notation which is used throughout this paper.

Notation (1) We denote by $\boldsymbol{N}$ the set of non-negative integers and by $\boldsymbol{Z}$ the set of integers.

(2) For a Noetherian $\boldsymbol{N}^u$-graded ring B with $B_{(0,\dots,0)}$ a local ring, we set

$$\operatorname{depth} B := \operatorname{depth} B_N,$$

where N is the unique $\boldsymbol{N}^u$-graded maximal ideal of B.

Now we recall the theorem of Huckaba-Marley.

Theorem 2.1 (**[HM, Theorem 3.10]**) *Let A be a local ring, I a proper ideal of A, $\mathcal{R} = A[IT]$ the Rees algebra with respect to I and $G = \bigoplus_{n\in\boldsymbol{N}} I^n/I^{n+1}$ the associated graded ring. Suppose* $\operatorname{depth} G < \operatorname{depth} A$. *Then* $\operatorname{depth}\mathcal{R} = \operatorname{depth} G + 1$.

Theorem 2.1 is generalized as the following form.

Theorem 2.2 *Let A be a local ring, $\mathcal{F} = \{I_n\}_{n\in\boldsymbol{N}}$ a filtration of proper ideals of A, $\mathcal{R} = A[I_nT^n \mid n \in \boldsymbol{N}]$ the Rees algebra with respect to $\mathcal{F}$ and $G = \bigoplus_{n\in\boldsymbol{N}} I_n/I_{n+1}$ the associated graded ring. Suppose that $\mathcal{R}$ is Noetherian and* $\operatorname{depth} G < \operatorname{depth} A$. *Then* $\operatorname{depth}\mathcal{R} = \operatorname{depth} G + 1$.

This result is known early by Goto-Nishida using the method of [GN] and later published in [CZ]. As a corollary, we see the following result.

Corollary 2.3 *Let $A = \bigoplus_{n\in\boldsymbol{N}} A_n$ be a Noetherian graded ring with A_0 a local ring, $\mathcal{F} = \{I_n\}_{n\in\boldsymbol{N}}$ a filtration of proper graded ideals of A, $\mathcal{R} = A[I_nT^n \mid n \in \boldsymbol{N}]$ the Rees algebra with respect to $\mathcal{F}$, and $G = \bigoplus_{n\in\boldsymbol{N}} I_n/I_{n+1}$ the associated graded ring. Assume that $\mathcal{R}$ is Noetherian and* $\operatorname{depth} G < \operatorname{depth} A$. *Then* $\operatorname{depth}\mathcal{R} = \operatorname{depth} G + 1$.

Proof Let M (resp. N) be the unique graded (resp. bigraded) maximal ideal of A (resp. $\mathcal{R}$). Then $\mathcal{R}_N = (\mathcal{R}_M)_N$, $G_N = (G_M)_N$ and

$$\mathcal{R}_M = A_M[I_nA_MT^n \mid n \in \boldsymbol{N}], \quad G_M = \bigoplus_{n\in\boldsymbol{N}} I_nA_M/I_{n+1}A_M.$$

So the result follows from Theorem 2.2. ∎

3 Depth of the ring of quotients modulo the initial ideal

Let K be a field and $S = K[X_1, \ldots, X_r]$ a polynomial ring with monomial order $<$. We assume that S is graded by a weight vector $w = (w_1, \ldots, w_r) \in (\boldsymbol{N} \setminus \{0\})^r$, that is $\deg X_i = w_i$ for $i = 1, \ldots, r$. And let J be a graded ideal of S.

The main result of this paper is the following

Theorem 3.1 *Assume that $S/\operatorname{in}(J)$ is reduced and Buchsbaum. Then* $\operatorname{depth} S/\operatorname{in}(J) = \operatorname{depth} S/J$. *In particular, if S/J is Cohen-Macaulay, then so is $S/\operatorname{in}(J)$.*

The most part of the rest of this paper is devoted to the proof of this theorem.

In order to prove Theorem 3.1, we first state the following

Notation (1) $A := S/J$.

(2) $x_i := X_i + J$.

(3) For $a = (a_1, \ldots, a_r) \in \boldsymbol{Z}^r$, we set

$$X^a := X_1^{a_1} \cdots X_r^{a_r} \quad \text{and} \quad x^a := x_1^{a_1} \cdots x_r^{a_r}.$$

(4) For $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r) \in \boldsymbol{Z}^r$, we set

$$(a, b) := \sum_i a_i b_i.$$

Let $g_1, \ldots, g_s$ be the reduced Gröbner basis of J and $\mathcal{M}$ the set of monomials which appear in at least one of $g_1, \ldots, g_s$. Note that each g_i is a homogeneous polynomial.

We define a binary relation $\prec$ on the set of monomials in S as follows.

Definition 3.2

$$X^a \prec X^b \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} \deg X^a < \deg X^b \\ \text{or} \\ \deg X^a = \deg X^b \text{ and } X^a > X^b. \end{array}$$

It is easy to see the following

Lemma 3.3 *$\prec$ is a monomial order.*

Since $\prec$ is another monomial order on S, we see the following

Fact 3.4 *There exists $\lambda = (\lambda_1, \ldots, \lambda_r) \in (\boldsymbol{N} \setminus \{0\})^r$ such that*

$$X^a \prec X^b \Leftrightarrow (\lambda, a) < (\lambda, b)$$

for any X^a, $X^b \in \mathcal{M}$.

It is well known that

$$B = \{x^a \mid X^a \notin \mathrm{in}(J)\}$$

is a K-vector space basis of A. And if $\alpha \in A$, one can obtain the expression of α as a linear combination of the elements of B by repeated application of rewriting with respect to g_i's. Since each g_i is homogeneous, we have the following

Lemma 3.5 *If $X^a \in \mathrm{in}(J)$, then there is a unique expression*

$$x^a = c_1 x^{a_1} + c_2 x^{a_2} + \cdots + c_u x^{a_u}$$

such that $c_i \in K$, $a_i \in \boldsymbol{Z}^r$, $X^{a_i} \notin \mathrm{in}(J)$ and $(\lambda, a_i) > (\lambda, a)$ for any i.

Set $J_n := (X^a \mid (\lambda, a) \geq n)$, $I_n := (J_n + J)/J$, $\mathcal{F} := \{I_n\}_{n \in \boldsymbol{N}}$, $\mathcal{R} := A[I_n T^n \mid n \in \boldsymbol{N}]$ and $G := \bigoplus_{n \in \boldsymbol{N}} I_n / I_{n+1}$. Then we see the following lemma which is a direct consequence of Lemma 3.5.

Lemma 3.6 *$\{x^a \mid X^a \notin \mathrm{in}(J),\ (\lambda, a) \geq n\}$ is a K-vector space basis of I_n.*

By Lemmas 3.5 and 3.6, we see the following

Lemma 3.7 *(1) $\mathcal{R} = K[x_i T^m \mid 1 \leq i \leq r, 0 \leq m \leq \lambda_i]$. In particular, $\mathcal{R}$ is Noetherian.*

(2) $\{x^a + I_{n+1} \mid X^a \notin \mathrm{in}(J),\ (\lambda, a) = n\}$ is a K-vector space basis of I_n / I_{n+1}.

(3) If $X^a \in \mathrm{in}(J)$, then $x^a \in I_{(\lambda, a)+1}$.

(4) K-algebra homomorphism $\Phi \colon S \to G$, $(X_i \mapsto x_i + I_{\lambda_i + 1} \in I_{\lambda_i} / I_{\lambda_i + 1})$ is surjective and $\ker \Phi = \mathrm{in}(J)$. In particular, $G \simeq S / \mathrm{in}(J)$.

Note that $\mathcal{R}$ and G are bigraded rings. We denote the degree inherited from S as the first entry and the degree defined by the Rees algebra structure as the second entry.

Let N be the unique bigraded maximal ideal of $\mathcal{R}$, M the irrelevant maximal ideal of A, $L := I_1T \oplus I_2T^2 \oplus I_3T^3 \oplus \cdots$ and $L' := I_1 \oplus I_2T \oplus I_3T^2 \oplus \cdots$. Then

Remark 3.8 (1) L and L' are bigraded ideals of $\mathcal{R}$.

(2) $L' = L(0,1)$.

(3) $A = \mathcal{R}/L$.

(4) $G = \mathcal{R}/L'$.

By the short exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{R} \longrightarrow A \longrightarrow 0$$

we get the following long exact sequence of local cohomology.

$$\begin{array}{ccccccc} & & & \cdots & \to & H_M^{i-1}(A) \\ \to & H_N^i(L) & \to & H_N^i(\mathcal{R}) & \to & H_M^i(A) \\ \to & \cdots & & & & \end{array}$$

Since $[H_M^i(A)]_{(u,n)} = 0$ for $n \neq 0$,

$$[H_N^i(L)]_{(u,n)} \simeq [H_N^i(\mathcal{R})]_{(u,n)} \quad \text{for } n \neq 0.$$

By the short exact sequence

$$0 \longrightarrow L' \longrightarrow \mathcal{R} \longrightarrow G \longrightarrow 0$$

we get the following long exact sequence of local cohomology.

$$\begin{array}{ccccccc} & & & \cdots & \to & H_N^{i-1}(G) \\ \to & H_N^i(L') & \to & H_N^i(\mathcal{R}) & \to & H_N^i(G) \\ \to & \cdots & & & & \end{array}$$

Here we recall the following result of Hochster.

Theorem 3.9 (Hochster) *Let Δ be a simplicial complex with vertex set $\{X_1, \ldots, X_r\}$. Then the $\boldsymbol{Z}^r$-graded Hilbert series of $H_N^i(K[\Delta])$ is*

$$\sum_{\sigma\in\Delta} \left(\dim_K \tilde{H}^{i-|\sigma|-1}(\mathrm{link}_\Delta(\sigma); K)\right) \prod_{X_j\in\sigma} \frac{t_j^{-1}}{1-t_j^{-1}}$$

where N is the unique graded maximal ideal. In particular, if $K[X_1, \ldots, X_r]$ is equiped with $\boldsymbol{N}^2$-grading such that $\deg X_j = (a_j, b_j)$ with $(a_j, b_j) \in \boldsymbol{N}^2 \setminus \{(0,0)\}$ for any j, then the $\boldsymbol{Z}^2$-graded Hilbert series of $H_N^i(K[\Delta])$ is

$$\sum_{\sigma\in\Delta} \left(\dim_K \tilde{H}^{i-|\sigma|-1}(\mathrm{link}_\Delta(\sigma); K)\right) \prod_{X_j\in\sigma} \frac{t_1^{-a_j}t_2^{-b_j}}{1-t_1^{-a_j}t_2^{-b_j}}.$$

Since $\mathrm{in}(J)$ is a square-free monomial ideal, $G \simeq S/\mathrm{in}(J) \simeq K[\Delta]$ for some simplicial complex Δ. Therefore,

Corollary 3.10 $[H_N^i(G)]_{(u,n)} = 0$ *for $n > 0$.*

So we see that

$$[H_N^i(L')]_{(u,n)} \simeq [H_N^i(\mathcal{R})]_{(u,n)} \quad \text{for } n > 0.$$

Summing up,

- $[H_N^i(L)]_{(u,n)} \simeq [H_N^i(\mathcal{R})]_{(u,n)}$ for $n \neq 0$.
- $[H_N^i(L')]_{(u,n)} \simeq [H_N^i(\mathcal{R})]_{(u,n)}$ for $n > 0$.
- $[H_N^i(L')]_{(u,n)} = [H_N^i(L)]_{(u,n+1)}$.
- $[H_N^i(G)]_{(u,n)} = 0$ for $n > 0$.

Therefore $[H_N^i(\mathcal{R})]_{(u,n+1)} \simeq [H_N^i(\mathcal{R})]_{(u,n)}$ for $n > 0$. Since $[H_N^i(\mathcal{R})]_{(u,n)} = 0$ for $n \gg 0$, we see that $[H_N^i(L)]_{(u,n)} \simeq [H_N^i(\mathcal{R})]_{(u,n)} = 0$ for $n > 0$.

Now assume that $e = \mathrm{depth} G < \mathrm{depth} A$. Then $\mathrm{depth}\mathcal{R} = e + 1$ by generalized Huckaba-Marley's theorem. By the exact sequence

$$[H_N^e(\mathcal{R})]_{(u,0)} \to [H_N^e(G)]_{(u,0)} \to [H_N^{e+1}(L)]_{(u,1)}$$

and the fact

$$[H_N^e(\mathcal{R})]_{(u,0)} = [H_N^{e+1}(L)]_{(u,1)} = 0,$$

we see that

$$[H_N^e(G)]_{(u,0)} = 0.$$

Since $[H_N^e(G)] \neq 0$, we see that $[H_N^e(G)]_{(u,n)} \neq 0$ for some u, n with $n < 0$. But then, by Theorem 3.9 we see that

$$\exists \sigma \in \Delta; \sigma \neq \emptyset,\ \tilde{H}^{e-|\sigma|-1}(\mathrm{link}_\Delta(\sigma); K) \neq 0.$$

This contradicts to the assumption that $G \simeq K[\Delta]$ is Buchsbaum and the proof of Theorem 3.1 is completed.

Remark 3.11 There is a counterexample if one drops the assumption that $S/\mathrm{in}(J)$ is reduced as the following example shows.

Example 3.12 Let $B = K[Y_1, Y_2, Y_3]$ be the polynomial ring over K with 3 variables and A' the second Veronese subring of B. Then A' is a Cohen-Macaulay domain of dimension 3. Set $A = A'/(Y_1^2, Y_3^2)$. Then A is a Cohen-Macaulay ring of dimension 1.

Let $\{X_{ij}\}_{1 \leq i \leq j \leq 3}$ be a family of indeterminates and set $S' = K[X_{ij} \mid 1 \leq i \leq j \leq 3]$ and $\deg X_{ij} = 1$ for any i and j. Define the monomial order on S' as the degree reverse lexicographic order induced by $X_{11} > X_{12} > X_{13} > X_{22} > X_{23} > X_{33}$. Then $A' \simeq S'/I_2(X)$, where $X = (X_{ij})$ is the 3×3 symmetric matrix with $X_{ij} := X_{ji}$ for $i > j$ and $I_2(X)$ is the ideal of S' generated by 2-minors of X.

Set $J' = I_2(X)$, $S = K[X_{ij} \mid 1 \leq i \leq j \leq 3, (i,j) \neq (1,1), (i,j) \neq (3,3)]$ and $J = (J' + (X_{11}, X_{33})) \cap S$. Then it is easily seen that $A \simeq S/J$. In particular, S/J is Cohen-Macaulay of dimension 1. On the other hand, it is verified that

$$\mathrm{in}(J) = (X_{12}^2, X_{12}X_{13}, X_{13}^2, X_{13}X_{22}, X_{13}X_{23}, X_{23}^2).$$

Since

$$\mathrm{in}(J) = (X_{12}, X_{13}, X_{22}, X_{23})^2 \cap (X_{12}^2, X_{13}, X_{23}^2)$$

and

$$\mathrm{in}(J) \colon (X_{12}, X_{13}, X_{22}, X_{23}) = (X_{12}^2, X_{13}, X_{23}^2)$$

is the primary component corresponding to the unique minimal prime ideal (X_{12}, X_{13}, X_{23}) of $\mathrm{in}(J)$, it is easily verified that

$$\mathrm{in}(J) \colon f = (X_{12}^2, X_{13}, X_{23}^2)$$

for any $f \in (X_{12}, X_{13}, X_{22}, X_{23}) \setminus (X_{12}, X_{13}, X_{23})$. So $S/\mathrm{in}(J)$ is Buchsbaum. Since the irrelevant maximal ideal of S is an associated prime ideal of $\mathrm{in}(J)$, we see that $\mathrm{depth} S/\mathrm{in}(J) = 0$.

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F-purity of homomorphisms, strong F-regularity, and F-injectivity

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1. Introduction

These notes are a summary of the results in [5].

In [6], a generalization of Matijevic–Roberts theorem was proved, and as a corollary, we have the following [6, Corollary 7.10].

1.1 Theorem. *Let p be a prime number, and A a $\mathbb{Z}^n$-graded noetherian ring. Let P be a prime ideal of A, and P^* be the prime ideal of A generated by the homogeneous elements of P. If A_{P^*} is excellent of characteristic p and is weakly F-regular (resp. F-regular, F-rational), then A_P is weakly F-regular (resp. F-regular, F-rational). If A_P is excellent of characteristic p and is weakly F-regular, then A_{P^*} is weakly F-regular.*

The proof relies on the smooth base change and the flat descent of (weak) F-regularity and F-rationality.

It is natural to ask the same problem for F-purity, strong F-regularity, and Cohen–Macaulay F-injectivity. This question was posed by Ken-ichi Yoshida. The purpose of these short notes is to give an answer to this question. On the way, we also consider the problem of the openness of the loci of strong F-regularity and CMFI (Cohen–Macaulay F-injective) property. The openness of the F-rational locus is discussed in [12]. Related to the smooth base change of F-purity, we define F-purity of a homomorphism between noetherian rings of characteristic p. It is not characterized by the F-purity of (geometric) fibers. We discuss when an F-pure homomorphism is flat.

Strong F-regularity was defined for an F-finite noetherian ring of characteristic p by Hochster and Huneke [8]. This notion was generalized to those for a general noetherian ring of characteristic p in different ways by Hochster [7] and Hochster–Huneke [10]. The author does not know if the two definitions agree. So we name Hochster's definition "strongly F-regular" and Hochster–Huneke's "very strongly F-regular." As the name shows, very strongly F-regular implies strongly F-regular in general. For local rings, F-finite rings, and algebras essentially of finite type over excellent local rings, these two notions coincide. We prove "F-pure base change" of strong F-regularity, generalizing the smooth base change.

2. F-purity of homomorphisms

It is sometimes useful to promote a property of a ring to that of a homomorphism of rings. This idea is due to Grothendieck.

2.1 Definition (Grothendieck). Let $\mathbb{P}$ be one of the properties: Cohen–Macaulay, Gorenstein, l.c.i., reduced, normal, and regular. We say that a homomorphism $f : A \to B$ of noetherian rings is $\mathbb{P}$, if f is flat, and the fiber ring $B \otimes_A \kappa(P)$ is geometrically $\mathbb{P}$ for any $P \in \operatorname{Spec} A$, that is, $B \otimes_A K$ satisfies $\mathbb{P}$ for any finite extension field K of $\kappa(P)$.

Weakening the flatness condition, Avramov and Foxby generalized this definition, see [2], [3], [4].

2.2 Remark. Let $\mathbb{P}$ be as in Definition 2.1. A composite of $\mathbb{P}$ morphisms is again $\mathbb{P}$. If $f : A \to B$ is $\mathbb{P}$ and A is $\mathbb{P}$, then B is $\mathbb{P}$. If $f : A \to B$ is $\mathbb{P}$ and A' is an A-algebra, then the base change $f' : A' \to B'$ is $\mathbb{P}$, provided both A' and B' are noetherian. If $f : A \to B$ is a homomorphism, A' is a faithfully flat A-algebra, and if the base change $f' : A' \to B'$ is $\mathbb{P}$, then f is $\mathbb{P}$.

It is natural to ask if Grothendieck's idea applies to F-singularities.

2.1 Theorem (Aberbach–Enescu [1]). *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat local homomorphism of noetherian local rings of characteristic p. If A is Cohen–Macaulay F-injective and $B/\mathfrak{m}B$ is geometrically Cohen–Macaulay F-injective, then B is Cohen–Macaulay F-injective.*

Thus we may define a Cohen–Macaulay F-injective homomorphism to be a flat homomorphism with CMFI geometric fibers.

2.3 Example (Singh [11]). There is a flat local homomorphism $f : A \to B$ with A a DVR, f has geometrically F-regular fibers, but B is not F-pure.

Because of the example, probably it is not appropriate to define an F-pure homomorphism to be a flat homomorphism with geometrically F-pure fibers.

(2.4) To define an F-pure homomorphism, we uitilize Radu–André homomorphisms.

Let R be a ring of characteristic p. Let $F_R^e : R \to {}^eR$ be the eth Frobenius map given by $F_R^e(x) = x^{p^e}$, where the ring R, considered as an R-algebra via the structure map F_R^e, is denoted by ${}^eR$. An element $a \in R$, viewed as an element of ${}^eR = R$, is denoted by ${}^ea$.

2.5 Definition. For a homomorphism $f : A \to B$ of commutative rings of characteristic p, we define

$$\Psi_e(f) = \Psi_e(A, B) : B \otimes_A {}^eA \to {}^eB$$

by $\Psi_e(f)(b \otimes {}^ea) = {}^e(b^{p^e}a)$, and call it the eth *Radu–André homomorphism* (or the eth relative Frobenius map).

2.2 Theorem (Radu–André). *Let $f : A \to B$ be a homomorphism of noetherian rings of characteristic p. Then the following are equivalent: 1) f is regular; 2) $\Psi_e(f)$ is flat for some $e \geq 1$; 3) $\Psi_e(f)$ is flat for every $e \geq 1$.*

The absolute case (i.e., the case that $A = \mathbb{F}_p$) is due to Kunz.

Using Radu–André homomorphism, we define the F-purity of homomorphisms.

2.6 Definition. A homomorphism $f : A \to B$ of rings of characteristic p is said to be *F-pure* if the Radu–André homomorphism $\Psi_1(f) : B \otimes_A {}^1A \to {}^1B$ is pure.

Thus a homomorphism $f : A \to B$ of rings of characteristic p is F-pure if it is regular. We list some consequences of the definition.

2.7 Lemma. *Let $f : A \to B$ and $g : B \to C$ be homomorphisms between $\mathbb{F}_p$-algebras.*

1) If f and g are F-pure, then so is gf. 2) A is F-pure if and only if the unique map $\mathbb{F}_p \to A$ is F-pure. 3) If gf is F-pure and g is pure, then f is F-pure. 4) If A is F-pure and f is F-pure, then B is F-pure. 5) A pure subring of an F-pure ring is F-pure. 6) Let A' be an A-algebra, and $B' = B \otimes_A A'$. If f is F-pure, then the base change $A' \to B'$ is also F-pure. 7) If $A \to A'$ is a pure homomorphism and $A' \to B' = B \otimes_A A'$ is F-pure, then f is F-pure.

F-purity over a field can be described as follows.

2.3 Theorem. *Let K be a field of characteristic p, and A a K-algebra. Then the following are equivalent.*

1. *B is noetherian, and $K \to B$ is F-pure.*

2. *For any $e \geq 1$, $B \otimes_K {}^eK$ is noetherian and F-pure.*

3. *For some $e \geq 1$, $B \otimes_K {}^eK$ is noetherian and F-pure.*

4. *B is noetherian, and B is geometrically F-pure over K.*

2.8 Corollary. *If $f : A \to B$ is an F-pure homomorphism between noetherian rings of characteristic p, then the fiber $B \otimes_A \kappa(P)$ is geometrically F-pure for each $P \in \operatorname{Spec} A$.*

The converse of the corollary is false, see (2.3).

2.9 Definition. A homomorphism $f : A \to B$ of rings of characteristic p is said to be *Dumitrescu* if $\Psi_1(f) : B \otimes_A {}^1A \to {}^1B$ is ${}^1A$-pure.

2.4 Theorem (Dumitrescu). *For a flat homomorphism $f : A \to B$ of noetherian rings of characteristic p, the following are equivalent.*

1. *f is Dumitrescu.*

2. *f is reduced.*

It is natural to ask, is a Dumitrescu homomorphism flat? The author does not know the answer. Some partial results follows.

2.5 Theorem. *Let $f : A \to B$ be a homomorphism of noetherian rings of characteristic p. If f is Dumitrescu and the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ contains all maximal ideals of A, then f is pure.*

2.10 Corollary. *A Dumitrescu local homomorphism between noetherian local rings of characteristic p is pure.*

2.11 Definition. Let $f : A \to B$ be a homomorphism of noetherian rings. We say that f is almost quasi-finite if each fiber $B \otimes_A \kappa(P)$ is finite over $\kappa(P)$. This is equivalent to say that for each $Q \in \operatorname{Spec} B$, $\kappa(Q)$ is a finite extension of $\kappa(P)$, where $P = Q \cap A$.

A quasi-finite homomorphism (finite-type homomorphism with zero dimensional fibers) is almost quasi-finite. A localization is almost quasi-finite. A composite of almost quasi-finite homomorphisms is almost quasi-finite. A base change of an almost quasi-finite homomorphism is again almost quasi-finite, provided it is a homomorphism between noetherian rings.

2.6 Theorem (Watanabe, H–). *Let $f : A \to B$ be an almost quasi-finite homomorphism between noetherian rings of characteristic p. Then the following are equivalent: 1) f is F-pure; 2) f is Dumitrescu; 3) f is regular.*

The case that both A and B are domains and f is finite is due to K.-i. Watanabe.

2.7 Theorem. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$, A is normally flat along tA, and $A/tA \to B/tB$ is flat, then f is flat.*

2.12 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If $t \in \mathfrak{m}$ is a nonzerodivisor of A and $A/tA \to B/tB$ is flat, then f is flat.*

2.13 Corollary. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a Dumitrescu local homomorphism between noetherian local rings of characteristic p. If A is regular, then f is flat.*

3. Strong F-regularity and Cohen–Macaulay F-injectivity

3.1 Definition. Let R be a noetherian ring of characteristic p. We say that R is

1. (Hochster, [7]) *strongly F-regular* if any R-submodule of any R-module is tightly closed.
2. (Hochster–Huneke, [10]) *very strongly F-regular* if for any $c \in R^\circ$, there exists some $e \geq 1$ such that ${}^e cF^e : R \to {}^e R \quad (x \mapsto {}^e(cx^{p^e}))$ is R-pure.

The name "very strongly F-regular" was new in [5]. This was introduced to distinguish the notion from the strong F-regularity by Hochster. The author does not know if these two definitions agree. They agree with the original definition of Hochster–Huneke [8], if the ring is F-finite.

3.2 Lemma. *Let R be a noetherian ring of characteristic p. Then the following are equivalent.*

1. *R is strongly F-regular.*
2. *For any multiplicatively closed subset S of R, R_S is strongly F-regular.*
3. *For any maximal ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is strongly F-regular.*

4. *For any maximal ideal $\mathfrak{m}$ of R, the R-submodule 0 of $E_R(R/\mathfrak{m})$ is tightly closed.*

5. *For any maximal ideal $\mathfrak{m}$ of R, the $R_\mathfrak{m}$-submodule 0 of $E_R(R/\mathfrak{m})$ is tightly closed.*

6. *For any maximal ideal $\mathfrak{m}$ of R, $R_\mathfrak{m}$ is very strongly F-regular.*

3.3 Remark. Let R be a noetherian ring of characteristic p.

1. If R is very strongly F-regular and S is a multiplicatively closed subset of R, then R_S is very strongly F-regular.

2. If R is very strongly F-regular, then it is strongly F-regular.

3. If R is strongly F-regular, then it is F-regular.

4. If R is F-rational Gorenstein, then it is strongly F-regular.

3.4 Lemma. *Let S be a noetherian normal ring, and R its cyclically pure subring. Then R is noetherian normal, and R is a pure subring of S.*

The following is a generalization of [9, (4.12)] (the assumption $R^\circ \subset S^\circ$ is dropped). For the proof, we use the lemma above.

3.5 Proposition. *Let S be a noetherian ring of characteristic p, and R its cyclically pure subring. If S is very strongly F-regular (resp. strongly F-regular, F-regular, weakly F-regular), then so is R.*

The following is proved using the Γ-construction developed in [10]. A similar result for F-rationality is proved by Vélez [12].

3.1 Theorem. *Let R be an excellent local ring of characteristic p, and A an R-algebra essentially of finite type. Then the strongly F-regular locus and the Cohen–Macaulay F-injective locus of A are Zariski open in* $\operatorname{Spec} A$.

Using a similar technique, Hoshi proved the following.

3.2 Theorem (Hoshi). *Let R be an excellent local ring of characteristic p, and A an R-algebra essentially of finite type. Then the F-pure locus of A is Zariski open in* $\operatorname{Spec} A$.

The following is an "F-pure base change" of strong F-regularity, which is stronger than the smooth base change.

3.3 Theorem. *Let $\varphi : A \to B$ be a homomorphism of noetherian rings of characteristic p. Assume that A is a strongly F-regular domain. Assume that the generic fiber $Q(A) \otimes_A B$ is strongly F-regular, where $Q(A)$ is the field of fractions of A. If φ is F-pure and B is locally excellent, then B is strongly F-regular.*

4. Matijevic–Roberts type theorem

M. Miyazaki and the author proved the following form of Matijevic–Roberts type theorem.

4.1 Theorem. *Let S be a scheme, G a smooth S-group scheme of finite type, X a noetherian G-scheme, $y \in X$, $Y := \overline{\{y\}}$, Y^* the smallest G-stable closed subscheme of X containing Y. Let η be the generic point of an irreducible component of Y^*. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings. Assume:*

1. (Smooth base change) *If $A \to B$ is a regular (i.e., flat with geometrically regular fibers) local homomorphism essentially of finite type and $A \in \mathcal{C}$, then $B \in \mathcal{D}$.*
2. (Flat descent) *If $A \to B$ is a regular local homomorphism essentially of finite type and $B \in \mathcal{D}$, then $A \in \mathcal{D}$.*

If $\mathcal{O}_{X,\eta} \in \mathcal{C}$, then $\mathcal{O}_{X,y} \in \mathcal{D}$.

4.1 Corollary. *Let R be a $\mathbb{Z}^n$-graded noetherian ring, and $P \in \operatorname{Spec} R$. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of noetherian local rings which satisfy 1 and 2 in the theorem. If $R_{P^*} \in \mathcal{C}$, then $R_P \in \mathcal{D}$, where P^* is the prime ideal of R generated by the all homogeneous elements in P.*

The smooth base change holds for F-purity (Lemma 2.7), strong F-regularity (Theorem 3.3), and Cohen–Macaulay F-injectivity (Theorem 2.1). Flat descent also holds, and we have the following.

4.2 Corollary. *Let R be a $\mathbb{Z}^n$-graded noetherian ring of characteristic p. Let P be a prime ideal of R, and P^* the prime ideal generated by the all homogeneous elements of P.*

1. *If R_{P^*} is F-pure, then R_P is F-pure.*
2. *If R_{P^*} is excellent and strongly F-regular, then R_P is strongly F-regular.*
3. *If R_{P^*} is Cohen–Macaulay F-injective, then R_P is Cohen–Macaulay F-injective.*

For applications, see [5].

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Regularity of the symbolic powers of space monomial curves

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1 Introduction

This is a joint work [5] with S. D. Cutkosky (University of Missouri).

Suppose that K is an algebraically closed field. Let $S = K[x_1, \ldots, x_m]$ be a polynomial ring, graded by a weighting $\mathrm{wt}(x_i) > 0$ for $0 \leq i \leq m$. Set $N = (x_1, \ldots, x_m)$. For a finitely generated graded S-module $M \neq 0$, we define the regularity of M by

$$\mathrm{reg}(M) = \max\{j + i \mid H^i_N(M)_j \neq 0\}.$$

For a homogeneous ideal J of S, we define

$$s(J) = \lim_{n\to\infty} \frac{\mathrm{reg}((J^n)^{\mathrm{sat}})}{n},$$

where $(J^n)^{\mathrm{sat}} = \cup_t (J^n :_S N^t)$ is the saturation of J^n.

It has very interesting properties as in

Theorem 1 (Cutkosky, Ein and Lazarsfeld [3]) *1. The limit $s(J)$ exists.*

2. The reciprical, $\frac{1}{s(J)}$, coincides with the Seshadri constant of the blow up along J.

The Seshadri constant is a very important invariant both in algebraic geometry and in commutative ring theory as in Remark 3.

In this note, we deal with the following two kinds of ideals.

Definition 2
- $I(H)$ denotes the defining ideal of H, where H is a set consisting of r closed points in $\mathbb{P}^2_K$.
- $I(a, b, c)$ denotes the defining ideal of the space monomial curve (t^a, t^b, t^c), where a, b, c are pairwise coprime positive integers.

Remark 3

- We have $s(I(H)) \geq \sqrt{r}$ in general.

  Assume that $r \geq 10$ and $d \leq \sqrt{r}n$. Let H be a set consisting of independent generic r points in $\mathbb{P}^2_{\mathbb{C}}$. In [12], Nagata conjectured that $[I(H)^{(n)}]_d = 0$. Nagata's conjecture is true if and only if $s(I(H)) = \sqrt{r}$.

  Nagata proved this conjecture [12] if r is a square. Using this result, he constructed the first counterexample to Hilbert's fourteenth problem.

- We have $s(I(a,b,c)) \geq \sqrt{abc}$. There is no negative curve if and only if $s(I(a,b,c)) = \sqrt{abc}$.

  Assume that $\sqrt{abc} \notin \mathbb{Q}$. If the symbolic Rees ring $R_s(I(a,b,c))$ is Noetherian, then there exists a negative curve [1]. The converse is also true when the characteristic of K is positive [1].

  Goto-Nishida-Watanabe [8] proved that there exists infinitely generated symbolic Rees ring $R_s(I(a,b,c))$ in the case where the characteristic of K is zero. In their example, there exists a negative curve.

  There is no example of (a,b,c) such that there is no negative curve except for $(a,b,c) = (1,1,1)$. We refer the reader to [11] for some observations for existence of negative curves.

  By Huneke's criterion [9], the symbolic Rees ring $R_s(I(a,b,c))$ is Noetherian if and only if there exist $f \in [I(a,b,c)^{(n_1)}]_{d_1}$ and $g \in [I(a,b,c)^{(n_2)}]_{d_2}$ such that

  $$\ell\left(S/(x,f,g)\right) = n_1 n_2 \ell\left(S/(x) + I(a,b,c)\right).$$

  In this case, $s(I(a,b,c)) = \max\{d_1/n_1, d_2/n_2\}$. If $d_1/n_1 > d_2/n_2$, then g is the equation of the image of the negative curve.

As in Remark 3, the Seshadri constant is deeply related to finite generation of symbolic Rees rings.

Motivation. We want to know $s(I(H))$ (or $s(I(a,b,c))$) using regularity. In order to do so, we investigate the asymptotic behavior of $\mathrm{reg}((I(H)^n)^{\mathrm{sat}})$ (or $\mathrm{reg}((I(a,b,c)^n)^{\mathrm{sat}})$).

The following is the main theorem in this note.

Theorem 4 *Let I be an ideal of the form $I(H)$ or $I(a,b,c)$. Set $\sigma_I(n) = \mathrm{reg}((I^n)^{\mathrm{sat}}) - \lfloor n \cdot s(I) \rfloor$.*

1. *$\sigma_I(n)$ is a bounded function on n.*
2. *Assume that $K \supset \mathbb{Q}$ or $K = \overline{\mathbb{F}_p}$. If $s(I) > \sqrt{r}$, then $\sigma_I(n)$ is an eventually periodic function on n. (Here, $r = \#H$ when $I = I(H)$, $r = abc$ when $I = I(a,b,c)$.)*

If the Rees ring $\oplus_{n\geq 0}(I^n)^{\text{sat}}$ is Noetherian, this theorem is easy to prove.

We do not know any homogeneous ideal J such that the function $\sigma_J(n)$ is not bounded.

In the rest, we assume that $S = K[x_1, \ldots, x_m]$ is a standard graded polynomial ring, that is, each variable has degree 1. We refer the reader to [5] for regularity over weighted polynomial rings.

2 a_i-invariant of powers of ideals

For a finitely generated graded S-module $M \neq 0$, we define

$$\begin{aligned} a_i(M) &= \max\{j \in \mathbb{Z} \mid H_N^i(M)_j \neq 0\} \text{ or } -\infty, \\ b_i(M) &= \max\{j \in \mathbb{Z} \mid \operatorname{Tor}_i^S(M, S/N)_j \neq 0\} \text{ or } -\infty. \end{aligned}$$

Then it is well-known that

$$\operatorname{reg}(M) = \max\{a_i(M) + i \mid 0 \leq i \leq \dim M\} = \max\{b_i(M) - i \mid 0 \leq i \leq \dim S\}.$$

Let J be a homogeneous ideal of S. Then, Cutkosky, Herzog and Trung [4], Kodiyalam [10] proved the following:

Theorem 5 *For each i, $b_i(J^n)$ is an eventually linear function on n. In particular, $\operatorname{reg}(J^n)$ is also eventually linear on n.*

By this theorem, we know that $b_i(J^n)$ is always eventually linear, but $a_i(J^n)$ is not always so as in Example 6 below. In the rest, we assume that $m \geq 2$.

Since

$$a_i((J^n)^{\text{sat}}) = \begin{cases} 0 & (i = 0, 1) \\ a_i(J^n) & (i \geq 2), \end{cases}$$

we obtain

$$\operatorname{reg}((J^n)^{\text{sat}}) = \max\{a_i(J^n) + i \mid i \geq 2\} \leq \max\{a_i(J^n) + i \mid i \geq 1\} = \operatorname{reg}(J^n).$$

If J is generated by homogeneous elements $f_1, \ldots, f_t$, then

$$\min\{\deg(f_i) \mid i\} \leq \lim_{n\to\infty} \frac{\operatorname{reg}(J^n)}{n} \leq \max\{\deg(f_i) \mid i\}. \tag{1}$$

by a proof of a theorem of Cutkosky, Herzog and Trung [4].

Example 6 Recall $\operatorname{reg}((J^n)^{\text{sat}}) = \max\{a_i(J^n)+i \mid i \geq 2\}$. If $a_i(J^n)$ were eventually linear for all i, then there would be i such that $\operatorname{reg}((J^n)^{\text{sat}}) = a_i(J^n) + i$ for $n \gg 0$ and, therefore, the limit $\lim_{n\to\infty} \frac{\operatorname{reg}((J^n)^{\text{sat}})}{n}$ would be an integer. However, there are examples that the limits are not integers as follows. (Therefore we know that $a_i(J^n)$ is not always eventually linear on n.)

(1) Consider the ring homomorphism $\varphi : B = K[y_1, y_2, y_3] \to S = K[x_1, x_2, x_3]$ such that $\varphi(y_1) = x_1^a$, $\varphi(y_2) = x_2^b$, $\varphi(y_3) = x_3^c$. We set $\deg(y_1) = a$, $\deg(y_2) = b$, $\deg(y_3) = c$ and $\deg(x_1) = \deg(x_2) = \deg(x_3) = 1$.

We think $I(a, b, c)$ as an ideal of B. Set $J = I(a, b, c)S$. As in [5], we have $s(J) = s(I(a, b, c))$.

For example, suppose that $(a, b, c) = (10, 11, 13)$. Then, there exist $f \in [I(a, b, c)]_{33}$ and $g \in [I(a, b, c)^{(3)}]_{130}$ satisfying Huneke's criterion. Since $33/1 < 130/3$, we know

$$\lim_{n\to\infty} \frac{\operatorname{reg}((J^n)^{\mathrm{sat}})}{n} = s(J) = s(I(a, b, c)) = \frac{130}{3} \notin \mathbb{Z}.$$

(2) There exists some example that $\lim_{n\to\infty} \frac{\operatorname{reg}((J^n)^{\mathrm{sat}})}{n}$ is irrational [2].

Remark that, if Nagata's conjecture is true, then $s(I(H)) = \sqrt{r}$. When r is not a square, $\sqrt{r}$ is irrational.

Here, we consider the ideal $I(H)$. Remark that

$$a_3(I(H)^n) + 3 = a_3(S) + 3 = 0 \quad \text{and} \quad \operatorname{reg}((I(H)^n)^{\mathrm{sat}}) > 0.$$

Therefore, in this case, we have

$$\begin{aligned} \operatorname{reg}((I(H)^n)^{\mathrm{sat}}) &= a_2(I(H)^n) + 2, \\ \operatorname{reg}(I(H)^n) &= \max\{a_1(I(H)^n) + 1, a_2(I(H)^n) + 2\}. \end{aligned}$$

Example 7 (1) Let H be a set of independent generic 16 points in $\mathbb{P}^2_{\mathbb{C}}$. Since Nagata's conjecture is true in this case [12], we have

$$\lim_{n\to\infty} \frac{\operatorname{reg}((I(H)^n)^{\mathrm{sat}})}{n} = s(I(H)) = \sqrt{16} = 4.$$

However, it is easy to see $[I(H)]_4 = 0$ and $[I(H)]_5 \neq 0$ since $\dim S_4 = \binom{4+2}{2} = 15$ and $\dim S_5 = \binom{5+2}{2} = 21$. So, we have

$$\lim_{n\to\infty} \frac{\operatorname{reg}(I(H)^n)}{n} \geq 5$$

by the inequality (1). Hence, $\lim_{n\to\infty} \frac{\operatorname{reg}((I(H)^n)^{\mathrm{sat}})}{n}$ does not satisfy the inequalities like (1).

(2) We consider H with multiplicity at each point. Then, there exists some example that $\operatorname{reg}((I(H)^n)^{\mathrm{sat}})$ is not an eventually quasi-linear polynomial function on n (Cutkosky-Herzog-Trung [4]). In this example, $a_2(I(H)^n)$ is not an eventually quasi-linear polynomial function on n. In this example, K is a field of positive characteristic that is transcendental over the prime field $\mathbb{F}_p$.

By this example, we guess that (2) of Theorem 4 would be false if we remove the assumption on K.

3 Proof of Theorem 4 (1) in the case $I(H)$

Set $H = \{p_1, \dots, p_r\}$. Let I_{p_i} be the defining ideal of p_i in $S = K[x_0, x_1, x_2]$. Put $N = (x_0, x_1, x_2)$. Then, I_{p_i} is generated by a regular sequence of length 2 and

$$I(H) = I_{p_1} \cap \cdots \cap I_{p_r}.$$

For the simplicity of notation, we denote $I(H)$ by I.

For each $n > 0$,

$$(I^n)^{\mathrm{sat}} = {I_{p_1}}^n \cap \cdots \cap {I_{p_r}}^n.$$

Since $\mathrm{reg}((I^n)^{\mathrm{sat}}) > 0 = a_3(S) + 3 = a_3(I^n) + 3$, we have

$$\mathrm{reg}((I^n)^{\mathrm{sat}}) = a_2(I^n) + 2.$$

Let $\pi : X \longrightarrow \mathbb{P}^2_K$ be the blow-up at H. Set $E_i = \pi^{-1}(p_i)$ and $E = E_1 + \cdots + E_r$. Let A be a Weil divisor on X such that $\mathcal{O}_X(A) = \pi^*(\mathcal{O}_{\mathbb{P}}(1))$. Then,

$$A^2 = 1, \quad E_i.E_j = -\delta_{ij}, \quad A.E_i = 0$$

is satisfied. Since $H^2_N(I^n)_d = H^1(X, \mathcal{O}_X(dA - nE))$, we have

$$\mathrm{reg}((I^n)^{\mathrm{sat}}) = \max\{d \mid H^1(X, \mathcal{O}_X(dA - nE)) \neq 0\} + 2.$$

In order to prove Theorem 4 (1), it is enough to show the following:

(I) $\exists t_0 \in \mathbb{N}, \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0, \forall d > s(I) \cdot n + t_0, H^1(X, \mathcal{O}_X(dA - nE)) = 0$.

(II) $\exists t_1 \in \mathbb{N}, \exists n_1 \in \mathbb{N}$ such that $\forall n > n_1, \exists d > s(I) \cdot n - t_1$ such that $H^1(X, \mathcal{O}_X(dA - nE)) \neq 0$.

Assume that both (I) and (II) are true. Then

$$s(I) \cdot n - t_1 < a_2(I^n) \leq s(I) \cdot n + t_0$$

for $n > \max\{n_0, n_1\}$. Therefore, $(a_2(I^n) + 2) - \lfloor s(I) \cdot n \rfloor$ is bounded.

First, we shall prove (I). The following Lemma is very important in this proof.

Lemma 8 (Fujita [7]) *Let Y be a projective variety over K and $\mathcal{M}$ be an ample line bundle on Y. Let $\mathcal{F}$ be a coherent sheaf on Y. Then, there exists $\mu \in \mathbb{N}$ such that, for all $i > 0$ and for any nef line bundle $\mathcal{L}$,*

$$H^i(Y, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{M}^{\otimes \mu}) = 0$$

is satisfied.

Since $\frac{1}{s(I)}$ coincides with the Seshadri constant,

- $dA - nE$ is ample iff $d > s(I) \cdot n$,
- $dA - nE$ is nef iff $d \geq s(I) \cdot n$.

Using Lemma 8, it is easy to show (I).

Next, we shall prove (II). Consider the following two cases:

Case 1. Assume that $s(I) = \sqrt{r}$.
In this case, we put $\alpha(n) = n\sqrt{r} - \lfloor n\sqrt{r} \rfloor + 2$. Then,

$$\begin{aligned} & \chi(\mathcal{O}_X((n\sqrt{r} - \alpha(n))A - nE)) \\ = \ & \frac{n\sqrt{r}}{2}(3 - \sqrt{r} - 2\alpha(n)) + \frac{1}{2}(\alpha(n)^2 - 3\alpha(n) + 2). \end{aligned}$$

Since $3 - \sqrt{r} - 2\alpha(n) < 0$, we have

$$-h^1((n\sqrt{r} - \alpha(n))A - nE) \leq \chi((n\sqrt{r} - \alpha(n))A - nE) < 0$$

for $n \gg 0$. In particular, $h^1 \neq 0$.

Case 2. Assume $s(I) > \sqrt{r}$.
We need the following lemma. We omit a proof.

Lemma 9 *There exists a curve C on X such that* $(s(I)A - E).C = 0$.

Corollary 10 $s(I) \in \mathbb{Q}$

Here, we set

$$\alpha(n) = \max\left\{0, \lceil \frac{1 - p_a(C)}{(C.A)} \rceil\right\} + s(I) \cdot n - \lfloor s(I) \cdot n \rfloor.$$

Then, we can prove $h^1((s(I) \cdot n - \alpha(n))A - nE) \neq 0$. **Q.E.D.**

Remark 11 In order to prove (2) of Theorem 4, we need the assumption that $K \supset \mathbb{Q}$ or $K = \overline{\mathbb{F}_p}$.

The key point is the following: Let C be a 1-dimensional projective scheme over K. Let $\mathcal{L}$ be an invertible sheaf on C with $\deg(\mathcal{L}) = 0$.

- If $K = \overline{\mathbb{F}_p}$, then $\exists \mu > 0$ such that $\mathcal{L}^{\otimes \mu} \simeq \mathcal{O}_C$.
- If $K \supset \mathbb{Q}$, then $h^0(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is eventually periodic on n for any coherent sheaf $\mathcal{F}$ (Cutkosky-Srinivas [6]).
- If K is transcendental over $\mathbb{F}_p$, then the eventual periodicity of $h^0(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is false as in [6].

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ARITHMETICAL RANK OF COHEN–MACAULAY SQUAREFREE MONOMIAL IDEALS OF HEIGHT TWO

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1. Introduction

Let S be a polynomial ring over a field K and I a squarefree monomial ideal of S. The *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min\left\{r \ : \ \text{there exist } a_1, \ldots, a_r \in S \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\right\}.$$

By the result of Lyubeznik [9], we have the following inequalities:

$$\operatorname{height} I \leq \operatorname{pd}_S S/I \leq \operatorname{ara} I,$$

where $\operatorname{pd}_S S/I$ denotes the projective dimension of S/I over S. We are interested in the problem when $\operatorname{ara} I = \operatorname{pd}_S S/I$ holds. Some classes of ideals those satisfy this equality are found in e.g., [1], [2], [4], [6], [7], [8], [10], [11]. If $\operatorname{ara} I = \operatorname{height} I$ holds, then I is said to be *set-theoretic complete intersection.* When this is the case, S/I is Cohen–Macaulay.

The main result of this report is the following theorem.

Theorem 1.1. *Let $I \subset S$ be a squarefree monomial ideal of height* 2. *Suppose that S/I is Cohen–Macaulay. Then*

$$\operatorname{ara} I = \operatorname{pd}_S S/I = 2.$$

In particular, I is set-theoretic complete intersection.

Note that there exist Cohen–Macaulay squarefree monomial ideals I of height 3, founded by Yan [12], Kimura, Terai and Yoshida [7], which do not satisfy $\operatorname{ara} I = \operatorname{pd}_S S/I$ when $\operatorname{char} K \neq 2$.

In this report, after recalling some definitions and properties of Stanley–Reisner ideals (Section 2), we state the motivation and pose questions (Section 3; Problem 3.5). In Section 4, we give answers for these questions (Propositions 4.1 and 4.2). In particular, Proposition 4.2 is the key lemma on the proof of Theorem 1.1, though we do not state the detailed proof of Theorem 1.1 in this report; see [5].

2. Preliminaries

In this section, we recall some definitions and properties of Stanley–Reisner ideals, especially Alexander duality.

Let Δ be a simplicial complex on the vertex set $X = \{x_1, \ldots, x_n\}$. That is, Δ is a collection of subsets of X satisfying the conditions (i) $\{x_i\} \in \Delta$ for all $i = 1, \ldots, n$; (ii) If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. If Δ consists of all subsets of X, then Δ is called a *simplex.* An elements of Δ is called a *face* of Δ, and it is called a *facet* of Δ if it is maximal among faces of Δ with respect to inclusion. The *dimension* of Δ is defined by $\dim \Delta := \max\{|F| - 1 : F \in \Delta\}$, where $|F|$ denotes the cardinality of F. The *Alexander dual complex* of Δ is a simplicial complex defined by $\Delta^* := \{F \subset X : X \setminus F \notin \Delta\}$. If $\dim \Delta < n-2$, then the vertex set of Δ^* is also X.

For a simplicial complex Δ on the vertex set $X = \{x_1, \dots, x_n\}$, we can associate a squarefree monomial ideal of $K[X] = K[x_1, \dots, x_n]$ by

$$I_\Delta = (x_{i_1} \cdots x_{i_s} \ : \ 1 \le i_1 < \cdots < i_s \le n, \ \{x_{i_1}, \dots, x_{i_s}\} \notin \Delta),$$

which is called the *Stanley–Reisner ideal* of Δ. The quotient ring $K[\Delta] := K[X]/I_\Delta$ is called the *Stanley–Reisner ring* of Δ. Conversely, for a squarefree monomial ideal I of $S = K[X]$ with $\operatorname{indeg} I := \min\{q : I_q \ne 0\} \ge 2$, there exists a simplicial complex Δ on X such that $I = I_\Delta$. In fact, this complex is given by $\Delta = \{F \subset X : m_F \notin I\}$, where $m_F = \prod_{x_i \in F} x_i$. Then the minimal prime decomposition of I is

$$I = I_\Delta = \bigcap_{F \in \Delta:\ \text{facet}} P_F,$$

where $P_F = (x_i : x_i \in X \setminus F)$. Moreover we assume that $\operatorname{height} I \ge 2$. The ideal $I^* := I_{\Delta^*}$ is called the *Alexander dual ideal* of I. Since $\Delta^{**} = \Delta$, we have $I^{**} = I$. The minimal set of monomial generators of I^* is

$$G(I^*) = \{m_{X \setminus F} \ : \ F \text{ is a facet of } \Delta\}.$$

Then it is easy to see that $\operatorname{indeg} I^* = \operatorname{height} I$. Moreover, Eagon–Reiner [3] proved that I^* has a linear resolution if and only if S/I is Cohen–Macaulay.

3. Motivation

In this section, we state our motivation. First, we survey results due to Barile and Terai [2], which are our starting point.

Let Γ be a simplicial complex on the vertex set $X = \{x_1, \dots, x_n\}$. Take an arbitrary face $F \in \Gamma$ and a new vertex x_0. The *cone from x_0 over F* is a simplex on the vertex set $F \cup \{x_0\}$. We denote it by $\mathrm{co}_{x_0} F$. Then $\Gamma' = \Gamma \cup \mathrm{co}_{x_0} F$ is a simplicial complex on the vertex set $X' := X \cup \{x_0\}$.

Barile and Terai [2] investigated the relations between arithmetical ranks of I_Γ and $I_{\Gamma'}$, and proved the following theorem.

Theorem 3.1 (Barile and Terai [2, Theorem 1, Theorem 2]). *We use notations as above.*

(1) $\operatorname{ara} I_{\Gamma'} \le \max\{\operatorname{ara} I_\Gamma + 1, n - |F|\}$.
(2) *If* $\operatorname{ara} I_\Gamma = \operatorname{pd} K[\Gamma]$ *holds, then* $\operatorname{ara} I_{\Gamma'} = \operatorname{pd} K[\Gamma']$ *also holds.*

As an application of Theorem 3.1 (2), they proved the following result, which was first proved by Morales on the different way.

Theorem 3.2 (Morales [10]). *Let I be a squarefree monomial ideal of S. If I has a 2-linear resolution, then*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

Remark 3.3. In fact, Barile and Terai [2] (resp. Morales [10]) proved Theorems 3.1 and 3.2 (resp. Theorem 3.2) with the assumption that the base field K is algebraically closed. But the author proved these theorems without this assumption by improving the proof due to Barile and Terai; see [5, Section 5].

For simplicial complexes, the notion of the *generalized tree* is defined by inductively: (i) a simplex is a generalized tree; (ii) if Γ is a generalized tree on the vertex set X, then for any face $F \in \Gamma$ and for any new vertex x_0, the simplicial complex $\Gamma \cup \mathrm{co}_{x_0} F$ on the vertex set $X \cup \{x_0\}$ is a generalized tree. Then we have the following lemma.

Lemma 3.4 (Barile and Terai [2, Lemma 2]). *Let Γ be a simplicial complex which is not a simplex. Then I_Γ has a 2-linear resolution if and only if Γ is a generalized tree.*

By virtue of Lemma 3.4, one can proceed the proof of Theorem 3.2 by indiction on $|X|$.

The assumption of Theorem 3.2, that is, I has a 2-linear resolution, is equivalent to that height $I^* = 2$ and S/I^* is Cohen–Macaulay, those are the assumption of Theorem 1.1. Then it is natural to think that if the Alexander dual of Theorem 3.1 holds, then Theorem 1.1 holds. Therefore we consider the Alexander dual of the above argument.

Let Δ be a simplicial complex on the vertex set $X = \{x_1, \dots, x_n\}$ with $\dim \Delta < n-2$. Set $\Gamma = \Delta^*$ and take a face $F \in \Gamma$ and a new vertex x_0. We set $\Gamma' = \Gamma \cup \mathrm{co}_{x_0} F$ as above, and $\Delta' = (\Gamma')^*$. We consider the following problem.

Problem 3.5. *We use the notations as above.*

(1) *Are there any relations between arithmetical ranks I_Δ and $I_{\Delta'}$?*
(2) *If $\mathrm{ara}\, I_\Delta = \mathrm{pd}\, K[\Delta]$ holds, then does $\mathrm{ara}\, I_{\Delta'} = \mathrm{pd}\, K[\Delta']$ hold?*

4. Answers for Problem 3.5

In this section, we give answers for Problem 3.5.

For Problem 3.5 (1), we obtain the following proposition.

Proposition 4.1. *We use the notations in the previous section. Then*

$$\mathrm{ara}\, I_{\Delta'} \le \mathrm{ara}\, I_\Delta + 1.$$

Proof. Set $S = K[X]$ and $S' = K[X']$. Let $G(I_\Delta) = \{m_1, \dots, m_\mu\}$ be the minimal set of monomial generators of I_Δ. Then the prime decomposition of $I_\Gamma = I_{\Delta^*}$ is

$$I_\Gamma = I_{\Delta^*} = \bigcap_{j=1}^{\mu} P_j \subset S, \tag{4.1}$$

where $P_j = (x_i \in X : x_i \text{ divides } m_j) \subset S$. Hence $I_{\Gamma'}$ can be written as

$$I_{\Gamma'} = P_0 \cap \left(\bigcap_{j=1}^{\mu} (P_j S' + (x_0)) \right) \subset S', \tag{4.2}$$

where $P_0 = P_{F\cup\{x_0\}} = (x_i : x_i \in X \setminus F) \subset S'$. Therefore

$$I_{\Delta'} = I_{(\Gamma')^*} = (m_0, x_0 m_1, \dots, x_0 m_\mu) \ \subset S',$$

where $m_0 = \prod_{x_i \in X\setminus F} x_i$.

Suppose that $q_1, \dots, q_h$ generate I_Δ up to radical. Then $x_0 q_1, \dots, x_0 q_h$ generate $x_0 I_\Delta$ up to radical. Therefore $m_0, x_0 q_1, \dots, x_0 q_h$ generate $I_{\Delta'}$ up to radical. This implies the desired inequality. □

Next, we provide a partial answer for Problem 3.5 (2).

Proposition 4.2. *We use the notations as above. If* $\operatorname{ara} I_\Delta = \operatorname{pd} K[\Delta] = 2$ *hold, then* $\operatorname{ara} I_{\Delta'} = \operatorname{pd} K[\Delta'] = 2$ *also hold.*

Proof. Set $I = I_\Delta$, $I' = I_{\Delta'}$, $S = K[X]$, and $S' = K[X']$. Let $G(I) = \{m_1, \dots, m_\mu\}$ be the minimal set of monomial generators of I. Then

$$I' = (m_0, x_0 m_1, \dots, x_0 m_\mu).$$

Let G be a facet of Γ containing F. Since the prime decomposition of $I_\Gamma = I^*$ is (4.1), we may assume $P_G = P_1$. Then m_0 is divisible by m_1 since $P_G S' \subset P_0$.

It is easy to see that $\operatorname{height} I' \geq 2$. Therefore inequalities

$$2 \leq \operatorname{height} I' \leq \operatorname{pd}_{S'} S'/I' \leq \operatorname{ara} I'$$

hold. Hence it is sufficient to prove that $\operatorname{ara} I' \leq 2$.

Let $q_1, q_2 \in S$ be elements which generate I up to radical. Then $q_1, q_2 \in I$. Since $m_1 \in I = \sqrt{(q_1, q_2)}$, there exist an integer $\ell > 0$ and elements $a_1, a_2 \in S$ such that

$$m_1^\ell = a_1 q_1 + a_2 q_2.$$

We prove that

$$q_1' = x_0 q_1 - a_2 m_0, \qquad q_2' = x_0 q_2 + a_1 m_0$$

generate I' up to radical.

Set $J = (q_1', q_2')$. It is clear that $J \subset I'$. We prove the opposite inclusion.

Since

$$a_1 q_1' + a_2 q_2' = x_0(a_1 q_1 + a_2 q_2) = x_0 m_1^\ell,$$

we have $x_0 m_1^\ell \in J$. Thus $x_0 m_1 \in \sqrt{J}$. Since $x_0 m_0$ is divisible by $x_0 m_1$, we have

$$x_0^2 q_1 = x_0 q_1' + a_2 x_0 m_0 \in \sqrt{J}, \qquad x_0^2 q_2 = x_0 q_2' - a_1 x_0 m_0 \in \sqrt{J}.$$

Thus $x_0 q_1, x_0 q_2 \in \sqrt{J}$. Therefore $x_0 I = \sqrt{(x_0 q_1, x_0 q_2)} \subset \sqrt{J}$.

It remains to prove $m_0 \in \sqrt{J}$. By $x_0 q_1, x_0 q_2 \in \sqrt{J}$, we also have $a_1 m_0, a_2 m_0 \in \sqrt{J}$. Note that

$$m_0 m_1^\ell = m_0(a_1 q_1 + a_2 q_2) = q_1(a_1 m_0) + q_2(a_2 m_0) \in \sqrt{J}.$$

Since m_1 divides m_0, this implies $m_0 \in \sqrt{J}$ as required. □

Although Proposition 4.2 is a partial answer for Problem 3.5 (2), it is sufficient to prove Theorem 1.1. The proof of Theorem 1.1 is done as an application of Proposition 4.2 by induction on $|X|$ using Lemma 3.4; see [5, Section 4].

Example 4.3. Let Γ be the simplicial complex on the vertex set $X = \{x_1, \dots, x_5\}$ whose facets are $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}$. Then the prime decomposition of I_Γ is

$$I_\Gamma = (x_4, x_5) \cap (x_1, x_5) \cap (x_1, x_2, x_3).$$

Let Δ be the Alexander dual complex of Γ. Then

$$I_\Delta = (x_4x_5, x_1x_5, x_1x_2x_3).$$

We choose the face $F = \{x_1\} \in \Gamma$. Let x_0 be a new vertex. Then $\operatorname{co}_{x_0} F = \{x_0, x_1\}$ and facets of $\Gamma' = \Gamma \cup \operatorname{co}_{x_0} F$ are $\{x_1, x_0\}$, $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{x_4, x_5\}$. Thus the prime decomposition of $I_{\Gamma'}$ is

$$I_{\Gamma'} = (x_2, x_3, x_4, x_5) \cap (x_0, x_4, x_5) \cap (x_0, x_1, x_5) \cap (x_0, x_1, x_2, x_3).$$

Hence

$$I_{\Delta'} = (x_2x_3x_4x_5, x_0x_4x_5, x_0x_1x_5, x_0x_1x_2x_3).$$

In this case, $G = \{x_1, x_2, x_3\}$, $m_1 = x_4x_5$, and $m_0 = x_2x_3x_4x_5$ with the notations in the proof of Proposition 4.2.

It is easy to see that I_Δ is generated by

$$q_1 = x_1x_5, \quad q_2 = x_4x_5 + x_1x_2x_3$$

up to radical; see Schmitt and Vogel [11, Lemma, p. 249]. Since

$$m_1^2 = (x_4x_5)^2 = -x_2x_3x_4q_1 + x_4x_5q_2,$$

$I_{\Delta'}$ is generated by following 2 elements up to radical:

$$\begin{cases} q_1' = x_0q_1 - x_4x_5m_0 = x_0x_1x_5 - x_2x_3x_4^2x_5^2, \\ q_2' = x_0q_2 - x_2x_3x_4m_0 = x_0x_4x_5 + x_0x_1x_2x_3 - x_2^2x_3^2x_4^2x_5. \end{cases}$$

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H-VECTORS OF SIMPLICIAL COMPLEXES WITH SERRE'S CONDITIONS

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ABSTRACT. The study of h-vectors of simplicial complexes is an interesting research area in combinatorics as well as in combinatorial commutative algebra. On h-vectors of simplicial complexes, one of fundamental problems is their non-negativity. For example, a classical result of Stanley guarantees that h-vectors of Cohen–Macaulay complexes are non-negative. We study the non-negativity of h-vectors in terms of Serre's condition (S_r).

1. INTRODUCTION

Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over an infinite field K. Let $I \subset S$ be a graded ideal and $R = S/I$. The *Hilbert series* of R is the formal power series $F(R, \lambda) = \sum_{q=0}^{\infty} (\dim_K R_q)\lambda^q$, where R_q is the graded component of degree q of R. It is known that $F(R, \lambda)$ is a rational function of the form $(h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1-\lambda)^d$, where each h_i is an integer with $h_s \neq 0$ and where $d = \dim R$. The vector $(h_0(R), h_1(R), \dots, h_s(R)) = (h_0, h_1, \dots, h_s)$ is called the *h-vector* of R. We say that $R = S/I$ satisfies *Serre's condition* (S_r) if

$$\operatorname{depth} R_P \geq \min\{r, \dim R_P\}$$

for all graded prime ideals $P \supset I$ of S.

Let Δ be a simplicial complex on $[n] = \{1, 2, \dots, n\}$. Thus Δ is a collection of subsets of $[n]$ satisfying that (i) $\{i\} \in \Delta$ for all $i \in [n]$ and (ii) if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$. The squarefree monomial ideal $I_\Delta \subset S$ generated by all squarefree monomials $x_F = \prod_{i \in F} x_i \in S$ with $F \notin \Delta$ is called the *Stanley-Reisner ideal* of Δ. The ring $K[\Delta] = S/I_\Delta$ is the *Stanley-Reisner ring* of Δ. The vector $h(\Delta) = h(K[\Delta])$ is called the *h-vector* of Δ.

We say that Δ satisfies Serre's condition (S_r) if $K[\Delta]$ satisfies Serre's condition (S_r). It is not hard to see that Δ satisfies (S_r) if and only if, for every $F \in \Delta$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); K) = 0$ for $i < \min\{r-1, \dim \mathrm{lk}_\Delta(F)\}$, where $\tilde{H}_i(\Delta; K)$ is the reduced homology groups of Δ over a field K and where $\mathrm{lk}_\Delta(F) = \{G \subset [n] \setminus F : G \cup F \in \Delta\}$ is the *link* of Δ with respect to a face $F \in \Delta$ (see [Te, p. 454]). A homological characterization of (S_r) is also known. It is know that a $(d-1)$-dimensional simplicial complex Δ satisfies (S_r) with $r \geq 2$ if and only if $\dim(\mathrm{Ext}_S^{n-i}(K[\Delta], \omega_S)) \leq i - r$ for $i = 0, 1, \dots, d-1$, where ω_S is the canonical module of S (see [Scl, Lemma 3.2.1]).

We remark some basic facts. Every simplicial complex satisfies (S_1). On the other hand, for $r \geq 2$, simplicial complexes satisfying (S_r) are pure and strongly connected. (S_2) states that Δ is pure and $\mathrm{lk}_\Delta(F)$ is connected for all faces $F \in \Delta$ with $|F| < \dim \Delta$. (S_d) is equivalent to the famous Cohen–Macaulay property of simplicial complexes.

A classical result of Stanley [St1] guarantees that if Δ is Cohen–Macaulay (that is, if it satisfies (S_d)) then $h_k(\Delta)$ is non-negative for all k. We generalize this classical result in the following way.

Theorem 1.1. *If a simplicial complex Δ satisfies (S_r) then $h_k(\Delta) \geq 0$ for $k = 0, 1, \ldots, r$.*

We also study what happens if $h_k = 0$ for some $1 \leq k \leq r$. We get the next result.

Theorem 1.2. *Let Δ be a simplicial complex which satisfies (S_r). If $h_t(\Delta) = 0$ for some $1 \leq t \leq r$ then $h_k(\Delta) = 0$ for all $k \geq t$ and Δ is Cohen–Macaulay.*

It is known that, for all integers $2 \leq r < d$, there exists a $(d-1)$-dimensional simplicial complex Δ which satisfies Serre's condition (S_r) but $h_{r+1}(\Delta) < 0$ ([TY, Example 3.5]). Thus we cannot expect that all the h_k are non-negative. However, we proved the following weak non-negative property for $h_k(\Delta)$ with $k \geq r$.

Theorem 1.3. *If a simplicial complex Δ satisfies (S_r) then $\sum_{k \geq r} h_k(\Delta) \geq 0$.*

To prove the above theorems, we prove the following algebraic result which might be itself of interest. For a finitely generated graded S-module M, let

$$\operatorname{reg} M = \max\{j : \operatorname{Tor}_i(M, K)_{i+j} \neq 0 \text{ for some } i\}$$

be the *(Castelnuovo-Mumford) regularity* of M.

Theorem 1.4. *Let $r \geq 1$ be an integer. Let $I \subset S$ be a graded ideal and d the Krull dimension of $R = S/I$. Suppose that $\operatorname{reg}(\operatorname{Ext}_S^{n-i}(R, \omega_S)) \leq i - r$ for $i = 0, 1, \ldots, d-1$. There exists a linear system of parameters $\Theta = \theta_1, \ldots, \theta_d$ of R such that*

$$h_k(R) = \dim_K(R/\Theta R)_k \quad \textit{for } k \leq r.$$

In this paper, we will give proofs of Theorems 1.1 and 1.4. The whole proofs of the results can be found in [MT].

2. Proof of Theorems

We first introduce some lemmas. The next result is well-known (see [St2]).

Lemma 2.1. *Let $I \subset S$ be a homogeneous ideal, d the Krull dimension of $R = S/I$ and $\Theta = \theta_1, \ldots, \theta_d$ a linear system of parameters of R. If the multiplication map*

$$\times\theta_i : \big(R/(\theta_1, \ldots, \theta_{i-1})R\big)_j \to \big(R/(\theta_1, \ldots, \theta_{i-1})R\big)_{j+1}$$

is injective for all $i = 1, 2, \ldots, d$ and for all $j \leq r - 1$ then

$$h_j(R) = \dim_K(R/\Theta R)_j \textit{ for } j \leq r.$$

Let $H_i(\mathbf{y}; M)$ (respectively, $H^i(\mathbf{y}; M)$) be the i-th Koszul homology (respectively, Koszul cohomology) of M with respect to a sequence $\mathbf{y} = y_1, \ldots, y_\ell$. A key lemma is the next result due to Aramova and Herzog [AH, Theorem 1.1].

Lemma 2.2 (Aramova-Herzog). *Let M be a finitely generated graded S-module of Krull dimension d and $\mathbf{y} = y_1, \ldots, y_\ell$ a generic linear form. Then $H_i(y_1, \ldots, y_k; M)$ has finite length and $H_i(y_1, \ldots, y_k; M)_{i+j} = 0$ for $j > \operatorname{reg} M$ in the following cases:*

(i) *$i \geq 1$ and $k = 1, 2, \ldots, \ell$;*
(ii) *$i = 0$ and $k \geq d$.*

Let $H^i_{\mathfrak{m}}(M)$ be the i-th local cohomology module of M. Another key lemma is the next result due to Schenzel [Sc2, Sc3].

Lemma 2.3 (Schenzel). *Let M be a finitely generated graded S-module of Krull dimension d and $\mathbf{y} = y_1, \ldots, y_p$ generic linear forms. Then, for all $j \in \mathbb{Z}$,*

$$\dim_K H^0_{\mathfrak{m}}(M/(y_1, \ldots, y_p)M)_j \leq \sum_{\ell=0}^{p} \dim_K H_\ell\big(\mathbf{y}; H^\ell_{\mathfrak{m}}(M)\big)_j.$$

Proof. (Sketch) Let $C^\bullet$ be the Čech complex and $K_\bullet(\mathbf{y}; M)$ the Koszul complex of M with respect to $\mathbf{y}$. Define the double complex $D^{\bullet\bullet}$ such that $D^{s,t} = C^s \bigotimes_S K_{p-t}(\mathbf{y}; M)$. There are two spectral sequences (we follow the notation of [Ei, Section A3])

$$\begin{aligned} {}^2_{\text{vert}}E^{s,t} &= H^s_{\mathfrak{m}}\big(H_{p-t}(\mathbf{y}; M)\big) &\Rightarrow\; & H^{s+t}\big(\text{tot}(D^{\bullet\bullet})\big) \\ {}^2_{\text{hor}}E^{s,t} &= H_{p-t}\big(\mathbf{y}; H^s_{\mathfrak{m}}(M)\big) &\Rightarrow\; & H^{s+t}\big(\text{tot}(D^{\bullet\bullet})\big). \end{aligned}$$

By Lemma 1.4(i), $H_{p-t}(\mathbf{y}; M)$ has finite length if $t \neq p$. Thus, by the basic properties of local cohomology,

$${}^2_{\text{vert}}E^{s,t} = 0 \quad \text{if } (s,t) \notin \big\{(0,0), (0,1), \ldots, (0,p), (1,p), (2,p), \ldots, (d,p)\big\}$$

and

$${}^2_{\text{vert}}E^{0,t} = H_{p-t}(\mathbf{y}; M) \quad \text{for } t = 0, 1, \ldots, p-1.$$

Then this spectral sequence degenerates at ${}^2E$ and $H^p(\text{tot}(D^{\bullet\bullet})) \cong H^0_{\mathfrak{m}}(H_0(\mathbf{y}; M)) \cong H^0_{\mathfrak{m}}(M/(y_1, \ldots, y_p)M)$. Since $\dim_K H^p(\text{tot}(D^{\bullet\bullet}))_j \leq \dim_K(\bigoplus_{s+t=p}\, {}^2_{\text{hor}}E^{s,t})_j$ for all $j \in \mathbb{Z}$, we get the desired inequality. □

Proof of Theorem 1.4. Let $\mathbf{y} = y_1, \ldots, y_d$ be generic linear forms. By Lemma 2.1, it is enough to prove that the multiplication map

$$\times y_i : \big(R/(y_1, \ldots, y_{i-1})R\big)_j \to \big(R/(y_1, \ldots, y_{i-1})R\big)_{j+1}$$

is injective for all $i = 1, 2, \ldots, d$ and $j \leq r-1$. To prove this, it is enough to prove that

$$H^0_{\mathfrak{m}}(R/(y_1, \ldots, y_i)R)_j = 0$$

for all $i = 1, 2, \ldots, d$ and $j \leq r-1$. Then, by Lemma 2.3, it is enough to prove that

$$H_\ell\big(y_1, \ldots, y_i; H^\ell_{\mathfrak{m}}(R)\big)_j = 0$$

for all $\ell \leq i \leq d$ and $j \leq r-1$.

Fix $\ell \leq i \leq d$ and $j \leq r-1$. By the local duality and the self duality of the Koszul complex,

$$\begin{aligned} H_\ell\big(y_1, \ldots, y_i; H^\ell_{\mathfrak{m}}(R)\big)_j &\cong H_\ell\big(y_1, \ldots, y_i; H^\ell_{\mathfrak{m}}(R)\big)^\vee_{-j} \\ &\cong H^\ell\big(y_1, \ldots, y_i; \text{Ext}^{n-\ell}_S(R, \omega_S)\big)_{-j} \\ &\cong H_{i-\ell}\big(y_1, \ldots, y_i; \text{Ext}^{n-\ell}_S(R, \omega_S)\big)_{(i-\ell)+\ell-j}. \end{aligned}$$

Since $\ell - j > \ell - r$, Lemma 2.2(i) and the assumption $\text{reg}\,(\text{Ext}^{n-\ell}_S(R, \omega_S)) \leq \ell - r$ show that the above vector space vanishes when $i > \ell$. Similarly, the above vector space also vanishes when $i = \ell$ by Lemma 2.2(ii). □

Now, we prove Theorem 1.1.

Proof. We may assume $r \geq 2$ since $h_1(\Delta)$ is always non-negative. Fix an integer $1 \leq i \leq d-1$. Let $N^i = \mathrm{Ext}_S^{n-i}(K[\Delta], \omega_S)$. By Theorem 1.4, it is enough to prove that $\mathrm{reg}\, N^i \leq i - r$. It follows from the result of Yanagawa [Ya] and Mustaţă [Mu], each N^i is a squarefree S-module. Then we have $\mathrm{reg}\, N^i \leq \dim N^i$. On the other hand, Serre's condition (S_r) implies $\dim N^i \leq i - r$. Then the statement follows. □

Remark 2.4. The key observation in the above proof is that $K[\Delta]$ is a squarefree S-module. Indeed, the proof of Theorem 1.1 works if we replace $K[\Delta]$ with any squarefree S-module M.

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Cohen–Macaulaynees for symbolic power ideals of edge ideals [1]

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0. Introduction

This is a joint work with Giancarlo Rinaldo.

Throughout this report, let $G = (V(G), E(G))$ be a graph, that is, G is a simple graph without loops and multiple edges, and $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. We put $V(G) = \{x_1, \ldots, x_n\}$ unless otherwise specified, and put $S = K[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field K. Then the *edge ideal* of G, denoted by $I(G)$, is defined by

$$I(G) = (x_i x_j \;:\; \{x_i, x_j\} \in E(G))S.$$

Since $I(G)$ is a squarefree monomial ideal, it can be regarded as a Stanley–Reisner ideal. In particular, there exists a simplicial complex $\Delta(G)$ on $V(G)$ such that $I(G) = I_{\Delta(G)}$. Then $\Delta(G)$ is called the *complementary simplicial complex* of G.

A subset $C \subseteq V(G)$ is called a *vertex cover* of G if $C \cap \{x_i, x_j\} \neq \emptyset$ holds whenever $\{x_i, x_j\} \in E(G)$. Then the irredundant primary (prime) decomposition of $I(G)$ is given by

$$I(G) = \bigcap_{C} P_C,$$

where C moves through all vertex covers which is minimal with respect to inclusion, and P_C is a prime ideal defined by $P_C = (x_i \;:\; i \in C)$. In pariticular, for each $\ell \geq 1$, the ℓth *symbolic power ideal* of $I(G)$ is given by

$$I(G)^{(\ell)} = I(G)S_W \cap S = \bigcap_{C} P_C^{\ell},$$

where $W = S \setminus \bigcup_{P \in \mathrm{Min}_S(S/I)} P$.

The aim of this report is to discuss Cohen–Macaulayness (and (FLC) properties) for symbolic or ordinary powers of edge ideals.

[1]This is an extended abstract. The final version will be published elsewhere.

1. Cohen–Macaulayness for symbolic power ideals

The following theorem is known as a special case of Cowsik–Nori Theorem.

Theorem 1.1 (**Cowsik–Nori** [1]). *Let S be a polynomial ring over a field K, and let I be a homogeneous radical ideal of S. Then the following conditions are equivalent:*

(1) *I is complete intersection.*
(2) *S/I^ℓ is Cohen–Macaulay for every integer $\ell \geq 1$.*
(3) *S/I^ℓ is Cohen–Macaulay for infinitely many integers $\ell \geq 1$.*

In [3], we gave a refinement of Cowsik-Nori Theorem as follows:

Theorem 1.2 (See [3, Theorem 2.1] **with M.Crupi and G.Rinaldo**). *Let $I(G)$ denote the edge ideal of a graph G. If $S/I(G)^\ell$ is Cohen-Macaulay for some $\ell \geq$ height I, then $I(G)$ is complete intersection.*

The main purpose of this report is to give more detailed version of Cowsik–Nori theorem. In order to do that, we consider the following questions:

Question 1.3. *Let $\ell \geq 1$ be an integer. Let $I(G)$ be the edge ideal of a graph G. Then:*

(1) *When is $S/I(G)^{(\ell)}$ Cohen-Macaulay?*
(2) *When does $I(G)^{(\ell)} = I(G)^\ell$ hold?*

1.1. **When is $S/I(G)^{(\ell)}$ Cohen-Macaulay?** Let G be a graph on the vertex set $V = [n]$ such that $\dim S/I(G) = 1$. Such a graph G is isomorphic to the complete graph K_n, and its edge ideal of G is

$$I(G) = (x_i x_j \,:\, 1 \leq i < j \leq n) = \bigcap_{i=1}^{n} (x_1, \ldots, \widehat{x_i}, \ldots, x_n).$$

Then $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every integer $\ell \geq 1$ because the symbolic power ideal has no embedded primes.

Now let us consider a disjoint union of two complete graphs K_m and K_n. Put $S_1 = K[x_1, \ldots, x_m]$, $S_2 = K[y_1, \ldots, y_n]$ and $S = S_1 \otimes_K S_2 = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Then the edge ideal of $G = K_m \coprod K_n$ is given by

$$I(G) = (x_i x_j \,:\, 1 \leq i < j \leq m) + (y_i y_j \,:\, 1 \leq i < j \leq n).$$

Moreover, we have

$$I(G)^{(\ell)} = \prod_{i \in [1,m], j \in [1,n]} (P_i + Q_j)^\ell,$$

where $P_i = (x_1, \ldots, \widehat{x_i}, \ldots, x_m)S$ and $Q_j = (y_1, \ldots, \widehat{y_j}, \ldots, y_n)S$. Then one can show that $x_1 + \cdots + x_m$ and $y_1 + \cdots + y_n$ form a regular sequence on $S/I(G)^{(\ell)}$, that is, $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every $\ell \geq 1$. This fact can be generalized as the following main theorem in this report.

Theorem 1.4 (Cohen–Macaulayness for symbolic powers). *Let $I(G)$ denote the edge ideal of a graph G. Then the following conditions are equivalent:*

(1) *$S/I(G)^{(\ell)}$ is Cohen–Macaulay for every integer $\ell \geq 1$.*
(2) *$S/I(G)^{(\ell)}$ is Cohen–Macaulay for some integer $\ell \geq 3$.*
(3) *G is a disjoint union of finitely many complete graphs.*

We can replace Cohen–Macaulayness with Serre's condition (S_2) in (1) or (2) of the theorem. On the other hand, in (2), we cannot replace "$\ell \geq 3$" with "$\ell \geq 2$" as the next example shows.

Example 1.5. For the pentagon G, $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Put $S = K[x_1, x_2, x_3, x_4, x_5]$. Then $S/I(G)$ and $S/I(G)^{(2)}$ are Cohen–Macaulay but $S/I(G)^{(3)}$ is not.

Remark 1.6. Recently, we found many examples of graph G for which $S/I(G)^2 = S/I(G)^{(2)}$ is Cohen–Macaulay. Moreover, Nakamura told us that he proved $S/I(G)^3$ is Buchsbaum for the pentagon G.

1.2. **When $I(G)^{(\ell)} = I(G)^\ell$ hold?** In what follows, we suppose that $I(G)$ is unmixed. Then $I(G)^{(\ell)} = I(G)^\ell$ if and only if $S/I(G)^\ell$ is unmixed.

For this question, we can give a complete answer. The following proposition can be proved by a similar method as in the proof of Simis–Vasconcelos–Villarreal [10, Lemma 5.8, Theorem 5.9].

Proposition 1.7. *Let $I(G)$ denote the edge ideal of a graph G. Let $\ell \geq 2$ be an integer. Then $I(G)^{(\ell)} = I(G)^\ell$ holds if and only if G contains no odd cycles of length $2s-1$ for any $1 \leq s \leq \ell$.*

In particular, if G contains a triangle (i.e., a 3-cycle), then $I(G)^{(2)} \neq I(G)^2$.

As an application of Theorem 1.4, we can obtain some result for Cohen–Macaulayness of ordinary powers, which gives an improvement of the main theorem in [3].

Theorem 1.8 (Cohen–Macaulayness of ordinary powers). *Let $I(G)$ be the edge ideal of a graph G. Then the following conditions are equivalent:*

(1) *$S/I(G)^\ell$ is Cohen–Macaulay for every integer $\ell \geq 1$.*
(2) *$S/I(G)^\ell$ is Cohen–Macaulay for some integer $\ell \geq 3$.*
(3) *$I(G)$ is complete intersection, that is, G is a disjoint union of finitely many complete 2-graphs and points.*

Proof. It suffices to show (2) $\Longrightarrow$ (3). Now suppose that $S/I(G)^\ell$ is Cohen–Macaulay for some $\ell \geq 3$. Then $S/I(G)^{(\ell)}$ is Cohen–Macaulay and thus G is a disjoint union of finitely many complete graphs (say, $K_{n_1}, \ldots, K_{n_d}$). Hence, $S/I(G)^{(m)}$ is Cohen–Macaulay for all $m \geq 1$ by Theorem 1.4. In particular, $I(G)^{(2)} = I(G)^2$.

On the other hand, if $\max\{n_1, \ldots, n_d\} \geq 3$, then G contains a triangle. This contradicts Proposition 1.7. Hence $\max\{n_1, \ldots, n_d\} \leq 2$. In other words, $I(G)$ is complete intersection, as required. □

The Cohen–Macaulayness for $S/I(G)^{(\ell)}$ does *not* imply the Cohen–Macaulayness for $S/I(G)^{\ell}$.

Example 1.9. Let $I(G)$ denote the edge ideal of a graph G. If G is the disjoint union of the d complete 3-graphs, then for every $\ell \geq 2$, $S/I(G)^{(\ell)}$ is Cohen–Macaulay but $S/I(G)^{\ell}$ is *not*. Moreover, $\dim S/I(G) = d$.

2. FLC PROPERTIES OF SYMBOLIC POWERS

Let $R = S/I$ be a homogeneous K-algebra, and let $\mathfrak{m}$ be the unique homogeneous maximal ideal of R. The ring R is said to have (FLC) if the local cohomology modules $H^i_{\mathfrak{m}}(R)$ has finite length for all $i < \dim R$. Note that R has (FLC) if and only if R is equidimensional and R_P is Cohen-Macaulay for every prime $P \subsetneq \mathfrak{m}$ in this situation.

We recall the notion of locally complete intersection complex which was introduced in [11]. A simplicial complex Δ on the vertex set V is *locally complete intersection* if $K[\mathrm{link}_\Delta\{v\}]$ is complete intersection for every $v \in V$, where $\mathrm{link}_\Delta\{v\} = \{F \in \Delta : v \notin F, F \cup \{v\} \in \Delta\}$. Note that a simplicial complex Δ is a generalized complete intersection complex, which was introduced by Goto and Takayama in [4], if and only if it is pure and a locally complete intersection complex.

Goto and Takayama [4] proved an analogous result of Cowsik-Nori theorem for Stanley–Reisner ideals.

Theorem 2.1 (Goto–Takayama [4]). *Let Δ be a simplicial complex. Then the following conditions are equivalent:*

(1) *S/I_Δ^{ℓ} has (FLC) for all integers $\ell \geq 1$.*
(2) *S/I_Δ^{ℓ} has (FLC) for infinitely many integers $\ell \geq 1$.*
(3) *Δ is pure and a locally complete intersection complex.*

In this section, we will give a refinement of this theorem in case of edge ideals. When $\dim S/I(G) \leq 2$, if $S/I(G)$ is unmixed, then $S/I(G)^{(\ell)}$ is unmixed, and thus it has (FLC) for every integer $\ell \geq 1$. But when $\dim S/I(G) \geq 3$, we can classify all graphs for which $S/I(G)^{\ell}$ has (FLC) for some (every) $\ell \geq 3$.

For complete graphs $K_{n_1}, \ldots, K_{n_d}$, we set $\Delta_{n_1,\ldots,n_d} = \Delta(K_{n_1} \coprod \cdots \coprod \Delta_{n_d})$.

Theorem 2.2 (FLC for symbolic powers). *Let $I(G)$ denote the edge ideal of G, and $\Delta = \Delta(G)$ the complementary simplicial complex of G. Let p denote the number of connected components of Δ. Suppose that Δ is pure and $d = \dim S/I(G) \geq 3$. Then the following conditions are equivalent:*

(1) *$S/I(G)^{(\ell)}$ has (FLC) for every $\ell \geq 1$.*
(2) *$S/I(G)^{(\ell)}$ has (FLC) for some $\ell \geq 3$.*
(3) *There exists $(n_{i1}, \ldots, n_{id}) \in \mathbf{N}^d$ for every $i = 1, \ldots, d$ such that Δ can be written as*

$$\Delta = \Delta_{n_{11},\ldots,n_{1d}} \coprod \Delta_{n_{21},\ldots,n_{2d}} \coprod \cdots \coprod \Delta_{n_{p1},\ldots,n_{pd}}.$$

The next corollary immediately follows from Theorems 1.4 and 2.2, which remains still open in case of Stanley–Reisner ideals.

Corollary 2.3. *Suppose that $\ell \geq 3$ and $d = \dim S/I(G) \geq 3$. Then the following conditions are equivalent:*

(1) *$S/I(G)^{(\ell)}$ is Cohen–Macaulay.*
(2) *$S/I(G)^{(\ell)}$ has (FLC) and $\Delta(G)$ is connected.*

The following theorem gives a refinement of Goto–Takayama theorem in case of edge ideals.

Theorem 2.4 (FLC for ordinary powers). *Let $I(G)$ denote the edge ideal of a graph G. Suppose that $\Delta(G)$ is pure. Put $d = \dim S/I(G) \geq 2$. Then the following conditions are equivalent:*

(1) *$S/I(G)^{\ell}$ has (FLC) for every $\ell \geq 1$.*
(2) *$S/I(G)^{\ell}$ has (FLC) for some $\ell \geq 3$.*
(3) *$\Delta(G)$ is locally complete intersection complex.*
(4) *$\Delta(G)$ is a disjoint union of complete intersection complexes if $d \geq 3$; $\Delta(G)$ is a disjoint union of finitely many m-gons ($m \geq 4$) and m'-pointed paths ($m' \geq 3$) if $d = 2$.*

Proof. (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) is clear. The equivalence of (1) and (4) follows from [4]. On the other hand, (2) $\Rightarrow$ (4) follows from Theorem 1.8 by a similar argument as in [4]. □

Example 2.5. Let $I(G)$ be the edge ideal and $\Delta(G)$ the complementary simplicial complex of a graph G.

(1) If G is the pentagon, then $S/I(G)^3$ is not Cohen–Macaulay but it has (FLC).
(2) If $G = K_{d,d}$ is the complete bipartite d-graph, then

$$I(K_{d,d}) = (x_i y_j \;:\; 1 \leq i, j \leq d)$$

and $S/I(G)^{\ell}$ has (FLC) but not Cohen–Macaulay if $d \geq 2$.

(3) If $\Delta(G) = \Delta_{3,3,3} \coprod \Delta_{3,3,3}$, then $S/I(G)^{(\ell)}$ has (FLC) for every $\ell \geq 1$ but $S/I(G)^{\ell}$ does not have (FLC). In particular, $\Delta(G)$ is not a locally complete intersection complex.

$$\begin{array}{ccccc}
\boxed{S/I^{\ell} : \mathbf{CM}} & \Longrightarrow & \boxed{S/I^{\ell} : \mathbf{(FLC)}} & \Longrightarrow & \boxed{I : \mathbf{pure,\ LCI}} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\boxed{S/I^{(\ell)} : \mathbf{CM}} & \Longrightarrow & \boxed{S/I^{(\ell)} : \mathbf{(FLC)}} & \Longrightarrow & \boxed{S/I : \mathbf{Buchsbaum}}
\end{array}$$

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Gröbner bases of contraction ideals

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We investigate Gröbner bases of contractions of ideals under some monomial homomorphisms. Our theorem provides many examples that have square-free initial ideals or quadratic Gröbner bases. As a consequence of our theorem, we obtain a generalization of the theorem of Aoki–Hibi–Ohsugi–Takemura and Hibi-Ohsugi ([1], [7]).

Let $R = K[x_1, \dots, x_r]$ and $S = K[y_1, \dots, y_s]$ be polynomial rings over a field K, I an ideal of S, and $\phi : R \to S$ a ring homomorphism. We call the ideal $\phi^{-1}(I)$ the *contraction ideal* of I under ϕ. The one of the most important ring homomorphisms in combinatorics and algebraic statistics are monomial homomorphisms: Let $\mathcal{A} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)})$ be a $d \times r$ integral matrix with column vectors $\mathbf{a}^{(i)} = {}^t(a_1^{(i)}, \dots, a_d^{(i)}) \in \mathbb{Z}^d$, and consider the ring homomorphism

$$\begin{aligned} \phi_{\mathcal{A}} : \ S = K[y_1, \dots, y_r] \ &\to \ K[z_1^{\pm 1}, \dots, z_d^{\pm 1}] \\ x_i \ &\mapsto \ \mathbf{z}^{\mathbf{a}^{(i)}} = \prod_{j=1}^{d} z_j^{a_j^{(i)}}. \end{aligned}$$

We call $\phi_{\mathcal{A}}$ a *monomial homomorphism.* Using abusive notation, we sometime confound the matrix $\mathcal{A} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)})$ and the set $\{\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)}\}$ if $\mathbf{a}^{(i)} \neq \mathbf{a}^{(j)}$ for all $i \neq j$. We denote by $\mathbb{N}\mathcal{A} = \{\sum_i n_i \mathbf{a}^{(i)} \mid n_i \in \mathbb{N}\}$ the semigroup generated by column vectors of $\mathcal{A}$, and by $K[\mathcal{A}]$ the monomial K-algebra $K[\mathbf{z}^{\mathbf{a}^{(1)}}, \dots, \mathbf{z}^{\mathbf{a}^{(r)}}]$. The ideal $P_{\mathcal{A}} = \operatorname{Ker} \phi_{\mathcal{A}}$ is called a *toric ideal.* It is known that the toric ideal $P_{\mathcal{A}}$ is generated by binomials $u - v$ where u and v are monomials of S with $\phi_{\mathcal{A}}(u) = \phi_{\mathcal{A}}(v)$. We call $\mathcal{A}$ a *configuration* if there exists a vector $0 \neq \lambda = (\lambda_1, \dots, \lambda_d) \in \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q}) \cong \mathbb{Q}^d$ such that $\lambda \cdot \mathbf{a}^{(i)} = 1$ for all i. If $\mathcal{A}$ is a configuration, then $P_{\mathcal{A}}$ is a homogeneous ideal in the usual sense and some algebraic properties of $K[\mathcal{A}] \cong S/P_{\mathcal{A}}$ can be derived from Gröbner bases of $P_{\mathcal{A}}$: If $\operatorname{in}_{\prec}(P_{\mathcal{A}})$ is generated by square-free monomials, then $K[\mathcal{A}]$ is normal. For a homogeneous ideal $I \subset S$, if I has a quadratic Gröbner basis with respect to some term order, then S/I is a Koszul algebra, that is, the residue field K has a linear minimal graded free resolution. In this paper, we consider the next problem.

Problem 1. Let $\phi_{\mathcal{A}} : R \to S$ be a monomial homomorphism and $I \subset S$ a homogeneous ideal. Then is it possible to find properties of Gröbner bases of $\phi_{\mathcal{A}}^{-1}(I)$ from these of I and $P_{\mathcal{A}} = \operatorname{Ker} \phi_{\mathcal{A}}$?

In some cases if both of I and $\operatorname{Ker} \phi_{\mathcal{A}}$ admit square-free or quadratic initial ideals, then so does the contraction ideal $\phi_{\mathcal{A}}^{-1}(I)$.

Thorem 1. *Let $d > 0$ and $\lambda_i \in \mathbb{N}$ for $i \in [d]$ be integers, $\mathbb{Z}^d = \bigoplus_{i=1}^{d} \mathbb{Z}\mathbf{e}_i$ a free $\mathbb{Z}$-module of rank d with a basis $\mathbf{e}_1, \dots, \mathbf{e}_d$, and $S = K\left[y_j^{(i)} \mid i \in [d],\ j \in [\lambda_i]\right]$ a* *polynomial ring. We regard S as a $\mathbb{Z}^d$-graded ring with* $\deg y_j^{(i)} = \mathbf{e}_i$ *for* $i \in [d]$, $j \in [\lambda_i]$. *Let* $\mathcal{A} \subset \mathbb{N}^d = \bigoplus_{i=1}^d \mathbb{N}\mathbf{e}_i$ *be a configuration. We set*

$$\tilde{\mathcal{A}} = \{\mathbf{a} \in \bigoplus_{i=1}^{d} \mathbb{Z}^{\lambda_i} \mid \deg_{\mathbb{Z}^d} \mathbf{y}^{\mathbf{a}} \in \mathcal{A}\},$$

$R = K[x_{\mathbf{a}} \mid \mathbf{a} \in \tilde{\mathcal{A}}]$, *and* $\phi_{\tilde{\mathcal{A}}} : R \to S$, $x_{\mathbf{a}} \mapsto \mathbf{y}^{\mathbf{a}}$. *Let* $I \subset S$ *be a* $\mathbb{Z}^d$*-graded ideal. Then the following hold.*

(1) *If both of* I *and* $P_{\mathcal{A}}$ *admit Gröbner bases of degree at most* m, *then so is* $\phi_{\tilde{\mathcal{A}}}^{-1}(I)$.

(2) *If both of* I *and* $P_{\mathcal{A}}$ *admit square-free initial ideals, then so is* $\phi_{\tilde{\mathcal{A}}}^{-1}(I)$.

Note that $\tilde{\mathcal{A}}$ is a nested configuration defined in [1]. This theorem is a generalization of the theorem of Aoki–Hibi–Ohsugi–Takemura [1] and Hibi-Ohsugi [7], and contains the result of Sullivant [9] (toric fiber products). This theorem also contain classical results: If $d = 1$ and $\mathcal{A} = \{r\}$, then $\phi_{\tilde{\mathcal{A}}}^{-1}(I)$ is the defining ideals of r-th Veronese subring of S/I. If $d = 2$ and $I_1 \subset S_1 := K\left[y_j^{(1)} \mid j \in [\lambda_1]\right]$ and $I_1 \subset S_2 := K\left[y_j^{(2)} \mid j \in [\lambda_2]\right]$ are homogeneous ideals, then $\phi_{\tilde{\mathcal{A}}}^{-1}(I_1 S + I_2 S)$ is the defining ideal of the Segre product of S_1/I_1 and S_2/I_2.

Our interest is primarily in the case where I is a toric ideal. In this case, $\phi_{\tilde{\mathcal{A}}}^{-1}(I)$ is also a toric ideal. Let $\mathcal{A} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)})$ be an $s \times r$ matrix with $\mathbf{a}^{(i)} \in \mathbb{N}^s$, and $\mathcal{B} = (\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(s)})$ a $d \times s$ matrix with $\mathbf{b}^{(j)} \in \mathbb{Z}^d$. Then we obtain monomial homomorphisms

$$R \xrightarrow{\phi_{\mathcal{A}}} S \xrightarrow{\phi_{\mathcal{B}}} K[z_1^{\pm 1}, \dots, z_d^{\pm 1}].$$

Note that the composition of monomial homomorphism is also monomial homomorphism defined by the product of the matrices; $\phi_{\mathcal{B}} \circ \phi_{\mathcal{A}} = \phi_{\mathcal{B} \cdot \mathcal{A}}$. In this case, Theorem 1 can be rephrased as follow.

Thorem 2. *Let the notation be as in Theorem 1. Let* μ *be a positive integer,* $\mathcal{B}_i = \{\mathbf{b}_j^{(i)} \mid j \in [\lambda_i]\}$ *a configuration of* $\mathbb{Z}^\mu$ *for* $i \in [d]$, *and* $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_d$. *We set* $\phi_{\mathcal{B}} : S \to K[z_1^{\pm 1}, \dots, z_\mu^{\pm 1}]$, $y_j^{(i)} \mapsto \mathbf{z}^{\mathbf{b}_j^{(i)}}$. *Assume that there exists an* $d \times \mu$ *rational matrix* $M \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^\mu, \mathbb{Q}^d)$ *such that* $M \cdot \mathbf{b}_j^{(i)} = \mathbf{e}_i$ *for all* $i \in [d]$ *and* $j \in [\lambda_i]$. *We set*

$$\mathcal{C} = \{ \sum_{i \in [d], j \in [\lambda_i]} a_j^{(i)} \mathbf{b}_j^{(i)} \mid \mathbf{a} = (a_j^{(i)} \mid i \in [d], j \in [\lambda_i]) \in \tilde{\mathcal{A}}\}.$$

Then the following hold.

(1) *If both of* $P_{\mathcal{B}}$ *and* $P_{\mathcal{A}}$ *admit Gröbner bases of degree at most* m, *then so is* $P_{\mathcal{C}}$.

(2) *If both of* $P_{\mathcal{B}}$ *and* $P_{\mathcal{A}}$ *admit square-free initial ideals, then so is* $P_{\mathcal{C}}$.

Suppose that there exists $\mu_1, \dots, \mu_d \in \mathbb{N}$ such that $\mu = \mu_1 + \dots + \mu_d$, $\mathbb{N}^\mu = \mathbb{N}^{\mu_1} \times \dots \times \mathbb{N}^{\mu_d}$ and $\mathcal{B}_i \subset 0 \times \dots \times \mathbb{N}^{\mu_i} \times \dots \times 0$ for all i. Then the existence of a matrix M in Theorem 2 is trivial, and the Gböbner basis of $P_{\mathcal{B}}$ is the union of

Gröbner bases of $P_{\mathcal{B}_i}$'s. In this case, Theorem 2 is equivalent to the theorems of Aoki–Hibi–Ohsugi–Takemura [1] and Hibi-Ohsugi [7].

Suppose that $\mathcal{A} = (\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(r)})$ is a matrix with $\mathbf{a}^{(i)} = \mathbf{a}^{(j)}$, and $\prec$ is a term order on R such that $x_i \prec x_j$. Let $\mathcal{A}' = (\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(i-1)}, \mathbf{a}^{(i+1)}, \ldots, \mathbf{a}^{(r)})$. Then the union of $\{x_j - x_i\}$ and a Gröbner basis of $P_{\mathcal{A}'} \subset K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r]$ with respect to the term order induced by $\prec$ is a Gröbner basis of $P_{\mathcal{A}}$. Therefore a Gröbner basis of $P_{\mathcal{A}}$ and that of $P_{\mathcal{A}'}$ are essentially equivalent. In particular, $\delta(\text{in}_\prec(P_{\mathcal{A}})) = \delta(\text{in}_\prec(P_{\mathcal{A}'}))$, and $\text{in}_\prec(P_{\mathcal{A}}))$ is generated by square-free monomials if and only if $\text{in}_\prec(P_{\mathcal{A}'})$ is.

1. Applications

1.1. Veronese configurations.

Let $S = K[y_1, \ldots, y_s] = \bigoplus_{i \in \mathbb{N}} S_i$ be a $\mathbb{N}$-graded ring with $\deg(y_i) = 1$ for all i. Let d be a positive integer, and $\mathcal{A} = \{{}^t(a_1, \ldots, a_s) \in \mathbb{N}^s \mid |\mathbf{a}| = d\}$ be the Veronese configuration, $R = K[x_\mathbf{a} \mid \mathbf{a} \in \mathcal{A}]$ be a polynomial ring, and $\phi_\mathcal{A} : R \to S$ $(x_\mathbf{a} \mapsto \mathbf{y}^\mathbf{a})$ the monomial homomorphism. It is known that there exist a lexicographic order on R such that $\text{in}_\prec(P_\mathcal{A})$ is generated by square-free monomial of degree two ([6]).

Corollary 1.1. *Let $I \subset S$ be a homogeneous ideal, ω a weight vector on S such that $\text{in}_\omega(I)$ is a monomial ideal, and $\prec$ a term order on R such that $\text{in}_\prec(P_\mathcal{A})$ is generated by square-free monomial of degree two. We denote the weight vector $\phi^*_\mathcal{A}\omega$ by ω'. Then the following hold:*

1. $\delta(\text{in}_{\prec_{\omega'}}(\phi_\mathcal{A}^{-1}(I)) \leq \max\{2, \delta(\text{in}_\omega(I))\}$.
2. *If $\text{in}_\omega(I)$ is generated by square-free monomials, then $\text{in}_{\prec_{\omega'}}(\phi_\mathcal{A}^{-1}(I))$ is generated by square-free monomials.*

Eisenbud–Reeves–Totaro proved in [5] that if K is an infinite field, the coordinates $y_1, \ldots, y_s$ of S are generic, and $\prec$ is a certain reversed lexicographic order, then it holds that $\delta(\text{in}_{\prec_{\omega'}}(\phi_\mathcal{A}^{-1}(I)) \leq \max\{2, \delta(\text{in}_\omega(I))/d\}$.

1.2. Toric fiber products.

We recall toric fiber products defined in [9]. Let $s_1, \ldots, s_d$, $t_1, \ldots, t_d$ and d be positive integers, and

$$S_1 = K[\mathbf{y}] = K\big[y_j^{(i)} \mid i \in [d], j \in [s_i]\big],\ S_2 = K[\mathbf{z}] = K\big[z_k^{(i)} \mid i \in [d], k \in [t_i]\big],$$

polynomial rings regarded as $\mathbb{Z}^d$-graded rings by assigning

$$\deg(y_j^{(i)}) = \deg(z_k^{(i)}) = \mathbf{e}_i$$

for all $i \in [d], j \in [s_i], k \in [t_i]$. Then

$$S = S_1 \otimes_K S_2 \cong K[y_j^{(i)}, z_k^{(i)} \mid i \in [d], j \in [s_i], k \in [t_i]]$$

carries an $\mathbb{Z}^d \times \mathbb{Z}^d$-graded ring structure by setting

$$\deg_S(y_j^{(i)}) = (\mathbf{e}_i, 0),\ \deg_S(z_k^{(i)}) = (0, \mathbf{e}_i)$$

for all $i \in [d], j \in [s_i], k \in [t_i]$ in S. Here, assuming S_1 and S_2 as subrings of S, we write $y_j^{(i)} \otimes 1$ and $1 \otimes z_k^{(i)}$ simply as $y_j^{(i)}$ and $z_k^{(i)}$. Let

$$\Delta = \{(\alpha, \alpha) \mid \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

be the subsemigroup of $\mathbb{Z}^d \times \mathbb{Z}^d$. Since Δ is generated by $\{(\mathbf{e}_i, \mathbf{e}_i) \mid i \in [d]\}$, we have

$$S^{(\Delta)} \cong K[y_j^{(i)} z_k^{(i)} \mid i \in [d], j \in [s_i], k \in [t_i]].$$

Let $R = K\big[x_{jk}^{(i)} \mid i \in [d], j \in [s_i], k \in [t_i]\big]$ be a polynomial ring, and $\varphi : R \to S$ the monomial homomorphism $\varphi(x_{jk}^{(i)}) = y_j^{(i)} z_k^{(i)}$.

Let $I_1 \subset S_1$ and $I_2 \subset S_2$ be $\mathbb{Z}^d$-graded ideals, and denote $I_1 \otimes S_2 + S_1 \otimes I_2$ simply by $I_1 + I_2$. The ideal

$$I_1 \times_{\mathbb{Z}^d} I_2 := \varphi^{-1}(I_1 + I_2)$$

is called the *toric fiber product* of I_1 and I_2. Originally, the assumptions in [9] are that $\deg(y_j^{(i)}) = \deg(z_k^{(i)}) = \mathbf{a}^{(i)} \in \mathbb{Z}^d$ with $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(d)}$ linearly independent, and I_1 and I_2 are $\mathbb{Z}^d$-graded ideals, which are equivalent to ours.

Let ω_1 and ω_2 weight vectors of S_1 and S_2 such that $\mathrm{in}_{\omega_1}(I_1)$ and $\mathrm{in}_{\omega_2}(I_2)$ are monomial ideals, and set $\omega = (\omega_1, \omega_2)$, the weight oder of S. Let G_1 and G_2 be Gröbner bases of I_1 and I_2 with respect to ω_1 and ω_2 respectively, and set $\omega = (\omega_1, \omega_2)$.

Corollary 1.2. *Let the notation be as above. Let $\prec$ be the lexicographic term order on R such that $\mathbf{x}_{j_1 k_1}^{i_1} \prec \mathbf{x}_{j_2 k_2}^{i_2}$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$ or $i_1 = i_2$ and $j_1 = j_2$ and $k_1 > k_2$. Then the following hold:*

1. $\delta(\mathrm{in}_{\prec_{\varphi^*\omega}}(I_1 \times_{\mathbb{Z}^d} I_2)) \le \max\{2, \delta(\mathrm{in}_{\omega_1}(I_1)), \delta(\mathrm{in}_{\omega_1}(I_2))\}$.
2. *([9] Corollary 2.11) If both of* $\mathrm{in}_{\omega_1}(I_1)$ *and* $\mathrm{in}_{\omega_2}(I_2)$ *are generated by square-free monomials, then* $\mathrm{in}_{\prec_{\varphi^*\omega}}(I_1 \times_{\mathbb{Z}^d} I_2)$ *is generated by square-free monomials.*

1.3. Nested configurations.

Let d and μ positive integers, and take $\lambda_i \in \mathbb{N}$ for $i \in [d]$. Let $\mathcal{A}$ be a configuration of $\mathbb{N}^d \subset \bigoplus_{i=1}^d \mathbb{Z}\mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with unity in the i-th position and zeros elsewhere. Let

$$\mathcal{B}_i = (\mathbf{b}_1^{(i)}, \dots, \mathbf{b}_{\lambda_i}^{(i)}),\ \mathbf{b}_j^{(i)} \in \mathbb{N}^\mu$$

be a configuration of $\mathbb{N}^\mu$ for$i = 1, 2, \dots, d$, and set $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_d)$. The *nested configuration* $\mathcal{A}[\mathcal{B}_1, \dots, \mathcal{B}_d]$ arising from $\mathcal{A}$ and $\mathcal{B}_1, \dots, \mathcal{B}_d$ is the configuration

$$\{\mathbf{b}_{j_1}^{(i_1)} + \cdots + \mathbf{b}_{j_r}^{(i_r)} \mid 1 \le r \in \mathbb{N}, \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_r} \in \mathcal{A},\ j_k \in [\lambda_{i_k}],\ i_k \in [d]\}.$$

Originally, Aoki–Hibi–Ohsugi–Takemura ([1]) define nested configurations in the case where there exists $0 < \mu_1, \dots, \mu_d \in \mathbb{N}$ such that $\mathbb{N}^\mu = \mathbb{N}^{\mu_1} \times \cdots \times \mathbb{N}^{\mu_d}$ and $\mathcal{B}_i \subset \mathbb{N}^{\mu_i}$. Let $\mathcal{E}_i = \{\mathbf{e}_1^{(i)}, \dots, \mathbf{e}_{\lambda_i}^{(i)}\}$ be a configuration of $\bigoplus_{j=1}^{\lambda_i} \mathbb{Z}\mathbf{e}_j^{(i)}$ where $\mathbf{e}_j^{(i)}$ is the vector with unity in the j-th position and zeros elsewhere. Let

$$S = K\big[(\mathcal{E}_1, \dots, \mathcal{E}_d)\big] \cong K\big[z_j^{(i)} \mid i \in [d],\ j \in [\lambda_i]\big]$$

be the $\mathbb{N}^d$-graded polynomial ring with $\deg_{\mathbb{N}^d} z_j^{(i)} = \mathbf{e}_i$. Then

$$S^{(\mathbb{N}\mathcal{A})} = K\bigl[\mathcal{A}[\mathcal{E}_1, \dots, \mathcal{E}_d]\bigr].$$

Corollary 1.3. *Let $R = K\bigl[x_m \mid m \in \mathcal{A}[\mathcal{E}_1, \dots, \mathcal{E}_d]\bigr]$ be a polynomial ring and I an $\mathbb{N}^d$-graded ideal of S. If I and $P_{\mathcal{A}}$ admit quadratic Gröbner bases with respect to some term orders, then so does $\phi^{-1}_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_d]}(I)$.*

Proof. If $P_{\mathcal{A}}$ admits a quadratic Gröbner basis, then so does $P_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_d]}$ ([1] Theorem 3.6). Therefore the assertion follows from Theorem 1. □

Theorem 1.4. *With the notation as above, assume in addition that there exists an $d \times \mu$ rational matrix $M \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^\mu, \mathbb{Q}^d)$ such that $M \cdot \mathbf{b}_j^{(i)} = \mathbf{e}_i$ for all $i \in [d]$ and $j \in [\lambda_i]$. If toric ideals $P_{\mathcal{A}}$ and $P_{\mathcal{B}_1 \cup \dots \cup \mathcal{B}_d}$ admit quadratic Gröbner bases with respect to some term orders, then $P_{\mathcal{A}[\mathcal{B}_1,\dots,\mathcal{B}_d]}$ admits a quadratic Gröbner basis.*

Proof. The set of column vectors of the product of matrices

$$\mathcal{B} \cdot \mathcal{A}[\mathcal{E}_1, \dots, \mathcal{E}_d]$$

coincides with $\mathcal{A}[\mathcal{B}_1, \dots, \mathcal{B}_d]$. Hence the Gröbner basis of $P_{\mathcal{A}[\mathcal{B}_1,\dots,\mathcal{B}_d]}$ is essentially equivalent to the Gröbnaer basis of $\phi^{-1}_{\mathcal{A}[\mathcal{E}_1,\dots,\mathcal{E}_d]}(P_{\mathcal{B}_1 \cup \dots \cup \mathcal{B}_d})$. Applying $I = P_{\mathcal{B}_1 \cup \dots \cup \mathcal{B}_d}$ in the above corollary, we conclude the assertion. □

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THE TORIC RING AND THE TORIC IDEAL ARISING FROM A NESTED CONFIGURATION

HIDEFUMI OHSUGI

ABSTRACT. This is a summary of the paper [7] with Takayuki Hibi (Osaka University). The toric ring together with the toric ideal arising from a nested configuration is discussed. Especially, the algebraic study of normality of the toric ring as well as of Gröbner bases of the toric ideal will be done in detail. In addition, as one of the combinatorial applications of our algebraic theory, toric ideals of multiples of the Birkhoff polytope will be investigated.

INTRODUCTION

This is a summary of the paper [7] with Takayuki Hibi (Osaka University). In [1], from a viewpoint of algebraic statistics, the concept of nested configurations is introduced. In the present paper, the toric ring together with the toric ideal arising from a nested configuration will be studied in detail.

Let $K[\mathbf{t}] = K[t_1, \dots, t_d]$ denote the polynomial ring in d variables over a field K. Recall that a *configuration* of $K[\mathbf{t}]$ is a finite set A of monomials belonging to $K[\mathbf{t}]$ such that there exists a vector $(w_1, \dots, w_d) \in \mathbb{R}^d$ with $\sum_{i=1}^{d} w_i a_i = 1$ for all $t_1^{a_1} \cdots t_d^{a_d} \in A$. We will associate each configuration A of $K[\mathbf{t}]$ with the homogeneous semigroup ring $K[A]$, called the *toric ring* of A, which is the subalgebra of $K[\mathbf{t}]$ generated by the monomials belonging to A. Let $K[X] = K[\{x_M \mid M \in A\}]$ denote the polynomial ring over K in the variables x_M with $M \in A$ with each $\deg(x_M) = 1$. The *toric ideal* I_A of A is the kernel of the surjective homomorphism $\pi : K[X] \to K[A]$ defined by setting $\pi(x_M) = M$ for all $M \in A$. It is known (e.g., [8, Section 4]) that the toric ideal I_A is generated by those homogeneous binomials $u - v$, where u and v are monomials of $K[X]$, with $\pi(u) = \pi(v)$.

Now, let $A = \{\mathbf{t}^{a_1}, \dots, \mathbf{t}^{a_n}\}$ be a configuration of $K[\mathbf{t}]$ with the properties that $\deg \mathbf{t}^{a_j} = r$ for each $1 \le j \le n$ and that, for each $1 \le i \le d$, there is $1 \le j \le n$ such that $\mathbf{t}^{a_j}$ is divided by t_i. Assume that, for each $1 \le i \le d$, a configuration $B_i = \{m_1^{(i)}, \dots, m_{\lambda_i}^{(i)}\}$ of a polynomial ring $K[\mathbf{u}^{(i)}] = K[u_1^{(i)}, \dots, u_{\mu_i}^{(i)}]$ in μ_i variables over K is given. Then the *nested configuration* [1] arising from A and $B_1, \dots, B_d$ is the configuration

$$A(B_1, \dots, B_d) := \left\{ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \;\middle|\; t_{i_1} \cdots t_{i_r} \in A, \;\; 1 \le j_k \le \lambda_{i_k} \text{ for } 1 \le k \le r \right\}$$

of the polynomial ring $K[\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(d)}]$ in $\sum_{i=1}^{d} \mu_i$ variables over K. One of the fundamental facts of the nested configuration is

Theorem 0.1 ([1]). *Work with the same notation as above. If each of the toric ideals $I_A, I_{B_1}, \ldots, I_{B_d}$ possesses a quadratic Gröbner basis, then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a quadratic Gröbner basis.*

In the present paper, first of all, in Section 1, we study normality of the toric ring arising from a nested configuration. It is natural to ask if each of the toric rings $K[A], K[B_1], \ldots, K[B_d]$ is normal if and only if the toric ring $K[A(B_1, \ldots, B_d)]$ is normal. Unfortunately, in general, the answer is negative. On the other hand, however, Corollary 1.8 guarantees that, when A consists of squarefree monomials, each of the toric rings $K[A], K[B_1], \ldots, K[B_d]$ is normal if and only if the toric ring $K[A(B_1, \ldots, B_d)]$ is normal.

Second, the topic of Section 2 is Gröbner bases of the toric ideal arising from a nested configuration. A natural generalization of Theorem 0.1 will be obtained. In fact, Theorem 2.4 together with Theorem 2.5 guarantees that if each of the toric ideals $I_A, I_{B_1}, \ldots, I_{B_d}$ possesses a Gröbner basis consisting of binomials of degree at most p, then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a Gröbner basis consisting of binomials of degree at most p. Moreover, if each of the toric ideals $I_A, I_{B_1}, \ldots, I_{B_d}$ possesses a Gröbner basis consisting of binomials whose initial monomial is squarefree, then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a Gröbner basis consisting of binomials whose initial monomial is squarefree.

Finally, in Section 3, as one of the combinatorial applications of our algebraic theory of nested configurations, we discuss the toric ideal of a multiple of the Birkhoff polytope $\mathcal{B}_3$. It seems to be known that the toric ideal of the multiple $2n\,\mathcal{B}_3$ possesses a quadratic Gröbner basis for each $n > 1$. We will prove that the toric ideal of the multiple $n\mathcal{B}_3$ possesses a quadratic Gröbner basis for each $n > 1$. See Theorem 3.4.

1. Normality of toric rings of nested configurations

The purpose of this section is to study normality of $K[A(B_1, \ldots, B_d)]$.

Theorem 1.1. *Work with the same notation as above. If $K[A]$, $K[B_1], \ldots, K[B_d]$ are normal, then $K[A(B_1, \ldots, B_d)]$ is normal.*

The converse of Theorem 1.1 is false in general.

Example 1.2. Let $A = \{t_1^2\}$ and $B_1 = \{v, uv, u^3v, u^4v\}$. Then $K[B_1]$ is very ample [6], but not normal. However, $I_{A(B_1)}$ has a squarefree quadratic initial ideal and hence $K[A(B_1)] = K[\{u^iv^2 \mid i = 0, 1, \ldots, 8\}]$ is normal.

Theorem 1.1 is not true if we replace "normal" with "very ample." (See [6] for the definition of very ample configurations.)

Example 1.3. Let $A = \{t_1, t_2\}$, $B_1 = \{v, uv, u^3v, u^4v\}$ and $B_2 = \{w\}$. Then $K[A]$ and $K[B_2]$ are polynomial rings. On the other hand, $K[B_1]$ is very ample, but not normal. However, $K[A(B_1, B_2)] = K[v, uv, u^3v, u^4v, w]$ is not very ample. In fact, the monomial u^2vw^α is a hole for all $\alpha \in \mathbb{Z}_{\geq 0}$.

Let P_A denote the convex hull of $\{\mathbf{a} \in \mathbb{Z}^d_{\geq 0} \mid \mathbf{t}^{\mathbf{a}} \in A\}$. For a subset $B \subset A$, $K[B]$ is called *combinatorial pure subring* ([5, 4]) of $K[A]$ if there exists a face F of P_A such that $\{\mathbf{b} \in \mathbb{Z}^d_{\geq 0} \mid \mathbf{t}^{\mathbf{b}} \in B\} = \{\mathbf{a} \in \mathbb{Z}^d_{\geq 0} \mid \mathbf{t}^{\mathbf{a}} \in A\} \cap F$. For example, if $B = A \cap \{t_{i_1}, \ldots, t_{i_s}\}$, then $K[B]$ is a combinatorial pure subring of $K[A]$. (This is the original definition of a combinatorial pure subring in [5].)

Lemma 1.4. *Work with the same notation as above. Then $K[A(B_1, \ldots, B_d)]$ has a combinatorial pure subring which is isomorphic to $K[A]$.*

Since every combinatorial pure subring of a normal (resp. very ample) semigroup ring is normal (resp. very ample), we have the following.

Theorem 1.5. *Work with the same notation as above. If $K[A(B_1, \ldots, B_d)]$ is normal (resp. very ample), then $K[A]$ is normal (resp. very ample).*

Lemma 1.6. *Work with the same notation as above. Let*

$$m = \max(i \mid t_1^i t_2^{a_2} \cdots t_d^{a_d} \in A) \ (\geq 1).$$

Then $K[A(B_1, \ldots, B_d)]$ has a combinatorial pure subring which is isomorphic to $K[A'(B_1)]$ where $A' = \{t_1^m\}$. In particular, if $m = 1$, then we have $K[A'(B_1)] \simeq K[B_1]$.

Thanks to Lemma 1.6, we have the following.

Theorem 1.7. *Work with the same notation as above. If A has no monomial divided by t_i^2 and if $K[A(B_1, \ldots, B_d)]$ is normal (resp. very ample), then $K[B_i]$ is normal (resp. very ample).*

Corollary 1.8. *Work with the same notation as above. If A consists of squarefree monomials, then the following conditions are equivalent:*
(i) *$K[A]$, $K[B_1], \ldots, K[B_d]$ are normal;*
(ii) *$K[A(B_1, \ldots, B_d)]$ is normal.*

2. Gröbner bases of toric ideals of nested configurations

In this section, we study Gröbner bases of the toric ideal of a nested configuration. Let, as before, $A = \{\mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_n}\}$ and $B_i = \{m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)}\}$ for $1 \leq i \leq d$. Let $K[\mathbf{x}]$ be a polynomial ring with the set of variables

$$\left\{ x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} \;\middle|\; \begin{array}{c} 1 \leq i_1 \leq \cdots \leq i_r \leq d, \ 1 \leq k \leq n \\ t_{i_1} \cdots t_{i_r} = \mathbf{t}^{a_k} \in A \\ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \in A(B_1, \ldots, B_d) \end{array} \right\}$$

and let $K[\mathbf{y}] = K[y_1, \ldots, y_n]$ and $K\left[\mathbf{z}^{(i)}\right] = K\left[z_1^{(i)}, \ldots, z_{\lambda_i}^{(i)}\right]$ $(i = 1, 2, \ldots, d)$ be polynomial rings. The toric ideal I_A is the kernel of the homomorphism $\pi_0 : K[\mathbf{y}] \longrightarrow K[\mathbf{t}]$ defined by setting $\pi_0(y_k) = \mathbf{t}^{a_k}$. The toric ideal I_{B_i} is the kernel of the homomorphism $\pi_i : K[\mathbf{z}^{(i)}] \longrightarrow K[\mathbf{u}^{(i)}]$ defined by setting $\pi_i(z_j^{(i)}) = m_j^{(i)}$. The toric ideal $I_{A(B_1,\ldots,B_d)}$ is the kernel of the homomorphism $\pi : K[\mathbf{x}] \longrightarrow K[\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(d)}]$ defined by setting $\pi\left(x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}\right) = m^{(i_1)}_{j_1}\cdots m^{(i_r)}_{j_r}$.

Lemma 2.1. *Let* $p_1 = x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} x^{(k)}_{(i_{r+1},j_{r+1})\cdots(i_{2r},j_{2r})}$ *be a quadratic monomial in* $K[\mathbf{x}]$. *Then,* $p_2 = x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})} x^{(k)}_{(i'_2,j'_2)(i'_4,j'_4)\cdots(i'_{2r},j'_{2r})}$ *where*

$$(i'_1,j'_1)\cdots(i'_{2r},j'_{2r}) = \mathrm{sort}((i_1,j_1)\cdots(i_{2r},j_{2r}))$$

with respect to the ordering

$$(1,1)\succ(1,2)\succ\cdots\succ(1,\lambda_1)\succ(2,1)\succ\cdots\succ(d,\lambda_d)$$

is a monomial belonging to $K[\mathbf{x}]$ *and, in particular, we have* $p_1 - p_2 \in I_{A(B_1,\ldots,B_d)}$.

Lemma 2.2. *Let* $y_{k_1}\cdots y_{k_p} - y_{k'_1}\cdots y_{k'_p}$ *be a binomial in* I_A *and let*

$$\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1})\cdots(i_{\ell r},j_{\ell r})}$$

be a monomial in $K[\mathbf{x}]$. *Then, there exists a binomial*

$$\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1})\cdots(i_{\ell r},j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1})\cdots(i'_{\ell r},j'_{\ell r})} \in I_{A(B_1,\ldots,B_d)},$$

where $\mathrm{sort}((i_1,j_1)\cdots(i_{pr},j_{pr})) = \mathrm{sort}((i'_1,j'_1)\cdots(i'_{pr},j'_{pr}))$.

Fix a monomial order $<_i$ on $K[\mathbf{z}^{(i)}]$ for each $1 \le i \le d$. Let $\mathcal{G}_i$ be a Gröbner basis of I_{B_i} with respect to $<_i$. For each $M \in A(B_1,\ldots,B_d)$, the expression $M = m^{(i_1)}_{j_1}\cdots m^{(i_r)}_{j_r}$ is called *standard* if

$$\prod_{i_\ell=j,\ 1\le\ell\le r} z^{(i_\ell)}_{j_\ell}$$

is a standard monomial with respect to $\mathcal{G}_j$ for all $1 \le j \le d$. In order to study the relation among I_A, I_{B_i} and $I_{A(B_1,\ldots,B_d)}$, we define homomorphisms

$$\varphi_0 : K[\mathbf{x}] \longrightarrow K[\mathbf{y}] \quad , \quad \varphi_0\left(x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}\right) = y_k,$$

$$\varphi_j : K[\mathbf{x}] \longrightarrow K[\mathbf{z}^{(j)}] \quad , \quad \varphi_j\left(x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}\right) = \prod_{i_\ell=j,\ 1\le\ell\le r} z^{(i_\ell)}_{j_\ell},$$

where $m^{(i_1)}_{j_1}\cdots m^{(i_r)}_{j_r}$ is the standard expression defined above.

Lemma 2.3 ([1]). *Let* f *be a binomial in* $K[\mathbf{x}]$. *Then the following conditions are equivalent:*

(i) $f \in I_{A(B_1,\ldots,B_d)}$;
(ii) $\varphi_i(f) \in I_{B_i}$ *for all* $1 \le i \le d$.

Moreover, if the above conditions hold, then we have $\varphi_0(f) \in I_A$.

2.1. **Polynomial ring case.** First, we study the case when all of $K[B_i]$ are polynomial rings.

Theorem 2.4. *Let $\mathcal{G}_0$ be a Gröbner basis of I_A with respect to a monomial order $<_0$. If each B_i is a set of variables, then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a Gröbner basis consisting of the following binomials:*

(1) $\underline{\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1})\cdots(i_{\ell r},j_{\ell r})}} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1})\cdots(i'_{\ell r},j'_{\ell r})}$

where $\underline{y_{k_1}\cdots y_{k_p}} - y_{k'_1}\cdots y_{k'_p} \in \mathcal{G}_0$ *and*

$$\mathrm{sort}((i_1,j_1)\cdots(i_{pr},j_{pr})) = \mathrm{sort}((i'_1,j'_1)\cdots(i'_{pr},j'_{pr})).$$

(2) $\underline{x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} x^{(k)}_{(i_{r+1},j_{r+1})\cdots(i_{2r},j_{2r})}} - x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})} x^{(k)}_{(i'_2,j'_2)(i'_4,j'_4)\cdots(i'_{2r},j'_{2r})}$

where $\mathrm{sort}((i_1,j_1)\cdots(i_{2r},j_{2r})) = (i'_1,j'_1)\cdots(i'_{2r},j'_{2r})$ *with respect to the ordering* $(1,1) \succ (1,2) \succ \cdots \succ (1,\lambda_1) \succ (2,1) \succ \cdots \succ (d,\lambda_d)$.

(3) $\underline{x^{(k)}_{(i_1,j_1)\cdots(i_\ell,j_\ell)\cdots(i_r,j_r)} x^{(k')}_{(i'_1,j'_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i'_r,j'_r)}} - x^{(k)}_{(i_1,j_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i_r,j_r)} x^{(k')}_{(i'_1,j'_1)\cdots(i_\ell,j_\ell)\cdots(i'_r,j'_r)}$

where $k < k'$, $i_\ell = i'_{\ell'}$ *and* $j_\ell > j'_{\ell'}$.

The initial monomial of each binomial is the first (underlined) monomial and, in particular, the initial monomial of each binomial in (2) and (3) is squarefree. Moreover, the initial monomial of each binomial in (1) is squarefree (resp. quadratic) if the corresponding monomial $y_{k_1}\cdots y_{k_p}$ is squarefree (resp. quadratic).

2.2. **General case.** We now study the general case.

Theorem 2.5. *Work with the same notation as above. Let $\mathcal{G}_0$ be a Gröbner basis of I_A and let $\mathcal{G}_i$ be a Gröbner basis of I_{B_i} with respect to $<_i$. Then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a Gröbner basis consisting of the binomials (1), (2) and (3) appearing in Theorem 2.4 together with the following binomials:*

(4) $\underline{\prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j_{\ell,1})\cdots(i,j_{\ell,q_\ell})M'_\ell}} - \prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j'_{\ell,1})\cdots(i,j'_{\ell,q_\ell})M'_\ell}$ *where the binomial*

$$0 \neq \underline{\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}}\cdots z^{(i)}_{j_{\ell,q_\ell}}} - \prod_{\ell=1}^{p} z^{(i)}_{j'_{\ell,1}}\cdots z^{(i)}_{j'_{\ell,q_\ell}} \text{ belongs to } \mathcal{G}_i.$$

The initial monomial of each binomial is the first (underlined) monomial and, in particular, the initial monomial of each binomial above is squarefree (resp. quadratic) if the corresponding monomial $\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}}\cdots z^{(i)}_{j_{\ell,q_\ell}}$ is squarefree (resp. quadratic).

If $\mathcal{G}_i$ possesses a binomial of degree 3, then we need the following binomials:

(a) $x^{(k_1)}_{M_1(i,j_1)M'_1} x^{(k_2)}_{M_2(i,j_2)M'_2} x^{(k_3)}_{M_3(i,j_3)M'_3} - x^{(k_1)}_{M_1(i,j'_1)M'_1} x^{(k_2)}_{M_2(i,j'_2)M'_2} x^{(k_3)}_{M_3(i,j'_3)M'_3}$
where $z^{(i)}_{j_1} z^{(i)}_{j_2} z^{(i)}_{j_3} - z^{(i)}_{j'_1} z^{(i)}_{j'_2} z^{(i)}_{j'_3} \in \mathcal{G}_i$.

(b) $x^{(k_1)}_{M_1(i,j_1)(i,j_2)M'_1}x^{(k_2)}_{M_2(i,j_3)M'_2} - x^{(k_1)}_{M_1(i,j'_1)(i,j'_2)M'_1}x^{(k_2)}_{M_2(i,j'_3)M'_2}$
where $z^{(i)}_{j_1}z^{(i)}_{j_2}z^{(i)}_{j_3} - z^{(i)}_{j'_1}z^{(i)}_{j'_2}z^{(i)}_{j'_3} \in \mathcal{G}_i$.

We do not need (b) if A has no monomial divided by t_i^2. In general, we have

$$\deg\left(\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}}\cdots z^{(i)}_{j_{\ell,q_\ell}}\right) = \sum_{\ell=1}^{p} q_\ell \geq p = \deg\left(\prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j_{\ell,1})\cdots(i,j_{\ell,q_\ell})M'_\ell}\right).$$

The binomials of type (a) are not always needed for a minimal Gröbner basis even if $\mathcal{G}_i$ has a cubic binomial. In such a case, $I_{A(B_1,\dots,B_d)}$ may have a quadratic Gröbner basis. In Section 3, we will show an example.

2.3. **Generators.** Thanks to a part of the argument in Proof of Theorem 2.5, we have the following.

Proposition 2.6. *Let $\mathcal{H}_0$ be a set of binomial generators of I_A and let $\mathcal{H}_i$ be a set of binomial generators of I_{B_i}. Then, the toric ideal $I_{A(B_1,\dots,B_d)}$ is generated by the following binomials:*

(1) $\displaystyle\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1})\cdots(i_{\ell r},j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1})\cdots(i'_{\ell r},j'_{\ell r})}$

where $y_{k_1}\cdots y_{k_p} - y_{k'_1}\cdots y_{k'_p} \in \mathcal{H}_0$ *and*

$$\mathrm{sort}((i_1,j_1)\cdots(i_{pr},j_{pr})) = \mathrm{sort}((i'_1,j'_1)\cdots(i'_{pr},j'_{pr})).$$

(2) $x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}x^{(k)}_{(i_{r+1},j_{r+1})\cdots(i_{2r},j_{2r})} - x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})}x^{(k)}_{(i'_2,j'_2)(i'_4,j'_4)\cdots(i'_{2r},j'_{2r})}$

where $\mathrm{sort}((i_1,j_1)\cdots(i_{2r},j_{2r})) = (i'_1,j'_1)\cdots(i'_{2r},j'_{2r})$ *with respect to the ordering* $(1,1) \succ (1,2) \succ \cdots \succ (1,\lambda_1) \succ (2,1) \succ \cdots \succ (d,\lambda_d)$.

(3) $x^{(k)}_{(i_1,j_1)\cdots(i_\ell,j_\ell)\cdots(i_r,j_r)}x^{(k')}_{(i'_1,j'_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i'_r,j'_r)} - x^{(k)}_{(i_1,j_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i_r,j_r)}x^{(k')}_{(i'_1,j'_1)\cdots(i_\ell,j_\ell)\cdots(i'_r,j'_r)}$

where $k < k'$, $i_\ell = i'_{\ell'}$ *and* $j_\ell > j'_{\ell'}$.

(4) $\prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j_{\ell,1})\cdots(i,j_{\ell,q_\ell})M'_\ell} - \prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j'_{\ell,1})\cdots(i,j'_{\ell,q_\ell})M'_\ell}$ *where the binomial*

$$0 \neq \prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}}\cdots z^{(i)}_{j_{\ell,q_\ell}} - \prod_{\ell=1}^{p} z^{(i)}_{j'_{\ell,1}}\cdots z^{(i)}_{j'_{\ell,q_\ell}} \text{ belongs to } \mathcal{H}_i.$$

3. Toric ideals of multiples of the Birkhoff polytope

Let $\mathbf{c} = (c_1,c_2,c_3) \in \mathbb{Z}^3_{\geq 0}$ and $\mathbf{r} = (r_1,r_2,r_3) \in \mathbb{Z}^3_{\geq 0}$ be vectors with $c_1+c_2+c_3 = r_1+r_2+r_3$. Then 3×3 *transportation polytope* $T_{\mathbf{rc}}$ is the set of all non-negative 3×3 matrices $A = (a_{ij})$ satisfying

$$\sum_{i=1}^{3} a_{ik} = c_k \text{ and } \sum_{j=1}^{3} a_{\ell j} = r_\ell$$

for $1 \leq k,\ell \leq 3$.

Example 3.1. Let $\mathbf{c} = \mathbf{r} = (1,1,1)$. Then the transportation polytope $\mathcal{B}_3 := T_{\mathbf{rc}}$ is called the *Birkhoff polytope*. The lattice points in $\mathcal{B}_3$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the toric ideal of $\mathcal{B}_3$ is a principal ideal generated by $z_1z_2z_3 - z_4z_5z_6$.

The following is proved by Haase–Paffenholz [2]:

- The toric ideal of 3×3 transportation polytope is generated by quadratic binomials except for $\mathcal{B}_3$.
- The toric ideal of 3×3 transportation polytope possesses a quadratic squarefree initial ideal if it is not a multiple of $\mathcal{B}_3$.

Thus, it is natural to ask whether the toric ideal of a multiple of $\mathcal{B}_3$ possesses a quadratic Gröbner basis except for $\mathcal{B}_3$. The following fact is due to Birkhoff:

- Every non-negative integer $p \times p$ matrix with equal row and column sums can be written as a sum of permutation matrices.

Thus, in order to study the toric ideal of n multiple of $\mathcal{B}_3$, we consider the following:

Example 3.2. Let $A = \{t_1^n\}$ and suppose that B_1 satisfies $\sharp|B_1| = 6$ and $I_{B_1} = \langle z_1z_2z_3 - z_4z_5z_6 \rangle$. If $n = 1$, then $A(B_1) = B_1$ and $\{x_1x_2x_3 - x_4x_5x_6\}$ is the reduced Gröbner basis of $I_{A(B_1)}$ with respect to any monomial order. If $n > 1$, then, by virtue of Theorem 2.5, $I_{A(B_1)}$ has a Gröbner bases consisting of the following binomials:

(a) $x_{1M_1}x_{2M_2}x_{3M_3} - x_{4M_1}x_{5M_2}x_{6M_3}$,
(b) $x_{j_1j_2M_1}x_{j_3M_2} - x_{j_4j_5M_1}x_{j_6M_2}$, where $\{j_1, j_2, j_3\} = \{1, 2, 3\}$ and $\{j_4, j_5, j_6\} = \{4, 5, 6\}$,
(c) $x_{j_1 \cdots j_n}x_{j_{n+1} \cdots j_{2n}} - x_{j'_1j'_3 \cdots j'_{2n-1}}x_{j'_2j'_4 \cdots j'_{2n}}$, where $\mathrm{sort}(j_1 \cdots j_{2n}) = j'_1 \cdots j'_{2n}$.

Since the Gröbner basis in Example 3.2 is not quadratic, we have to consider another monomial order to find a quadratic Gröbner basis.

Remark 3.3. In [2], they say that L. Piechnik and C. Haase proved that the toric ideal of the multiple $2n\mathcal{B}_3$ possesses a squarefree quadratic initial ideal for $n > 1$. This fact is directly obtained by Theorem 2.5 since the toric ideal of the multiple $2\mathcal{B}_3$ possesses a squarefree quadratic initial ideal. Similarly, since the toric ideal of the multiple $3\mathcal{B}_3$ possesses a squarefree quadratic initial ideal, Theorem 2.5 guarantees that the toric ideal of the multiple $3n\mathcal{B}_3$ possesses a squarefree quadratic initial ideal for $n > 1$.

Theorem 3.4. *Let $A = \{t_1^n\}$ with $n > 1$ and suppose that B_1 satisfies $\sharp|B_1| = 6$ and $I_{B_1} = \langle z_1z_2z_3 - z_4z_5z_6 \rangle$. Then, $I_{A(B_1)}$ has a quadratic Gröbner basis consisting of the following binomials:*

(i) $\underline{x_{j_1 j_2 M_1} x_{j_3 M_2}} - x_{j_4 j_5 M_1} x_{j_6 M_2}$ *where* $\{j_1, j_2, j_3\} = \{1,2,3\}$ *and* $\{j_4, j_5, j_6\} = \{4,5,6\}$,

(ii) $\underline{x_{j_1 \cdots j_n} x_{j_{n+1} \cdots j_{2n}}} - x_{1 \cdots 1 j'_1 \cdots j'_\alpha} x_{1 \cdots 1 j'_{\alpha+1} \cdots j'_{2\alpha}}$ *where* $\text{sort}(j_1 \cdots j_{2n}) = 1 \cdots 1 j'_1 \cdots j'_{2\alpha}$ *and* $j'_2 > 1$.

Acknowledgement. This research was supported by JST, CREST.

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Set-theoretic complete intersection monomial curves II

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We will give recent results for a question which asks whether every monomial curves in affine space is set-theoretic complete intersection. We use the following notations: Let $N > 2$ be a natural number, $n_1, \dots, n_N$ natural numbers satisfying $\gcd(n_1, \dots, n_N) = 1$. And let k be a field, and $A = k[X_1, \dots, X_N]$ a polynomial ring over k. We denote the ring of integers by $\mathbb{Z}$. For $v \in \mathbb{Z}^N = \sum_{i=1}^N \mathbb{Z}e_i$, put $\sigma_i(v)$ be the i-th entry of v for each i, $v^+ = \sum_{i=1}^N \max\{\sigma_i(v), 0\}e_i$, and $v^- = \sum_{i=1}^N \max\{-\sigma_i(v), 0\}e_i$. For $v \in \mathbb{Z}^N$, we put $F(v) = X^{v^-} - X^{v^+} \in A$ called a binomial, and for $V \subset \mathbb{Z}^N$, we set $I(V) = (F(v))_{v \in V}$ called a lattice ideal.

Definition 1. The curve

$$C = \{(t^{n_1}, t^{n_2}, \dots, t^{n_N}) : t \in k\}$$

is called a **monomial curve** in affine N-space defined by $n_1, n_2, \dots, n_N$. And the kernel of the ring homomorphism

$$A \longrightarrow k[t], \qquad X_i \longmapsto t^{n_i} \quad \text{for each } i$$

is called the defining ideal of the monomial curve C.

Definition 2. In general, for an ideal I in a ring R, if

$$\exists f_1, \dots, f_r \in I \text{ s.t. } \sqrt{I} = \sqrt{(f_1, \dots, f_r)}, \quad r = \operatorname{ht} I,$$

then I is called a **set-theoretic complete intersection**.

Question 1. Is every monomial curve C in affine N-space a set-theoretic complete intersection? Namely, if we put I be the defining ideal of C, then are there

$$f_1, \dots, f_{N-1} \in I \text{ s.t. } \sqrt{I} = \sqrt{(f_1, \dots, f_{N-1})}?$$

We give the historical notes. If $\operatorname{char} k > 0$, the above question is valid. Further, there exist binomials $f_1, \ldots, f_{N-1}$ satisfying the above ([9]). If $N = 3$, the question is valid by Bresinsky [1], Herzog [8] and Valla [11]. If $N = 4$ and if A/I is Gorenstein, then every monomial curve is a set-theoretic complete intersection proved by Bresinsky [2]. There are other examples in which the question is valid; the case that $(n_1, n_2, \ldots, n_N)$ is an arithmetical sequence [10], and the case that the defining ideal is an almost complete intersection [3, 4]. In 2008, we proved that the question is valid, if $N = 4$ and $n_1 + n_4 = n_2 + n_3$, which is defined by a "balanced" semigroup ([7]).

Now we give the sketch of the proof of the last case. From now, assume $\operatorname{char} k = 0$. Let $V = \operatorname{Ker}(n_1, n_2, n_3, n_4) \subset \mathbb{Z}^4$ and $w = {}^t(-1, 1, 1, -1) \in V$. We choose $V_1, V_2 \subset V$ submodules of rank 2 satisfying $V_1 + V_2 = V$ and $V_1 \cap V_2 = \mathbb{Z}w$ and both $V_1/\mathbb{Z}w$ and $V_2/\mathbb{Z}w$ are torsion free. Assume that $I(V)$ is stci on $I(V_l)$ for $l = 1, 2$, i.e. $I(V)/I(V_l)$ is generated by one element up to radical. Then we prove that, if we choose suitable V_1 and V_2, then $I(V)$ is stci on $I(V_1) \cap I(V_2)$ and $I(V_1) \cap I(V_2)$ is stci on $I(\mathbb{Z}w)$. In this case, $I(V)$ is a set-theoretic complete intersection. From this proof, we present two questions in general case.

Question 2. Q2-1. Assume $\operatorname{rank} V_1 = \operatorname{rank} V_2 = \operatorname{rank} V_1 \cap V_2 + 1$.
When $I(V_1) \cap I(V_2)$ is stci on $I(V_1 \cap V_2)$?

Q2-2. Assume $\operatorname{rank} V_1 = \operatorname{rank} V_2 = \operatorname{rank} V_1 + V_2 - 1$
and that $I(V_1 + V_2)$ is stci on $I(V_l)$ for $l = 1, 2$.
When $I(V_1 + V_2)$ is stci on $I(V_1) \cap I(V_2)$?

From now, we consider the above questions. We assume that any submodule in $\mathbb{Z}^N$ is saturated and contained in $\operatorname{Ker}(n_1, \ldots, n_N)$. Then $I(V)$ is prime and a toric ideal.

Definition 3. For a non empty subset S in $\{1, \ldots, N\}$ satisfying $F(v) \in (X_i)_{i \in S}$ or $\operatorname{supp} v \cap S = \emptyset$ for any $v \in V$, put

$$\mathfrak{p}_S = I(V_S) + (X_i)_{i \in S}$$

where $V_S = \{v \in V : \operatorname{supp} v \cap S = \emptyset\}$. We call a minimal $\mathfrak{p}_S$ w.r.t inclusion a **lattice divisor** of $I(V)$.

If $\mathfrak{p}_S$ is a lattice divisor, then $\operatorname{ht} \mathfrak{p}_S = \operatorname{ht} I(V) + 1$ and $\mathfrak{p}_S \supset I(V)$.

Definition 4. Let $V \subset \mathbb{Z}^N$ and $v_1, \ldots, v_r \in V$ with $V = \sum_{j=1}^r \mathbb{Z}v_j$ where $\operatorname{rank} V = r$. Assume $r = N - 1$. We may assume that the determinant

$$|\sigma_i(v_j)|_{i,j=1,\ldots N-1}$$

is positive. For $v_N \in \mathbb{Z}^N$, put $\tau = \tau_V : \mathbb{Z}^N \to \mathbb{Z}$ sending v_N to

$$|\sigma_i(v_j)|_{i,j=1,\ldots N}.$$

Then τ is defined by positive integers.

Assume $r < N-1$. Let $\mathfrak{p}_{S_1}, \ldots, \mathfrak{p}_{S_L}$ be the lattice divisors of $I(V)$ and $\rho_l : \mathbb{Z}^N \to \mathbb{Z}^{|S_l|} = \sum_{i \in S_l} \mathbb{Z}e_i$ the projection for each l. Then $\operatorname{rank} \rho_l(V) = |S_l| - 1$ and we may define $\tau_l : \mathbb{Z}^{|S_l|} \to \mathbb{Z}$ as above. Put

$$\tau = (\tau_1\rho_1, \ldots, \tau_L\rho_L) : \mathbb{Z}^N \to \mathbb{Z}^L.$$

In any case, the map τ does not depend on the choice of a basis $v_1, \ldots, v_r$ of V and $\operatorname{Ker} \tau = V$. We call τ the **defining map** of V.

We give the examples of the defining maps.

Example 1. (1) If $V = \operatorname{Ker}(n_1, \ldots, n_N)$, then the defining map is

$$\tau = (n_1, \ldots, n_N) : \mathbb{Z}^N \to \mathbb{Z}.$$

(2) If $V = \operatorname{Ker}\begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix}$, then the defining map is

$$\tau = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix} : \mathbb{Z}^4 \to \mathbb{Z}^2.$$

(3) If $V = \mathbb{Z}^t(-1, 1, 1, -1)$, then the defining map is

$$\tau = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} : \mathbb{Z}^4 \to \mathbb{Z}^4.$$

We also give a few propositions without proof.

Proposition 1. *The defining map of V is surjective over $\mathbb{Q}$ if and only if V is simplicial, i.e. $A/I(V)$ is a semigroup ring associated with a simplicial semigroup.*

Lemma 2. *Let $W \subset V$ be submodules in $\mathbb{Z}^N$. Assume $V = W + \mathbb{Z}v$. Put τ be the defining map of W and $\widetilde{\tau} : A \to k[t] = k[t_1, \ldots, t_L]$ the induced algebra map by τ. Then $\operatorname{Ker} \widetilde{\tau} = I(W)$ and $\widetilde{\tau}^{-1}(F(\tau(v)) = I(V)$.*

Proposition 3. *Let $W \subset V$ be submodules in $\mathbb{Z}^N$. Assume $V = W + \mathbb{Z}v$. And let τ, $\widetilde{\tau}$ be as in Lemma 2. If there is $g \in \operatorname{Im}\widetilde{\tau}$ satisfying $\sqrt{(g)} = (F(\tau(v)))$, then $I(V)$ is stci on $I(W)$.*

Definition 5. Let W be a submodules in $\mathbb{Z}^N$, τ the defining map of W and $\widetilde{\tau} : A \to k[t]$ the induced algebra map by τ. For a non zero element $g \in k[t]$, we call the ideal $\widetilde{\tau}^{-1}(g)$ in A, the **lattice closure** of g.

Example 2. If $V_1/W \cong V_2/W \cong V_1 + V_2/V_1 \cong \mathbb{Z}$, then $I(V_1) \cap I(V_2)$ is a lattice closure. Indeed,

$$I(V_1) \cap I(V_2) = \widetilde{\tau}^{-1}(F(\tau(v_1))F(\tau(v_2))),$$

where $V_1 = W + \mathbb{Z}v_1$ and $V_2 = W + \mathbb{Z}v_2$.

By Example 2, we generalize the question **Q2-1** to

Question 3. When the lattice closure $\widetilde{\tau}^{-1}(g)$ is stci on $I(W)$, for non zero $g \in k[t]$?

We consider the following two conditions:

(C1) the ideal generated by $M \in k[t]$ satisfying $Mg \in \operatorname{Im}\widetilde{\tau}$ is a monomial ideal containing a power of g of height ≥ 2,

(C2) there is a monomial $M \in \operatorname{Im}\widetilde{\tau}$ satisfying $Mg \in \operatorname{Im}\widetilde{\tau}$.

Proposition 4. *Assume that g is not a monomial. Then $\widetilde{\tau}^{-1}(g)$ is generated by one element up to radical on $I(W)$ if both conditions (C1) and (C2) are satisfied.*

Proof. Let $g = \sum c_l M_l$ where M_l is a monomial and $c_l \in k$ is non zero for each l. By (C1) and (C2), for each l, there is $b_l > 0$ satisfying $M_l^{b_l} M_{l'} \in \operatorname{Im}\widetilde{\tau}$ for each l' (this part is crucial). This implies that there is $b > 0$ with $g^b \in \operatorname{Im}\widetilde{\tau}$. Therefore $\widetilde{\tau}^{-1}(g)$ is stci on $I(W)$. □

Note that the converse of Proposition 4 is valid, if g is irreducible, or a binomial $F(v)$.

From Proposition 4, we obtain the following result.

Corollary 5. *Let d_i be a positive integer for each i, $W = \mathbb{Z}^t(-d_1, d_2, d_3, -d_4)$, and τ the defining map of W. And let V_1, V_2 submodules in $\mathbb{Z}^4$ of rank 2 containing W with $V_1/W \cong V_2/W \cong V_1 + V_2/V_1 \cong \mathbb{Z}$. If both $F(\tau(v_1))$ and $F(\tau(v_2))$ satisfy the condition (C1) and if*

$$\begin{vmatrix} \sigma_1(v_1) & \sigma_1(w) \\ \sigma_4(v_1) & \sigma_4(w) \end{vmatrix} \begin{vmatrix} \sigma_2(v_1) & \sigma_2(w) \\ \sigma_3(v_1) & \sigma_3(w) \end{vmatrix} \begin{vmatrix} \sigma_1(v_2) & \sigma_1(w) \\ \sigma_4(v_2) & \sigma_4(w) \end{vmatrix} \begin{vmatrix} \sigma_2(v_2) & \sigma_2(w) \\ \sigma_3(v_2) & \sigma_3(w) \end{vmatrix} < 0,$$

then $I(V_1) \cap I(V_2)$ *is stci on* $I(W)$, *where* $v_l \in V_l$ *with* $V_l = W + \mathbb{Z}v_l$ *for* $l = 1, 2$.

Example 3. Let $w = {}^t(-1, 1, 1, -1)$, $W = \mathbb{Z}w$, and τ the defining map of W. And put $v_1 = {}^t(-3, 4, -1, 0)$ and $v_2 = {}^t(7, -1, -4, 0)$. Then $\tau(v_1) = {}^t(1, -4, 4, -1)$, $\tau(v_2) = {}^t(6, 3, -1, -4)$. Since $t_1^3 F(\tau(v_1)), t_4^3 F(\tau(v_1)) \in \operatorname{Im} \widetilde{\tau}$ (resp. $t_2^3 F(\tau(v_2)), t_3^3 F(\tau(v_2)) \in \operatorname{Im} \widetilde{\tau}$), $F(\tau(v_1))$ (resp. $F(\tau(v_2))$) satisfies the condition (C1). Since the inequality in Corollary 5 is satisfied, we conclude that $I(V_1) \cap I(V_2)$ is stci on $I(W)$, where $V_l = \mathbb{Z}v_l + \mathbb{Z}w$ for $l = 1, 2$.

For the question **Q2-2**, we have

Lemma 6. *Let* $V \subset \mathbb{Z}^4$ *be of rank* 3 *and* $V_l \subset V$ *with* $V/V_l \cong \mathbb{Z}$ *for* $l = 1, 2$. *If* $I(V)$ *is stci on both* $I(V_1)$ *and* $I(V_2)$ *and if* $\sqrt{I(V_1) + I(V_2)} = I(V)$, *then* $I(V)$ *is stci on* $I(V_1) \cap I(V_2)$.

Proof. There is $g_l \in I(V)$ with $I(V) = \sqrt{(g_l) + I(V_l)}$ for each l. We may assume that g_1 and g_2 are homogeneous of the same degree. Since $g_1 - g_2 \in I(V)$ and since $I(V)$ is homogeneous of height 3, there is $m > 0$ satisfying $g_1^m - g_2^m \in I(V_1) + I(V_2)$. We write $g_1^m - g_2^m = h_1 + h_2$ where $h_1 \in I(V_1)$ and $h_2 \in I(V_2)$. Then

$$I(V) = \sqrt{(g_1^m - h_1) + I(V_1) \cap I(V_2)}.$$

□

Example 4. Let V_1, V_2 be as in Example 3 and put $V = V_1 + V_2$. Then $V = \operatorname{Ker}(17, 19, 25, 27)$ and $\sqrt{I(V_1) + I(V_2)} = I(V)$. By Lemma 6, $I(V)$ is stci on $I(V_1) \cap I(V_2)$. Combining with Example 3, the monomial curve associated with 17, 19, 25 and 27 is a set-theoretic complete intersection.

Theorem 7. *Let* $V = \operatorname{Ker}(n_1, n_2, n_3, n_4)$. *If* $n_1 + n_4$ *is contained in the semigroup generated by* n_2 *and* n_3, *then* $I(V)$ *is a set-theoretic complete intersection.*

By the conditon of the theorem, there are natural numbers d_2, d_3 satisfying $n_1 + n_4 = d_2 n_2 + d_3 n_3$. Put $V = \operatorname{Ker}(n_1, n_2, n_3, n_4)$. Then $w = {}^t(-1, d_2, d_3, -1) \in V$. Now, we may find $v_1, v_2 \in V$ with $V = \mathbb{Z}w + \mathbb{Z}v_1 + \mathbb{Z}v_2$ such that both $I(V_1)$ and $I(V_2)$ satisfy the condition (C1) where $V_l = \mathbb{Z}w + \mathbb{Z}v_l$ for each l and that v_1, v_2 satisfy the inequality in Corollary 5. By Corollary 5, $I(V_1) \cap I(V_2)$ is stci on $I(\mathbb{Z}w)$. Further, if $I(V)$ is stci on $I(V_1) \cap I(V_2)$, then we conlude that $I(V)$ is a set-theoretic complete intersection, and this is possible. This is a sketch of the proof of Theorem 7.

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ON THE k-BUCHSBAUM PROPERTY OF SYMBOLIC POWERS OF STANLEY-REISNER IDEALS

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1. Introduction

Let Δ be a simplicial complex on a vertex set $[n] = \{1, 2, ..., n\}$. Let $S = K[x_1, x_2, ..., x_n]$ be a polynomial ring of n-variables over a field K. Stanley-Reisner ideal I is defined as;

$$I = I_\Delta = (\prod_{i \in F} x_i \mid F \notin \Delta),$$

which is a square-free monomial ideal of S being associated to Δ. The residue class ring S/I is called the Stanley-Reisner ring. Throughout this report, we assume that Δ is pure and $\dim(\Delta) = 1$, which means that any maximal element of Δ consists of two element.

It is known that S/I is always a Buchsbaum ring, and that S/I is Cohen-Macaulay if and only if Δ is connected (see [BH],[S]). For the case of symbolic powers $I^{(r)}$ of I, the first author and N. V. Trung gave the characterization for $S/I^{(r)}$ to be Cohen-Macaulay in terms of the graphical property of Δ ([MT]). After that the authors get the characterization of Buchsbaumness of $S/I^{(r)}$ in [MN].

In this report, we study the k-Buchsbaum property of $S/I^{(r)}$ for all $r > 0$ and all Δ. In our situation, $S/I^{(r)}$ is a generalized Cohen-Macaulay ring with $\dim S/I^{(r)} = 2$ and $\operatorname{depth} S/I^{(r)} > 0$. The condition for $S/I^{(r)}$ to be k-Buchsbaum is equivalent to saying that k is the minimal number satisfying $\mathfrak{m}^k H^1_\mathfrak{m}(S/I^{(r)}) = (0)$. We put

$$k(r) = \min\{k \in \mathbb{N} | \mathfrak{m}^k H^1_\mathfrak{m}(S/I^{(r)}) = (0)\}.$$

Our purpose can be said to determine the value $k(r)$ for any $r > 0$ and any Δ. The main result is the following theorem.

Theorem 1.1. *Let $r > 1$ be an integer. Assume that $S/I^{(r)}$ is not Cohen-Macaulay. Then*

$$k(r) = \mathrm{d}(H^1_\mathfrak{m}(S/I^{(r)})) = \begin{cases} r-2 & \text{if } \operatorname{diam}(\Delta) \leq 2 \\ r-1 & \text{if } 3 \leq \operatorname{diam}(\Delta) < \infty \\ 2r-1 & \text{if } \operatorname{diam}(\Delta) = \infty \end{cases}$$

2000 *Mathematics Subject Classification.* 13H10, 13F55, 05E99.

Key words and phrases. k-Buchsbaum, monomial ideal, connected graph, symbolic power.

Here, we put

$$\mathrm{d}(M) = \max\{n | M_n \neq 0\} - \min\{n | M_n \neq 0\} + 1$$

for the finitely generated $\mathbb{Z}$-graded module M with $M \neq (0)$ and $\mathrm{d}(M) = 0$ if $M = (0)$. It is clear that $k(r) \leq \mathrm{d}(H^1_{\mathfrak{m}}(S/I^{(r)}))$. $\mathrm{diam}(\Delta)$ denotes the diameter of simplical complex Δ, that is defined as;

$$\mathrm{diam}(\Delta) = \max_{i,j \in [n]} \mathrm{dist}(i,j),$$

where $\mathrm{dist}(i,j)$ is the minimal length of the path between nodes i and j. $\mathrm{dist}(i,j)$ is infinite if there is no paths connecting i and j. Thus, $\mathrm{diam}(\Delta) < \infty$ is equivalent to saying that Δ is connected.

From Theorem 1.1, we immediately get the characterization of the Buchsbaumness of $S/I^{(r)}$.

Corollary 1.2. ([MN, Theorem 3.7]) *Let I be the Stanley-Reisner ideal of a pure simplicial complex Δ of dimension one. Let $r > 0$ be an integer. Then the following statements hold true.*

(1) *$S/I^{(2)}$ is Buchsbaum if and only if Δ is connected.*
(2) *$S/I^{(3)}$ is Buchsbaum if and only if* $\mathrm{diam}(\Delta) \leq 2$.
(3) *Let $r > 3$. If $S/I^{(r)}$ is Buchsbaum, then it is Cohen-Macaulay.*

This report consists of three sections. In Section 2, we set up the notation and terminology. We quote some fundamental results from [MT] and [MN]. In Section 3, we prepare auxiliary arguments with respect to the cone of complexes, and then give the proof of the main result.

2. Preliminaries

We begin with the notation on a simplicial complex. A simplicial complex Δ on a finite set $[n] = \{1, 2, ..., n\}$ is a collection of subsets of $[n]$ such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. Notice that, for the convenience in the later discussions, we do *not* assume the condition that $\{i\} \in \Delta$ for $i = 1, 2, ..., n$. We put $\dim F = |F| - 1$, where $|F|$ means the cardinality of F, and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which is called the dimension of Δ. When we assume a linear order on $[n]$, say $<$, Δ is called an oriented simplicial complex. In such a case, we denote $F = \{i_1, ..., i_r\}$ for $F \in \Delta$ with the order sequence $i_1 < ... < i_r$. Let Δ be an oriented simplicial complex with $\dim \Delta = d$. We denote by $\mathcal{C}(\Delta)_\bullet$ the augmented oriented chain complex of Δ:

$$\mathcal{C}(\Delta)_\bullet : 0 \to C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to C_{-1} \to 0$$

where

$$C_t = \bigoplus_{\substack{F \in \Delta \\ \dim F = t}} \mathbb{Z}F \qquad \text{and} \qquad \partial F = \sum_{j=0}^{t} (-1)^j F_j$$

for all $F \in \Delta$. Here we denote $F_j = \{i_0, ..., \hat{i_j}, ..., i_t\}$ for $F = \{i_0, ..., i_t\}$. For any field K, we define the i-th reduced simplicial homology group $\widetilde{H}_i(\Delta; K)$ of Δ to be the i-th homology group of the complex $\mathcal{C}(\Delta)_\bullet \otimes K$. Further we define the i-th reduced simplicial cohomology group $\widetilde{H}^i(\Delta; K)$ of Δ to be the i-th cohomology group of the dual chain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta)_\bullet, K)$ for all i. Then it follows that

$$\dim_k \widetilde{H}_i(\Delta; K) = \dim_k \widetilde{H}^i(\Delta; K) \quad \text{for all } i \in \mathbb{Z} \qquad \text{and}$$

$$\widetilde{H}_{-1}(\Delta; K) \cong \widetilde{H}^{-1}(\Delta; K) \cong \begin{cases} K & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{otherwise} \end{cases} .$$

We also note that $\widetilde{H}_i(\Delta; K) = \widetilde{H}^i(\Delta; K) = 0$ for all $i \in \mathbb{Z}$ if $\Delta = \emptyset$. Moreover, it is known that

$$\dim_K(\widetilde{H}_0(\Delta; K)) = \text{the number of connected components of } \Delta - 1$$

when $\Delta \neq \emptyset$ (see [V, Proposition 5.2.3]). Let $\Gamma \subseteq \Delta$ be a simplicial subcomplex of Δ. Then $\mathcal{C}(\Gamma)_\bullet$ is a subcomplex $\mathcal{C}(\Delta)_\bullet$, which yields the quotient complex $\mathcal{C}(\Delta)_\bullet/\mathcal{C}(\Gamma)_\bullet$. The cohomology module

$$\widetilde{H}^i(\Delta, \Gamma; k) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta)_\bullet/\mathcal{C}(\Gamma)_\bullet; k))$$

is called the *i-th reduced relative simplicial cohomology* of the pair (Δ, Γ). Let Γ and Δ be simplicial complexes on disjoint vertex sets V and W, respectively. The *join* $\Gamma * \Delta$ is the simplicial complex on the vertex set $V \cup W$ consists of faces $F \cup G$ where $F \in \Gamma$ and $G \in \Delta$. The *cone*

$$\mathrm{Cone}(\Delta) = x * \Delta$$

of Δ is the join of a point $\{x\}$ with Δ.

Let I be a monomial ideal of a polynomial ring $S = K[x_1, x_2, ..., x_n]$ over K. For $\mathbf{a} = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$ we put the subset $G_{\mathbf{a}} = \{i | a_i < 0\}$ of $[n]$. The degree complex (see [T]) is a simplicial complex denoted by $\Delta_{\mathbf{a}}(I)$ and consists of all $F \subseteq [n]$ such that

(1) $F \cap G_{\mathbf{a}} = \emptyset$,
(2) For every minimal generator $x^{\mathbf{b}}$ of I there exists an index $i \notin F \cup G_{\mathbf{a}}$ with $b_i > a_i$.

Here we pick up important results stated in [MT] and [MN], which will be applied several times in our argument.

Lemma 2.1. *Let I be the Stanley-Reisner ideal of a pure simplicial complex Δ of dimension one. Then, the following assertions hold true for all $0 < r \in \mathbb{N}$.*

(1) *Let $\mathbf{a} \in \mathbb{N}^n$ and $\Delta_{\mathbf{a}}(I^{(r)}) \neq \emptyset$. Then $\Delta_{\mathbf{a}}(I^{(r)})$ is a subcomplex of Δ of pure dimension one.*
(2) *Let $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{N}^n$. For $i, j \in [n]$, we put $\sigma_{ij} = |\mathbf{a}| - (a_i + a_j)$, where $|\mathbf{a}| = \sum_{k=1}^{n} a_k$. Then we have the following equivalent conditions:*
   (a) $\{i, j\} \in \Delta_{\mathbf{a}}(I^{(r)})$.
   (b) $\sigma_{ij} < r$ *and* $\{i, j\} \in \Delta$.

Next is the behaviour of the first local cohomology modules of $S/I^{(r)}$.

Lemma 2.2 ([MT],[MN]). *Let I be the Stanley-Reisner ideal of a pure simplicial complex Δ of dimension one. Let $r > 0$ be an integer. The following assertions hold true.*

(1) *Let $\mathbf{a} \in \mathbb{Z}^n$. If $G_{\mathbf{a}} \neq \emptyset$ then $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} = 0$.*
(2) $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = 0$ *for all $j > 2r - 2$.*
(3) *Let $0 \leq j < r$. Then $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = 0$ if and only if G is connected.*
(4) *Assume $r > 1$. Then $[H^1_{\mathfrak{m}}(S/I^{(r)})]_r = 0$ if and only if $\operatorname{diam}(G) \leq 2$.*
(5) *Assume $r > 2$ and $r + 1 \leq j \leq 2r - 2$. Then $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = (0)$ if and only if any pair of disjoint edges of Δ is contained in a cycle of length 4.*

At the end of the section, we recall a formula between the local cohomology modules and reduced cohomology modules, due to Takayama.

Lemma 2.3. ([BH, Lemma 5.3.7], [T, Lemma 2]) *Let I be a monomial ideal of S. For all $t \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^n$, there is an isomorphism of K-vector spaces*

$$H^t_{\mathfrak{m}}(S/I)_{\mathbf{a}} \cong \widetilde{H}^{t-|G_{\mathbf{a}}|-1}(\Delta_{\mathbf{a}}(I); K).$$

The above isomorphism gives us more information. Let $\mathbf{b} \in \mathbb{N}^n$ and take the monomial $\mathbf{x^b} = \prod_{j=1}^n x_j^{b_j} \in S$. The multiplicative map $S/I \ni f \mapsto \mathbf{x^b} f \in S/I$ induces the homomorphism

$$H^t_{\mathfrak{m}}(S/I)_{\mathbf{a}} \xrightarrow{\mathbf{x^b}} H^t_{\mathfrak{m}}(S/I)_{\mathbf{a}+\mathbf{b}}.$$

Lemma 2.4. ([MN, Lemma 2.3]) *Let I be a monomial ideal of S and $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. For any integers $j \geq 0$, we have the following commutative diagram:*

$$\begin{array}{ccc} H^j_{\mathfrak{m}}(S/I)_{\mathbf{a}} & \xrightarrow{\mathbf{x^b}} & H^j_{\mathfrak{m}}(S/I)_{\mathbf{a}+\mathbf{b}} \\ \downarrow & & \downarrow \\ \widetilde{H}^{j-1}(\Delta_{\mathbf{a}}(I); K) & \longrightarrow & \widetilde{H}^{j-1}(\Delta_{\mathbf{a}+\mathbf{b}}(I); K) \end{array}$$

where the vertical maps are isomorphisms as in Lemma 2.3 and the bottom map is induced from the natural embedding $\Delta_{\mathbf{a}+\mathbf{b}}(I) \subseteq \Delta_{\mathbf{a}}(I)$ of simplicial complexes.

3. Proof of the main result

We begin by establishing the following assertion.

Lemma 3.1. *Let Δ be an arbitrary simplicial complex over $[n]$ and $\Gamma \subseteq \Delta$ a simplicial subcomplex. Then there is an isomorphism of K-vector spaces*

$$\widetilde{H}^j(\Delta, \Gamma; K) \cong \widetilde{H}^j(\Delta \cup \operatorname{Cone}(\Gamma); K),$$

for all $j \in \mathbb{Z}$, where $\operatorname{Cone}(\Gamma) = \operatorname{Cone}_x(\Gamma)$ with a new vertex $x \notin [n]$.

Proof. By definition, there is an isomorphism between the chain complexes

$$\mathcal{C}(\Delta)_\bullet/\mathcal{C}(\Gamma)_\bullet \cong \mathcal{C}(\Delta \cup \mathrm{Cone}(\Gamma))_\bullet/\mathcal{C}(\mathrm{Cone}(\Gamma))_\bullet.$$

Therefore, $\widetilde{H}^j(\Delta, \Gamma; K) \cong \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma), \mathrm{Cone}(\Gamma); K)$ for all $j \in \mathbb{Z}$. On the other hand, the short exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta \cup \mathrm{Cone}(\Gamma))_\bullet/\mathcal{C}(\mathrm{Cone}(\Gamma))_\bullet, K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta \cup \mathrm{Cone}(\Gamma))_\bullet, K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\mathrm{Cone}(\Gamma))_\bullet, K) \longrightarrow 0$$

yields the following long exact sequence

$$\cdots \longrightarrow \widetilde{H}^{j-1}(\mathrm{Cone}(\Gamma); K) \longrightarrow \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma), \mathrm{Cone}(\Gamma); K) \longrightarrow \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma); K) \longrightarrow \widetilde{H}^j(\mathrm{Cone}(\Gamma)); K) \longrightarrow \cdots$$

Since $\mathrm{Cone}(\Gamma)$ is acyclic, $\widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma); K) \cong \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma), \mathrm{Cone}(\Gamma); K)$ for all $j \in \mathbb{Z}$. This implies our assertion.

□

We now come to the point to prove the main theorem.

Proof of Theorem 1.1

The proof is divided into three cases. Let $t = \mathrm{d}(H^1_{\mathfrak{m}}(S/I^{(r)}))$. We shall prove that $\mathfrak{m}^{t-1}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$.

Case 1: Suppose that $2 < \mathrm{diam}(\Delta) < \infty$. Then $t = r - 1$. We will show that $\mathfrak{m}^{r-2}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$. It is clear if $r = 2$ by Lemma 2.2 (4). Assume that $r \geq 3$. Since $\mathrm{diam}(\Delta) > 2$, there exists $1 \leq i < j \leq n$ such that $\mathrm{dist}(i, j) \geq 3$. Hence $\mathrm{star}_\Delta(i) \cup \mathrm{star}_\Delta(j)$ is not connected. Put

$$\mathbf{a} = (r-1)\mathbf{e}_i + \mathbf{e}_j \quad \text{and} \quad \mathbf{b} = (r-2)\mathbf{e}_j,$$

where $\mathbf{e}_i$ is i-th unit vector in $\mathbb{Z}^n$. Then one can check that

$$\Delta_{\mathbf{a}}(I^{(r)}) = \Delta_{\mathbf{a}+\mathbf{b}}(I^{(r)}) = \mathrm{star}_\Delta(i) \cup \mathrm{star}_\Delta(j),$$

by Lemma 2.1 (2). Hence

$$0 \neq H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \xrightarrow{x^{\mathbf{b}}} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}+\mathbf{b}}$$

is isomorphic, which implies that $\mathfrak{m}^{r-2}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$. Therefore, we get $k(r) = r - 2$.

Case 2: Suppose that $\mathrm{diam}(\Delta) \leq 2$. Then $t = r - 2$. We will show that $\mathfrak{m}^{r-3}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$. Since $S/I^{(r)}$ is not Cohen-Macaulay, we have $r \geq 3$ by Lemma 2.2, besides, there exists a pair of disjoint edges of Δ, say $\{1, 2\}, \{3, 4\}$, which is not contained in any cycle of length 4 by [MT, Theorem 2.4]. We may assume that $\{1, 3\}, \{1, 4\} \notin \Delta$. It is clear if $r = 3$ by Lemma 2.2 (4). It is enough to check the assertion in the case that $r \geq 4$. Put

$$\mathbf{a} = (r-1)\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4 \quad \text{and} \quad \mathbf{b} = (r-3)\mathbf{e}_3.$$

Then one can check that

$$\Delta_{\mathbf{a}}(I^{(r)}) = \Delta_{\mathbf{a}+\mathbf{b}}(I^{(r)}) = \mathrm{star}_\Delta(1) \cup \{3,4\},$$

which is not connected since $\{1,3\}, \{1,4\} \notin \Delta$. Therefore,

$$0 \neq H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \xrightarrow{x^{\mathbf{b}}} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}+\mathbf{b}}$$

is isomorphic, which implies that $\mathfrak{m}^{r-3}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$ as required.

Case 3: Suppose that $\mathrm{diam}(\Delta) = \infty$, i. e., Δ is not connected. Then $t = 2r-1$. We will show that $\mathfrak{m}^{2r-2}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$. Since Δ is not connected, we may assume that $\{1,2\}, \{3,4\}$ belong to different components of Δ. Put

$$\mathbf{a} = (r-1)\mathbf{e}_1 + (r-1)\mathbf{e}_3.$$

Applying Lemma 2.1, one can check

$$\Gamma = \Delta_{\mathbf{a}}(I^{(r)}) = \mathrm{star}_\Delta(1) \cup \mathrm{star}_\Delta(3).$$

Then $\widetilde{H}^{-1}(\Gamma; K) = 0$ and Γ is not connected. Hence, we have the following long exact sequence of reduced cohomology modules

$$0 \longrightarrow \widetilde{H}^0(\Delta,\Gamma;K) \longrightarrow \widetilde{H}^0(\Delta;K) \longrightarrow \widetilde{H}^0(\Gamma;K) \longrightarrow \widetilde{H}^1(\Delta,\Gamma;K) \longrightarrow \cdots$$

Note that $\widetilde{H}^0(\Delta;K) \longrightarrow \widetilde{H}^0(\Gamma;K)$ in the above sequence is induced from the natural embedding $\Gamma \subseteq \Delta$. On the other hand, by Lemma 3.1,

$$\begin{aligned}\dim_K(\widetilde{H}^0(\Delta,\Gamma;K)) &= \dim_K(\widetilde{H}^0(\Delta \cup \mathrm{Cone}(\Gamma);K)) \\ &= \text{the number of connected components of } \Delta \cup \mathrm{Cone}(\Gamma) \ - \ 1 \\ &< \text{the number of connected components of } \Delta \ - \ 1 \\ &= \dim_K(\widetilde{H}^0(\Delta;K)).\end{aligned}$$

Hence the natural map $\widetilde{H}^0(\Delta;K) \longrightarrow \widetilde{H}^0(\Gamma;K)$ is never zero map. By Lemma 2.4, we have the following commutative diagram:

$$\begin{array}{ccc} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{0}} & \xrightarrow{\mathbf{x}^{\mathbf{a}}} & H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \\ \downarrow & & \downarrow \\ \widetilde{H}^0(\Delta_{\mathbf{0}}(I^{(r)});K) & \longrightarrow & \widetilde{H}^0(\Delta_{\mathbf{a}}(I^{(r)});K) \end{array}$$

Moreover, since $\Delta_{\mathbf{0}}(I^{(r)}) = \Delta$ we obtain $\mathbf{x}^{\mathbf{a}}H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{0}} \neq (0)$. It implies

$$\mathfrak{m}^{2r-2}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0),$$

which is the desired conclusion.

Acknowledgements. This research was carried out while the first author visited Meiji University. He thanks the support from JSPS Ronpaku (Dissertation PhD) program for this visit.

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On F-thresholds of some determinantal rings

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This is a joint work with Ken-ichi Yoshida.

Throughout this article, let k be a perfect field of characteristic $p > 0$, and let R be an F-finite (i.e. the Frobenius map is finite) reduced Noetherian ring containing k.

For a ring R, let R° be the complement of the union of all minimal prime ideals of R, and let $\mathfrak{m}$ be a maximal ideal of R. For each integer $e \geq 1$ and each ideal I of R, $I^{[p^e]}$ denotes the ideal generated by the p^eth powers of the elements of I.

Suppose that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. For each integer $e \geq 1$ and each ideal J such that $\mathfrak{a} \subseteq \sqrt{J}$, put

$$\nu_{\mathfrak{a}}^J(p^e) = \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\}.$$

Definition 1 (**F-threshold, [HMTW]**) Let $R, \mathfrak{a}$ and J be as above. Then the *F-threshold* of $(R, \mathfrak{a})$ with respect to J, denoted by $c^J(\mathfrak{a})$, is defined by

$$c^J(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

if the limit exists. In particular, we call $c^{\mathfrak{a}}(\mathfrak{a})$ the *diagonal F-threshold* of R with respect to $\mathfrak{a}$.

Definition 2 (**F-pure threshold, [TW]**) Let R and $\mathfrak{a}$ be as above. Let $t \geq 0$ be a real number. Then the pair $(R, \mathfrak{a}^t)$ is *F-pure* if for all large $q = p^e$, there exists an element $d \in \mathfrak{a}^{\lfloor t(q-1) \rfloor}$ such that $d^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-homomorphism.

The *F-pure threshold* of R with respect to $\mathfrak{a}$, denoted by $\mathrm{fpt}(\mathfrak{a})$, as

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-pure}\}.$$

If R is local, then $c^{\mathfrak{m}}(\mathfrak{m}) \geq \mathrm{fpt}(\mathfrak{m})$ holds ([HMTW]).

We ask the following question:

Question 3 (1) What is the value of $c^{\mathfrak{m}}(\mathfrak{m})$?

(2) What is the value of $\mathrm{fpt}(\mathfrak{m})$?

(3) When does $c^{\mathfrak{m}}(\mathfrak{m})$ coincide with $\mathrm{fpt}(\mathfrak{m})$?

1 F-threshold

We gave an answer of Question 3(1) for some determinantal rings.

Theorem 4 Let $R = \bigoplus_{n\geq 0} R_n, S = \bigoplus_{n\geq 0} S_n$ be Noetherian graded rings. Assume that $k = R_0 = S_0$ and $R = k[R_1], S = k[S_1]$. Put $\mathfrak{m} = \bigoplus_{n>0} R_n, \mathfrak{n} = \bigoplus_{n>0} S_n$.

Let $T = R\#S = \bigoplus_{n\geq 0} R_n \otimes S_n$ be the Segre product of R and S. Let M be the graded maximal ideal of T. Then

$$c^M(M) = \max\{c^{\mathfrak{m}}(\mathfrak{m}), c^{\mathfrak{n}}(\mathfrak{n})\}.$$

proof) First, note that $M^r = \mathfrak{m}^r\#\mathfrak{n}^r$ and $M^{[p^e]} = \mathfrak{m}^{[p^e]}\#\mathfrak{n}^{[p^e]}$.

Assume that $c^{\mathfrak{m}}(\mathfrak{m}) \geq c^{\mathfrak{n}}(\mathfrak{n})$ and $\mathfrak{m}^r \not\subseteq \mathfrak{m}^{[p^e]}$. Then we can take $x \in \mathfrak{m}^r \backslash \mathfrak{m}^{[p^e]}$ where $\deg x = d$.

For all $y \in S_d$, we have $x \otimes y \notin M^{[p^e]}$. Indeed, consider an exact sequence

$$0 \longrightarrow [\mathfrak{m}^{[p^e]}]_d \longrightarrow R_d \longrightarrow [R/\mathfrak{m}^{[p^e]}]_d \longrightarrow 0.$$

By Applying the functor $\otimes S_d$, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\mathfrak{m}^{[p^e]}]_d & \longrightarrow & R_d & \longrightarrow & [R/\mathfrak{m}^{[p^e]}]_d & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & [\mathfrak{m}^{[p^e]}]_d \otimes S_d & \longrightarrow & R_d \otimes S_d & \longrightarrow & [R/\mathfrak{m}^{[p^e]}]_d \otimes S_d & \longrightarrow & 0 \end{array}$$

as above. Since $x \notin \mathfrak{m}^{[p^e]}$, for all $y \in S_d$,

$$x \otimes y \notin \mathrm{Ker}(R_d \otimes S_d \longrightarrow [R/\mathfrak{m}^{[p^e]}]_d \otimes S_d).$$

Therefore

$$x \otimes y \notin [\mathfrak{m}^{[p^e]}]_d \otimes S_d \supset [\mathfrak{m}^{[p^e]}]_d \otimes [\mathfrak{n}^{[p^e]}]_d = [M^{[p^e]}]_d.$$

So we have $c^{\mathfrak{m}}(\mathfrak{m}) \leq c^M(M)$.

Next, we will show $c^M(M) \leq c^{\mathfrak{m}}(\mathfrak{m})$. Set

$$\tilde{\nu}_{\mathfrak{m}}^{\mathfrak{m}}(p^e) = \inf\{r \in \mathbb{N} \mid \mathfrak{m}^r \subseteq \mathfrak{m}^{[p^e]}\}.$$

Assume $\mathfrak{m}^r \subseteq \mathfrak{m}^{[p^e]}$. Since $M^r \subseteq M^{[p^e]}$, so we have

$$\nu_M^M(p^e) < \tilde{\nu}_M^M(p^e) \leq \tilde{\nu}_{\mathfrak{m}}^{\mathfrak{m}}(p^e).$$

Therefore, we have $c^M(M) \leq c^{\mathfrak{m}}(\mathfrak{m})$. □

Corollary 5 For integers $2 \leq r \leq s$, let $\mathbf{X}$ be a generic $r \times s$ matrix. Set

$$R = k[\mathbf{X}]/I_2(\mathbf{X}) = k[X_1, \ldots, X_r]\#k[Y_1, \ldots, Y_s].$$

And let $\mathfrak{m}$ be a maximal ideal of R. Then

$$c^{\mathfrak{m}}(\mathfrak{m}) = \max\{r, s\}.$$

2 F-pure threshold

We do not have any satisfying answer. However we can give a partial answer.

Proposition 6 Let

$$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{bmatrix}$$

be a generic 2×3 matrix. Set $R = k[[\mathbf{X}]]/I_2(\mathbf{X})$, and put $\mathfrak{m} = (X_1, X_2, X_3, Y_1, Y_2, Y_3)R$. Then $\mathrm{fpt}(\mathfrak{m}) = 2 \neq 3 = c^{\mathfrak{m}}(\mathfrak{m})$.

To prove this proposition, we use two lemmas.

Lemma 7 Let $R, \mathfrak{m}$ be the same as in Proposition 6. Then

$$2 \leq \mathrm{fpt}(\mathfrak{m}) \leq 2 + \frac{4}{p}.$$

Lemma 8 Let $R, \mathfrak{m}$ be the same as in Proposition 6. Then the F-pure threshold $\mathrm{fpt}(\mathfrak{m})$ does not depend on p.

proof of Lemma 8) First, R has three ring-theoretic properties: that is strong F-regular, $\mathbb{N}$-graded and toric. Second, the following is known: if R is strongly F-regular, then

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid \tau(\mathfrak{a}^t) = R\}.$$

By [Ta, Corollary 3.5], R is strongly F-regular if and only if $\mathrm{Ann}_R(0)^{*\mathfrak{a}^t}_{E_R} = R$. By the $\mathbb{N}$-gradedness of R and the definition of the generalized test ideal $\tau(\mathfrak{a}^t)$, $\mathrm{Ann}_R(0)^{*\mathfrak{a}^t}_{E_R} = R$ if and only if $\tau(\mathfrak{a}^t) = R$. And by an argument of [B], $\tau(\mathfrak{a}^t)$ does not depend on p. □

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F-coherent rings and related results

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Throughout, all rings are commutative and of characteristic $p > 0$. The notation $(R, \mathfrak{m})$ denotes a local Noetherian ring. After tight closure theory was invented by Hochster and Huneke, certain classes of Noetherian rings defined via the Frobenius map have been studied in connection with those singularities that appear in birational geometry. In this article, we would like to introduce a new class of Noetherian rings via the Frobenius map. Let R be a Noetherian ring of characteristic $p > 0$ and we define the ring R^{∞} as the perfect closure of R_{red}, where R_{red} is the reduced part of R. Then we say that R is *F-coherent* if the perfect closure R^{∞} is a coherent ring. Recall that a commutative ring is coherent if its every finitely generated ideal is finitely presented. As the ring R^{∞} is usually not Noetherian, it makes sense to consider when R^{∞} is coherent. In general, it is hard to check that a given ring is coherent. For example, let $R := k[X^2, X^3, Y_1, Y_2, \ldots, XY_1, XY_2, \ldots] \subseteq k[X, Y_1, Y_2, \ldots]$ be a subring of a polynomial algebra in infinitely many variables over a field k. Then we have $((X^3)R :_R X^2) = (X^3, X^4, XY_1, XY_2, \ldots)$, which is not finitely generated. Hence R is not coherent.

Question 1. What are the necessary (or sufficient) conditions to assure that R is F-coherent?

Question 2. Is there any relationship between F-coherent rings and rings studied typically in tight closure theory?

Answers to these natural questions are given under various assumptions. However, the following question still seems unclear at the time of writing this article and it will be an important study to find out a practical way of constructing many examples of F-coherent rings.

Question 3. A regular ring is F-coherent. Is there any example of an F-coherent ring whose perfect closure does not coincide with the perfect closure of a regular ring?

The coherence of R^{∞} or of R^{+}, which is the integral integral clsoure of R in an algebraic closure of the field of fractions of R, was first investigated by Aberbach and Hochster in an attempt to relate the so-called localization problem in tight closure to the plus closure of rings. It is very clear from the definition that plus closure commutes with localization. However, their attempt did not succeed, because it was recently shown by Brenner and Monsky [1] that tight clsoure does not commute with localization, at least in dimesnion three.

1 Basic properties of F-coherent rings

As a simple observation, if $R \to S$ is a purely inseparable extension of Noetherian rings such that either R or S is F-coherent, then so is the other. This is quite useful, because it is usually hard to see the coherent property of rings. We state the following proposition without proof (see [4] for a proof).

Proposition 4. *Let R be a Noetherian ring of characteristic $p > 0$. Then the following hold.*

(1) *Any regular ring is F-coherent.*

(2) *Let S be a multiplicative subset of an F-coherent ring R. Then the localization $S^{-1}R$ is F-coherent as well.*

(3) *Let U denote the F-coherent locus of* Spec R, *i.e. the set of all $P \in$* Spec R *for which R_P is F-coherent. Then, if U is constructible, U is a non-empty Zariski open subset.*

(4) *Let $R \subseteq S$ be a faithfully flat extension. Then, if S is F-coherent, so is R.*

In relation to the above proposition, we do not know whether R is F-coherent or not, assuming that the ring extension $R \subseteq S$ is pure and S is F-coherent. We also point out that (3) in the proposition is equivalent to asking the topological property of the set of all $P \in \text{Spec } R^{\infty}$ such that R^{∞}_P is coherent.

Example 5. F-coherent rings which are not regular may be constructed as follows. Let us consider $k[t^3, t^5, t^7]$ for a field k of characteristic $p = 3, 5, 7$. Then the map $k[t^3, t^5, t^7] \subseteq k[t]$ is obviously purely inseparable. Hence $k[t^3, t^5, t^7]$ is F-coherent if $p = 3, 5, 7$, but it is not normal.

Here is another example. Let $k[x^4, x^3y, xy^3, y^4]$ for a field of characteristic $p = 2$. Then we have a tower $k[x^4, y^4] \subseteq k[x^4, x^3y, xy^3, y^4] \subseteq k[x, y]$, which is purely inseparable, due to the assumption $p = 2$. Hence $k[x^4, x^3y, xy^3, y^4]$ is F-coherent, but it is not Cohen-Macaulay.

As seen in the examples above, when a ring is purely inseparable (sub)extension of a regular ring, it is immediate that the ring is F-coherent. In the following section, we will give an example of a non-F-coherent ring using the Segre product. But this requires some preliminaries from the theory of valuations.

2 Cohen-Macaulay property of R^∞

In general, the following fact is known (the proof is found in [5]).

Theorem 6 (Roberts-Singh-Srinivas). *Let $(R, \mathfrak{m})$ be a complete local domain of characteristic $p > 0$. Then there exists a nonzero element $c \in R$ such that*

$$c^{1/p^n}((x_1, \ldots, x_i) :_{R^\infty} x_{i+1}) \subseteq (x_1, \ldots, x_i)$$

for all $n \in \mathbb{N}$, $0 \le i < d = \dim R$, and a system of parameters $x_1, \ldots, x_d$ of R.

To obtain more results on F-coherent rings, we need some preliminaries from valuation theory. Let A be a domain and let P be its prime ideal. Then by Zorn's lemma, there exits a valuation ring (V, Q) such that $A \subseteq V \subseteq K$ and $P = A \cap Q$ for the field of fractions K of A. We assume that A is a Noetherian domain of characteristic $p > 0$. Then we can take the valuation ring (V, Q) to be discrete together with its valuation $v : V \to \mathbb{Z} \cup \{\infty\}$, which naturally defines a valuation $v : A^\infty \to \mathbb{Q} \cup \{\infty\}$.

We prove the following.

Theorem 7. *Let $(R, \mathfrak{m})$ be an F-coherent complete local domain of characteristic $p > 0$. Then every system of parameters of R is a regular sequence on R^∞. In other words, R^∞ is a big Cohen-Macaulay R-algebra.*

Proof. For a contradiciton, assume that R^∞ is not a big Cohen-Macaulay algebra. Then there exists a system of parameters $x_1, \ldots, x_d$ of R with $d = \dim R$ such that the kernel, which we denote by N, of the multiplication map:

$$R^\infty/(x_1, \ldots, x_i)R^\infty \xrightarrow{x_{i+1}} R^\infty/(x_1, \ldots, x_i)R^\infty$$

is nonzero for some i. Let $R^\infty \cdot z \subseteq N$ be a nonzero cyclic module. Then we find that $R^\infty \cdot z \simeq R^\infty/J$ for a finitely generated ideal J by assumption and Theorem 6 yields that $c^{1/p^n} \cdot R^\infty \subseteq J$ for all $n > 0$ and some $0 \neq c \in R$. Let $v : R^\infty \to \mathbb{Q} \cup \{\infty\}$ be a valuation with center $P \subseteq R^\infty$ such that $J \subseteq P$. Then we see that $v(c^{1/p^n}) = \frac{1}{p^n} \cdot v(c) \to 0$ as $n \to \infty$, while $v(J)$ is bounded from below by a strictly positive number, because J is finitely generated and v is positive on J. So we get a contradiction and thus, every system of parameters of R is regular on R^∞. □

Rings which are not F-coherent are constructed by the Segre product. Let p be a prime such that $p \equiv 1 (\mathrm{mod}\, 3)$ and let $R\#S$ be the Segre product of $R = k[X, Y, Z]/(X^3 + Y^3 + Z^3)$ and $S = k[s, t]$ for a field k of characteristic p. Then $R\#S = k[xs, ys, zs, xt, yt, zt]$, where x, y, z are the respective images of X, Y, Z in $R\#S$. Denoting by T the localization of $R\#S$ at the maximal ideal (xs, ys, zs, xt, yt, zt), we find that $\dim T = 3$ and $ys, xt, xs - yt$ forms a system of parameters. But then there is a relation $(zt)(zs)(xs - yt) - (zs)^2(xt) + (zt)^2(ys) = 0$ and so T is not Cohen-Macaulay.

Proposition 8. *Let T be as above and assume that $p \equiv 1 (\bmod 3)$ for a prime p. Then T is not F-coherent.*

Let R^0 denote the complement of the union of all minimal primes of R. We use some tight closure theory in the proof of the proposition.

Proof. In fact, it is seen that T is F-pure (see [3] P. 162 and P. 269 for example). Now suppose that T is F-coherent. Then we show that $I^F = I^*$ for any ideal $I \subseteq T$, where I^F is the Frobenius closure of I and I^* is the tight clsoure of R. The following discussion holds for any reduced Noetherian ring T.

We first recall that $I^F \subseteq I^*$. For a contradiction, let $u \in T$ be such that $u \in I^*$, but $u \notin I^F$. Then since T is F-coherent, it follows that $J := IT^\infty :_{T^\infty} u$ is a finitely generated non-unit ideal, and we have $cu^q \in I^{[q]}$ for $q = p^e \gg 0$ and $c \in T^0$ by our assumption. Hence $c^{1/q}u \in IT^{1/q} \subseteq IT^\infty$, or equivalently $c^{1/q} \in J$ for $q = p^e \gg 0$. Since $c \in T^0$, there is a minimal prime P of T^∞ such that $(J + P)$ is a proper ideal and thus $c \notin P$. Now it is easy to see that $I^F = I^*$ by choosing an optimal valuation as previously.

In conclusion, it follows that T is weakly F-regular. But then T is Cohen-Macaulay, which is a contradiction to the preceding remark. □

The next corollary relates the F-coherent property to the F-purity [4].

Corollary 9. *Suppose that R is a reduced F-coherent excellent ring of characteristic $p > 0$. Then if R is F-pure, it is Cohen-Macaulay.*

Let $R \longrightarrow \widehat{R}$ be the completion map of local rings. Then we do not know if R is F-coherent if and only if $\widehat{R}$ is so. However, the F-coherent property behaves well under henselization [4]. The proof consists in applying the fact that the henselization of a local ring $(R, \mathfrak{m})$ is obtained as the direct limit of various localizations of module-finite étale R-algebras, so we skip the proof.

Proposition 10. *Let $(R, \mathfrak{m})$ be a reduced local ring of characteristic $p > 0$. Then R is F-coherent if and only if the henselization R^{h} is so.*

For an injective ring map $f : R \longrightarrow S$ such that there is a ring map $h : S \longrightarrow R$ satisfying the property that $h \circ f$ is the identity map on R, we say that R is an *algebra retract of S*. Then the following result may help us construct interesting examples of F-coherent rings.

Theorem 11. *Let $R \longrightarrow S$ be a ring extension of reduced Noetherian rings of characteristic $p > 0$ and suppose that R is an algebra retract of S and S is F-coherent. Then R is F-coherent.*

Proof. Let $\phi : S \to R$ be a retraction map. Then we may extend ϕ to a ring map $\phi_\infty : S^\infty \to R^\infty$ such that the restriction $\phi_\infty|_{R^\infty}$ is the identity map as follows. The composition of ring maps:

$$S^{1/q} \xrightarrow{\simeq} S \xrightarrow{\phi} R \xrightarrow{\simeq} R^{1/q},$$

where the first and the third maps are the Frobenius bijections, shows a compatible sequence of retraction maps ϕ_e with $\phi_0 = \phi$ for every $q = p^e$ and taking its direct limit, we find that $\varinjlim_e \phi_e$ is the desired map. Now applying ([2], Theorem 4.1.5), we conclude that R^∞ is coherent, as desired. □

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a-INVARIANT OF NORMAL GRADED GORENSTEIN RINGS AND VARIETIES WITH EVEN CANONICAL CLASS

KEI-ICHI WATANABE

INTRODUCTION

Let $R = \oplus_{n\geq 0} R_n$ be a normal Noetherian graded ring with $R_0 = k$ a field. Then we can attach to R a normal projective variety $X =$ Proj (R). In this paper, we always assume that R is a normal graded ring and we also assume that $R_m \neq 0$ for all sufficiently large n. Let K_R be the canonical module of R. We say that R is *quasi Gorenstein* if K_R is a free R module. R is a Gorenstein ring if and only if R is Cohen-Macaulay and quasi Gorenstein. If R is quasi Gorenstein, then $K_R \cong R(a)$ as graded R modules and this a is called the a invariant of R and denoted by $a(R)$. This concept was first defined in [GW] and known to be a very important invariant of a graded ring.

In this paper, we study the following two questions.

Question 0.1. If R is a (quasi) Gorenstein ring, what kind of projective variety is Proj (X)?

Question 0.2. Given a normal projective variety X over k, what are the possible values of $a(R)$ for a (quasi) Gorenstein ring R with Proj $(R) \cong X$.

If R is generated by elements of degree 1, then $K_R \cong R(a)$ if and only if $\omega_X \cong \mathcal{O}_X(a)$, where ω_X is the dualizing module of X. Hence either ω_X is ample, ω_X^{-1} is ample or $\omega_X \cong \mathcal{O}_X$. But since we do not assume R is generated by elements of degree 1, the answer is quite different. The following theorems are main results of this paper.

Theorem 0.3. *Let X be any normal projective variety over k and α be any positive odd number.*

(1) *There is a quasi Gorenstein ring R with* Proj $(R) = X$ *and $a(R) = \alpha$.*
(2) *If, moreover, X satisfies the condition (CM) below, there is a Gorenstein ring R with* Proj $(R) = X$ *and $a(R) = \alpha$.*

The following condition is a necessary and sufficient condition for X to have a Cohen-Macauly graded ring R with Proj $(R) = X$.

Proposition 0.4. *Let X be a normal projective variety over a field k with* $\mathrm{H}^0(X, \mathcal{O}_x) =$ *k. Then there is a Cohen-Macaulay normal graded ring R with* Proj $(R) \cong X$ *if and only if X satisfies the following condition (CM).*

(CM) X is a Cohen-Macaulay variety and $H^i(X, \mathcal{O}_X) = 0$ for every i, $1 \leq i \leq$ $\dim X - 1$.

If there is some quasi Gorenstein normal ring with Proj $(R) = X$ with *even* $a(R)$, X must satisfy an extra condition.

Definition 0.5. We say that a normal variety X has *even canonical class* if the canonical divisor K_X is linearly equivalent to $2L$, where L is a (Weil) divisor on X.

We will write $D \sim E$ for (Weil) divisors on X if $E = D + \mathrm{div}_X(f)$ for some $f \in k(X)$. In this case we say that E is linearly equivalent to D.

Theorem 0.6. *Let X be any normal projective variety over k and α be any positive even number.*

(1) *There is a quasi Gorenstein ring R with* Proj $(R) = X$ *and $a(R) = \alpha$ if and only if X has even canonical class.*
(2) *If, moreover, X satisfies the condition (CM) and if there is a divisor with $2L \sim K_X$ and $H^i(X, L) = 0$ for $0 < i < dim X$, then there is a Gorenstein ring R with* Proj $(R) = X$ *and $a(R) = \alpha$.*

1. Preliminaries

Our method to prove our assertions is based on so called DPD (Dolgacev- Pinkham - Demazure) construction of normal graded rings.

Theorem 1.1. [D] *Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian normal graded ring over $R_0 = k$. We put $X =$ Proj (R). If we fix a homogeneous element $T \neq 0$ of degree 1 in the fraction field of R, then there exists a unique $\mathbb{Q}$ divisor D on X such that ND is an ample Cartier divisor for some positive integer N and*

$$R = R(X, D) := \oplus_{n\geq 0} \mathrm{H}^0(X, \mathcal{O}_X(nD))T^n,$$

where we put $\mathrm{H}^0(X, \mathcal{O}_X(nD)) = \{f \in k(X) \mid \mathrm{div}_X(f) + nD \geq 0\} \cup \{0\}$.

Note that for every $n \in \mathbb{Z}$, $\mathcal{O}_X(nD) = \mathcal{O}_X([nD])$, where $[nD]$ is the largest integral divisor (element of $\mathrm{div}(X)$) with $[nD] \leq nD$.

Definition 1.2. Let D be a $\mathbb{Q}$ divisor on a normal projective variety X. We denote by $\mathrm{div}(X)$ the divisor group of X and by $\mathrm{Cl}(X)$ the divisor class group of X. We will write D as

$$D = E + \sum_V \frac{p_V}{q_V} V,$$

where $E \in \mathrm{div}(X)$, the V's are irreducible codimension 1 subvarieties of X and we always assume $q_V > p_V \geq 1, (p_V.q_V) = 1$. Then we put

$$\mathrm{frac} D = \sum_V \frac{q_V - 1}{q_V}.$$

Cohen-Macaulay and Gorenstein property of $R = R(X, D)$ is characterized in terms of cohomology class and divisor class group ot X.

Theorem 1.3. [W] *Let X be a normal projective variety over k with $\mathrm{H}^0(X, \mathcal{O}_X) = k$ and $R = R(X, D)$ with D as above.*

(1) *We have an isomorphism of graded R modules $\mathrm{H}^i_{\mathfrak{m}}(R) \cong \oplus_{n\in\mathbb{Z}} \mathrm{H}^{i-1}(X, \mathcal{O}_X(nD))T^n$ for $2 \leq i \leq \dim R$. In particular, R is Cohen-Macaulay if and only if $\mathrm{H}^i(X, \mathcal{O}_X(nD) = 0$ for $1 \leq i < \dim X$ and for all $n \in \mathbb{Z}$.*
(2) *$K_R \cong R(a)$ if and only if $K_X + \mathrm{frac} D \sim aD$.*

2. The proof of the main results

Theorem 2.1. *Let X be any normal projective variety over k and α be any positive odd number.*

(1) *There is a quasi Gorenstein ring R with* Proj $(R) = X$ *and* $a(R) = \alpha$.

(2) *If, moreover, X satisfies the condition (CM) below, there is a Gorenstein ring R with* Proj $(R) = X$ *and* $a(R) = \alpha$.

Proof. Recall that by 1.1, it suffices to find $D \in \text{div}(X) \otimes \mathbb{Q}$ such that

(1) $\alpha D \sim K_X + \text{frac}D$,

(2) ND is ample Cartier divisor for some positive integer N.

First let $\alpha = 1$. Then we put $D = K_X + \frac{1}{2}V$, where V is an irreducible subvariety of X of codimension 1 of X such that $H = 2D = V + 2K_X$ is an ample Cartier divisor. This D satisfies the conditions above.

Next, let $\alpha \geq 3$. then put $q = \frac{\alpha+1}{2}$ and $D = -H + \frac{1}{q}V + \frac{1}{\alpha+1}W$, where H is an ample Cartier divisor and V, W are integral divisors satisfying the conditions

(a) $V \sim \alpha H + K_X$ and

(b) $W \sim sH - 2K_X$ for some positive integer s.

Then we can easily see that $\alpha D \sim K_X + \text{frac}D$ and $(\alpha+1)D \sim (\alpha - 1 + s)H$ is an ample Cartier divisor. This proves our assertion (1).

(2) If we look carefully at $[nD]$ for every $n \in \mathbb{Z}$, taking H sufficiently ample we see that either $[nD] = 0$, $[nD] = K_X$, $[nD]$ is sufficiently ample or $-[nD]$ is sufficiently ample. In any case, since we assume the condition (CM), we can assert $\mathrm{H}^i(X, \mathcal{O}_X(nD)) = 0$ for every i, $0 < i < \dim X$ and for every $n \in \mathbb{Z}$. This asserets that R is Cohen-Macaulay by 1.3 (2).

Before showing (2), let us review the structure of the ring $R(X, D)$ we have constructed above. □

Remark 2.2. In our construction of 2.1, $\alpha = 1$, we find that $[2nD] = nH$ and $[(2n+1)D] = K_X + nH$ for every $n \in \mathbb{Z}$. Thus if we put

$$S = \oplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(nH))T^{2n},$$

we see that

$$R = S \oplus K_S T$$

as graded S modules.

Now let us investigate the case where $a(R)$ is even.

Theorem 2.3. *Let X be a normal projective variety over k. If there is a normal graded quasi Gorenstein ring R with* Proj $(R) = X$ *and even $a(R)$, then X has even canonical class.*

Proof. Write $D = E + \sum_V \frac{p_V}{q_V}V$ as in 1.2 and put $\alpha = a(R)$. Since $\alpha D \sim K_X + \text{frac}D$, we have

$$\alpha p_V \equiv -1 \pmod{q_V}.$$

In particular, since α is even, every q_V should be odd and if we write $\alpha p_V = m_V q_V + (q_V - 1)$, then m_V should be even. Now, $\alpha D \sim K_X + \text{frac} D$ is equivalent to $K_X \sim \sum_V m_V V$, which is even. □

Remark 2.4. What normal projective varieties have even canonical class?

(1) If k is algebraically closed, then every curve has even canonical class.
(2) Assume $\dim X = 2$ and X is smooth. Then X is minimal since exceptional curve C must satisfy $K_X \cdot C = -1$, which is impossible if K_X is even. Hence, if moreover X is rational, then $X = \Sigma_n$, a Hirzebruch surface with even n.

Now we will show our existence theorem with even $a(R)$.

Theorem 2.5. *labelmain even Let X be any normal projective variety over k and α be any positive even number.*

(1) *There is a quasi Gorenstein ring R with* Proj $(R) = X$ *and* $a(R) = \alpha$ *if and only if X has even canonical class.*
(2) *If, moreover, X satisfies the condition (CM) and if there is a divisor with $2L \sim K_X$ and $H^i(X, L) = 0$ for $0 < i < \dim X$, then there is a Gorenstein ring R with* Proj $(R) = X$ *and* $a(R) = \alpha$.

Proof. Assume α is a positive even integer. Since X has even canonical class, take $L \in \text{div}(X)$ such that $2L \sim K_X$. We fix this divisor L.

First we consider the case $\alpha = 2$. Take

$$D = L + \frac{1}{3}V,$$

such that $V \sim H - 3L$, where H is a sufficiently ample Cartier divisor. Then $2D \sim K_X + \text{frac} D$ and $3D \sim H$ is an ample Cartier divisor.

Note that we have $3nD = nH, [(3n+1)D] = L + nH$ and $[(3n+2)D] = K_X + nH$. Hence if we put $S = \oplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(nH)).T^{3n}$ and $M = \oplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(L + nH)).T^{3n}$, then

$$R = S \oplus MT \oplus K_S T^2$$

as graded S modules.

If $\alpha \geq 4$, then we put $D = -(\alpha - 2)H + L + \dfrac{\alpha - 2}{\alpha - 1}V + \dfrac{1}{\alpha + 1}W$, where H is a sufficiently ample Cartier divisor and $V \sim \alpha H - L$. Then $\alpha D \sim K_X + \text{frac} D$ and We can arrange W so that $(\alpha^2 - 1)D$ is an ample Cartier divisor. We can also show (2) as in 2.1. □

3. Some Concluding Remarks

Definition 3.1. Given a normal projective variety X over k, we define $\mathcal{A}(X)$ to be the set of $a(R)$, where R varies on the class of normal quasi Gorenstein rings with Proj $(R) = X$.

Remark 3.2. It is easy to show that if $0 \in \mathcal{A}(X)$, then $K_X \sim 0$ and in this case $\mathcal{A}(X)$ coincides with the set of non-negative integers.

We can show the following result concerning $\mathcal{A}(X)$.

Theorem 3.3. *Let X be a normal projective variety whose canonical devisor K_X is $\mathbb{Q}$-cartier and assume that either K_X or $-K_X$ is ample or $K_X \sim 0$. Then $\mathcal{A}(X)$ is the one of the following sets.*

(1) *If $K_X \sim 0$, then $\mathcal{A}(X) = \mathbb{Z}\geq 0$, the set of all non-negative integers.*
(2) *$\mathcal{A}(X) = \mathbb{Z} \setminus \{0\}$ if $-K_X$ is ample and X has even canonical class.*
(3) *$\mathcal{A}(X)$ is the set of all odd integers if $-K_X$ is ample and X does not have even canonical class.*
(4) *$\mathcal{A}(X)$ is the set of all positive integers if K_X is ample and X has even canonical class.*
(5) *$\mathcal{A}(X)$ is the set of all positive odd integers if K_X is ample and X does not have even canonical class.*

Remark 3.4. When there is a quasi Gorenstein ring with Proj $(R) = X$ and *negative* $a(R)$, then $-K_X - \text{frac}D$ is ample and thus $-K_X$ is "big", that is,
$\dim_k \mathrm{H}^0(X, \mathcal{O}_X(-nK_X)) \geq cn^{\dim X}$ when n gets very big for some positive constant c. Does the converse hold ?

Remark 3.5. If we require R to be a complete intersection or a hypersurface, then $X =$ Proj (R) will be very limited. For example assume $R = k[X, Y, Z, W]/(f)$ be a graded hypersurface with even $a(R)$ and assume $X =$ Proj (R) is a smooth rational surface. Then I believe that $X = \mathbb{P}^1 \times \mathbb{P}^1$.

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GORENSTEIN REES ALGEBRAS OVER RINGS OF DEPTH ONE HAVING FINITE LOCAL COHOMOLOGY

SHIN-ICHIRO IAI

This is a joint work with Shiro Goto [5]. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. For each ideal I of A, we set $\mathcal{R}(I) := \oplus_{i \geq 0} I^i$, which is called the Rees algebra of I. Let $\mathfrak{q}$ be a parameter ideal of A. In this paper, we consider a question of when the Rees algebra $\mathcal{R}(\mathfrak{q}^d)$ is Gorenstein.

A motivation for considering this question comes from an example of Hochster and Roberts [7]. Let $k[[s^2, t, s^3, st]]$ be a subring of the formal power series ring $k[[s,t]]$ over a field k in two variables s and t. Let $\mathfrak{q} = (s^2, t)$ be an ideal of the ring $k[[s^2, t, s^3, st]]$ generated by elements s^2 and t. Then they showed that the Rees algebra $\mathcal{R}(\mathfrak{q}^2)$ is Gorenstein but the base ring $k[[s^2, t, s^3, st]]$ is not Cohen-Macaulay. It is one of the most important examples of Rees algebras and has provided the impetus for large amount of research. That is not only an example of a non-Cohen-Macaulay ring that is direct summand of a Gorenstein ring, but also an example of an arithmetic Gorensteinfication. Our main result is a generalization of their example about the Gorenstein Rees algebra.

Suppose that A is a generalized Cohen-Macaulay ring (referred to in this paper as a ring of finite local cohomology), namely the ith local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(A)$ of A with respect to $\mathfrak{m}$ is finitely generated for all integers $i < d$. Assume that $\operatorname{depth} A > 0$. We denote the S_2-fication of A by $\widetilde{A}$ (cf. [2] and [6]). Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A and $\mathrm{K}_{\widehat{A}}$ the canonical module of $\widehat{A}$. For each finitely generated A-module M, we denote the length of M by $\ell_A(M)$ and set $\mu_A(M) := \dim_{A/\mathfrak{m}} M/\mathfrak{m}M$, $\mathrm{r}_A(M) := \dim_{A/\mathfrak{m}} \operatorname{Ext}_A^{\operatorname{depth}_A M}(A/\mathfrak{m}, M)$, and

$$\Sigma(x_1, x_2, \dots, x_s; M) := \sum_{i=1}^{s} [(x_1, \dots, x_{i-1}, x_{i+1}, \dots x_s)M :_M x_i] + (x_1, x_2, \dots, x_s)M,$$

where $x_1, x_2, \dots, x_s \in A$. Let $a_1, a_2, \dots, a_d$ be a system of parameters for A and $\mathfrak{q} = (a_1, a_2, \dots, a_d)A$. We say that the system $a_1, a_2, \dots, a_d$ of parameters for A is standard if the equality

$$\ell_A(A/\mathfrak{q}) - \mathrm{e}_{\mathfrak{q}}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$$

holds, where $\mathrm{e}_{\mathfrak{q}}(A)$ denotes the multiplicity of A with respect to $\mathfrak{q}$ (cf. [9] and [10]). There always exists a standard system of parameters for a ring of finite local cohomology. We put $\mathfrak{a} = \Sigma(a_1, a_2, \dots, a_d; A)$. Then $\mathfrak{a}$ is a common ideal of A and $\widetilde{A}$ whenever the system $a_1, a_2, \dots, a_d$ of parameters for A is standard, so that $\mathfrak{a} \subseteq A : \widetilde{A}$. With this notation, the main result in this paper can be stated as follows.

Theorem 1. *Assume that $d \geq 2$ and the system $a_1, a_2, \ldots, a_d$ of parameters for A is standard. Then the following three conditions are equivalent.*

(1) *The Rees algebra $\mathcal{R}(\mathfrak{q}^d)$ is Gorenstein.*

(2) $\operatorname{depth} A = 1$, $\mathrm{r}_A(A) = 1$, $\mu_A(\widetilde{A}) = \mu_{\widehat{A}}(\mathrm{K}_{\widehat{A}})$, *and* $\mathfrak{a} = A : \widetilde{A}$.

(3) $\operatorname{depth} A = 1$, $\mathrm{r}_A(A) = 1$, $\mu_A(\widetilde{A}) = \mu_{\widehat{A}}(\mathrm{K}_{\widehat{A}})$, *and* $\ell_A(\widetilde{A}/\mathfrak{a}) = 2\ell_A(A/\mathfrak{a})$.

The equality $\mathfrak{a} = A : \widetilde{A}$ means that the multiplicity $\mathrm{e}_{\mathfrak{q}}(A)$ of A with respect to the standard parameter ideal $\mathfrak{q}$ must be as small as possible, because we always have the inclusion $\mathfrak{a} \subseteq A : \widetilde{A}$ and the equality

$$\mathrm{e}_{\mathfrak{q}}(A) = \ell_A(A/\mathfrak{a}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$$

(see [6], (3.15)). For example, the ring $k[[s^2, t, s^3, st]]$ is Buchsbaum and then the sequence s^3, t is also a standard system of parameters for $k[[s^2, t, s^3, st]]$, but the Rees algebra $\mathcal{R}((s^3, t)^2)$ is not Gorenstein.

When the base ring A is Buchsbaum, we have the following explicit result.

Corollary 2. *Assume $d \geq 2$. Let A be a Buchsbaum local ring of depth one. Then the Rees algebra $\mathcal{R}(\mathfrak{q}^d)$ is Gorenstein if and only if $\mathrm{e}_{\mathfrak{m}}(A) = 2$ and $\mathfrak{q}$ is a reduction of $\mathfrak{m}$. When this is the case, one has $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ if $i \neq 1, d$.*

The last assertion $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ if $i \neq 1, d$ means that the ring $\widetilde{A}$ is Cohen-Macaualy (cf. [2]). In this case, we can omit the equality $\mu_A(\widetilde{A}) = \mu_{\widehat{A}}(\mathrm{K}_{\widehat{A}})$ from the condition (2) in the theorem.

Corollary 3. *Assume that $d \geq 2$ and $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ if $i \neq 1, d$. Then the following two conditions are equivalent.*

(1) *The Rees algebra $\mathcal{R}(\mathfrak{q}^d)$ is Gorenstein.*

(2) $\operatorname{depth} A = 1$, $\mathrm{r}_A(A) = 1$, *and* $\mathfrak{a} = A : \widetilde{A}$.

When this is the case, the system $a_1, a_2, \ldots, a_d$ of parameters for A is standard.

In general, the theorem needs the equality $\mu_A(\widetilde{A}) = \mu_{\widehat{A}}(\mathrm{K}_{\widehat{A}})$ in the condition (2). For example, let R be a 3-dimensional complete regular local ring with the maximal ideal $\mathfrak{n} = (a_1, a_2, a_3)$. Let $0 \to F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} R/\mathfrak{n} \to 0$ denote a minimal free resolution of $R/\mathfrak{n}$. We put $E_1 = \mathfrak{n}$ and $E_2 = \operatorname{Coker} \varphi_3$. Let $A = R \ltimes (E_1 \oplus E_2)$ denote the idealization of $E_1 \oplus E_2$ over R. Then A is a Buchsbaum local ring of depth one and $\widetilde{A} = R \ltimes (R \oplus E_2)$ (cf. [1]). Put $\mathfrak{q} = (a_1, a_2, a_3)A$ and $\mathfrak{a} = \Sigma(a_1, a_2, a_3; A)$. Then we have equalites $\mathrm{r}_A(A) = 1$ and $\mathfrak{a} = A : \widetilde{A}$, but the Rees algebra $\mathcal{R}(\mathfrak{q}^2)$ is not Gorenstein, as $\mathrm{e}_{\mathfrak{q}}(A) \neq 2$.

Let us close this paper with the following example.

Example 4. *Let $(R, \mathfrak{n})$ be a complete local ring and $d = \dim R \geq 2$. Assume that R has finite local cohomology and that $\mathrm{K}_R \cong R$ as R-modules. Let $a_1, a_2, \ldots, a_d$ be a standard system of parameters for R. Put $\mathfrak{b} = \Sigma(a_1, a_2, \ldots, a_d; R)$ and $A = R \ltimes \mathfrak{b}$, which is the idealization of $\mathfrak{b}$ over R. Then* $\operatorname{depth} A = 1$ *and $\widetilde{A} = R \ltimes R$. Since $R/\mathfrak{b}$ is a Gorenstein ring, $\mathrm{r}_A(A) = 1$. Set $\mathfrak{q} = (a_1, a_2, \ldots, a_d)A$ and $\mathfrak{a} = \Sigma(a_1, a_2, \ldots, a_d; A)$. Then $\mathfrak{a} = \mathfrak{b} \times \mathfrak{b}$, and hence $\ell_A(\widetilde{A}/\mathfrak{a}) = 2\ell_A(A/\mathfrak{a})$. Therefore the Rees algebra $\mathcal{R}(\mathfrak{q}^d)$ is Gorenstein by the theorem.*

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QUASI-SOCLE IDEALS IN BUCHSBAUM RINGS

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ABSTRACT. Quasi-socle ideals, that is, ideals of the form $I = Q : \mathfrak{m}^q$ ($q \geq 2$) with Q parameter ideals in a Buchsbaum local ring $(A, \mathfrak{m})$ are explored in connection to the question of when I is integral over Q and when the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is Buchsbaum. The assertions obtained by H.-J. Wang [Wan] in the Cohen-Macaulay case holds true after necessary modifications of the conditions on parameter ideals Q and integers q. Examples are explored.

1. INTRODUCTION

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, $d = \dim A > 0$, and infinite residue class field $A/\mathfrak{m}$. Let

$$\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

be the associated graded ring of $\mathfrak{m}$. For each $\mathfrak{m}$-primary ideal I in A we denote by $\mathrm{e}_I^i(A)$ $(0 \leq i \leq d)$ the i-th Hilbert coefficient of A with respect I, whence the Hilbert polynomial of I is given by the following formula

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

for all $n \gg 0$, where $\ell_A(*)$ denotes the length.

With this notation the purpose of this paper is to prove the following.

Theorem 1.1. *Suppose that A is a Buchsbaum ring and* $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$. *Let $q \geq 2$ be an integer and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^{q+2}$. Assume that $a_d = ab$ for some $a \in \mathfrak{m}^q, b \in \mathfrak{m}$ and put $I = Q : \mathfrak{m}^q$. Then*

$$\mathfrak{m}^q I = \mathfrak{m}^q Q, \quad I \subseteq \mathfrak{m}^{q+2}, \quad \text{and} \quad I^2 = QI$$

and the following assertions hold true.

(1) $\mathrm{e}_I^1(A) = \mathrm{e}_I^0(A) + \mathrm{e}_Q^1(A) - \ell_A(A/I)$.

(2) *The Hilbert function of I is given by* $\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \sum_{i=2}^d (-1)^i \left[\mathrm{e}_Q^{i-1}(A) + \mathrm{e}_Q^i(A)\right]\binom{n+d-i}{d-i}$ *for all $n \geq 0$.*

(3) *The associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is a Buchsbaum ring with*

$$\mathrm{H}_M^i(\mathrm{G}(I)) = [\mathrm{H}_M^i(\mathrm{G}(I))]_{1-i} \cong \mathrm{H}_{\mathfrak{m}}^i(A)$$

Key words and phrases: quasi-socle ideal, Cohen-Macaulay ring, Buchsbaum ring, associated graded ring, local cohomology, multiplicity.

2000 Mathematics Subject Classification: 13H10, 13A30, 13B22, 13H15.

as A-modules for all $i < d$ and

$$\max\{n \in \mathbb{Z} \mid [\mathrm{H}^d_M(\mathrm{G}(I))]_n \neq (0)\} \leq 1 - d.$$

Here $M = \mathfrak{m}\mathrm{G}(I) + \mathrm{G}(I)_+$ and $[\mathrm{H}^i_M(\mathrm{G}(I))]_n$ $(i, n \in \mathbb{Z})$ denotes the homogeneous component with degree n in the i-th graded local cohomology module $\mathrm{H}^i_M(\mathrm{G}(I))$.

Thus the quasi-socle ideals $I = Q : \mathfrak{m}^q$ behave very well, inside Buchsbaum rings also, under the conditions stated in Theorem 1.1. Notice that, because A is a Buchsbaum ring, the Hilbert coefficients $\mathrm{e}^i_Q(A)$ of the parameter ideal Q are given by the following formula

$$(-1)^i \mathrm{e}^i_Q(A) = \begin{cases} \mathrm{e}^0_Q(A) & \text{if } i = 0, \\ \ell_A(\mathrm{H}^0_\mathfrak{m}(A)) & \text{if } i = d, \\ \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \ell_A(\mathrm{H}^j_\mathfrak{m}(A)) & \text{if } 1 \leq i \leq d-1 \end{cases}$$

and one has the equality $\ell_A(A/Q^{n+1}) = \sum_{i=0}^d (-1)^i \mathrm{e}^i_Q(A)\binom{n+d-i}{d-1}$ for all $n \geq 0$ ([Sch, Korollar 3.2]), so that $\{\mathrm{e}^i_Q(A)\}_{1 \leq i \leq d}$ are independent of the choice of Q and are invariants of A. The crucial point in Theorem 1.1 is the equality $I^2 = QI$; assertions (1), (2), and (3) readily follow from this fact via [GO, Section 2] and [GN, Section 5], since $(a_1, \cdots, \check{a}_i, \cdots, a_d) : \mathfrak{m} \subseteq I$ for all $1 \leq i \leq d$. Here we should also note the condition in Theorem 1.1 that $a_d = ab$ for some $a \in \mathfrak{m}^q$ and $b \in \mathfrak{m}$ is rather technical but, at this moment, we do not know whether this additional condition is superfluous or not.

We now briefly explain the background of our theorem 1.1. Our researches date back to the works of A. Corso, C. Polini, C. Huneke, W. V. Vasconcelos, and the first author, where they explored the socle ideals $Q : \mathfrak{m}$ for parameter ideals Q in Cohen-Macaulay rings A and proved the following result.

Theorem 1.2 ([CHV, CP1, CP2, CPV, G1]). *Let Q be a parameter ideal in a Cohen-Macaulay ring A and let $I = Q : \mathfrak{m}$. Then the following conditions are equivalent.*

(1) $I^2 \neq QI$.
(2) *Q is integrally closed in A.*
(3) *A is a regular local ring and the A-module $\mathfrak{m}/Q$ is cyclic.*

Hence, if A is a Cohen-Macaulay ring which is not regular, then $I^2 = QI$ for every parameter ideal Q in A, so that $\mathrm{G}(I)$ and $\mathrm{F}(I) = \mathrm{G}(\mathrm{I})/\mathfrak{m}\mathrm{G}(\mathrm{I})$ are both Cohen-Macaulay rings, where $I = Q : \mathfrak{m}$. The Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ is also a Cohen-Macaulay ring, if $\dim A \geq 2$.

This result has led to two directions of researches for better understanding of quasi-socle ideals $I = Q : \mathfrak{m}^q$ in arbitrary local rings. One direction is to weaken the assumption on base rings A, which was performed by the first and the third authors [GSa1, GSa2, GSa3]. They explored the socle ideals $I = Q : \mathfrak{m}$ inside Buchsbaum local rings A and showed that $I^2 = QI$ and $\mathrm{G}(I)$ is a Buchsbaum ring, if $\mathrm{e}^0_\mathfrak{m}(A) \geq 2$ and if Q is contained in a sufficiently high power of the maximal ideal $\mathfrak{m}$. The other direction was independently performed by H.-J. Wang [Wan] and the first author, N. Matsuoka, R. Takahashi, S. Kimura, T. T. Phuong, and H. L. Truong [GMT, GKM, GKMP, GKPT]. In [GMT] the quasi-socle ideals $Q : \mathfrak{m}^2$ in Gorenstein local rings A with $\dim A > 0$ and $\mathrm{e}^0_\mathfrak{m}(A) \geq 3$ are explored, and in [GKM, GKMP, GKPT] the quasi-socle ideals $Q : \mathfrak{m}^q$ $(q \geq 1)$ in Cohen-Macaulay local rings of dimension 1 are closely studied. However, at least in the case where $\dim A \geq 2$, Wang [Wan] gave a great achievement in these topics, settling affirmatively a conjecture of C. Polini and B. Ulrich [PU]. Let us note one of his results in the following form.

Theorem 1.3 ([Wan]). *Suppose that A is a Cohen-Macaulay ring and let $q \geq 1$ be an integer. Let Q be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^{q+1}$ and put $I = Q : \mathfrak{m}^q$. Then*

$$\mathfrak{m}^q I = \mathfrak{m}^q Q, \quad I \subseteq \mathfrak{m}^{q+1}, \quad \text{and} \quad I^2 = QI,$$

provided $\operatorname{depth} G(\mathfrak{m}) \geq 2$.

Since Buchsbaum rings are very akin to Cohen-Macaulay rings, it seems quite natural to expect that similar results of the Cohen-Macaulay case, such as Theorem 1.3, should be true also in the Buchsbaum case after mild modifications of the corresponding conditions, which we now report in Theorem 1.1.

The proof of Theorem 1.1 shall be given in Section 2, which we will divide into two parts. The first part is to show that $\mathfrak{m}^q I = \mathfrak{m}^q Q$. The second part is to prove that $I^2 = QI$. Since A is not necessarily a Cohen-Macaulay ring, the equality $I^2 = QI$ does not readily follow from the fact that $\mathfrak{m}^q I = \mathfrak{m}^q Q$. We shall carefully analyze this phenomenon in Section 2. A similar but more restricted result also holds true in the case where $G(\mathfrak{m})$ is a Buchsbaum ring with $\operatorname{depth} G(\mathfrak{m}) = 1$, which we will discuss in Section 3. In Section 4 we will give examples of Buchsbaum rings A with $\operatorname{depth} G(\mathfrak{m}) = d-1$, which satisfy the conditions required in Theorems 1.1 and 3.1

2. Proof of Theorem 1.1

For each f $(\neq 0)$ $\in A$, let f^* denote the initial form of f in $G(\mathfrak{m})$. The aim of this section is to prove Theorem 1.1. Let us begin with the following.

Lemma 2.1. *Suppose* $\operatorname{depth} G(\mathfrak{m}) \geq 1$. *Then $\mathfrak{m}^\alpha : \mathfrak{m}^\beta = \mathfrak{m}^{\alpha-\beta}$ for all $\alpha, \beta \in \mathbb{Z}$ with $\beta \geq 0$.*

After mild modifications of conditions on parameter ideals Q and integers q, Wang's technique [Wan] still works in the case where A is a FLC ring. Let us note a detailed proof in order to clarify where and why we need such modifications.

Proposition 2.2. *Suppose* $\operatorname{depth} G(\mathfrak{m}) \geq 2$. *Assume that A is a FLC ring and choose an integer $\ell > 0$ so that $\mathfrak{m}^\ell$ is standard* (cf. [T, Section 3]). *Let $q \geq 2$ be an integer and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^{q+\ell+1}$. We put $I = Q : \mathfrak{m}^q$. Then $\mathfrak{m}^q I = \mathfrak{m}^q Q$ and $I \subseteq \mathfrak{m}^{q+\ell+1}$, whence $I^2 \subseteq Q$.*

Proof. Once we have $\mathfrak{m}^q I = \mathfrak{m}^q Q$, by Lemma 2.1 we get that $I \subseteq \mathfrak{m}^{q+\ell+1}$. We shall show $\mathfrak{m}^q I \subseteq \mathfrak{m}^q Q$.

Let F denote the set of all the products $\prod_{i=1}^q f_i$, where $f_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ for all $1 \leq i \leq q$ and f_i^*, f_j^* form a regular sequence in $G(\mathfrak{m})$ for all integers $1 \leq i < j \leq q$. Then $\mathfrak{m}^q = (F)$. Let $\alpha \in I$ and let $f = \prod_{i=1}^q f_i \in F$, Let us write

$$\alpha f = \sum_{i=1}^{d} a_i x_i$$

with $x_i \in A$. It suffices to show that $x_i \in \mathfrak{m}^q$ for all $1 \leq i \leq d$.

We put $g_j = \prod_{1 \le k \le q,\ k \ne j} f_k$ for each $1 \le j \le q$ and choose $g \in \mathfrak{m} \setminus \mathfrak{m}^2$ so that g^*, f_j^* is a regular sequence in $\mathrm{G}(\mathfrak{m})$ for all $1 \le j \le q$. Let

$$\alpha(g_j g) = \sum_{i=1}^{d} a_i x_{ij}$$

with $x_{ij} \in A$. Then, since $f = f_j g_j$, we have

$$\sum_{i=1}^{d} a_i (f_j x_{ij}) = \sum_{i=1}^{d} a_i (g x_i),$$

whence, for all $1 \le i \le d$ and $1 \le j \le q$, we get

$$g x_i - f_j x_{ij} \in (a_1, \cdots, \check{a}_i, \cdots a_d) : a_i.$$

Therefore, since $(a_1, \cdots, \check{a}_i, \cdots a_d) : a_i = (a_1, \cdots, \check{a}_i, \cdots a_d) : \mathfrak{m}^\ell$ by [T, Proposition 3.1] (recall that $\mathfrak{m}^\ell$ is standard) and $Q : \mathfrak{m}^\ell \subseteq \mathfrak{m}^{q+\ell+1} : \mathfrak{m}^\ell = \mathfrak{m}^{q+1}$ by Lemma 2.1, we get

$$g x_i - f_j x_{ij} \in \mathfrak{m}^{q+1}.$$

Consequently, since g^*, f_j^* form a regular sequence in $\mathrm{G}(\mathfrak{m})$, we have

$$g x_i - f_j x_{ij} \in (g, f_j) \cap \mathfrak{m}^{q+1} = (g, f_j) {\cdot} \mathfrak{m}^q$$

(cf. [VV]), so that $g x_i - f_j x_{ij} = g x_i' - f_j x_{ij}'$ with $x_i', x_{ij}' \in \mathfrak{m}^q$. Hence

$$x_i - x_i' \in (f_j) : g = (f_j)$$

and so,

$$x_i \in \bigcap_{j=1}^{q} [\mathfrak{m}^q + (f_j)]\,.\ (1 \le i \le d)$$

Claim 1. *$\bigcap_{j=1}^{k} [\mathfrak{m}^q + (f_j)] \subseteq \mathfrak{m}^q + (\prod_{j=1}^{k} f_j)$ for all $1 \le k \le q$.*

Proof of Claim 1. We can prove this Claim by induction on k ($1 \le k \le q$), and use the fact that $(\prod_{j=1}^{k-1} f_j)^*$ *and* $f_k{}^*$ form a regular sequence in $\mathrm{G}(\mathfrak{m})$. □

Thanks to Claim 1 we get $x_i \in \mathfrak{m}^q + (\prod_{j=1}^{q} f_j) = \mathfrak{m}^q$ for all $1 \le i \le d$. □

We need the following to show the equality $I^2 = QI$.

Lemma 2.3 (cf. [GSa3, Lemma 2.3]). *Let W, L and M be ideals in a commutative ring R and $a, b \in R$. Assume that $a \in M, aW = (0), L : a = L : a^2$, and $L : ab = L : b$. Then*

$$(L + (ab) + W) : M = [(L + W) : M] + [(L + (ab)) : M]\,.$$

The heart of our proof of Theorem 1.1 is the following.

Proposition 2.4. *Suppose that A is a FLC ring and choose an integer $\ell > 0$ so that $\mathfrak{m}^\ell$ is standard. Let $q \ge 1$ be an integer and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Let $I = Q : \mathfrak{m}^q$ and assume that the following three conditions are satisfied:*

(1) *$\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I^2 \subseteq Q$, and $a_i \in \mathfrak{m}^\ell$ for all $1 \le i \le d-1$;*
(2) *There exist elements $a \in \mathfrak{m}^q$ and $b \in \mathfrak{m}$ such that $a_d = ab$ and both systems $\{a_1, a_2, \cdots, a_{d-}, a\}$ and $\{a_1, a_2, \cdots, a_{d-1}, b\}$ of parameters in A are standard;*

(3) *either* $d=1$ *or* $q \geq \ell$.

We then have $I^2 = QI$.

Proof. We notice that the system $\{a_1, a_2, \cdots, a_d\}$ of parameters is standard, because so is $\{a_1, a_2, \cdots, a_{d-1}, a\}$ ([T, Corollary 3.3]). We put $W = \mathrm{H}^0_{\mathfrak{m}}(A), L = (a_1, a_2, \cdots, a_{d-1})$, and $M = \mathfrak{m}^q$. Then $a \in M$, $aW = (0)$, and

$$L : a = L : a^2 = L : ab = L : b = \bigcup_{n \geq 0} [L : \mathfrak{m}^n],$$

since all the systems $\{a_1, a_2, \cdots, a_{d-1}, a\}$, $\{a_1, a_2, \cdots, a_{d-1}, b\}$, and $\{a_1, a_2, \cdots, a_{d-1}, ab\}$ of parameters are standard. On the other hand, we have $\bigcup_{n \geq 0}[L : \mathfrak{m}^n] = L : \mathfrak{m}^\ell$ by [T, Proposition 3.1], because $L \subseteq \mathfrak{m}^\ell$ and $\mathfrak{m}^\ell$ is standard. Hence $L : a = L : M$, if $q \geq \ell$. Consequently, since $W = W : M$, by Lemma 2.3 we get

$$(Q+W) : \mathfrak{m}^q = W + [Q : \mathfrak{m}^q] = W + I, \ (d=1)$$

and

$$(Q+W) : \mathfrak{m}^q = Q : \mathfrak{m}^q = I, \ (q \geq \ell)$$

Suppose now that $d = 1$ and let $\overline{A} = A/W$, $\overline{\mathfrak{m}} = \mathfrak{m}/W$, $\overline{I} = I\overline{A}$, and $\overline{Q} = Q\overline{A}$. Then $\overline{I} = \overline{Q} : \overline{\mathfrak{m}}^q$ and $\overline{\mathfrak{m}}^q \cdot \overline{I} = \overline{\mathfrak{m}}^q \cdot \overline{Q}$. Let $x \in \overline{I}^2$. Then, since $\overline{I}^2 \subseteq \overline{Q}$, we have $x = a_1 y$ with $y \in \overline{A}$. Let $\alpha \in \overline{\mathfrak{m}}^q$. Then, since $a_1(\alpha y) = \alpha x \in \overline{\mathfrak{m}}^q \cdot \overline{I}^2 = \overline{\mathfrak{m}}^q \cdot \overline{Q}^2$, we get $a_1(\alpha y) = a_1^2 z$ for some $z \in \overline{A}$. Therefore $\alpha y \in \overline{Q}$ (notice that a_1 is $\overline{A}$-regular), so that $x = a_1 y \in \overline{Q} \cdot \overline{I}$, because $y \in \overline{Q} : \overline{\mathfrak{m}}^q = \overline{I}$. Thus $\overline{I}^2 = \overline{Q} \cdot \overline{I}$, so that $I^2 \subseteq QI + W$. Since $W \cap Q = (0)$ and $I^2 \subseteq Q$, we get $I^2 \subseteq (QI + W) \cap Q = QI$ as required.

Suppose now that $d \geq 2$ and that our assertion holds true for $d-1$. Let $B = A/(a_1)$. Then all the conditions (1), (2), and (3) are satisfied for the parameter ideal $Q/(a_1)$ in B and we get $I^2 \subseteq QI + (a_1)$. Let $x \in I^2$ and write $x = y + a_1 z$ with $y \in QI$ and $z \in A$. Let $\alpha \in \mathfrak{m}^q$. We then have

$$\alpha x = \alpha y + a_1(\alpha z) \in Q^2,$$

because $x \in I^2$ and $\mathfrak{m}^q I = \mathfrak{m}^q Q$. Consequently $a_1(\alpha z) \in Q^2$ (notice that $\alpha y \in Q^2$), so that $a_1(\alpha z) \in (a_1) \cap Q^2 = a_1 Q$, because $a_1, a_2, \cdots, a_d$ form a d-sequence in A (cf. [T, Proposition 3.1]). Hence $\alpha z - v \in (0) : a_1 \subseteq W$ for some $v \in Q$, which guarantees $z \in (Q+W) : \mathfrak{m}^q = I$, since $q \geq \ell$. Thus $x = y + a_1 z \in QI$, so that $I^2 = QI$. □

Summarizing Propositions 2.2 and 2.5, we have the following. Taking $\ell = 1$ in the case where A is a Buchsbaum ring, Theorem 1.1 now follows from Theorem 2.5.

Theorem 2.5. *Suppose that A is a FLC ring and* $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$. *Choose an integer $\ell \geq 1$ so that $\mathfrak{m}^\ell$ is standard. Let $Q = (a_1, a_2, \cdots, a_d)$ be a system of parameters in A and put $I = Q : \mathfrak{m}^q$, where q is an integer such that $q \geq \max\{\ell, 2\}$. Assume that the following two conditions are satisfied:*

(i) $Q \subseteq \mathfrak{m}^{q+\ell+1}$;

(ii) *There exist elements $a \in \mathfrak{m}^q$ and $b \in \mathfrak{m}$ such that $a_d = ab$ and the system $a_1, \cdots, a_{d-1}, b$ of parameters in A is standard.*

Then $I^2 = QI$ and the following assertions hold true.

(1) $\mathrm{e}_I^1(A) = \mathrm{e}_I^0(A) + \mathrm{e}_Q^1(A) - \ell_A(A/I)$.

(2) *The Hilbert function of I is given by*
$\ell_A(A/I^{n+1}) = e_I^0(A)\binom{n+d}{d} - e_I^1(A)\binom{n+d-1}{d-1} + \sum_{i=2}^{d}(-1)^i\left[e_Q^{i-1}(A) + e_Q^i(A)\right]\binom{n+d-i}{d-i}$
for all $n \geq 0$.

(3) *The graded ring* $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ *is Buchsbaum, if so is* A.

(4) $H_M^i(G(I)) = [H_M^i(G(I))]_{1-i} \cong H_{\mathfrak{m}}^i(A)$ *as* A*-modules for all* $i < d$ *and*

$$\max\left\{n \in \mathbb{Z} \mid [H_M^d(G(I))]_n \neq (0)\right\} \leq 1 - d,$$

where $M = \mathfrak{m}G(I) + G(I)_+$.

Proof. The equality $I^2 = QI$ follows directly from Propositions 2.2 and 2.5. See [GO, Section 2] (resp. [GN, Section 5]) for assertions (1), (2) (resp. (3), (4)). □

3. The case where $\operatorname{depth} G(\mathfrak{m}) = 1$

In this section we study what happens if $\operatorname{depth} G(\mathfrak{m}) = 1$. Our goal is the following.

Theorem 3.1. *Let A be a Buchsbaum ring with $d = \dim A \geq 2$ and suppose that* $G(\mathfrak{m})$ *is a Buchsbaum ring with* $\operatorname{depth} G(\mathfrak{m}) = 1$. *Let*

$$n = \min\{n \in \mathbb{Z} \mid [H_M^1(G(\mathfrak{m}))]_n \neq (0)\}$$

where $M = G(\mathfrak{m})_+$. Then $n \geq 0$ and for every integer $1 \leq q \leq n+1$ and for every parameter ideal $Q = (a_1, a_2, \cdots, a_d)$ of A such that $Q \subseteq \mathfrak{m}^{q+2}$, we have

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$ *and*

(2) $I \subseteq \mathfrak{m}^{q+2}$,

where $I = Q : \mathfrak{m}^q$. Consequently, $I^2 = QI$, so that assertions (1), (2), (3) *in* Theorem 1.1 *hold true also in the present setting, provided $a_d = ab$ for some $a \in \mathfrak{m}^q$ and $b \in \mathfrak{m}$.*

The proof of Theorem 3.1 is essentially the same as that of Theorem 1.1. However let us note a detailed proof to show where we use the assumption that $1 \leq q \leq n+1$.

Proof of Theorem 3.1. Choose $f \in \mathfrak{m}\backslash\mathfrak{m}^2$ so that f^* is $G(\mathfrak{m})$-regular. Then, since $G(\mathfrak{m})$ is a Buchsbaum ring, we get

$$H_M^0(G(\overline{\mathfrak{m}})) \cong \left[H_M^1(G(\mathfrak{m}))\right](-1)$$

as graded $G(\mathfrak{m})$-modules, where $\overline{\mathfrak{m}} = \mathfrak{m}/(f)$. Hence

$$n + 1 \geq \min\{n \in \mathbb{Z} \mid [H_M^0(G(\overline{\mathfrak{m}}))]_n \neq (0)\} \geq 1,$$

so that $n \geq 0$.

To show assertion (1), we may assume that $q \geq 2$ (see [GSa1, GSa2, GSa3] for the case where $q = 1$). Let F denote the set of all the products $\prod_{i=1}^q f_i$, where $f_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ for all $1 \leq i \leq q$ and f_i^*, f_j^* form, for all integers $1 \leq i < j \leq q$, a part of a homogeneous system of parameters in $G(\mathfrak{m})$. Then $\mathfrak{m}^q = (F)$. Let $\alpha \in I$ and let $f = \prod_{i=1}^q f_i \in F$, Let us write

$$\alpha f = \sum_{i=1}^{d} a_i x_i$$

with $x_i \in A$. We will show that $x_i \in \mathfrak{m}^q$ for all $1 \leq i \leq d$. As exactly same way and same notation in the proof of Prop 2.2, we have

$$g x_i - f_j x_{ij} \in \mathfrak{m}^{q+1}.$$

Let $\overline{A} = A/(f_j)$ and let $\overline{*}$ denote the image in $\overline{A}$. We then have

$$\overline{g}{\cdot}\overline{x_i} \in \overline{\mathfrak{m}}^{q+1},$$

where $\overline{\mathfrak{m}} = \mathfrak{m}/(f_j)$. Hence $\overline{x_i} \in \overline{\mathfrak{m}}^q$. In fact, assume that $\overline{x_i} \notin \overline{\mathfrak{m}}^q$ and let $\ell = \mathrm{o}_{\overline{\mathfrak{m}}}(\overline{x_i})$. Then $\ell \leq q-1$, while

$$0 \neq \overline{x_i}^* \in \mathrm{H}^0_M(\mathrm{G}(\overline{\mathfrak{m}})) \cong \left[\mathrm{H}^1_M(\mathrm{G}(\mathfrak{m}))\right](-1).$$

Hence $[\mathrm{H}^1_M(\mathrm{G}(\mathfrak{m}))]_{\ell-1} \neq (0)$ and so, $n \leq \ell-1 \leq q-2$. This is impossible, since $q \leq n+1$ by our assumption.

Thus $x_i \in \mathfrak{m}^q + (f_j)$ for all $1 \leq i \leq d$ and $1 \leq j \leq q$, so that the proof of Claim 1 shows $x_i \in \mathfrak{m}^q$ for all $1 \leq i \leq d$. In fact, with the same notation as in the proof of Claim 1, the crucial point is to check that $\overline{y} \in \overline{\mathfrak{m}}^{q-(k-1)}$. Suppose that $\overline{y} \notin \overline{\mathfrak{m}}^{q-(k-1)}$. Then, since $\overline{\prod_{j=1}^{k-1} f_j{\cdot}\overline{y}} \in \overline{\mathfrak{m}}^q$ and $\left(\overline{\prod_{j=1}^{k-1} f_j}\right)^*$ is a part of a homogeneous system of parameters in the Buchsbaum ring $\mathrm{G}(\overline{\mathfrak{m}})$, we get $\overline{y}^* \in \mathrm{H}^0_M(\mathrm{G}(\overline{\mathfrak{m}}))$, so that $n+1 \leq \mathrm{o}_{\overline{\mathfrak{m}}}(\overline{y}) \leq q-k \leq q-2$, which is impossible, since $q \leq n+1$. Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$, so that $I \subseteq \mathfrak{m}^{q+2}$ by Lemma 2.1. The other assertions follow similarly as in the proof of Theorem 1.1. □

4. Example

Let $d > 0$ and $n \geq 0$ be integers. We look at the graded ring

$$R = k[X_1, X_2, \cdots, X_d, Y_1, Y_2, \cdots, Y_d]/[(Y_i \mid 1 \leq i \leq d)^{n+2} + (\sum_{i=1}^{d} X_i Y_i^{n+1})],$$

where $U = k[X_1, X_2, \cdots, X_d, Y_1, Y_2, \cdots, Y_d]$ denotes the polynomial ring with $2d$ indeterminates over a field k. Let $M = R_+$, $A = R_M$, and $\mathfrak{m} = MR_M$.

Example 4.1. The following assertions hold true.

(1) $\dim R = d$ and $\operatorname{depth} R = d-1$.
(2) $\mathrm{H}^{d-1}_M(R) \cong [R/M](-(n+2-d))$ as graded R-modules.
(3) R is a Buchsbaum ring.
(4) $\mathrm{e}^0_{\mathfrak{m}}(A) = \binom{d+n+1}{d} - 1$.

Since $R \cong \mathrm{G}(\mathfrak{m})$, Example 4.1 provides Buchsbaum rings A which satisfy the conditions required in Theorem 1.1 (take $d \geq 3$) and Theorem 3.1 (take $d = 2$).

Remark 4.2. Taking $n = 0$ in Example 4.1, we have $\mathrm{e}^0_{\mathfrak{m}}(A) = d = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h^i(A)$, where $h^i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$. Hence our Buchsbaum local ring A has minimal multiplicity in the sense of [G2, Section 4].

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QUASI-SOCLE IDEALS AND GOTO NUMBERS OF PARAMETERS

TRAN THI PHUONG

1. Introduction and the main results

This report records my talk at the 31-st Symposium on Commutative Algebra in Japan. My talk is based on the joint work [GKPT], which will appear in J. Pure App. Algebra. I refer the readers to [GKPT] for the detail of arguments .

In what follows, let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let Q be a parameter ideal in A and let $q > 0$ be an integer. We put $I = Q : \mathfrak{m}^q$ and refer to those ideals as quasi-socle ideals in A. In this report we are interested in the following question about quasi-socle ideals I, which are also the main subject of the researches [GMT, GKM, GKMP].

Question 1.1.

(1) Find the conditions under which $I \subseteq \overline{Q}$, where $\overline{Q}$ stands for the integral closure of Q.

(2) When $I \subseteq \overline{Q}$, estimate or describe the reduction number

$$\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$$

of I with respect to Q in terms of some invariants of Q or A.

(3) Clarify what kind of ring-theoretic properties do the graded rings

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n, \ \mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}, \text{ and } \mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

associated to the ideal I enjoy.

The present research is a continuation of [GMT, GKM, GKMP] and aims mainly at the analysis of the case where A is a complete intersection with $\dim A = 1$. Following W. Heinzer and I. Swanson [HS], for each parameter ideal Q in a Noetherian local ring $(A, \mathfrak{m})$ we define

$$\mathrm{g}(Q) = \max\{q \in \mathbb{Z} \mid Q : \mathfrak{m}^q \subseteq \overline{Q}\}$$

and call it the Goto number of Q. In this report we are also interested in computing Goto numbers $\mathrm{g}(Q)$ of parameter ideals. In [HS] one finds, among many interesting results, that if the base local ring $(A, \mathfrak{m})$ has dimension one, then there exists an integer $k \gg 0$ such that the Goto number $\mathrm{g}(Q)$ is constant for every parameter ideal Q contained in $\mathfrak{m}^k$. We will show that this is not true if $\dim A > 1$, explicitly computing Goto numbers $\mathrm{g}(Q)$ for certain parameter ideals Q in a Noetherian local ring $(A, \mathfrak{m})$ with Gorenstein associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

To state the main results of this report, let us fix some notation. Let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\{a_i\}_{1 \leq i \leq d}$ be positive integers and let $\{x_i\}_{1\le i\le d}$ be elements of A with $x_i \in \mathfrak{m}^{a_i}$ for each $1 \le i \le d$ such that the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1\le i\le d}$ constitute a homogeneous system of parameters in $\mathrm{G}(\mathfrak{m})$. Hence $\mathfrak{m}^\ell = \sum_{i=1}^d x_i \mathfrak{m}^{\ell-a_i}$ for $\ell \gg 0$, so that $Q = (x_1, x_2, \cdots, x_d)$ is a parameter ideal in A. Let $q \in \mathbb{Z}$, $I = Q : \mathfrak{m}^q$,

$$\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i, \quad \text{and} \quad \ell = \rho + 1 - q,$$

where $\mathrm{a}(*)$ denote the a-invariants of graded rings ([GW, (3.1.4)]). We put

$$\ell_1 = \inf\{n \in \mathbb{Z} \mid \mathfrak{m}^n \subseteq I\} \quad \text{and} \quad \ell_2 = \sup\{n \in \mathbb{Z} \mid I \subseteq Q + \mathfrak{m}^n\}.$$

With this notation our main result is stated as follows.

Theorem 1.2. *Suppose that* $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n\ge 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ *is a Cohen-Macaulay ring and consider the following four conditions:*

(1) $\ell_1 \ge a_i$ *for all* $1 \le i \le d$.
(2) $I \subseteq \overline{Q}$.
(3) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.
(4) $\ell_2 \ge a_i$ *for all* $1 \le i \le d$.

Then one has the implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). *If* $\mathrm{G}(\mathfrak{m})$ *is a Gorenstein ring, then one has the equality* $I = Q + \mathfrak{m}^\ell$, *so that* $\ell_1 \le \ell \le \ell_2$, *whence conditions* (1), (2), (3), *and* (4) *are equivalent to the following:*

(5) $\ell \ge a_i$ *for all* $1 \le i \le d$.

Consequently, the Goto number $\mathrm{g}(Q)$ *of* Q *is given by the formula*

$$\mathrm{g}(Q) = \left[\mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i + 1\right] - \max\{a_i \mid 1 \le i \le d\},$$

provided $\mathrm{G}(\mathfrak{m})$ *is a Gorenstein ring; in particular* $\mathrm{g}(Q) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1$, *if* $d = 1$.

Let $R = k[R_1]$ be a homogeneous ring over a field k with $d = \dim R > 0$. We choose a homogeneous system $f_1, f_2, \cdots, f_d$ of parameters of R and put $\mathfrak{q} = (f_1, f_2, \cdots, f_d)$. Let $M = R_+$. Then, applying Theorem 1.2 to the local ring $A = R_M$, we readily get the following, where $\mathrm{g}(\mathfrak{q}) = \max\{n \in \mathbb{Z} \mid \mathfrak{q} : M^n \text{ is integral over } \mathfrak{q}\}$.

Corollary 1.3. *Suppose that* R *is a Gorenstein ring. Then*

$$\mathrm{g}(\mathfrak{q}) = \left[\mathrm{a}(R) + \sum_{i=1}^d \deg f_i + 1\right] - \max\{\deg f_i \mid 1 \le i \le d\}.$$

Hence $\mathrm{g}(\mathfrak{q}) = \mathrm{a}(R) + 1$, *if* $d = 1$.

Corollary 1.4. *With the same notation as is in* Theorem 1.2 *let* $d = 1$ *and put* $a = a_1$. *Assume that* $\mathrm{G}(\mathfrak{m})$ *is a reduced ring. Then the following conditions are equivalent:*

(1) $I \subseteq \overline{Q}$.
(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.
(3) $I \subseteq \mathfrak{m}^a$.

(4) $\ell_2 \geq a$.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A > 0$. We denote by $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A)$ the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Let $J \subseteq K$ ($\subsetneq A$) be ideals in A. We denote by $\overline{J}$ the integral closure of J. When $K \subseteq \overline{J}$, let

$$\mathrm{r}_J(K) = \min \{n \in \mathbb{Z} \mid K^{n+1} = JK^n\}$$

denote the reduction number of K with respect to J. For each finitely generated A-module M let $\mu_A(M)$ and $\ell_A(M)$ be the number of elements in a minimal system of generators for M and the length of M, respectively. We denote by $\mathrm{v}(A) = \ell_A(\mathfrak{m}/\mathfrak{m}^2)$ the embedding dimension of A.

2. The case where $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring

The purpose of this section is to prove Theorem 1.2. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\{a_i\}_{1 \leq i \leq d}$ be positive integers and let $\{x_i\}_{1 \leq i \leq d}$ be elements of A such that $x_i \in \mathfrak{m}^{a_i}$ for each $1 \leq i \leq d$. Assume that the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1 \leq i \leq d}$ constitute a homogeneous system of parameters in $\mathrm{G}(\mathfrak{m})$. Let $q \in \mathbb{Z}$ and $Q = (x_1, x_2, \cdots, x_d)$. We put $I = Q : \mathfrak{m}^q$.

Let us begin with the following.

Proposition 2.1. *Let $\ell_3 \in \mathbb{Z}$ and suppose that $\mathfrak{m}^{\ell_3} \subseteq \overline{Q}$. Then $\ell_3 \geq a_i$ for all $1 \leq i \leq d$.*

We put $\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i$ (cf. [GW, (3.1.6)]) and $\ell = \rho + 1 - q$. Let $\ell_1 = \inf\{n \in \mathbb{Z} \mid \mathfrak{m}^n \subseteq I\}$ and $\ell_2 = \sup\{n \in \mathbb{Z} \mid I \subseteq Q + \mathfrak{m}^n\}$.

We are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. (4) $\Rightarrow$ (3) We may assume $\ell_2 < \infty$. Then, since $I \subseteq Q + \mathfrak{m}^{\ell_2}$, we have $\mathfrak{m}^q I \subseteq \mathfrak{m}^q Q + \mathfrak{m}^{q+\ell_2}$, whence $\mathfrak{m}^q I = \mathfrak{m}^q Q + [Q \cap \mathfrak{m}^{q+\ell_2}]$. Notice that

$$Q \cap \mathfrak{m}^{q+\ell_2} = \sum_{i=1}^{d} x_i \mathfrak{m}^{q+\ell_2-a_i},$$

because the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1 \leq i \leq d}$ constitute a homogeneous system of parameters in the Cohen-Macaulay ring $\mathrm{G}(\mathfrak{m})$, and we have $\mathfrak{m}^{q+\ell_2-a_i} \subseteq \mathfrak{m}^q$, since $\ell_2 \geq a_i$ for all $1 \leq i \leq d$. Thus $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(3) $\Rightarrow$ (2) See [NR, Section 7, Theorem 2].

(2) $\Rightarrow$ (1) This follows from Proposition 2.1.

We now assume that $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring. Then $I = Q + \mathfrak{m}^\ell$ by [Wat] (see [O, Theorem 1.6] also), whence $\ell_1 \leq \ell \leq \ell_2$, so that the implication (1) $\Rightarrow$ (4) follows. Therefore, $I \subseteq \overline{Q}$ if and only if $\ell = \rho + 1 - q \geq a_i$ for all $1 \leq i \leq d$, or equivalently

$$q \leq \left[\mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i + 1\right] - \max\{a_i \mid 1 \leq i \leq d\}.$$

Thus $g(Q) = \left\lceil a(G(\mathfrak{m})) + \sum_{i=1}^{d} a_i + 1 \right\rceil - \max\{a_i \mid 1 \le i \le d\}$, so that

$$g(Q) = a(G(\mathfrak{m})) + 1,$$

if $d = 1$. □

Remark 2.2 (cf. Example 3.4). Unless $G(\mathfrak{m})$ is a Gorenstein ring, the implication (1) ⇒ (4) in Theorem 1.2 does not hold true in general, even when A is a complete intersection and $G(\mathfrak{m})$ is a Cohen-Macaulay ring. For example, let $V = k[[t]]$ be the formal power series ring over a field k and look at the numerical semigroup ring $A = k[[t^5, t^8, t^{12}]] \subseteq V$. Then $A \cong k[[X,Y,Z]]/(Y^3 - Z^2, X^4 - YZ)$, while $G(\mathfrak{m}) \cong k[X,Y,Z]/(Y^4, YZ, Z^2)$, whence $G(\mathfrak{m})$ is a Cohen-Macaulay ring but not a Gorenstein ring. Let $Q = (t^{20})$ in A and let $I = Q : \mathfrak{m}^3$; hence $a_1 = 4$ and $q = 3$. Then $I = (t^{20}, t^{22}, t^{23}, t^{26}, t^{29}) \subseteq \mathfrak{m}^3$ and $I^3 = QI^2$, so that $I \subseteq \overline{Q}$, while $I^2 = QI + (t^{44}) \subseteq Q$ but $t^{44} \notin QI$, since $t^{24} \notin I$. Thus $I^2 = Q \cap I^2 \neq QI$, so that $r_Q(I) = 2$ and the ring $G(I)$ is not Cohen-Macaulay. It is direct to check that $\mathfrak{m}^4 \subseteq I$, $\mathfrak{m}^3 \nsubseteq I$, and $I \nsubseteq Q + \mathfrak{m}^4 = \mathfrak{m}^4$ since $t^{22} \in I$ but $t^{22} \notin \mathfrak{m}^4$. Thus $\ell_1 = 4$ and $\ell_2 = 3$.

Thanks to Theorem 1.2, similarly as in [GKMP] we have the following complete answer to Question 1.1 for the parameter ideals $Q = (x_1, x_2, \cdots, x_d)$.

Theorem 2.3. *With the same notation as is in* Theorem1.2 *assume that* $G(\mathfrak{m})$ *is a Gorenstein ring. Suppose that* $\ell \ge a_i$ *for all* $1 \le i \le d$*. Then the following assertions hold true.*

1. $G(I)$ *is a Cohen-Macaulay ring,* $r_Q(I) = \lceil \frac{q}{\ell} \rceil$*, and* $a(G(I)) = \lceil \frac{q}{\ell} \rceil - d$*, where* $\lceil \frac{q}{\ell} \rceil = \min\{n \in \mathbb{Z} \mid \frac{q}{\ell} \le n\}$.
2. $F(I)$ *is a Cohen-Macaulay ring.*
3. $\mathcal{R}(I)$ *is a Cohen-Macaulay ring if and only if* $q \le (d-1)\ell$.
4. *Suppose that* $q > 0$*. Then* $G(I)$ *is a Gorenstein ring if and only if* $\ell \mid q$.
5. *Suppose that* $q > 0$*. Then* $\mathcal{R}(I)$ *is a Gorenstein ring if and only if* $q = (d-2)\ell$.

We now discuss Goto numbers. For each Noetherian local ring A let

$$\mathcal{G}(A) = \{g(Q) \mid Q \text{ is a parameter ideal in } A\}.$$

We explore the value $\min \mathcal{G}(A)$ in the setting of Theorem 1.2 with $\dim A = 1$. For the purpose the following result is fundamental.

Theorem 2.4 ([HS, Theorem 3.1]). *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension one. Then there exists an integer* $k \gg 0$ *such that* $g(Q) = \min \mathcal{G}(A)$ *for every parameter ideal* Q *of* A *contained in* $\mathfrak{m}^k$.

Thanks to Theorem 1.2 and Theorem 2.4, we then have the following.

Corollary 2.5. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim A = 1$*. Then* $\min \mathcal{G}(A) = a(G(\mathfrak{m})) + 1$*, if* $G(\mathfrak{m})$ *is a Gorenstein ring.*

We close this section with the following.

Proposition 2.6. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $\dim A = 1$*. Then* $v(A) \le 2$ *if and only if* $\min \mathcal{G}(A) = e(A) - 1$.

Proof. Suppose that $\mathrm{v}(A) \leq 2$. Then $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathfrak{m})) = \mathrm{e}(A) - 2$. Hence $\min \mathcal{G}(A) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1 = \mathrm{e}(A) - 1$ by Corollary 2.5. Conversely, assume that $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$. To prove the assertion, enlarging the field $A/\mathfrak{m}$ if necessary, we may assume that the field $A/\mathfrak{m}$ is infinite (use Theorem 2.4). Let $x \in \mathfrak{m}$ and assume that $Q = (x)$ is a reduction of $\mathfrak{m}$. We put $e = \mathrm{e}(A)$ and $q = \mathrm{g}(Q)$. Then $q \geq e - 1$. Let $B = A/Q$ and $\mathfrak{n} = \mathfrak{m}/Q$. Then $Q : \mathfrak{m}^q \subseteq \overline{Q} \subsetneq A$. Hence $\mathfrak{n}^q \neq (0)$, so that $\mathfrak{n}^i \neq \mathfrak{n}^{i+1}$ for any $0 \leq i \leq q$. Consequently, because $q + 1 \geq e$ and

$$e = \ell_A(A/Q) = \sum_{i \geq 0} \ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \geq \sum_{i=0}^{q} \ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \geq q + 1,$$

we get $\mathfrak{n}^{q+1} = (0)$ and $\ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) = 1$ for all $0 \leq i \leq q$. Hence $\ell_A(\mathfrak{n}/\mathfrak{n}^2) \leq 1$, so that $\mathrm{v}(A) \leq 2$. □

3. The case where $A = B/yB$ and B is not a regular local ring

In the following two sections 3 and 4 we shall restrict our attention on quasi-socle ideals in the ring A of the form $A = B/yB$, where $(B, \mathfrak{n})$ is a Cohen-Macaulay local ring of dimension 2 and y is part of a system of parameters in B. This class of local rings contains all the local complete intersections of dimension one. Typical examples we have in mind are numerical semigroup rings and the main purpose is to go beyond the restriction in [GKMP] that parameter ideals be generated by monomials.

In this section assume that B is *not* a regular local ring; we do not assume that $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a Gorenstein ring. Our result is the following.

Theorem 3.1. *Let $(B, \mathfrak{n})$ be a Cohen-Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let n, q be integers such that $n \geq q > 0$. Let $y \in \mathfrak{n}^n$ and assume that y is regular in B. We put $A = B/yB$ and $\mathfrak{m} = \mathfrak{n}/yB$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true, where $m = n - q$.*

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, *and* $Q \cap I^2 = QI$. *Hence* $\mathrm{g}(Q) \geq n$.

(2) $I^2 = QI$, *if one of the following conditions is satisfied.*

  (i) $m \geq q - 1$;

  (ii) $m < q - 1$ *and* $Q \subseteq \mathfrak{m}^{q-m}$;

  (iii) $m > 0$ *and* $Q \subseteq \mathfrak{m}^{q-1}$.

(3) *Suppose that B is a Gorenstein ring. Then $I^3 = QI^2$ and $\mathrm{G}(I)$ is a Cohen-Macaulay ring, if one of the following conditions is satisfied.*

  (i) $m < q - 1$ *and* $Q \subseteq \mathfrak{m}^{q-(m+1)}$;

  (ii) $Q \subseteq \mathfrak{m}^{q-1}$.

Let us note here some concrete examples. Let $n \geq 0$ be an integer and put $a = 6n+5$, $b = 6n + 8$, and $c = 9n + 12$. Then $0 < a < b < c$ and $\mathrm{GCD}(a, b, c) = 1$. Let $A = k[[t^a, t^b, t^c]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal power series ring over a field k. Then

$$A \cong k[[X, Y, Z]]/(Y^3 - Z^2,\ X^{3n+4} - Y^{3n+1}Z),$$

where $k[[X,Y,Z]]$ denotes the formal powers series ring. Let $\mathfrak{m}$ be the maximal ideal in A. Then

$$\mathrm{G}(\mathfrak{m}) \cong k[X,Y,Z]/(Y^{3n+4}, Y^{3n+1}Z, Z^2).$$

Hence A is a complete intersection with $\dim A = 1$, whose associated graded ring $\mathrm{G}(\mathfrak{m})$ is not a Gorenstein ring but Cohen-Macaulay. We put

$$B = k[[X,Y,Z]]/(Y^3 - Z^2)$$

and let y denote the image of $X^{3n+4} - Y^{3n+1}Z$ in B. Let $\mathfrak{n} = (X,Y,Z)B$ be the maximal ideal in B. Then B is not a regular local ring and $A = B/yB$. We have $y \in \mathfrak{n}^{3n+2}$ and y is part of a system of parameters of B. Therefore by Theorem 3.1 (1), (2), and (3) we have the following.

Example 3.2. Let $0 < q \leq 3n+2$ be an integer and put $m = (3n+2) - q$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true.

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, and $Q \cap I^2 = QI$. Hence $\mathrm{g}(Q) \geq 3n+2$.
(2) $I^2 = QI$, if one of the following conditions is satisfied.
  (i) $m \geq q-1$;
  (ii) $m < q-1$ and $Q \subseteq \mathfrak{m}^{q-m}$;
  (iii) $m > 0$ and $Q \subseteq \mathfrak{m}^{q-1}$.
(3) $I^3 = QI^2$ and the ring $\mathrm{G}(I)$ is Cohen-Macaulay, if one of the following conditions is satisfied.
  (i) $m < q-1$ and $Q \subseteq \mathfrak{m}^{q-(m+1)}$;
  (ii) $Q \subseteq \mathfrak{m}^{q-1}$.

Remark 3.3. In Example 3.2 (3) the equality $I^2 = QI$ does not necessarily hold true. For example, let $n = 0$; hence $A = k[[t^5, t^8, t^{12}]]$. Let $Q = (t^5)$ in A and $I = Q : \mathfrak{m}^2$. Then $I = (t^5, t^{12}, t^{16}) \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2$.

As we see in the following examples, the assumption that $y \in \mathfrak{n}^q$ in Theorem 3.1 is crucial in order to control Cohen-Macaulayness in $\mathrm{G}(I)$ for quasi-socle ideals $I = Q : \mathfrak{m}^q$.

Example 3.4. In Example 3.2 take $n = 0$ and look at the local ring $A = k[[t^5, t^8, t^{12}]]$. Hence

$$A \cong k[[X,Y,Z]]/(Y^3 - Z^2, X^4 - YZ).$$

Let $0 < s \in \langle 5, 8, 12 \rangle := \{5\alpha + 8\beta + 12\gamma \mid 0 \leq \alpha,\ \beta,\ \gamma \in \mathbb{Z}\}$ and $Q = (t^s)$ in A, *monomial* parameters. Let us consider the quasi-socle ideal $I = Q : \mathfrak{m}^3$. Then we always have $I \subseteq \overline{Q}$, but $\mathrm{G}(I)$ is Cohen-Macaulay (resp. the equality $\mathfrak{m}^3 I = \mathfrak{m}^3 Q$ holds true) if and only if $s \in \{5, 10, 12, 15, 17\}$ (resp. $s \in \{5, 12, 17\}$), or equivalently $Q \cap I^2 = QI$. Thus Cohen-Macaulayness in $\mathrm{G}(I)$ is rather wild, as we summarize in the following table.

4. The case where $A = B/yB$ and B is a regular local ring

In this section let us assume that $(B, \mathfrak{n})$ is a regular local ring of dimension 2 and $A = B/yB$, where y is part of a system of parameters in B. Hence $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$

s	I	$\mathfrak{m}^3 I = \mathfrak{m}^3 Q$	$\mathrm{G}(I)$ is CM	$\mathrm{r}_Q(I)$
5	$\mathfrak{m} = (t^5, t^8, t^{12})$	Yes	Yes	3
8	(t^8, t^{10}, t^{17})	No	No	3
10	$(t^{10}, t^{12}, t^{13}, t^{16})$	No	Yes	2
12	$(t^{12}, t^{15}, t^{18}, t^{21})$	Yes	Yes	1
13	$(t^{13}, t^{15}, t^{16}, t^{22})$	No	No	2
15	$(t^{15}, t^{17}, t^{18}, t^{21}, t^{24})$	No	Yes	2
16	$(t^{16}, t^{18}, t^{22}, t^{25})$	No	No	2
17	$(t^{17}, t^{20}, t^{23}, t^{24}, t^{26})$	Yes	Yes	1
18	$(t^{18}, t^{20}, t^{21}, t^{24}, t^{27})$	No	No	2
≥ 20	$(t^s, t^{s+2}, t^{s+3}, t^{s+6}, t^{s+9})$	No	No	2

is a Gorenstein ring, so that the basic assumption in Theorem 1.2 is satisfied. Recall that $\mathrm{v}(A) \leq 2$ and $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$ (Proposition 2.6).

Our result of this time is the following.

Theorem 4.1. *Let $(B, \mathfrak{n})$ be a regular local ring of dimension 2. Let n, q be integers such that $n > q > 0$ and put $m = n - q$. Let $0 \neq y \in \mathfrak{n}^n$ and put $A = B/yB$ and $\mathfrak{m} = \mathfrak{n}/yB$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true.*

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, *and* $Q \cap I^2 = QI$.

(2) $I^2 = QI$, *if one of the following conditions is satisfied.*
  - (i) $m \geq q$;
  - (ii) $m < q$ *and* $Q \subseteq \mathfrak{m}^{q-(m-1)}$.

(3) $I^3 = QI^2$ *and the ring* $\mathrm{G}(I)$ *is Cohen-Macaulay, if one of the following conditions is satisfied.*
  - (i) $m < q$ *and* $Q \subseteq \mathfrak{m}^{q-m}$;
  - (ii) $Q \subseteq \mathfrak{m}^{q-1}$.

Remark 4.2. To see that the results of Theorem 4.1 are sharp, the reader may consult [GKM, GKMP] for examples of monomial parameter ideals $Q = (t^s)$ $(0 < s \in H)$ in numerical semigroup rings $A = k[[H]]$. See [GKMP, Proposition 10] for the case where $H = \langle a, b \rangle$ with $\mathrm{GCD}(a, b) = 1$. Let us pick up the simplest ones.

(1) The equality $I^2 = QI$ does not necessarily hold true. Let $A = k[[t^3, t^4]]$, $Q = (t^3)$, and $I = Q : \mathfrak{m}^2$. Then $I = \mathfrak{m} \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2$.

(2) The reduction number $\mathrm{r}_Q(I)$ could be not less than 3. Let $A = k[[t^4, t^5]]$, $Q = (t^4)$, and $I = Q : \mathfrak{m}^3$. Then $I = \mathfrak{m} \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 3$.

(3) The ring $\mathrm{G}(I)$ is not necessarily Cohen-Macaulay. Let $A = k[[t^5, t^6]]$, $Q = (t^{11})$, and $I = Q : \mathfrak{m}^4$. Then $I = (t^{11}, t^{12}, t^{15}) \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 3$. However, since $t^{36} \in Q \cap I^3$ but $t^{36} \notin QI^2$, we have $Q \cap I^3 \neq QI^2$, so that $\mathrm{G}(I)$ is not a Cohen-Macaulay ring.

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On quasi-socle ideals in a Gorenstein local ring

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1 Introduction and Main Theorems

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let Q be a parameter ideal for A (that is, Q is generated by a system of parameters for A) and $p > 0$ a positive integer. Then we call the quotient ideal $Q :_A \mathfrak{m}^p$ the p-th quasi-socle ideal of Q in A (or shortly the p-socle ideal of Q) and put $I = Q :_A \mathfrak{m}^p$.

The purpose of this article is to answer the following two problems:

- When $I \subseteq \mathfrak{m}^p$?
- When does the equality $I^2 = QI$ hold?

When $p = 1$, $I = Q :_A \mathfrak{m}$ is so-called the socle ideal of Q in A, and there are many studies of the socle ideals. In the Cohen-Macaulay case, for example, there is the following.

Theorem ([CHV, CP, CPV, G]). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim A = d$. Let Q be a parameter ideal in A and let $I = Q :_A \mathfrak{m}$. Then the following three conditions are equivalent to each other.*

(1) $I^2 \neq QI$.

(2) *Q is integrally closed in A.*

(3) *A is a regular local ring, which contains a regular system $a_1, a_2, \cdots, a_d$ of parameters such that $Q = (a_1, \cdots, a_{d-1}, a_d^q)$ for some $1 \leq q \in \mathbb{Z}$.*

Therefore, we have $I^2 = QI$ for every parameter ideal Q in A, if A is a Cohen-Macaulay local ring but not regular.

Then what is studied in the case when $p \geq 2$? When $p \geq 2$ and A is a Cohen-Macaulay local ring with $d = \dim A \geq 2$, there is a remarkable and excellent theorem given by H.-J. Wang.

Theorem A ([W]). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A \geq 2$. Let $p > 0$ be a positive integer and Q a parameter ideal for A. Assume that $Q \subseteq \mathfrak{m}^p$ and put $I = Q :_A \mathfrak{m}^p$. Then we have*

$$\mathfrak{m}^p I = \mathfrak{m}^p Q, \quad I \subseteq \mathfrak{m}^p, \quad \text{and} \quad I^2 = QI$$

provided that A is not regular if $d \geq 2$ and that $p \geq 2$ if $d \geq 3$.

Then our question is what about one-dimensional case. As S. Goto, S. Kimura, N. Matsuoka, and R. Takahashi suggested in [GKM, GTM], you can see that one-dimensional case is different from higher-dimensional cases and more complicated to control even though A is Cohen-Macaulay. For example, $Q :_A \mathfrak{m}^p$ is not necessarily contained in $\mathfrak{m}^p$ for a parameter ideal $Q \subseteq \mathfrak{m}^p$ in a one-dimensional Cohen-Macaulay local ring.

Example 1. *Let $A = k[[X, Y]]/(X^2)$, where $k[[X, Y]]$ is the formal power series ring with two indeterminates X and Y over a field k. Put $\mathfrak{m} = (x, y) \subset A$, where x and y are the images of X and Y in A respectively. Then $\mathfrak{m}^n = (xy^{n-1}, y^n)$ for all positive integers $n > 0$. Now let $p \geq 2$ be an integer and put $Q = (y^{2p-2})$. Then $Q \subseteq \mathfrak{m}^{2p-2} \subseteq \mathfrak{m}^p$ and $Q :_A \mathfrak{m}^p = (xy^{p-2}, y^{p-1}) = \mathfrak{m}^{p-1} \nsubseteq \mathfrak{m}^p$.*

Then the first theorem of this article is the following.

Main Theorem 1. *Let $(A, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring, and $p > 0$ a positive integer and $q \geq 0$ a non-negative integer. Then $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^q$ for all parameter ideals $Q \subseteq \mathfrak{m}^{p+q}$. Moreover if A is not regular, then $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$ for all parameter ideals $Q \subseteq \mathfrak{m}^{p+q}$.*

Therefore we get the following, taking $q = p - 1$ in Main Theorem 1.

Corollary. *Assume that A is a one-dimensional Cohen-Macaulay local ring, but not regular. Let $p > 0$ be a positive integer. Then $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^p$ for all parameter ideals $Q \subseteq \mathfrak{m}^{2p-1}$*

We shall remark that Example 1 shows that the value $2p - 1$ of an order of parameter ideals $Q \subseteq \mathfrak{m}^{2p-1}$ in this corollary is the best possible.

Here we should mention a very interesting paper [GKPT] given by S. Goto, S. Kimura, T. T. Phuong, and H. L. Truong. The author was inspired by discussions with Satoru Kimura. In fact our technique of proof is similar to theirs (see [GKPT, Lemma 3.2, Proposition 3.3.]) and they give more interesting results in rather special settings. But their results do not cover Main Theorem 1.

So next, we would like to talk about the reduction number of quasi-socle ideal in a one-dimensional Cohen-Macaulay local ring. Some results are given for one-dimensional (or arbitrarily dimensional) Cohen-Macaulay local rings. S. Goto, S. Kimura and N. Matsuoka explored quasi-socle ideals in numerical semigroup rings (see [GKM]). S. Goto, S. Kimura, N. Matsuoka and T. T. Phuong, and also, S. Goto, S. Kimura, T. T. Phuong, and H. L. Truong explored quasi-socle ideals in a certain one-dimensional (or an arbitrarily dimensional) Cohen-Macaulay local ring and got some interesting results, but in rather special settings (see [GKMP, GKPT]). When $p = 2$ and the base ring A is Gorenstein, there is the following remarkable result given by S. Goto, N. Matsuoka, and R. Takahashi [GTM].

Theorem B ([GTM]). *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A > 0$, Q a parameter ideal for A and $I = Q :_A \mathfrak{m}^2$. Then the following assertions hold.*

(*i*) *Assume that $n = \mu_A(\mathfrak{m}/Q) \geq 2$, where $\mu_A(*)$ denotes the minimal number of generators. Then the following conditions are equivalent.*

(1) $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$.

(2) $I \subseteq \overline{Q}$, *where $\overline{Q}$ denotes the integral closure of Q.*

(3) $\mathfrak{m} I \cap Q = \mathfrak{m} Q$.

(4) $\mu_A(I) = n + d$.

(*ii*) *Assume that $\mu_A(\mathfrak{m}) \geq 2$, $Q \subseteq \mathfrak{m}^2$, and $\mathfrak{m} I \cap Q = \mathfrak{m} Q$. Then we have that $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$, $I \subseteq \mathfrak{m}^2$, and $I^2 = QI$.*

(*iii*) *Assume that $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 3$. Then $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ and $I^3 = QI^2$. Moreover $I^2 = QI$ if $Q \subseteq \mathfrak{m}^2$.*

When $d = \dim A \geq 2$, the assertions (ii) and (iii) of Theorem B are completely covered by Wang's theorem, Theorem A. But Goto-Takahashi-Matsuoka's technique of proof is independent form Wang's one, and works for one-dimensional case.

Here our question is the following.

Question: *Can we generalize Theorem B to $p \geq 2$?*

But the following example shows difficulty solving the question.

Example 2 ([GKPT, Example 5.3.]). *Let $A = k[[t^5, t^8, t^{12}]] \subseteq k[[t]]$ and $\mathfrak{m} = (t^5, t^8, t^{12})$, where k is a field. Then $(A, \mathfrak{m})$ is a one-dimensional Gorenstein local ring and $G_{\mathfrak{m}}(A)$ is Cohen-Macaulay, where $G_{\mathfrak{m}}(A)$ is the associated graded ring of A with respect to $\mathfrak{m}$. Let $0 < \alpha \in < 5, 8, 12 >= \{5a + 8b + 12c \mid 0 \leq a, b, c \in \mathbb{Z}\}$, and $Q = (t^\alpha)$ and $I = Q :_A \mathfrak{m}^3$. Assume that $\alpha \geq 20$. Then $I = (t^\alpha, t^{\alpha+2}, t^{\alpha+3}, t^{\alpha+6}, t^{\alpha+9}) \subseteq \overline{Q}$, hence $\mathfrak{m} I \cap Q = \mathfrak{m} Q$, but $\mathfrak{m}^3 I \neq \mathfrak{m}^3 Q$, $I^2 \neq QI$ and $I^3 = QI^2$. Finally, you should notice $\mathfrak{m}^2 I \cap Q \neq \mathfrak{m}^2 Q$.*

Then the second and third theorem of this article are the following.

Main Theorem 2 (a generalization of Theorem B (i)). *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A$ and $p \geq 2$ be an integer. Let Q be a parameter ideal, $I = Q :_A \mathfrak{m}^p$, $n = \mu_A(\mathfrak{m}^{p-1} + Q/Q)$, and $s = \ell_A(\mathfrak{m}/\mathfrak{m}^{p-1} + Q)$, where $\ell_A(*)$ denotes the length. Assume that the following three conditions hold.*

(*a*) $\mu_A(I/Q) = n \geq 2$.

(*b*) $I \subseteq \mathfrak{m}^{p-1}$.

(*c*) $\mathfrak{m}^{p-1} I + Q = Q :_A \mathfrak{m}$.

Then the following conditions are equivalent.

(1) $\mathfrak{m}^{2p-2}I = \mathfrak{m}^{2p-2}Q$ *and* $\ell_A(\mathfrak{m}I/\mathfrak{m}^{p-1}I) = s(d+1)$.

(2) $I \subseteq \overline{Q}$ *and* $\ell_A(\mathfrak{m}I/\mathfrak{m}^{p-1}I) = s(d+1)$.

(3) $\mathfrak{m}I \cap Q = \mathfrak{m}Q$ *and* $\ell_A(\mathfrak{m}I/\mathfrak{m}^{p-1}I) = s(d+1)$.

(4) $\mu_A(I) = n + d$ *and* $\ell_A(\mathfrak{m}I/\mathfrak{m}^{p-1}I) = s(d+1)$.

(5) $\mathfrak{m}^{p-1}I \cap Q = \mathfrak{m}^{p-1}Q$.

When $p = 2$, you can check that the condition $\mu_A(I/Q) = n$ is automatically satisfied, and the assumption $\mu_A(I/Q) \geq 2$ implies the conditions (b) and (c). Also, if $p = 2$, then $s = 0$ and hence the equality $\ell_A(\mathfrak{m}I/\mathfrak{m}^{p-1}I) = s(d+1)$ is obvious. Therefore this theorem gives a kind of generalization of the assertion (i) of Theorem B.

Main Theorem 3 (a generalization of Theorem B (ii)). *Let* $(A, \mathfrak{m})$ *be a Gorenstein local ring with* $d = \dim A > 0$, *and assume that* $\operatorname{depth} G_{\mathfrak{m}}(A) > 0$. *Let* $p \geq 2$ *be an integer,* Q *a parameter ideal and* $I = Q :_A \mathfrak{m}^p$. *Assume that* $\mu_A(\mathfrak{m}) \geq 2$, $Q \subseteq \mathfrak{m}^{2p-2}$, *and* $\mathfrak{m}^{p-1}I \cap Q = \mathfrak{m}^{p-1}Q$. *Then we have that* $\mathfrak{m}^pI = \mathfrak{m}^pQ$, $I \subseteq \mathfrak{m}^{2p-2}$, *and* $I^2 = QI$.

The next target is to generalize the assertion (iii) of Theorem B. For example, the author want to control all parameter ideals using an invariant depends only on the base ring A, for exaple $\mathrm{e}^0_{\mathfrak{m}}(A)$ or something.

In the next section, we shall give a proof of Main Theorem 1. The author skip a proof of Main Theorem 2 and 3 for a limit the number of pages.

2 Proof of Main Theorem 1

In this section we shall give a proof of Main Theorem 1. In fact, we shall show Theorem 2.3 which shows more general assertion and get Main Theorem 1 as a corollary (Corollary 2.4) to Theorem 2.3.

Now let us begin with the following.

Lemma 2.1. *Let* A *be a commutative ring,* $\mathfrak{a}$, $\mathfrak{b}$, *and* $\mathfrak{c}$ *ideals of* A. *Assume that* $\mathfrak{a}$ *contains a non-zero divisor and* $\mathfrak{a} \subseteq \mathfrak{b}$. *Furthermore assume that there exists a subset* $F \subseteq \mathfrak{a}$ *of* $\mathfrak{a}$ *such that* $\mathfrak{a}$ *is generated by* F *and* $(f) :_A \mathfrak{b} \subseteq \mathfrak{c}$ *for all elements* $f \in F$. *Then we have that* $(a) :_A \mathfrak{b} \subseteq \mathfrak{c}$ *for all non-zero divisors* $a \in \mathfrak{a}$.

Proof. Take any non-zero divisor $a \in \mathfrak{a}$ and fix it. Take any elements $x \in (a) :_A \mathfrak{b}$ and $f \in F$. Since $\mathfrak{a} \subseteq \mathfrak{b}$, we can write $fx = ay$ for some $y \in A$. On the other hand, take any element $b \in \mathfrak{b}$ and write $bx = az$ for some $z \in A$. Then we have that

$$bay = b(fx) = f(bx) = faz.$$

Hence $by = fz$ because a is a non-zero divisor. Thus $y \in (f) :_A \mathfrak{b}$. Then we have that $y \in \mathfrak{c}$ by the assumption. Therefore $fx = ay \in a\mathfrak{c}$, and thus, $x \in a\mathfrak{c} :_A \mathfrak{a}$ because $\mathfrak{a}$ is generated by the set F. Now, because $a \in \mathfrak{a}$, we have that $ax \in a\mathfrak{c}$. Therefore $x \in \mathfrak{c}$, since a is a non-zero divisor. Then we get that $(a) :_A \mathfrak{b} \subseteq \mathfrak{c}$. □

Lemma 2.2. *Let $(A, \mathfrak{m})$ be a commutative local ring and assume that $\mathfrak{m}$ contains a non-zero divisor. Let $p > 0$ be a positive integer and $q \geq 0$ an integer. Let $a_1, a_2, \ldots, a_{p+q} \in \mathfrak{m}$ be non-zero divisors and assume that $(a_1) \neq \mathfrak{m}$. Then $(a_1 a_2 \cdots a_{p+q}) :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$.*

Proof. First suppose that $q > 0$. Now it is easy to show that

$$(a_1 a_2 \cdots a_{p+q}) :_A \mathfrak{m}^p \subseteq (a_1 a_2 \cdots a_{p+q}) :_A a_1 a_2 \cdots a_p \subseteq [(0) :_A a_1 \cdots a_p] + (a_{p+1} \cdots a_{p+q}).$$

Since $a_1 \cdots a_p$ is a non-zeor divisor, we have that $(a_1 a_2 \cdots a_{p+q}) :_A \mathfrak{m}^p \subseteq (a_{p+1} \cdots a_{p+q})$. Take any element $x \in (a_1 a_2 \cdots a_{p+q}) :_A \mathfrak{m}^p$ and write $x = a_{p+1} \cdots a_{p+q} y$ for some $y \in A$. Now we claim the following.

Claim 1. $y \in (a_1) :_A \mathfrak{m}$.

Proof of Claim 1. Take any element $\alpha \in \mathfrak{m}$. Then

$$\alpha a_2 \cdots a_p x = \alpha a_2 \cdots a_p a_{p+1} \cdots a_{p+q} y$$

because $x = a_{p+1} \cdots a_{p+q} y$. On the other hand, since $\alpha a_2 \cdots a_p \in \mathfrak{m}^p$, we can write $\alpha a_2 \cdots a_p x = a_1 \cdots a_{p+q} z$ for some $z \in A$. Thus we have that $\alpha y = a_1 z$, since $a_2 \cdots a_{p+q}$ is a non-zero divisor. Therefore $y \in (a_1) :_A \mathfrak{m}$. □

Because of the assumption that $(a_1) \neq \mathfrak{m}$, we have that $(a_1) :_A \mathfrak{m} \subseteq \mathfrak{m}$. Then $y \in (a_1) :_A \mathfrak{m} \subseteq \mathfrak{m}$. Therefore $x = a_{p+1} \cdots a_{p+q} y \in \mathfrak{m}^{q+1}$, and thus, we get that $(a_1 a_2 \cdots a_{p+q}) :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$.

In fact, this proof works in the case when $q = 0$ considering an element x itself instead of y, that is, the above proof of Claim 1 shows that $(a_1 a_2 \cdots a_p) :_A \mathfrak{m}^p \subseteq (a_1) :_A \mathfrak{m}$. □

Then we have the following.

Theorem 2.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with* depth $A > 0$ *and $p > 0$ a positive integer. Let $q \geq 0$ be an integer. Assume that $\mathfrak{m}$ is not principal. Then we have that $(a) :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$ for all non-zero divisors $a \in \mathfrak{m}^{p+q}$.*

Proof. First it is easy to check that $\mathfrak{m}^{p+q}$ is generated by the following set F:

$$F = \{a_1 a_2 \cdots a_{p+q} \mid a_1, a_2, \ldots, a_{p+q} \in \mathfrak{m} \text{ are non-zero divisors}\}.$$

Since $\mathfrak{m}$ is not principal, we have that $(a_1 a_2 \cdots a_{p+q}) :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$ for all elements $a_1 a_2 \cdots a_{p+q} \in F$, by Lemma 2.2. Therefore, by Lemma 2.1, we get that $(a) :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$ for all non-zero divisors $a \in \mathfrak{m}^{p+q}$. □

Applying Theorem 2.3 to one-dimensional Cohen-Macaulay local ring, we get the following.

Corollary 2.4 (Main Theorem 1). *Let $(A, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring, $p > 0$ a positive integer, and $q \geq 0$ an integer. Then we have that $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^q$ for all parameter ideals $Q \subseteq \mathfrak{m}^{p+q}$. Moreover if A is not regular, then we have that $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$ for all parameter ideals $Q \subseteq \mathfrak{m}^{p+q}$.*

Proof. When A is regular, that is A is a DVR, it is clear that $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^q$ for all parameter ideals $Q \subseteq \mathfrak{m}^{p+q}$. So we may assume that A is not regular. Then our assertion readily follows form Theorem 2.3. □

Finally the author would like to give the following, although the following assertion is almost coverd by Wang's theorem (Theorem A) in the case when $\dim A \geq 2$.

Corollary 2.5. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. Let $p > 0$ be a positive integer and $q \geq 0$ an integer, and suppose that $p+q \geq 2$. Assume that $\mathfrak{m}$ is not principal. Then we have that $Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1}$ for all parameter ideals $Q \subseteq \mathfrak{m}^{p+q}$.*

Proof. When $d = 1$, it readily follows from Theorem 2.3. Suppose that $d \geq 2$ and our assertion holds for $d - 1$. Let $Q = (a_1, a_2, \ldots, a_d) \subseteq \mathfrak{m}^{p+q}$ be a parameter ideal for A. Now we shall remark that $A/(a_1)$ is not regular, since $a_1 \in \mathfrak{m}^{p+q} \subseteq \mathfrak{m}^2$. Then, passing to $A/(a_1)$, by the hypothesis of induction on d, we have that

$$Q :_A \mathfrak{m}^p \subseteq \mathfrak{m}^{q+1} + (a_1) \subseteq \mathfrak{m}^{q+1}$$

as is claimed. □

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COHEN-MACAULAYNESS VERSUS THE VANISHING OF THE FIRST HILBERT COEFFICIENT OF PARAMETERS

SHIRO GOTO AND KAZUHO OZEKI

1. Introduction

This is based on [GhGHOPV, GO] a joint work with L. Ghezzi, J. Hong, T. T. Phuong, and W. V. Vasconcelos.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then, for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all integers $n \gg 0$, which we call the Hilbert coefficients of A with respect to I. We say that A is unmixed, if $\dim \widehat{A}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A. With this notation Wolmer V. Vasconcelos posed, exploring the vanishing of $\mathrm{e}_Q^1(Q)$ for parameter ideals Q, in his lecture at the conference in Yokohama of March, 2008 the following conjecture.

Conjecture 1.1 ([V]). Assume that A is unmixed. Then A is a Cohen-Macaulay local ring, once $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q of A.

In Section 2 of the present paper we shall settle Conjecture 1.1 affirmatively. Here we should note that Conjecture 1.1 is already solved partially by [GhHV] and [MV]. Let us call those local rings A with $\mathrm{e}_Q^1(A) = 0$ for some parameter ideals Q *Vasconcelos.* In Section 3 we shall explore basic properties of Vasconcelos rings. In Section 4 we will study the problem of when $\mathrm{e}_Q^1(A)$ is constant and independent of the choice of parameter ideals Q in A. We shall show that A is a Buchsbaum ring, if A is unmixed and $\mathrm{e}_Q^1(A)$ is constant (Theorem 4.3).

In what follows, unless otherwise specified, let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let $\{\mathrm{H}_\mathfrak{m}^i(*)\}_{i \in \mathbb{Z}}$ be the local cohomology functors of A with respect to the maximal ideal $\mathfrak{m}$. Let $\mu_A(M)$ denote, for an A-module M, the number of generators.

Let $\operatorname{Assh} A = \{\mathfrak{p} \in \operatorname{Ass} A \mid \dim A/\mathfrak{p} = d\}$ and let $(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} A} \mathrm{I}(\mathfrak{p})$ be a primary decomposition of (0) in A with $\mathfrak{p}$-primary ideals $\mathrm{I}(\mathfrak{p})$ in A. We put

$$\mathrm{U}_A(0) = \bigcap_{\mathfrak{p} \in \operatorname{Assh} A} \mathrm{I}(\mathfrak{p})$$

and call it the unmixed component of (0) in A.

2. Proof of the conjecture of Vasconcelos

The purpose of this section is to prove the following, which settles Conjecture 1.1 affirmatively. One of the main results of this paper is the following.

Theorem 2.1. *Let A be unmixed. Then the following four conditions are equivalent to each other.*

(1) *A is a Cohen-Macaulay local ring.*
(2) *$\mathrm{e}_I^1(A) \geq 0$ for every $\mathfrak{m}$-primary ideal I in A .*
(3) *$\mathrm{e}_Q^1(A) \geq 0$ for some parameter ideal Q in A.*
(4) *$\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q in A.*

In our proof of Theorem 2.1 the following facts are the key. See [GNa, Section 3] for the proof.

Proposition 2.2 ([GNa]). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A \geq 2$, possessing the canonical module K_A. Assume that $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p} \in \mathrm{Ass}A \setminus \{\mathfrak{m}\}$. Then the following assertions hold true.*

(1) *The local cohomology module $\mathrm{H}_\mathfrak{m}^1(A)$ is finitely generated.*
(2) *The set $\mathcal{F} = \{\mathfrak{p} \in \mathrm{Spec}A \mid \dim A_\mathfrak{p} > \mathrm{depth}A_\mathfrak{p} = 1\}$ is finite.*
(3) *Suppose that the residue class field $k = A/\mathfrak{m}$ of A is infinite and let I be an $\mathfrak{m}$-primary ideal in A. Then one can choose an element $a \in I \setminus \mathfrak{m}I$ so that a is superficial for I and $\dim A/\mathfrak{p} = d-1$ for every $\mathfrak{p} \in \mathrm{Ass}_A A/aA \setminus \{\mathfrak{m}\}$.*

Proof of Theorem 2.1. We have only to check the implication (3) $\Rightarrow$ (1). Let $Q = (a_1, a_2, \cdots, a_d)$ with a system $a_1, a_2, \cdots, a_d$ of parameters in A. Enlarging the residue class field $k = A/\mathfrak{m}$ of A and passing to the $\mathfrak{m}$-adic completion of A, we may assume that the field $k = A/\mathfrak{m}$ is infinite and that A is complete. Since the assertion is obvious in the case where $d \leq 2$ (recall that for any Noetherian local ring $(A, \mathfrak{m})$ of dimension one, we have $\mathrm{e}_Q^1(A) = -\ell_A(\mathrm{H}_\mathfrak{m}^0(A))$; see [GNi, Lemma 2.4 (1)], and the two-dimensional case is readily deduced from this fact via the reduction modulo some superficial element $x = a_1$ of Q; see [GNi, Lemma 2.2] and notice that x is A-regular), we may assume that $d \geq 3$ and that our assertion holds true for $d-1$. Then we are able to choose, thanks to Proposition 2.2 (3), the element $x = a_1$ so that x is a superficial element of the parameter ideal Q and (the ring A/xA is *not necessarily* unmixed but) the unmixed component $U = \mathrm{U}_B(0)$ of (0) in $B = A/xA$ has finite length, whence $U = \mathrm{H}_\mathfrak{m}^0(B)$. Then the $d-1$ dimensional local ring B/U is Cohen-Macaulay by the hypothesis of induction on d, because

$$\mathrm{e}_{Q\cdot(B/U)}^1(B/U) = \mathrm{e}_{QB}^1(B) = \mathrm{e}_Q^1(A) \geq 0$$

(cf. [GNi, Lemma 2.2]). Hence $\mathrm{H}_\mathfrak{m}^i(B) = (0)$ for all $i \neq 0, d-1$. We now look at the long exact sequence

$$\begin{array}{rcl} \cdots & \to & \mathrm{H}_\mathfrak{m}^1(A) \xrightarrow{x} \mathrm{H}_\mathfrak{m}^1(A) \to \mathrm{H}_\mathfrak{m}^1(B) \to \\ \cdots \to \mathrm{H}_\mathfrak{m}^{i-1}(B) & \to & \mathrm{H}_\mathfrak{m}^i(A) \xrightarrow{x} \mathrm{H}_\mathfrak{m}^i(A) \to \cdots \\ \cdots \to \mathrm{H}_\mathfrak{m}^{d-2}(B) & \to & \mathrm{H}_\mathfrak{m}^{d-1}(A) \xrightarrow{x} \mathrm{H}_\mathfrak{m}^{d-1}(A) \to \cdots \end{array}$$

of local cohomology modules, derived from the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

of A-modules. We then have $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $2 \le i \le d-1$, since $\mathrm{H}^i_{\mathfrak{m}}(B) = (0)$ for all $1 \le i \le d-2$, while $\mathrm{H}^1_{\mathfrak{m}}(A) = x\mathrm{H}^1_{\mathfrak{m}}(A)$, since $\mathrm{H}^1_{\mathfrak{m}}(B) = (0)$. Consequently $\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, because the A-module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated by Proposition 2.1 (1). Thus A is a Cohen-Macaulay ring. □

Let us give one consequence of Theorem 2.1.

Corollary 2.3 ([MV]). *We have $\mathrm{e}^1_Q(A) \le 0$ for every parameter ideals Q in A.*

3. Vasconcelos rings

The purpose of this section is to develop a theory of Vasconcelos rings. Let us begin with the definition.

Definition 3.1. We say that A is a *Vasconcelos* ring, if either $d = 0$, or $d > 0$ and $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal Q in A.

Here is a basic characterization of Vasconcelos rings.

Theorem 3.2. *Suppose that $d = \dim A > 0$. Then the following four conditions are equivalent.*

(1) *A is a Vasconcelos ring.*
(2) *$\mathrm{e}^1_Q(A) = 0$ for every parameter ideal Q in A.*
(3) *$\widehat{A}/\mathrm{U}_{\widehat{A}}(0)$ is a Cohen-Macaulay ring and $\dim_{\widehat{A}} \mathrm{U}_{\widehat{A}}(0) \le d-2$, where $\mathrm{U}_{\widehat{A}}(0)$ denotes the unmixed component of (0) in the $\mathfrak{m}$-adic completion $\widehat{A}$ of A.*
(4) *The $\mathfrak{m}$-adic completion $\widehat{A}$ of A contains an ideal $I \ne \widehat{A}$ such that $\widehat{A}/I$ is a Cohen-Macaulay ring and $\dim_{\widehat{A}} I \le d-2$.*

When this is the case, $\widehat{A}$ is a Vasconcelos ring, $\mathrm{H}^{d-1}_{\mathfrak{m}}(A) = (0)$, and the canonical module $\mathrm{K}_{\widehat{A}}$ of $\widehat{A}$ is a Cohen-Macaulay $\widehat{A}$-module.

Proof. See [GhGHOPV, Theorem 3.3]. □

Notice that condition (3) of Theorem 3.2 is free from parameters. Therefore, since $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal, then $\mathrm{e}^1_Q(A) = 0$ for every parameter ideals Q in A. This is what the theorem says.

In the rest of this section, let us give some consequences of Theorem 3.2.

Corollary 3.3. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let Q be a parameter ideal in A. Assume that $\mathrm{e}^i_Q(A) = 0$ for all $1 \le i \le d$. Then A is a Cohen-Macaulay ring.*

Suppose that $d > 0$ and let Q be a parameter ideal in A. We denote by $R = \mathcal{R}(Q)$ (resp. $G = \mathrm{G}(Q)$) the Rees algebra (resp. the associated graded ring) of Q. Hence

$$R = A[Qt] \quad \text{and} \quad G = \mathcal{R}'(Q)/t^{-1}\mathcal{R}'(Q),$$

where t is an indeterminate over A and $\mathcal{R}'(Q) = A[Qt, t^{-1}]$. Let $\mathfrak{M} = \mathfrak{m}R + R_+$ be the graded maximal ideal in R. With this notation we have the following.

Corollary 3.4. *The following assertions hold true.*

(1) *A is a Vasconcelos ring if and only if $G_{\mathfrak{M}}$ is a Vasconcelos ring.*
(2) *Suppose that A is a homomorphic image of a Cohen-Macaulay ring. Then $R_{\mathfrak{M}}$ is a Vasconcelos ring, if A is a Vasconcelos ring.*

Thus Vasconcelos rings enjoy very nice properties.

4. Buchsbaumness in local rings possessing constant first Hilbert coefficients of parameters

In this section we study the problem of when $\mathrm{e}_Q^1(A)$ is constant and independent of the choice of parameter ideals Q in A.

Here let us briefly recall the definition of Buchsbaum local rings. The readers may consult the monumental book [SV] of J. Stückrad and W. Vogel for a detailed theory, some of which we shall note here for the use in this paper.

We say that our local ring A is Buchsbaum, if the difference

$$\ell_A(A/Q) - \mathrm{e}_Q^0(A)$$

is independent of the choice of parameter ideals Q in A and is an invariant of A, which we denote by $\mathbb{I}(A)$. As is well-known, A is a Buchsbaum ring if and only if every system $a_1, a_2, \cdots, a_d$ of parameters in A forms a d-sequence in the sense of C. Huneke ([H]). When A is a Buchsbaum local ring, one has

$$\mathfrak{m}{\cdot}\mathrm{H}_{\mathfrak{m}}^i(A) = (0)$$

for all $i \neq d$, whence the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(A)\}_{i \neq d}$ are finite-dimensional vector spaces over the field $A/\mathfrak{m}$, and the equality

$$\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_{\mathfrak{m}}^i(A))$$

holds true.

We say that A is a generalized Cohen-Macaulay local ring, if all the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(A)\}_{i \neq d}$ are finitely generated. Hence every Cohen-Macaulay local ring is Buchsbaum with $\mathbb{I}(A) = 0$ and Buchsbaum local rings are generalized Cohen-Macaulay. A given Noetherian local ring A with $d = \dim A > 0$ is a generalized Cohen-Macaulay local ring if and only if

$$\mathbb{I}(A) := \sup_Q \{\ell_A(A/Q) - \mathrm{e}_Q^0(A)\} < \infty,$$

where Q runs through parameter ideals in A ([STC]). When this is the case, one has

$$\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_{\mathfrak{m}}^i(A)).$$

Suppose that A is a generalized Cohen-Macaulay local ring and let Q be a parameter ideal in A. Then Q is called standard, if

$$\mathbb{I}(A) = \ell_A(A/Q) - \mathrm{e}_Q^0(A).$$

This condition is equivalent to saying that Q is generated by a system $a_1, a_2, \cdots, a_d$ of parameters which forms a strong d-sequence in any order ([STC]).

Let

$$\Lambda = \Lambda(A) = \{\mathrm{e}_Q^1(A) \mid Q \text{ be a parameter ideal in } A\ \}.$$

Then we can ask the following questions.

Question (1) When is Λ a finite set?
(2) When is Λ a singleton?

For example, our characterization of Vasconcelos rings says that $0 \in \Lambda$ if and only if $\Lambda = \{0\}$.

Let us summarize what is known about the questions, where we put $h^i(A) = \ell_A(\mathrm{H}_\mathfrak{m}^i(A))$ for each $i \in \mathbb{Z}$.

Proposition 4.1 ([GNi, Sch]). *Suppose that A is a generalized Cohen-Macaulay local ring and $d \geq 2$. Let Q be a parameter ideal in A. Then we have the following.*

(1) $\mathrm{e}_Q^1(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$.

(2) *We have* $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$, *if Q is standard.*

Thanks to Proposition 4.1, if A is a generalized Cohen-Macaulay ring then we have

$$0 \geq \mathrm{e}_Q^1(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A. Hence Λ is finite. If A is a Buchsbaum ring then, since all parameter ideals in A are standard, we have

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A. Thus, we have

$$\Lambda = \{-\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)\},$$

so that Λ is a singleton It is natural to ask the converse is also true.

Our answer is the following.

Theorem 4.2. *Suppose that $d \geq 2$ and A is unmixed. Assume that Λ is a finite set and put $\ell = -\min \Lambda$. Then $\mathfrak{m}^\ell \mathrm{H}_\mathfrak{m}^i(A) = (0)$ for every $i \neq d$. Hence $\mathrm{H}_\mathfrak{m}^i(A)$ is a finitely generated A-module for every $i \neq d$, so that A is a generalized Cohen-Macaulay local ring.*

The main result of this section is stated as follows.

Theorem 4.3. *Suppose that $d = \dim A \geq 2$ and A is unmixed. Then the following two conditions are equivalent.*

(1) *A is a Buchsbaum local ring.*
(2) *The first Hilbert coefficients $\mathrm{e}_Q^1(A)$ of A are constant and independent of the choice of parameter ideals Q in A.*

When this is the case, one has the equality

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} \ell_A(\mathrm{H}_{\mathfrak{m}}^i(A))$$

for every parameter ideal Q in A.

Thus Buchsbaum rings are characterized in terms of consistency of the first Hilbert coefficients of parameters. This is a new characterization of Buchsbaum rings.

The following result is a key for the proof of Theorem 4.3.

Theorem 4.4. *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 2$ and* $\mathrm{depth} A > 0$. *Let Q be a parameter ideal in A. Then the following two conditions are equivalent.*

(1) *Q is a standard parameter ideal in A.*
(2) $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$.

Proof of Theorem 4.3. We have only to show the implication (2) $\Rightarrow$ (1). Since $\sharp\Lambda = 1$, by Theorem 4.2 A is a generalized Cohen-Macaulay local ring, so that $\Lambda = \{-\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)\}$ by [Sch, Korollar 3.2]. Hence by Theorem 4.4 every parameter ideal Q is standard in A, because $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$, so that A is a Buchsbaum local ring. □

Unless A is unmixed, Theorem 4.3 is no more true, even if $\mathrm{e}_Q^1(A) = 0$ for every parameter ideal Q in A (cf. [GhGHOPV, Theorem 2.7]). Let us note one example.

Example 4.5. Let R be a regular local ring with the maximal ideal $\mathfrak{n}$ and $d = \dim R \geq 3$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. We put $\mathfrak{p} = (X_1, X_2, \cdots, X_{d-1})$ and $D = R/\mathfrak{p}$. Then D is a DVR. Let $A = R \ltimes D$ denote the idealization of D over R. Then A is a Noetherian local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times D$ and $\dim A = d$. Let Q be a parameter ideal in A and put $\mathfrak{q} = \varphi(Q)$, where $\varphi : A \to R, \varphi(a, x) = a$ denotes the projection map. We then have

$$\begin{aligned}
\ell_A(A/Q^{n+1}) &= \ell_R(R/\mathfrak{q}^{n+1}) + \ell_D(D/\mathfrak{q}^{n+1}D) \\
&= \ell_R(R/\mathfrak{q})\cdot\binom{n+d}{d} + \ell_D(D/\mathfrak{q}D)\cdot\binom{n+1}{1} \\
&= \mathrm{e}_{\mathfrak{q}}^0(R)\binom{n+d}{d} + \mathrm{e}_{\mathfrak{q}D}^0(D)\binom{n+1}{1}
\end{aligned}$$

for all integers $n \geq 0$, so that $\mathrm{e}_Q^0(A) = \mathrm{e}_{\mathfrak{q}}^0(R), \mathrm{e}_Q^{d-1}(A) = (-1)^{d-1}\mathrm{e}_{\mathfrak{q}D}^0(D)$, and $\mathrm{e}_Q^i(A) = 0$ if $i \neq 0, d-1$. Hence $\mathrm{e}_Q^1(A)$ is constant but A is not even a generalized Cohen-Macaulay local ring, because $\mathrm{H}^1_{\mathfrak{m}}(A)$ ($\cong \mathrm{H}^1_{\mathfrak{n}}(D)$) is not a finitely generated A-module. The local ring A is not unmixed, although $\operatorname{depth} A = 1$.

5. Characterization of local rings with constant $\mathrm{e}^1_Q(A)$

We close this paper with a characterization of Noetherian local rings A possessing $\sharp\Lambda = 1$. Let us note the following.

Proposition 5.1 ([GhGHOPV, Proposition 4.7]). *Suppose that $d = \dim A \geq 2$ and let U be the unmixed component of the ideal (0) in A. Assume that there exists an integer $t \geq 0$ such that $\mathrm{e}^1_Q(A) = -t$ for every parameter ideal Q in A. Then $\dim_A U \leq d-2$ and $\mathrm{e}^1_{\mathfrak{q}}(A/U) = -t$ for every parameter ideal $\mathfrak{q}$ in A/U.*

The goal of this paper is the following.

Theorem 5.2. *Suppose that $d = \dim A \geq 2$. Then the following two conditions are equivalent.*

(1) $\sharp\Lambda = 1$.

(2) *Let $U = \mathrm{U}_{\widehat{A}}(0)$ be the unmixed component of the ideal (0) in the $\mathfrak{m}$-adic completion $\widehat{A}$ of A. Then $\dim_{\widehat{A}} U \leq d-2$ and $\widehat{A}/U$ is a Buchsbaum local ring.*

When this is the case, one has the equality

$$\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(\widehat{A}/U)$$

for every parameter ideal Q in A.

Proof. (1) $\Rightarrow$ (2) For every parameter ideal $\mathfrak{q}$ of $\widehat{A}$ we have $\mathfrak{q} = (\mathfrak{q}\cap A)\widehat{A}$, so that $\mathfrak{q}\cap A$ is a parameter ideal in A. Hence $\Lambda(\widehat{A}) = \Lambda$ and so the implication follows from Theorem 4.3 and Proposition 5.1.

(2) $\Rightarrow$ (1) Since $\dim_{\widehat{A}} U \leq d-2$ and $\widehat{A}/U$ is a Buchsbaum local ring, we get $\#\Lambda(\widehat{A}) = 1$ by [GhGHOPV, Lemma 2.4 (c)], whence $\#\Lambda = 1$.

See Proposition 5.1 and [Sch, Korollar 3.2] for the last assertion. □

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A NOTE ON THE BUCHSBAUM-RIM FUNCTION OF A PARAMETER MODULE

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This note is a summary of a part of the paper [11] with Eero Hyry (University of Tampere). In this note we prove that the Buchsbaum-Rim function $\ell_A(\mathcal{S}_{\nu+1}(F)/N^{\nu+1})$ of a parameter module N in F is bounded above by $e(F/N)\binom{\nu+d+r-1}{d+r-1}$ for every integer $\nu \geq 0$. Moreover, it turns out that the base ring A is Cohen-Macaulay once the equality holds for some integer ν. As a direct consequence, we observe that the first Buchsbaum-Rim coefficient $e_1(F/N)$ of a parameter module N is always non-positive.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $F = A^r$ be a free module of rank $r > 0$, and let $S = \mathcal{S}_A(F)$ be the symmetric algebra of F, which is a polynomial ring over A. For a submodule M of F, let $\mathcal{R}(M)$ denote the image of the natural homomorphism $\mathcal{S}_A(M) \to \mathcal{S}_A(F)$, which is a standard graded subalgebra of S. Assume that the quotient F/M has finite length and $M \subseteq \mathfrak{m}F$. Then we can consider the function

$$\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \ ; \quad \nu \mapsto \ell_A(S_\nu/M^\nu)$$

where S_ν and M^ν denote the homogeneous components of degree ν of S and $\mathcal{R}(M)$, respectively. Buchsbaum and Rim studied this function in [4] in order to generalize the notion of the usual Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal. They proved that $\lambda(\nu)$ eventually coincides with a polynomial $P(\nu)$ of degree $d + r - 1$. This polynomial can then be written in the form

$$P(\nu) = \sum_{i=0}^{d+r-1} (-1)^i e_i(F/M) \binom{\nu + d + r - 2 - i}{d + r - 1 - i}$$

with integer coefficients $e_i(F/M)$. The coefficients $e_i(F/M)$ are called the *Buchsbaum-Rim coefficients* of F/M. The *Buchsbaum-Rim multiplicity of* F/M, denoted by $e(F/M)$, is now defined to be the leading coefficient $e_0(F/M)$.

In their article Buchsbaum and Rim also introduced the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module N in F is said to be *a parameter module in F*, if the following three conditions are satisfied: (i) F/N has finite length, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d+r-1$, where $\mu_A(N)$ is the minimal number of generators of N.

A starting point of this note is the characterization of the Cohen-Macaulay property of A given in [4, Corollary 4.5] by means of the equality $\ell_A(F/N) = e(F/N)$ for every parameter module N of rank r in $F = A^r$. Brennan, Ulrich and Vasconcelos observed in [1, Theorem 3.4] that if A is Cohen-Macaulay, then in fact

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

for all integers $\nu \geq 0$. Our main result is now as follows:

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the inequality*

$$\ell_A(S_{\nu+1}/N^{\nu+1}) \geq e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

always holds true for every parameter module N in $F = A^r$ and for every integer $\nu \geq 0$.

(2) *The following statements are equivalent:*

(i) *A is a Cohen-Macaulay local ring;*

(ii) *There exists an integer $r > 0$ and a parameter module N of rank r in $F = A^r$ such that the equality*

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

holds true for some integer $\nu \geq 0$.

This generalizes our previous result [10, Theorem 1.3] where we assumed that $\nu = 0$. The equivalence of (i) and (ii) in (2) seems to contain some new information even in the ideal case. Indeed, it improves a recent observation that the ring A is Cohen-Macaulay if there exists a parameter ideal Q in A such that $\ell_A(A/Q^{\nu+1}) = e(A/Q)\binom{\nu+d}{d}$ for all $\nu \gg 0$ (see [8, 12]). Moreover, as a direct consequence of (1), we have the non-positivity of the first Buchsbaum-Rim coefficient of a parameter module.

Corollary 1.2. *For any rank $r > 0$, the inequality*

$$e_1(F/N) \leq 0$$

always holds true for every parameter module N in $F = A^r$.

Mandal and Verma have recently proved that $e_1(A/Q) \leq 0$ for any parameter ideal Q in A (see [15], and also [8]). Corollary 1.2 can be viewed as the module version of this fact. However, our proof based on the inequality in Theorem 1.1 (1) is completely different from theirs and is considerably more simpler.

2. Preliminaries

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $F = A^r$ be a free module of rank $r > 0$. Let $S = \mathcal{S}_A(F)$ be the symmetric algebra of F. Let N be a parameter module in F, that is, N is a submodule of F satisfying the conditions: (i) $\ell_A(F/N) < \infty$, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d + r - 1$. We put $n = d + r - 1$. Let N^ν be the homogeneous component of degree ν of the graded subalgebra $\mathcal{R}(N) = \mathrm{Im}(\mathcal{S}_A(N) \to S)$ of S. Let $\widetilde{N} = (c_{ij})$ be the matrix associated to a minimal free presentation

$$A^n \xrightarrow{\widetilde{N}} F \to F/N \to 0$$

of F/N. Let $X = (X_{ij})$ be a generic matrix of the same size $r \times n$. We denote by $I_s(X)$ the ideal in the polynomial ring $A[X] = A[X_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n]$ generated by the s-minors of X. Let $B = A[X]_{(\mathfrak{m}, X)}$ be the ring localized at the graded maximal ideal $(\mathfrak{m}, X)$ of $A[X]$. The substitution map $A[X] \to A$ where $X_{ij} \mapsto c_{ij}$ now induces a map $\varphi : B \to A$. We consider the ring A as a B-algebra via the map φ. Let

$$\mathfrak{b} = \operatorname{Ker} \varphi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)B.$$

Set $G = B^r$, and let L denote the submodule $\mathrm{Im}(B^n \xrightarrow{X} G)$ of G. Let G_ν and L^ν be the homogeneous components of degree ν of the graded algebras $\mathcal{S}_B(G)$ and $\mathcal{R}(L)$, respectively. Then one can check the following.

Lemma 2.1. *For any integers $\nu \geq 0$, we have the following:*

(1) $(G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b}) \cong S_{\nu+1}/N^{\nu+1}$;
(2) $\mathrm{Supp}_B(G_{\nu+1}/L^{\nu+1}) = \mathrm{Supp}_B(B/I_r(X)B)$;
(3) *The ideal $\mathfrak{b}$ is generated by a system of parameters of the module $G_{\nu+1}/L^{\nu+1}$.*

The following fact concerning $G_{\nu+1}/L^{\nu+1}$ is known by [3, Corollary 3.2] (see also [13, Proposition 3.3]).

Lemma 2.2. *For any integer $\nu \geq 0$, we have $G_{\nu+1}/L^{\nu+1}$ is a perfect B-modules of grade d.*

The following plays a key role in the proof of Theorem 1.1.

Proposition 2.3. *For any $\mathfrak{p} \in \mathrm{Min}_B(B/I_r(X)B)$, the equality*

$$\ell_{B_{\mathfrak{p}}}\left((G_{\nu+1}/L^{\nu+1})_{\mathfrak{p}}\right) = \ell_{B_{\mathfrak{p}}}\left((B/I_r(X)B)_{\mathfrak{p}}\right)\binom{\nu+d+r-1}{d+r-1}$$

holds true for all integers $\nu \geq 0$.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need to introduce more notation. For any matrix $\mathfrak{a}$ of size $r \times n$ over an arbitrary ring, we denote by $K_{\bullet}(\mathfrak{a})$ its Eagon-Northcott complex [6]. When $r = 1$, the complex $K_{\bullet}(\mathfrak{a})$ is just the ordinary Koszul complex of the sequence $\mathfrak{a}$. See [7, Appendix A2] for the definition and more details of complexes of this type. Recall in particular that if N is a parameter module in a free module F as in section 2, then

$$e(F/N) = \chi(K_{\bullet}(\widetilde{N})),$$

where $\chi(K_{\bullet}(\widetilde{N}))$ denotes the Euler-Poincaré characteristic of the complex $K_{\bullet}(\widetilde{N})$ (see [4] and [14]). Moreover, one can check the following by computing $\mathrm{Tor}_p^B(B/IB, A)$ for all $p \geq 0$ (see [5]).

Lemma 3.1. *Using the setting and notation of section 2, we have*

$$\chi(K_{\bullet}(\mathfrak{b}) \otimes_B (B/I_r(X)B)) = \chi(K_{\bullet}(\widetilde{N})).$$

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. We use the same notation as in section 2. Put $I = I_r(X)$.

(1): Fix integers $\nu \geq 0$. The ideal $\mathfrak{b}$ being generated by a system of parameters of the module $G_{\nu+1}/L^{\nu+1}$, we get

$$\begin{aligned}
& \ell_A(S_{\nu+1}/N^{\nu+1}) \\
= \;& \ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b})) \\
\geq \;& e(\mathfrak{b}; G_{\nu+1}/L^{\nu+1}) \\
= \;& \sum_{\mathfrak{p} \in \mathrm{Assh}_B(G_{\nu+1}/L^{\nu+1})} e(\mathfrak{b}; B/\mathfrak{p}) \cdot \ell_{B_\mathfrak{p}}((G_{\nu+1}/L^{\nu+1})_\mathfrak{p}) \\
= \;& \sum_{\mathfrak{p} \in \mathrm{Assh}_B(B/IB)} e(\mathfrak{b}; B/\mathfrak{p}) \cdot \ell_{B_\mathfrak{p}}((B/IB)_\mathfrak{p}) \binom{\nu+d+r-1}{d+r-1} \\
= \;& e(\mathfrak{b}; B/IB) \binom{\nu+d+r-1}{d+r-1} \\
= \;& \chi(K_\bullet(\mathfrak{b}) \otimes_B (B/IB)) \binom{\nu+d+r-1}{d+r-1} \\
= \;& \chi(K_\bullet(\widetilde{N})) \binom{\nu+d+r-1}{d+r-1} \\
= \;& e(F/N) \binom{\nu+d+r-1}{d+r-1}
\end{aligned}$$

as desired, where $e(\mathfrak{b}; *)$ denotes the multiplicity of $*$ with respect to $\mathfrak{b}$.

(2): The other implication being clear, by the ideal case, for example, it is enough to show that (ii) implies (i). Assume thus that

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N) \binom{\nu+d+r-1}{d+r-1}$$

for some $\nu \geq 0$. The above argument then gives

$$\ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b})) = e(\mathfrak{b}; G_{\nu+1}/L^{\nu+1}).$$

It follows that $G_{\nu+1}/L^{\nu+1}$ is a Cohen-Macaulay B-module of dimension rn ([2, (5.12) Corollary]). By Lemma 2.2, $G_{\nu+1}/L^{\nu+1}$ is a perfect B-module of grade d. Thus, by the Auslander-Buchsbaum formula,

$$\begin{aligned}
\operatorname{depth} B &= \operatorname{depth}_B(G_{\nu+1}/L^{\nu+1}) + \operatorname{pd}_B(G_{\nu+1}/L^{\nu+1}) \\
&= \dim_B(G_{\nu+1}/L^{\nu+1}) + \operatorname{grade}_B(G_{\nu+1}/L^{\nu+1}) \\
&= rn + d \\
&= \dim B.
\end{aligned}$$

Therefore B is Cohen-Macaulay so that A is Cohen-Macaulay, too. □

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GORENSTEINNESS OF A RING WHICH ADMITS A MODULE OF FINITE HOMOLOGICAL DIMENSION

TOKUJI ARAYA

This ia a join work with Ryo Takahashi.

Through in this talk, we denote by R a noetherian local ring with the unique maximal ideal $\mathfrak{m}$ and the residue class field k. We also denote by $\operatorname{mod} R$ the category of finitely generated R-modules and by $\mathcal{D}(R)$ the derived category.

In the 1970s, Foxby [5], verifying a conjecture of Vasconcelos [12], proved the following theorem.

Theorem 1. [5, (4.4)] *If there exists a non-zero finitely generated R-module M such that both projective dimension of M and injective dimension of M are finite, then R is Gorenstein.*

As a natural generalization of this statement, Takahashi and White [11] asked the following.

Question 2. If there exists a non-zero finitely generated R-module M such that both C-projective dimension of M and C-injective dimension of M are finite for some semidualizing module C, then must R be Gorenstein ?

Recently Sather-Wagstaff and Yassemi [10] answered that this question has an affirmative answer in the case where the C-projective dimension is equal to zero. The main purpose of this talk is to give a complete answer to the question. To see this, we give some definitions.

Definition 3. For $C \in \operatorname{mod} R$, we say C is *semidualizing* if the homothety map $R \to \operatorname{Hom}(C, C)$ is an isomorphism and $\operatorname{Ext}^i(C, C) = 0$ for all $i \geq 1$.

The followings are typical examples of semidualizing module.

Example 4. (1) The free module R is semidualizing.

(2) If R is Cohen-Macaulay with a canonical module ω, then ω is semidualizing.

Definition 5. Let C be a semidualizing module. For $M \in \operatorname{mod} R$, the *$C$-projective dimension* $C\text{-}\mathrm{pd}_R\, M$ of M is defined to be the infimum of integers n such that there exists an exact sequence

$$0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0$$

of R-modules where P_i is projective for $0 \leq i \leq n$. Dually, the *C-injective dimension* $C\text{-id}_R\, M$ of M is defined to be the infimum of integers n such that there exists an exact sequence

$$0 \to M \to \mathrm{Hom}_R(C, I^0) \to \mathrm{Hom}_R(C, I^1) \to \cdots \to \mathrm{Hom}_R(C, I^n) \to 0$$

of R-modules where I^i is injective for $0 \leq i \leq n$. The C-projective and C-injective dimensions of the zero module are defined as $-\infty$.

Remark 6. (1) Since $R \otimes_R P_i \cong P_i$ and $\mathrm{Hom}_R(R, I^i) \cong I^i$, we have $R\text{-pd}_R\, M = \mathrm{pd}_R\, M$ and $R\text{-id}_R\, M = \mathrm{id}_R\, M$.

(2) If R is Cohen-Macaulay with a canonical module ω, then we can see that $\omega\text{-pd}_R\, M$ is finite if and only if $\mathrm{id}_R\, M$ is finite. Similarly, we can also see that $\omega\text{-id}_R\, M$ is finite if and only if $\mathrm{pd}_R\, M$ is finite.

(3) For any semidualizing module C, we can check that $C\text{-pd}_R\, M = \mathrm{pd}_R\, \mathrm{Hom}_R(C, M)$ and $C\text{-id}_R\, M = \mathrm{id}_R\, C \otimes_R M$ (c.f. [?, cdim].

Now we can give a main theorem of this talk.

Theorem 7. *If there exists a non-zero finitely generated R-module M such that both C-projective dimension of M and C-injective dimension of M are finite for some semidualizing module C, then R is Gorenstein.*

As a corollay of Theorem 7, we get the following.

Corollary 8. *If there exists a semidualizing module C such that C-injective dimension of C is finite, then R is Gorenstein.*

Let R be a Cohen-Macaulay with a canonical module ω. It is known that if $\mathrm{pd}_R\, \omega$ is finite, than R is Gorenstein. On the other hand, if $\omega\text{-id}_R\, \omega$ is finite, then $\mathrm{pd}_R\, \omega$ is finite by Remark 6 (2). Thus Corollary 8 is a generalization of this fact.

To prove our theorem, we prepare a lemma and a proposition.

Lemma 9. *Let X, Y, Z be R-complexes. Assume the following:*
(1) *$H^i(X)$ and $H^i(Z)$ are finitely generated for all $i \in \mathbb{Z}$,*
(2) *$H^i(X)$ and $H^i(Z)$ are zero for all $i \gg 0$,*
(3) $\mathrm{pd}_R\, Z < \infty$.
Then there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}_R(X, Y) \otimes^{\mathbf{L}}_R Z \cong \mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathbf{L}}_R Z).$$

in $\mathcal{D}(R)$.

Proposition 10. [7, Theorem 3.2] *Let X be an R-complex. Assume the following:*
(1) *$H^i(X)$ are finitely generated for all $i \in \mathbb{Z}$,*
(2) *$H^i(X)$ are zero for all $i \gg 0$,*

(3) $\mathbf{R}\mathrm{Hom}(X \otimes^{\mathbf{L}}_R X, X \otimes^{\mathbf{L}}_R X) \cong R$.

Then X is isomorphic to $R[n]$ for some $n \in \mathbb{Z}$ in $\mathcal{D}(R)$.

Now we can prove our main theorem.

Proof of Theorem 7. Note from [11, (2.9)–(2.11)] that M is in both the Auslander class $\mathcal{A}_C(R)$ and the Bass class $\mathcal{B}_C(R)$, and that $\mathrm{Hom}_R(C, M)$ (respectively, $C \otimes_R M$) is a nonzero finitely generated R-module of finite projective (respectively, injective) dimension by Remark 6 (3). We have isomorphisms

$$\begin{aligned} C \otimes_R M &\cong C \otimes^{\mathbf{L}}_R M \\ &\cong C \otimes^{\mathbf{L}}_R (C \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(C, M)) \\ &\cong (C \otimes^{\mathbf{L}}_R C) \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(C, M) \end{aligned}$$

in $\mathcal{D}(R)$. Using Lemma 9, we get isomorphisms

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_R(k, C \otimes_R M) &\cong \mathbf{R}\mathrm{Hom}_R(k, (C \otimes^{\mathbf{L}}_R C) \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(C, M)) \\ &\cong \mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R C) \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(C, M). \end{aligned}$$

By [1, (A.7.9)], we obtain:

$$\begin{aligned} \sup(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R C)) &= \sup(\mathbf{R}\mathrm{Hom}_R(k, C \otimes_R M)) \\ &\qquad - \sup(k \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(C, M)) \\ &= \mathrm{id}_R(C \otimes_R M) < \infty \\ \inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R C)) &= \inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes_R M)) \\ &\qquad - \inf(k \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(C, M)) \\ &= \mathrm{depth}_R(C \otimes_R M) + \mathrm{pd}_R \mathrm{Hom}(C, M) \\ &> -\infty. \end{aligned}$$

Hence the R-complex $\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R C)$ is bounded, and so is $C \otimes^{\mathbf{L}}_R C$ by [6, (2.5)]. Thus we get $\mathrm{id}_R(C \otimes^{\mathbf{L}}_R C) = \sup(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R C)) < \infty$ by [1, (A.5.7.4)]. It follows from [2, (4.4) and (4.6)(a)] that there is a natural isomorphism $C \cong \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R C)$, and so we have natural isomorphisms $\mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R C, C \otimes^{\mathbf{L}}_R C) \cong \mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R C)) \cong \mathbf{R}\mathrm{Hom}_R(C, C) \cong R$. It follows from Proposition 10 that C is isomorphic to R and therefore $C \otimes^{\mathbf{L}}_R C$ is isomorphic to R. Since $\mathrm{id}_R(C \otimes^{\mathbf{L}}_R C) < \infty$, R is a Gorenstein ring. □

Our method in the proof of Theorem 7 actually gives a more simple proof of Theorem 1 than the proof due to Foxby. In fact, let R and M be as in Theorem 1. Then we have

$$\mathbf{R}\mathrm{Hom}_R(k, M) \cong \mathbf{R}\mathrm{Hom}_R(k, R \otimes^{\mathbf{L}}_R M) \cong \mathbf{R}\mathrm{Hom}_R(k, R) \otimes^{\mathbf{L}}_R M,$$

which gives

$$\begin{aligned}\operatorname{id}_R R &= \sup \mathbf{R}\mathrm{Hom}_R(k, R)\\ &= \sup \mathbf{R}\mathrm{Hom}_R(k, M) - \sup(k \otimes^{\mathbf{L}}_R M) = \operatorname{id}_R M < \infty,\end{aligned}$$

namely, R is Gorenstein.

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On hypersurfaces of countable Cohen-Macaulay type

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This is a joint work with Tokuji Araya and Ryo Takahashi.

Throughout this proceeding, let k be an algebraically closed field of characteristic zero, and let R be a complete Gorenstein local ring with coefficient field k. We denote by $\mathrm{mod}(R)$ the category of finitely generated R-modules, by $\mathrm{CM}(R)$ the full subcategory of $\mathrm{mod}(R)$ consisting of all maximal Cohen-Macaulay R-modules, and by $\mathcal{P}(R)$ the full subcategory of $\mathrm{CM}(R)$ consisting of all modules that are locally free on the punctured spectrum of R. The stable categories of $\mathrm{CM}(R)$ and $\mathcal{P}(R)$ are denoted by $\underline{\mathrm{CM}}(R)$ and $\underline{\mathcal{P}}(R)$, respectively. Let $\mathcal{M}(R)$ be the set of non-isomorphic indecomposable maximal Cohen-Macaulay R-modules X with $X \notin \mathcal{P}(R)$, and let $\mathcal{V}(M)$ be the non-free locus of M for each maximal Cohen-Macaulay R-module M.

When R has finite Cohen-Macaulay representation type, R is isomorphic to $k[[x_0, x_1, x_2, \ldots, x_d]]/(f)$, where

$$f = \begin{cases} x_0^2 + x_1^{n+1} + x_2^2 + \cdots + x_d^2 & (A_n) \\ x_0^2 x_1 + x_1^{n-1} + x_2^2 + \cdots + x_d^2 & (D_n) \\ x_0^3 + x_1^4 + x_2^2 + \cdots + x_d^2 & (E_6) \\ x_0^3 + x_0 x_1^3 + x_2^2 + \cdots + x_d^2 & (E_7) \\ x_0^3 + x_1^5 + x_2^2 + \cdots + x_d^2 & (E_8) \end{cases}.$$

In this case, all objects and morphisms in $\mathrm{CM}(R)$ have been classified completely, namely, the AR-quiver of $\underline{\mathrm{CM}}(R)$ has been obtained; see [1],[3],[5],[8],[9]. When R has infinite but countable Cohen-Macaulay representation type, R is isomorphic to $k[[x_0, x_1, x_2, \ldots, x_d]]/(f)$, where

$$f = \begin{cases} x_0^2 + x_2^2 + \cdots + x_d^2 & (A_\infty^d) \\ x_0^2 x_1 + x_2^2 + \cdots + x_d^2 & (D_\infty^d) \end{cases}.$$

In this case, all objects in $\mathrm{CM}(R)$ have been classified completely (see [3],[4]), but morphisms in $\mathrm{CM}(R)$ have not. The purpose of this proceeding is to investigate the relationships among objects in $\mathrm{CM}(R)$.

The main result of this proceeding is the following theorem.

Theorem 1 *Let $R = k[[x_0, x_1, x_2, \dots, x_d]]/(f)$, where f is either (A^d_∞) or (D^d_∞), and let $\mathfrak{p}_R = (x_0, x_2, \dots, x_d)$ and $\mathfrak{m}_R = (x_0, x_1, x_2, \dots, x_d)$ be ideals of R. Then the following hold.*

(1) *There exist an indecomposable maximal Cohen-Macaulay module X_R such that*

(a) $\mathcal{M}(R) = \{X_R, \Omega(X_R)\}$,

(b) $\mathcal{V}(X_R) = \{\mathfrak{p}_R, \mathfrak{m}_R\} = \mathcal{V}(\Omega(X_R))$.

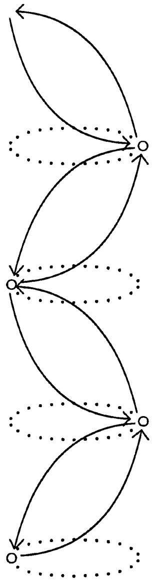
(2) *For each $M \in \mathcal{P}(R)$, there is an exact sequence*

$$0 \to L \to M \oplus R^n \to N \to 0$$

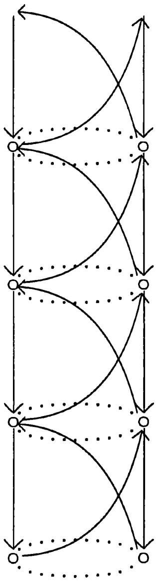
such that $L, N \in \mathcal{M}(R)$ and $n \in \mathbb{N}_0$.

(3) *[Schreyer, [7]] We are able to draw the AR-quiver of $\underline{\mathcal{P}}(R)$.*

$f = (A^d_\infty), (D^{2n}_\infty)$



$f = (D^{2n-1}_\infty)$



The proof of Theorem 1 will use Knörrer's periodicity. Let us recall here Knörrer's periodicity.

Proposition 2 (Knörrer's periodicity) *For the hypersurfaces $S = k[[x_0, x_1, \dots, x_d]]/(f)$ and $S^{\sharp\sharp} = k[[x_0, x_1, \dots, x_d, y, z]]/(f + yz)$, there is a triangle equivalence from $\underline{\mathrm{CM}}(S)$ to $\underline{\mathrm{CM}}(S^{\sharp\sharp})$.*

Now we shall give the outline of the proof of Theorem 1.
(Outline of Proof) (i) The (A^1_∞) case, we have $\mathrm{CM}(R) = \mathrm{add}\{R, R/(x_0)R, \mathrm{Coker}(\varphi_n) | \ n \geq 1\}$, where

$$\varphi_n = \begin{pmatrix} x_0 & x_1^n \\ 0 & x_0 \end{pmatrix},$$

by [3]. In this case $X_R = R/(x_0)R$ satisfies (1) and (2).

(ii) The (D^1_∞) case, we have $\mathrm{CM}(R) = \mathrm{add}\{R, R/(x_0)R, R/(x_0x_1)R, R/(x_0^2)R, R/(x_1)R, \mathrm{Coker}(\varphi_n^+), \mathrm{Coker}(\varphi_n^-), \mathrm{Coker}(\psi_n^+), \mathrm{Coker}(\psi_n^-) | \ n \geq 1\}$, where

$$\varphi_n^+ = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix}, \varphi_n^- = \begin{pmatrix} x_0x_1 & x_1^{n+1} \\ 0 & -x_0x_1 \end{pmatrix}, \psi_n^+ = \begin{pmatrix} x_0x_1 & x_1^n \\ 0 & -x_0 \end{pmatrix}, \psi_n^- = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0x_1 \end{pmatrix},$$

by [3]. In this case $X_R = R/(x_0)R$ satisfies (1) and (2).

(iii) The (A^2_∞) case: R is isomorphic to $k[[x_0, x_1, x_2]]/(x_0x_2)$ by exchanging the variables. We have $\mathrm{CM}(R) = \mathrm{add}\{R, R/(x_0)R, R/(x_2)R, \mathrm{Coker}(\varphi_n^+), \mathrm{Coker}(\varphi_n^-) | \ n \geq 1\}$, where

$$\varphi_n^+ = \begin{pmatrix} x_0 & x_1^n \\ 0 & x_0 \end{pmatrix}, \varphi_n^- = \begin{pmatrix} x_0 & -x_1^n \\ 0 & x_2 \end{pmatrix},$$

by [4]. In this case $X_R = R/(x_0)R$ satisfies (1) and (2).

(iv) The (D^2_∞) case, R is isomorphic to $k[[x_0, x_1, x_2]]/(x_0^2x_1 - x_2^2)$ by exchanging the variables. We have $\mathrm{CM}(R) = \mathrm{add}\{R, \mathrm{Coker}(\alpha^+), \mathrm{Coker}(\alpha^-), \mathrm{Coker}(\beta^+), \mathrm{Coker}(\beta^-), \mathrm{Coker}(\varphi_n^+), \mathrm{Coker}(\varphi_n^-), \mathrm{Coker}(\psi_n^+), \mathrm{Coker}(\psi_n^-) | \ n \geq 1\}$, where

$$\alpha^+ = \begin{pmatrix} x_2 & x_0x_1 \\ x_0 & x_2 \end{pmatrix}, \alpha^- = \begin{pmatrix} -x_2 & x_0x_1 \\ x_0 & -x_2 \end{pmatrix},$$

$$\beta^+ = \begin{pmatrix} x_0^2 & x_2 \\ x_2 & x_1 \end{pmatrix}, \beta^- = \begin{pmatrix} x_1 & -x_2 \\ -x_2 & x_0^2 \end{pmatrix},$$

$$\varphi_n^+ = \begin{pmatrix} x_2 & x_0x_1 & 0 & -x_1^{n+1} \\ x_0 & x_2 & x_1^n & 0 \\ 0 & 0 & x_2 & x_0x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix}, \varphi_n^- = \begin{pmatrix} -x_2 & x_0x_1 & 0 & -x_1^{n+1} \\ x_0 & -x_2 & x_1^n & 0 \\ 0 & 0 & -x_2 & x_0x_1 \\ 0 & 0 & x_0 & -x_2 \end{pmatrix},$$

$$\psi_n^+ = \begin{pmatrix} x_2 & x_0x_1 & -x_1^n & 0 \\ x_0 & x_2 & 0 & x_1^n \\ 0 & 0 & x_2 & x_0x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix}, \psi_n^- = \begin{pmatrix} -x_2 & x_0x_1 & -x_1^n & 0 \\ x_0 & x_2 & 0 & x_1^n \\ 0 & 0 & -x_2 & x_0x_1 \\ 0 & 0 & x_0 & -x_2 \end{pmatrix},$$

by [4]. In this case $X_R = \mathrm{Coker}(\alpha^+)$ satisfies (1) and (2).

(v) The general case, there is a unique hypersurface T of type (A^1_∞), (D^1_∞), (A^2_∞) or (D^2_∞) and the composition of Knörrer's periodicity

$$F : \underline{\mathrm{CM}}(T) \longrightarrow \underline{\mathrm{CM}}(R)$$

is a triangle equivalent functor. In this case $X_R = FX_T$ satisfies (1) and (2). □

As an application of Theorem 1, we get the following result.

Corollary 3 *With the notation of Theorem* 1 *the following hold.*

(1) *The dimension of* $\underline{\mathrm{CM}}(R)$ *(in the sense of Rouquier) is equal to one.*

(2) *The Grothendieck group of* $\mathrm{CM}(R)$ *is generated by* $[R]$ *and* $[X_R]$.

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CLASSIFYING THICK SUBCATEGORIES OF COHEN-MACAULAY MODULES

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One of the principal approaches to the understanding of the structure of a given category is classifying its subcategories having a specific property. It has been studied in many areas of mathematics which include stable homotopy theory, ring theory, algebraic geometry and modular representation theory. A landmark result in this context was obtained in the definitive work due to Gabriel [15] in the early 1960s. He proved a classification theorem of the localizing subcategories of the category of modules over a commutative noetherian ring by making a one-to-one correspondence between the set of those subcategories and the set of specialization-closed subsets of the prime ideal spectrum of the ring. A lot of analogous classification results of subcategories of modules have been obtained by many authors; see [22, 32, 29, 16, 17, 18] for instance.

For a triangulated category, a high emphasis has been placed on classifying its *thick* subcategories, namely, full triangulated subcategories which are closed under taking direct summands. The first classification theorem was obtained in the deep work on stable homotopy theory due to Devinatz, Hopkins and Smith [12, 21]. They classified the thick subcategories of the category of compact objects in the p-local stable homotopy category. Hopkins [20] and Neeman [31] provided a corresponding classification result of the thick subcategories of the derived category of perfect complexes (i.e., bounded complexes of finitely generated projective modules) over a commutative noetherian ring by making a one-to-one correspondence between the set of those subcategories and the set of specialization-closed subsets of the prime ideal spectrum of the ring. Thomason [36] generalized the theorem of Hopkins and Neeman to quasi-compact and quasi-separated schemes, in particular, to arbitrary commutative rings and algebraic varieties. Recently, Avramov, Buchweitz, Christensen, Iyengar and Piepmeyer [2] gave a classification of the thick subcategories of the derived category of perfect differential modules over a commutative noetherian ring. On the other hand, Benson, Carlson and Rickard [6] classified the thick subcategories of the stable category of finitely generated representations of a finite p-group in terms of closed homogeneous subvarieties of the maximal ideal spectrum of the group cohomology ring. Friedlander and Pevtsova [14] extended this classification theorem to finite group schemes. A recent work of Benson, Iyengar and Krause [8] gives a new proof of the theorem of Benson, Carlson and Rickard. A lot of other related results concerning thick subcategories of a triangulated category have been obtained. For example, see [3, 4, 5, 28, 10, 7, 24, 9, 13, 34].

Here we mention that in most of the classification theorems of subcategories stated above, the subcategories are classified in terms of certain sets of prime ideals. Each of them establishes an assignment corresponding each subcatgory to a set of prime ideals, which is (or should be) called the *support* of the subcategory.

In the present article, as a higher dimensional version of the work of Benson, Carlson and Rickard, we consider classifying thick subcategories of the stable category of Cohen-Macaulay modules over a Gorenstein local ring, through defining a suitable support for those subcategories. Over a hypersurface we shall give a complete classification of them in terms of specialization-closed subsets of the prime ideal spectrum of the base ring contained in its singular locus.

CONVENTION. In the rest of this article, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a local ring of Krull dimension d. The unique maximal ideal of R and the residue field of R are denoted by $\mathfrak{m}$ and k, respectively. By a *subcategory*, we always mean a full subcategory which is closed under isomorphism.

Let us make several definitions of subcategories.

Definition 1. (1) Let $\mathcal{C}$ be a category.

(i) We call the subcategory of $\mathcal{C}$ which has no object the *empty subcategory* of $\mathcal{C}$.

(ii) Suppose that $\mathcal{C}$ admits the zero object 0. We call the subcategory of $\mathcal{C}$ consisting of all objects that are isomorphic to 0 the *zero subcategory* of $\mathcal{C}$.

(2) A subcategory of a triangulated category is called *thick* if it is closed under direct summands and triangles.

(3) A subcategory of $\operatorname{mod} R$ is called *resolving* if it contains R and if it is closed under direct summands, extensions and syzygies.

Next we recall the definitions of the nonfree loci.

Definition 2. (1) We denote by $\mathcal{V}(X)$ the *nonfree locus* of an R-module X, namely, the set of prime ideals $\mathfrak{p}$ of R such that $X_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module.

(2) We denote by $\mathcal{V}(\mathcal{X})$ the *nonfree locus* of a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, namely, the union of $\mathcal{V}(X)$ where X runs through all nonisomorphic R-modules in $\mathcal{X}$.

We denote by $\operatorname{Sing} R$ the *singular locus* of R, namely, the set of prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is not a regular local ring. For each ideal I of R, we denote by $\mathrm{V}(I)$ the set of prime ideals of R containing I. Recall that a subset Z of $\operatorname{Spec} R$ is called *specialization-closed* provided that if $\mathfrak{p} \in Z$ and $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$ then $\mathfrak{q} \in Z$. Note that every closed subset of $\operatorname{Spec} R$ is specialization-closed. For a subset Φ of $\operatorname{Spec} R$, we denote by $\mathcal{V}^{-1}(\Phi)$ the subcategory of $\operatorname{mod} R$ consisting of all R-modules M such that $\mathcal{V}(M)$ is contained in Φ.

We recall the definition of the stable category of Cohen-Macaulay modules over a Cohen-Macaulay local ring.

Definition 3. (1) Let M, N be R-modules. We denote by $\mathcal{F}_R(M,N)$ the set of R-homomorphisms $M \to N$ factoring through free R-modules. It is easy to observe that $\mathcal{F}_R(M,N)$ is an R-submodule of $\operatorname{Hom}_R(M,N)$. We set $\underline{\operatorname{Hom}}_R(M,N) = \operatorname{Hom}_R(M,N)/\mathcal{F}_R(M,N)$.

(2) Let R be a Cohen-Macaulay local ring. The *stable category* of $\mathrm{CM}(R)$, which is denoted by $\underline{\mathrm{CM}}(R)$, is defined as follows.

(i) $\mathrm{Ob}(\underline{\mathrm{CM}}(R)) = \mathrm{Ob}(\mathrm{CM}(R))$.

(ii) $\operatorname{Hom}_{\underline{\mathrm{CM}}(R)}(M,N) = \underline{\operatorname{Hom}}_R(M,N)$ for $M, N \in \mathrm{Ob}(\underline{\mathrm{CM}}(R))$.

It is known that $\underline{\mathrm{CM}}(R)$ is a triangulated category if R is Gorenstein; see [4, 19].

Now, we define the notion of a support for objects and subcategories of the stable category of Cohen-Macaulay modules.

Definition 4. Let R be a Cohen-Macaulay local ring.

(1) For an object M of $\underline{\mathrm{CM}}(R)$, we denote by $\underline{\operatorname{Supp}}\, M$ the set of prime ideals $\mathfrak{p}$ of R such that the localization $M_\mathfrak{p}$ is not isomorphic to the zero module 0 in the category $\underline{\mathrm{CM}}(R_\mathfrak{p})$. We call it the *stable support* of M.

(2) For a subcategory $\mathcal{Y}$ of $\underline{\mathrm{CM}}(R)$, we denote by $\underline{\operatorname{Supp}}\, \mathcal{Y}$ the union of $\underline{\operatorname{Supp}}\, M$ where M runs through all nonisomorphic objects in $\mathcal{Y}$. We call it the *stable support* of $\mathcal{Y}$.

(3) For a subset Φ of $\operatorname{Spec} R$, we denote by $\underline{\operatorname{Supp}}^{-1}\, \Phi$ the subcategory of $\underline{\mathrm{CM}}(R)$ consisting of all objects $M \in \underline{\mathrm{CM}}(R)$ such that $\underline{\operatorname{Supp}}\, M$ is contained in Φ.

The notion of a stable support is essentially the same thing as that of a nonfree locus.

Proposition 5. *Let R be a Cohen-Macaulay local ring.*

(1) *Let M be a Cohen-Macaulay R-module. Then* $\underline{\mathrm{Supp}}\, M = \mathcal{V}(M)$.

(2) *Let $\mathcal{X}$ be a subcategory of* $\mathrm{CM}(R)$. *Then* $\underline{\mathrm{Supp}}\, \underline{\mathcal{X}} = \mathcal{V}(\mathcal{X})$.

(3) *Let $\mathcal{Y}$ be a subcategory of* $\underline{\mathrm{CM}}(R)$. *Then* $\underline{\mathrm{Supp}}\, \mathcal{Y} = \mathcal{V}(\overline{\mathcal{Y}})$.

(4) *Let Φ be a subset of* $\mathrm{Spec}\, R$. *Then* $\underline{\mathrm{Supp}}^{-1}\, \Phi = \underline{\mathcal{V}^{-1}(\Phi)}$.

Now we can state our main result.

Theorem 6. (1) *Let R be a local hypersurface. Then one has the following one-to-one correspondences:*

$$\begin{array}{c} \{\textit{nonempty thick subcategories of } \underline{\mathrm{CM}}(R)\} \\ {\scriptstyle \underline{\mathrm{Supp}}}\big\downarrow \quad \big\uparrow{\scriptstyle \underline{\mathrm{Supp}}^{-1}} \\ \{\textit{specialization-closed subsets of } \mathrm{Spec}\, R \textit{ contained in } \mathrm{Sing}\, R\} \\ {\scriptstyle \mathcal{V}^{-1}}\big\downarrow \quad \big\uparrow{\scriptstyle \mathcal{V}} \\ \{\textit{resolving subcategories of } \mathrm{mod}\, R \textit{ contained in } \mathrm{CM}(R)\}. \end{array}$$

(2) *Let R be a d-dimensional Gorenstein singular local ring with residue field k which is a hypersurface on the punctured spectrum. Then one has the following one-to-one correspondences:*

$$\begin{array}{c} \{\textit{thick subcategories of } \underline{\mathrm{CM}}(R) \textit{ containing } \Omega^d k\} \\ {\scriptstyle \underline{\mathrm{Supp}}}\big\downarrow \quad \big\uparrow{\scriptstyle \underline{\mathrm{Supp}}^{-1}} \\ \{\textit{nonempty specialization-closed subsets of } \mathrm{Spec}\, R \textit{ contained in } \mathrm{Sing}\, R\} \\ {\scriptstyle \mathcal{V}^{-1}}\big\downarrow \quad \big\uparrow{\scriptstyle \mathcal{V}} \\ \{\textit{resolving subcategories of } \mathrm{mod}\, R \textit{ contained in } \mathrm{CM}(R) \textit{ containing } \Omega^d k\}. \end{array}$$

Remark 7. Very recently, after the work in this article was completed, Iyengar announced in his lecture [25] that thick subcategories of the bounded derived category of finitely generated modules over a locally complete intersection which is essentially of finite type over a field are classified in terms of certain subsets of the prime ideal spectrum of the Hochschild cohomology ring. This provides a classification of thick subcategories of the stable category of Cohen-Macaulay modules over such a ring, which is a different classification from ours.

A singular local hypersurface and a Cohen-Macaulay singular local ring with an isolated sigularity are trivial examples of a ring which satisfies the assumption of Theorem 6(2). We make here some nontrivial examples.

Example 8. Let k be a field. The following rings R are Cohen-Macaulay singular local rings which are hypersurfaces on the punctured spectrums.

(1) Let $R = k[[x,y,z]]/(x^2, yz)$. Then R is a 1-dimensional local complete intersection which is neither a hypersurface nor with an isolated singularity. All the prime ideals of R are $\mathfrak{p} = (x,y)$, $\mathfrak{q} = (x,z)$ and $\mathfrak{m} = (x,y,z)$. It is easy to observe that both of the local rings $R_\mathfrak{p}$ and $R_\mathfrak{q}$ are hypersurfaces.

(2) Let $R = k[[x,y,z,w]]/(y^2 - xz, yz - xw, z^2 - yw, zw, w^2)$. Then R is a 1-dimensional Gorenstein local ring which is neither a complete intersection nor with an isolated singularity. All the prime ideals are $\mathfrak{p} = (y,z,w)$ and $\mathfrak{m} = (x,y,z,w)$. We easily see that $R_\mathfrak{p}$ is a hypersurface.

(3) Let $R = k[[x,y,z]]/(x^2, xy, yz)$. Then R is a 1-dimensional Cohen-Macaulay local ring which is neither Gorenstein nor with an isolated singularity. All the prime ideals are $\mathfrak{p} = (x,y)$, $\mathfrak{q} = (x,z)$ and $\mathfrak{m} = (x,y,z)$. We have that $R_\mathfrak{p}$ is a hypersurface and that $R_\mathfrak{q}$ is a field.

Applying Theorem 6(1), we observe that over a hypersurface R having an isolated singularity there are only trivial resolving subcategories of $\operatorname{mod} R$ contained in $\mathrm{CM}(R)$ and thick subcategories of $\mathrm{CM}(R)$.

Corollary 9. *Let R be a hypersurface with an isolated singularity.*

(1) *All resolving subcategories of* $\operatorname{mod} R$ *contained in* $\mathrm{CM}(R)$ *are* $\operatorname{add} R$ *and* $\mathrm{CM}(R)$.

(2) *All thick subcategories of* $\underline{\mathrm{CM}}(R)$ *are the empty subcategory, the zero subcategory, and* $\underline{\mathrm{CM}}(R)$.

As another application of Theorem 6, we obtain a vanishing result of homological and cohomological δ-functors from the category of finitely generated modules over a hypersurface.

Proposition 10. *Let R be a hypersurface and M an R-module. Let $\mathcal{A}$ be an abelian category.*

(1) *Let $T : \operatorname{mod} R \to \mathcal{A}$ be a covariant or contravariant homological δ-functor with $T_i(R) = 0$ for $i \gg 0$. If there exists an R-module M with $\mathrm{pd}_R M = \infty$ and $T_i(M) = 0$ for $i \gg 0$, then $T_i(k) = 0$ for $i \gg 0$.*

(2) *Let $T : \operatorname{mod} R \to \mathcal{A}$ be a covariant or contravariant cohomological δ-functor with $T^i(R) = 0$ for $i \gg 0$. If there exists an R-module M with $\mathrm{pd}_R M = \infty$ and $T^i(M) = 0$ for $i \gg 0$, then $T^i(k) = 0$ for $i \gg 0$.*

Proof. (1) We easily see that for any R-module N and any integers $n \geq 0$ and $i \gg 0$ we have

$$T_i(\Omega^n N) \cong \begin{cases} T_{i+n}(N) & \text{if } T \text{ is covariant,} \\ T_{i-n}(N) & \text{if } T \text{ is contravariant.} \end{cases}$$

Consider the subcategory $\mathcal{X}$ of $\mathrm{CM}(R)$ consisting of all Cohen-Macaulay R-modules X with $T_i(X) = 0$ for $i \gg 0$. Then it is easily observed that $\mathcal{X}$ is a thick subcategory of $\mathrm{CM}(R)$ containing R. Since $T_i(\Omega^d M)$ is isomorphic to $T_{i+d}(M)$ (respectively, $T_{i-d}(M)$) for $i \gg 0$ if T is covariant (respectively, contravariant), the nonfree Cohen-Macaulay R-module $\Omega^d M$ belongs to $\mathcal{X}$. Hence the maximal ideal $\mathfrak{m}$ belongs to $\mathcal{V}(\Omega^d M)$, which is contained in $\mathcal{V}(\mathcal{X})$, and we have $\mathcal{V}(\Omega^d k) \subseteq \{\mathfrak{m}\} \subseteq \mathcal{V}(\mathcal{X})$. Therefore $\Omega^d k$ belongs to $\mathcal{V}^{-1}(\mathcal{V}(\mathcal{X}))$, which coincides with $\mathcal{X}$ by Theorem 6(1). Thus we obtain $T_i(\Omega^d k) = 0$ for $i \gg 0$. Since $T_i(\Omega^d k)$ is isomorphic to $T_{i+d}(k)$ (respectively, $T_{i-d}(k)$) for $i \gg 0$ if T is covariant (respectively, contravariant), we have $T_i(k) = 0$ for $i \gg 0$, as desired.

(2) This is shown similarly to (1). □

As a corollary of Proposition 10, we obtain the following vanishing result of Tor and Ext modules.

Corollary 11. *Let R be an abstract hypersurface. Let M, N be R-modules. If $\mathrm{Tor}_i^R(M,N) = 0$ for $i \gg 0$, then either $\mathrm{pd}_R M < \infty$ or $\mathrm{pd}_R N < \infty$. Dually, if $\mathrm{Ext}_R^i(M,N) = 0$ for $i \gg 0$, then either $\mathrm{pd}_R M < \infty$ or $\mathrm{id}_R N < \infty$.*

The first assertion of Corollary 11 gives another proof of a theorem of Huneke and Wiegand [23, Theorem 1.9].

Corollary 12 (Huneke-Wiegand)**.** *Let R be an abstract hypersurface. Let M and N be R-modules. If $\mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_{i+1}^R(M,N) = 0$ for some $i \geq 0$, then either M or N has finite projective dimension.*

Remark 13. Several generalizations of Corollaries 11(1) and 12 to complete intersections have been obtained by Jorgensen [26, 27], Miller [30] and Avramov and Buchweitz [1].

The assertions of Corollary 11 do not necessarily hold if the ring R is not an abstract hypersurface.

Example 14. Let k be a field. Consider the artinian complete intersection local ring $R = k[[x,y]]/(x^2,y^2)$. Then we can easily verify $\mathrm{Tor}_i^R(R/(x),R/(y)) = 0$ and $\mathrm{Ext}_R^i(R/(x),R/(y)) = 0$ for all $i > 0$. But both $R/(x)$ and $R/(y)$ have infinite projective dimension, and infinite injective dimension.

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TRIANGULATED CATEGORIES FOR ISOLATED HYPERSUFACE SINGULARITIES AND MIRROR SYMMETRY

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1. Introduction

Mirror symmetry is now understood as a categorical duality between algebraic geometry and symplectic geometry. One of our motivations is to apply some idea of mirror symmetry to singularity theory in order to understand various mysterious correpondences among isolated singularities, root systems, Weyl groups, Lie algebras, discrete groups, finite dimensional algebras and so on.

In this paper, we describe an algebro-geometric aspect of the homological mirror symmetry of isolated hypersurface singularities. We shall show that the stable categories of finitely generated graded Cohen-Macaulay modules are equivalent to the bounded derived categories of finite dimensional modules over finite dimensional algebras.

2. Categories of singularities

Let k be an algebraically closed field of characteristic zero. Our main interest is a weighted homogeneous polynomial $f \in S := k[x_1, \dots, x_n]$, where we set $\deg(x_i) =: r_i \in \mathbb{Z}_{>0}$, $\deg(f) =: h \in \mathbb{Z}_{>0}$. Assume that f defines at most an isolated singularity at the origin.

2.1. **The maximal abelian grading.** Define an abelian group L_f by

$$L_f := \oplus_{i=1}^{n} \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{f}/I,$$

where I is the subgroup generated by

$$\vec{f} - \sum_{i=1}^{n} k_i \vec{x_i}, \quad \text{for} \quad a_{k_1,\dots,k_n} \neq 0,$$

where $a_{k_1,\dots,k_n}$ are coefficients of the monomial $x_1^{k_1} \dots x_n^{k_n}$ in f. Note that the quotient ring $R_f := S/(f)$ is L_f-graded.

A group homomorphism $\deg : L_f \to \mathbb{Z}, \vec{x_i} \mapsto r_i$ is called the **degree map**.

Date: January 11, 2010.

2.2. **Categories of singularities.** Denote by $\mathrm{gr}^{L_f}\text{-}R_f$ the category of finitely generated L_f-graded R_f-modules and by $\mathrm{proj}^{L_f}\text{-}R_f \subset \mathrm{gr}^{L_f}\text{-}R_f$ the full subcategory of projective modules.

Definition 2.1. The triangulated category

$$D^{L_f}_{Sg}(R_f) := D^b(\mathrm{gr}^{L_f}\text{-}R_f)/K^b(\mathrm{proj}^{L_f}\text{-}R_f)$$

is called the **triangulated category of L_f-graded singularity**.

Remark 2.2. If the ring R_f is regular, then we have the equivalence $D^b(\mathrm{gr}^{L_f}\text{-}R_f) \simeq K^b(\mathrm{proj}^{L_f}\text{-}R_f)$. Hence, we see that the category $D^{L_f}_{Sg}(R_f)$ measures the complexity of the singularity f.

Remark 2.3. The objects

$$k(\vec{l}) := (R_f/\mathfrak{m})(\vec{l}) \in D^{L_f}_{Sg}(R_f), \quad \vec{l} \in L_f.$$

will play an essential role in our story.

2.3. **Categories of CM modules.** Although the category $D^{L_f}_{Sg}(R_f)$ is easy to define, it is too difficult to study since it is defined as a localization. Therefore, we replace it by the equivalent category which is more natural from the mirror symmetry point of view.

Definition 2.4. $M \in \mathrm{gr}^{L_f}\text{-}R_f$ is a L_f-graded **Cohen-Macauley** module if

$$\mathrm{Ext}^i_{R_f}(R_f/\mathfrak{m}, M) = 0, \quad i < \dim R_f.$$

Note that R_f is L_f-graded **Gorenstein** ring, i.e., we have

$$K_{R_f} \simeq R_f(-\vec{\epsilon}_f), \quad \vec{\epsilon}_f := \sum_{i=1}^{n} \vec{x}_i - \vec{f},$$

where $(\vec{l})$ is the grading shift by $\vec{l} \in L_f$. Therefore, we have the following:

Lemma 2.5 (Auslander). *The category of Cohen-Macauley R_f-modules $\mathrm{CM}^{L_f}(R_f) \subset \mathrm{gr}^{L_f}\text{-}R_f$ is a* **Frobenius category**, *i.e., an exact category with enough injectives and projectives and its class of injectives coincides with that of projectives.* □

Definition 2.6. Define a category $\underline{\mathrm{CM}}^{L_f}(R_f)$ as follows:

$$Ob(\underline{\mathrm{CM}}^{L_f}(R_f)) = Ob(\mathrm{CM}^{L_f}(R_f)),$$

$$\underline{\mathrm{CM}}^{L_f}(R_f)(M,N) := \mathrm{Hom}_{\mathrm{gr}^{L_f}\text{-}R_f}(M,N)/\mathcal{P}(M,N),$$

where $g \in \mathcal{P}(M,N)$ if and only if there exist a projective object P and homomorphisms $g' : M \to P$ and $g'' : P \to N$ such that $g = g'' \circ g'$. The category $\underline{\mathrm{CM}}^{L_f}(R_f)$ is called the **stable category** of a category $\mathrm{CM}^{L_f}(R_f)$

Then, the following facts are well-known.

Proposition 2.7 (Happel[H]). *The category* $\underline{\mathrm{CM}}^{L_f}(R_f)$ *is a triangulated category.* □

Proposition 2.8. $\underline{\mathrm{CM}}^{L_f}(R_f)$ *is finite*

$$\sum_i \dim_k \underline{\mathrm{CM}}^{L_f}(R_{f_W})(M, T^i N) < \infty,$$

and **Krull-Schmidt**, *i.e., any object is a finite direct sum of indecomposable objects.* □

The triangulated category $\underline{\mathrm{CM}}^{L_f}(R_f)$ has the following special property:

Proposition 2.9 (Auslander-Reiten[AR]). *The functor* $\mathcal{S} = T^{n-2} \circ (-\vec{\epsilon}_f)$ *defines the* **Serre functor** *on* $\underline{\mathrm{CM}}^{L_f}(R_f)$, *i.e., there exists a bi-functorial isomorphism*

$$\underline{\mathrm{CM}}^{L_f}(R_{f_W})(M, N) \simeq \mathrm{Hom}_k(\underline{\mathrm{CM}}^{L_f}(R_{f_W})(N, \mathcal{S}M), k).$$

□

2.4. **Categories of matrix factorizations.** Since R_f is a hypersurface and L_f-graded local ring, there exists an L_f-graded free resolution of $M \in \mathrm{CM}^{L_f}(R_f)$ in gr^{L_f}-S:

$$0 \to F_1 \xrightarrow{f_1} F_0 \to M \to 0.$$

However, the multiplication of f on M is zero, we have a morphism (homotopy) $f_0 : F_0 \to F_1$ such that

$$f_1 f_0 = f \cdot \mathrm{id}_{F_0}, \quad f_0 f_1 = f \cdot \mathrm{id}_{F_1}.$$

Based on this observation, Eisenbud introduced the following notion of matrix factorizations:

Definition 2.10 (Eisenbud[E]). Let F_0, F_1 be L_f-graded free modules and $f_0 : F_0 \to F_1$, $f_1 : F_1 \to F_0$ be morphisms such that $f_1 f_0 = f \cdot \mathrm{id}_{F_0}, f_0 f_1 = f \cdot \mathrm{id}_{F_1}$. The tuple (F_0, F_1, f_0, f_1) is called a L_f-graded **matrix factorization** of f and denoted by

$$\overline{F} := \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right).$$

Example 2.11. There exist $f_i \in \mathfrak{m}$, $i = 1, \ldots, n$ such that

$$f = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n.$$

This decomposition defines matrix factorizations which will be isomorphic to $k(\vec{l})$ in $D_{Sg}^{L_f}(R_f)$.

Lemma 2.12. *The category* $\mathrm{MF}_S^{L_f}(f)$ *of graded matrix factorizations of* f *is a Frobenius category. Therefore, its stable category*

$$\mathrm{HMF}_S^{L_f}(f) := \underline{\mathrm{MF}}_S^{L_f}(f)$$

is triangulated. □

Lemma 2.13. *On the category* $\mathrm{HMF}_S^{L_f}(f)$*, we have* $T^2 = (\vec{f})$*. In particular,* $\mathrm{HMF}_S^{L_f}(f)$ *is fractional Calabi–Yau triangulated category of dimension* $(n-2)-2\frac{\epsilon_f}{h_f}$*, where* $\epsilon_f := \deg(\vec{\epsilon_f})$ *and* $h_f := \deg(\vec{f})$. □

2.5. **Triangulated equivalences and semi-orthogonal decompositions.** For any matrix factorization $\overline{F} = \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right)$, the L_f-graded R_f-module $\mathrm{Coker}(f_1)$ is in $\mathrm{CM}^{L_f}(R_f)$. Furthermore, we have the L_f-graded version of the famous triangulated equivalences:

Theorem 2.14 (c.f., Buchweitz, Orlov[O]). *There exists a triangulated equivalence*

$$\mathrm{HMF}_S^{L_f}(f) \simeq \underline{\mathrm{CM}}^{L_f}(R_f) \simeq D_{Sg}^{L_f}(R_f).$$

□

In order to have L_f-graded generalization of Orlov's semi-orthogonal decomposition, we first introduce the quotient stack

$$X_{L_f} := [\mathrm{Spec}(R_f)\backslash\{0\} / \mathrm{Spec}(k \cdot L_f)].$$

Then, we have the triangulated equivalence $D^b\mathrm{coh}(X_{L_f}) \simeq D^b(\mathrm{gr}^{L_f}\text{-}R_f)/D^b(\mathrm{tor}^{L_f}\text{-}R_f)$.

Proposition 2.15 (c.f., Orlov[O]). *There exists the following triangulated equivalence:*

(i) *If* $\epsilon_f > 0$,

$$D^b\mathrm{coh}(X_{L_f}) \simeq \left\langle D_{Sg}^{L_f}(R_f), \mathcal{A}(0), \dots, \mathcal{A}(\epsilon_f - 1) \right\rangle,$$

where $\mathcal{A}(i) := \left\langle \mathcal{O}_{X_{L_f}}(\vec{l}) \right\rangle_{\deg(\vec{l})=i}$.

(ii) *If* $\epsilon_f = 0$, $D^b\mathrm{coh}(X_{L_f}) \simeq D_{Sg}^{L_f}(R_f)$.

(iii) *If* $\epsilon_f < 0$,

$$D_{Sg}^{L_f}(R_f) \simeq \left\langle D^b\mathrm{coh}(X_{L_f}), \mathcal{K}(0), \dots, \mathcal{K}(-\epsilon_f + 1) \right\rangle,$$

where $\mathcal{K}(i) := \left\langle k(\vec{l}) \right\rangle_{\deg(\vec{l})=i}$.

□

3. Structure of $\mathrm{HMF}_S^{L_f}(f)$

Now, we assume that f is a sum of the following polynomials:
Fermat type: $x_1^{p_1} + x_2^{p_2} + \cdots + x_k^{p_k}$,
chain: $x_1^{p_1} + x_1 x_2^{p_2} + \cdots + x_{l-1} x_l^{p_l}$,
loop: $x_n x_1^{p_1} + x_1 x_2^{p_2} + \cdots + x_{m-1} x_m^{p_m}$.
Then, from the mirror symmetry conjecture for the above singularities, we may expect the following:

Conjecture 3.1 (c.f. T: arXiv:0711.3907). *There exists a triangulated equivalence*

$$\mathrm{HMF}_S^{L_f}(f) \simeq D^b(\text{mod-}k\vec{\Delta}/I)$$

for some quiver $\vec{\Delta}$ and relations I. □

From the next section, we shall give some results for the above conjecture.

4. Examples

4.1. Simplest case.

Theorem 4.1. *There exists a triangulated equivalence $\mathrm{HMF}_{k[x]}^{\mathbb{Z}}(x^{l+1}) \simeq D^b(\text{mod-}k\vec{\Delta}_{A_l})$ where $\vec{\Delta}_{A_l}$ is the Dynkin quiver of type A_l.*

4.2. Curve singularities.

Consider polynomials of the following types:
Type I: $f = x^p + y^q$, Type II: $f = x^p + xy^q$, Type III: $f = yx^p + xy^q$.

Theorem 4.2. *For any f of type I, II and III, $\exists$ a quiver $\vec{\Delta}$ and relations I such that*

$$\mathrm{HMF}_S^{L_f}(f) \simeq D^b(\text{mod-}k\vec{\Delta}/I)$$

□

Corollary 4.3 (ADE). *Let f be one of polynomials in the table*

f	*Type*
$x^{l+1} + y^2$	A_l $(l \geq 1)$
$yx^{l-1} + y^2$	D_l $(l \geq 4)$
$x^4 + y^3$	E_6
$yx^3 + y^3$	E_7
$x^5 + y^3$	E_8.

Then, we have the triangulated equivalence

$$\mathrm{HMF}_S^{L_f}(f) \simeq D^b(\text{mod-}k\vec{\Delta}),$$

where $\vec{\Delta}$ is the Dynkin quiver of corresponding type. □

4.3. **Surface singularities.**

Theorem 4.4 (Kajiura-Saito-Takahashi[KST1]). *Let f be one of ADE singularities. Then, we have the triangulated equivalence*

$$\mathrm{HMF}_S^{\mathbb{Z}}(f) \simeq D^b(\text{mod-}k\vec{\Delta}),$$

where $\vec{\Delta}$ is the Dynkin quiver of corresponding type.

4.4. **Another important class.**

Theorem 4.5 (Kajiura–Saito–Takahashi[KST2]). *For Arnold's 14 exceptional singularities, Conjecture holds.*

5. Sketch of Proof

All the above trinagulated equivalences are proven in the following way:

(i) We find enough "good" matrix factorizations.

(ii) We show that these matrix factorizations form a strongly exceptional collection.

(iii) We use the **"category generating lemma"** in order to prove the above strongly exceptional collection is full.

First two steps are done by case-by-case study by hand. Therefore, we shall explain the category generating lemma:

Theorem 5.1 (Category Generating Lemma). *Suppose a full triangulated subcategory $\mathcal{T}'$ of $\mathrm{HMF}_S^{L_f}(f)$ generated by an exceptional collection $(E_1, \dots, E_n)$ satisfies the following:*

(i) $\mathcal{T}'$ is closed under the shift $(\vec{l})$ for all $\vec{l} \in L_f$,

(ii) There exists an object $E \in \mathcal{T}'$ isomorphic to $k(\vec{0})$ in $D_{Sg}^{L_f}(R_f)$.

Then $\mathcal{T}' \simeq \mathrm{HMF}_S^{L_f}(f)$.

Proof. First, we note that $\mathcal{T}'$ is right admissible:

Lemma 5.2. *For any $X \in \mathrm{HMF}_S^{L_f}(f)$ there is an exact triangle*

$$N \to X \to M \to TN$$

where $N \in \mathcal{T}'$ and $\mathrm{Hom}(N, M) = 0$. □

Then, we only have to show that the right orthogonal is zero. But, this follows from

$$\begin{aligned} & \mathrm{HMF}_S^{L_f}(f)(E(\vec{l}), T^i M) = 0 \quad {}^\forall \vec{l} \in L_f, \quad {}^\forall i \in \mathbb{Z} \\ \Longleftrightarrow \quad & \mathrm{Ext}^i_{R_f}(R_f/\mathfrak{m}, M) = 0 \ (i \neq d) \\ \Longleftrightarrow \quad & M \in \mathrm{CM}^{L_f}(R_f) \text{ is Gorenstein} \\ \Longleftrightarrow \quad & M \in \mathrm{CM}^{L_f}(R_f) \text{ is free} \\ \Longleftrightarrow \quad & M \simeq 0 \text{ in } \underline{\mathrm{CM}}^{L_f}(R_f). \end{aligned}$$

Therefore, $\mathcal{T}' \simeq \mathrm{HMF}_S^{L_f}(f)$. □

6. A Full Exceptional Collection Exists

Consider polynomials of the following types:
Type I: $f = x^{p_1} + y^{p_2} + z^{p_3}$, Type II: $f = x^{p_1} + y^{p_2} + yz^{\frac{p_3}{p_2}}$, Type III: $f = x^{p_1} + y^{q_3+1}z + yz^{q_2+1}$, Type IV: $f = x^{p_1} + xy^{\frac{p_2}{p_1}} + yz^{\frac{p_3}{p_2}}$, Type V: $f = zx^k + xy^l + yz^m$.

Theorem 6.1. *For any f of type I, II, III, IV and V, there exists a full exceptional collection in $\mathrm{HMF}_S^{L_f}(f)$.*

Proof. First, we describe the category $D^b\mathrm{coh}(X_{L_f})$.

Proposition 6.2. *X_{L_f} is isomorphic to $\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}$, an orbifold $\mathbb{P}^1$ with 3-isotropic points of order $\alpha_1, \alpha_2, \alpha_3$:*

Type	$A_f := (\alpha_1, \alpha_2, \alpha_3)$
I	(p_1, p_2, p_3)
II	$(p_1, \frac{p_3}{p_2}, (p_2 - 1)p_1)$
III	$(p_1, p_1 q_2, p_1 q_3)$
IV	$(p_1, (\frac{p_3}{p_2} - 1)p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1)$
V	$(lm - m + 1, lk - k + 1, km - m + 1).$

Remark 6.3. If $L_f \simeq \mathbb{Z}$, A_f is called the **Dolgachev number**.

Remark 6.4. If $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$, then we have $\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3} \simeq [\mathbb{P}^1/G_{\alpha_1,\alpha_2,\alpha_3}]$, where

$$G_{\alpha_1,\alpha_2,\alpha_3} := < g_1, g_2, g_3 \mid g_1^{\alpha_1} = g_2^{\alpha_2} = g_3^{\alpha_3} = g_1 g_2 g_3 >$$

is a binary polyhedral group.

Proof (of Proposition). Set $R_{A_f} := k[X_1, X_2, X_3]/(X_1^{\alpha_1} + X_2^{\alpha_2} + X_3^{\alpha_3})$ and

$$L_{A_f} := \bigoplus_{i=1}^{3} \mathbb{Z}\vec{X}_i \Big/ \left(\alpha_i \vec{X}_i - \alpha_j \vec{X}_j; 1 \leq i < j \leq 3\right).$$

Note that we have the equivalence of abelian categories

$$\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}) \simeq \mathrm{gr}^{L_{A_f}}\text{-}R_{A_f}/\mathrm{tor}^{L_{A_f}}\text{-}R_{A_f}.$$

We can show (see [T]) that there exists a natural embedding

$$R_f \hookrightarrow R_{A_f}, \quad L_f \hookrightarrow L_{A_f},$$

which induces an equivalence

$$\mathrm{gr}^{L_f}\text{-}R_f/\mathrm{tor}^{L_f}\text{-}R_f \simeq \mathrm{gr}^{L_{A_f}}\text{-}R_{A_f}/\mathrm{tor}^{L_{A_f}}\text{-}R_{A_f}.$$

□

Proposition 6.5 (Geigle–Lenzing[GL]). *$D^b\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3})$ has a full exceptional collection.* □

Hence, Theorem follows from the semi-orthogonal decomposition of $\mathrm{HMF}^{L_f}_S(f)$. □

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SUBFUNCTORS OF IDENTITY FUNCTOR AND T-STRUCTURES

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Let R be a commutative noetherian ring. We denote the category of all R-modules by R-Mod and also denote the derived category consisting of all left bounded complexes of R-modules by $\mathcal{D}^+(R\text{-Mod})$.

The aim of this paper is to characterize the section functor Γ_W (resp. the right derived functor $\mathbf{R}\Gamma_W$ of Γ_W) as elements of the set of all functors on R-Mod (resp. $\mathcal{D}^+(R\text{-Mod})$).

1. The definition of abstract local cohomology functors

Let us recall some definitions for functors from the category theory.

Definition 1.1. Let γ be a functor on R-Mod.

(1) A functor γ is called a preradical functor if γ is a subfunctor of identity functor $\mathbf{1}$.
(2) A preradical functor γ is called a radical functor if $\gamma(M/\gamma(M)) = 0$ for every R-module M.
(3) A functor γ is said to preserve injectivity if $\gamma(I)$ is an injective R-module whenever I is an injective R-module.

Example 1.2. Let W be a subset of Spec(R). Recall that W is said to be specialization-closed if $\mathfrak{p} \in W$ and $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(R)$ imply $\mathfrak{q} \in W$.

When W is closed under specialization, we can define the section functor Γ_W with support in W as

$$\Gamma_W(M) = \{x \in M \mid \mathrm{Supp}(Rx) \subseteq W\},$$

for all $M \in R$-Mod. Then it is easy to see that Γ_W is a left exact radical functor that preserves injectivity.

The notion of stable t-structure is introduced by J. Miyachi.

Definition 1.3. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of a triangulated category $\mathcal{T}$ is called a stable t-structure on $\mathcal{T}$ if it satisfies the following conditions:

(1) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$.
(2) $\mathcal{U} = \mathcal{U}[1]$ and $\mathcal{V} = \mathcal{V}[1]$.

(3) For any $X \in \mathcal{T}$, there is a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

For a triangle functor δ on triangulated category $\mathcal{T}$, we define two full subcategories of $\mathcal{T}$

$$\begin{aligned} \mathrm{Im}(\delta) &= \{X \in \mathcal{T} \mid X \cong \delta(Y) \text{ for some } Y \in \mathcal{T}\}, \\ \mathrm{Ker}(\delta) &= \{X \in \mathcal{T} \mid \delta(X) \cong 0\}. \end{aligned}$$

The following theorem proved by J. Miyachi is a key to our argument. We shall refer to this theorem as Miyachi's Theorem.

Theorem 1.4. [2, Proposition 2.6] *Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ a full triangulated subcategory of $\mathcal{T}$. Then the following conditions are equivalent for $\mathcal{U}$.*

(1) *There is a full subcategory $\mathcal{V}$ of $\mathcal{T}$ such that $(\mathcal{U}, \mathcal{V})$ is a stable t-structure on $\mathcal{T}$.*
(2) *The natural embedding functor $i : \mathcal{U} \to \mathcal{T}$ has a right adjoint $\rho : \mathcal{T} \to \mathcal{U}$.*

If it is the case, setting $\delta = i \circ \rho : \mathcal{T} \to \mathcal{T}$, we have the equalities

$$\mathcal{U} = \mathrm{Im}(\delta) \quad \text{and} \quad \mathcal{V} = \mathcal{U}^{\perp} = \mathrm{Ker}(\delta).$$

Remark 1.5. Let $(\mathcal{U}, \mathcal{V})$ be a stable t-structure on $\mathcal{T}$, and let ρ be a right adjoint functor of $i : \mathcal{U} \to \mathcal{T}$. Set $\delta = i \circ \rho$ as in the theorem. The functor ρ, hence δ as well, is unique up to isomorphisms, by the uniqueness of right adjoint functors.

Now we can define an abstract local cohomology functor.

Definition 1.6. We denote $\mathcal{T} = \mathcal{D}^+(R\text{-Mod})$ in this definition. Let $\delta : \mathcal{T} \to \mathcal{T}$ be a triangle functor. We call that δ is an abstract local cohomology functor if the following conditions are satisfied:

(1) The natural embedding functor $i : \mathrm{Im}(\delta) \to \mathcal{T}$ has a right adjoint $\rho : \mathcal{T} \to \mathrm{Im}(\delta)$ and $\delta \cong i \circ \rho$. (Hence, by Miyachi's Theorem, $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ is a stable t-structure on $\mathcal{T}$.)
(2) The t-structure $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ divides indecomposable injective R-modules, by which we mean that each indecomposable injective R-module belongs to either $\mathrm{Im}(\delta)$ or $\mathrm{Ker}(\delta)$.

Example 1.7. We denote by $E_R(R/\mathfrak{p})$ the injective hull of an R-module $R/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$.

Let W be a specialization-closed subset of $\mathrm{Spec}(R)$. We claim that $\mathbf{R}\Gamma_W$ is an abstract local cohomology functor on $\mathcal{D}^+(R\text{-Mod})$.

In fact, it is known that $\mathcal{D}^+(R\text{-Mod})$ is triangle-equivalent to the triangulated category $\mathcal{K}^+(\mathrm{Inj}(R))$, which is the homotopy category consisting of all left-bounded injective complexes over R. Through this equivalence, for any injective complex $I \in \mathcal{K}^+(\mathrm{Inj}(R))$, $\mathbf{R}\Gamma_W(I) = \Gamma_W(I)$ is the subcomplex of I consisting of injective modules supported in W. Hence every object of $\mathrm{Im}(\mathbf{R}\Gamma_W)$ (resp. $\mathrm{Ker}(\mathbf{R}\Gamma_W)$) is an injective complex whose components are direct sums of $E_R(R/\mathfrak{p})$ with $\mathfrak{p} \in W$ (resp. $\mathfrak{p} \in \mathrm{Spec}(R)\backslash W$). In particular, if $\mathfrak{p} \in W$ (resp. $\mathfrak{p} \in \mathrm{Spec}(R)\backslash W$), then $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ (resp. $E_R(R/\mathfrak{p}) \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$). Since $\mathrm{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$ for $\mathfrak{p} \in W$ and $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, we can see that

$$\mathrm{Hom}_{\mathcal{K}^+(\mathrm{Inj}(R))}(I, J) = \mathrm{Hom}_{\mathcal{K}^+(\mathrm{Inj}(R))}(I, \Gamma_W(J))$$

for any $I \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ and $J \in \mathcal{K}^+(\mathrm{Inj}(R))$. Hence it follows from the above equivalence that $\mathbf{R}\Gamma_W$ is a right adjoint of the natural embedding $i : \mathrm{Im}(\mathbf{R}\Gamma_W) \to \mathcal{D}^+(R\text{-Mod})$.

2. Main result

The main result of this paper is the following.

Theorem 2.1. (1) *The following conditions are equivalent for a left exact preradical functor γ on R-Mod.*

(i) *γ is a radical functor.*
(ii) *γ preserves injectivity.*
(iii) *γ is a section functor with support in a specialization closed subset of* $\mathrm{Spec}(R)$.
(iv) *$\mathbf{R}\gamma$ is an abstract local cohomology functor.*

(2) *Given an abstract local cohomology functor δ on $\mathcal{D}^+(R\text{-Mod})$, there exists a specialization closed subset $W \subseteq \mathrm{Spec}(R)$ such that δ is isomorphic to the right derived functor $\mathbf{R}\Gamma_W$ of the section functor Γ_W.*

We shall prove the statement (2) in Theorem 2.1. To do this, we introduce several lemmas.

Lemma 2.2. *Let $X \in \mathcal{D}^+(R\text{-Mod})$ and let W be a specialization-closed subset of* $\mathrm{Spec}(R)$.

(1) $X \cong 0 \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, X)_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.
(2) $X \in \mathrm{Im}(\mathbf{R}\Gamma_W) \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{q}, X)_\mathfrak{q} = 0$ *for all* $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$.
(3) $X \in \mathrm{Ker}(\mathbf{R}\Gamma_W) \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, X)_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in W$.

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let $X \ncong 0 \in \mathcal{D}^+(R\text{-Mod})$. If $X \in \mathrm{Im}(\mathbf{R}\Gamma_\mathfrak{m})$, then $\mathbf{R}\mathrm{Hom}_R(E_R(k), X) \ncong 0$.*

It follows from above results that we can show the following lemma.

Lemma 2.4. *Let* $X \in \mathcal{D}^+(R\text{-Mod})$ *and let* W *be a specialization-closed subset of* $\mathrm{Spec}(R)$.

(1) *If* $X \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$ *and* $\mathbf{R}\mathrm{Hom}_R(X, E_R(R/\mathfrak{q})) = 0$ *for all* $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, *then* $X \cong 0$.

(2) *If* $X \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ *and* $\mathbf{R}\mathrm{Hom}_R(E_R(R/\mathfrak{p}), X) = 0$ *for all* $\mathfrak{p} \in W$, *then* $X \cong 0$.

Now we can prove the statement (2) in our main theorem.

Proof of Theorem 2.1(2). In this proof we denote $\mathcal{T} = \mathcal{D}^+(R\text{-Mod})$. Suppose that $\delta : \mathcal{T} \to \mathcal{T}$ is an abstract local cohomology functor. We divides the proof into several steps.

(1st step) : Consider the subset $W = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)\}$ of $\mathrm{Spec}(R)$. Then it is easy to see that W is a specialization-closed subset. ∎

Our final goal is, of course, to show the isomorphism $\delta \cong \mathbf{R}\Gamma_W$. Notice that, since the both functors δ and $\mathbf{R}\Gamma_W$ are abstract local cohomology functors, we have two stable t-structures $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ and $(\mathrm{Im}(\mathbf{R}\Gamma_W), \mathrm{Ker}(\mathbf{R}\Gamma_W))$ on $\mathcal{T}$.

(2nd step) : Note that if $\mathfrak{p} \in W$, then $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta) \cap \mathrm{Im}(\mathbf{R}\Gamma_W)$. On the other hand, if $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, then $E_R(R/\mathfrak{q}) \in \mathrm{Ker}(\delta) \cap \mathrm{Ker}(\mathbf{R}\Gamma_W)$. ∎

(3rd step) : To prove the theorem, it is enough to show that $\mathrm{Im}(\delta) = \mathrm{Im}(\mathbf{R}\Gamma_W)$ by Miyachi's Theorem 1.4. (See also Remark 1.5.) ∎

(4th step) : Now we prove the inclusion $\mathrm{Im}(\delta) \subseteq \mathrm{Im}(\mathbf{R}\Gamma_W)$.

To do this, assume $X \in \mathrm{Im}(\delta)$. Then there is a triangle in $\mathcal{T}$; $\mathbf{R}\Gamma_W(X) \to X \to V \to \mathbf{R}\Gamma_W(X)[1]$, where $V \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$. Let $\mathfrak{q}$ be an arbitrary element of $\mathrm{Spec}(R)\backslash W$. Since $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ and $(\mathrm{Im}(\mathbf{R}\Gamma_W), \mathrm{Ker}(\mathbf{R}\Gamma_W))$ are stable t-structures and since $E_R(R/\mathfrak{q})$ belongs to $\mathrm{Ker}(\delta) \cap \mathrm{Ker}(\mathbf{R}\Gamma_W)$, it follows that

$$\mathrm{Hom}_{\mathcal{T}}(X, E_R(R/\mathfrak{q})[n]) = \mathrm{Hom}_{\mathcal{T}}(\mathbf{R}\Gamma_W(X), E_R(R/\mathfrak{q})[n]) = 0$$

for any integer n. Then by the above triangle we have

$$\mathrm{Hom}_{\mathcal{T}}(V, E_R(R/\mathfrak{q})[n]) = 0$$

for any integer n. This is equivalent to that $\mathbf{R}\mathrm{Hom}_R(V, E_R(R/\mathfrak{q})) \cong 0$. In fact, the n-th cohomology module of $\mathbf{R}\mathrm{Hom}_R(V, E_R(R/\mathfrak{q}))$ is just $\mathrm{Hom}_{\mathcal{T}}(V, E_R(R/\mathfrak{q})[n]) = 0$. Since $V \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$, Lemma 2.4(1) forces $V \cong 0$, therefore $X \cong \mathbf{R}\Gamma_W(X)$. Hence we have $X \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ as desired. ∎

(5th step) : For the final step of the proof, we show the inclusion $\mathrm{Im}(\delta) \supseteq \mathrm{Im}(\mathbf{R}\Gamma_W)$.

Let $X \in \mathrm{Im}(\mathbf{R}\Gamma_W)$. Then there are triangles $\delta(X) \to X \to Y \to \delta(X)[1]$ with $Y \in \mathrm{Ker}(\delta)$, and $\mathbf{R}\Gamma_W(Y) \to Y \to V \to \mathbf{R}\Gamma_W(Y)[1]$ with $V \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$. Let $\mathfrak{p}$ be an arbitrary prime ideal belonging to W. Similarly to the 4th step, since $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta) \cap \mathrm{Im}(\mathbf{R}\Gamma_W)$, we see that $\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], Y) = \mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], V) = 0$ for any integer n, hence we have $\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], \mathbf{R}\Gamma_W(Y)) = 0$ for any n. This shows $\mathbf{R}\mathrm{Hom}_R(E_R(R/\mathfrak{p}), \mathbf{R}\Gamma_W(Y)) = 0$, then by Lemma 2.4(2) we have $\mathbf{R}\Gamma_W(Y) = 0$. Thus $Y \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$. Then, since $(\mathrm{Im}(\mathbf{R}\Gamma_W), \mathrm{Ker}(\mathbf{R}\Gamma_W))$ is a stable t-structure, the morphism $X \to Y$ in the triangle $\delta(X) \to X \to Y \to \delta(X)[1]$ is zero. It then follows that $\delta(X) \cong X \oplus Y[-1]$. Since there is no nontrivial morphisms $\delta(X) \to Y[-1]$ in $\mathcal{T}$, it is concluded that $\delta(X) \cong X$, hence $X \in \mathrm{Im}(\delta)$ as desired, and the proof is completed. □

Next, we consider the following sets.

Definition 2.5. (1) We denote by $\mathbb{S}(R)$ the set of all left exact radical functors on R-Mod.

(2) We denote by $\mathbb{A}(R)$ the set of the isomorphism classes $[\delta]$ where δ ranges over all abstract local cohomology functors on $\mathcal{D}^+(R\text{-Mod})$.

(3) We denote by $\mathrm{sp}(R)$ the set of all specialization closed subsets of $\mathrm{Spec}(R)$.

If $\{W_\lambda \mid \lambda \in \Lambda\}$ is a set of specialization-closed subsets of $\mathrm{Spec}(R)$, then $\bigcap_\lambda W_\lambda$ and $\bigcup_\lambda W_\lambda$ are also closed under specialization. By this reason $\mathrm{sp}(R)$ is a complete lattice. In view of Theorem 2.1, the complete lattice structure on $\mathrm{sp}(R)$ induces complete lattice structures on $\mathbb{S}(R)$ and $\mathbb{A}(R)$.

Corollary 2.6. *The mapping $\mathbb{S}(R) \to \mathbb{A}(R)$ which maps γ to $[\mathbf{R}\gamma]$ (resp. $\mathrm{sp}(R) \to \mathbb{A}(R)$ which sends W to $[\mathbf{R}\Gamma_W]$) gives an isomorphism of complete lattices.*

3. Characterization of Γ_I and $\Gamma_{I,J}$

We are concerned with the following two types of subsets $V(I)$ and $W(I,J)$ in $\mathrm{Spec}(R)$ which are closed under specialization, and their corresponding left exact radical functors Γ_I and $\Gamma_{I,J}$. The aim of this section is to characterize Γ_I and $\Gamma_{I,J}$ as elements of $\mathbb{S}(R)$.

Definition 3.1. Let I, J be ideals of R.

(1) We set $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I\}$ and set $\Gamma_I := \Gamma_{V(I)}$ the corresponding left exact radical functor. See [1].

(2) We set $W(I,J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some } n > 0\}$. The corresponding left exact radical functor $\varGamma_{W(I,J)}$ is denoted by $\varGamma_{I,J}$, which is called the section functor defined by the pair of ideals I, J.

We define the 'multiplication' and 'quotient' in $\mathbb{S}(R)$.

Lemma 3.2. *If γ_1, $\gamma_2 \in \mathbb{S}(R)$, then $\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_1 = \gamma_1 \cap \gamma_2 \in \mathbb{S}(R)$.*

Lemma 3.3. *Let γ_1, $\gamma_2 \in \mathbb{S}(R)$. Suppose that $\gamma_1 \subseteq \gamma_2$. Then the set $S_{\gamma_1,\gamma_2} = \{\gamma \in \mathbb{S}(R) \mid \gamma \cdot \gamma_2 = \gamma_1\}$ has a unique maximal element with respect to inclusion relation.*

Definition 3.4. For γ_1, $\gamma_2 \in \mathbb{S}(R)$ with $\gamma_1 \subseteq \gamma_2$, we denote by γ_1/γ_2 the unique maximal element of S_{γ_1,γ_2} in Lemma 3.3 and call it the quotient of γ_1 by γ_2.

Now we characterize $\varGamma_I$ and $\varGamma_{I,J}$ as elements of $\mathbb{S}(R)$.

Theorem 3.5. *The following conditions are equivalent for $\gamma \in \mathbb{S}(R)$.*

(1) *$\gamma = \varGamma_I$ for an ideal I of R.*

(2) *γ satisfies the ascending chain condition in the following sense: If there is an ascending chain of left exact radical functors*

$$\gamma_1 \subseteq \gamma_2 \subseteq \cdots \subseteq \gamma_n \subseteq \cdots \subseteq \gamma$$

with $\bigcup_n \gamma_n = \gamma$, then there is an integer $N > 0$ such that $\gamma_N = \gamma_{N+1} = \cdots = \gamma$.

Theorem 3.6. *The following conditions are equivalent for $\gamma \in \mathbb{S}(R)$.*

(1) *$\gamma = \varGamma_{I,J}$ for a pair of ideals I, J of R.*

(2) *$\gamma = \gamma_1/\gamma_2$ for left exact radical functors $\gamma_1 \subseteq \gamma_2$, the both of which satisfy the ascending chain condition in Theorem 3.5.*

We note that same theorems in terms of $\mathbb{A}(R)$ hold.

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Gorenstein orders associated with modules

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For a ring Λ we denote by Mod-Λ the category of right Λ-modules. We denote by Λ^{op} the opposite ring of Λ and consider left Λ-modules as right Λ^{op}-modules.

1 Gorenstein orders

Throughout this note, we will work over a commutative Noether ring R. We denote by $(-)_{\mathfrak{p}}$ the localization at a prime ideal $\mathfrak{p}$ of R. An R-algebra Λ is a ring Λ endowed with a ring homomorphism $R \to \Lambda$ whose image is contained in the center of Λ, and a Noether R-algebra Λ is an R-algebra Λ which is finitely generated as an R-module. If Λ, Γ are R-algebras, every Γ-Λ-bimodule M is assumed to be a right $(\Gamma^{\mathrm{op}} \otimes_R \Lambda)$-module, i.e., to satisfy $ax = xa$ for all $a \in R, x \in M$.

Recall that a finitely generated R-module M is said to have Gorenstein dimension zero provided M is reflexive, i.e., the canonical homomorphism

$$\varepsilon_M : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(\mathrm{Hom}(M, R), R) = 0$ for $i \geq 1$ (see [4]). Note that if R is a Gorenstein ring then a finitely generated R-module M has Gorenstein dimension zero whenever $\mathrm{Ext}^i_R(M, R) = 0$ for $i \geq 1$.

Definition 1.1 (cf. [2]). A Noether R-algebra Λ is said to be a Gorenstein R-order provided that R is a Gorenstein ring, and that Λ has Gorenstein dimension zero as an R-module and $\Lambda \cong \mathrm{Hom}_R(\Lambda, R)$ as Λ-bimodules.

Assume R is a Gorenstein ring. Then Gorenstein R-orders are Gorenstein algebras in the sense of [6] in which the theory of Gorenstein algebras is studied in detail. For instance, (a) for any finite group G, the group ring $R[G]$ is a Gorenstein R-order; (b) for any Noether R-algebra A having Gorenstein dimension zero as an R-module, the trivial extension ring $A \ltimes \mathrm{Hom}_R(A, R)$ is a Gorenstein R-order (see [5]); and (c) if R is a 2-dimensional normal domain, for any finitely generated torsionfree R-module M, the endomorphism ring $\mathrm{End}_R(M)$ is a Gorenstein R-order (see [3, Lemma 5.4]).

Lemma 1.2. *Let Λ be a Gorenstein R-order. Then the following hold.*

The detailed version of this note has been accepted for publication in *Comm. Algebra*.

(1) For any $M \in \text{Mod-}\Lambda$ *we have* $\text{Ext}^i_\Lambda(M,\Lambda) \cong \text{Ext}^i_R(M,R)$ *in* $\text{Mod-}\Lambda^{\text{op}}$ *for all* $i \geq 0$.

(2) For any idempotent $e \in \Lambda$, $e\Lambda e$ *is a Gorenstein R-order.*

Let Δ be a Gorenstein R-order and $e \in \Delta$ an idempotent. Then $\Lambda = e\Delta e$, $\Gamma = (1-e)\Delta(1-e)$ are Gorenstein R-orders and, setting $M = (1-e)\Delta e$ and $N = e\Delta(1-e)$, we can decompose Δ into a matrix ring

$$\begin{pmatrix} \Gamma & M \\ N & \Lambda \end{pmatrix},$$

where M has Gorenstein dimension zero as an R-module and $N \cong \text{Hom}_R(M,R)$ as Λ-Γ-bimodules. Conversely, for a Gorenstein R-order Λ we ask when there exist a Gorenstein R-oder Γ and a Γ-Λ-bimodule M having Gorenstein dimension zero as an R-module such that, setting $N = \text{Hom}_R(M,R)$, we have a matrix ring

$$\begin{pmatrix} \Gamma & M \\ N & \Lambda \end{pmatrix}$$

which is a Gorenstein R-order.

2 Derived equivalent Gorenstein orders

We start by formulating a lemma on derived equivalences for endomorphism rings. For an object X of an additive category $\mathfrak{A}$ we denote by $\text{add}(X)$ the additive full subcategory of $\mathfrak{A}$ consisting of direct summands of finite direct sums of copies of X.

Lemma 2.1. *Let* $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ *be an exact sequence in an abelian category* $\mathcal{A}$ *and* P *an object of* $\mathcal{A}$. *Assume* $E \in \text{add}(P)$ *and both* $\text{Hom}_{\mathcal{A}}(P,\varepsilon)$ *and* $\text{Hom}_{\mathcal{A}}(\mu,P)$ *are epic. Then* $\text{End}_{\mathcal{A}}(X \oplus P)$ *and* $\text{End}_{\mathcal{A}}(Y \oplus P)$ *are derived equivalent to each other.*

By [1, Theorem 4.3] we have the following.

Lemma 2.2. *Every ring derived equivalent to a Gorenstein R-order is a Gorenstein R-order.*

Let Λ be a Gorenstein R-order, $M \in \text{Mod-}\Lambda$ a finitely generated module having Gorenstein dimension zero as an R-module and $\Omega^n M \in \text{Mod-}\Lambda$ the nth syzygy of M. We set

$$\Delta_n = \text{End}_\Lambda(\Omega^n M \oplus \Lambda) \quad \text{and} \quad \Gamma_n = \text{End}_\Lambda(\Omega^n M)$$

for $n \in \mathbb{Z}$. Note that the Δ_n are uniquely determined up to Morita equivalence and that

$$\Delta_n \cong \begin{pmatrix} \Gamma_n & \Omega^n M \\ \text{Hom}_R(\Omega^n M, R) & \Lambda \end{pmatrix}$$

for all $n \in \mathbb{Z}$. By Lemmas 2.1 and 2.2 we have the following.

Theorem 2.3. *Every Δ_n is derived equivalent to Δ_0 and hence if Δ_0 is a Gorenstein R-order then Δ_n and Γ_n are Gorenstein R-orders for all $n \in \mathbb{Z}$.*

We need to ask when Δ_0 is a Gorenstein R-order.

Proposition 2.4. *The ring Δ_0 is a Gorenstein R-order if $M_\mathfrak{p}$ is projective as a right $\Lambda_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}$ of R with $\operatorname{ht}\mathfrak{p} \leq 1$ and $\operatorname{Ext}^i_\Lambda(M, M)_\mathfrak{p} = 0$, $1 \leq i \leq \operatorname{ht}\mathfrak{p} - 2$, for every prime ideal $\mathfrak{p}$ of R with $\operatorname{ht}\mathfrak{p} \geq 3$.*

Corollary 2.5. *If $\operatorname{ht}\mathfrak{p} = 2$ for every maximal ideal $\mathfrak{p}$ of R and $M = \Omega^2 L$ with L a right Λ-module of finite length, then Δ_0 is a Gorenstein R-order*

3 Matrix rings associated with ideals

Throughout this section, M is a two-sided ideal of Λ with $\operatorname{Ext}^i_R(\Lambda/M, R) = 0$ for $i \geq 2$. Note that $\operatorname{Ext}^i_R(M, R) = 0$ for $i \geq 1$ and M has Gorenstein dimension zero as an R-module.

Proposition 3.1. *Set $N = \operatorname{Hom}_R(M, R)$. Then for any idempotent $e \in \Lambda$ we have matrix rings*

$$\begin{pmatrix} e\Lambda e & eM \\ Ne & \Lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e\Lambda e & eN \\ Me & \Lambda \end{pmatrix}$$

which are Gorenstein R-orders.

If either $\operatorname{Ext}^i_R(\Lambda/M, R) = 0$ for $i \neq 1$ or $\operatorname{Ext}^i_R(\Lambda/M, R) = 0$ for $i \geq 1$, we can describe $\operatorname{Hom}_R(M, R)$ concretely.

Consider first the case where $\operatorname{Ext}^i_R(\Lambda/M, R) = 0$ for $i \neq 1$, this is the case if $\operatorname{ht}\mathfrak{p} = 1$ for every maximal ideal $\mathfrak{p}$ of R and Λ/M is an Artin ring. Let S be the set of regular elements of R and denote by $(-)_S$ the quotient by S. Set $N = \{q \in \Lambda_S \mid qM \subset \Lambda\}$.

Proposition 3.2. *We have $N = \{q \in \Lambda_S \mid qM \subset \Lambda\} = \{q \in \Lambda_S \mid Mq \subset \Lambda\}$ and hence we have a matrix ring*

$$\begin{pmatrix} \Lambda & M \\ N & \Lambda \end{pmatrix}$$

which is a Gorenstein R-order. Furthermore, we have $M = \{\lambda \in \Lambda \mid \lambda N \subset \Lambda\} = \{\lambda \in \Lambda \mid N\lambda \subset \Lambda\}$.

Next, assume $\operatorname{Ext}^i_R(\Lambda/M, R) = 0$ for $i \geq 1$, this is always the case if $\dim R = 0$. Set $L = \{\lambda \in \Lambda \mid \lambda M = 0\}$.

Proposition 3.3. *We have $L = \{\lambda \in \Lambda \mid \lambda M = 0\} = \{\lambda \in \Lambda \mid M\lambda = 0\}$ and hence we have a matrix ring*

$$\begin{pmatrix} \Lambda & M \\ \Lambda/L & \Lambda \end{pmatrix}$$

which is a Gorenstein R-order. Furthermore, we have $M = \{\lambda \in \Lambda \mid \lambda L = 0\} = \{\lambda \in \Lambda \mid L\lambda = 0\}$.

Let $\Gamma = \Lambda/L \ltimes M$, the trivial extension of Λ/L by M (see [5]). Note that we have a homomorphism of R-algebras $\Lambda \to \Gamma, \lambda \mapsto (\lambda + L, 0)$ and Γ is a Λ-bimodule. Also, identifying $x \in M$ with $(0, x) \in \Gamma$, we consider M as a two-sided ideal of Γ.

Proposition 3.4. *Let $e \in \Lambda$ be an idempotent such that $e \notin L$ and $eMe \subset L$. Then we have matrix rings*

$$\begin{pmatrix} e\Gamma e & eM \\ (\Lambda/L)e & \Lambda \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} e\Gamma e & e(\Lambda/L) \\ Me & \Lambda \end{pmatrix}$$

which are Gorenstein R-orders.

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TILTING AND CLUSTER TILTING FOR QUOTIENT SINGULARITIES

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1. Introduction

This note is a report of [IT].[1] Our aim is to discuss tilting theoretic aspect of Cohen-Macaulay modules. Tilting theory is a generalization of Morita theory, and

- Morita theory realizes abelian categories as module categories of rings, while
- tilting theory realizes triangulated categories as derived categories of rings.

The representation theory of Cohen-Macaulay modules was initiated by the school of Auslander (see the book [Y]). Recently there is development in this theory stimulated by the connection with

- cluster tilting theory e.g. [BIKR, KR, KMV, Iy, IR, IY],
- tilting theory e.g. [A, KST1, KST2, LP, U].

The key role is played by the stable category defined as follows:

Definition 1.1. Let $R = \bigoplus_{i \geq 0} R_i$ be a commutative graded k-algebra. We denote by

$$\mathrm{CM}^{\mathbb{Z}}(R)$$

the category of graded Cohen-Macaulay R-modules. We denote by

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$$

the *stable category* of $\mathrm{CM}^{\mathbb{Z}}(R)$, i.e.

- $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ has the same objects as $\mathrm{CM}^{\mathbb{Z}}(R)$,
- The morphism set is given by

$$\underline{\mathrm{Hom}}^{\mathbb{Z}}_R(X, Y) := \mathrm{Hom}^{\mathbb{Z}}_R(X, Y)/P(X, Y),$$

  where $\mathrm{Hom}^{\mathbb{Z}}_R(X, Y)$ consists of graded homomorphisms and $P(X, Y)$ consists of graded homomorphisms which factor through graded free R-modules.

During the Conference this category is discussed by other people from a lot of viewpoint:

- Hopkins-Neeman type results motivated by homotopy theory [Tr, YY],
- representation theory of Cohen-Macaulay modules [Bur, Ii],
- geometry of resolutions of singularities [Bur],
- mirror symmetry [Ta].

Let us recall two of the most fundamental properties of $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$.

Theorem 1.2. *Assume that R is a Gorenstein isolated singularity.*

(1) [H, Buc] $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ *forms a triangulated category,*

[1] The detailed version of this paper will be submitted for publication elsewhere.

(2) [AR] $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ *satisfies Auslander-Reiten-Serre duality, i.e. there exists a functorial isomorphism*

$$\underline{\mathrm{Hom}}_R^{\mathbb{Z}}(X,Y) \simeq D\,\underline{\mathrm{Hom}}_R^{\mathbb{Z}}(Y, X(a)[d-1])$$

for any $X, Y \in \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$, *where* d *is the Krull dimension of* R *and* a *is the a-invariant of* R *(i.e.* $\omega \simeq R(a)$ *in* $\mathrm{CM}^{\mathbb{Z}}(R)$*).*

The aim of this note is to give an example of the following question, which is also studied by a lot of authors, e.g. [A, KST1, KST2, LP, U].

Question 1.3. *Find* R *such that the category* $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ *is triangle equivalent to the derived category of a ring.*

An important notion to approach this Question is the following:

Definition 1.4. Let $\mathcal{T}$ be a triangulated category. We say that an object $U \in \mathcal{T}$ is *tilting* if

- $\mathrm{Hom}_{\mathcal{T}}(U, U[n]) = 0$ for any $n \neq 0$,
- $\mathcal{T}$ is a unique triangulated subcategory of $\mathcal{T}$ containing U and closed under direct summands.

One can easily check that, for any ring Λ, the homotopy category

$$\mathbf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$$

of bounded complexes of finitely generated projective Λ-modules has a tilting object

$$\Lambda = (\cdots \to 0 \to \overset{0}{\Lambda} \to 0 \to \cdots).$$

The importance of the notion of tilting objects comes from the following Morita-Rickard-type Theorem showing a certain converse of the above statement.

Theorem 1.5. [Ke] *Let* U *be a tilting object in an* algebraic *triangulated category* $\mathcal{T}$. *Then* $\mathcal{T}$ *is triangle equivalent to* $\mathbf{K}^{\mathrm{b}}(\mathrm{proj}\,\mathrm{End}_{\mathcal{T}}(U))$ *up to direct summands.*

A triangulated category is called *algebraic* [Kr] if it is triangle equivalent to the stable category of a Frobenius category [H]. For example our triangulated category $\mathcal{T} = \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ is algebraic.

2. Our results

Now we are ready to state our results. Let

$$S = k[x_1, \cdots, x_d]$$

be a polynomial algebra over a field k of characteristic 0. We regard S as a $\mathbb{Z}$-graded k-algebra by putting $\deg x_i = 1$ for any i. Let G be a finite subgroup of $\mathrm{SL}_d(k)$ acting on $k^d \backslash \{0\}$ freely.. Since the action of G on S preserves the grading, the invariant subalgebra

$$R := S^G$$

forms a $\mathbb{Z}$-graded k-subalgebra of S. Since R is Gorenstein, we have a triangulated category $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. The following is our main result.

Theorem 2.1. [IT] *The R-module*

$$U := \bigoplus_{i=1}^{d} [\Omega_S^i k(i)]_{\mathrm{CM}}$$

is a tilting object of $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$, *where* Ω_S *is the kernel of the graded free cover and* $[-]_{\mathrm{CM}}$ *is the maximal direct summand which is a Cohen-Macaulay R-module.*

We also give an explicit description of $\underline{\mathrm{End}}_R^{\mathbb{Z}}(U)$. Let

$$E = \bigoplus_{i \geq 0} E_i := \bigoplus_{i \geq 0} \bigwedge^i V$$

be the exterior algebra of the dual vector space $V := \mathrm{Hom}_k(S_1, k)$ of the degree 1 part S_1 of S. We denote by $E^{(d)}$ the k-algebra defined by

$$E^{(d)} := \begin{bmatrix} E_0 & E_1 & \cdots & E_{d-1} \\ 0 & E_0 & \cdots & E_{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_0 \end{bmatrix}.$$

The action of G on V determine an action of G on $E^{(d)}$, and we denote by

$$G * E^{(d)}$$

the *skew group algebra*, i.e. $G * E^{(d)} = kG \otimes_k E^{(d)}$ as a k-vector space and the multiplication is given by

$$(g \otimes a)(g' \otimes a') = gg' \otimes a^{g'} a'$$

for any $g, g' \in G$ and $a, a' \in E^{(d)}$. Let

$$e := 1 - \frac{1}{\#G} \sum_{g \in G} g \in G * E^{(d)}$$

be an idempotent.

Theorem 2.2. [IT] *We have an isomorphism* $\underline{\mathrm{End}}_R^{\mathbb{Z}}(U) \simeq e(G * E^{(d)})e$.

By Theorems 1.5, 2.1 and 2.2 we have the following result.

Corollary 2.3. *There exists a triangle equivalence*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \simeq \mathbf{K}^{\mathrm{b}}(\mathrm{proj}\, e(G * E^{(d)})e).$$

The key ingredient of the proof comes from cluster tilting theory. We denote by $\mathrm{CM}(R)$ the category of (ungraded) maximal Cohen-Macaulay R-modules.

Proposition 2.4. [Iy] *S is a $(d-1)$-cluster tilting R-module, i.e. we have*

$$\begin{aligned} \mathrm{add}\, S &= \{X \in \mathrm{CM}(R) \mid \mathrm{Ext}_R^i(S, X) = 0 \ \textit{for any} \ 0 < i < d-1\} \\ &= \{X \in \mathrm{CM}(R) \mid \mathrm{Ext}_R^i(X, S) = 0 \ \textit{for any} \ 0 < i < d-1\}. \end{aligned}$$

Using this we can calculate the vanishing of selfextensions of U. For the case of $d = 2$ the above equality means $\mathrm{CM}(R) = \mathrm{add}\, S$, which is a classical result due to Herzog and Auslander [Y].

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MAXIMAL COHEN-MACAULAY MODULES OVER QUOTIENT SURFACE SINGULARITIES

IGOR BURBAN

ABSTRACT. In this note I discuss a relationship between the McKay Correspondence for two-dimensional quotient singularities and the theory of maximal Cohen-Macaulay modules.

1. McKay's observation

Let $G \subseteq \mathsf{SL}_2(\mathbb{C})$ be a finite group. Then one can attach to it the following pair of combinatorial objects.

First object. By Maschke's theorem, the category of finite dimensional representations of G over the field $\mathbb{C}$ is semi-simple. Let $\{V_0, V_1, \dots, V_n\}$ be the set of the isomorphy classes of the *irreducible* representations of G, where $V_0 = \mathbb{C}$ is the trivial representation, and $W = \mathbb{C}^2$ be the *fundamental* representation of G induced by the embedding $G \subseteq \mathsf{SL}_2(\mathbb{C})$. For any $0 \le i \le n$, we set $m_i = \dim_{\mathbb{C}}(V_i)$. For any $0 \le i \le n$ we have decompositions

$$V_i \otimes_{\mathbb{C}} W \cong \bigoplus_{j=0}^{n} V_j^{a_{ij}}.$$

One can show that $a_{ii} = 0$ and $a_{ij} = a_{ji}$ for all $0 \le i, j \le n$.

Definition 1. The McKay graph $\mathsf{MK}(G)$ of a finite group $G \subseteq \mathsf{SL}_2(\mathbb{C})$ is defined as follows.

(1) The set of vertices of $\mathsf{MK}(G)$ is $\{0, 1, \dots, n\}$.
(2) For any $0 \le i \ne j \le n$ the vertex i is connected with the vertex j by a_{ij} arrows.
(3) The vertex i has "weight" m_i.

Second object. Let $A = \mathbb{C}[\![x, y]\!]^G$ be the quotient singularity defined by G, $X = \mathsf{Spec}(A)$, $o \in X$ the closed point of X and $\widetilde{X} \xrightarrow{\pi} X$ a minimal resolution of singularities. Let $E = \pi^{-1}(o)$ be the exceptional divisor of the resolution. It is well-known that E is a tree of projective lines.

In 1978 John McKay made [8] the following striking

Observation. Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup. Then we have:

(1) The number of the irreducible components of E is equal to the number of non-trivial irreducible representations of G.
(2) Let $\mathsf{MK}(G)' = \mathsf{MK}(G) \setminus \{0\}$ be the graph obtained from $\mathsf{MK}(G)$ by excluding the vertex 0 and all arrows connected with it. Then $\mathsf{MK}(G)'$ is isomorphic to the dual intersection graph Γ_E of the curve E. In other words, there exists a labeling of the irreducible components $E_1, \dots, E_n$ such that for any $1 \le i \ne j \le n$ we have:

$$a_{ij} = \#(E_i \cap E_j) =: c_{ij}.$$

(3) The cycle $Z = \sum_{i=1}^{n} m_i [E_i] \in H_2(\widetilde{X}, \mathbb{Z})$ is the *fundamental cycle* of the resolution $\widetilde{X}$. This can be expressed in plain words as follows. For any $1 \le i \le n$ let $c_{ii} := -2 = E_i^2$ be the self-intersection index of E_i and $C = (c_{ij}) \in \mathsf{Mat}_{n \times n}(\mathbb{Z})$ be the *intersection matrix* of E.

Then Z is the smallest vector $\underline{l} = (l_1, l_2, \dots, l_n)$ with non-negative integral entries such that

$$\langle \underline{e}_i, \underline{l} \rangle_C := \langle \underline{e}_i^t \, C \, \underline{l} \rangle \leq 0$$

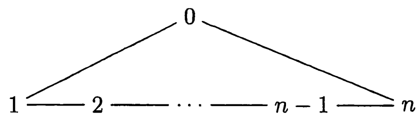
for all $1 \leq i \leq n$, where $e_i = (0, \dots, 0, 1, 0, \dots 0)$ is the i-th basic vector of $\mathbb{Z}^n$.

Example 2. Let $\mathbb{Z}/(n+1)\mathbb{Z} \cong G = \langle g \rangle \subset \mathsf{SL}_2(\mathbb{C})$ be a cyclic subgroup of order $n+1$ generated by the element

$$g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix},$$

where ξ is a primitive $(n+1)$-st root of 1. Then we have:

- G has $n+1$ irreducible representations $\{V_0, V_1, \dots, V_n\}$, where all $V_i = \mathbb{C}$ and the action of g is given by the multiplication with ξ^i. It is easy to see that $W = V_1 \oplus V_n$ and the McKay's graph $\mathsf{MK}(G)$ is a cycle



- Next, we have:

$$A := \mathbb{C}[\![x, y]\!]^G = \mathbb{C}[\![x^{n+1}, xy, y^{n+1}]\!] \cong \mathbb{C}[\![u, v, w]\!]/(uw - v^{n+1})$$

is a simple surface singularity of type A_n. It is well-known that the exceptional divisor of a minimal resolution of singularities of $\mathsf{Spec}(A)$ is a chain of n projective lines. Hence, the intersection matrix of the exceptional divisor is just

$$C = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

It is easy to show that in this case the fundamental cycle Z is equal to $\sum_{i=1}^{n} [E_i] = (1, 1, \dots, 1)$, in a full accordance with McKay's observation.

Explanation. McKay himself has verified his observation using Klein's classification of finite subgroups in $\mathsf{SL}_2(\mathbb{C})$ by a tedious case-by-case analysis [8]. It turns out, however, that the McKay correspondence can be explained in a more conceptual way by introducing the third intermediate object: the *stable category of the maximal Cohen-Macaulay A–modules* $\underline{\mathsf{CM}}(A)$. Namely, there exist natural bijections

$$\mathsf{MK}(G)' \xleftarrow{\sim} \mathrm{ind}\big(\underline{\mathsf{CM}}(A)\big) \xrightarrow{\sim} \Gamma_E,$$

where $\mathrm{ind}\big(\underline{\mathsf{CM}}(A)\big)$ is the set of the isomorphy classes of indecomposable objects in $\underline{\mathsf{CM}}(A)$. The statement about the fundamental cycle and the dimensions of irreducible representations of G can be derived using the Auslander-Reiten theory of the category $\underline{\mathsf{CM}}(A)$.

2. Algebraic McKay Correspondence

Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup. Then the ring of invariants $A = \mathbb{C}[\![x, y]\!]^G$ is a normal surface singularity. Recall the following standard facts about Cohen-Macaulay modules over surface singularities.

Theorem 3. *Let $(A, \mathfrak{m})$ be a local Noetherian ring of Krull dimension two.*

- *A is normal if and only if it is Cohen-Macaulay and regular in codimension one.*

- *Assume A to be Cohen-Macaulay. Then for any maximal Cohen-Macaulay module M and any Noetherian module N the module $\mathsf{Hom}_A(N, M)$ is maximal Cohen-Macaulay.*
- *Assume additionally that A is Gorenstein in codimension one (for instance, A is a normal singularity). Then a Noetherian module M is maximal Cohen-Macaulay if and only if it is reflexive. Moreover, the functor $M \mapsto M^{\vee\vee}$ is left adjoint to the forgetful functor $\mathsf{CM}(A) \longrightarrow A-\mathsf{mod}$.*
- *Let $(A, \mathfrak{m}) \subseteq (B, \mathfrak{n})$ be a finite extension of Cohen-Macaulay surface singularities, which are Gorenstein in codimension one. Then for any Noetherian B–module M we have an isomorphism of A–modules $M^{\vee\vee_A} \cong M^{\vee\vee_B}$.*
- *Let A be regular. Then any maximal Cohen-Macaulay module over A is free.*

For a proof one may consult [3, Section 3] and references therein.

The following theorem of Herzog [7] was the starting point of an extensive study of maximal Cohen-Macaulay modules over surface singularities.

Theorem 4. *Let k be an algebraically closed field, $G \subset \mathsf{GL}_2(k)$ be a finite subgroup such that $\gcd(|G|, \mathrm{char}(k)) = 1$, $R = k[\![x, y]\!]$ and $A = R^G$. Then we have: $\mathsf{CM}(A) = \mathsf{add}_A(R)$. In other words, any maximal Cohen-Macaulay module over A is isomorphic to a direct sum of direct summands of R viewed as an A-module.*

Proof. The embedding $i : A \to R$ has a left inverse $p : R \to A$ given by the Reinold's operator

$$p(r) = \frac{1}{|G|}\sum_{g\in G} g(r).$$

It is easy to see that the map p is A-linear. Hence, we have an isomorphism $R \cong A \oplus A'$ in the category of A–modules. Next, for any Noetherian A–module M we have:

$$R \otimes_A M \cong M \oplus (A' \otimes_A M).$$

If M is maximal Cohen-Macaulay over A then there exists a positive integer t such that

$$R^t \cong (R \otimes_A M)^{\vee\vee_R} \cong (R \otimes_A M)^{\vee\vee_A} \cong M \oplus (A' \otimes_A M)^{\vee\vee_A}.$$

Hence, M is a direct summand of R^t as stated. □

From Herzog's result we get the following corollary.

Corollary 5. *Let $\Lambda = \mathsf{End}_A(R)$. Then the functor*

$$\mathsf{Hom}_A(R, -) : \mathsf{CM}(A) = \mathsf{add}_A(R) \longrightarrow \mathsf{pro}(\Lambda)$$

is an equivalence of categories, where $\mathsf{pro}(\Lambda)$ is the category of the finitely generated projective right Λ–modules.

The following result is due to Auslander [2].

Theorem 6. *Let k be an algebraically closed field and $G \subset \mathsf{GL}_2(k)$ be a small finite subgroup such that $\gcd(|G|, \mathrm{char}(k)) = 1$ (note that any subgroup in $\mathsf{SL}_2(k)$ is automatically small). Let $R = k[\![x, y]\!]$ and $A = R^G$. Then the algebra homomorphism*

$$\theta : R * G \longrightarrow \mathsf{End}_A(R), \quad t[g] \overset{\theta}{\mapsto} \bigl(r \mapsto tg(r)\bigr)$$

is an isomorphism of algebras.

As a corollary, we obtain the following "algebraic" version of the McKay Correspondence, which is due to Auslander [2], see also [10].

Theorem 7. *Let k be an algebraically closed field and $G \subset \mathsf{GL}_2(k)$ be a small finite subgroup such that $\gcd(|G|, \mathrm{char}(k)) = 1$, $R = k[\![x, y]\!]$ and $A = R^G$.*

(1) *The functor* $\mathsf{pro}(R*G) \to \mathsf{CM}(A)$ *assigning to a projective module* P *its* A*-submodule of invariants* P^G*, is an equivalence of categories quasi-inverse to the functor*

$$\mathsf{Hom}_A(R, -) : \mathsf{CM}(A) \longrightarrow \mathsf{pro}(R*G).$$

(2) *Since we have an isomorphism* $R*G/\mathsf{rad}(R*G) \cong k[G]$*, a bijection between the projective and the semi-simple right* $R*G$*–modules* $P \mapsto P/\mathsf{rad}(P)$ *yields a bijection between the isomorphy classes of irreducible representations of the group* G *and indecomposable projective right modules over* $R*G$*. If* V *is an irreducible representation of* G*, then the corresponding projective* $R*G$*–module is just* $R\otimes_k V$*, where the action of an element* $t[g] \in R*G$ *on a simple tensor* $r\otimes h \in R\otimes_k V$ *is given by:*

$$t[g]\circ(r\otimes h) = tg(r)\otimes gh.$$

(3) *The correspondence between the irreducible representations of* G *and the indecomposable maximal Cohen-Macaulay modules over* A *is given by the functor*

$$\mathsf{Rep}(G) \ni V \mapsto \left(R\otimes_k V\right)^G \in \mathsf{CM}(A).$$

In the notations of the above theorem, consider the Koszul resolution of the trivial representation $V_0 = k$ of the group G, viewed as an $R*G$–module:

$$0 \to R\otimes_k \wedge^2(W) \xrightarrow{\alpha} R\otimes_k W \xrightarrow{\beta} R \xrightarrow{\phi} k \to 0$$

where $\alpha\big(p\otimes(f_1\otimes f_2 - f_2\otimes f_1)\big) = p\tilde{f_1}\otimes f_2 - p\tilde{f_2}\otimes f_1$, $\quad\beta(q\otimes f) = q\tilde{f}$ and $\phi(t) = t(0,0)$.

Remark 8. Let V be a non-trivial irreducible $k[G]$–module. Then its minimal free projective resolution in the category of $R*G$–modules is

$$0 \to R\otimes_k(\wedge^2(W)\otimes_k V) \longrightarrow R\otimes_k(W\otimes_k V) \longrightarrow R\otimes_k V \longrightarrow V \to 0. \tag{1}$$

Since the functor of taking G-invariants is exact, we obtain a short exact sequence of Cohen-Macaulay A–modules

$$0 \to \left(R\otimes_k(\wedge^2(W)\otimes_k V)\right)^G \longrightarrow \left(R\otimes_k(W\otimes_k V)\right)^G \longrightarrow \left(R\otimes_k V\right)^G \to 0, \tag{2}$$

which is precisely the Auslander-Reiten sequence sequence ending at the indecomposable Cohen-Macaulay module $\left(R\otimes_k V\right)^G$, see [2] and [10].

Corollary 9. *If* G *is a finite subgroup of* $\mathsf{SL}_2(k)$ *then we have:* $\wedge^2 W \cong V_0 = k$*. Hence, the Auslander-Reiten quiver of the category* $\mathsf{CM}(A)$ *is obtained from the McKay' graph* $\mathsf{MK}(G)$ *by "doubling" all the arrows.*

Example 10. Let $\mathbb{Z}/(n+1)\mathbb{Z} \cong G \subset \mathsf{SL}_2(\mathbb{C})$ be as in Example 2, $A = \mathbb{C}[\![x,y]\!]^G$ and $\{V_0, V_1, \dots, V_n\}$ be the set of the isomorphy classes of irreducible representations of G, where $V_i = \mathbb{C}$ and $g\cdot 1 = \xi^i$ for $0 \le i \le n$. Then the corresponding indecomposable Cohen-Macaulay A–modules are

$$\mathbb{C}[\![x,y]\!] \supseteq I_l := \left(\mathbb{C}[\![x,y]\!]\otimes_k V_l\right)^G = \Big\{\sum_{i,j=0}^{\infty} a_{ij}x^iy^j \;\Big|\; a_{ij}\in\mathbb{C},\ i-j\equiv l \bmod n\Big\}, \quad 0\le l\le n.$$

The following result is due to Auslander [2].

Theorem 11. *Let* $(A,\mathfrak{m})$ *be a normal surface singularity with a canonical module* K*.*

- *Let* $\omega \in \mathsf{Ext}^2_A(k,K) \cong k$ *be a generator and*

$$0 \longrightarrow K \longrightarrow D \longrightarrow A \longrightarrow k \longrightarrow 0 \tag{3}$$

  be the corresponding extension class. Then the module D *is maximal Cohen-Macaulay.*
- *Let* $G \subset \mathsf{GL}_2(k)$ *be a finite subgroup,* $R = k[\![x,y]\!]$ *and* $A = R^G$ *be the corresponding quotient singularity. Then the sequence (3) is obtained from the sequence (1) by taking* G*-invariants. In particular, we have:* $K \cong (R\otimes_k \wedge^2 W)^G$ *(see [9]) and* $D \cong (R\otimes_k W)^G$*.*

- *For a non-regular indecomposable Cohen-Macaulay module M, the complex*

$$0 \longrightarrow (K \otimes_A M)^{\vee\vee} \longrightarrow (D \otimes_A M)^{\vee\vee} \longrightarrow M \longrightarrow 0 \tag{4}$$

induced by the short exact sequence (3), is exact. Moreover, it is an Auslander-Reiten sequence, ending at M.
- *For $G \subset \mathsf{SL}_2(k)$ holds: $D \cong (\Omega^1_A)^{\vee\vee}$, where Ω^1_A is the module of Kähler differentials of A.*

3. Geometric McKay Correspondence

Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup, $A = \mathbb{C}[\![x,y]\!]^G$, $X = \mathsf{Spec}(A)$ and $\widetilde{X} \xrightarrow{\pi} X$ be a minimal resolution of singularities. Let $E = \pi^{-1}(o)$ be the exceptional divisor of π. The following facts are well-known.

(1) $E = E_1 \cup \cdots \cup E_n$ is a tree of projective lines.
(2) We have: $H_2(\widetilde{X}, \mathbb{Z}) = \cup_{i=1}^n \mathbb{Z}[E_i] \cong \mathbb{Z}^n$.
(3) For any $1 \le i \le n$ there exists a unique element $E_i^* \in H_2(\widetilde{X}, \mathbb{Z})$ such that $E_i^* \cdot E_j = \delta_{ij}$ for all $1 \le j \le n$.

The following result is due to Artin and Verdier [1], see also [6] and [4].

Theorem 12. *Let M be a maximal Cohen-Macaulay module over A and $\widetilde{M} = \pi^*(M)/\mathsf{tor}$ be the corresponding torsion free sheaf on $\widetilde{X}$. Then we have:*

(1) *The torsion free coherent sheaf $\widetilde{M}$ is locally free.*
(2) *The isomorphy class of M is uniquely determined by the pair*

$$\big(\mathrm{rk}(\widetilde{M}), c_1(\widetilde{M})\big) \in \mathbb{Z}_+ \times H^2(\widetilde{X}, \mathbb{Z}).$$

(3) *If M is indecomposable than either $M \cong A$ or there exists $1 \le i \le n$ such that $c_1(\widetilde{M}) = E_i^*$. In that case we have: $\mathrm{rk}(\widetilde{M}) = c_1(\widetilde{M}) \cdot Z$, where Z is the fundamental cycle of $\widetilde{X}$.*

Hence, combining the Theorem 11 and Theorem 12, we get a bijection between the set of the isomorphy classes of non-trivial irreducible representations of G, the set of indecomposable objects of the stable category of the maximal Cohen-Macaulay modules $\underline{\mathsf{CM}}(A)$ and the set of the irreducible components of the exceptional divisor E.

If V is a representation of G and $M = (\mathbb{C}[\![x,y]\!] \otimes V)^G$ is the corresponding Cohen-Macaulay module, then $\mathrm{rk}(\widetilde{M}) = \mathsf{dim}_{\mathbb{C}}(V)$. Thus, the last part of Theorem 12 implies that the fundamental cycle Z is equal to $\sum_{i=1}^n m_i[E_i]$, where $m_i = \mathsf{dim}_{\mathbb{C}}(V_i)$ for $1 \le i \le n$.

The following result is due to Esnault and Knörrer [5].

Theorem 13. *Let V be a non-trivial irreducible representation of G, $M = (\mathbb{C}[\![x,y]\!] \otimes_{\mathbb{C}} V)^G$ be the corresponding indecomposable Cohen-Macaulay module, $\widetilde{M} = \pi^*(M)/\mathsf{tor}$ the corresponding vector bundle on $\widetilde{X}$ and F the irreducible component of E such that $c_1(\widetilde{M}) = F^*$. Let $N = (M \otimes_A \Omega^1_A)^{\vee\vee}$ and $\widetilde{N}$ be the corresponding vector bundle on $\widetilde{X}$. Then we have:*

$$\det(\widetilde{N}) \cong \det(\widetilde{M})^{\otimes 2} \otimes \mathcal{O}_{\widetilde{X}}(F).$$

Using this result, the isomorphism of the McKay graph $\mathsf{MK}(G)'$ and the dual intersection graph Γ_E follows from Theorem 11 and Theorem 12.

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COHEN-MACAULAY TAME AND COUNTABLE NON-ISOLATED SURFACE SINGULARITIES

IGOR BURBAN AND YURIY DROZD

ABSTRACT. This is a short report on our joint work in progress about a classification of the maximal Cohen-Macaulay modules over certain non-isolated surface singularities.

Let k be an algebraically closed field of characteristic zero and $(A, \mathfrak{m})$ a complete local Noetherian Cohen-Macaulay k-algebra of Krull dimension two. The following result is due to Herzog [6], Auslander [1] and Esnault [5], see also [3] and [11].

Theorem 1. *The ring A has finite Cohen-Macaulay representation type (i.e. there exists only finitely many indecomposable maximal Cohen-Macaulay modules) if and only if A is a quotient singularity (i.e. there exists a finite group $G \subset \mathsf{GL}_2(k)$ such that $A \cong k[\![x, y]\!]^G$).*

In a work of Buchweitz, Greuel and Schreyer [2], the case of non-isolated hypersurface singularities was considered.

Theorem 2. *A non-isolated hypersurface singularity $A = k[\![x, y, z]\!]/(f)$ has discrete (or countable) Cohen-Macaulay representation type (meaning that there are only countably many indecomposable maximal Cohen-Macaulay modules over A) if and only if $A \cong k[\![x, y, z]\!]/(xy)$ (A_∞-singularity) or $A \cong k[\![x, y, z]\!]/(x^2y - z^2)$ (D_∞-singularity).*

The next theorem is due to Kahn [7] and Drozd, Greuel and Kashuba [4].

Theorem 3. *Let $(A, \mathfrak{m})$ be a simply elliptic or a cusp surface singularity. Then the category of the maximal Cohen-Macaulay modules $\mathsf{CM}(A)$ is representation tame.*

Remark 4. The only cases when a simply elliptic or a cusp singularity is a complete intersection, are the following:

(1) $T_{p,q,r}(\lambda)$–singularities given by the equation $x^p + y^q + z^r - \lambda xyz$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ and $\lambda \in k^* \setminus D_{p,q,r}$ for a certain finite set of values $D_{p,q,r}$.

(2) $T_{p,q,r,t}(\lambda)$–singularities given by two equations $x^p + y^q = uv$, $u^r + v^t = \lambda xy$, where $p, q, r, t \geq 2$, $\max(p, q, r, t) \geq 3$ and $\lambda \in k^* \setminus D_{p,q,r,t}$.

This work grew up from an attempt to answer the following questions.

Question 5. *Let $(A, \mathfrak{m})$ be a non-isolated Cohen-Macaulay surface singularity.*

(1) *Assume that A has countable Cohen-Macaulay representation type. Is it true that there exists a finite group G of ring automorphisms of B such that $A \cong B^G$, where B is*

$$k[\![x, y, z]\!]/(xy) \quad or \quad k[\![x, y, z]\!]/(x^2y - z^2)?$$

(2) *Can A have tame Cohen-Macaulay representation type?*

It turns out that the answer on the first question (posed in 1987 by F.-O. Schreyer in [9]) is negative, whereas the answer on the second question is positive. In other words, we show there exist wide classes of non-isolated Cohen-Macaulay surface singularities of countable and tame Cohen-Macaulay representation type. The key role of our approach is plaid by the following construction.

Let $(A, \mathfrak{m})$ be a *reduced* complete Cohen-Macaulay k–algebra of Krull dimension two, which is not an isolated singularity, and let R be its normalization. Then R is again complete and the ring extension $A \subset R$ is finite. Moreover, the ring R is isomorphic to the product of a finite number of normal local rings:

$$R \cong (R_1, \mathfrak{n}_1) \times (R_1, \mathfrak{n}_1) \times \cdots \times (R_t, \mathfrak{n}_t).$$

By a theorem of Serre, all rings R_i are automatically Cohen-Macaulay.

Let $I = \mathsf{ann}(R/A) \cong \mathsf{Hom}_A(R, A)$ be the conductor ideal. Note that I is also an ideal in R, denote $\bar{A} = A/I$ and $\bar{R} = R/I$. Observe that I is Cohen-Macaulay, both as A- and R-module. Moreover, $V(I) \subset \mathsf{Spec}(A)$ is exactly the locus where the ring A is not normal. It is not difficult to show that both rings $\bar{A}$ and $\bar{R}$ have Krull dimension one and are Cohen-Macaulay (but not necessary reduced). Let $Q(\bar{A})$ and $Q(\bar{R})$ be the corresponding total rings of fractions, then the inclusion $\bar{A} \to \bar{R}$ induces an inclusion $Q(\bar{A}) \to Q(\bar{R})$.

For a maximal Cohen-Macaulay A-module M, let $R \boxtimes_A M = (R \otimes_A M)^{\vee\vee}$ be the corresponding maximal Cohen-Macaulay module over R. It is not difficult to see that the canonical morphism

$$\theta_M : Q(\bar{R}) \otimes_A M = Q(\bar{R}) \otimes_{Q(\bar{A})} \big(Q(\bar{A}) \otimes_A M\big) \longrightarrow Q(\bar{R}) \otimes_R \big(R \otimes_A M\big) \longrightarrow Q(\bar{R}) \otimes_R \big(R \boxtimes_A M\big)$$

is an epimorphism. Moreover, one can show that the canonical morphism

$$\eta_M : Q(\bar{A}) \otimes_A M \to Q(\bar{R}) \otimes_A M \xrightarrow{\theta_M} Q(\bar{R}) \otimes_R \big(R \boxtimes_A M\big)$$

is a monomorphism in the category of $Q(A)$-modules.

Definition 6. In the notations of this section, consider the following *category of triples* $\mathsf{Tri}(A)$. Its objects are triples $(\widetilde{M}, V, \theta)$, where $\widetilde{M}$ is a maximal Cohen-Macaulay R–module, V is a Noetherian $Q(\bar{A})$–module and $\theta : Q(\bar{R}) \otimes_{Q(\bar{A})} V \to Q(\bar{R}) \otimes_R \widetilde{M}$ is an epimorphism of $Q(\bar{R})$–modules such that the induced morphism of $Q(\bar{A})$–modules

$$V \to Q(\bar{R}) \otimes_{Q(\bar{A})} V \xrightarrow{\theta} Q(\bar{R}) \otimes_R \widetilde{M}$$

is an monomorphism. A morphism between two triples $(\widetilde{M}, V, \theta)$ and $(\widetilde{M}', V', \theta')$ is given by a pair (F, f), where $F : \widetilde{M} \to \widetilde{M}'$ is a morphism of R–modules and $f : V \to V'$ is a morphism of $Q(\bar{A})$–modules such that the following diagram

$$\begin{array}{ccc}
Q(\bar{R}) \otimes_{Q(\bar{A})} V & \longrightarrow & Q(\bar{R}) \otimes_R \widetilde{M} \\
\Big\downarrow{\scriptstyle 1\otimes f} & & \Big\downarrow{\scriptstyle 1\otimes F} \\
Q(\bar{R}) \otimes_{Q(\bar{A})} V' & \longrightarrow & Q(\bar{R}) \otimes_R \widetilde{M}'
\end{array}$$

is commutative in the category of $Q(\bar{R})$–modules.

The definition is motivated by the following theorem.

Theorem 7. *Let k be an algebraically closed field and $(A, \mathfrak{m})$ be a reduced complete non-isolated Cohen-Macaulay surface singularity. Then the functor $\mathbb{F} : \mathsf{CM}(A) \to \mathsf{Tri}(A)$, mapping a maximal Cohen-Macaulay module M to the triple $\big(R \boxtimes_A M, Q(\bar{A}) \otimes_A M, \theta_M\big)$, is an equivalence of categories.*

Moreover, the full subcategory $\mathsf{CM}^{\mathsf{lf}}(A)$ consisting of the maximal Cohen-Macaulay modules which are locally free on the punctured spectrum of A, is equivalent to the full subcategory $\mathsf{Tri}^{\mathsf{lf}}(A)$ consisting of those triples $(\widetilde{M}, V, \theta)$ for which the morphism θ is an isomorphism.

We illustrate our approach by the following example. Let $A = k[\![x, y, z]\!]/(x^3 + y^2 - xyz)$. Its normalization R is $k[\![u, v]\!]$, where $u = \frac{y}{x}$ and $v = \frac{xz - y}{x}$. Next, $I = (x, y)A = (uv)R$ is the conductor ideal, hence $\bar{A} = A/I = k[\![z]\!]$, whereas $\bar{R} = k[\![u, v]\!]$. The canonical map $\bar{A} \to \bar{R}$ maps z to $u + v$. We have: $Q(\bar{A}) \cong k((z))$ and $Q(\bar{R}) \cong k((u)) \times k((v))$.

Let $(\widetilde{M}, V, \theta)$ be an object of $\mathsf{Tri}(A)$. Since R is regular, we have: $M \cong R^m$ for some integer $m \geq 1$. Next, V is just a vector space over the field $k((z))$, hence $V \cong k((z))^n$ for some $n \geq 1$. Hence, the gluing map θ is given by a pair of matrices of full row rank and the same size:

$$\theta = (\theta_u, \theta_v) \in \mathsf{Mat}_{m\times n}\big(k((u))\big) \times \mathsf{Mat}_{m\times n}\big(k((v))\big).$$

The transformation rule is

$$(\theta_u, \theta_v) \mapsto (S_1^{-1}\theta_u T, S_2^{-1}\theta_v T), \tag{1}$$

where $S_1 \in \mathsf{GL}(m, k[\![u]\!])$, $S_2 \in \mathsf{GL}(m, k[\![v]\!])$ are such that $S_1(0) = S_2(0)$ and $T \in \mathsf{GL}(n, k((z)))$. The canonical form of an indecomposable pair (θ_u, θ_v) is one of the following.

Continuous series. Let $l, t \geq 1$ be positive integers, $\omega = \big((m_1, n_1), \dots, (m_t, n_t)\big) \in (\mathbb{Z}_+^2)^t$ be a *"non-periodic sequence"* such that $\min(m_i, n_i) = 1$ for all $1 \leq i \leq t$ and $\lambda \in k^*$. Then we have the corresponding canonical form:

$$\theta_u = \begin{pmatrix} u^{m_1} I_l & 0 & 0 & \dots & 0 \\ 0 & u^{m_2} I_l & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & u^{m_{t-1}} I_l & 0 \\ 0 & 0 & \dots & 0 & u^{m_t} I_l \end{pmatrix} \quad \theta_v = \begin{pmatrix} 0 & v^{n_2} I_l & 0 & \dots & 0 \\ 0 & 0 & v^{n_3} I_l & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v^{n_t} I_l \\ v^{n_1} J_l(\lambda) & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2}$$

The triple $\big(R^{tl}, k((z))^{tl}, (\theta_u, \theta_v)\big)$ defines an indecomposable maximal Cohen-Macaulay module $M(\omega, l, \lambda)$, which is locally free of rank tl on the punctured spectrum of A. Moreover, any indecomposable maximal Cohen-Macaulay module which is locally free on the punctured spectrum, is described by a triple of the above form.

Discrete series. Indecomposable Cohen-Macaulay A–modules which are not locally free on the punctured spectrum, are described by a single discrete parameter $\omega = \big(m_0, (m_1, n_1), \dots, (m_t, n_t), n_{t+1}\big)$, where $m_0 = n_{t+1} = 1$ and $\min(m_i, n_i) = 1$ for all $1 \leq i \leq t$. Consider the matrices $\theta_u(\omega)$ and $\theta_v(\omega)$ of the size $(t+1) \times (t+2)$ defined as follows:

$$\theta_u = \begin{pmatrix} u^{m_0} & 0 & 0 & \dots & 0 \\ 0 & u^{m_1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & u^{m_t} & 0 \end{pmatrix} \quad \text{and} \quad \theta_v = \begin{pmatrix} 0 & v^{n_1} & 0 & \dots & 0 \\ 0 & 0 & v^{n_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & v^{n_{t+1}} \end{pmatrix}. \tag{3}$$

In the case $t = 0$, we set $\theta_u = (1\ 0)$ and $\theta_v = (0\ 1)$. The triple $\big(R^{t+1}, k((z))^{t+1}, (\theta_u, \theta_v)\big)$ defines an indecomposable Cohen-Macaulay A–module $N(\omega)$ of rank $t+1$, which is not locally free on the punctured spectrum. Moreover, any indecomposable maximal Cohen-Macaulay A–module, which is not locally free on the punctured spectrum, is isomorphic to some $N(\omega)$.

Our classification allows to deduce the following result.

Proposition 8. *For the ring $A = k[\![x, y, z]\!]/(x^3 + y^2 - xyz)$ the the category of maximal Cohen-Macualy modules over A is representation tame. Moreover, the maximal Cohen-Macaulay A–modules of rank one are the following.*

1. *There exists exactly one such module $N = N\big(1(,)1\big)$, which is not locally free on the punctured spectrum. We have the following A-module isomorphisms: $N \cong I \cong R$.*
2. *The rank one maximal Cohen-Macaulay A–modules, which are locally free on the punctured spectrum, have the following shape:*

$$M\big((1, m), \lambda\big) \cong I_{m,\lambda} \quad \textit{and} \quad M\big((m, 1), 1, \lambda\big) \cong J_{m,\lambda},$$

where $m \geq 1$, $\lambda \in k^*$, $I_{m,\lambda} = \langle x^{m+1}, yx^{m-1} + \lambda(xz - y)^m \rangle \subset A$ and $J_{m,\lambda} = \langle x^{m+1}, y^m + \lambda x^{m-1}(xz - y) \rangle \subset A$.

Using the technique of matrix problems and Theorem 7 one can show the following result.

Theorem 9. *For any $t \in \mathbb{Z}_{>0}$, let $R = R_t := k[\![x_1, y_1]\!] \times k[\![x_2, y_2]\!] \times \cdots \times k[\![x_t, y_t]\!]$ and*

$$A = A_t := \big\{(p_1, p_2, \dots, p_t) \,\big|\, p_i(0, z) = p_{i+1}(z, 0) \textit{ for all } 0 \leq i \leq t-1\big\} \subset R.$$

Then A a non-isolated Cohen-Macaulay surface singularity of discrete Cohen-Macaulay representation type. Moreover, any indecomposable maximal Cohen-Macaulay modules is isomorphic to an ideal in A.

Remark 10. Let $A = A_t$ be as in Theorem 9. Then the affine scheme $\mathsf{Spec}(A)$ has t irreducible components. On the other hand, the schemes $\mathsf{Spec}(A_\infty^G)$ and $\mathsf{Spec}(D_\infty^H)$ have at most two irreducible components. Hence, for $t \geq 3$ the ring A_t can not be isomorphic to the ring of invariants of the ring A_∞ or D_∞ with respect to a finite group action. This provides a negative answer on Schreyer's question.

Theorem 11. *Let $(A, \mathfrak{m})$ be a degenerate cusp in the sense of* [10]. *Then the category* $\mathsf{CM}(A)$ *is representation tame.*

Remark 12. Degenerate cusps which are complete intersections are the following rings

(1) $k[\![x,y,z]\!]/(x^p + y^q + z^r - xyz)$, where $\max(p,q,r) = \infty$,
(2) $k[\![x,y,u,v]\!]/(x^p + y^q - uv, u^r + v^t - xy)$, where $\max(p,q,r,t) = \infty$.

Corollary 13. *The non-reduced curve singularity $A = k[\![x,y]\!]/(xy)^2$ has tame Cohen-Macaulay representation type.*

Proof. By Knörrer's correspondence [8], the category $\mathsf{CM}(A)$ has the same representation type as $\mathsf{CM}(B)$, where B is the surface singularity $k[\![x,y,z]\!]/(x^2y^2 + z^2)$. It remains to note that

$$k[\![x,y,z]\!]/(x^2y^2 + z^2) = k[\![x,y,z]\!]/(xy + z)^2 - 2xyz \cong k[\![u,v,w]\!]/(u^2 - uvw)$$

is a degenerate cusp (it is a $T_{2\infty\infty\infty}$-singularity), which is representation tame by Theorem 11. Hence, $\mathsf{CM}(A)$ is representation tame, too. □

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KOSZUL HOMOLOGY AND SYZYGIES OF VERONESE SUBALGEBRAS

TIM RÖMER

The results presented in the following are joint work with Winfried Bruns and Aldo Conca and will appear in [2].

Let K be a field. Green and Lazarsfeld [5, 6] defined the property N_p for a graded ring as follows. A finitely generated $\mathbb{N}$-graded K-algebra $R = \bigoplus_{i \in \mathbb{N}} R_i$ satisfies property N_0 if R is generated in degree 1. In the following we will always assume that this is the case. Then R can be presented as a quotient $R = S/I$ where S is a standard graded polynomial ring and $I \subset S$ is a graded ideal. Let $\beta^S_{i,j}(R) = \dim_K \operatorname{Tor}^S_i(R, K)$ be the graded Betti numbers of R as an S-module. We are interested in the following property:

Definition 1.1. The K-algebra R satisfies *property* N_p for some $p > 0$ if $\beta^S_{i,j}(R) = 0$ for $j > i + 1$ and $1 \leq i \leq p$.

Example 1.2.

(1) The property N_1 is equivalent to say that R is defined by quadrics, i.e. if we write $R = S/I$ for a graded ideal I containing no linear forms, then I is generated by homogeneous polynomials of degree 2.

(2) The property N_p for $p > 1$ means that $R = S/I$ is defined by quadrics and that the minimal graded free resolutions of R is of the form

$$\cdots \to F_{p+1} \to S(-p+1)^{\beta_p} \to \cdots \to S(-2)^{\beta_1} \to S \to R \to 0.$$

If R satisfies N_p for some $p > 1$, then R satisfies $N_{p'}$ for every $1 \leq p' \leq p$. This motivates the following definition.

Definition 1.3. Let R be a standard graded K-algebra. We define the *Green-Lazarsfeld index* $\operatorname{index}(R)$ of R to be

$$\operatorname{index}(R) = \sup\{p \geq 0 : R \text{ satisfies } N_p\}.$$

Determining $\operatorname{index}(R)$ in general seems to be a difficult problem. Here we focus on the case of a Veronese subring

$$R^{(c)} = \bigoplus_{i \in \mathbb{N}} R_{ic}, d \geq 1$$

of a standard graded K-algebra R. Observe that we consider $R^{(c)}$ as a standard graded K-algebra with homogeneous component of degree i equal to R_{ic}. Already the case of a polynomial ring $S = K[X_1, \dots, X_n]$ is interesting.

Example 1.4. If $n \leq 2$ or $c \leq 2$, then $S^{(c)}$ is a determinantal ring. In this case the minimal free resolution of $S^{(c)}$ is well-known and one is able to determine the Green-Lazarsfeld index.

At first assume that $n = 2$. Then the minimal free graded resolution of $S^{(c)}$ is given by the Eagon-Northcott complex which implies that $\operatorname{index}(S^{(c)}) = \infty$.

Next we consider the case $c = 2$. The resolution of $S^{(2)}$ in characteristic 0 is known by work of Jozefiak, Pragacz and Weyman [7]. We get that $\operatorname{index}(S^{(2)}) = 5$ if $n > 3$ and $\operatorname{index}(S^{(2)}) = \infty$ if $n \leq 3$.

For $n \leq 6$ we get from results of Andersen [1] that $\operatorname{index}(S^{(2)})$ is independent on $\operatorname{char} K$. For $n > 6$ and $\operatorname{char} K = 5$ she showed that $\operatorname{index}(S^{(2)}) = 4$.

For $n > 2$ and $c > 2$ the following is known:

$$c \leq \operatorname{index}(S^{(c)}) \leq 3c - 3. \tag{1}$$

The lower bound follows from results of Green [4] for any c and n. Ottaviani and Paoletti [8] proved the upper bound in characteristic 0. They also showed that $\operatorname{index}(S^{(c)}) = 3c - 3$ for $n = 3$. Motivated by these results they conjectured:

Conjecture 1.5. We have

$$\operatorname{index}(S^{(c)}) = 3c - 3 \text{ for every } n \geq 3 \text{ and } c \geq 3.$$

For $n = 4$ and $c = 3$ the conjecture is true by [8, Lemma 3.3]. See also Eisenbud, Green, Hulek and Popescu [3] for related results. Rubei [9] proved that $\operatorname{index}(S^{(3)}) \geq 4$ if $\operatorname{char} K = 0$.

One way to attack this problem is to study the Koszul complex associated to the c-th power of the maximal ideal of S which is closely related to the problems described so far. Let $\mathfrak{m}$ the maximal graded ideal of S. Let $K(\mathfrak{m}^c)$ denote the Koszul complex associated to $\mathfrak{m}^c$, $Z_t(\mathfrak{m}^c)$ the module of cycles of homological degree t and $H_t(\mathfrak{m}^c)$ the corresponding homology module. Let T be the symmetric algebra on vector space S_c. Then it is easy to see that:

Lemma1.6. *For $i \in \mathbb{N}$, $j \in \mathbb{Z}$ and $0 \leq k < c$ we have*

$$\beta^T_{i,j}(S^{(c)}) = \dim_K H_i(\mathfrak{m}^c)_{jc}.$$

Thus studying the N_p-property of $S^{(c)}$ is equivalent to study vanishing theorems of $H_i(\mathfrak{m}^c, S)$. Considering $Z_t(\mathfrak{m}^c)$ carefully allowed us to prove a result of Green [4, Theorem 2.2]:

Theorem1.7. *We have:*

$$H_i(\mathfrak{m}^c)_j = 0 \text{ for every } j \geq ic + i + c.$$

In particular, on gets that $S^{(c)}$ satisfies N_c.

Moreover, a generalizations of this theorem is proved in [2] since we can give upper bounds on the degrees of a minimal system of generators of $Z_t(\mathfrak{m}^c)$. Using the last two results and some further arguments allows us to give a proof of one of our main results:

Theorem1.8. *([2]) We have:*

(1) $c + 1 \leq \operatorname{index}(S^{(c)})$ *if* $\operatorname{char} K = 0$ *or* $> c + 1$.

(2) *If $R = S/I$ for a graded ideal $I \subset S$, then*

$$\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)}) \text{ for every } c \geq \operatorname{rate}_S(R).$$

In particular, if R is Koszul then $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$ *for every $c \geq 2$,*

Using an Avramov-Golod type of duality we can show Ottaviani and Paoletti's upper bound $\operatorname{index}(S^{(c)}) \leq 3c - 3$ in arbitrary characteristic. It is also not difficult to prove that for $n = 3$ one has $\operatorname{index}(S^{(c)}) = 3c-3$ independently of the characteristic. We refer to [2] for details and more related results.

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GENERIC TROPICAL VARIETIES

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The results presented in the following are joint work with Kirsten Schmitz; see [10] and [11].

Tropical geometry is a relatively new area of mathematics and has received a lot of attention in the recent years; see, e.g., [3, 5, 6, 7, 8, 12]. One of the possible approaches to tropical geometry is to associate a combinatorial object to a projective algebraic variety which provides a very useful method for studying problems in algebraic geometry and related areas. For this let K be an algebraically closed field of characteristic 0 with a non-archimedean valuation $v\colon K \to \mathbb{R} \cup \{\infty\}$ which might be trivial. We think of the following two cases:

Example 1.1.

(i) The field of Puiseux-series

$$\mathbb{C}((t^{\mathbb{R}})) = \{\sum c_\alpha t^\alpha : c_\alpha \in \mathbb{R} \text{ and } \{\alpha : c_\alpha \neq 0\} \text{ is well-ordered}\}$$

has a non-archimedean valuation v given by

$$v(\sum c_\alpha t^\alpha) = \min\{\alpha : c_\alpha \neq 0\}.$$

(ii) For $\mathbb{C}$ the trivial valuation is also non-archimedean.

Assume for a moment that v is surjective and K is complete like in the case of the field of Puiseux-series. The tropical variety $T(X)$ of an algebraic variety X is the real-valued image of X under the valuation map; see, e.g., [4, 9, 12]. More precisely:

Definition 1.2. Let I be a graded ideal in $K[x_1, \dots, x_n]$. The tropical variety of I is

$$T(I) = \{v(p) : p \in V(I) \cap (K^*)^n\} \subseteq \mathbb{R}^n.$$

One is for example interested to understand $V(I)$ via the approximation $T(I)$. As said above, tropical geometry is a very active area of research, but at the moment there are not too many results related to commutative algebra.

In certain settings $T(X)$ has the structure of a polyhedral complex as was observed, e.g., in [1, 9] and there is a practical characterization in terms of initial ideals given in [12] and [4, Theorem 4.2]. For this we consider $S = K[x_1, \dots, x_n]$. Let $f = \sum_\mu a_\mu x^\mu \in S$, $\omega \in \mathbb{R}^n$. We set

$$\mathrm{in}_\omega(f) = \sum_{v(a_\mu)+\omega\cdot\mu=c} a_\mu x^\mu$$

where $c = \min\{v(a_\mu) + \omega \cdot \mu : a_\mu \neq 0\}$. The following theorem follows from results of Speyer–Sturmfels [12] and Draisma [4, Theorem 4.2].

Theorem 1.3. *Let L/K be a field extension of valued fields such that L is algebraically closed, $v(L) = \mathbb{R} \cup \{\infty\}$ and L is complete. Let $I \subset K[x_1, \dots, x_n]$ be a graded ideal. Then:*

$$\begin{aligned} & T(IL[x_1, \dots, x_n]) \\ = \ & \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(f) \text{ is not a monomial for every } f \in I\}. \end{aligned}$$

From now on we assume that K has a trivial valuation (e.g. $\mathbb{C}$). For a graded ideal $I \subset S$ we define the tropical variety of I to be

$$T(I) = \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(f) \text{ is not a monomial for every } f \in I\}.$$

This is also called the *constant coefficient case.*

From the view point of commutative algebra one might ask what we can say about algebraic properties of I knowing $T(I)$.

Example 1.4.

(i) If $P \subset S$ is a graded monomial free prime ideal, then a result of Bieri and Groves implies that $\dim S/P = \dim T(P)$.

(ii) This can easily be extended since

$$T(I) = T(\sqrt{I}) = T(\bigcap_{I\subseteq P} P) = \bigcup_{I\subseteq P} T(P).$$

It would be interesting to have more results in this direction, but not too much is known.

A standard construction in commutative algebra is the one of a Gröbner fan. For this let $I \subset S$ be a graded ideal and $\omega \in \mathbb{R}^n$. We set

$$\mathrm{in}_\omega(I) = (\mathrm{in}_\omega(f) : f \in I).$$

Now let $C[\omega]$ be the closure of

$$\{\omega' \in \mathbb{R}^n : \mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)\}.$$

Then it is known that $C[\omega]$ is a finitely generated cone in $\mathbb{R}^n$ and the set of all such cones is a complete fan in $\mathbb{R}^n$ which is called the *Gröbner fan $GF(I)$* of I.

Bogart, Jensen, Speyer, Sturmfels and Thomas [2] observed that $T(I)$ has a natural fan structure. More precisely:

Theorem 1.5. *$T(I)$ is the subfan of $GF(I)$ consisting of all cones C such that $\mathrm{in}_C(I)$ contains no monomial.*

Usually one is interested in the maximal cones of $GF(I)$ which correspond to monomial initial ideals of I. Somehow the construction of $T(I)$ leads to the "opposite part" to these ideas. The conclusion is that one can study $T(I)$ in the constant coefficient case as a subfan of a well-known object in commutative algebra.

Remark 1.6. We have that

$$T((x_1)) = \emptyset \text{ and } T((x_1 + x_2)) = \mathbb{R}(1,1) \neq \emptyset.$$

Algebraically the ideals (x_1) and $(x_1 + x_2)$ are the same and differ only by a coordinate transformation.

Thus we see that tropical varieties depend on the chosen coordinates. One can avoid this as we will see. Like considering generic initial ideals it is an interesting problem to consider tropical varieties generically. For this let $g = (g_{ij}) \in GL_n(K)$ and consider the K-algebra homomorphism on S induced by

$$x_j \mapsto \sum_{i=1}^{n} g_{ij} x_i.$$

The question is to understand $T(gI)$ for $g \in GL_n(K)$. (To avoid trivial cases we always assume $0 < \dim S/I < n$.) Before studying $T(gI)$ one can start to consider the Gröbner fan $GF(gI)$. Then we can prove:

Theorem 1.7. *There exists a Zariski-open set $\emptyset \neq U \subset GL_n(K)$ and a fan $\mathcal{F}$ such that*

$$GF(gI) = \mathcal{F} \text{ as a fan for every } g \in U.$$

We write $gGF(I) = \mathcal{F}$ and call this the generic Gröbner fan *of I.*

So one already knows that $T(gI)$, $g \in U$ can only be one of the finitely many subfans of $gGF(I)$. The difficulty is now to show that $T(gI)$ is constant on a non-empty Zariski-open set.

One of our main result in [10] is:

Theorem 1.8. *Let $I \subset S$ be a graded ideal. Then there exists Zariski-open set $\emptyset \neq U \subset GL_n(K)$ and a subfan $\mathcal{F}$ of $gGF(I)$ such that*

$$T(gI) = \mathcal{F} \text{ as a fan for every } g \in U.$$

We write $gT(I) = \mathcal{F}$ and call this the generic tropical variety *of I.*

Note that this result also applies to ideals which contain monomials.

Surprisingly (at least for us) $gT(I)$ as a set does not contain more information than we know from the result of Bieri and Groves. More precisely, for $A \subseteq [n]$ let

$$C_A = \{\omega \in \mathbb{R}^n : \omega_i = \min_k\{\omega_k\} \text{ for all } i \in A\}$$

and consider the induced fan $\mathcal{W}_n$ in $\mathbb{R}^n$. Let $\mathcal{W}_n^m$ be the m-skeleton of $\mathcal{W}_n$. We can prove that:

Theorem 1.9. *([10]) Let $I \subset S$ be a graded ideal with $0 < \dim S/I = m < n$. Then*

$$gT(I) = \mathcal{W}_n^m \text{ as a set.}$$

For example $gT(f)$ for $f \in S$ is independent of the degree of f, or knowing $gT(I)$ one can not decide whether I is prime or not. Thus $gT(I)$ as a set is not too interesting and one has to add some information. It is easy to see that as fans: $gT(I) = \mathcal{W}_n^1$ if $\dim S/I = 1$ and $gT(I) = \mathcal{W}_n^2$ if $\dim S/I = 2$. Interestingly we can prove:

Theorem 1.10. *([11]) Let $I \subset S$ be a graded ideal with $\dim I = m < n$ and $0 < \operatorname{depth} I = t$. Then the following statements are equivalent:*

(i) $gT(I) = \mathcal{W}_n^m$ *as a fan;*
(ii) $t \geq m - 1$.

Thus we can decide whether I is Cohen-Macaulay or almost Cohen-Macaulay knowing $gT(I)$ as a fan (provided $\operatorname{depth} I > 0$). The proof of the result shows in particular, that $\mathcal{W}_n^m$ is the coarsest fan structure on the set $gT(I)$ and $gT(I)$ always refines this structure. If $\operatorname{depth} I < \dim I - 1$ it is still possible to get some information on $\operatorname{depth} I$, but this is a little bit more technical.

There exists a definition of multiplicities in tropical geometry. For this let C be a maximal cone of $T(I)$. Then the *multiplicity* $m(C)$ of C is defined as

$$\sum \text{lenght}((S/\operatorname{in}_C(I))_P)$$

where the sum is taken over all minimal primes of $\operatorname{in}_C(I)$ of maximal dimension and which do not contain a monomial. We can prove:

Theorem 1.11. *([11]) Let $I \subset S$ be a graded ideal with $\dim S/I = m$ and let C be a maximal cone of $gT(I)$. Then*

$$m(C) = m(I)$$

where $m(I)$ is the multiplicity of I.

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