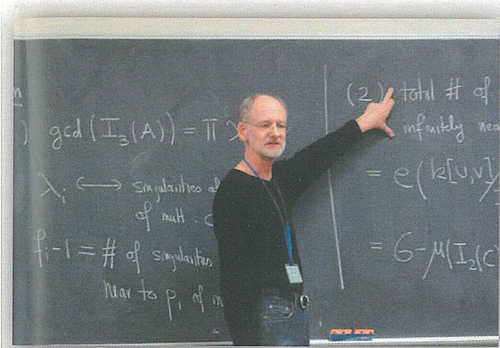
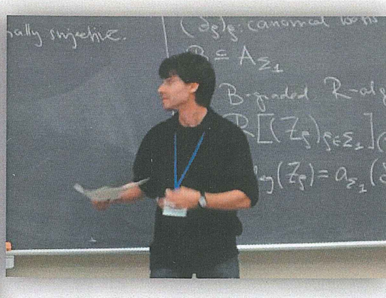
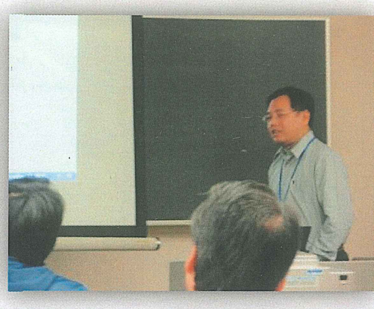
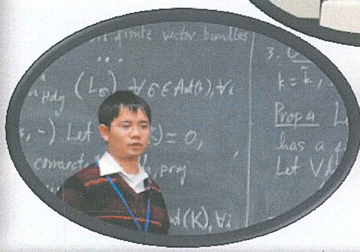
The 32nd Symposium
The 6th Japan-Vietnam Joint Seminar
on Commutative Algebra

Proceedings

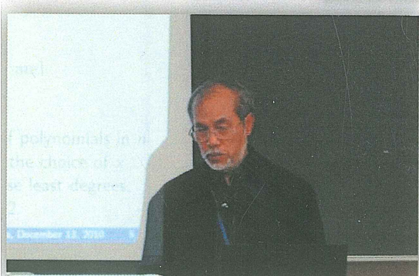
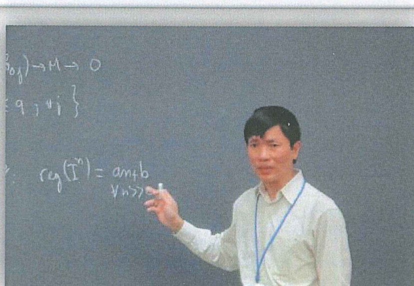
December 11-16, 2010

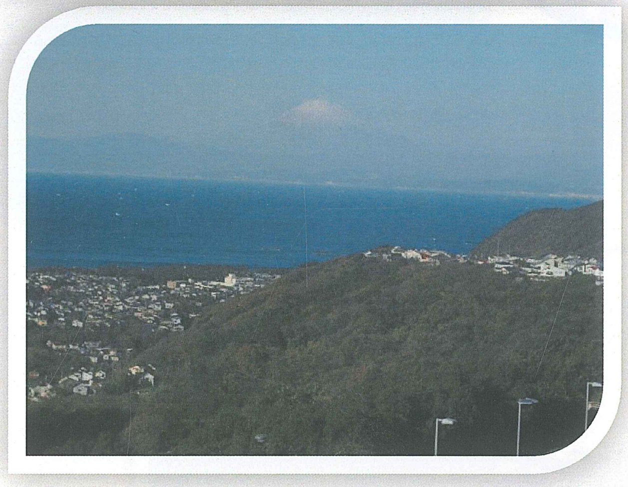
International Productivity Center, Hayama



MIMS
Meiji Institute for Advanced Study of Mathematical Sciences

Proceedings

of

The 32–nd Symposium
The 6–th Japan-Vietnam Joint Seminar

on

Commutative Algebra

IPC (Hayama, Japan)

December 11–16, 2010

Preface

This is the proceedings of the two conferences (the 32-nd Symposium on Commutative Algebra in Japan and the 6-th Japan-Vietnam Joint Seminar on Commutative Algebra), which were jointly held at IPC (International Productivity Center) located in Hayama of Japan during December 11–16, 2010.

There were more than 65 participants, 11 persons coming from abroad (B. Ulrich, J. Hong, H. L. Dao from USA, N. V. Trung, N. T. Cuong, L. T. Hoa, D. T. Cuong, L. T. Nhan, F. Rohrer from Vietnam, H.-J. Chiang-Hsieh from Taiwan, and M. Mandal from India). All of them were financially supported by the following grants of Meiji University (S. Goto and K. Kurano) and Grant-in-Aid for Scientific Researches in Japan, Ministry of Education (Yuji Yoshino, S. Goto, and K. Watanabe):

- 明治大学 GP 大学院教育改革プログラム　（代表者　後藤四郎）
- 明治大学大学院研究科共同研究「永田予想とイデアルの symbolic power の研究」（研究代表者　蔵野和彦）
- 平成 22 年度科学研究費基盤研究 (B) 研究課題番号 21340008,「三角圏の研究とその Cohen-Macaulay 加群への応用」（研究代表者　吉野雄二）
- 平成 22 年度科学研究費基盤研究 (C) 研究課題番号 22540054,「可換環論：非コーエンマコーレイ環解析の視点から」（研究代表者　後藤四郎）
- 平成 22 年度科学研究費基盤研究 (C) 研究課題番号 20540050,「特異点論における正標数の手法」（研究代表者　渡辺敬一）

On behalf of the organizers I would like to express my hearty thanks for their generous support.

I would like to thank all the speakers for their excellent lectures and thank also all the participants for their exciting discussions and comments during the conferences, which I profoundly enjoyed. I heartily wish these conferences are a very good opportunity for fruitful developments in our mathematical researches, and in our friendship as well, despite of the huge distance among our home lands.

We are now organizing so that the 33-rd Symposium on Commutative Algebra in Japan will be held in November, 2011 in Japan and the 7-th Japan-Vietnam Joint Seminar on Commutative Algebra in December, 2011 in Vietnam. I look forward to seeing all the people whom I met at Hayama also the next time. January, 2011 Shiro Goto

Host Institution
Meiji Institute for Advanced Study of Mathematical Sciences (**MIMS**)
Graduate School of Science and Technology, Meiji University
Institute of Mathematics, Vietnamese Academy of Science and Technology

Organizers
Shiro Goto, Kei-ichi Watanabe, Koji Nishida, Kazuhiko Kurano, Ngo Viet Trung, Nguyen Tu Cuong, Le Tuan Hoa

Contents

The 32nd Symposium on Commutative Algebra in Japan
The 6th Japan-Vietnam Joint Seminar on Commutative Algebra

Time Table

11 (Sat)	12 (Sun)	13 (Mon)	14 (Tue)	15 (Wed)	16 (Thu)
	9:00–9:45 S. Goto, K.Ozeki	9:00–9:45 D. T. Cuong	9:00–9:45 M. Hashimoto	9:00–9:45 S. Kuroda	9:00–9:45 K. Kurano
	10:00–10:30 H. L. Truong	10:00–10:30 T. Shibuta	10:00–10:30 H.-J. Chiang-Hsieh	10:00–10:20 K. Kawasaki	10:00–10:30 A. Sannai
	10:40–11:10 M. Miyazaki	10:40–11:10 K. Kimura	10:40–11:10 F. Rohrer	10:30–11:00 N. Terai, K. Yoshida	11:00–11:45 K. Yoshida, K. Watanabe
	11:25–12:15 B. Ulrich	11:25–12:10 J. Hong	11:20–12:10 B. Ulrich	11:20–12:05 N. V. Trung	
14:55–15:00 *Opening*	14:00–14:45 L. T. Hoa	14:00–14:45 N. T. Cuong		13:45–14:30 L. T. Nhan	
15:00–15:45 Y. Yoshino	15:00–15:30 K. Shimomoto	15:00–15:30 F. Hayasaka		14:45–15:15 K. Kato	
16:00–16:20 N. Hiramatsu	16:00–16:20 K. Matsuda	16:00–16:20 T. Yoshizawa		15:45–16:15 K. Iima	
16:30–17:00 K. Nishida	16:30–16:50 H. Nari	16:30–16:50 M. Ohtani		16:25–16:55 T. Araya	
17:15–18:00 D. H. Long	17:10–17:40 M. Mousumi	17:05–17:50 K. Yanagawa		17:05–17:50 R. Takahashi	
		19:00–21:00 *Banquet*			

The 32nd Symposium on Commutative Algebra in Japan
The 6th Japan-Vietnam Joint Seminar on Commutative Algebra

Schedule of Scientific Program

December 11 (Sat)

14:55 ~ 15:00	Opening
15:00 ~ 15:45	Yuji Yoshino (Okayama University) Stable analogue of degenerations of Cohen-Macaulay modules
16:00 ~ 16:20	Naoya Hiramatsu (Okayama University) Examples of degenerations of Cohen-Macaulay modules
16:30 ~ 17:00	Koji Nishida (Chiba University) Noetherian and non-Noetherian symbolic Rees algebras
17:15 ~ 18:00	Dao Hai Long (University of Kansas) The homological conjectures revisited

December 12 (Sun)

9:00 ~ 9:45	Shiro Goto (Meiji University), Kazuho Ozeki (Meiji University) Uniform bounds for Hilbert coefficients of parameters
10:00 ~ 10:30	Hoang Le Truong (Institute of Mathematics, Hanoi) Uniform bounds and Hilbert coefficients for sequentially generalized Cohen-Macaulay modules
10:40 ~ 11:10	Mitsuhiro Miyazaki (Kyoto University of Education) A criterion of a Hibi ring to be of type 2
11:25 ~ 12:15	Bernd Ulrich (Purdue University) Multiplicities, integral dependence, and equisingularity
14:00 ~ 14:45	Le Tuan Hoa (Institute of Mathematics, Hanoi) Non-linear behavior of Castelnuovo-Mumford regularity
15:00 ~ 15:30	Kazuma Shimomoto (Meiji University) Local Bertini theorem with applications to Iwasawa Main Conjecture
16:00 ~ 16:20	Kazunori Matsuda (Nagoya University) Diagonal F-thresholds of Hibi rings
16:30 ~ 16:50	Hirokatsu Nari (Nihon University) Pseudo symmetric semigroups generated by 3 elements

17:10 ~ 17:40 Mousumi Mandal (Indian Institute of Technology)
The positivity of the first coefficients of normal Hilbert polynomials

December 13 (Mon)

9:00 ~ 9:45 Doan Trung Cuong (Institute of Mathematics, Hanoi)
Hodge cohomology of some classes of Nori finite vector bundles

10:00 ~ 10:30 Takafumi Shibuta (Rikkyo University/JST CREST)
Irreducibility criterion for algebroid curves

10:40 ~ 11:10 Kyouko Kimura (Shizuoka University)
On the Betti numbers of edge ideals of chordal graphs

11:25 ~ 12:10 Jooyoun Hong (Southern Connecticut State University)
The first Hilbert coefficients and Euler characteristics

14:00 ~ 14:45 Nguyen Tu Cuong (Institute of Mathematics, Hanoi)
On the limit closure of a sequence of elements in local rings

15:00 ~ 15:30 Futoshi Hayasaka (Kagoshima National College of Technology)
Asymptotic behavior of the grade associated to multigraded modules

16:00 ~ 16:20 Takeshi Yoshizawa (Okayama University)
On Gorenstein injectivity of top local cohomology modules

16:30 ~ 16:50 Masahiro Ohtani (Nagoya University)
Binomial edge ideals of complete r-partite graphs

17:05 ~ 17:50 Kohji Yanagawa (Kansai University)
Sliding functor and polarization functor for multigraded modules

19:00 ~ 21:00 Banquet

December 14 (Tue)

9:00 ~ 9:45 Mitsuyasu Hashimoto (Nagoya University)
Equivariant total ring of fractions and factoriality of rings generated by semiinvariants

10:00 ~ 10:30 Hung-Jen Chiang-Hsieh (National Chung Cheng University)
Some properties of the matroidal ideals

10:40 ~ 11:10 Fred Rohrer (Institute of Mathematics, Hanoi)
On Toric Schemes

11:20 ~ 12:10 Bernd Ulrich (Purdue University)
Rees algebras and singularities

December 15 (Wed)

9:00 ~ 9:45	Shigeru Kuroda (Tokyo Metropolitan University) Local slice construction and wild automorphisms
10:00 ~ 10:20	Ken-ichiroh Kawasaki (Nara University of Education) On a characterization of cofinite complexes
10:30 ~ 11:00	Naoki Terai (Saga University), Ken-ichi Yoshida (Nagoya University) The second power of Stanley-Reisner ideals
11:20 ~ 12:05	Ngo Viet Trung (Institute of Mathematics, Hanoi) Cohen-Macaulayness of powers of Stanley-Reisner ideals
13:45 ~ 14:30	Le Thanh Nhan (Thai Nguyen University) On Cohen-Macaulayfication of certain local rings
14:45 ~ 15:15	Kiriko Kato (Osaka Prefecture University) Symmetric Auslander and Bass categories
15:45 ~ 16:15	Kei-ichiro Iima (Nara National College of Technology) On modules of finite projective dimension with respect to a semidualizing module
16:25 ~ 16:55	Tokuji Araya (Nara University of Education) A homological dimension over AB ring
17:05 ~ 17:50	Ryo Takahashi (Shinshu University) Classifying resolving subcategories over a Cohen-Macaulay local ring

December 16 (Thu)

9:00 ~ 9:45	Kazuhiko Kurano (Meiji University) Canonical modules of multi-section rings
10:00 ~ 10:30	Akiyoshi Sannai (University of Tokyo) Galois extension and maps on local cohomology
11:00 ~ 11:45	Ken-ichi Yoshida (Nagoya University), Kei-ichi Watanabe (Nihon University) A positive characteristic approach to Wang's theorem

Stable analogue of degenerations of Cohen-Macaulay modules

Yuji Yoshino
Okayama University
e-mail: yoshino@math.okayama-u.ac.jp

This lecture is based on my recent papers [10, 3] and will continue to the lecture by Hiramatsu in this conference.

1 Degenerations

Let us recall the definition of degeneration of finitely generated modules over a noetherian algebra, which is given in our previous paper [8, Definition 2.1].

Let R be an associative k-algebra where k is any field. We take a discrete valuation ring (V, tV, k) which is a k-algebra and t is a prime element. We denote by K the quotient field of V. We denote by $\text{mod}(R)$ the category of all finitely generated left R-modules and R-homomorphisms. Then we have the natural functors

$$\text{mod}(R) \xleftarrow{\ r\ } \text{mod}(R \otimes_k V) \xrightarrow{\ \ell\ } \text{mod}(R \otimes_k K),$$

where $r = - \otimes_V V/tV$ and $\ell = - \otimes_V K$.

Definition 1.1 For modules $M, N \in \text{mod}(R)$, we say that M degenerates to N if there is a module $Q \in \text{mod}(R \otimes_k V)$ that is V-flat such that $\ell(Q) \cong M \otimes_k K$ and $r(Q) \cong N$.

The module Q, regarded as a bimodule ${}_RQ_V$, is a flat family of R-modules with parameter in V. At the closed point in the parameter space $\text{Spec}V$, the fiber of Q is N, which is a meaning of the isomorphism $r(Q) \cong N$. On the other hand, the isomorphism $\ell(Q) \cong M \otimes_k K$ means that the generic fiber of Q is essentially given by M.

In the previous paper([8, Theorem 2.2]) we have proved the following theorem.

Theorem 1.2 *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*

(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

By virtue of this theorem together with a theorem of Zwara [12, Theorem 1], we see that if R is a finite-dimensional algebra over k, then our definition of degeneration agrees with the classical (geometric) definition of degenerations using module varieties of R-module structures. We also remark from this theorem that we can always take $k[t]_{(t)}$ as V in Definition 1.1. (See [8, Corollary 2.4].)

In the rest of the paper we mainly treat the case when R is a commutative ring.

The following lemma is easily observed. See [8, Remark 2.5].

Lemma 1.3 *If there is an exact sequence in* $\mathrm{mod}(R)$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

then M degenerates to $L \bigcirc N$.

Such a degeneration given as in the lemma will be called a degeneration by an extension. However there is a degeneration which is not a degeneration by an extension. We give one of the easiest examples.

Example 1.4 Let $R = k[[x, y]]/(x^2)$, where k is a field. In this case, a pair of matrices

$$(\varphi, \psi) = \left(\begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)$$

over $k[[x, y]]$ is a matrix factorization of x^2, giving a Cohen-Macaulay R-module N that is isomorphic to an ideal $(x, y^2)R$. Thus there is a periodic free resolution of N;

$$\cdots \longrightarrow R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \longrightarrow N \longrightarrow 0.$$

Now we deform the matices to

$$(\Phi, \Psi) = \left(\begin{pmatrix} x + ty & y^2 \\ -t^2 & x - ty \end{pmatrix}, \begin{pmatrix} x - ty & -y^2 \\ t^2 & x + ty \end{pmatrix} \right)$$

over $R \otimes_k V$. Since this is a matrix factorization of x^2 again, we have a free resolution

$$\cdots \xrightarrow{\Phi} (R \otimes_k V)^2 \xrightarrow{\Psi} (R \otimes_k V)^2 \xrightarrow{\Phi} (R \otimes_k V)^2 \longrightarrow Q \longrightarrow 0.$$

It is obvious to see that $r(Q) = Q/tQ \cong N$, since $\Phi \otimes_V V/tV = \varphi$. On the other hand, since t^2 is a unit in $R \otimes_k K$, we have $\Phi \otimes_V K \cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ after several elementary transformations of matrices. Hence, $\ell(Q) = Q_t \cong R \otimes_k K$. As a conclusion, we see that R degenerates to $(x, y^2)R$!

Since it is known that $N = (x, y^2)R$ is an indecomposable module, this degeneration is not a degeneration by an extension.

Remark 1.5 Let R be a commutative noetherian algebra over k, and suppose that a finitely generated R-module M degenerates to a finitely generated R-module N. Then the following hold.

(1) The modules M and N give the same class in the Grothendieck group, i.e. $[M] = [N]$ as elements of $K_0(\mathrm{mod}(R))$. This is actually a direct consequence of Theorem 1.2. In particular, rank M = rank N if the ranks are defined for R-modules. Furthermore, if $(R, \mathfrak{m})$ is a local ring, then $e(I, M) = e(I, N)$ for any $\mathfrak{m}$-primary ideal I, where $e(I, M)$ denotes the multiplicity of M along I.

(2) If L is an R-module of finite length, then we have the following inequalities of lengths for any integer i:

$$\begin{cases} \mathrm{length}_R(\mathrm{Ext}^i_R(L, M)) \leqq \mathrm{length}_R(\mathrm{Ext}^i_R(L, N)), \\ \mathrm{length}_R(\mathrm{Ext}^i_R(M, L)) \leqq \mathrm{length}_R(\mathrm{Ext}^i_R(N, L)). \end{cases}$$

See [7, Lemma 4.5]. In particular, when R is a local ring, then

$$\nu(M) \leqq \nu(N), \quad \beta_i(M) \leqq \beta_i(N) \quad \text{and} \quad \mu^i(M) \leqq \mu^i(N) \quad (i \geqq 0),$$

where ν, β_i and μ^i denote the minimal number of generators, the ith Betti number and the ith Bass number respectively.

(3) Let us denote by $\mathcal{F}_i^R(M)$ the ith Fitting ideal of the R-module M. Then we have the inclusions $\mathcal{F}_i^R(M) \supseteqq \mathcal{F}_i^R(N)$ for all $i \geqq 0$. See [10, Theorem 2.5]

2 Remarks for the rings $R \otimes_k K$.

In this lecture we are interested in the stable analogue of degenerations of Cohen-Macaulay modules over a commutative Gorenstein local ring. For this purpose, $(R, \mathfrak{m}, k)$ always denotes a Gorenstein local ring which is a k-algebra, and $V = k[t]_{(t)}$ and $K = k(t)$ where t is a variable.

Remark 2.1 We note that $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein as well as R and we have the equality of Krull dimension;

$$\dim R \otimes_k V = \dim R + 1, \quad \dim R \otimes_k K = \dim R.$$

If $\dim R = 0$ (i.e. R is artinian), then the rings $R \otimes_k V$ and $R \otimes_k K$ are local. In fact, the ideal $\mathfrak{m}(R \otimes_k V)$ of $R \otimes_k V$ is nilpotent, and $(R \otimes_k V)/\mathfrak{m}(R \otimes_k V) \cong V$, hence $(\mathfrak{m}, t)(R \otimes_k V)$ is a unique maximal ideal of $R \otimes_k V$. By the same reason, $\mathfrak{m}(R \otimes_k K)$ is a unique maximal ideal of $R \otimes_k K$.

However we should note that $R \otimes_k V$ and $R \otimes_k K$ will never be local rings if $\dim R > 0$. Actually, if there is a prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} = 1$, then taking an $x \in R$ so that $x \notin \mathfrak{p}$, we have a maximal ideal $(\mathfrak{p}, xt - 1)R \otimes_k V$ (resp. $(\mathfrak{p}, xt - 1)R \otimes_k K$), which is distinct from the maximal ideal $(\mathfrak{m}, t)R \otimes_k V$ (resp. $\mathfrak{m}(R \otimes_k K)$).

Since $R \otimes_k K$ is non-local, there may be a lot of projective modules which are not free. The following example gives such one of them.

Example 2.2 Let $R = k[[x, y]]/(x^3 - y^2)$. It is known that the maximal ideal $\mathfrak{m} = (x, y)$ is a unique non-free indecomposable Cohen-Macaulay module over R. See [6, Proposition 5.11]. In fact it is given by a matrix factorization of the polynomial $x^3 - y^2$;

$$(\varphi, \psi) = \left(\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}, \begin{pmatrix} y & -x \\ -x^2 & y \end{pmatrix} \right).$$

Therefore there is an exact sequence

$$\cdots \longrightarrow R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Now we deform these matrices and consider the pair of matrices over $R \otimes_k K$;

$$(\Phi, \Psi) = \left(\begin{pmatrix} y - xt & x - t^2 \\ x^2 & y + xt \end{pmatrix}, \begin{pmatrix} y + xt & -x + t^2 \\ -x^2 & y - xt \end{pmatrix} \right).$$

Define the $R \otimes_k K$-module P by the following exact sequence;

$$\cdots \longrightarrow (R \otimes_k K)^2 \xrightarrow{\Psi} (R \otimes_k K)^2 \xrightarrow{\Phi} (R \otimes_k K)^2 \longrightarrow P \longrightarrow 0.$$

In this case we can prove that P is a projective module of rank one over $R \otimes_k K$ but non-free. (Hence the Picard group of $R \otimes_k K$ is non-trivial.)

3 Stable degenerations

Let A be a commutative Gorenstein ring which is not necessarily local. We say that a finitely generated A-module M is Cohen-Macaulay if $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. We consider the category of all Cohen-Macaulay modules over A with all A-module homomorphisms:

$$\mathrm{CM}(A) := \{M \in \mathrm{mod}(A) \mid M \text{ is a Cohen-Macaulay module over } A\},$$

where $\mathrm{mod}(A)$ denotes the category of all finitely generated A-modules. We can then consider the stable category of $\mathrm{CM}(A)$, which we denote by $\underline{\mathrm{CM}}(A)$. Recall that the objects of $\underline{\mathrm{CM}}(A)$ is Cohen-Macaulay modules over A, and the morphisms of $\underline{\mathrm{CM}}(A)$ are elements of $\underline{\mathrm{Hom}}_A(M, N) := \mathrm{Hom}_A(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(A)$, where $P(M, N)$ denotes the set of morphisms from M to N factoring through projective A-modules. For a Cohen-Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(A)$. Note that $\underline{M} \cong \underline{N}$ in $\underline{\mathrm{CM}}(A)$ if and only if there are projective A-modules P_1 and P_2 such that $M \oplus P_1 \cong N \oplus P_2$ in $\mathrm{CM}(A)$.

Under such circumstances it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category. In fact, if $L \in \mathrm{CM}(A)$ then we can embed L into a projective A-module

P such that the quotient P/L, which we denote by $\Omega^{-1}L$, is Cohen-Macaulay as well. We define the shift functor in $\underline{\mathrm{CM}}(A)$ by $\underline{L}[1] = \underline{\Omega^{-1}L}$. If there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{CM}(A)$, then we have the following commutative diagram by taking the pushout;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & \Omega^{-1}L & \longrightarrow & 0. \end{array}$$

We define the triangles in $\underline{\mathrm{CM}}(A)$ are the sequences

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1],$$

obtained in such a way.

Let $x \in A$ be a non-zero divisor on A. Note that x is a non-zero divisor on every Cohen-Macaulay module over A. Thus the functor $-\otimes_A A/xA$ sends a Cohen-Macaulay module over A to that over A/xA. Therefore it yields a functor $\mathrm{CM}(A) \to \mathrm{CM}(A/xA)$. Since this functor maps projective A-modules to projective A/xA-modules, it induces the functor $\mathcal{R} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(A/xA)$. It is easy to verify that $\mathcal{R}$ is a triangle functor.

Now let $S \subset A$ be a multiplicative subset of A. Then, by a similar reason to the above, we have a triangle functor $\mathcal{L} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(S^{-1}A)$ which maps $\underline{M}$ to $\underline{S^{-1}M}$.

As before, let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring that is a k-algebra and let $V = k[t]_{(t)}$ and $K = k(t)$. Since $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein rings, we can apply the observation above. Actually, $t \in R \otimes_k V$ is a non-zero divisor on $R \otimes_k V$ and there are isomorphisms of k-algebras; $(R \otimes_k V)/t(R \otimes_k V) \cong R$ and $(R \otimes_k V)_t \cong R \otimes_k K$. Thus there are triangle functors $\mathcal{L} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R \otimes_k K)$ defined by the localization by t, and $\mathcal{R} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R)$ defined by taking $-\otimes_{R \otimes_k V} (R \otimes_k V)/t(R \otimes_k V) = -\otimes_V V/tV$. Now we define the stable degeneration of Cohen-Macaulay modules.

Definition 3.1 Let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$. We say that $\underline{M}$ **stably degenerates to** $\underline{N}$ if there is a Cohen-Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

The following lemmas are easily observed. See [10] for the proof.

Lemma 3.2 *[10, Lemma 4.2, Proposition 4.3]*

(1) *Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$.*

(2) *Suppose that there is a triangle in $\underline{\mathrm{CM}}(R)$;*

$$\underline{L} \xrightarrow{\alpha} \underline{M} \xrightarrow{\beta} \underline{N} \xrightarrow{\gamma} \underline{L}[1].$$

Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$.

Lemma 3.3 *[10, Proposition 4.4] Let* $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$ *and suppose that* $\underline{M}$ *stably degenerates to* $\underline{N}$. *Then the following hold.*

(1) $\underline{M}[1]$ *(resp.* $\underline{M}[-1]$*) stably degenerates to* $\underline{N}[1]$ *(resp.* $\underline{N}[-1]$*).*

(2) $\underline{M^*}$ *stably degenerates to* $\underline{N^*}$, *where* M^* *denotes the* R*-dual* $\mathrm{Hom}_R(M, R)$.

Lemma 3.4 *[10, Proposition 4.5] Let* $\underline{M}$, $\underline{N}$, $\underline{X} \in \underline{\mathrm{CM}}(R)$. *If* $\underline{M} \oplus \underline{X}$ *stably degenerates to* $\underline{N}$, *then* $\underline{M}$ *stably degenerates to* $\underline{N} \oplus \underline{X}[1]$.

Remark 3.5 The zero object in $\underline{\mathrm{CM}}(R)$ can stably degenerate to a non-zero object. In fact, in Example 1.4 the free module R degenerates to an ideal N. Hence it follows from Proposition 3.2(1) that $\underline{0} = \underline{R}$ stably degenerates to $\underline{N}$.

For another example, note that there is a triangle

$$\underline{X} \longrightarrow \underline{0} \longrightarrow \underline{X}[1] \xrightarrow{\ 1\ } \underline{X}[1],$$

for any $\underline{X} \in \underline{\mathrm{CM}}(R)$. Hence $\underline{0}$ stably degenerates to $\underline{X} \oplus \underline{X}[1]$ by Proposition 3.2(2).

4 Main results

Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra and assume for simplicity that k is an infinite field. For Cohen-Macaulay R-modules M and N we consider the following four conditions:

(1) $R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.

(2) There is a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$ in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.

(3) $\underline{M}$ stably degenerates to $\underline{N}$.

(4) There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.

In [10] we proved the following implications and equivalences of these conditions:

Theorem 4.1 *(i) In general,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *holds.*

(ii) If $\dim R = 0$, *then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *holds.*

(iii) If R *is an isolated singularity of any dimension, then* (2) $\Leftrightarrow$ (3) *holds.*

(iv) There is an example of isolated singularity of $\dim R = 1$ *for which* (2) $\Rightarrow$ (1) *fails.*

(v) There is an example of $\dim R = 0$ *for which* (4) $\Rightarrow$ (3) *fails.*

We give here an outline of some of the proofs.

Proof of (1) $\Rightarrow$ (2) : By Theorem 1.2, there exists an exact sequence

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} (R^m \oplus M) \oplus Z \to (R^n \oplus N) \to 0,$$

where ψ is nilpotent. In such a case Z ia a Cohen-Macaulay module as well. Then converting this into a triangle in $\underline{\mathrm{CM}}(R)$, and noting that the nilpotency of $\psi \in \mathrm{End}_R(Z)$ forces the nilpotency of $\underline{\psi} \in \underline{\mathrm{End}}_R(Z)$, we can see that (2) holds. □

Proof of (2) $\Rightarrow$ (3): Suppose that there exists a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$, where $\underline{\psi}$ is nilpotent. Then we have a triangle of the form;

$$\underline{Z \otimes_k V} \xrightarrow{\binom{\phi}{t+\psi}} \underline{M \otimes_k V} \oplus \underline{Z \otimes_k V} \longrightarrow \underline{Q} \longrightarrow \underline{Z \otimes_k V}[1],$$

for a $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$. Note $\mathcal{L}(t + \underline{\psi})$ is an isomorphism in $\underline{\mathrm{CM}}(R \otimes_k K)$. Thus $\mathcal{L}(\underline{Q}) \cong \mathcal{L}(\underline{M \otimes_k V}) = \underline{M \otimes_k K}$. On the other hand, since $\mathcal{R}(t + \underline{\psi}) = \underline{\psi}$, $\mathcal{R}(\underline{Q}) \cong N$. Thus $\underline{M}$ stably degenerates to $\underline{N}$. □

Proof of (3) $\Rightarrow$ (1) when $\dim R = 0$: In this proof we assume $\dim R = 0$. Suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then there is a $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ with $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$. By definition, we have isomorphisms $Q_t \oplus P_1 \cong (M \otimes_k K) \oplus P_2$ in $\mathrm{CM}(R \otimes_k K)$ for some projective $R \otimes_k K$-modules P_1, P_2, and $Q/tQ \oplus R^a \cong N \oplus R^b$ in $\mathrm{CM}(R)$ for some $a, b \in \mathbb{N}$. Since $R \otimes_k K$ is a local ring, P_1 and P_2 are free. Thus $Q_t \oplus (R \otimes_k K)^c \cong (M \otimes_k K) \oplus (R \otimes_k K)^d$ for some $c, d \in \mathbb{N}$. Setting $\tilde{Q} = Q \oplus (R \otimes_k V)^{a+c}$, we have isomorphisms

$$\tilde{Q}_t \cong (M \oplus R^{a+d}) \otimes_k K, \quad \tilde{Q}/t\tilde{Q} \cong N \oplus R^{b+c}.$$

Since $\tilde{Q}$ is V-flat, $M \oplus R^{a+d}$ degenerates to $N \oplus R^{b+c}$. □

The difficult part of the proof is to show the implications (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2). Actually it is technically difficult to show the existence of a Cohen-Macaulay module Z and X in each case. To get over this difficulty, we use the following lemma called Swan's Lemma in Algebraic K-Theory.

Lemma 4.2 *[4, Lemma 5.1] Let R be a noetherian ring and t a variable. Assume that an $R[t]$-module L is a submodule of $W \otimes_R R[t]$ with W being a finitely generated R-module. Then there is an exact sequence of $R[t]$-modules;*

$$0 \longrightarrow X \otimes_R R[t] \longrightarrow Y \otimes_R R[t] \longrightarrow L \longrightarrow 0,$$

where X and Y are finitely generated R-modules.

By virtue of Swan's lemma we can prove the following proposition that will play an essential role in the proof of Theorem 4.1.

Proposition 4.3 *Let R be a Gorenstein local k-algebra, where k is an infinite field. Suppose we are given a Cohen-Macaulay $R \otimes_k V$-module P' satisfying that the localization $P = P'_t$ by t is a projective $R \otimes_k K$-module. Then there is a Cohen-Macaulay R-module X with a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$ of the following form:*

$$\underline{X \otimes_k V} \longrightarrow \underline{X \otimes_k V} \longrightarrow \underline{P'} \longrightarrow \underline{X \otimes_k V}[1]. \tag{4.1}$$

As a direct consequence of Theorem 4.1, we have the following corollary.

Corollary 4.4 *Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras. Assume that the both R_1 and R_2 are isolated singularities, and that k is an infinite field. Suppose there is a k-linear equivalence $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ of triangulated categories. Then, for $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R_1)$, $\underline{M}$ stably degenerates to $\underline{N}$ if and only if $F(\underline{M})$ stably degenerates to $F(\underline{N})$.*

Remark 4.5 Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras as above. Then it hardly occurs that there is a k-linear equivalence of categories between $\mathrm{CM}(R_1)$ and $\mathrm{CM}(R_2)$. In fact, if it occurs, then R_1 is isomorphic to R_2 as a k-algebra. (See [2, Proposition 5.1].)

On the other hand, an equivalence between $\underline{\mathrm{CM}}(R_1)$ and $\underline{\mathrm{CM}}(R_2)$ may happen for non-isomorphic k-algebras. For example, let $R_1 = k[[x, y, z]]/(x^n + y^2 + z^2)$ and $R_2 = k[[x]]/(x^n)$ with characteristic of k not being 2 and $n \in \mathbb{N}$. Then, by Knoerrer's periodicity ([6, Theorem 12.10]), we have an equivalence $\underline{\mathrm{CM}}(k[[x, y, z]]/(x^n + y^2 + z^2)) \cong \underline{\mathrm{CM}}(k[[x]]/(x^n))$. Since $k[[x]]/(x^n)$ is an artinian Gorenstein ring, the stable degeneration of modules over $k[[x]]/(x^n)$ is equivalent to a degeneration up to free summands by Theorem 4.1(*ii*). Moreover the degeneration problem for modules over $k[[x]]/(x^n)$ is known to be equivalent to the degeneration problem for Jordan canonical forms of square matrices of size n. Thus by virtue of Corollary 4.4, it is easy to describe the stable degenerations of Cohen-Macaulay modules over $k[[x, y, z]]/(x^n + y^2 + z^2)$.

See the lecture of Hiramatsu for more detail.

The following example is the one for (v) in Theorem 4.1.

Example 4.6 Let $R = k[[x, y]]/(x^2, y^2)$. Note that R is an artinian Gorenstein local ring. Now consider the modules $M_\lambda = R/(x - \lambda y)R$ for all $\lambda \in k$. We denote by k the unique simple module $R/(x, y)R$ over R. In this case, it is known by [5, Example 3.1] that $R \oplus k^2$ degenerates to $M_\lambda \oplus M_\mu \oplus k^2$ for any choice of $\lambda, \mu \in k$.

We claim that $\underline{R}$ never stably degenerates to $\underline{M_\lambda \oplus M_\mu}$ if $\lambda + \mu \neq 0$.

In fact, if there is such a stable degeneration, then it follows from Theorem 4.1 that R^m degenerates to $M_\lambda \oplus M_\mu \oplus R^n$ for some $m, n \in \mathbb{N}$. Since $[R^m] = [M_\lambda \oplus M_\mu \oplus R^n]$ in the Grothendieck group, we have $m > n \geqq 0$. Now we apply Remark 1.5(3) to obtain an inclusion of Fitting ideals; $\mathcal{F}_n^R(M_\lambda \oplus M_\mu \oplus R^n) \subseteq \mathcal{F}_n^R(R^m)$. We note that $\mathcal{F}_n^R(R^m) = 0$ since $n < m$, and an easy computation shows that

$$\mathcal{F}_n^R(M_\lambda \oplus M_\mu \oplus R^n) = \mathcal{F}_0^R(M_\lambda)\mathcal{F}_0^R(M_\mu)\mathcal{F}_n^R(R^n) = (x - \lambda y)(x - \mu y)R = (\lambda + \mu)xyR.$$

Hence we must have $\lambda + \mu = 0$.

As an application of Theorem 4.1 we can define the stable degeneration order for Cohen-Macaulay modules.

Definition 4.7 Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra as before, and let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$. If there is a sequence of objects $\underline{L_0}, \underline{L_1}, \underline{L_2}, \dots, \underline{L_n}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{L_0} = \underline{M}$, $\underline{L_n} = \underline{N}$ and $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $i = 0, 1, \cdots, n-1$, then we write $\underline{M} \leq_{st} \underline{N}$.

Theorem 4.1 shows that the relation $\leq_{st}$ gives a partial order on the set of isomorphism classes of objects in $\underline{\mathrm{CM}}(R)$. In fact we can prove the antisymmetric law for $\leq_{st}$.

Theorem 4.8 *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local algebra over an infinite field k, and let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$. If $\underline{M} \leq_{st} \underline{N}$ and $\underline{N} \leq_{st} \underline{M}$, then $\underline{M} \cong \underline{N}$.*

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EXAMPLES OF DEGENERATIONS OF COHEN-MACAULAY MODULES

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1. Introduction

The aim of this article is to give an outline of the paper [3], which is a joint work with Yuji Yoshino and will be submitted for publication elsewhere.

In this note, we would like to give several examples of degenerations of maximal Cohen-Macaulay modules and to show how we can describe them (Theorem 3.1). This result depends heavily on the recent work by Yoshino about the stable analogue of degenerations for Cohen-Macaulay modules over a Gorenstein local algebra [10]. In Section 4 we also investigate the relation among the extended versions of the degeneration order, the extension order and the AR order (Theorem 4.4).

2. The first examples

In this section, we recall the definition of degeneration and state several known results on degenerations.

Definition 2.1. Let R be a noetherian algebra over a field k, and let M and N be finitely generated left R-modules. We say that M degenerates to N, or N is a degeneration of M, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module.
(2) $Q/tQ \cong N$ as a left R-module.
(3) $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V[1/t]$-module.

The following characterization of degenerations has been proved by Yoshino [8].

Theorem 2.2. [8, Theorem 2.2] *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \longrightarrow Z \xrightarrow{\binom{\varphi}{\psi}} M \oplus Z \longrightarrow N \longrightarrow 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Remark 2.3. Assume that there is an exact sequence of finitely generated left R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then M degenerates to $L \oplus N$. See [8, Remark 2.5].

We are mainly interested in degenerations of modules over commutative rings. Henceforth, in the rest of the paper, all the rings are assumed to be commutative.

Definition 2.4. Let M and N be finitely generated modules over a commutative noetherian k-algebra R.

(1) We denote by $M \leq_{deg} N$ if N is obtained from M by iterative degenerations, i.e. there is a sequence of finitely generated R-modules $L_0, L_1, \ldots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates to L_{i+1} for $0 \leq i < r$.

(2) We say that M degenerates by an extension to N if there is a short exact sequence $0 \to U \to M \to V \to 0$ of finitely generated R-modules such that $N \cong U \oplus N$.

We denote by $M \leq_{ext} N$ if N is obtained from M by iterative degenerations by extensions, i.e. there is a sequence of finitely generated R-modules $L_0, L_1, \ldots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates by an extension to L_{i+1} for $0 \leq i < r$.

If R is a local ring, then $\leq_{deg}$ and $\leq_{ext}$ are known to be partial orders on the set of isomorphism classes of finitely generated R-modules, which are called the degeneration order and the extension order respectively. See [7] for the detail.

Remark 2.5. By virtue of Remark 2.3, if $M \leq_{ext} N$ then $M \leq_{deg} N$. However the converse is not necessarily true.

For example, consider a ring $R = k[[x, y]]/(x^2)$. A pair of matrices over $k[[x, y]]$;

$$(\varphi, \psi) = \left(\begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)$$

is a matrix factorization of the equation x^2, hence it gives a maximal Cohen-Macaulay R-module N that is isomorphic to the ideal $(x, y^2)R$. It is known that N is indecomposable. Then we can show that R degenerates to $(x, y^2)R$ in this case, and hence $R \leq_{deg} (x, y^2)R$. See [3, Remark 2.5.].

In general if $M \leq_{ext} N$ and if $M \not\cong N$, then N is a non-trivial direct sum of modules. Since $N \cong (x, y^2)R$ is indecomposable, we see that $R \leq_{ext} (x, y^2)R$ can never happen.

Remark 2.6. We remark that if finitely generated R-modules M and N satisfy the relation $M \leq_{ext} N$, then M degenerates to N.

Now we note that the following lemma holds.

Lemma 2.7. *As in the lemma, let I be an ideal of a noetherian k-algebra R, and let M and N be finitely generated R/I-modules. Then $M \leq_{deg} N$ (resp. $M \leq_{ext} N$) as R-modules if and only if so does as R/I-modules.*

We make several other remarks on degenerations for the later use.

Remark 2.8. Let R be a noetherian k-algebra, and let M and N be finitely generated R-modules. Suppose that M degenerates to N. Then the following hold.

(1) The modules M and N give the same class in the Grothendieck group, *i.e.* $[M] = [N]$ as an element of $K_0(\mathrm{mod}(R))$, where $\mathrm{mod}(R)$ denotes the category of finitely generated R-modules and R-homomorphisms. (See [10, Remark 2.3 (1)]).

(2) The ith Fitting ideal of M contains that of N for all $i \geq 0$. Namely, denoting the ith Fitting ideal of an R-module M by $\mathcal{F}_i^R(M)$, we have $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all $i \geqq 0$. (See [10, Theorem 2.5]).

Now we give an example of modules of finite length for which we can easily describe the degeneration.

Let $R = k[[x]]$ be a formal power series ring over a field k with one variable x and let M be an R-module of length n. It is easy to see that there is an isomorphism

$$M \cong R/(x^{p_1}) \oplus \cdots \oplus R/(x^{p_n}), \tag{2.1}$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = n. \tag{2.2}$$

In this case the finite presentation of M is given as follows:

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} x^{p_1} & & \\ & \ddots & \\ & & x^{p_n} \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0.$$

Note that we can easily compute the ith Fitting ideal of M from this presentation;

$$\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots+p_n}) \quad (i \geq 0).$$

We denote by p_M the sequence $(p_1, p_2, \cdots, p_n)$ of non-negative integers. Recall that such a sequence satisfying (2.2) is called a partition of n.

Conversely, given a partition $p = (p_1, p_2, \cdots, p_n)$ of n, we can associate an R-module of length n by (2.1), which we denote by $M(p)$. In such a way we see that there is a one-one correspondence between the set of partitions of n and the set of isomorphism classes of R-modules of length n.

Definition 2.9. Let n be a positive integer and let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Then we denote $p \succeq q$ if it satisfies $\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i$ for all $1 \leq j \leq n$.

We note that $\succeq$ is known to be a partial order on the set of partitions of n and called the dominance order (see for example [4, page 7]).

In the following proposition we show the degeneration order for R-modules of length n coincides with the opposite of the dominance order of corresponding partitions. Note that if M and N are R-modules of finite length and if M degenerates to N, then the length of M equals the length of N, since $[M] = [N]$ in the Grothendieck group.

Proposition 2.10. *Let $R = k[[x]]$ as above, and let M, N be R-modules of length n. Then the following conditions are equivalent:*

(1) $M \leq_{deg} N$,
(2) $M \leq_{ext} N$,
(3) $p_M \succeq p_N$.

Proof. First of all, we assume M degenerates to N, and let $p_M = (p_1, p_2, \cdots, p_n)$ and $p_N = (q_1, q_2, \cdots, q_n)$. Then, by definition, we have the equalities of the Fitting ideals; $\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots+p_n})$ and $\mathcal{F}_i^R(M) = (x^{q_{i+1}+\cdots+q_n})$ for all $i \geq 0$. Since M degenerates to N, it follows from Remark 2.8(2) that $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all i. Thus $p_{i+1} + \cdots + p_n \leq q_{i+1} + \cdots + q_n$. Since $\sum_{i=1}^n p_i = n = \sum_{i=1}^n q_i$, it follows that $p_1 + \cdots + p_i \geq q_1 + \cdots + q_i$ for all $i \geq 0$. Therefore $p_M \succeq p_N$.

Secondly, assume $M \leq_{deg} N$. Then there are R-modules $L_0, L_1, \ldots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates to L_{i+1} for $0 \leq i < r$. It then follows from the above that $p_{L_0} \succeq p_{L_1} \succeq \cdots \succeq p_{L_r}$. Since $\succeq$ is a partial order, we have $p_M \succeq p_N$. Thus we have proved the implication (1) $\Rightarrow$ (3).

Finally we shall prove (3) $\Rightarrow$ (2). To this end let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Note that it is enough to prove that the corresponding R-module $M(p)$ degenerates by an extension to $M(q)$ whenever q is a predecessor of p under the dominance order. (Recall that q is called a predecessor of p if $p \succeq q$ and there are no partitions r with $p \succeq r \succeq q$ other than p and q.)

Assume that q is a predecessor of p under the dominance order. Then it is easy to see that there are numbers $1 \leq i < j \leq n$ with $p_i - p_j \geq 2$, $p_i > p_{i+1}$, $p_{j-1} > p_j$ such that the equality $q = (p_1, \cdots, p_i - 1, p_{i+1}, \cdots, p_j + 1, \cdots, p_n)$ holds. In this case, setting $L = M((p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, p_j, \cdots, p_n))$, we have $M(p) = L \oplus M((p_i, p_j))$ and $M(q) = L \oplus M((p_i - 1, p_j + 1))$. Note that, in general, if M degenerates by an extension to N, then $M \oplus L$ degenerates by an extension to $N \oplus L$, for any R-modules L. Hence it is enough to show that $M((a, b))$ degenerates by an extension to $M((a-1, b+1))$ if $a \geq b+2$. However there is a short exact sequence of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/(x^{a-1}) & \longrightarrow & R/(x^a) \oplus R/(x^b) & \longrightarrow & R/(x^{b+1}) & \longrightarrow & 0 \\ & & 1 & \longrightarrow & (x, 1) & & & & \end{array}$$

Thus $M((a, b)) = R/(x^a) \oplus R/(x^b)$ degenerates by an extension to $M((a-1, b+1)) = R/(x^{a-1}) \oplus R/(x^{b+1})$. $\square$

Combining Proposition 2.10 with Lemma 2.7, we have the following corollary which will be used in the next section.

Corollary 2.11. *Let $R = k[[x]]/(x^m)$, where k is a field and m is a positive integer, and let M, N be finitely generated R-modules. Then $M \leq_{deg} N$ holds if and only if $M \leq_{ext} N$ holds.*

3. The Second examples

Let k be a field and $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(f)$, where f is a polynomial of the form

$$f = x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2 \qquad (n \geq 1).$$

Recall that such a ring R is call the ring of simple singularity of type (A_n). Note that R is a Gorenstein complete local ring and has finite Cohen-Macaulay representation type (cf. [6]). The main result of this section is the following whose proof will be given in the last part of this section.

Theorem 3.1. *Let k be an algebraically closed field of characteristic 0 and let $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$ as above, where we assume that d is even. For maximal Cohen-Macaulay R-modules M and N, if $M \leq_{deg} N$, then $M \leq_{ext} N$.*

To prove the theorem, we need several results concerning the stable degeneration which was introduced by Yoshino in [10].

Let A be a commutative Gorenstein ring. We denote by $\mathrm{CM}(A)$ the category of all maximal Cohen-Macaulay A-module with all A-homomorphisms. And we also denote by $\underline{\mathrm{CM}}(A)$ the stable category of $\mathrm{CM}(A)$. For a maximal Cohen-Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(A)$. Since A is Gorenstein, it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category.

The following theorem proved by Yoshino [10] shows the relation between stable degenerations and ordinary degenerations.

Theorem 3.2. [10, Theorem 5.1, 6.1, 7.1] *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra, where k is an infinite field. Consider the following four conditions for maximal Cohen-Macaulay R-modules M and N:*

(1) *$R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.*
(2) *There is a triangle*

$$\underline{Z} \xrightarrow{\binom{\varphi}{\psi}} \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.
(3) *$\underline{M}$ stably degenerates to $\underline{N}$.*
(4) *There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.*

Then, in general, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold. If R is an isolated singularity, then (2) and (3) are equivalent. Furthermore, if R is an artinian ring, then the conditions (1), (2) and (3) are equivalent.

Corollary 3.3. [10, Corollary 6.6] *Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras. Assume that the both R_1 and R_2 are isolated singularities, and that k is an infinite field. Suppose there is a k-linear equivalence $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ of triangulated categories. Then, for $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R_1)$, $\underline{M}$ stably degenerates to $\underline{N}$ if and only if $F(\underline{M})$ stably degenerates to $F(\underline{N})$.*

We now consider the stable analogue of the degeneration by an extension.

Definition 3.4.

(1) We denote by $\underline{M} \leq_{st} \underline{N}$ if $\underline{N}$ is obtained from $\underline{M}$ by iterative stable degenerations, i.e. there is a sequence of objects $\underline{L_0}, \underline{L_1}, \ldots, \underline{L_r}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $0 \leq i < r$.

(2) We say that $\underline{M}$ stably degenerates by a triangle to $\underline{N}$, if there is a triangle of the form $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{U} \oplus \underline{V} \cong \underline{N}$. We denote by $\underline{M} \leq_{tri} \underline{N}$ if there is a finite sequence of modules $\underline{L_0}, \underline{L_1}, \cdots, \underline{L_r}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates by a triangle to $\underline{L_{i+1}}$ for $0 \leq i < r$.

Remark 3.5. Let R be a Gorenstein local ring that is a k-algebra.

(1) Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$. Therefore that $M \leq_{deg} N$ forces that $\underline{M} \leq_{st} \underline{N}$. (See [10, Lemma 4.2].)

(2) Suppose that there is a triangle

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1],$$

in $\underline{\mathrm{CM}}(R)$. Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$, thus $\underline{M} \leq_{st} \underline{L} \oplus \underline{N}$. (See [10, Proposition 4.3].)

We need the following proposition to prove Theorem 3.1.

Proposition 3.6. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local ring and let $M, N \in \mathrm{CM}(R)$. Assume $[M] = [N]$ in $K_0(\mathrm{mod}(R))$. Then $\underline{M} \leq_{tri} \underline{N}$ if and only if $M \leq_{ext} N$.*

The following lemma is known as the Knörrer's periodicity (cf. [6]).

Lemma 3.7. *Let k be an algebraically closed field of characteristic 0 and let $S = k[[x_0, x_1, \cdots, x_n]]$ be a formal power series ring. For a non-zero element $f \in (x_0, x_1, \cdots, x_n)S$, we consider the two rings $R = S/(f)$ and $R^\sharp = S[[y, z]]/(f + y^2 + z^2)$. Then the stable categories $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$ are equivalent as triangulated categories.*

Now we proceed to the proof of Theorem 3.1.

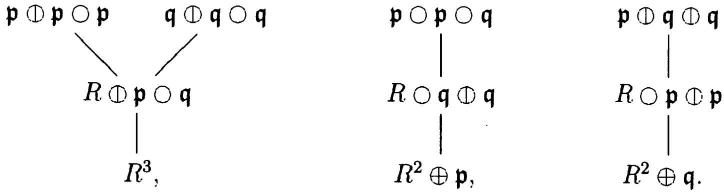
Let k be an algebraically closed field of characteristic 0 and let

$$R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$$

as in the theorem, where we assume that d is even. Suppose that $M \leq_{deg} N$ for maximal Cohen-Macaulay R-modules M and N. We want to show $M \leq_{ext} N$.

Since $M \leq_{deg} N$, we have $\underline{M} \leq_{st} \underline{N}$ in $\underline{\mathrm{CM}}(R)$ and $[M] = [N]$ in $K_0(\mathrm{mod}(R))$, by Remarks 3.5(1) and 2.8(1). Now let us denote $R' = k[[x_0]]/(x_0^{n+1})$, and we note that $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R')$ are equivalent to each other as triangulated categories. In fact this equivalence is given by using $d/2$-times of Lemma 3.7, since d is even. Let $\Omega : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R')$ be a triangle functor which gives the equivalence. Then, by virtue of Corollary 3.3, we have $\Omega(\underline{M}) \leq_{st} \Omega(\underline{N})$ in $\underline{\mathrm{CM}}(R')$. Since R' is an artinian algebra, the equivalence (1) $\Leftrightarrow$ (3) holds in Theorem 3.2, and thus we have $\tilde{M} \oplus R'^m \leq_{deg} \tilde{N} \oplus R'^n$, where $\tilde{M}$ (resp. $\tilde{N}$) is a module in $\mathrm{CM}(R')$ with $\underline{\tilde{M}} \cong \Omega(\underline{M})$ (resp. $\underline{\tilde{N}} \cong \Omega(\underline{N})$) and m, n are non-negative integers. It then follows from Corollary 2.11 that $\tilde{M} \oplus R'^m \leq_{ext} \tilde{N} \oplus R'^n$. Hence, by Proposition 3.6, we have that $\Omega(\underline{M}) \leq_{tri} \Omega(\underline{N})$ in $\underline{\mathrm{CM}}(R')$. Noting that the partial order $\leq_{tri}$ is preserved under a triangle functor, we see that $\underline{M} \leq_{tri} \underline{N}$ in $\underline{\mathrm{CM}}(R)$. Since $[M] = [N]$ in $K_0(\mathrm{mod}(R))$, applying Proposition 3.6, we finally obtain that $M \leq_{ext} N$. □

Example 3.8. Let $R = k[[x_0, x_1, x_2]]/(x_0^3 + x_1^2 + x_2^2)$, where k is an algebraically closed field of characteristic 0. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the ideals generated by $(x_0, x_1 - \sqrt{-1}\ x_2)$ and $(x_0^2, x_1 + \sqrt{-1}\ x_2)$ respectively. It is known that the set $\{R, \mathfrak{p}, \mathfrak{q}\}$ is a complete list of the isomorphism classes of indecomposable maximal Cohen-Macaulay modules over R. The Hasse diagram of degenerations of maximal Cohen-Macaulay R-modules of rank 3 is a disjoint union of the following diagrams:



4. Extended orders

In the rest of this paper R denotes a (commutative) Cohen-Macaulay complete local k-algebra, where k is any field.

We shall show that any extended degenerations of maximal Cohen-Macaulay R-modules are generated by extended degenerations of Auslander-Reiten (abbr.AR) sequences if R is of finite Cohen-Macaulay representation type. For the theory of AR sequences of maximal Cohen-Macaulay modules, we refer to [6]. First of all we recall the definitions of the extended orders generated respectively by degenerations, extensions and AR sequences.

Definition 4.1. [7, Definition 4.11, 4.13] The relation $\leq_{DEG}$ on $\mathrm{CM}(R)$, which is called the extended degeneration order, is a partial order generated by the following rules:

(1) If $M \leq_{deg} N$ then $M \leq_{DEG} N$.
(2) $M \leq_{DEG} N$ if and only if $M \oplus L \leq_{DEG} N \oplus L$ for all $L \in \mathrm{CM}(R)$.
(3) $M \leq_{DEG} N$ if and only if $M^n \leq_{DEG} N^n$ for all natural numbers n.

Definition 4.2. [7, Definition 3.6] The relation $\leq_{EXT}$ on $\mathrm{CM}(R)$, which is called the extended extension order, is a partial order generated by the following rules:

(1) If $M \leq_{ext} N$ then $M \leq_{EXT} N$.
(2) $M \leq_{EXT} N$ if and only if $M \oplus L \leq_{EXT} N \oplus L$ for all $L \in \mathrm{CM}(R)$.
(3) $M \leq_{EXT} N$ if and only if $M^n \leq_{EXT} N^n$ for all natural numbers n.

Definition 4.3. [7, Definition 5.1] The relation $\leq_{AR}$ on $\mathrm{CM}(R)$, which is called the extended AR order, is a partial order generated by the following rules:

(1) If $0 \to X \to E \to Y \to 0$ is an AR sequence in $\mathrm{CM}(R)$, then $E \leq_{AR} X \oplus Y$.
(2) $M \leq_{AR} N$ if and only if $M \oplus L \leq_{AR} N \oplus L$ for all $L \in \mathrm{CM}(R)$.
(3) $M \leq_{AR} N$ if and only if $M^n \leq_{AR} N^n$ for all natural numbers n.

The following is the main theorem of this section. We say that a Cohen-Macaulay complete local k-algebra is of finite Cohen-Macaulay representation type if there are only a finite number of isomorphism classes of objects in $\mathrm{CM}(R)$.

Theorem 4.4. *Let R be a Cohen-Macaulay complete local k-algebra as above. Adding to this, we assume that R is of finite Cohen-Macaulay representation type. Then the following conditions are equivalent for $M, N \in \mathrm{CM}(R)$:*

(1) $M \leq_{DEG} N$,
(2) $M \leq_{EXT} N$,
(3) $M \leq_{AR} N$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear from the definitions.

To prove (1) $\Rightarrow$ (2), it suffices to show that $M \leq_{EXT} N$ whenever M degenerates to N. If M degenerates to N, then, by virtue of Theorem 2.2, we have a short exact sequence $0 \to Z \to M \oplus Z \to N \to 0$ with $Z \in \mathrm{CM}(R)$. Thus $M \oplus Z \leq_{ext} N \oplus Z$, hence $M \leq_{EXT} N$.

It remains to prove that (2) $\Rightarrow$ (3), for which we need the following lemma which is essentially due to Auslander and Reiten [1].

Lemma 4.5. *Under the same assumptions on R as in Theorem 4.4, let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathrm{CM}(R)$. Then there are a finite number of AR sequences in $\mathrm{CM}(R)$;*

$$0 \to X_i \to E_i \to Y_i \to 0 \quad (1 \leq i \leq n),$$

such that there is an equality in $\mathrm{G}(\mathrm{CM}(R))$;

$$L - M + N = \sum_{i=1}^{n} (X_i - E_i + Y_i).$$

Here, $\mathrm{G}(\mathrm{CM}(R)) = \bigoplus \mathbb{Z} \cdot X$ where X runs through all isomorphism classes of indecomposable objects in $\mathrm{CM}(R)$.

To prove this lemma, we consider the functor category $\mathrm{Mod}(\mathrm{CM}(R))$ and the Auslander category $\mathrm{mod}(\mathrm{CM}(R))$ of $\mathrm{CM}(R)$. □

Remark 4.6. In the paper [7], Yoshino introduced the order relation $\leq_{hom}$ as well. Adding to the assumption that R is of finite Cohen-Macaulay representation type, if we assume further conditions on R, such as R is an integral domain of dimension 1 or R is of dimension 2, then he showed that $\leq_{hom}$ is also equal to any of $\leq_{AR}$, $\leq_{EXT}$ and $\leq_{DEG}$.

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Noetherian and Non-Noetherian Symbolic Rees Algebras

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1 Criteria for finite generation

Let us recall Huneke's criterion for finite generation of symbolic Rees algebras.

Theorem 1.1 [3, Theorem 3.1] *Let $(A, \mathfrak{m})$ be a 3-dimensional regular local ring and $\mathfrak{p}$ a prime ideal such that $\dim A/\mathfrak{p} = 1$. If there exist positive integers $k, \ell > 0$ and elements $f \in \mathfrak{p}^{(k)}, g \in \mathfrak{p}^{(\ell)}$ such that*

$$\ell_A(A/(x, f, g)A) = k\ell \cdot \ell_A(A/xA + \mathfrak{p})$$

for some/any $x \in A \setminus \mathfrak{p}$, then the symbolic Rees algebra $\mathrm{R}_s(\mathfrak{p}) = \oplus_{n \geq 0} \mathfrak{p}^{(n)}$ is finitely generated. The converse holds if $A/\mathfrak{m}$ is infinite.

Applying this criterion, a lot of symbolic Rees algebras of prime ideals defining space monomial curves are proved to be finitely generated. However, computing the length is not always easy. Sometimes it is quite difficult. So, under a little bit more general situation, we would like to consider another criterion, which is also essentially due to Huneke [3, Theorem 3.25].

In the rest of this report we assume that $(A, \mathfrak{m})$ is a 3-dimensional Cohen-Macaulay local ring and I is an ideal of A such that A/I is a 1-dimensional Cohen-Macaulay local ring (I is not necessarily prime). Moreover we assume that $I_\mathfrak{p}$ is generated by 2 elements for any $\mathfrak{p} \in \mathrm{Min}_A A/I$. The n-th symbolic power of I and the symbolic Rees algebra of I are defined by

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}_A A/I} (I^n A_\mathfrak{p} \cap A) \quad \text{and} \quad \mathrm{R}_s(I) = \sum_{n \geq 0} I^{(n)} t^n \subset A[t],$$

where t is an indeterminate. The another criterion we focus in this report can be stated as follows.

Theorem 1.2 *Suppose there exist positive integers $k, \ell > 0$ and elements $f \in I^{(k)}, g \in I^{(\ell)}$ such that*

(i) $I \subseteq \sqrt{(f, g)A}$ *and*

(ii) $\mathrm{R}(I_\mathfrak{p})_+ \subseteq \sqrt{(ft^k, gt^\ell)\mathrm{R}(I_\mathfrak{p})}$ *for any* $\mathfrak{p} \in \mathrm{Min}_A A/I$.

Then $\mathrm{R}_s(I)$ is finitely generated. The converse holds if $A/\mathfrak{m}$ is infinite.

Proof. We put $m = k\ell$ and $\mathfrak{a} = I^{(m)}$. Then f^{ℓ}, g^{k} is a regular sequence contained in $\mathfrak{a}$. Let $J = (f^{\ell}, g^{k})A \subseteq \mathfrak{a}$.

Claim 1 depth $A/J^{n-1}\mathfrak{a} = 1$ for any $n \in \mathbb{Z}$.

Claim 1 is obvious if $n \leq 1$. So, we assume $n \geq 2$ and look at the exact sequence

$$0 \to J^{n-1}/J^{n-1}\mathfrak{a} \to A/J^{n-1}\mathfrak{a} \to A/J^{n-1} \to 0\,.$$

As $J^{n-1}/J^{n-1}\mathfrak{a} \cong J^{n-1}/J^{n} \otimes_A A/\mathfrak{a}$ and J^{n-1}/J^{n} is A/J-free, $J^{n-1}/J^{n-1}\mathfrak{a}$ is $A/\mathfrak{a}$-free. Hence $\operatorname{depth}_A J^{n\ 1}/J^{n\ 1}\mathfrak{a} = 1$. Then depth $A/J^{n\ 1}\mathfrak{a} = 1$ since depth $A/J^{n\ 1} = 1$.

By Claim 1 we get $\operatorname{Ass}_A A/J^{n-1}\mathfrak{a} = \operatorname{Min}_A A/J^{n-1}\mathfrak{a}$. Condition (i) means $\operatorname{Min}_A A/J^{n-1}\mathfrak{a} = \operatorname{Min}_A A/I$. We take any $\mathfrak{p} \in \operatorname{Min}_A A/I = \operatorname{Ass}_A A/J^{n-1}\mathfrak{a}$. We want to show

Claim 2 ${\mathfrak{a}_\mathfrak{p}}^{n} = {J_\mathfrak{p}}^{n\ 1} \cdot \mathfrak{a}_\mathfrak{p}$ for any $n \geq 1$.

If this is true, for any $n \geq 1$, we get

$$\begin{aligned} I^{(mn)} &= \bigcap_{\mathfrak{p} \in \operatorname{Min}_A A/I} ({\mathfrak{a}_\mathfrak{p}}^{n} \cap A) \\ &= \bigcap_{\mathfrak{p} \in \operatorname{Ass}_A A/J^{n}\mathfrak{a}} ({J_\mathfrak{p}}^{n\ 1} \cdot \mathfrak{a}_\mathfrak{p} \cap A) \\ &= J^{n-1}\mathfrak{a}\,, \end{aligned}$$

which implies $I^{(mn)} = \mathfrak{a}^{n}$, and so

$$\mathrm{R}_s(I)^{(m)} = \oplus_{n \geq 0} I^{(mn)} t^{mn} = \oplus_{n \geq 0} \mathfrak{a}^{n} t^{mn} = A[t]\,.$$

Therefore $\mathrm{R}_s(I)^{(m)}$ is Noetherian. Then we see that $\mathrm{R}_s(I)$ itself is also Noetherian.

Now we prove Claim 2. If $k = \ell = 1$, the condition (i) means $J_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$, so $I_\mathfrak{p} = J_\mathfrak{p}$ as I is generically a complete intersection. Hence we may assume $k \geq 2$ or $\ell \geq 2$. Then $m \geq 2$. As $\mathrm{G}(I_\mathfrak{p})$ is isomorphic to a polynomial ring with two variables over $A_\mathfrak{p}/I_\mathfrak{p}$, $\mathrm{G}(I_\mathfrak{p})$ is Cohen-Macaulay and $\mathrm{a}(\mathrm{G}(I_\mathfrak{p})) = -2$. Then, as $\mathfrak{a}_\mathfrak{p} = {I_\mathfrak{p}}^{m}$, $\mathrm{G}(\mathfrak{a}_\mathfrak{p})$ is also Cohen-Macaulay and

$$\mathrm{a}(\mathrm{G}(\mathfrak{a}_\mathfrak{p})) = [-\frac{2}{m}] = -1\,,$$

where $[\,*\,]$ denotes the largest integer less than or equal to $*$ (cf. [2, Lemma 2.4]). By the condition (ii), we see that $J_\mathfrak{p}$ is a reduction of $\mathfrak{a}_\mathfrak{p}$. Then

$$\mathrm{r}_{J_\mathfrak{p}}(\mathfrak{a}_\mathfrak{p}) = \dim A_\mathfrak{p} + \mathrm{a}(\mathrm{G}(\mathfrak{a}_\mathfrak{p})) = 1\,.$$

Hence ${\mathfrak{a}_\mathfrak{p}}^{n} = {J_\mathfrak{p}}^{n-1} \cdot \mathfrak{a}_\mathfrak{p}$ for any $n \geq 0$.

We omit the proof of the converse.

2 Noetherian symbolic Rees algebras

In this section, applying Theorem 1.2, we prove the following.

Example 2.1 *Let x, y, z be an sop for A and I an ideal of A generated by the maximal minors of the matrix*

$$\begin{pmatrix} x^{\alpha} & y^5 & z \\ y^2 & z & x^{\alpha'} \end{pmatrix},$$

where α, α' are positive integers with $3\alpha = 2\alpha'$. Then $\mathrm{R}_s(I)$ is finitely generated.

For example, if $A = K[X, Y, Z]$ (K is a field) and I is the defining ideal of the space monomial curve : $X = t^9, Y = t^7, Z = t^{31}$, then I is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^2 & Y^5 & Z \\ Y^2 & Z & X^3 \end{pmatrix}.$$

Proof of Example 1.3. We put $a = z^2 - x^{\alpha'}y^5$, $b = x^{\alpha+\alpha'} - y^2z$, $c = y^7 - x^{\alpha}z$. Then we have the following relations:

$$(1)\quad x^{\alpha}a + y^5b + zc = 0 \qquad (2)\quad y^2a + zb + x^{\alpha'}c = 0\,.$$

Furthermore, there exist $d_2 \in I^{(2)}$, $d_3 \in I^{(3)}$, $d_5 \in I^{(5)}$, $d_7 \in I^{(7)}$ and $d_{10} \in I^{(10)}$ satisfying the following relations:

$$(3)\quad zd_2 = a^2 - x^{\alpha'-\alpha}y^3bc \qquad (4)\quad x^{\alpha'-\alpha}d_3 = -yb^3 - cd_2$$

$$(5)\quad y^2d_3 = -x^{2\alpha-\alpha'}bd_2 - c^3 \qquad (6)\quad cd_5 = y\,(d_3)^2 - b^4d_2$$

$$(7)\quad bd_2d_7 = (d_5)^2 + c\,(d_3)^3 \qquad (8)\quad d_5d_{10} = b\,(d_7)^2 - (d_3)^5\,.$$

As $a \equiv z^2 \bmod (y)$, by (3) we have

$$d_2 \equiv z^3 \bmod (y)\,.$$

As $c \equiv -x^{\alpha}z \bmod (y)$ and $b \equiv x^{\alpha+\alpha'} \bmod (y)$, (4) and (5) respectively imply

$$d_3 \equiv x^{2\alpha-\alpha'}z^4 \bmod (y) \quad \text{and} \quad d_5 \equiv x^{3\alpha+4\alpha'}z^2 \bmod (y)\,.$$

Then, using (7) we see

$$(9)\quad d_7 \equiv x^{5\alpha+7\alpha'}z - z^{10} \bmod (y)\,.$$

Hence, by (8) we get

$$x^{3\alpha+4\alpha'}z^2 \cdot d_{10} \equiv x^{\alpha+\alpha'} \cdot (x^{5\alpha+7\alpha'}z)^2 \bmod (y, z^3)\,,$$

and so

$$(10)\quad d_{10} \equiv x^{8\alpha+11\alpha'} \bmod (y, z)\,.$$

In order to prove Example 2.1,we want to show

(i) $I \subseteq \sqrt{(d_7, d_{10})A}$,

(ii) $\mathrm{R}(I_{\mathfrak{p}})_+ \subseteq \sqrt{(d_7 t^7, d_{10} t^{10})\mathrm{R}(I_{\mathfrak{p}})}$ for any $\mathfrak{p} \in \mathrm{Min}_A A/I$.

First, we prove (i). Let us take any $\mathfrak{q} \in \mathrm{Min}_A A/(d_7, d_{10})A$. It is enough to show $a, b, c \in \mathfrak{q}$. By (8), we get $d_3 \in \mathfrak{q}$. Then $d_5 \in \mathfrak{q}$ follows from (7), and so by (6) we have $b \in \mathfrak{q}$ or $d_2 \in \mathfrak{q}$. In both cases we see $c \in \mathfrak{q}$ by (5). If $x \in \mathfrak{q}$ and $y \in \mathfrak{q}$, then (9) means $z \in \mathfrak{q}$, which is impossible as $\mathrm{ht}_A \mathfrak{q} = 2$. Hence $x \notin \mathfrak{q}$ or $y \notin \mathfrak{q}$. Then, if $b \in \mathfrak{q}$, we get $a \in \mathfrak{q}$ by (1) or (2). So, let us consider the case where $d_2 \in \mathfrak{q}$. In this case, (3) implies $a \in \mathfrak{q}$. On the other hand, by (10) we see $y \notin \mathfrak{q}$ or $z \notin \mathfrak{q}$, and hence we get $b \in \mathfrak{q}$ by (1) or (2).

Next, we prove (ii). Take any $Q \in \mathrm{Spec}\,\mathrm{R}(I_{\mathfrak{p}})$ containing d_7t^7 and $d_{10}t^{10}$. It is enough to show that Q contains two elements among at, bt and ct since $I_{\mathfrak{p}}$ is generated by any two elements among a, b and c. From (3) $\sim$ (8), we get the following relations in $\mathrm{R}(I_{\mathfrak{p}})$:

$$(3')\quad z \cdot d_2t^2 = (at)^2 - x^{\alpha'-\alpha} y^3 \cdot bt \cdot ct$$

$$(5')\quad y^2 \cdot d_3t^3 = -x^{2\alpha-\alpha'} \cdot bt \cdot d_2t^2 - (ct)^3$$

$$(6')\quad ct \cdot d_5t^5 = y \cdot (d_3t^3)^2 - (bt)^4 \cdot d_2t^2$$

$$(7')\quad bt \cdot d_2t^2 \cdot d_7t^7 = (d_5t^5)^2 + ct \cdot (d_3t^3)^3$$

$$(8')\quad d_5t^5 \cdot d_{10}t^{10} = bt \cdot (d_7t^7)^2 - (d_3t^3)^5 .$$

By (8') we get $d_3t^3 \in Q$. Then $d_5t^5 \in Q$ follows from (7'), and so by (6') we have $bt \in Q$ or $d_2t^2 \in Q$. In both cases we see $ct \in Q$ by (5'). If $d_2t^2 \in Q$, we see $at \in Q$ by (3'), and so the proof is complete.

3 Non-Noetherian symbolic Rees algebras

Let us recall the following example due to Goto-Nishida-Watanabe.

Example 3.1 ([1]) *Let $A = \mathbb{Q}[X, Y, Z]$. We denote by $\mathrm{P}_{\mathbb{Q}}(k, \ell, m)$ the defining ideal of the space monomial curve $X = t^k, Y = t^\ell, Z = t^m$. $\mathrm{R}_s(\mathfrak{p})$ is not finitely generated if $\mathfrak{p}$ is one of the following prime ideals.*

(1) $\mathfrak{p} = \mathrm{P}_{\mathbb{Q}}(7n-3, 5n^2-2n, 8n-3)$, *where n is an integer such that $4 \le n$ and $3 \nmid n$. In this case $\mathfrak{p}$ is generated by the maximal minors of the matrix*

$$\begin{pmatrix} X^n & Y^2 & Z^{2n\;\,1} \\ Y & Z^n & X^{2n\;\,1} \end{pmatrix}$$

(2) $\mathfrak{p} = \mathrm{P}_{\mathbb{Q}}(7n - 10, 5n^2 - 7n + 1, 8n - 3)$, *where n is an integer such that* $5 \leq n$, $3 \nmid 7n - 10$ *and* $59 \nmid n + 7$. *In this case* $\mathfrak{p}$ *is generated by the maximal minors of the matrix*

$$\begin{pmatrix} X^n & Y^2 & Z^{2n\ 3} \\ Y & Z^{n\ 1} & X^{2n\ 1} \end{pmatrix}.$$

The next result includes the example stated above as special cases.

Example 3.2 *Let* $A = \mathbb{Q}[X, Y, Z]$ *and* I *an ideal generated by the maximal minors of the matrix*

$$\begin{pmatrix} X^{\alpha} & Y^2 & Z^{\gamma'} \\ Y & Z^{\gamma} & X^{\alpha'} \end{pmatrix},$$

where $\alpha < \alpha' < 2\alpha$, $\gamma < \gamma' < 2\gamma$ *and* $2\alpha'\gamma' < \alpha(4\gamma + \gamma')$. *Then* $\mathrm{R}_s(I)$ *is not finitely generated.*

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Some homological conjectures revisited

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1 Introduction

The late 1950s and early 1960s witnessed significant success of homological methods in commutative algebra, highlighted by the solutions of several open questions on regular local rings by the works of Auslander, Buchsbaum, Serre and others. The most famous one was perhaps the proof that regular local rings are unique factorization domains. Of fundamental importance to these solutions was the property that finitely generated modules over regular local rings have finite projective (in fact free) resolutions.

Various interesting generalizations were suggested by these classical works, and they can be described roughly as follows: if R is a Noetherian local ring, and M is a finitely generated R module of finite projective dimension, then M shares many properties enjoyed by modules over regular local rings. These questions are collectively known as the homological conjectures, and they have had enormous impact on commutative algebra. For some excellent surveys, see [14, 19]. Some of the conjectures are still open today, for example

Conjecture 1.1 (Peskine-Szpiro). *Let (R, m) be a local ring and M, N be finitely generated R-modules such that* $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) = \{m\}$ *and M has finite projective dimension. Then* $\dim M + \dim N \leq \dim R$.

To understand why finite projective dimension is relevant here, recall Serre's definition of intersection multiplicity. In 1961, Serre defined a notion of intersection multiplicity for two finitely generated modules M, N over a regular local ring R with $\ell(M \otimes_R N) < \infty$ as:

Definition 1.2.

$$\chi^R(M, N) = \sum_{i \geq 0} (-1)^i \ell(\operatorname{Tor}_i^R(M, N))$$

Of course, over regular rings all modules have finite projective dimension, so this sum makes sense. Serre proved that Conjecture 1.1 is true over regular local rings and, at least in the geometric case, this definition gives what one expects from intersection theory (vanishing, positivity, etc). Serre's result can be viewed as local analogue of (and in fact it implies) the following: in projective space $\mathbb{P}^n$, two subvarieties U, V such that $\dim U + \dim V \geq n$ must intersect. For example, two lines in projective space must always meet. Definition 1.2 reveals a breathtaking connection between homological algebra and intersection theory, a theme that will hopefully be supported in this note.

Another noteworthy conjecture was

Conjecture 1.3 (Auslander). *Let R be a local ring and M a finite R-module of finite projective dimension. Then R is* Tor*-rigid, that is for any finitely generated R-module M and any integer i,$\operatorname{Tor}_i^R(M,N)=0$ implies $\operatorname{Tor}_j^R(M,N)=0$ for all $j>i$.*

This is a theorem for regular local rings by works of Auslander and Lichtenbaum ([1, 16]). Its relevance to intersection theory can be seen as follows: suppose $M = R/I$ and $N = R/J$ define subschemes of $\operatorname{Spec}(R)$. If $\operatorname{Tor}_i^R(R/I, R/J) = 0$ then all the higher Tor vanish, so by Serre's formula, the intersection multiplicity of $U = \operatorname{Spec} R/I$ and $V = \operatorname{Spec}(R/J)$ would just be the length of $R/(I+J)$. This fact can be viewed as an extension of Bezout theorem.

In this talk, we will survey several novel questions which look similar to the classical homological conjectures, but have different motivations and applications. We hope that they will provide a connection between the classical topics and some recent developments in commutative algebra and related areas.

2 A dimension inequality and lengths of Tor modules over hypersurfaces with isolated singularities

In this Section, we will focus on extensions of Conjecture 1.1 in the hypersurface case. That is, we assume that $R = S/(f)$, where S is regular local and f is an element in the maximal ideal of S. For simplicity, we shall assume that R has *isolated singularity*, i.e. R_P is regular for all $P \in \operatorname{Spec} R - \{m\}$.

In this situation one can study our dimensional inequality using a pairing due to Hochster. For a pair of modules M, N over R, the modules $\operatorname{Tor}_i^R(M,N)$ will only be supported at m for $i \geq \dim R$. So one (Hochster in [15]) can define a function:

Definition 2.1. *(Hochster)*

$$\theta^R(M,N) = \ell(\operatorname{Tor}_{2e+2}^R(M,N)) - \ell(\operatorname{Tor}_{2e+1}^R(M,N))$$

Here e is any integer at least half $\dim R$*. This is well-defined by a result of Eisenbud which asserts that projective resolutions over hypersurfaces become periodic of period at most 2 after* $\dim R$ *steps.*

Hochster also observed that for M, N such that $\ell(M \otimes_R N) < \infty$ one has:

$$\theta^R(M,N) = \chi^S(M,N)$$

Furthermore, θ is biadditive on short exact sequence. So it is a pairing on the *Grothendieck group* of finitely generated modules over R. Let $\mathcal{M}(R)$ be the category of finitely generated R-modules. Recall that the Grothendieck group of finitely generated modules over R is defined as:

$$\mathrm{G}(R) = \frac{\bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z}[M]}{\langle [M_2] - [M_1] - [M_3] \mid 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is exact} \rangle}$$

Now if $\operatorname{pd} M < \infty$, one has a resolution:

$$0 \to R^{n_p} \to \cdots \to R^{n_0} \to M \to 0$$

which shows that in $G(R)$, one has $[M] = (\operatorname{rk} M)[R]$. So $\theta^R(M,N) = (\operatorname{rk} M)\theta^R(R,N) = 0$. Thus $\chi^S(M,N) = 0$, and by positivity property of Serre intersection multiplicity, $\dim M + \dim N < \dim S = \dim R + 1$, so $\dim M + \dim N \leq \dim R$.

In fact the proof just presented shows a lot more: we only need to assume that the class $[M]$ is *torsion* in $G(R)/([R]) = \overline{G}(R)$. In other words $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. In summary:

Proposition 2.2. *(Hochster) Let $R = S/(f)$ be a hypersurface such that S is regular on which positivity property of Serre's intersection multiplicity holds (this is true if R contains a field, for example). Suppose M is a finitely generated R-module such that $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. If N is another finite module such that $\ell(M \otimes_R N) < \infty$. Then $\theta^R(M,N) = 0$ and consequently*

$$\dim M + \dim N \leq \dim R$$

This suggests we can use K-theoretic insights to understand when the dimensional inequality holds. Such general idea has been used quite successfully in works of Kurano, Roberts, Srinivas, and many others.

Let us go back to the geometric case for more insights. One can look at a smooth projective hypersurface X over the complex numbers. There is a curious observation: if $\dim X$ is *odd*, then for any two subvarieties U, V of X such that $\dim U + \dim V \geq \dim X$ must intersect non-trivially.

The reason is basically that the intersection product factors through the cup product in singular cohomology. But one of the subvariety, say U, must have dimension $\dim U > \dim X/2$ (as $\dim X$ is odd). But the Lefschetz hyperplane theorem tells us that the class of U must equal to a power of the hyperplane section, so the cup product $cl(U).cl(V)$ can not be 0. The statement is also true in positive characteristics, but one needs to use l-adic cohomology (see Theorem 4.10 of [5]).

Translating to the local case, we take R to be the local ring at the origin of the cone of X, one then has R is a hypersurface with isolated singularity and $\dim R = \dim X + 1$. Recall that the dimension inequality is tied to the vanishing of Hochster's function, together with some other Conjectures about Chow groups of hypersurfaces, we are encouraged to make (see [4, 5])

Conjecture 2.3 (-). *Let R be a hypersurface with isolated singularity such that $\dim R$ is even. Then $\theta^R(M,N) = 0$ for any pair of modules M, N.*

In other words, if M, N are any two maximal Cohen-Macaulay modules over R, one must have:

$$\ell(\mathrm{Tor}_1^R(M,N)) = \ell(\mathrm{Tor}_2^R(M,N))$$

The above Conjecture has been proved when R is excellent and equicharacteristic of dimension at most 4 ([4]), and when R is standard graded over a field (Moore-Piepmeyer-Spiroff-Walker [17]). Recent results by Polishchuk-Vaintrob implies that the characteristic 0 case is true. Polischuk-Vaintrob defined an Euler characteristic function on the category of *matrix factorizations* of a pair $f \in R = k[[x_1 \cdots, x_n]]$. It comes out to be the following, for any pair of maximal Cohen-Macaulay modules over R:

$$\chi(M,N) = \ell(\mathrm{Ext}_R^1(M,N)) - \ell(\mathrm{Ext}_R^2(M,N))$$

(It is a well-known fact that the category of reduced matrix factorizations of f and the stable category of maximal Cohen-Macaulay modules are equivalent, see [21])

It is not hard to show that the above pairing agrees with $\theta^R(\mathrm{Hom}_R(M,R),N)$ up to sign. Then general results in [18], which use Hochschild cohomology and residue theory, imply the vanishing of $\theta^R(M,N)$ in the characteristic 0 case. Their work and that of Moore-Piepmeyer-Spiroff-Walker suggested the following:

Conjecture 2.4 (Walker). *Let R be a local hypersurface with isolated singularity and $\dim R$ is odd. Then $(-1)^{\frac{n+1}{2}}\theta^R(-,-)$ is semi-positive definite. That is $(-1)^{\frac{n+1}{2}}\theta^R(M,M) \geq 0$.*

Walker's conjecture is easy in dimension 1. The Grothendieck group is generated by the minimal primes of R and the residue field k. It is not hard to see that $\theta^R(k,-) = 0$, and one can compute directly $\theta^R(R/P, R/Q)$ for any pair of minimal primes P, Q. The dimension 3 can be also proved, albeit with slightly more efforts:

Theorem 2.5 (-, Walker). *Let R be a local hypersurface with isolated singularity and $\dim R = 3$. Then $\theta^R(M,M) \geq 0$.*

The proof combines of the following technical statements, which modify similar results in [7]:

Theorem 2.6 (-, Walker). *Let R be local hypersurface of dimension 3. Let M, N be a reflexive R-modules which are locally free on $U_R = \mathrm{Spec}(R) - \{m\}$, the punctured spectrum of R. Suppose $\mathrm{Hom}_R(M,N)$ is a maximal Cohen-Macaulay R-module. Then $\theta^R(M^*,N) \leq 0$.*

Lemma 2.7. *Let R be a local hypersurface of dimension 3 and M, N be a reflexive R-modules which are locally free of constant rank on U_R. Let $[I], [J] \in \mathrm{Cl}(R)$ represent $c_1(M), c_1(N)$ respectively. We have $\theta^R(M, N) = \theta^R(I, J)$.*

Here $c_1(M)$ is the codimension one local Chern class of M, see [3, 11].

3 Gabber's conjecture and rigidity of Tor

The results at the end of the last Section was inspired by this technical result in [7]

Theorem 3.1 (-). *Let R be local hypersurface of dimension 3. Let M be a reflexive R-modules which is locally free on $U_R = \mathrm{Spec}(R) - \{m\}$, the punctured spectrum of R. Suppose $\mathrm{Hom}_R(M, M)$ is a maximal Cohen-Macaulay R-module and $c_1(M)$ is torsion in the codimenison one Chow group of R. Then M is free.*

The above theorem implies that for a local hypersuraface of dimension 3, the Picard group of U_R is torsion-free. This is a special case of

Conjecture 3.2 (Gabber, [12]). *Let R be a local complete intersection of dimension 3. Then $\mathrm{Pic}(U_R)$ is torsion-free.*

Again, this conjecture was inspired by known geometric facts:

Theorem 3.3 (Grothendieck-Lefschetz). *Let X be a complex projective variety and Y be a complete intersection in X such that $\dim Y \geq 2$. Then $\pi_1^{et}(X) \cong \pi_1^{et}(Y)$.*

Here the fundamental groups was constructed using *étale* topology (covering spaces are replaced by finite étale (unramified, flat) maps. They agree with the topological ones for smooth complex projective varieties! In particular, this theorem implies that if X is a complete intersection in complex projective space and $\dim X \geq 2$, then $\pi_1^{et}(X) = 0$. Thisin turn implies $\mathrm{Pic}(X)$ is torsion-free, since a torsion element gives a cyclic cover of X which is étale.

Conjecture 3.2 is known when the order of the torsion element is prime to the characteristic of R by Grothendieck's techniques on local Leftschetz theorems (cf. [2, 20]), and the positive characteristic case can be found in [9] (it is probably known to experts, though we can not find an exact reference. It was claimed in [12] that it is known in positive characteristic). We also note that when U_R is replaced by a smooth projective complete intersection over afield of positive characteristic the analogous result on the Picard group is contained in [8, Theorem 1.8]. In any case, the main difficulty is when R is of mixed characteristic.

With Claudia Miller and Jinjia Li ([9, Theorem 2.9]), we gave a simple proof of the positive characteristic case (as mentioned above, it was claimed by Gabber, but we do not know of any references):

Theorem 3.4 (-, Miller, Li). *Let R be a local complete intersection of characteristic $p > 0$. Then $\mathrm{Pic}(U_R)$ has no p-torsion.*

Actually, we show some thing quite stronger: If I is an ideal such that $\tilde{I}$ is an element of $\mathrm{Pic}(U_R)$, then the *depth* of the p-th symbolic power of I has to be 2, unless I is principal.

The proof used a very interesting result on homological property of the Frobenius map. For a ring R of characteristic $p > 0$, the Frobenius homomorphism $F : R \to R$ takes an element $a \in R$ to a^p.

Theorem 3.5 (Avramov-Miller, Dutta). *Let R be a local complete intersection of characteristic $p > 0$. Let ${}^F R$ be R as a module over itself via the Frobenius homomorphism. Then ${}^F R$ is Tor-rigid, that is for any finitely generated R-module M and any integer i,$\mathrm{Tor}_i^R({}^F R, M) = 0$ implies $\mathrm{Tor}_j^R({}^F R, M) = 0$ for all $j > i$.*

This reminds us of another famous conjecture, Conjecture 1.3 in the Introduction:

Conjecture 3.6 (Auslander). *Let R be a local ring and M a finite R-module of finite projective dimension. Then R is* Tor-*rigid, that is for any finitely generated R-module N and any integer i,$\mathrm{Tor}_i^R(M, N) = 0$ implies $\mathrm{Tor}_j^R(M, N) = 0$ for all $j > i$.*

Auslander's Conjecture is known for regular local rings by works of Auslander and Lichtenbaum ([1, 16]). Unfortunately, it is false in general! Ray Heitmann gave a counter example of projective dimension 2 over a Cohen-Macaulay ring ([13]).

Based on our proof of the hypersurface case of Gabber's conjecture (3.1), we formulate an alternative conjecture for complete intersections in [7].

Conjecture 3.7 (-). *Let R be local complete intersection (of arbitrary dimension). Let M, N be R-modules such that M is locally free of constant rank on U_R and $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. Then (M, N) is* Tor-*rigid.*

A special case of this Conjecture is when M, N have finite lengths. Then all the conditions are automatically satisfied, and the Conjecture asserts that M, N are Tor-rigid. This is false if R is only Gorenstein!

Here is a counter-example for a Gorenstein ring by Hochster-Huneke which we learned from private communication with Huneke. Let $R = \mathbb{C}[[t^9, t^{11}, t^{13}, t^{15}, t^{17}, t^{19}, t^{21}, t^{23}]]$, $I = (t^9, t^{11}, t^{13}, t^{21})$ and $J = (t^{15}, t^{17}, t^{19}, t^{23})$. Then $\mathrm{Tor}_1^R(R/I, R/J) = 0$ but $\mathrm{Tor}_2^R(R/I, R/J) \neq 0$.

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UNIFORM BOUNDS FOR HILBERT COEFFICIENTS OF PARAMETERS

SHIRO GOTO AND KAZUHO OZEKI

1. Introduction

The purpose of this paper is to study the problem of when the Hilbert coefficients of parameter ideals in a Noetherian local ring have uniform bounds, and when this is the case, to ask for their sharp bounds.

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all $n \gg 0$, which we call the Hilbert coefficients of A with respect to I.

With this notation our first purpose is to study the problem of when the sets

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}$$

are finite for all $1 \le i \le d$.

Then the first main result is stated as follows. We say that our local ring is a generalized Cohen-Macaulay ring, if the local cohomology modules $\mathrm{H}_{\mathfrak{m}}^i(A)$ are finitely generated for all $i \neq d$.

Theorem 1.1. *Let A be a Noetherian local ring with $d = \dim A \ge 2$. Then the following conditions are equivalent.*

(1) *A is a generalized Cohen-Macaulay ring.*
(2) *The set $\Lambda_i(A)$ is finite for all $1 \le i \le d$.*

Although the finiteness problem of $\Lambda_i(A)$ is settled affirmatively, we need to ask for the sharp bounds for the values of $\mathrm{e}_Q^i(A)$ of parameter ideals Q, which is our second purpose of the present research.

When A is a generalized Cohen-Macaulay ring with $d = \dim A \ge 2$, one has the inequalities

$$0 \ge \mathrm{e}_Q^1(A) \ge -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A ([8, Theorem 8], [3, Lemma 2.4]), where the equality $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$ holds true if and only if Q is a standard parameter ideal in A ([9, Korollar 3.2], [4, Theorem 2.1]), provided $\operatorname{depth} A > 0$. The reader may consult [2] for the characterization of local rings which contain parameter ideals Q with $\mathrm{e}_Q^1(A) = 0$. Thus the behavior of the first Hilbert coefficients $\mathrm{e}_Q^1(A)$ for parameter ideals Q are rather satisfactorily understood.

The second purpose is to study the natural question of how about $e_Q^2(A)$. First, we will show that in the case where $\dim A = 2$ and $\operatorname{depth} A > 0$, even though A is not necessarily a generalized Cohen-Macaulay ring, the inequality

$$-h^1(A) \leq e_Q^2(A) \leq 0$$

holds true for every parameter ideal Q in A. We will also show that $e_Q^2(A) = 0$ if and only if the ideal Q is generated by a system a, b of parameters which forms a d-sequence in A. When A is a generalized Cohen-Macaulay ring with $\dim A \geq 3$, we shall show that the inequality

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} h^j(A) \leq e_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$$

holds true for every parameter ideal Q (Proposition 3.6). The following theorem which is the second main result of this paper shows that the upper bound

$$e_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$$

is sharp, clarifying when the equality $e_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$ holds true.

Theorem 1.2. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A. Then the following two conditions are equivalent.*

(1) $e_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$.

(2) *There exist elements $a_1, a_2, \cdots, a_d \in A$ such that*
- (a) $Q = (a_1, a_2, \cdots, a_d)$,
- (b) *the sequence $a_1, a_2, \cdots, a_d$ is a d-sequence in A, and*
- (c) $Q \cdot H_{\mathfrak{m}}^j(A/(a_1, a_2, \cdots, a_k)) = (0)$ *for all $j \geq 1$ and $k \geq 0$ with $j + k \leq d-2$.*

When this is the case, we furthermore have the following :

(i) $e_Q^i(A) = (-1)^i \cdot \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} h^j(A)$ *for $3 \leq i \leq d-1$ and*

(ii) $e_Q^d(A) = 0$.

At this moment we do not know the sharp uniform bound for $e_Q^3(A)$ for parameter ideals Q in a generalized Cohen-Macaulay ring A with $\dim A \geq 3$.

Let us briefly note how this paper is organized. We shall prove Theorem 1.1 in Section 2. Theorem 1.2 will be proven in Section 4. Section 3 is devoted to some preliminary steps for the proof of Theorem 1.2. We will closely study in Section 3 the problem of when $e_Q^2(A) = 0$ in the case where $\dim A = 2$.

In what follows, unless otherwise specified, let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We throughout assume that the field $A/\mathfrak{m}$ is infinite. For each $\mathfrak{m}$-primary ideal I in A, we put

$$R(I) = A[It], \quad R'(I) = A[It, t^{-1}], \quad \text{and} \quad G(I) = R'(I)/t^{-1}R'(I),$$

where t is an indeterminate over A. Let $\mathcal{M} = \mathfrak{m}R + R_+$ be the unique graded maximal ideal in $R = R(I)$. We denote by $H_{\mathcal{M}}^i(*)$ ($i \in \mathbb{Z}$) the $i^{\underline{th}}$ local cohomology functor of $R(I)$ with respect to $\mathcal{M}$. Let L be a graded R-module. For each $n \in \mathbb{Z}$ let $[H_{\mathcal{M}}^i(L)]_n$

stand for the homogeneous component of $\mathrm{H}^i_{\mathcal{M}}(L)$ with degree n. We denote by $L(\alpha)$, for each $\alpha \in \mathbb{Z}$, the graded R-module whose grading is given by $[L(\alpha)]_n = L_{\alpha+n}$ for all $n \in \mathbb{Z}$.

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

The heart of the proof of the implication (1) $\Rightarrow$ (2) is, in the case where A is a generalized Cohen-Macaulay ring, the existence of uniform bounds of the Castelnuovo-Mumford regularity $\operatorname{reg} \mathrm{G}(Q)$ of the associated graded rings $\mathrm{G}(Q)$ of parameter ideals Q. So, let us briefly recall the definition of the Castelnuovo-Mumford regularity.

Let Q be a parameter ideal in A and let

$$\mathrm{R}(Q) = A[Qt], \quad \mathrm{R}'(Q) = A[Qt, t^{-1}], \quad \text{and} \quad \mathrm{G}(Q) = \mathrm{R}'(Q)/t^{-1}\mathrm{R}'(Q)$$

respectively denote the Rees algebra, the extended Rees algebra, and the associated graded ring of Q, where t is an indeterminate over A. Let $\mathcal{M} = \mathfrak{m}R + R_+$ be the unique graded maximal ideal in $R = \mathrm{R}(Q)$. For each $i \in \mathbb{Z}$ let

$$\mathrm{a}_i(\mathrm{G}(Q)) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathcal{M}}(\mathrm{G}(Q))]_n \neq (0)\}$$

and put

$$\mathrm{reg}\mathrm{G}(Q) = \max\{\mathrm{a}_i(\mathrm{G}(Q)) + i \mid i \in \mathbb{Z}\},$$

which we call the Castelnuovo-Mumford regularity of the graded ring $\mathrm{G}(Q)$.

Let us now note the following result of Linh and Trung [7], which gives a uniform bound for $\operatorname{reg} \mathrm{G}(Q)$ for parameter ideals Q in a generalized Cohen-Macaulay ring.

Theorem 2.1 ([7], Theorem 2.3). *Suppose that A is a generalized Cohen-Macaulay ring and let Q be a parameter ideal in A. Then*

(1) $\operatorname{reg} \mathrm{G}(Q) \leq \max\{\mathrm{I}(A) - 1, 0\}$, *if* $d = 1$.

(2) $\operatorname{reg} \mathrm{G}(Q) \leq \max\{(4\mathrm{I}(A))^{(d-1)!} - \mathrm{I}(A) - 1, 0\}$, *if* $d \geq 2$.

Thus, the following result is the key for our proof of the implication (1) $\Rightarrow$ (2) in Theorem 1.1, where $h_i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ and $\mathrm{I}(A) = \sum_{j=0}^{d-1} \binom{d-1}{j} h^j(A)$.

Theorem 2.2. *Suppose that A is a generalized Cohen-Macaulay ring. Let Q be a parameter ideal in A and put $r = \operatorname{reg} \mathrm{G}(Q)$. Then*

(1) $|\mathrm{e}^1_Q(A)| \leq \mathrm{I}(A)$.

(2) $|\mathrm{e}^i_Q(A)| \leq 3 \cdot 2^{i-2}(r+1)^{i-1}\mathrm{I}(A)$ *for* $2 \leq i \leq d$.

Proof. See [5, Section 2]. □

Therefore, thanks to the uniform bounds [7, Theorem 2.3] of $\operatorname{reg} \mathrm{G}(Q)$ for parameter ideals Q in a generalized Cohen-Macaulay ring A, we readily get the finiteness in the set $\Lambda_i(A)$ for all $1 \leq i \leq d$.

We are now in a position to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. We may assume that A is complete. Also we may assume A is not unmixed, because $\Lambda_1(A)$ is a finite set (cf. [2, Proposition 4.2]). Let U denote the unmixed component of the ideal (0) in A. We put $B = A/U$ and $\ell = \dim_A U$ $(\leq d-1)$. We must show that B is a generalized Cohen-Macaulay ring and $\ell = 0$.

Let Q be a parameter ideal in A. We then have

$$\ell_A(A/Q^{n+1}) = \ell_A(B/Q^{n+1}B) + \ell_A(U/Q^{n+1} \cap U)$$

for all integers $n \geq 0$. Therefore, the function $\ell_A(U/Q^{n+1} \cap U)$ is a polynomial in $n \gg 0$ with degree t and there exist integers $\{\mathrm{s}_Q^i(U)\}_{0 \leq i \leq t}$ with $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U)$ such that

$$\ell_A(U/Q^{n+1} \cap U) = \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}$$

for all $n \gg 0$, whence

$$\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(B) \binom{n+d-i}{d-i} + \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}.$$

Consequently

$$(-1)^{d-i}\mathrm{e}_Q^{d-i}(A) = \begin{cases} (-1)^{d-i}\mathrm{e}_Q^{d-i}(B) + (-1)^{t-i}\mathrm{s}_Q^{t-i}(U) & \text{if } \ 0 \leq i \leq t, \\ (-1)^{d-i}\mathrm{e}_Q^{d-i}(B) & \text{if } \ t+1 \leq i \leq d. \end{cases}$$

Therefore, if $t < d-1$, we have $\mathrm{e}_Q^1(A) = \mathrm{e}_Q^1(B)$, so that $\Lambda_1(B) = \Lambda_1(A)$ is a finite set. If $t = d-1$, we get $-\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B) + \mathrm{s}_Q^0(U)$. Since $\mathrm{e}_Q^1(A), \mathrm{e}_Q^1(B) \leq 0$ and $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U) \geq 1$, $\Lambda_1(B)$ is a finite set also in this case. Thus the set $\Lambda_1(B)$ is finite in any case, so that the ring B a generalized Cohen-Macaulay ring.

We now assume that $t \geq 1$ and choose a system $a_1, a_2, \cdots, a_d$ of parameters in A so that $(a_{t+1}, a_{t+2}, \cdots, a_d)U = (0)$. Let $\ell \geq 1$ be an integer such that $\mathfrak{m}^\ell$ is standard for the ring B and choose integers $n \geq \ell$. We look at parameter ideals $Q = (a_1^n, a_2^n, \cdots, a_d^n)$ of A. Then

$$(-1)^{d-t}\mathrm{e}_Q^{d-t}(B) = \sum_{j=1}^{t} \binom{t-1}{j-1} h^j(B)$$

by [9, Korollar 3.2], which is independent of the integers $n \geq \ell$. Therefore, since

$$\mathrm{s}_Q^0(U) = \mathrm{e}^0_{(a_1^n, a_2^n, \cdots, a_t^n)}(U) = n^t \cdot \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \geq n^t,$$

we see

$$\begin{aligned} (-1)^{d-t}\mathrm{e}_Q^{d-t}(A) &= (-1)^{d-t}\mathrm{e}_Q^{d-t}(B) + \mathrm{s}_Q^0(U) \\ &= \sum_{j=1}^{t} \binom{t-1}{j-1} h^j(B) + n^t \cdot \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \geq n^t, \end{aligned}$$

whence the set $\Lambda_{d-t}(A)$ cannot be finite. Thus $t = 0$ and A a generalized Cohen-Macaulay ring. □

3. The second Hilbert coefficients $e_Q^2(A)$ of parameters

In this section we study the second Hilbert coefficients $\mathrm{e}_Q^2(A)$ of parameter ideals Q. The purpose is to find the sharp bound for $\mathrm{e}_Q^2(A)$. The bound $|\mathrm{e}_Q^2(A)| \leq 3(r+1)\mathrm{I}(A)$ given by Theorem 1.1 is too huge in general and far from the sharp bound.

Let us begin with the following.

Lemma 3.1. *Suppose that* $d = 2$ *and* $\operatorname{depth} A > 0$. *Let* $Q = (x, y)$ *be a parameter ideal in* A *and assume that* x *is superficial with respect to* Q. *Then*

$$\mathrm{e}_Q^2(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap Q^\ell}{(x^\ell)}\right) \leq 0$$

for all $\ell \gg 0$.

Proof. Let $\ell \gg 0$ be an integer which is sufficiently large and put $I = Q^\ell$. Let $G = \mathrm{G}(I)$ and $R = \mathrm{R}(I)$ be the associated graded ring and the Rees algebra of I, respectively. We put $\mathcal{M} = \mathfrak{m}R + R_+$. Then $[\mathrm{H}_{\mathcal{M}}^i(G)]_n = (0)$ for all integers $i \in \mathbb{Z}$ and $n > 0$, thanks to [6, Lemma 2.4]. We put $a = x^\ell$ and $b = y^\ell$. Then the element a remains superficial with respect to I and the equality $I^2 = (a, b)I$ holds true, whence $\mathrm{a}_2(G) < 0$.

We furthermore have the following.

Claim 1. $[\mathrm{H}_{\mathcal{M}}^i(R)]_0 \cong [\mathrm{H}_{\mathcal{M}}^i(G)]_0$ *as* A*-modules for all* $i \in \mathbb{Z}$. *Hence* $\mathrm{H}_{\mathcal{M}}^0(G) = (0)$, *so that* $f = at \in R$ *is* G*-regular.*

Proof of Claim 1. See [5, Proof of Lemma 3.1]. □

Thanks to [1, Theorem 4.1], Claim 1 shows that

$$\mathrm{e}_Q^2(A) = \sum_{i=0}^{2}(-1)^i \ell_A([\mathrm{H}_{\mathcal{M}}^i(G)]_0) = -\ell_A([\mathrm{H}_{\mathcal{M}}^1(G)]_0),$$

since $a_2(G) < 0$. Therefore to prove

$$\mathrm{e}_Q^2(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap Q^\ell}{(x^\ell)}\right),$$

it is suffices to check that

$$[\mathrm{H}_{\mathcal{M}}^1(G)]_0 \cong \frac{[(a) : b] \cap I}{(a)}$$

as A-modules.

Let $\overline{A} = A/(a)$ and $\overline{I} = I\overline{A}$. Then $G/fG \cong \mathrm{G}(\overline{I})$, because $f = at$ is G-regular (cf. Claim 1). We now look at the exact sequence

$$0 \to \mathrm{H}_{\mathcal{M}}^0(\mathrm{G}(\overline{I})) \to \mathrm{H}_{\mathcal{M}}^1(G)(-1) \xrightarrow{f} \mathrm{H}_{\mathcal{M}}^1(G)$$

of local cohomology modules which is induced from the exact sequence

$$0 \to G(-1) \xrightarrow{f} G \to \mathrm{G}(\overline{I}) \to 0$$

of graded G-modules. Then, since $[\mathrm{H}_{\mathcal{M}}^1(G)]_n = (0)$ for all $n \geq 1$, we have an isomorphism

$$[\mathrm{H}_{\mathcal{M}}^0(\mathrm{G}(\overline{I}))]_1 \cong [\mathrm{H}_{\mathcal{M}}^1(G)]_0$$

of A-modules and the vanishing $[\mathrm{H}_{\mathcal{M}}^0(\mathrm{G}(\overline{I})]_n = (0)$ for $n \geq 2$.

Look now at the homomorphism

$$\rho : \frac{[(a) : b] \cap I}{(a)} \to [\mathrm{H}_{\mathcal{M}}^0(\mathrm{G}(\overline{I}))]_1$$

of A-modules defined by $\rho(\overline{x}) = \overline{\overline{x}t}$ for each $x \in [(a) : b] \cap I$, where $\overline{x}$ and $\overline{\overline{x}t}$ denote the images of x in $\overline{A}$ and $\overline{x}t \in [\mathrm{R}(\overline{I})]_1$ in $\mathrm{G}(\overline{I})$, respectively. We will show that the map ρ is an isomorphism. Take $\varphi \in [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1$ and write $\varphi = \overline{\overline{x}t}$ with $x \in I$. Since $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(I))]_2 = (0)$, we have $bt \cdot \overline{\overline{x}t} = \overline{\overline{bx}t^2} = 0$ in $\mathrm{G}(\overline{I})$, whence $bx \in [(a) + I^3] \cap I^2 = [(a) \cap I^2] + I^3 = aI + bI^2$ (recall that $I^2 = (a,b)I$ and that a is super-regular with respect to I). So, we write $bx = ai + bj$ with $i \in I$ and $j \in I^2$. Then, since $b(x-j) = ai \in (a)$, we have $x - j \in [(a) : b] \cap I$, whence $\varphi = \overline{\overline{x}t} = \overline{\overline{(x-j)}t}$. Thus the map ρ is surjective.

To show that the map ρ is injective, take $x \in [(a) : b] \cap I$ and suppose that $\rho(\overline{x}) = \overline{\overline{x}t} = 0$ in $\mathrm{G}(\overline{I})$. Then

$$x \in [(a) : b] \cap [(a) + I^2] = (a) + [((a) : b) \cap I^2].$$

To conclude that $x \in (a)$, we need the following.

Claim 2. *Let $n \geq 2$ be an integer. Then $[(a) : b] \cap I^n \subseteq (a) + [((a) : b) \cap I^{n+1}]$.*

Proof of Claim 2. See [5, Proof of Lemma 3.1]. □

Since $x \in (a) + [((a) : b) \cap I^2]$, thanks to Claim 2, we get $x \in (a) + I^{n+1}$ for all $n \geq 1$, whence $x \in (a)$, so that the map ρ is injective. Thus

$$[\mathrm{H}^1_{\mathcal{M}}(G)]_0 \cong \frac{[(a) : b] \cap I}{(a)}$$

as A-modules. □

Theorem 3.2. *Suppose that $d = 2$ and* $\operatorname{depth} A > 0$. *Let $Q = (x, y)$ be a parameter ideal in A and assume that x is superficial with respect to Q. Then*

$$-h^1(A) \leq \mathrm{e}^2_Q(A) \leq 0$$

and the following three conditions are equivalent.

(1) $\mathrm{e}^2_Q(A) = 0$.

(2) *x, y forms a d-sequence in A.*

(3) *x^ℓ, y^ℓ forms a d-sequence in A for all integers $\ell \geq 1$.*

Passing to the ring $A/\mathrm{H}^0_{\mathfrak{m}}(A)$, thanks to Theorem 3.2, we readily get the following.

Corollary 3.3. *Suppose that $d = 2$ and let Q be a parameter ideal in A. Then*

$$h^0(A) - h^1(A) \leq \mathrm{e}^2_Q(A) \leq h^0(A).$$

Proof of Theorem 3.2. By Lemma 3.1 we have

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap (x,y)^\ell}{(x^\ell)}\right) \leq 0$$

for all integers $\ell \gg 0$. To show that $-h^1(A) \leq \mathrm{e}^2_Q(A)$, we may assume that $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated. Take the integer $\ell \gg 0$ so that the system $a = x^\ell$, $b = y^\ell$ of parameters of A is standard. Then since

$$\frac{[(a) : b] \cap Q^\ell}{(a)} \subseteq \frac{(a) : b}{(a)} \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a)) \cong \mathrm{H}^1_{\mathfrak{m}}(A),$$

we get $-h^1(A) \leq \mathrm{e}^2_Q(A)$.

Let us consider the second assertion.

(1) $\Rightarrow$ (3). Take an integer $N \geq 1$ so that

$$\mathrm{e}_Q^2(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap (x, y)^\ell}{(x^\ell)}\right)$$

for all $\ell \geq N$ (cf. Lemma 3.1); hence

$$[(x^\ell) : y^\ell] \cap (x, y)^\ell = (x^\ell).$$

Claim 3. *$[(x^\ell) : y^\ell] \cap (x, y)^\ell = (x^\ell)$ for all $\ell \geq 1$.*

Proof of Claim 3. We may assume that $1 \leq \ell < N$. Take $\tau \in [(x^\ell) : y^\ell] \cap (x, y)^\ell$. Then, since $y^N(x^{N-\ell}\tau) = y^{N-\ell}x^{N-\ell}(y^\ell\tau) \in (x^N)$, we have $x^{N-\ell}\tau \in [(x^N) : y^N] \cap (x, y)^N = (x^N)$. Thus $\tau \in (x^\ell)$, because x is A-regular (recall that $\operatorname{depth} A > 0$ and x is superficial with respect to Q). ☐

Since x^ℓ is A-regular and $[(x^\ell) : y^\ell] \cap (x^\ell, y^\ell) = (x^\ell)$ by Claim 3, we readily see that x^ℓ, y^ℓ is a d-sequence in A.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) It is well-known that $\mathrm{e}^2_{(x,y)}(A) = 0$, if $\operatorname{depth} A > 0$ and the system x, y of parameters forms a d-sequence in A; see Proposition 3.4 below. ☐

The results in the following proposition are, more or less, known.

Proposition 3.4. *Suppose that $d > 0$ and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Let $G = \mathrm{G}(Q)$ and $R = \mathrm{R}(Q)$. Let $f_i = a_i t \in R$ for $1 \leq i \leq d$. Assume that the sequence $a_1, a_2, \cdots, a_d$ forms a d-sequence in A. Then we have the following, where $Q_i = (a_1, a_2, \cdots, a_i)$ for $0 \leq i \leq d$.*

1. *$\mathrm{e}_Q^0(A) = \ell_A(A/Q) - \ell_A([Q_{d-1} : a_d]/Q_{d-1})$.*
2. *$(-1)^i\mathrm{e}_Q^i(A) = h^0(A/Q_{d-i}) - h^0(A/Q_{d-i-1})$ for $1 \leq i \leq d-1$ and $(-1)^d\mathrm{e}_Q^d(A) = h^0(A)$.*
3. *$\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d}(-1)^i\mathrm{e}_Q^i(A)\binom{n+d-i}{d-i}$ for all $n \geq 0$, whence $\ell_A(A/Q) = \sum_{i=0}^{d}(-1)^i\mathrm{e}_Q^i(A)$.*
4. *$f_1, f_2, \cdots, f_d$ forms a d-sequence in G.*
5. *$\mathrm{H}^0_{\mathcal{M}}(G) = [\mathrm{H}^0_{\mathcal{M}}(G)]_0 \cong \mathrm{H}^0_{\mathfrak{m}}(A)$, where $\mathcal{M} = \mathfrak{m}R + R_+$*
6. *$[\mathrm{H}^i_{\mathcal{M}}(G)]_n = (0)$ for all $n > -i$ and $i \in \mathbb{Z}$, whence $\operatorname{reg} G = 0$.*

Let us note one example of local rings A which are not generalized Cohen-Macaulay rings but every parameter ideal in A is generated by a system of parameters that forms a d-sequence in A.

Example 3.5. Let R be a regular local ring with the maximal ideal $\mathfrak{n}$ and $d = \dim R \geq 2$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. We put $\mathfrak{p} = (X_1, X_2, \cdots, X_{d-1})$ and $D = R/\mathfrak{p}$. Then D is a DVR. Let $A = R \ltimes D$ denote the idealization of D over R. Then A is a Noetherian local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times D$, $\dim A = d$, and $\operatorname{depth} A = 1$. We furthermore have the following.

1. $\Lambda_i(A) = \{0\}$ for all $1 \leq i \leq d$ such that $i \neq d-1$.
2. $\Lambda_0(A) = \{n \mid 0 < n \in \mathbb{Z}\}$ and $\Lambda_{d-1}(A) = \{(-1)^{d-1}n \mid 0 < n \in \mathbb{Z}\}$.
3. After renumbering, every system of parameters in A forms a d-sequence.

The ring A is not a generalized Cohen-Macaulay ring, because $\mathrm{H}^1_{\mathfrak{m}}(A)$ ($\cong \mathrm{H}^1_{\mathfrak{n}}(D)$) is not a finitely generated A-module.

We now consider the case where $\dim A \geq 3$

Theorem 3.6. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Then*

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} h^j(A) \leq \mathrm{e}_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A).$$

We have $Q{\cdot}\mathrm{H}_\mathfrak{m}^j(A/(a_1, a_2, \cdots, a_k)) = (0)$ for all $k \geq 0$ and $j \geq 1$ with $j + k \leq d - 2$, if $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$ and if $a_1, a_2, \cdots, a_d$ forms a superficial sequence with respect to Q.

Proof. See [5, Theorem 3.6]. □

4. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Thanks to Proposition 3.4 and Theorem 3.6, we have only to show the following.

Theorem 4.1. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A and assume that $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$. Then Q is generated by a system of parameters which forms a d-sequence in A.*

For each ideal $\mathfrak{a}$ in A ($\mathfrak{a} \neq A$) let $\mathrm{U}(\mathfrak{a})$ denote the unmixed component of $\mathfrak{a}$. When $\mathfrak{a} = (a)$ with $a \in A$, we write $\mathrm{U}(\mathfrak{a})$ simply by $\mathrm{U}(a)$. We have

$$\mathrm{U}(a) = \bigcup_{n \geq 0} [(a) :_A \mathfrak{m}^n],$$

if A is a generalized Cohen-Macaulay ring with $\dim A \geq 2$ and a is a part of a system of parameters in A (cf. [10, Section 2]). The following result is the key in our proof of Theorem 4.1.

Proposition 4.2. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 2$ and $\operatorname{depth} A > 0$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Assume that $a_d \mathrm{H}_\mathfrak{m}^1(A) = (0)$ and that the sequence $a_1, a_2, \cdots, a_{d-1}$ forms a d-sequence in the generalized Cohen-Macaulay ring $A/\mathrm{U}(a_d)$. Then*

$$\mathrm{U}(a_1) \cap [Q + \mathrm{U}(a_d)] = (a_1).$$

We are now ready to prove Theorem 4.1.

Proof of the Theorem 4.1. We proceed by induction on d. Choose $a_1, a_2, \cdots, a_d \in A$ so that $Q = (a_1, a_2, \cdots, a_d)$ and for each $1 \leq i \leq d-2$, the $i+2$ elements $a_1, a_2, \cdots, a_i, a_{d-1}, a_d$ form a superficial sequence with respect to Q. We will show that there exist $b_2, b_3, \cdots, b_d \in A$ such that $b_1 = a_{d-1}, b_2, b_3, \cdots, b_d$ forms a d-sequence in A and $Q = (b_1, b_2, \cdots, b_d)$. We put $\overline{A} = A/(a_1)$, $\overline{Q} = Q\overline{A}$, and $C = \overline{A}/\mathrm{H}_\mathfrak{m}^0(\overline{A})$ $(= A/\mathrm{U}(a_1))$.

Suppose that $d = 3$. Then

$$\mathrm{e}_{QC}^2(C) = \mathrm{e}_{\overline{Q}}^2(\overline{A}) - h^0(\overline{A}) = \mathrm{e}_Q^2(A) - h^0(\overline{A}) = h^1(A) - h^0(\overline{A}) = 0,$$

because $h^1(A) = h^0(\overline{A})$ (recall that $Q\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ by Proposition 3.6). Hence, thanks to Proposition 3.2, a_2, a_3 forms a d-sequence in C, because a_2 is superficial for the ideal $QC = (a_2, a_3)C$. Therefore, since $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, we have

$$\mathrm{U}(a_2) \cap [Q + \mathrm{U}(a_1)] = (a_2),$$

by Proposition 4.2. Let $Q = (a_2, a_3, b_3)$ and $B = A/\mathrm{U}(a_2)$. Then since $\mathrm{e}^2_{QB}(B) = 0$, by Proposition 3.2 the sequence $b_2 = a_3, b_3$ forms a d-sequence in B, because b_2 is superficial for QB. Therefore, since $\mathrm{U}(a_2) \cap Q \subseteq \mathrm{U}(a_2) \cap [Q + \mathrm{U}(a_1)] = (a_2)$, the sequence b_2, b_3 forms a d-sequence in $A/(a_2)$, so that $b_1 = a_2, b_2, b_3$ forms a d-sequence in A, because b_1 is A-regular.

Assume that $d \geq 4$ and that our assertion holds true for $d - 1$. Then, thanks to Theorem 3.6 and its proof, we have

$$\begin{aligned}\mathrm{e}^2_Q(A) = \mathrm{e}^2_{\overline{Q}}(\overline{A}) = \mathrm{e}^2_{QC}(C) &\leq \sum_{j=1}^{d-3}\binom{d-4}{j-1}h^j(C)\\ &= \sum_{j=1}^{d-3}\binom{d-4}{j-1}h^j(\overline{A})\\ &= \sum_{j=1}^{d-2}\binom{d-3}{j-1}h^j(A) = \mathrm{e}^2_Q(A),\end{aligned}$$

because $Q{\cdot}\mathrm{H}^j_{\mathfrak{m}}(A) = (0)$ for $1 \leq j \leq d-3$. Hence

$$\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3}\binom{d-4}{j-1}h^j(C).$$

Therefore, because $QC = (\overline{a_2}, \overline{a_3}, \cdots, \overline{a_d})C$ and the sequence $\overline{a_2}, \overline{a_3}, \cdots, \overline{a_i}, \overline{a_{d-1}}, \overline{a_d}$ is superficial in the ideal QC for all $1 \leq i \leq d-2$ where $\overline{a_j}$ denotes the image of a_j in C, the hypothesis of induction on d yields that there exist $\gamma_2, \gamma_3, \cdots, \gamma_{d-1} \in C$ such that the sequence $\gamma_1 = \overline{a_{d-1}}, \gamma_2, \gamma_3, \cdots, \gamma_{d-1}$ forms a d-sequence in C and $QC = (\gamma_1, \gamma_2, \cdots, \gamma_{d-1})C$. Let us write $\gamma_j = \overline{c_j}$ for each $2 \leq j \leq d-1$ with $c_j \in Q$, where $\overline{c_j}$ denote the image of c_j in C. We put $\mathfrak{q} = (a_1, a_{d-1}, c_2, c_3, \cdots, c_{d-1})$. Then $\mathfrak{q}$ is a parameter ideal in A, $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, and $a_{d-1}, c_2, c_3, \cdots, c_{d-1}$ forms a d-sequence in C. Therefore

$$\mathrm{U}(a_{d-1}) \cap [Q + \mathrm{U}(a_1)] = \mathrm{U}(a_{d-1}) \cap [\mathfrak{q} + \mathrm{U}(a_1)] = (a_{d-1})$$

by Proposition 4.2, whence $\mathrm{U}(a_{d-1}) \cap Q = (a_{d-1})$.

Let $B = A/\mathrm{U}(a_{d-1})$. We then have

$$\mathrm{e}^2_{QB}(B) = \sum_{j=1}^{d-3}\binom{d-4}{j-1}h^j(B)$$

for the same reason as for the equality

$$\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3}\binom{d-4}{j-1}h^j(C)$$

(in fact, to show $\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C)$, we only need that a_1 is superficial with respect to Q). Therefore, by the hypothesis of induction on d, we may choose elements $\beta_2, \beta_3, \cdots, \beta_d \in B$ so that $QB = (\beta_2, \beta_3, \cdots, \beta_d)B$ and the sequence $\beta_2, \beta_3, \cdots, \beta_d$ forms a d-sequence in B. We put $b_1 = a_{d-1}$ and write $\beta_j = \overline{b_j}$ with $b_j \in Q$ for $2 \leq j \leq d$, where $\overline{b_j}$ denotes the image of b_j in B. We now put $\mathfrak{q}' = (b_1, b_2, \cdots, b_d)$. Then $\mathfrak{q}'$ is a parameter ideal in A and because $\mathrm{U}(b_1) \cap Q = (b_1)$, we get

$$Q \subseteq [\mathfrak{q}' + \mathrm{U}(b_1)] \cap Q = \mathfrak{q}' + [\mathrm{U}(b_1) \cap Q] \subseteq \mathfrak{q}' + (b_1) = \mathfrak{q}';$$

hence $Q = \mathfrak{q}'$. Thus the sequence $b_2, b_3, \cdots, b_d$ forms a d-sequence in $A/(b_1)$, so that $b_1, b_2, \cdots, b_d$ forms a d-sequence in A, because b_1 is A-regular. This complete the proof of Theorem 4.1 and that of Theorem 1.2 as well. □

Acknowledgements

The authors are most grateful to Hoang Le Truong and Ngo Viet Trung for their inspiring discussions during the 5-th Japan-Vietnam Joint Seminar on Commutative Algebra (January 5-9, 2010, Institute of Mathematics Hanoi). Our Theorem 1.1 is deep in debt from their suggestions.

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A criterion of a Hibi ring to be of type 2

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1 Introduction

Hibi ring is an algebra with straightening law (ASL for short) on a distributive lattice with straightening relation $\alpha\beta = (\alpha \wedge \beta)(\alpha \vee \beta)$ for any incomparable α and β. It has a structure of an affine semigroupring and is normal Cohen-Macaulay if so is the base ring. Furthermore in many important cases in practice, a ring in question can be deformed into a Hibi ring. So it is important to know the ring theoretical properties of Hibi rings.

Let H be a finite distributive lattice and P the set of join-irreducible elements of H. As one can reconstruct H from P, the Hibi ring on H is thoroughly determined by P. Hibi [Hib] showed that the Hibi ring on H is Gorenstein if and only if P is pure.

As Gorenstein ring is a Cohen-Macaulay ring with type 1, the simplest Hibi ring next to Gorenstein is Hibi ring of type 2. In this note we characterize whether the Hibi ring defined by P is of type 2 in terms of the combinatorial property of P.

2 Preliminary

In this note all rings and algebras are assumed to be commutative with identity element.

We denote by $\boldsymbol{N}$ the set of non-negative integers, by $\boldsymbol{Z}$ the set of integers, by $\boldsymbol{R}$ the set of real numbers and by $\boldsymbol{R}_{\geq 0}$ the set of non-negative real numbers.

First we recall some definitions concerning partially ordered sets (poset for short).

Definition 2.1 Let Q be a finite poset.

- A chain in Q is a totally ordered subuset of Q.

- For a chain X in Q, we define the length of X as $\#X - 1$.

- The maximum length of chains in Q is called the rank of Q and denoted as $\mathrm{rank}Q$.

- If every maximal chain has the same length, we say that Q is pure.

- If x, $y \in Q$, $x < y$ and there is no $z \in Q$ with $x < z < y$, we say y covers x and denote $x \lessdot y$ or $y \gtrdot x$.

- For x, $y \in Q$ with $x \leq y$, we set $[x, y]_Q := \{z \in Q \mid x \leq z \leq y\}$ and for x, $y \in Q$ with $x < y$, we set $[x, y)_Q := \{z \in Q \mid x \leq z < y\}$ and $(x, y]_Q := \{z \in Q \mid x < z \leq y\}$.

- Let ∞ be a new element which is not contained in Q. We set $Q^+ := Q \cup \{\infty\}$ with the order $x < \infty$ for $\forall x \in Q$.

- For a chain X in Q, we set $\mathrm{star}_Q(X) := \{z \in Q \mid X \cup \{z\} \text{ is a chain}\}$.

- If $I \subset Q$ and $x \in I$, $y \in Q$, $y \leq x \Rightarrow y \in I$, then we say that I is a poset ideal of Q.

- For a non-empty subset S of Q, we say the set $\{x \in Q \mid \exists s \in S; s \geq x\}$ the poset ideal generated by S.

- Q is a lattice if for any x, $y \in Q$, $\{x, y\}$ has both minimum upper bound and maximum lower bound in Q. These elements are denoted by $x \vee y$ and $x \wedge y$ respectively.

- Q is a distributive lattice if Q is a lattice and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any x, y, $z \in Q$.

Next we recall the definition of a Hibi ring. Let K be a field, H a finite distributive lattice, x_0 the unique minimal element of H and P the set of join-irreducible elements of H, i.e., $P = \{x \in H \mid x = \alpha \vee \beta \Rightarrow x = \alpha$ or $x = \beta\}$. Note that we treat x_0 as a join-irreducible element. It is known that H is isomorphic to the set of non-empty poset ideals of P ordered by inclusion.

Let $\{T_x\}_{x \in P}$ be a family of indeterminates indexed by P.

Definition 2.2 ([Hib]) $\mathcal{R}_K(H) := K[\prod_{x \leq \alpha} T_x \mid \alpha \in H]$.

$\mathcal{R}_K(H)$ is called the Hibi ring over K on H. By setting $\deg T_{x_0} = 1$ and $\deg T_x = 0$ for any $x \in Q \setminus \{x_0\}$, $\mathcal{R}_K(H)$ is a standard graded algebra.

Definition 2.3 For a map $\nu \colon P \to \boldsymbol{N}$, we set $T^{\nu} := \prod_{x \in P} T_x^{\nu(x)}$. We set $\overline{\mathcal{T}}(P) := \{\nu \colon P \to \boldsymbol{N} \mid x \leq y \Rightarrow \nu(x) \geq \nu(y)\}$ and $\mathcal{T}(P) := \{\nu \colon P \to \boldsymbol{N} \setminus \{0\} \mid x < y \Rightarrow \nu(x) > \nu(y)\}$.

With this notation,

Theorem 2.4 ([Hib]) $\mathcal{R}_K(H) = \bigoplus_{\nu \in \overline{\mathcal{T}}(P)} KT^{\nu}$. *In particular, by the result of Hochster [Hoc], $\mathcal{R}_K(H)$ is a normal affine semigroupring and is Cohen-Macaulay.*

Note that $\deg T^{\nu} = \nu(x_0)$.

Here we recall the description of the canonical module of a normal affine semigroupring by Stanley [Sta].

Theorem 2.5 (Stanley) *Let S be a finitely generated additive submonoid of $\boldsymbol{N}^n$ and $X_1, \ldots, X_n$ indeterminates. If the affine semigroupring $\bigoplus_{s \in S} KX^s$ in $K[X_1, \ldots, X_n]$ is normal, then the canonical module of $\bigoplus_{s \in S} KX^s$ is $\bigoplus_{s \in S \cap \mathrm{relint} \boldsymbol{R}_{\geq 0} S} KX^s$.*

Corollary 2.6 *The canonical module of $\mathcal{R}_K(H)$ is $\bigoplus_{\nu \in \mathcal{T}(P)} KT^{\nu}$.*

In order to describe the generators of the canonical module of $\mathcal{R}_K(H)$, we state the following

Definition 2.7 We define the order on $\mathcal{T}(P)$ by setting $\nu \leq \nu' \overset{\text{def}}{\iff} \nu' - \nu \in \overline{\mathcal{T}}(P)$ for ν, $\nu' \in \mathcal{T}(P)$, where $(\nu' - \nu)(x) := \nu'(x) - \nu(x)$.

Then the following fact is easily verified.

Corollary 2.8 *Let ν be an element of $\mathcal{T}(P)$. Then T^{ν} is a generator of the canonical module of $\mathcal{R}_K(H)$ if and only if ν is a minimal element of $\mathcal{T}(P)$.*

Finally, we recall the following characterization of Gorenstein property of $\mathcal{R}_K(H)$ by Hibi [Hib].

Theorem 2.9 (Hibi) *$\mathcal{R}_K(H)$ is Gorenstein if and only if P is pure.*

3 Characterizations of type 2 Hibi rings

In this section we state our main results. As in the previous section, let K be a field, H a finite distributive lattice, x_0 the minimal element of H and P the set of join-irreducible elements of H. Set $r = \mathrm{rank} P^{+}$. Then

Theorem 3.1 *$\mathcal{R}_K(H)$ is level and* $\text{type}\mathcal{R}_K(H) = 2$ *if and only if there exists $x \in P$ such that*

(1) $P^+ \setminus \{x\}$ is pure of rank r and

(2) $P^+ \cup \{x^*\}$ is pure of rank r,

where x^* is a new element not contained in P^+ and we define the order of $P^+ \cup \{x^*\}$ by extending that of P^+ by $z < x^* \iff z \leq x$, $z > x^* \iff z > x$ for $z \in P^+$

Remark 3.2 In the notation of Theorem 3.1, if both $P^+ \setminus \{x\}$ and $P^+ \cup \{x^*\}$ are pure and one of them is of rank r, then the other one is of rank r automatically. Furthermore, this condition 'rank is r' is equivalent to $P^+ \neq \text{star}_{P^+}(x)$.

Theorem 3.3 *$\mathcal{R}_K(H)$ is not level and* $\text{type}\mathcal{R}_K(H) = 2$ *if and only if there exist x, $y \in P$ such that*

(1) $x \lessdot y$,

(2) $P^+ = [x_0, y]_{P^+} \cup [x, \infty]_{P^+}$,

(3) $(P^+, < \setminus\{(x, y)\})$ is pure of rank r and

(4) $P^+ \cup \{x^*\}$ is pure of rank r,

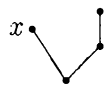
where in (3), we identify the binary relation $<$ with the subset $\{(z, w) \mid z < w\}$ of $P^+ \times P^+$ and denote by $(P^+, < \setminus\{(x, y)\})$ the poset with base set P^+ and strict order relation $\{(z, w) \mid z < w\} \setminus \{(x, y)\}$ and in (4), x^* is a new element not contained in P^+ and we define the order of $P^+ \cup \{x^*\}$ by extending that of P^+ by $z < x^* \iff z \leq x$, $z > x^* \iff z \geq y$ for $z \in P^+$.

Remark 3.4 In the notation of Theorem 3.3, if both $(P^+, < \setminus\{(x, y)\})$ and $P^+ \cup \{x^*\}$ are pure and one of them is of rank r, then the other one is of rank r automatically. Furthermore, this condition 'rank is r' is equivalent to $P^+ \neq \text{star}_{P^+}(\{x, y\})$.

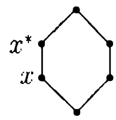
4 Examples

In this section, we state some examples concerning Theorems 3.1 and 3.3. First we state examples concerning Theorem 3.1. Note that H is uniquely determined by P since H is isomorphic to the set of non-empty poset ideals of P ordered by inclusion.

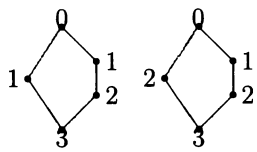
Example 4.1 Let P be the following poset.

$P =$ 

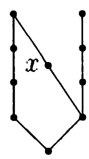
Then

$P^+ \setminus \{x\} =$ and $P^+ \cup \{x^*\} =$  .

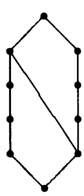
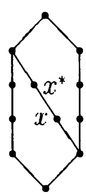
Therefore, type$\mathcal{R}_K(H) = 2$ by Theorem 3.1. In fact, there are 2 minimal elements in $\mathcal{T}(P)$ as follows.



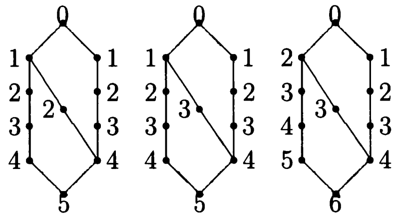
Example 4.2 Let P be the following poset.

$P =$  .

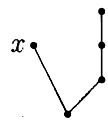
Then

$P^+ \setminus \{x\} =$  and $P^+ \cup \{x^*\} =$  .

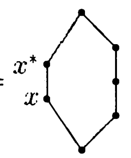
Therefore, type$\mathcal{R}_K(H) \neq 2$ by Theorems 3.1 and 3.3. In fact there are 3 minimal elements of $\mathcal{T}(P)$ as follows.



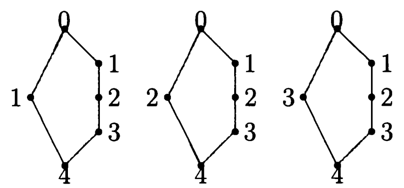
Example 4.3 Let P be the following poset.

$P =$  .

Then

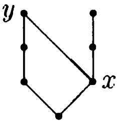
$P^+ \setminus \{x\} =$ and $P^+ \cup \{x^*\} =$  .

Therefore type$\mathcal{R}_K(H) \neq 2$ by Theorems 3.1 and 3.3. In fact there are 3 minimal elements in $\mathcal{T}(P)$.



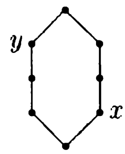
Next we state examples concerning Theorem 3.3.
We set $(P^+, < \setminus\{(x,y)\}) = \tilde{P}$.

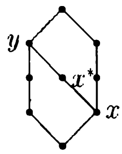
Example 4.4 Let P be the following poset.

$P =$ 

.

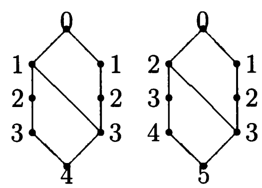
Then

$\tilde{P} =$ 

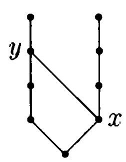
and $P^+ \cup \{x^*\} =$ 

.

Therefore type$\mathcal{R}_K(H) = 2$ by Theorem 3.3. In fact the minimal elements of $\mathcal{T}(P)$ are

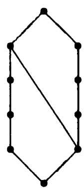
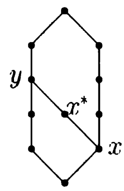


Example 4.5 Let P be the following poset.

$P =$ 

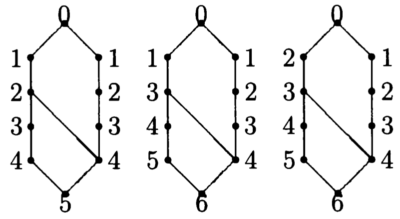
.

Then

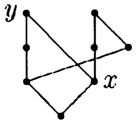
$\tilde{P} =$ 
and $P^+ \cup \{x^*\} =$ 

.

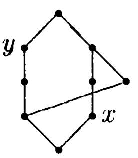
Therefore type$\mathcal{R}_K(H) \neq 2$ by Theorems 3.1 and 3.3. In fact there are 3 minimal elements in $\mathcal{T}(P)$.

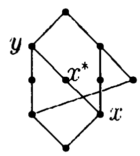


Example 4.6 Let P be the following poset.

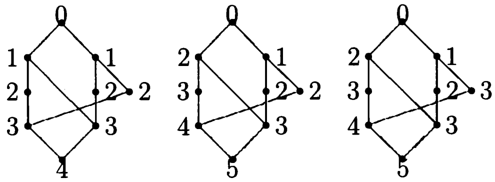
$P =$  .

Then

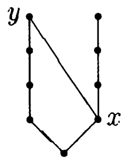
$\tilde{P} =$ 

and $P^+ \cup \{x^*\} =$  .

Therefore type$\mathcal{R}_K(H) \neq 2$ by Theorems 3.1 and 3.3. In fact there are 3 minimal elements in $\mathcal{T}(P)$.



Example 4.7 Let P be the following poset.

$P =$ 

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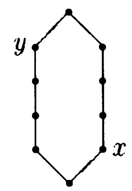
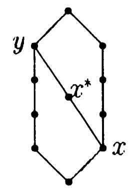
Then

$\tilde{P} =$ 

and $P^+ \cup \{x^*\} =$ 

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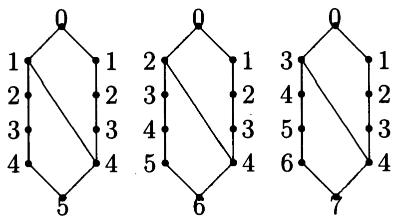
Therefore type$\mathcal{R}_K(H) \neq 2$ by Theorems 3.1 and 3.3. In fact there are 3 minimal elements in $\mathcal{T}(P)$.



5 A sketch of proof

In this section, we sketch the proofs of Theorems 3.1 and 3.3. We use the notation of these theorems.

First we make the following

Definition 5.1 Let $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ be a (possibly empty) sequence of elements in $P \setminus \{x_0\}$. We say the sequence $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ satisfies the condition N if

(1) $y_1 > x_1 < y_2 > x_2 < \cdots < y_t > x_t$.

(2) For any i, j with $1 \leq i < j \leq t$, x_j is not contained in the poset ideal generated by $\{y_1, y_2, \ldots, y_i\}$.

Definition 5.2 For a sequence $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ in P with condition N, we set

$$r(y_1, x_1, y_2, x_2, \ldots, y_t, x_t) := \sum_{i=1}^{t+1} \operatorname{rank}[x_{i-1}, y_i]_{P^+} - \sum_{i=1}^{t} \operatorname{rank}[x_i, y_i]_{P^+},$$

where we set $y_{t+1} := \infty$ and $r() = r$. We also set $r_{\max} := \max\{r(y_1, x_1, \ldots, y_t, x_t) \mid y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ satisfies the condition N$\}$.

Definition 5.3 A sequence $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ of elements in P is called a special sequence if the following conditions are satisfied.

(1) $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ satisfies condition N.

(2) $r(y_1, x_1, \ldots, y_t, x_t) = r_{\max}$.

(3) If $y'_1, x'_1, y'_2, x'_2, \ldots, y'_{t'}, x'_{t'}$ is a sequence of elements in P with condition N and $r(y'_1, x'_1, \ldots, y'_{t'}, x'_{t'}) = r_{\max}$, then $t \leq t'$.

(4) For any i with $1 \leq i \leq t$, $\operatorname{rank}[z, y_{i+1}]_{P^+} - \operatorname{rank}[z, y_i]_{P^+} < \operatorname{rank}[x_i, y_{i+1}]_{P^+} - \operatorname{rank}[x_i, y_i]_{P^+}$ for any $z \in (x_i, y_{i+1}]_{P^+} \cap (x_i, y_i]_{P^+}$ and $\operatorname{rank}[x_{i-1}, z]_{P^+} - \operatorname{rank}[x_i, z]_{P^+} < \operatorname{rank}[x_{i-1}, y_i]_{P^+} - \operatorname{rank}[x_i, y_i]_{P^+}$ for any $z \in [x_{i-1}, y_i)_{P^+} \cap [x_i, y_i)_{P^+}$.

Here we recall our previous result [Miy].

Theorem 5.4 *(1) If ν is a minimal element of $\mathcal{T}(P)$, then there is a sequence $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ in P with condition N such that $r(y_1, x_1, \ldots, y_t, x_t) \geq \nu(x_0)$. In particular, $\nu(x_0) \leq r_{\max}$.*

(2) There is a minimal element ν of $\mathcal{T}(P)$ such that $\nu(x_0) = r_{\max}$. In particular, $\mathcal{R}_K(H)$ is level if and only if $r_{\max} = r$.

In fact we have shown the following

Proposition 5.5 *If $y_1, x_1, y_2, x_2, \ldots, y_t, x_t$ is a special sequence, then there is a minimal element ν of $\mathcal{T}(P)$ such that*

$$\nu(y_i) = \sum_{j=i}^{t} \operatorname{rank}[x_j, y_{j+1}]_{P^+} - \sum_{j=i}^{t} \operatorname{rank}[x_j, y_j]_{P^+}$$

and

$$\nu(x_{i-1}) = \sum_{j=i-1}^{t} \operatorname{rank}[x_j, y_{j+1}]_{P^+} - \sum_{j=i}^{t} \operatorname{rank}[x_j, y_j]_{P^+}$$

for $i = 1, 2, \ldots, t+1$.

Set $Q := \{z \in P \mid \text{rank}[x_0, z]_{P^+} + \text{rank}[z, \infty]_{P^+} < r\}$.

Lemma 5.6 *If* $r_{\max} \geq r + 2$, *then* $\text{type}\mathcal{R}_K(H) \geq 3$.

Lemma 5.7 *If* $r_{\max} = r + 1$ *and* $Q \neq \emptyset$, *then* $\text{type}\mathcal{R}_K(H) \geq 3$.

Lemma 5.8 *If* $\#Q \geq 2$ *then* $\text{type}\mathcal{R}_K(H) \geq 3$.

Lemma 5.9 *If* $Q = \{z\}$ *and* $\text{rank}[x_0, z]_{P^+} + \text{rank}[z, \infty]_{P^+} \leq r - 2$, *then* $\text{type}\mathcal{R}_K(H) \geq 3$.

Lemma 5.10 *If* $r_{\max} = r$ *and* $Q = \emptyset$, *then* P *is pure and therefore* $\mathcal{R}_K(H)$ *is Gorenstein.*

Lemma 5.11 *Set* $\nu_\downarrow(x) = \text{rank}[x, \infty]_{P^+}$ *and* $\nu_\uparrow(x) = r - \text{rank}[x_0, x]_{P^+}$ *for* $x \in P$. *Then* $\nu_\downarrow$ *and* $\nu_\uparrow$ *are minimal elements of* $\mathcal{T}(P)$.

First assume that $\text{type}\mathcal{R}_K(H) = 2$. Then by the Lemmas above, we see that there are two cases.

Case 1 $r_{\max} = r$, $Q = \{x\}$ and $\text{rank}[x_0, x]_{P^+} + \text{rank}[x, \infty]_{P^+} = r - 1$.

In this case P satisfies the conditions of Theorem 3.1.

Case 2 $r_{\max} = r + 1$, $Q = \emptyset$.

Take a special sequence $y_1, x_1, \ldots, y_t, x_t$. Then one sees that $t = 1$ and $y_1 > x_1$. Set $y = y_1$ and $x = x_1$. Then one verifies (1), (3) and (4) of Theorem 3.3. If $z \in P \setminus ([x_0, y]_{P^+} \cup [x, \infty]_{P^+})$, then one can verify that there are minimal elements ν_1 and ν_2 of $\mathcal{T}(P)$ such that $\nu_1(z) = \text{rank}[z, \infty]_{P^+}$ and $\nu_2(z) = r + 1 - \text{rank}[x_0, z]_{P^+}$. So there are at least 3 minimal elements in $\mathcal{T}(P)$.

Now assume that P satisfies the conditions of Theorem 3.1. Then $\nu_\downarrow$ and $\nu_\uparrow$ are precisely the 2 minimal elements of $\mathcal{T}(P)$.

Next assume that P satisfies the conditions of Theorem 3.3. Then one can easily verify that y, x satisfy condition N and $r(y, x) = r + 1$. Moreover by (3) and (4) of Theorem 3.3, one sees that

$$\text{rank}[x_0, z_1]_{P^+} + \text{rank}[z_1, z_2]_{P^+} + \text{rank}[z_2, \infty]_{P^+} = r$$

for any $z_1, z_2 \in P$ with $z_1 < z_2$ and $(z_1, z_2) \neq (x, y)$. In particular, y, x is the unique special sequence.

So by Proposition 5.5, one sees that there is a minimal element ν of $\mathcal{T}(P)$ such that $\nu(x_0) = r+1$, $\nu(x) = \text{rank}[x,\infty]_{P^+}$ and $\nu(y) = \text{rank}[y,\infty]_{P^+} + 1 = r+1-\text{rank}[x_0,y]_{P^+}$.

By (3) of Theorem 3.3, one sees that $[x,\infty]_{P^+} \setminus \{y\}$ and $[x_0,y]_{P^+} \setminus \{x\}$ are pure. Therefore, for any $z \in ([x,\infty]_{P^+} \setminus \{y\}) \cup ([x_0,y]_{P^+} \setminus \{x\}) = P^+$, the value of $\nu'(z)$ is uniquely determined for any $\nu' \in \mathcal{T}(P)$ with $\nu'(x_0) = r+1$, $\nu'(x) = \text{rank}[x,\infty]_{P^+}$ and $\nu'(y) = \text{rank}[y,\infty]_{P^+} + 1 = r+1-\text{rank}[x_0,y]_{P^+}$.

So $\nu_\downarrow = \nu_\uparrow$ and ν above are the only minimal elements of $\mathcal{T}(P)$.

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Local Bertini theorem with applications to the Iwasawa Main Conjecture

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1 Local Bertini theorem and characteristic ideals

Recall that a classical Bertini theorem for a smooth complex projective variety asserts that a generic hyperplane section of a variety is smooth. This has a local analogue and can be proved in many cases. Let $(R, m, \mathbf{k})$ be a local Noetherian ring and let $x \in m$ be a nonzero divisor. Then the local Bertini problem for a ring-theoretic property $\mathcal{P}$ is formulated in the following way.

Question 1. Assume that $(R, m, \mathbf{k})$ has $\mathcal{P}$. Then is it true that R/xR has $\mathcal{P}$ for a generic choice of a nonzero divisor $x \in m$?

We are interested in the case where $\mathcal{P}$=normal and our final goal is to answer the above question and develop useful techniques to study the Iwasawa Main Conjecture, which relates the characteristic ideal attached to a Selmer group to an ideal generated by a p-adic L-function, which are both defined for a certain big Galois representation. Suppose R is a Noetherian normal domain and M is a finitely generated torsion R-module. Then one can associate the characteristic ideal denoted by $\mathrm{char}_R(M)$. When R has large Krull-dimension, it is not so practical to study characteristic ideals over R directly. Instead it is necessary to reduce to the case where R/xR is normal and M/xM is a torsion R/xR-module and study $\mathrm{char}_{R/xR}(M/xM)$. This is the main reason why a positive answer to the above question is required. Since we are dealing with the mixed characteristic local ring, the Zariski topology is not appropriate to make sense of a "generic" element. Instead, we need to use a valuation topology introduced in the study of rigid analytic geometry. It is natural to believe that this topology is the best choice for the formulation of Bertini-type theorems in the mixed characteristic case, and this will be a task in the future.

The local Bertini problem for a local ring containing a field was studied extensively by Flenner [2] and thus, he solved the problem for normality when the local ring contains a field. However, his proof in the mixed characteristic case is not clear. For applications, we need a strong version of Bertini-type theorem. Our main result in [6] is the following:

Theorem 2 (Local Bertini Theorem). *Let $(R, m, \mathbf{k})$ be a complete local domain of mixed characteristic $p > 0$ and suppose the following conditions:*

(1) *let $A \to R$ be a coefficient ring map, where $(A, \pi_A, \mathbf{k})$ is a complete discrete valuation ring;*

(2) *let $x_0, x_1, \ldots, x_d \in m$ be a fixed set of minimal generators of m;*

(3) *R is normal, the depth of R is at least 3, and the residue field $\mathbf{k}$ is infinite.*

Then there exists a Zariski dense open subset $U \subseteq \mathbb{P}^d(\mathbf{k})$ satisfying the following properties. For any $a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(U)$, the quotient R/x_aR is a normal domain of mixed characteristic $p > 0$, where we put

$$x_a := \sum_{i=0}^{d} a_i x_i.$$

Notation: $\mathbb{P}^d(\mathbf{k})$ is a d-dimensional projective space with coordinates in $\mathbf{k}$. Let us assume that any point $a = (a_0 : \cdots : a_d) \in \mathbb{P}^d(A)$ in the projective space is normalized so that some $a_i \in A$ is a unit. The *specialization map* $\mathrm{Sp}_A : \mathbb{P}^d(A) \to \mathbb{P}^d(\mathbf{k})$ is given by reducing the point a by the unique maximal ideal of A. Then this map is well-defined. Quite roughly, the topology is introduced into $\mathbb{P}^d(A)$ in such a way that Sp_A is continuous and $\mathbb{P}^d(\mathbf{k})$ comes with the Zariski topology.

Lemma 3. *Let R be a Noetherian ring and let P be a prime ideal of R. Fix a nonzero divisor $x \in R$ such that $x \in P$. Then R_P/xR_P is regular if and only if R_P is regular and $x \notin P^{(2)}$, where $P^{(2)}$ is the second symbolic power of P.*

Proof. Assume that R_P/xR_P is regular. Then $x \in R_P$ is a nonzero divisor by assumption. So R_P is regular and x is part of a regular system of parameters for R_P and thus, $x \notin P^{(2)}$. Conversely, if R_P is regular, then $x \notin P^{(2)}$ implies that x is part of a regular system of parameters for R_P, say R_P/xR_P is regular. □

To prove the "local Bertini theorem", one need to check the normality of R/x_aR via Serre's criterion. So the above lemma is important for checking (R_1). However, a direct computation for the second symbolic power of a prime ideal P is daunting, we would like to relate this problem to "basic elements" (due to Swan) via the following lemma.

Lemma 4 (Flenner). *Let $A \to M$ be a derivation for a commutative ring A and an A-module M. Fix a prime ideal P and $x \in A$. If $dx \notin PM_P$, then $x \notin P^{(2)}$.*

Proof. We may assume that A is a local ring with P its unique maximal ideal. For a contradiction, assume that $x \in P^{(2)} = P^2$. So we may write $x = \sum_{i=1}^{n} a_i b_i$ with $a_i, b_i \in P$. It then follows that $dx = a_i \sum_{i=1}^{n} db_i + b_i \sum_{i=1}^{n} da_i \in PM$, which is false. □

This lemma plays a role in the proof of the local Bertini theorem, but several steps are required for its completion and we refer the reader to [6].

Definition 5. Let A be a Noetherian normal domain and let M be a finitely generated torsion A-module. Then we define the *characteristic ideal* of M as

$$\mathrm{char}_A(M) := \Big(\prod_{\mathrm{ht}\ P=1} P^{\ell_{A_P}(M_P)} \Big)^{**},$$

where $(-)^* = \mathrm{Hom}_A(-, A)$ and $\ell_{A_P}(M_P)$ denotes the length.

When A is a UFD, one need not take the reflexive closure $(-)^{**}$, since every height-one prime ideal is principal. Thus, the characteristic ideal over a UFD is always principal. In taking the reflexive closure, we notice the following fact. Let M and N be finitely generated torsion A-modules. Then $\mathrm{char}_A(M) \subseteq \mathrm{char}_A(N) \iff \ell_{A_P}(M_P) \geq \ell_{A_P}(N_P)$ for every height-one prime ideal P of A. Here is another main theorem on characteristic ideals, whose proof is via the local Bertini theorem. Let R be a local Cohen-Macaulay normal domain which is module-finite over $\mathbb{Z}_p[[z_1, \dots, z_n]]$ with its coefficient ring $\mathcal{O}$ (a complete discrete valuation ring).

Theorem 6 (Control Theorem for Characteristic Ideals). *Assume that M and N are finitely generated torsion R-modules. Then the following statements are equivalent:*

(1) $\mathrm{char}_R(M) \subseteq \mathrm{char}_R(N)$.

(2) *For any height-one prime:*

$$xR_{\widehat{\mathcal{O}^{\mathrm{ur}}}} \in \mathcal{L}_{\widehat{\mathcal{O}^{\mathrm{ur}}}}(M_{\widehat{\mathcal{O}^{\mathrm{ur}}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}^{\mathrm{ur}}}}(N_{\widehat{\mathcal{O}^{\mathrm{ur}}}}),$$

which is not contained in a fixed finite subset, there exists a finite étale extension of discrete valuation rings $\mathcal{O} \to \mathcal{O}'$ such that we have $x \in R_{\mathcal{O}'}$ and

$$\mathrm{char}_{R_{\mathcal{O}'}/xR_{\mathcal{O}'}}(M_{\mathcal{O}'}/xM_{\mathcal{O}'}) \subseteq \mathrm{char}_{R_{\mathcal{O}'}/xR_{\mathcal{O}'}}(N_{\mathcal{O}'}/xN_{\mathcal{O}'}).$$

(3) *For all but finitely many height-one primes:*

$$xR_{\widehat{\mathcal{O}^{\mathrm{ur}}}} \in \mathcal{L}_{\widehat{\mathcal{O}^{\mathrm{ur}}}}(M_{\widehat{\mathcal{O}^{\mathrm{ur}}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}^{\mathrm{ur}}}}(N_{\widehat{\mathcal{O}^{\mathrm{ur}}}}),$$

we have

$$\mathrm{char}_{R_{\widehat{\mathcal{O}^{\mathrm{ur}}}}/xR_{\widehat{\mathcal{O}^{\mathrm{ur}}}}}(M_{\widehat{\mathcal{O}^{\mathrm{ur}}}}/xM_{\widehat{\mathcal{O}^{\mathrm{ur}}}}) \subseteq \mathrm{char}_{R_{\widehat{\mathcal{O}^{\mathrm{ur}}}}/xR_{\widehat{\mathcal{O}^{\mathrm{ur}}}}}(N_{\widehat{\mathcal{O}^{\mathrm{ur}}}}/xN_{\widehat{\mathcal{O}^{\mathrm{ur}}}}).$$

For a proof and the notation of the theorem, see [6]. Let me add to say that a general strategy in proving $(LHS) = (RHS)$ in the statement of the main conjecture (see below) is to prove $(LHS) \subseteq (RHS)$ and $(LHS) \supseteq (RHS)$ separately. Hence the above theorem helps to take care of this issue. In the next section, we give a formulation of the main conjecture for Galois representations in some detail with comments on known results.

2 Iwasawa Main Conjecture for nearly-ordinary deformations

In the previous section, we discussed characteristic ideals. In number theory, characteristic ideals appear as algebraic invariants which represent ideal class groups, Tate-Shafarevich groups, or Mordell-Weil groups of Abelian varieties over number fields. These number-theoretic invariants are generalized as *Selmer groups*. As mentioned briefly, the Iwasawa Main Conjecture describes the relation between the characteristic ideal associated to a Selmer group, which is an "algebraic object", and the special value of a certain zeta function. Here is a simple example for this relation.

Example 7. In college calculus, we learn the following mysterious formula (due to Leibniz):

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}.$$

In number theory, this reads as

$$\text{(special value of zeta function)} = \text{(period} \times \text{algebraic number)},$$

which is also the presentation of the class number formula for the field $\mathbb{Q}(\sqrt{-1})$. Let $\rho : \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \to \{+1,-1\}$ be a group homomorhism (Dirichlet character), where we put $\rho(\sigma) = -1$ for the unique non-trivial element $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q})$. Then there is an isomorphism $\mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \simeq (\mathbb{Z}/4\mathbb{Z})^\times$. Since there is a surjection $\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, we may extend the map ρ from $\mathbb{Z}$ to the set $\{+1,-1,0\}$. Then the zeta function attched to ρ is

$$L(1,\rho)=\sum_{n=1}^{\infty}\frac{\rho(n)}{n}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots.$$

On the other hand, the "4" in $\frac{\pi}{4}$ reflects the fact that the number of roots of unity in $\mathbb{Q}(\sqrt{-1})$ is four; $\{+1,-1,+\sqrt{-1},-\sqrt{-1}\}$. Finally, π is known to be a "period", which means that

$$\pi = 2\int_{-1}^{1}\sqrt{1-x^2}dx.$$

The set of periods forms a subring of $\mathbb{C}$, which contain $\overline{\mathbb{Z}}$, the set of all algebraic integers. What is a period?

Let us give a set-up for the main conjecture. We need to leave out the detail due to the limit of space, so the interested reader should look at [4], [5] for the detail and the references listed there. For a fixed prime number $p \geq 3$, let $\mathbb{Q}_\infty/\mathbb{Q}$ be the unique cyclotomic $\mathbb{Z}_p$-extension and let C_∞ be the Galois group of $\mathbb{Q}_\infty/\mathbb{Q}$. Then there is a canonical Galois character $\chi : C_\infty \simeq 1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$. Let

$$\cdots \to Y_1(p^{t+1}) \to Y_1(p^t) \to \cdots \to Y_1(p)$$

be a tower of modular curves, where $Y_1(p^{t+1}) \to Y_1(p^t)$ is finite flat map induced by the inclusion of congruence subgroups $\Gamma_1(p^{t+1}) \subseteq \Gamma_1(p^t)$. Then for $d \in (\mathbb{Z}/p^t\mathbb{Z})^\times$ with $d \equiv 1 \pmod p$, the diamond operator $\langle d \rangle$ maps $[E, P]$ to $[E, dP]$ (E is a complex elliptic curve and $P \in E$ is a point of order p^t). The group of diamond operators then induces a group of diamond operators on the projective system $\{Y_1(p^t)\}_{t \geq 1}$, which we denote by D_∞. Then we have a canonical character $\eta : D_\infty \simeq 1+p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$.

Let $\mathbb{H}^{\mathrm{ord}}_{Np^\infty}$ be the (universal) ordinary Hecke algebra for tame level N with $(N, p) = 1$. This algebra is constructed as follows. Let $h^{\mathrm{ord}}_k(\Gamma_1(Np^r), \mathcal{O}_K)$ be an $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S^{\mathrm{ord}}_k(\Gamma_1(Np^r), \mathcal{O}))$ generated by all Hecke operators $T(n)$, where $S^{\mathrm{ord}}_k(\Gamma_1(Np^r), \mathcal{O})$ is an $\mathcal{O}$-module generated by all ordinary cuspidal forms of weight $k \geq 2$ with respect to $\Gamma_1(Np^r)$ and $\mathbb{Z}_p \to \mathcal{O}$ is a finite flat extension of discrete valuation rings. Then $h^{\mathrm{ord}}_k(\Gamma_1(Np^r), \mathcal{O}_K)$ is a commutative ring and let

$$\mathbb{H}^{\mathrm{ord}}_{Np^\infty} := \varprojlim_r h^{\mathrm{ord}}_k(\Gamma_1(Np^r), \mathcal{O}),$$

where the map $h^{\mathrm{ord}}_k(\Gamma_1(Np^{r+1}), \mathcal{O}) \to h^{\mathrm{ord}}_k(\Gamma_1(Np^r), \mathcal{O})$ is induced by the duality between Hecke algebras and the space of modular forms.

A Hecke operator, acting on the space of modular forms, may be described as a "double coset operator", which is very useful for actual computations. For more on Hecke operators, see a nice book [1]. One of the important facts in Hida theory is that the resulting Hecke algebra $\mathbb{H}^{\mathrm{ord}}_{Np^\infty}$ does not depend on the choice of $k \geq 2$ and it is a semi-local finite flat $\mathbb{Z}_p[[D_\infty]]$-algebra ($\mathbb{Z}_p[[D_\infty]]$ is a complete regular local ring of dimension 2). Here is a natural question for commutative algebraists.

Question 8. The set of Hecke operators forms a commutative ring and it can be Noetherian and Cohen-Macaulay in many cases. Then when are they Gorenstein, or complete intersection?

Say that a $\mathbb{Z}_p$-algebra homomorphism $\kappa : \mathbb{H}^{\mathrm{ord}}_{Np^\infty} \to \overline{\mathbb{Q}}_p$ is an *arithmetic character of weight* $w(\kappa)$, if there exists an open subgroup $\Gamma \subseteq D_\infty$ for which the restriction map $\kappa|_{\mathbb{Z}_p[[\Gamma]]} : \mathbb{Z}_p[[\Gamma]] \to \overline{\mathbb{Q}}_p$ coincides with the ring homomorphism induced by the character $\eta^{w(\kappa)}$ for some $w(\kappa) \in \mathbb{Z}$. The *nearly-ordinary Hecke algebra* (over $\mathbb{Q}$) is defined as the completed tensor product:

$$\mathbb{H}^{\mathrm{n.ord}}_{Np^\infty} := \mathbb{H}^{\mathrm{ord}}_{Np^\infty} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[G_\infty]],$$

finite flat over the complete regular local ring $\mathbb{Z}_p[[D_\infty \times C_\infty]]$ of dimension 3.

Let $\mathbf{f}$ be a fixed $\mathbb{I}_{\mathbf{f}}$-adic cuspidal newform with tame level N whose coefficients with respect to q-expansion are in a module-finite $\mathbb{Z}_p[[D_\infty]]$-algebra $\mathbb{I}_{\mathbf{f}}$. Then $\mathbb{H}^{\mathrm{n.ord}}_{Np^\infty}$

maps onto a local domain $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$ and there exists a continuous Galois representation (due to Hida [3]): $\rho_{\mathbf{f}} : G_{\mathbb{Q}} \to \mathrm{Aut}(T_{\mathrm{n.ord}})$, where $T_{\mathrm{n.ord}}$ is a torsion-free $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$-module of generic rank 2 with a continous action of $G_{\mathbb{Q}}$ compatible with the $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$-module structure. Here the topology of $\mathrm{Aut}(T_{\mathrm{n.ord}})$ is induced by the m-adic topology of the local domain $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$. Assume that $T_{\mathrm{n.ord}}$ is a free module. Then we have a (big) Galois representation $\rho_{\mathbf{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}})$. Let $(-)^{PD} = \mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ and let $\mathrm{Sel}(\rho_{\mathbf{f}})$ be the Greenberg-type Selmer group associated to the representation $\rho_{\mathbf{f}}$.

Conjecture 9 (Iwasawa Main Conjecture). *Under the notation as above, assume that* $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$ *is a normal domain and the residual representation of* $\rho_{\mathbf{f}}$ *is irreducible. Then,* $\mathrm{Sel}(\rho_{\mathbf{f}})^{PD}$ *is a finitely generated torsion* $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$*-module and*

$$\mathrm{char}_{\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}}(\mathrm{Sel}(\rho_{\mathbf{f}})^{PD}) = (L_p(\rho_{\mathbf{f}})).$$

The p-adic L-function $L_p(\rho_{\mathbf{f}}) \in \mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$ was constructed first by Kitagawa as having the following interpolation property:

$$\frac{L_p(\rho_{\mathbf{f}})(\chi^{j-1}, \kappa)}{\mathbf{C}_{p,\kappa}} = (-1)^{j\ 1}(j-1)!\left(1 - \frac{\omega^{i-j}(p)p^{j-1}}{a_p(\mathbf{f}_\kappa)}\right)\left(\frac{p^{j-1}}{a_p(\mathbf{f}_\kappa)}\right)^{\mathrm{ord}_p(\omega^{i-j})}$$

$$\times G(\omega^{j-i})\frac{L(\mathbf{f}_\kappa, \omega^{i-j}, j)}{(2\pi\sqrt{-1})^{j-1}\mathbf{C}_{\infty,\kappa}^{(\ 1)^i}}$$

where $\mathbf{C}_{p,\kappa} \in \overline{\mathbb{Q}}_p$ is a p-adic period, $\mathbf{C}_{\infty,\kappa} \in \mathbb{C}$ is a complex period, $G(\omega^{j-i})$ is the Gauss sum, $\chi : \mathbb{Z}_p[[C_\infty]] \to \overline{\mathbb{Q}}_p$ is induced by the cyclotomic character, and κ is an arithmetic character. $L(\mathbf{f}_\kappa, \omega^{i-j}, j)$ is a complex L-function of the cupidal form $\mathbf{f}_\kappa$ twisted by $\omega^{i\ j}$, a finite order character of D_∞. The mysterious fact is

$$\frac{L(\mathbf{f}_\kappa, \omega^{i\ j}, j)}{(2\pi\sqrt{-1})^{j\ 1}\mathbf{C}_{\infty,\kappa}} \in \overline{\mathbb{Q}},$$

because $L(\mathbf{f}_\kappa, \omega^{i\ j}, j)$ is a priori a complex-valued function. Everything in the above formula may be regarded as p-adic numbers under a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Comments: It was shown in [4] that the main conjecture satisfies at least an inclusion relation in one direction when the Hecke algebra $\mathbb{H}_{\mathbf{f}}^{\mathrm{n.ord}}$ is isomorphic to a power series ring in two variables, using Euler system theory and Beilinson-Kato elements in K_2 of modular curves. The next objective in this research project is to expand the basic part of Euler system theory over general normal domains.

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Non-linear behaviour of Castelnuovo-Mumford regularity

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Abstract

It is shown that the Castelnuovo-Mumford regularity of lex-segment ideals of ordinary powers as well as of Frobenious powers of a homogeneous polynomial ideal is a function of polynomial type of huge degree.

Key words: Castelnuovo-Mumford regularity, lex-segment ideal, powers of ideals.

Introduction

Let I be a homogeneous ideal of a polynomial ring $R = K[x_1, ..., x_r]$ over a field K. More than ten years ago Cutkosky, Herzog and Trung [3] and Kodiyalam [9] independently proved that the Castelnuovo-Mumford regularity $\text{reg}(I^n)$ is a linear function of n when $n \gg 0$. The reason for this phenomenon comes from the fact that the Rees algebra $R[It] = \oplus_{n\geq 0} I^n t^n$ is a Noetherian bigraded algebra. This raises the problem to study the asymptotic behaviour of the Castelnuovo-Mumford regularity of some other filtrations of ideals. Although in most cases, the underlying bigraded "Rees algebra" need not to be Noetherian and hence the method of [3] and [9] could not be applied directly, there were given in [5] some particular cases, where the Castelnuovo-Mumford regularity is still bounded by a linear function.

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The first aim of this paper is to study the Castelnuovo-Mumford regularity of the lex-segment ideals $\text{lex}(I^n)$ of a power I^n of I. Lex-segment ideals play an important role in many problems in Commutative Algebra, see e.g. [10, Chapter 2]. Although it is well-known that $\text{reg}(I)$ is always less than or equal to $\text{reg}(\text{lex}(I))$ and the difference between them could be huge (see [8, Proposition 12]), one may hope that $\text{reg}(\text{lex}(I^n))$ still well behaves when $n \gg 0$. One of our main result shows that in contrast to the ordinary powers, $\text{reg}(\text{lex}(I^n))$ is asymptotically a polynomial of huge degree (see Theorem 3). This is somewhat a surprise.

However the sequence $\{\text{lex}(I^n)\}_{n\geq 1}$ does not form a filtration of ideals. For an arbitrary filtration of ideals it was eventually shown in [5, Proposition 3.6] that a partial Castelnuovo-Mumford regularity, namely the so-called a-invariant of R/I^n, is always bounded by a linear function. It is natural to ask, whether there is a general approach to bound the Castelnuovo-Mumford regularity of a filtration of ideals. We will show that this is impossible. Moreover, in such a general setting, there is no bounding function for the Castelnuovo-Mumford regularity, see Proposition 8.

Our second aim is to study the Frobenious powers of ideals $I^{[n]}$ ($p = \text{char}\, K$ and $n = p^k$), which also appear very often in Commutative Algebra. We will show that in this case the Castelnuovo-Mumford regularity behaves very well: except few initial values, this is a linear function of n (see Theorem 5). Like the case of ordinary powers, when lex-segment ideals of Frobenious powers are considered, the Castelnuovo-Mumford regularity is also a function of polynomial type of huge degree (see Theorem 7).

Our approach is based on a closed formula for computing the Castelnuovo-Mumford regularity of lex-segment ideals given by Chardin and Moreno-Socias in [4]. Another important result used here is the polynomial nature of Hilbert coefficients recently given by Herzog, Puthenpurakal and Verma for ordinary powers in [6]. It turns out that this property also holds for the Frobenious powers (see Proposition 6).

The paper will be divided in three sections. In Section 1 we study lex-segment ideals of powers of ideals. Section 2 is devoted to the Frobenious powers. In Section 3 we construct a class of filtrations of ideals for which there is no bounding function on the Castelnuovo-Mumford regularity.

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1 Lex-segment ideals of powers of ideals

Let us recall some notions. Let $R = K[x_1, \dots, x_r]$ be a standard graded algebra over a field K and $\mathfrak{m} = (x_1, ..., x_r)$. Let M be a finitely graded R-module. For every integer $i \geq 0$ we set

$$a_i(M) := \max\{t |\ H^i_{\mathfrak{m}}(M)_t \neq 0\},$$

with $a_i(M) = -\infty$ if $H^i_{\mathfrak{m}}(M) = 0$, where $H^i_{\mathfrak{m}}(M)$ denotes the i-th local cohomology module of M with respect to $\mathfrak{m}$. The *Castelnuovo-Mumford regularity* of M is defined by

$$\operatorname{reg}(M) = \max\{a_i(M) + i |\ i \geq 0\}.$$

It is clear that $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$. The number $a_i(M)$ can be considered as a partial Castelnuovo-Mumford regularity of M. The number $a_d(M)$ ($d = \dim M$) is well known under the name a-invariant of M.

The lex-segment ideal $\operatorname{lex}(I)$ associated to (the Hilbert function of) I is the ideal generated by all the first $H_I(m)$ monomials of degree m with respect to the lexicographic order, when m runs through all positive integers. This ideal has the same Hilbert function as I.

It is well known that $\operatorname{reg}(I) \leq \operatorname{reg}(\operatorname{lex}(I))$. Inspired of results in [3] and [9] about the asymptotic linear property of $\operatorname{reg}(I^n)$, it is natural to ask if $\operatorname{reg}(\operatorname{lex}(I^n))$ can be also bounded by a linear function of n. It is clear that $\operatorname{reg}(\operatorname{lex}(I^n)) = \operatorname{reg}(I^n)$ in the case $d := \dim R/I = 0$. Hence we always assume that $d \geq 1$. First we observe

Proposition 1 *Let $I \subset R$ be a homogeneous ideal of dimension $d \geq 1$ and codimension c generated by polynomials of degrees at most Δ. Then for all $n \geq 1$*

$$\operatorname{reg}(\operatorname{lex}(I^n)) < (2\Delta n)^{cd2^{d-1}}.$$

PROOF. Since I^n and $\operatorname{lex}(I^n)$ have the same Hilbert function and I^n is generated by homogeneous polynomials of degrees at most $n\Delta$, this immediately follows by [7, Theorem 6.4].

In order to study $\operatorname{reg}(\operatorname{lex}(I^n))$ more closely we need to use a formula given by Chardin and Moreno-Socias in [4]. Recall that if we write the Hilbert polynomial of R/I as

$$P_{R/I}(t) = e_0\binom{t+d-1}{d-1} - e_1\binom{t+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}, \quad (1)$$

then $e_0, ..., e_{d-1}$ are called the Hilbert coefficients of R/I. Let $d \geq 1$. We recursively define $B_0 = e_0$ and for all $1 \leq i \leq d-1$,

$$B_i = (-1)^i e_i + \binom{B_{i-1}+1}{2} - \binom{B_{i-2}+1}{3} + \cdots + (-1)^{i+1}\binom{B_0+1}{i+1}. \quad (2)$$

Note that $B_0 \leq B_1 \leq \cdots \leq B_{d-1}$ can be defined by using the so-called Gotzmann's representation of $P_{R/I}(t)$ and the above recursive formula is given in [1, Proposition 3.9] (see also [8, Lemma 10]). In order to emphasize the dependence of e_i, B_i on I we will write $e_i(I)$, $B_i(I)$. Another important invariant is the so-called *regularity index* of the Hilbert function defined by the formula:

$$\operatorname{ri}(I) := \max\{m |\ H_{R/I}(m) \neq P_{R/I}(m)\}.$$

Note that the Grothendieck-Serre formula

$$H_{R/I}(t) - P_{R/I}(t) = \sum_{i=0}^{d}(-1)^i \ell(H^i_{\mathfrak{m}}(R/I)_t \quad (3)$$

yields $\operatorname{ri}(I) \leq \operatorname{reg}(I)$.

With the above notation one can formulate [4, Theorem 2.5(i)] as follows:

Lemma 2 *Let $I \subset R$ be a homogeneous ideal of dimension $d \geq 1$. Then*

$$\operatorname{reg}(\operatorname{lex}(I)) = \max\{B_{d-1}(I),\ \operatorname{ri}(I)+1\}.$$

Following [6], a numerical function $h(t)$ is said to be of polynomial type if there exists a polynomial $p(t)$ such that $h(n) = p(n)$ for all $n \gg 0$. Now we can prove the main result of this paper which says that $\operatorname{reg}(\operatorname{lex}(I^n))$ is always of polynomial type of huge degree.

Theorem 3 *Let $I \subset R$ be a homogeneous ideal of dimension $d \geq 1$ and codimension c. Then* $\operatorname{reg}(\operatorname{lex}(I^n))$ *is a function of polynomial type. Moreover, if $c \geq 2$ then this polynomial has degree $c2^{d-1}$ and a rational leading coefficient which is equal to*

$$\frac{1}{2^{2^{d-1}-1}}\Big(\lim_{n\to\infty}\frac{e_0(I^n)}{n^c}\Big)^{2^{d-1}}.$$

PROOF. By the main result of [13], $\operatorname{ri}(I^n)$ is asymptotically a linear function. On the other hand, by [6, Theorem 1.1], all Hilbert coefficients $e_i(I^n)$ are functions of polynomial type. Using the recursive formula (2), it is immediate to see that all $B_i(I^n)$ are functions of polynomial type. In particular, $B_{d-1}(I^n)$ is of polynomial type. Hence, by Lemma 2, $\operatorname{reg}(\operatorname{lex}(I^n))$ is a function of polynomial type.

Now let $c \geq 2$. By [6, Proposition 2.1], there exists

$$\lim_{n\to\infty}\frac{e_0(I^n)}{n^c} = \alpha \in \mathbb{Q}_+ \setminus \{0\}. \tag{4}$$

Since $c \geq 2$ and $B_{d-1}(I^n) \geq B_0(I^n) = e_0(I^n)$ is of polynomial type of degree at least 2, we must have $B_{d-1}(I^n) > \operatorname{ri}(I^n)$ for all $n \gg 0$. Hence, by Lemma 2,

$$\operatorname{reg}(\operatorname{lex}(I^n)) = B_{d-1}(I^n) \ \text{ for all } n \gg 0. \tag{5}$$

In order to calculate the degree of this function, it suffices to show that

$$\lim_{n\to\infty}\frac{B_i(n)}{n^{c2^i}} = \frac{\alpha^{2^i}}{2^{2^i-1}}, \tag{6}$$

for all $0 \leq i \leq d-1$.

We do by induction on i. Since $B_0(I) = e_0(I)$, the case $i = 0$ is just (4). Assume that (6) holds for $0 \leq i < d-1$. We show that it also holds for $i+1$. By [6, Theorem 1.1], $e_{i+1}(I^n)$ is a function of polynomial type of degree at most $c+i+1$. Therefore one can find a larger number β such that $|e_{i+1}(I^n)| < \beta n^{c+i+1}$. Since $c2^{i+1} > c+i+1$ for all $c \geq 2$ and $i \geq 0$, this implies

$$0 \leq \lim_{n\to\infty}\frac{|e_{i+1}(I^n)|}{n^{c2^{i+1}}} \leq \beta \lim_{n\to\infty}\frac{n^{c+i+1}}{n^{c2^{i+1}}} = 0.$$

By induction hypothesis, for all $0 \leq j \leq i-1$ we have

$$0 \leq \lim_{n\to\infty} \frac{\binom{B_j(I^n)+1}{i+2-j}}{n^{c2^{i+1}}} \leq \lim_{n\to\infty} \frac{(B_j(I^n))^{i+2-j}}{n^{c2^{i+1}}} = \lim_{n\to\infty} \left(\frac{(B_j(I^n))}{n^{c2^j}}\right)^{i+2-j} \lim_{n\to\infty} \frac{n^{(i+2-j)c2^j}}{n^{c2^{i+1}}} = 0,$$

where the last equality follows from the fact that $2^a > a+1$ for all $a \geq 2$. Putting the above calculation into (2), we then get

$$\lim_{n\to\infty} \frac{B_{i+1}(I^n)}{n^{c2^{i+1}}} = \lim_{n\to\infty} \frac{\binom{B_i(I^n)+1}{2}}{n^{c2^{i+1}}} = \frac{1}{2}\left(\frac{B_i(I^n)}{n^{c2^i}}\right)^2 = \frac{1}{2}\left(\frac{\alpha^{2^i}}{2^{2^i-1}}\right)^2 = \frac{\alpha^{2^{i+1}}}{2^{2^{i+1}-1}}.$$

The induction is completed and hence also the proof of the theorem.

Example 1. We have $\operatorname{reg}(\operatorname{lex}((x_1^n))) = \operatorname{reg}((x_1^n)) = n$. This shows that the statement about the degree in Theorem 3 does not hold in the case $c = 1$.

Example 2. Let $I = (x_1, ..., x_c) \subset R := K[x_1, ..., x_{c+d}]$, where $c \geq 2$, $d \geq 1$. Then $e_0(R/I^n) = \binom{c+n-1}{c}$. Hence, for $c \geq 2$, we have

$$\lim_{n\to\infty} \frac{\operatorname{reg}(\operatorname{lex}(I^n))}{n^{c2^{d-1}}} = \frac{1}{2^{2^{d-1}-1}(c!)^{2^{d-1}}}.$$

Moreover, one can easily show that

$$H_{R/I^n}(t) = \sum_{i=0}^{n-1} \binom{c+i-1}{c-1}\binom{d+t-i-1}{d-1},$$

for all $t \geq n$. This implies $\operatorname{ri}(I^n) = n-1$. In particular, if $d = 1$, from Lemma 2 we can conclude that $\operatorname{reg}(\operatorname{lex}(I^n)) = \binom{n+c-1}{c}$ for all n.

2 Frobenious powers of ideals

The purpose of this section is to consider Frobenious powers of ideals in positive characteristic.

Our method is to reduce to the case of monomial ideals. Therefore in the first paragraph we consider Frobenious powers of monomial ideals. In the next paragraph we will consider Frobenious powers of arbitrary ideals. Let $\operatorname{char}(K) = p$ be a prime number and $n = p^k$. Recall that the n-th Frobenious power of an $I \subseteq R$ is defined by

$$I^{[n]} = (u^n |\ u \in I).$$

Note that the definition of $I^{[p^n]}$ does not depend on a choice of a generating set of I. From the Peskine-Szpiro Theorem on the exactness of the Frobenious functor, we get

Lemma 4 *Let $n = p^k$ and*

$$0 \to \oplus_j R(-b_{qj}) \to \cdots \to \oplus_j R(-b_{1j}) \to \oplus_j R(-b_{0j}) \to I \to 0$$

be the minimal free resolution of I. Then

$$0 \to \oplus_j R(-nb_{qj}) \to \cdots \to \oplus_j R(-nb_{1j}) \to \oplus_j R(-nb_{0j}) \to I^{[n]} \to 0$$

is the minimal free resolution of $I^{[n]}$.

By the main result of [9] and [5], $\operatorname{reg}(I^n)$ is a linear function of n, when $n \gg 0$. However, it is unclear, for which n_0 then $\operatorname{reg}(I^n)$ is a linear function for all $n \geq n_0$. In contrast to the ordinary powers, the following result says that the Castelnuovo-Mumford regularity of Frobenious powers behaves quite good.

Theorem 5 *With the notation in Lemma 4, denote by $b_i(I)$ the maximum generating degree of the i-th syzygy module of I, i.e. $b_i(I) = \max_j\{b_{ij}\}$. Then*

$$n \operatorname{reg}(I) \leq \operatorname{reg}(I^{[n]}) \leq n \operatorname{reg}(I) + (r-1)(n-1)$$

for all n. Moreover, let Q be the smallest index such that $b_Q(I) = \max\{b_i(I) \mid i \leq q\}$. Then

$$\operatorname{reg}(I^{[n]}) = b_Q(I)n - Q$$

for all $n \geq r-2$.

PROOF. For short, we write $b_i := b_i(I)$. By [2, Proposition 1.1 and Theorem 1.2] and Lemma 4,

$$\operatorname{reg}(I^{[n]}) = \max\{nb_i - i \mid 0 \leq i \leq q\} \geq \max\{n(b_i - i) \mid 0 \leq i \leq q\} = n \operatorname{reg}(I),$$

and since $q \leq r-1$,

$$\operatorname{reg}(I^{[n]}) = \max\{n(b_i - i) + (n-1)i \mid 0 \leq i \leq q\} \leq n \operatorname{reg}(I) + (r-1)(n-1).$$

Let $i \neq Q$. If $b_Q = b_i$, then $i > Q$ and so $b_Q n - Q > b_i n - i$.
If $b_i < b_Q$, then $b_Q n - Q - b_i n + i = n(b_q - b_i) + i - Q > 0$, provided $i \geq Q$. If $i < Q$, then $Q \geq 1$, and since $i \leq q \leq r-1$, we have $b_Q n - Q - b_i n + i = n(b_q - b_i) + i - Q \geq n - (r-2) \geq 0$, provided $n \geq r-2$.
Thus, we always have $b_Q n - Q \geq b_i n - i$ for all $i \geq 0$ and $n \geq r-2$. Hence $\operatorname{reg}(I^{[n]}) = b_Q n - Q$ for all $n \geq r-2$.

The following example shows that $r-2$ in the above proposition is the optimal value.

Example 3. Let $I = (x^3, x^2y, x^2z, x^2u, x^2v, x^2w, xy^4, y^6) \subset R := K[x, y, z, u, v, w]$. This is a so-called stable ideal. This ideal has the following minimal free resolution

$$0 \to R(-8) \to R^6(-7) \to R^{15}(-6) \to R^{20}(-5) \to R^{15}(-4) \oplus R(-6) \oplus R(-7)$$
$$\to R^6(-3) \oplus R(-5) \oplus R(-6) \to I \to 0.$$

It gives the following minimal free resolution of $I^{[n]}$:

$$0 \to R(-8n) \to R^6(-7n) \to R^{15}(-6n) \to R^{20}(-5n) \to R^{15}(-4n) \oplus R(-6n) \oplus R(-7n)$$
$$\to R^6(-3n) \oplus R(-5n) \oplus R(-6n) \to I^{[n]} \to 0.$$

If $p = 2$, we have the following table of values of $\text{reg}(I^{[n]})$:

n	1	2	≥ 4
$\text{reg}(I^{[n]})$	6	13	$8n - 5$

Thus, in this example, $\text{reg}(I^{[n]})$ agrees with a linear function only starting from $n = 4$.

In order to study the Castelnuovo-Mumford regularity of lex-segment ideals of Frobenious powers, we also need the following result which similar to [6, Theorem 1.1 and Proposition 2.1].

Proposition 6 *Let I be a an ideal of codimension c and dimension d. Let $n = P^k$. Then the Hilbert coefficient $e_i(I^{[n]})$ is a function of n of polynomial type of degree at most $c+i$, for all $i = 0, ..., d-1$. Moreover $\lim_{n\to\infty} \frac{e_0(I^{[n]})}{n^c}$ exists and is a positive rational number.*

PROOF. Since the Hilbert polynomial is additive on exact sequences, from Lemma 4 we see that the Hilbert polynomial $P_{R/I^{[n]}}(t)$ is a sum of a fixed number of terms of the type $\pm\binom{t+nb_{ij}+r-1}{r-1}$. In this algebraic sum, coefficients of t^δ cancel for $\delta > d-1$. For $\delta \leq d-1$, the coefficient $a_\delta(n)$ of t^δ in $P_{R/I^{[n]}}(t)$ is an algebraic sum of coefficients of t^δ in all terms of type $\pm\binom{t+nb_{ij}+r-1}{r-1}$. Clearly such a coefficient is a sum of products of $r-1-\delta$ numbers of the type $nb_{ij} + r - \ell$, $1 \leq \ell \leq r-2$. This implies that $a_\delta(n)$ is a function of polynomial type of degree at most $r - 1 - \delta$, whose coefficients are rational numbers. In particular $e_0(I^{[n]}) = (d-1)!a_{d-1}(n)$ is a function of polynomial type of degree at most c and of rational coefficients. Comparing the coefficients of the standard expression of $P_{R/I^{[n]}}(t) = \sum_{\delta=0}^{d-1} a_\delta(n)t^\delta$ with the one in (1), we get

$$(-1)^{i+1}e_{i+1}(I^{[n]}) = a_{d-i-2}(n) - \sum_{j=0}^{i}(-1)^j c_j e_j(I^{[n]}),$$

where c_j is the coefficient of t^{d-i-2} in $\binom{t+d-j-1}{d-j-1}$. Since $a_{d-i-2}(n)$ is of polynomial type of degree at most $r-1-(d-i-2) = c+i+1$, by induction one get that $e_{i+1}(I^{[n]})$ is of polynomial type of degree at most $c+i+1$ and of rational coefficients.
Further, by [6, Proposition 2.1], $e_0(I^n)$ is of polynomial type of degree c. Since $\dim R/I^{[n]} = \dim R/I = \dim R/I^n$ and $H_{R/I^n}(t) \leq H_{R/I^{[n]}}(t)$, we must have $e_0(I^{[n]}) \geq e_0(I^n)$. This implies that $e_0(I^{[n]})$ is also of polynomial type of degree exactly c. As mentioned above, the corresponding polynomial of $e_0(I^{[n]})$ has rational coefficients. Hence $\lim_{n\to\infty} \frac{e_0(I^{[n]})}{n^c}$ exists and is a positive rational number.

Using the above proposition and the same of arguments as in the proof of Theorem 3 we deduce that the Castelnuovo-Mumford regularity of lex-segment ideals of Frobenious powers also badly behave.

Theorem 7 *Let* $\operatorname{char} K = p$ *and* $n = p^k$. *Then for any homogeneous ideal* I *of dimension* $d \geq 1$ *and codimension* $c \geq 2$, $\operatorname{reg}(\operatorname{lex}(I^{[n]}))$ *is a function of* n *of polynomial type of degree* $c2^{d-1}$ *and the leading coefficient of this polynomial is equal to*

$$\frac{1}{2^{2^{d-1}-1}}\Big(\lim_{n\to\infty}\frac{e_0(I^{[n]})}{n^c}\Big)^{2^{d-1}}.$$

We would like to mention that it is of interest to study the ideals $I(n) := (f_1^n, ..., f_s^n)$, where $f_1, ..., f_s$ are some given homogeneous polynomials. Of course, $I(n)$ depends on the choice of a generating system. Let us consider the case $\operatorname{char}(K) = p$. By Theorem 5, $\operatorname{reg}(I(n))$ is a linear function if $n = p^e \geq r - 2$. Easy examples show that this does not hold for all $n \geq r - 2$. But it is unclear if this will hold for $n \gg 0$.

Note that if I is a monomial ideal generated by monomials $m_1, ..., m_s$, then all results in this section can be formulated for $I(n)$. The main reason is that in this case Lemma 4 holds for (n). This was mentioned in [10, Exercise 1.7].

3 Castelnuovo-Mumford regularity of filtrations of ideals

Consider again Example 2 in the case $d = 1$, that is, let $I = (x_1, ..., x_c) \subset R := K[x_1, ..., x_{c+1}]$, $c \geq 2$. The Grothendieck-Serre formula (3) implies that

$$\operatorname{reg}(R/\operatorname{lex}(I^n)) = \max\{a_1(R/\operatorname{lex}(I^n))+1, \operatorname{ri}(\operatorname{lex}(I^n))\} = \max\{a_0(R/\operatorname{lex}(I^n)), \operatorname{ri}(\operatorname{lex}(I^n))\}.$$

Since $\operatorname{ri}(\operatorname{lex}(I^n)) = \operatorname{ri}(I^n) = n - 1 < \binom{n+c-1}{c} = \operatorname{reg}(R/\operatorname{lex}(I^n))$, we get

$$a_0(R/\operatorname{lex}(I^n)) = \operatorname{reg}(\operatorname{lex}(I^n)) - 1 = \binom{c+n-1}{c} - 1,$$

and

$$a_1(R/\operatorname{lex}(I^n)) = \operatorname{reg}(\operatorname{lex}(I^n)) - 2 = \binom{c+n-1}{c} - 2.$$

In particular, $a_1(R/\operatorname{lex}(I^n))$ cannot be bounded by a linear function.

Note that in this example, even in the case $c = 2$, the sequence $R \supset \operatorname{lex}(I) \supset \operatorname{lex}(I^2) \supset \cdots$ is not a filtration of ideals, because $(\operatorname{lex}(I))^2 \not\subseteq \operatorname{lex}(I^2)$.

On the other hand, if $R \supset I_1 \supset I_2 \supset \cdots$ is a filtration of ideals of dimension d, that is, $I_iI_j \subseteq I_{i+j}$ for all $i, j \geq 0$, then a similar proof of [5, Proposition 3.6] shows that $a_d(R/I_n)$ is bounded by a linear function of n. In particular, in the 0-dimensional case, $\operatorname{reg}(I_n)$ is bounded by a linear function of n. One may ask if this still holds for higher dimensions? Unfortunately this is not true even in the case $d = 1$, as shown by the following result.

Proposition 8 *Let* $I \subseteq (x_1, ..., x_{r-1}) \subset I_0 := R = K[x_1, ..., x_r]$ *be any homogeneous ideal. Let* $a_1 \leq a_2 \leq a_3 \leq \cdots$ *be an arbitrary positive integers. We recursively define the ideal*

I_n, $n \geq 1$, as follow: let $I_1 = (I, x_1x_r^{a_1})$, and for $n \geq 2$,

$$I_n = \Big(\sum_{i+j=n;\ i,j\geq 1} I_iI_j,\ x_1x_r^{a_n}\Big).$$

Then the sequence $\{I_n\}_{n\geq 0}$ is a filtration of ideals and $\operatorname{reg}(I_n) \geq a_n + 1$ for $n \geq 2$. In particular, there is no bounding function for the Castelnuovo-Mumford regularity which only depends on a finitely many numbers of the filtration.

PROOF. By the construction $I_iI_j \subseteq I_{i+j}$ for all $i, j \geq 0$. We show by induction that $I_n \supseteq I_{n+1}$. This is trivial for $n = 0$. Let $n > 0$. Then, by the induction hypothesis,

$$\sum_{i+j=n+1;\ i,j\geq 1} I_iI_j \subseteq \sum_{i+j=n+1;\ i,j\geq 1} I_{i-1}I_j \subseteq I_n.$$

Since $a_{n+1} \geq a_n$, we also have $x_1x_r^{a_{n+1}} \in I_n$. Hence $I_{n+1} \subseteq I_n$. Thus $\{I_n\}_{n\geq 0}$ is a filtration of ideals.
Let $n \geq 2$. It is clear that $I_i \subseteq (x_1, ..., x_{r-1})$ for all $i \geq 1$. Hence

$$\sum_{i+j=n;\ i,j\geq 1} I_iI_j \subseteq (x_1, ..., x_{r-1})^2.$$

Since $x_1x_r^{a_n} \notin (x_1, ..., x_{r-1})^2$, $x_1x_r^{a_n} \notin \sum_{i+j=n;\ i,j\geq 1} I_iI_j$. This implies that $x_1x_r^{a_n}$ is a minimal generating polynomial of I_n. Since $\operatorname{reg}(I_n)$ is bounded below by the maximal generating degree of I_n, we get $\operatorname{reg}(I_n) \geq a_n + 1$.

Remark. (i) If $\{I_n\}_{n\geq 0}$ is a good filtration, i.e. the Rees algebra $\oplus_{n\geq 0} I_nt^n$ is finitely generated, then from [3] or [9] we see that $\operatorname{reg}(I_n)$ is a linear function of n for $n \gg 0$. Moreover, by [12], for any homogeneous ideal J of R, $\operatorname{reg}(J + I_n)$ is still a linear function of n for all $n \gg 0$. Note that $\{J + I_n\}_{n\geq 0}$ is no more a good filtration, but one can say that it is obtained from a good filtration by a small perturbation.
(ii) Let $\{I_n\}_{n\geq 0}$ and $\{J_n\}_{n\geq 0}$ be two filtrations of R. Assume that $\operatorname{reg}(J_n)$ is bounded by a linear function of n. If $J_n \subseteq I_n$ and $\ell(I_n/J_n) < \infty$ for all n, then the exact sequence

$$0 \to I_n/J_n \to R/J_n \to R/I_n,$$

implies that $\operatorname{reg}(R/I_n) \leq \operatorname{reg}(R/J_n)$. Hence $\operatorname{reg}(I_n)$ is bounded by a linear function of n, too. Again a small perturbation does not change $\operatorname{reg}(I_n)$ very much.
On the other hand, the conditions $I_n \subseteq J_n$ and $\ell(J_n/I_n) < \infty$ for all n do not guarantee the existence of a linear bound for $\operatorname{reg}(I_n)$. In fact, the filtration $\{I_n\}_{n\geq 0}$ in the above proposition with $I = (x_1, ..., x_{r-1})$ $(r \geq 3)$ and the filtration $\{J_n = (x_1, (x_2, ..., x_{r-1})^n)\}$, satisfy these conditions, but $\operatorname{reg}(I_n)$ is not linearly bounded if the sequence $\{a_n\}$ is increasing quickly. This means the perturbation way can impose a big change on $\operatorname{reg}(I_n)$.

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DIAGONAL F-THRESHOLDS OF HIBI RINGS

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Throughout this article, let k be a field of characteristic $p > 0$, and let R be a Noetherian domain containing k. For an ideal I of R and positive integer e, let $I^{[p^e]}$ be an ideal generated by the p^eth powers of the elements of I.

1. Hibi ring

Let $P = (p_1, \cdots, p_n)$ be a finite partially ordered set(poset for short) and $J(P)$ the set of poset ideals of P.

For a distributive lattice $D = J(P)$, we define a map:

$$\begin{array}{ccc} \varphi : D(= J(P)) & \longrightarrow & K[T, X_1, \ldots, X_n] \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ I & \longmapsto & T\prod_{p_i \in I} X_i. \end{array}$$

In [Hib], Hibi defined the *Hibi ring* $\mathcal{R}_k[D]$ by

$$\mathcal{R}_k[D] := k[\{\varphi(I)\} \mid I \in D].$$

The Hibi ring has been studied by many reseachers.

Remark 1. (1) The Hibi ring is a graded ASL and normal Cohen-Macaulay domain([Hib]).
(2) $\dim \mathcal{R}_k[D] = \#P + 1$.
(3) $\mathcal{R}_k[D]$ is Gorenstein if and only if P is pure([Hib]).
(4) Hashimoto, Hibi and Noma([HaHibN]) computed divisor class groups of Hibi rings.
(5) Miyazaki gave criteria of a Hibi ring to be level and to be of type 2([Mi]).

2. F-threshold

Suppose that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of R such that $\mathfrak{a} \neq 0$. For each integer $e \geq 1$ and each ideal J such that $\mathfrak{a} \subseteq \sqrt{J}$, we put

$$\nu_{\mathfrak{a}}^J(p^e) = \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\}.$$

Definition 2 (F-threshold, [HMTW]). Let $R, \mathfrak{a}$ and J be as above. Then the *F-threshold* of $(R, \mathfrak{a})$ with respect to J, denoted by $c^J(\mathfrak{a})$, is defined by

$$c^J(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

if the limit exists. In particular, we call $c^{\mathfrak{a}}(\mathfrak{a})$ the *diagonal F-threshold* of R with respect to $\mathfrak{a}$.

In this article, we will focus on $c^{\mathfrak{m}}(\mathfrak{m})$, that is, the diagonal F-threshold of the maximal ideal with respect to the maximal ideal itself.

Example 3. (1) Let $(S, \mathfrak{n})$ be a regular local ring of positive characteristic. Then $c^{\mathfrak{n}}(\mathfrak{n}) = \dim S$.

(2) ([MOY]) Let $R = k[X_1, \ldots, X_d]^{(r)}$ be the rth Veronese subring of $S = k[X_1, \ldots, X_d]$. Put $\mathfrak{m} = (X_1, \ldots, X_d)^r$. Then $c^{\mathfrak{m}}(\mathfrak{m}) = \frac{r+d-1}{r}$.

(3) ([MOY]) Let $S = k[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a polynomial ring and put $\mathfrak{n} = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)S$. Take a binomial $f = X_1^{a_1} \cdots X_m^{a_m} - Y_1^{b_1} \cdots Y_n^{b_n} \in S$, where $a_1 \geq a_2 \geq \cdots \geq a_m$, $b_1 \geq b_2 \geq \cdots \geq b_n$. Let $R = S_{\mathfrak{n}}/(f)$ be a binomial hypersurface local ring with the unique maximal ideal $\mathfrak{m}$. Then

$$c^{\mathfrak{m}}(\mathfrak{m}) = m + n - 2 + \frac{\max\{a_1 + b_1 - \min\{\sum_{i=1}^{m} a_i, \sum_{j=1}^{n} b_j\}, 0\}}{\max\{a_1, b_1\}}.$$

3. Main Theorem

In this article, we will give a formura of diagonal F-thresholds $c^{\mathfrak{m}}(\mathfrak{m})$ of Hibi rings.

$C = (q_1, \ldots, q_t)$ is called *path* of P if C satisfies the following conditions: (1) $q_1, \ldots, q_t \in P$, (2) if $i \neq j$ then $q_i \neq q_j$ and (3) $q_i \gtrdot q_{i+1}$ or $q_i \lessdot q_{i+1}$, where $q_i \gtrdot q_{i+1}$ means q_i covers q_{i+1}.

For a path $C = (q_1, \ldots, q_t)$, q_i is called *locally maximal element* of C if $q_{i-1} \lessdot q_i$ and $q_i \gtrdot q_{i+1}$, and *locally minimal element* of C if $q_{i-1} \gtrdot q_i$ and $q_i \lessdot q_{i+1}$.

For a path $C = (q_1, \ldots, q_t)$, if $q_1 \leq \cdots \leq q_t$ then we call C *ascending chain* and if $q_1 \geq \cdots \geq q_t$ then we call C *descending chain.* We denote an ascending chain A and a descending chain D. Moreover, for an ascending chain $A = (q_1, \ldots, q_t)$, we put $t(A) = q_t$ and $< A >= \{q \in P \mid q \leq q_t\}$.

For a path $C = (q_1, \ldots, q_t)$ such that q_1 is a minimal element of P and q_t is a maximal element of P, we introduce the notion of the chain decomposition. We decompose C as follows:

$$C = A_1 + D_1 + A_2 + \cdots + D_{n-1} + A_n$$

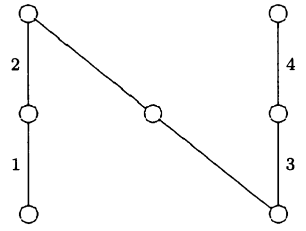
such that

$$\begin{aligned} A_1 &= (q_1, \ldots, q_{a(1)}), \\ D_1 &= (q_{a(1)+1}, \ldots, q_{d(1)}), \\ A_2 &= (q_{d(1)+1}, \ldots, q_{a(2)}), \\ &\vdots \\ D_{n-1} &= (q_{a(n-1)+1}, \ldots, q_{d(n-1)}), \\ A_n &= (q_{d(n-1)+1}, \ldots, q_{a(n)} = q_t) \end{aligned}$$

and $q_{a(i)}$ are locally maximal elements of C and $q_{d(j)}$ are locally minimal elements of C.

For a path $C : (q_1, \ldots, q_t)$, we define length$^* C = \#\{(q_i, q_{i+1}) \in E(C) \mid q_i \lessdot q_{i+1}\}$, where $E(C)$ is the set of edges of C.

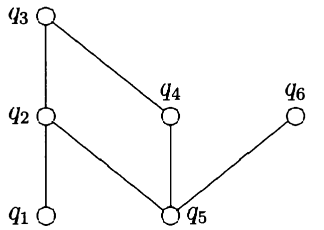
Example 4. (1) If C is a chain, then $\text{length}^* C = \text{length}\, C$.
(2) Consider the following path C:



Then, $\text{length}^* C = 4$.

For a path $C = (q_1, \ldots, q_t)$, we introduce the condition (*):
(1) q_1 is a minimal element of P and q_t is a maximal element of P.
For a chain decomposition $C = A_1 + D_1 + \ldots + D_{n-1} + A_n$,
(2) $V(D_i) \cap \left(\bigcup_{m=1}^{i-1} < A_m > \cup < A_i \setminus t(A_i) > \cup \{t(A_i)\} \right) = \emptyset$.
(3) $V(A_{i+1}) \cap \left(\bigcup_{m=1}^{i} < A_m > \right) = \emptyset$.

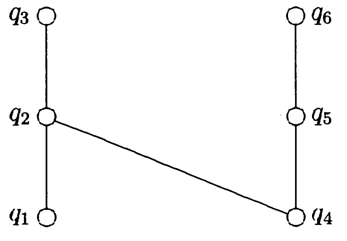
Example 5. Consider the folloing poset P:



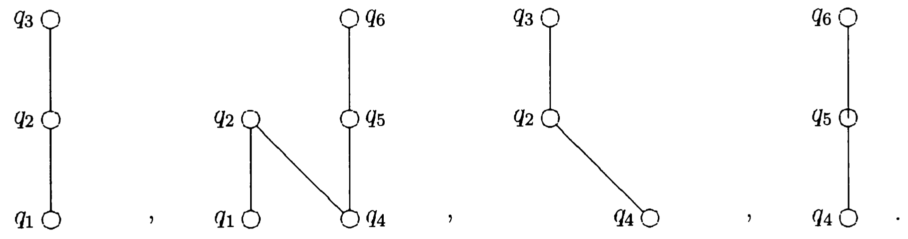
Then, $C_1 = (q_1, q_2, q_5, q_6)$ satisfies the condition (*), but $C_2 = (q_1, q_2, q_3, q_4, q_5, q_6)$ does not satisfy the condition (*).

We define $\text{rank}^* P = \max\{\text{length}^* C \mid C \text{ satisfies a condition } (*)\}$.

Example 6. Consider the following poset P:



Then, the following paths satisfy the condition (*):



Hence we have $\operatorname{rank}^* P = 3$.

The main theorem of this article is the following:

Theorem 7. Let $R = \mathcal{R}_k[D]$ be the Hibi ring of a poset P, and $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then

$$c^{\mathfrak{m}}(\mathfrak{m}) = \operatorname{rank}^* P + 2.$$

Corollary 8. Let X be an $m \times n$ matrix. Put $R = k[X]/I_2(X)$, where $I_2(X)$ is the set of 2-minors of X. Let $\mathfrak{m}$ be a maximal ideal of R. Then

$$c^{\mathfrak{m}}(\mathfrak{m}) = \max\{m, n\}.$$

Remark 9. Hirose gave a formula of F-thresholds of monomial ideals on toric rings([Hir]).

Last year, at the 31th Symposium on Commutative Algebra, Chiba, Ohtani and I gave a formula of $c^{\mathfrak{m}}(\mathfrak{m})$ of Segre products, and proved Corollary 8 as a corollary of it. In this article, we consider Hibi rings as another generarization.

4. Comparison between F-thresholds and other invariants

In the final section, we compute F-pure thresholds $\operatorname{fpt}(\mathfrak{m})$ and a-invariants $a(R)$ of Hibi rings, and compare them with $c^{\mathfrak{m}}(\mathfrak{m})$.

First, we recall definitions of the a-invariant and the F-pure threshold.

Definition 10. ([GW], Definition(3.1.4)) Let $R = \oplus_{n=0}^{+\infty} R_n$ be a standard graded Noetherian ring over a field $R_0 = k$, and $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Let $H^i_{\mathfrak{m}}(R)$ be the ith local cohomology mudule of R with respect to $\mathfrak{m}$. For a graded module M, let $[M]_n$ be the nth graded component of M.

Then we define

$$a(R) = \max\{n \in \mathbb{Z} \mid [H^{\dim R}_{\mathfrak{m}}(R)]_n \neq 0\},$$

and call it the *a-invariant* of R.

Definition 11. Let R be an F-finite Noetherian reduced ring of characteristic $p > 0$. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. Let $t \geq 0$ be a real number. Then,

(1) The pair $(R, \mathfrak{a}^t)$ is *F-pure* if for all large $q = p^e \gg 0$, there exists an element $d \in \mathfrak{a}^{\lfloor t(q-1) \rfloor}$ such that $d^{1/q} \hookrightarrow R^{1/q}$ splits as an R-homomorphism.

(2) The pair $(R, \mathfrak{a}^t)$ is *strongly F-pure* if for every $c \in R^\circ$, there exist $q = p^e$ and $d \in \mathfrak{a}^{\lceil tq \rceil}$ such that $(cd)^{1/q} \hookrightarrow R^{1/q}$ splits as an R-homomorphism.

(3) Suppose that R is strongly F-regular. Then

$$\operatorname{fpt}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-pure}\}$$

is said to be the *F-pure threshold of R* with respect to $\mathfrak{a}$.

In [BH], Bruns and Herzog computed $a(R)$ for an ASL R.

Theorem 12. ([BH], Theorem 1.1) Let R be a monotonely graded ASL over a field k on an upper semi-modular lattice Π with principal chain $\mathcal{P}(\Pi) = \xi_1, \dots, \xi_m$. Then

$$a(R) = -\sum_{i=1}^{m} \deg \xi_i.$$

By the above theorem, we give the following.

Theorem 13. Under the same notation as in Theorem 7, then

$$-a(R) = \operatorname{rank} P + 2.$$

Moreover, Chiba computed F-pure thresholds $\operatorname{fpt}(\mathfrak{m})$ of Hibi rings. For a poset P, we put $\operatorname{rank}_* P = \min\{\operatorname{length}^* C \mid C \text{ satisfies the condition } (*) \}$. Note that $\operatorname{rank}_* P = \min\{\operatorname{length} C \mid C \text{ is a maximal chain of } P \} \leq \operatorname{rank} P$.

Theorem 14. Under the same notation as in Theorem 7, then

$$\operatorname{fpt}(\mathfrak{m}) = \operatorname{rank}_* P + 2.$$

Therefore, for a Hibi ring R, we have

$$c^{\mathfrak{m}}(\mathfrak{m}) \geq -a(R) \geq \operatorname{fpt}(\mathfrak{m}).$$

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PSEUDO SYMMETRIC SEMIGROUPS GENERATED BY 3 ELEMENTS

HIROKATSU NARI

1. Introduction

This is a joint work with Takahiro Numata and Kei-ichi Watanabe.

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* H is a subset of $\mathbb{N}$ which is closed under addition and $\mathbb{N} \setminus H$ is a finite set. We always assume $0 \in H$.

We define $\mathrm{F}(H) := \max\{n \mid n \notin H\}$, and $\mathrm{g}(H) := \mathrm{Card}(\mathbb{N} \setminus H)$. We call $\mathrm{F}(H)$ the *Frobenius number* of H, and we call $\mathrm{g}(H)$ the *genus* of H. Then it is known that $2\,\mathrm{g}(H) \geq \mathrm{F}(H)+1$. We denote by $H = \langle a_1, a_2, ..., a_n \rangle$ the numerical semigroup generated by $a_1, a_2, ..., a_n$. Namely, $H = \sum_{i=1}^{n} a_i\mathbb{N}$. Moreover, every numerical semigroup admits a unique minimal system of generators.

We say that H is *symmetric* if $\mathrm{F}(H)$ is odd and for every $a \in \mathbb{Z}$, either $a \in H$ or $\mathrm{F}(H) - a \in H$, or equivalently, $2\,\mathrm{g}(H) = \mathrm{F}(H) + 1$. We say that H is *pseudo-symmetric* if $\mathrm{F}(H)$ is even and for every $a \in \mathbb{Z}, a \neq \mathrm{F}(H)/2$, either $a \in H$ or $\mathrm{F}(H) - a \in H$, or equivalently, $2\,\mathrm{g}(H) = \mathrm{F}(H) + 2$. It is known that a numerical semigroup is symmetric (resp. pseudo-symmetric) if and only if its semigroup ring is a Gorenstein (resp. Kunz) ring (see [BDF]). The a-invariant of the semigroup ring R ([GW]) is defined to be $a(R) = \max\{n \mid [H^1_{\mathfrak{m}}(R)]_n \neq 0\}$. Since $H^1_{\mathfrak{m}}(R) \cong k[t, t^{-1}]/R$, $a(R) = \max\{m \mid m \notin H\}$, that is, $\mathrm{F}(H) = a(R)$.

We say that an integer x is a *pseudo-Frobenius number* of H if $x \notin H$ and $x+s \in H$ for all $s \in H, s \neq 0$. We denote by $\mathrm{PF}(H)$ the set of pseudo-Frobenius numbers of H. The cardinality in $\mathrm{PF}(H)$ is called the *type* of H, denoted by $\mathrm{t}(H)$. Since $\mathrm{PF}(H)$ corresponds to the socle of $H^1_{\mathfrak{m}}(k[H])$, $\mathrm{t}(H) = \mathrm{r}(k[H])$, the Cohen-Macaulay type of $k[H]$. As $\mathrm{F}(H) \in \mathrm{PF}(H)$, $\mathrm{t}(H) = 1$ if and only if H is symmetric.

In this paperwe investigate numerical semigroups generated by three elements $H = \langle a, b, c \rangle$. We always assume that semigroup $H = \langle a, b, c \rangle$ is *not* symmetric. We now let $R = k[H] = k[t^a, t^b, t^c] \cong k[X, Y, Z]/\mathfrak{p}(a, b, c)$ be its semigroup ring over a field k. We denote by $\mathfrak{p} = \mathfrak{p}(a, b, c)$ the kernel of the homomorphism φ: $S = k[X, Y, Z] \to R$ of k-algebras defined by $\varphi(X) = t^a$, $\varphi(Y) = t^b$, and $\varphi(Z) = t^c$. Then it is known that if H is not symmetric, then the ideal $\mathfrak{p} = \mathrm{Ker}(\varphi)$ is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$$

for some positive integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$. The main goal of this paper is the following Theorem.

Theorem. Let H be a numerical semigroup. Then
(1) if $\beta' b > \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$,

(2) if $\beta' b < \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H)+1) = \alpha'\beta'\gamma'$.

2. Non-symmetric semigroups generated by 3 elements

We now let $H = \langle a, b, c\rangle$ be a non-symmetric numerical semigroup and $R = k[H] \cong k[X, Y, Z]/\mathfrak{p}$ be its semigroup ring over a field k. Then it is known that the ideal $\mathfrak{p}$ of $S = k[X, Y, Z]$ is generated by the maximal minors of the matrix $\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$, where α, β, γ, α', β', and γ' are positive integers. Comparing the degree of this matrix, we have the following.

Proposition 2.1. *If $H = \langle a, b, c\rangle$ is not symmetric, then*

(1) $(\alpha+\alpha')a = \beta' b + \gamma c$ *and* $\alpha + \alpha' = \min\{n \mid an \in \langle b, c\rangle\}$,
(2) $(\beta+\beta')b = \alpha a + \gamma' c$ *and* $\beta + \beta' = \min\{n \mid bn \in \langle a, c\rangle\}$,
(3) $(\gamma+\gamma')c = \alpha' a + \beta b$ *and* $\gamma + \gamma' = \min\{n \mid cn \in \langle a, b\rangle\}$.

Since the defining ideal of $k[H]/(t^a)$ is generated by the maximal minors of the matrix $\begin{pmatrix} 0 & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & 0 \end{pmatrix}$, we get $a = \dim_k k[H]/(t^a) = \beta\gamma + \beta'\gamma + \beta'\gamma'$. Similarly, we obtain that $b = \gamma\alpha + \gamma'\alpha + \gamma'\alpha'$ and $c = \alpha\beta + \alpha'\beta + \alpha'\beta'$.

Since $\mathrm{pd}_S(R) = 2$, we get a free resolution of R

$$0 \to S(-s) \oplus S(-t) \to S(-p) \oplus S(-q) \oplus S(-r) \to S \to R \to 0,$$

where $p = \deg(Z^{\gamma+\gamma'})$, $q = \deg(X^{\alpha+\alpha'})$, $r = \deg(Y^{\beta+\beta'})$, $s = \deg(X^{\alpha}) + p$, $t = \deg(Y^{\beta'}) + p$. Taking $\mathrm{Hom}_S(*, K_S) = \mathrm{Hom}_S(*, S(-x))$, we get

$$0 \to S(-x) \to S(p-x) \oplus S(q-x) \oplus S(r-x) \to S(s-x) \oplus S(t-x) \to K_R \to 0,$$

where $x = a+b+c$ and $K_R = \mathrm{Ext}^2_S(R, K_S)$. We put $f = s - x$ and $f' = t - x$. From this exact sequence and Proposition 2.1, we have the following.

Proposition 2.2. *If $H = \langle a, b, c\rangle$ is not symmetric, then* $\mathrm{PF}(H) = \{f, f'\}$ *and*

(1) $f = \alpha a + (\gamma+\gamma')c - (a+b+c)$,
(2) $f' = \beta' b + (\gamma+\gamma')c - (a+b+c)$.

3. Main results

The following is the key lemma to prove our theorem.

Lemma 3.1. *Let $H = \langle a, b, c\rangle$ be as in the previous section. We assume that $\beta' b > \alpha a$, or $f' > f$. Then*

(1) for $p, q, r \in \mathbb{N}$, $f' - f + pa + qb + rc \notin H$ if and only if $p < \alpha$ and $q < \beta$ and $r < \gamma$.

(2) $\mathrm{Card}\{h \in H \mid f' - f + h \notin H\} = \alpha\beta\gamma$.

(3) $\mathrm{Card}[[(f - H) \cap \mathbb{N}] \setminus (f' - H)] = \alpha\beta\gamma$.

Proof. Since $f'-f+\alpha a$, $f'-f+\beta b = \gamma' c$, $f'-f+\gamma c = \alpha' a \in H$, $f'-f+pa+qb+rc \in H$ if $p \geq \alpha$ or $q \geq \beta$ or $r \geq \gamma$. Conversely, assume $p < \alpha$ and $q < \beta$ and $r < \gamma$ and $f' - f + pa + qb + rc = ua + vb + wc \in H$ for some $u, v, w \in \mathbb{N}$. Then we have $(\beta' + q - v)b = (\alpha - p + u)a + (v - r)c$. If $v \geq r$, then this contradicts Proposition 2.1

(2). If $r > v$, we have $(\alpha - p + u)a = (\beta' + q - v)b + (r - v)c$. Then by Proposition 2.1 (1), we must have $p - u \geq \alpha'$ and again we have a contradiction since $r - v < \gamma$. This finishes the proof of (1) and (2) follows easily from (1).

To show (3), it suffices to note that for $h \in H$, $f - h \notin f' - H$ if and only if $f' - (f - h) \notin H$. Thus we have $\mathrm{Card}[(f - H) \setminus (f' - H)] = \mathrm{Card}\{h \in H \mid f' - f + h \notin H\} = \alpha\beta\gamma$. □

Theorem 3.2. *Let $H = \langle a, b, c\rangle$ be a numerical semigroup. Then*

(1) *if $\beta' b > \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$,*
(2) *if $\beta' b < \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha'\beta'\gamma'$.*

Proof. We may assume $\beta' b > \alpha a$. Then by Proposition 2.2, $\mathrm{F}(H) = f'$. Since $\mathbb{N} \setminus H = ((f' - H) \cap \mathbb{N}) \cup ((f - H) \cap \mathbb{N})$, we get

$$g(H) = \mathrm{Card}[(f' - H) \cap \mathbb{N}] + \mathrm{Card}[[(f - H) \cap \mathbb{N}] \setminus (f' - H)]$$

hence

$$g(H) = (F(H) + 1 - g(H)) + \alpha\beta\gamma.$$

□

As a corollary, we find another characterization of pseudo-symmetric numerical semigroups.

Corollary 3.3. *H is pseudo symmetric if and only if*

(1) *if $\beta' b > \alpha a$, then $\alpha = \beta = \gamma = 1$ and*
(2) *if $\beta' b < \alpha a$, then $\alpha' = \beta' = \gamma' = 1$.*

Proof. First suppose that $\beta' b > \alpha a$. By Theorem 3.2, $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$. Since H is pseudo-symmetric if and only if $2\,\mathrm{g}(H) = \mathrm{F}(H) + 2$, we obtain that $\alpha\beta\gamma = 1$, that is, $\alpha = \beta = \gamma = 1$. Similarly, H is pseudo-symmetric if and only if condition (2) is hold. □

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THE POSITIVITY OF THE FIRST COEFFICIENTS OF NORMAL HILBERT POLYNOMIALS

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ABSTRACT. Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. If R is unmixed, then $\overline{\mathrm{e}}_I^1(R) \geq 0$ for every $\mathfrak{m}$-primary ideal I in R, where $\overline{\mathrm{e}}_I^1(R)$ denotes the first coefficient of the normal Hilbert polynomial of R with respect to I. Thus the positivity conjecture on $\overline{\mathrm{e}}_I^1(R)$ posed by Wolmer V. Vasconcelos [V] is settled affirmatively.

1. INTRODUCTION

Throughout this paper let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Assume that R is analytically unramified, whence the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is reduced. We fix an $\mathfrak{m}$-primary ideal I in R and denote by $\overline{I^{n+1}}$ (resp. $\lambda_R(R/\overline{I^{n+1}})$) the integral closure of I^{n+1} (resp. the length of $R/\overline{I^{n+1}}$) for each $n \geq 0$. Then the normal Hilbert function

$$\lambda_R(R/\overline{I^{n+1}})$$

of R with respect to I is of polynomial type with degree d and we have integers $\{\overline{\mathrm{e}}_I^i(R)\}_{0 \leq i \leq d}$ such that the equality

$$\lambda_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}}_I^0(R)\binom{n+d}{d} - \overline{\mathrm{e}}_I^1(R)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{\mathrm{e}}_I^d(R)$$

holds true for all $n \gg 0$. We call these integers $\overline{\mathrm{e}}_I^i(R)$ the coefficients of the normal Hilbert polynomial of R with respect to I.

In this paper we are interested in the analysis of the first coefficient $\overline{\mathrm{e}}_I^1(R)$ of the normal Hilbert polynomial. The main purpose is to study the positivity conjecture on $\overline{\mathrm{e}}_I^1(R)$ posed by Wolmer V. Vasconcelos [V] and our result is stated as follows.

Key words and phrases: associated graded ring, Rees algebra, normal ideal, normal Hilbert polynomial

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Theorem 1.1. *Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. If R is unmixed, then*

$$\overline{e}_I^1(R) \geq 0$$

for every $\mathfrak{m}$-primary ideal I in R.

Here we should note that the conjecture holds true in the case where R is a Cohen-Macaulay local ring ([PUV, Theorem 2.2]). In fact, generally we have

$$\overline{e}_I^0(R) = e_I^0(R),$$

where $e_I^0(R)$ stands for the ordinary Hilbert-Samuel multiplicity of R with respect to I. Therefore $\overline{e}_I^1(R) \geq e_I^1(R)$ and so, if R is a Cohen-Macaulay local ring, we get

$$\overline{e}_I^1(R) \geq e_I^1(R) \geq 0,$$

because $e_I^1(R) \geq 0$ ([Nr, Corollary 1]). Mainly based on this fact, the third author M. Mandal, B. Singh, and J. Verma [MSV] gave several interesting answers in certain special cases and our Theorem 1.1 now affirmatively settles the conjecture in full generality.

We shall prove Theorem 1.1 in Section 2. In Section 3 we will discuss a few results related to the positivity conjecture. We expect that the integral closure $\overline{R}$ of R is a regular ring and $I\overline{R}$ is normal, that is, $I^n\overline{R}$ is integrally closed for all $n \geq 1$, once $\overline{e}_I^1(R) = 0$ for some $\mathfrak{m}$-primary ideal I in R. We shall give an affirmative answer in the case where $\overline{R}$ is a Cohen-Macaulay ring.

Throughout this paper, unless otherwise specified, we denote by R a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let $\overline{R}$ be the integral closure of R in its total quotient ring. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\lambda_R(M)$) stand for the number of elements in a minimal system of generators (resp. the length) of M.

2. Proof of Theorem 1.1

The main purpose of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. We have $\bar{\mathrm{e}}^1_{I\widehat{R}}(\widehat{R}) = \bar{\mathrm{e}}^1_I(R)$, since $\overline{\mathfrak{a}\widehat{R}} = \overline{\mathfrak{a}}\widehat{R}$ for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in R. Therefore, passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, without loss of generality we may assume that R is complete. If $d = 1$, we then have

$$\bar{\mathrm{e}}^1_I(R) = \lambda_R(\overline{R}/R) \geq 0.$$

Suppose that $d \geq 2$ and let $S = \overline{R}$. For each $\mathfrak{p} \in \mathrm{Ass}R$ we put $S(\mathfrak{p}) = \overline{R/\mathfrak{p}}$. Then $S(\mathfrak{p})$ is a module-finite extension of $R/\mathfrak{p}$ and we get

$$S = \prod_{\mathfrak{p}\in \mathrm{Ass}R} S(\mathfrak{p}) \quad \text{and} \quad \overline{I^{n+1}} = \overline{I^{n+1}S} \cap R$$

for all $n \geq 0$. Hence

$$\begin{aligned} \lambda_R(R/\overline{I^{n+1}}) \leq \lambda_R(S/\overline{I^{n+1}S}) &= \sum_{\mathfrak{p}\in \mathrm{Ass}R} \lambda_R(S(\mathfrak{p})/\overline{I^{n+1}S(\mathfrak{p})}) \\ &= \sum_{\mathfrak{p}\in \mathrm{Ass}R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})})\cdot\lambda_{S(\mathfrak{p})}(S(\mathfrak{p})/\overline{I^{n+1}S(\mathfrak{p})}), \end{aligned}$$

where $\mathfrak{m}_{S(\mathfrak{p})}$ denotes the maximal ideal of $S(\mathfrak{p})$. Notice that, since $\dim S(\mathfrak{p}) = d$ for each $\mathfrak{p} \in \mathrm{Ass}R$, we have

$$\begin{aligned} \bar{\mathrm{e}}^0_I(R) = \mathrm{e}^0_I(R) = \mathrm{e}^0_I(S) &= \sum_{\mathfrak{p}\in \mathrm{Ass}R} \mathrm{e}^0_I(S(\mathfrak{p})) \\ &= \sum_{\mathfrak{p}\in \mathrm{Ass}R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})})\cdot\mathrm{e}^0_{IS(\mathfrak{p})}(S(\mathfrak{p})) \\ &= \sum_{\mathfrak{p}\in \mathrm{Ass}R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})})\cdot\bar{\mathrm{e}}^0_{IS(\mathfrak{p})}(S(\mathfrak{p})), \end{aligned}$$

whence

$$\begin{aligned} 0 &\leq \lambda_R(S/\overline{I^{n+1}S}) - \lambda_R(R/\overline{I^{n+1}}) \\ &= \left[\bar{\mathrm{e}}^1_I(R) - \sum_{\mathfrak{p}\in \mathrm{Ass}R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})})\cdot\bar{\mathrm{e}}^1_{IS(\mathfrak{p})}(S(\mathfrak{p}))\right]\binom{n+d-1}{d-1} \\ &+ \ (\text{terms of lower degree}), \end{aligned}$$

so that

$$\bar{\mathrm{e}}^1_I(R) \geq \sum_{\mathfrak{p}\in \mathrm{Ass}R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})})\cdot\bar{\mathrm{e}}^1_{IS(\mathfrak{p})}(S(\mathfrak{p})).$$

Thus, in order to see $\overline{e}_I^1(R) \geq 0$, it suffices to show that $\overline{e}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq 0$ for each $\mathfrak{p} \in \mathrm{Ass}R$. If $d = 2$, we get

$$\overline{e}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq e_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq 0,$$

because $S(\mathfrak{p})$ is a Cohen-Macaulay local ring. Hence $\overline{e}_I^1(R) \geq 0$.

Suppose that $d \geq 3$ and that our assertion holds true for $d-1$. Then thanks to the above observation, passing to the ring $S(\mathfrak{p})$, we may assume that R is a normal complete local ring. Let $I = (a_1, a_2, \ldots, a_\ell)$ with $a_i \in R$, where $\ell = \mu_R(I)$. Let

$$T = R[Z_1, Z_2, \ldots, Z_\ell], \quad \mathfrak{q} = \mathfrak{m}T, \quad x = \sum_{i=1}^{\ell} a_i Z_i, \quad \text{and} \quad D = T/xT,$$

where $Z_1, Z_2, \ldots, Z_\ell$ are indeterminates over R. Let

$$R' = T_{\mathfrak{q}}, \quad I' = IR', \quad \text{and} \quad D' = D_{\mathfrak{q}}.$$

We then have $\overline{I^{n+1}R'} = \overline{I^{n+1}}R'$ for all $n \geq 0$, so that $\lambda_{R'}(R'/\overline{I^{n+1}R'}) = \lambda_R(R/\overline{I^{n+1}})$, whence

$$\overline{e}_I^1(R) = \overline{e}_{I'}^1(R').$$

Here we notice that $\mathrm{Ass}D' = \mathrm{Assh}D'$, because R' is catenary and normal; hence D' is unmixed, as D' is a homomorphic image of a Cohen-Macaulay ring. The ring D' is analytically unramified. To see this, since D' is a Nagata local ring, by [M, Theorem 70] it suffices to show that D is reduced, that is, $D_P = T_P/xT_P$ is an integral domain for every $P \in \mathrm{Ass}_T D$. Let $\mathfrak{p} = P \cap R$. Then since $\mathrm{ht}_T P = 1$, we have $\mathrm{ht}_R \mathfrak{p} \leq 1$, so that $I \not\subseteq \mathfrak{p}$, because $\mathrm{ht}_R \mathfrak{p} \leq 1 < d = \dim R$. Without loss of generality we may assume that $a_\ell \notin \mathfrak{p}$. Then, because $x = \sum_{i=1}^{\ell} a_i Z_i$ and a_ℓ is a unit of $R_\mathfrak{p}$, we get

$$T_\mathfrak{p} = R_\mathfrak{p}[Z_1, Z_2, \ldots, Z_\ell] = R_\mathfrak{p}[Z_1, Z_2, \ldots, Z_{\ell-1}, x],$$

whence the ring

$$T_\mathfrak{p}/xT_\mathfrak{p} = R_\mathfrak{p}[Z_1, Z_2, \ldots, Z_\ell]/xR_\mathfrak{p}[Z_1, Z_2, \ldots, Z_\ell] = R_\mathfrak{p}[Z_1, Z_2, \ldots, Z_{\ell-1}]$$

is an integral domain, as it is the polynomial ring with $\ell - 1$ indeterminates over $R_\mathfrak{p}$. Therefore for all $P \in \mathrm{Ass}_T D$ the ring $D_P = T_P/xT_P$ is an integral domain, because it is a localization of $R_\mathfrak{p}[Z_1, Z_2, \ldots, Z_{\ell-1}]$. Thus D is reduced, whence D' is analytically unramified and unmixed.

Let us denote by $\mathcal{A}$ the extended Rees ring of IT and by $\overline{\mathcal{A}}$ the integral closure of $\mathcal{A}$ in $T[t, t^{-1}]$, where t denotes an indeterminate. Similarly, let us denote by T the extended Rees ring of ID and by $\overline{\mathsf{T}}$ the integral closure of T in $D[t, t^{-1}]$. We put $N = (t^{-1}, It)$ in $\mathcal{A}$. We look at the homomorphism

$$\psi : T[t, t^{-1}] \to D[t, t^{-1}]$$

of graded T-algebras such that $\psi(t) = t$. Since $\psi(\mathcal{A}) = \mathsf{T}$ and $\overline{\mathsf{T}}$ is a module-finite extension of T, the homomorphism ψ gives rise to the finite homomorphism

$$\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \longrightarrow \overline{\mathsf{T}}$$

of graded T-algebras. Let $\overline{\mathcal{B}}$ (resp. $\overline{\mathsf{U}}$) denote the integral closure of $\mathcal{B} = \mathcal{A}_{\mathfrak{q}}$ (resp. $\mathsf{U} = \mathsf{T}_{\mathfrak{q}}$). Then we get the homomorphism

$$\varphi_{\mathfrak{q}} : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \to \overline{\mathsf{U}}$$

of graded R'-algebras and, thanks to Proof of [HU, Theorem 2.1], we furthermore have the following. Let us include a brief proof for the sake of completeness.

Claim 1. *The homomorphism*

$$\varphi_P : [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_P \longrightarrow [\overline{\mathsf{T}}]_P$$

is an isomorphism for all $P \in \operatorname{Spec}\mathcal{A} \setminus V(N)$. Hence the kernel and the cokernel of the homomorphism $\varphi_{\mathfrak{q}} : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \longrightarrow \overline{\mathsf{U}}$ of graded $\mathcal{B}$-modules are of finite length, so that they are finitely graded.

Proof. Because $\overline{\mathcal{A}}[t] = T[t, t^{-1}]$ and $xt\overline{\mathcal{A}}[t] = xT[t, t^{-1}]$, the homomorphism $\varphi_{t^{-1}}$ is an isomorphism, whence so is the homomorphism φ_P, if $t^{-1} \notin P$.

Suppose now that $It \not\subseteq P$. We may assume $a_\ell t \notin P$. Notice that

$$\begin{aligned}
[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_{a_\ell t} &= \left[\overline{R[It, t^{-1}][Z_1, Z_2, \ldots, Z_\ell]}/xt\cdot\overline{R[It, t^{-1}][Z_1, Z_2, \ldots, Z_\ell]}\right]_{a_\ell t} \\
&= \left(\overline{R[It, t^{-1}][\frac{1}{a_\ell t}]}\right)[Z_1, Z_2, \ldots, Z_\ell]/\left(\sum_{i=1}^{\ell-1} \frac{a_i Z_i t}{a_\ell t} + Z_\ell\right) \\
&= \left(\overline{R[It, t^{-1}][\frac{1}{a_\ell t}]}\right)[Z_1, Z_2, \ldots, Z_{\ell-1}]
\end{aligned}$$

and that

$$\begin{aligned}
D[t,t^{-1}]_{a_\ell t} &= T[t,t^{-1},\frac{1}{a_\ell t}]/x{\cdot}T[t,t^{-1},\frac{1}{a_\ell t}] \\
&= T[t,t^{-1},\frac{1}{a_\ell}]/x{\cdot}T[t,t^{-1},\frac{1}{a_\ell}] \\
&= R[\frac{1}{a_\ell},Z_1,Z_2,\ldots,Z_\ell,t,t^{-1}]/x{\cdot}R[\frac{1}{a_\ell},Z_1,Z_2,\ldots,Z_\ell,t,t^{-1}] \\
&= \left(R[\frac{1}{a_\ell},t,t^{-1}]\right)[Z_1,Z_2,\ldots,Z_\ell]/\left(\sum_{i=1}^{\ell-1}\frac{a_iZ_i}{a_\ell}+Z_\ell\right) \\
&= \left([R[t,t^{-1}]]\,[\frac{1}{a_\ell t}]\right)[Z_1,Z_2,\ldots,Z_{\ell-1}].
\end{aligned}$$

Then we get the following commutative diagram

$$\begin{array}{ccccc}
[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_{a_\ell t} & \xrightarrow{\varphi_{a_\ell t}} & [\overline{\mathsf{T}}]_{a_\ell t} & \longrightarrow & D[t,t^{-1}]_{a_\ell t} \\
\downarrow\simeq & & & & \downarrow\simeq \\
\left([\overline{R[It,t^{-1}]}][\frac{1}{a_\ell t}]\right)[Z_1,\ldots,Z_{\ell-1}] & & \longrightarrow & & \left([R[t,t^{-1}]]\,[\frac{1}{a_\ell t}]\right)[Z_1,\ldots,Z_{\ell-1}],
\end{array}$$

where the vertical homomorphisms are isomorphisms, so that the horizontal homomorphism $\varphi_{a_\ell t}$ is injective. Because $\left([\overline{R[It,t^{-1}]}][\frac{1}{a_\ell t}]\right)[Z_1,Z_2,\ldots,Z_{\ell-1}]$ is integrally closed in $\left([R[t,t^{-1}]]\,[\frac{1}{a_\ell t}]\right)[Z_1,Z_2,\ldots,Z_{\ell-1}]$ and $\varphi_{a_\ell t}$ is finite, $\varphi_{a_\ell t}$ is an isomorphism, whence

$$\varphi_P : [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_P \longrightarrow [\overline{\mathsf{T}}]_P$$

is an isomorphism too. This proves Claim 1. □

The normal ring $\overline{\mathcal{B}}$ is catenary, since it is a finitely generated R'-algebra, while we get

$$\dim \overline{\mathcal{B}}/(xt,t^{-1})\overline{\mathcal{B}} = \dim \overline{\mathsf{U}}/t^{-1}\overline{\mathsf{U}} = d-1$$

by Claim 1. Therefore t^{-1}, xt forms a regular sequence in the normal ring $\overline{\mathcal{B}}$. Hence xt is a non-zerodivisor in the associated graded ring

$$\overline{\mathcal{B}}/t^{-1}\overline{\mathcal{B}} = \bigoplus_{n\geq 0}\overline{I^n R'}/\overline{I^{n+1}R'}$$

of the filtration $\{\overline{I^nR'}\}_{n\in Z}$ of integrally closed ideals in R'. Consequently, we have

$$\overline{\mathrm{e}}_I^1(R) = \overline{\mathrm{e}}_{I'}^1(R') = \overline{\mathrm{e}}_{ID'}^1(D'),$$

since $\dim D' = \dim R' - 1 = d - 1 \geq 2$ and since the kernel and the cokernel of the homomorphism

$$\overline{\varphi}_{\mathfrak{q}} : \overline{\mathcal{B}}/(xt, t^{-1})\overline{\mathcal{B}} \longrightarrow \overline{\mathrm{U}}/t^{-1}\overline{\mathrm{U}}$$

induced from $\varphi_{\mathfrak{q}}$ are finitely graded. Thus the hypothesis of induction on d yields the assertion that $\overline{\mathrm{e}}_I^1(R) \geq 0$, which completes the proof of Theorem 1.1. □

The condition in Theorem 1.1 that R is unmixed is not superfluous. Let us note the simplest example. See [MSV, Example 2.4] for more examples.

Example 2.1. We look at the local ring

$$R = k[[X, Y, Z]]/\mathfrak{a},$$

where $k[[X, Y, Z]]$ is the formal power series ring over a field k and $\mathfrak{a} = (X) \cap (Y, Z)$. Then $\dim R = 2$, R is mixed, and $\overline{\mathrm{e}}_{\mathfrak{m}}^1(R) = \overline{\mathrm{e}}_{\mathfrak{m}}^2(R) = -1$. Hence the famous bad example [N, p. 203, Example 2] of Nagata which is a non-regular local integral domain $(A, \mathfrak{n})$ of dimension 2 with $\mathrm{e}_{\mathfrak{n}}^0(A) = 1$ possess $\overline{\mathrm{e}}_{\mathfrak{n}}^1(A) = \overline{\mathrm{e}}_{\mathfrak{n}}^2(A) = -1$, because

$$\widehat{A} \cong k[[X, Y, Z]]/[(X) \cap (Y, Z)]$$

for some field k.

Proof. We put $T = k[[X, Y, Z]]$ and $\mathfrak{q} = (X, Y, Z)$ in T. Then $\overline{R} = T/(X) \oplus T/(Y, Z)$ and we have the exact sequence

$$(E) \quad 0 \to R \to T/(X) \oplus T/(Y, Z) \to T/\mathfrak{q} \to 0$$

of T-modules; hence $\mathfrak{m}\overline{R} \subseteq R$. Recall that $\mathfrak{m}$ is a normal ideal in R, that is, $\overline{\mathfrak{m}^n} = \mathfrak{m}^n$ for all $n \geq 1$, since the associated graded ring

$$\mathrm{gr}_{\mathfrak{m}}(R) = k[X, Y, Z]/[(X) \cap (Y, Z)]$$

of $\mathfrak{m}$ is reduced. Therefore, as

$$\mathfrak{m}^{n+1} = \overline{\mathfrak{m}^{n+1}} = \overline{\mathfrak{m}^{n+1}\overline{R}} \cap R = \mathfrak{m}^{n+1}\overline{R} \cap R,$$

thanks to exact sequence (E) above, we get

$$0 \to R/\overline{\mathfrak{m}^{n+1}} \to T/[(X) + \mathfrak{q}^{n+1}] \oplus T/[(Y, Z) + \mathfrak{q}^{n+1}] \to T/\mathfrak{q} \to 0$$

for all $n \geq 0$. Hence

$$\lambda_R(R/\overline{\mathfrak{m}^{n+1}}) = \binom{n+2}{2} + \binom{n+1}{1} - 1,$$

so that $\overline{\mathrm{e}}^1_{\mathfrak{m}}(R) = \overline{\mathrm{e}}^2_{\mathfrak{m}}(R) = -1$. □

Let us note a consequence of Theorem 1.1.

Corollary 2.2 ([MTV, Theorem 1]). *Let R be an analytically unramified unmixed local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let I be a parameter ideal in R. If $\overline{\mathrm{e}}^1_I(R) = \mathrm{e}^1_I(R)$, then R is a regular local ring with $\mu_R(\mathfrak{m}/I) \leq 1$, whence I is normal.*

Proof. We get $\mathrm{e}^1_I(R) \geq 0$ by Theorem 1.1, whence by [GhGHOPV, Theorem 2.1] R is a Cohen-Macaulay local ring with $\mathrm{e}^1_I(R) = 0$. Because $\overline{\mathrm{e}}^1_{\overline{I}}(R) \geq \mathrm{e}^1_{\overline{I}}(R)$ and

$$\mathrm{e}^1_{\overline{I}}(R) \geq 0$$

([Nr, Corollary 1]), we furthermore have $\mathrm{e}^1_{\overline{I}}(R) = 0$, whence $\overline{I}$ is a parameter ideal in R ([Nr, Corollary 2]). Because parameter ideals contain no proper reductions ([NR]), we get $\overline{I} = I$, whence by [G, Theorem (3.1)] R is a regular local ring with $\mu_R(\mathfrak{m}/I) \leq 1$ and I is normal. □

Remark 2.3. In Corollary 2.2, unless I is a parameter ideal, R is not necessarily a regular local ring, even though $\overline{\mathrm{e}}^1_I(R) = \mathrm{e}^1_I(R)$. Let us note an example. We look at the local ring

$$R = k[[X, Y, Z]]/(Z^2 - XY),$$

where $k[[X, Y, Z]]$ is the formal power series ring over a field k of characteristic 0. Then R is a rational singularity, so that $\overline{\mathrm{e}}^1_I(R) = \mathrm{e}^1_I(R)$ for every integrally closed $\mathfrak{m}$-primary ideal I in R.

3. A further problem

Let R be an analytically unramified unmixed local ring and I an $\mathfrak{m}$-primary ideal in R. We then expect that $\overline{R}$ is a regular ring and $I\overline{R}$ is normal, that is all the powers $I^n\overline{R}$ are integrally closed, once $\overline{\mathrm{e}}^1_I(R) = 0$. This is the case when $\overline{R}$ is a Cohen-Macaulay ring, as we will show in the following.

Theorem 3.1. *Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let S be an overring of R and assume that S is a finitely generated R-module with $\dim_R S/R < d$. Let I be an $\mathfrak{m}$-primary ideal in R such that $\bar{e}_I^1(R) = 0$. If $\operatorname{depth}_R S = d$, then S is a regular ring, $S = \overline{R}$, and $I\overline{R}$ is normal.*

Proof. We may assume that R is complete. Let $\mathrm{Q}(R)$ be the total quotient ring of R. We notice that S is a Cohen-Macaulay R-module with $\dim_R S = d$; hence R is unmixed. Therefore $S \subseteq \mathrm{Q}(R)$, as $\dim_R S/R < d$, so that $S \subseteq \overline{R}$. Since R is complete, we get a decomposition $S = \prod_{i=1}^{\ell} S_i$ of S where S_i is a Cohen-Macaulay local ring with $\dim S_i = d$. Consequently, for the same reason as in Proof of Theorem 1.1 we have

$$\bar{e}_I^1(R) \geq \sum_{i=1}^{\ell} \lambda_R(S_i/\mathfrak{m}_i)\cdot \bar{e}_{IS_i}^1(S_i) \geq 0,$$

where $\mathfrak{m}_i$ is the maximal ideal in S_i; hence $\bar{e}_{IS_i}^1(S_i) = 0$ for each $1 \leq i \leq \ell$. As $\bar{e}_{IS_i}^1(S_i) \geq e_{\overline{IS_i}}^1(S_i) \geq 0$, we have $e_{\overline{IS_i}}^1(S_i) = 0$, so that $\overline{IS_i}$ is a parameter ideal in S_i. Hence $\overline{IS_i} = IS_i$. Therefore by [G, Theorem (3.1)] S_i is a regular local ring and IS_i is normal. Thus S is regular and IS is normal, whence $S = \overline{R}$. □

Corollary 3.2. *Let R be a two-dimensional analytically unramified unmixed local ring with maximal ideal $\mathfrak{m}$ and let I be an $\mathfrak{m}$-primary ideal in R. If $\bar{e}_I^1(R) = 0$, then $\overline{R}$ is a regular ring and $I\overline{R}$ is normal.*

Proof. Notice that $\overline{R}$ is a finitely generated R-module and $\operatorname{depth}_R \overline{R} = 2$, because R is analytically unramified and unmixed with $\dim R = 2$, whence the assertion follows from Theorem 3.1, taking $S = \overline{R}$. □

Remark 3.3. The ring R itself is, however, not necessarily a regular local ring even if $\dim R = 2$. Let us note an example. We look at the local ring

$$R = k[[X, Y, Z, W]]/[(X, Y) \cap (Z, W)],$$

where $k[[X, Y, Z, W]]$ is the formal power series ring over a field k. We then have $\bar{e}_{\mathfrak{m}}^1(R) = 0$ and $\bar{e}_{\mathfrak{m}}^2(R) = -1$. The ring R is Buchsbaum but not Cohen-Macaulay, while

$$\overline{R} = k[[X, Y]] \times k[[Z, W]]$$

is a regular ring.

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HODGE COHOMOLOGY OF ÉTALE NORI FINITE VECTOR BUNDLES

ĐOÀN TRUNG CƯỜNG

ABSTRACT. Étale Nori finite vector bundles are those bundles defined by representations of a finite étale group scheme in the usual way. In this note we show that in many cases the dimensions of the Hodge cohomology groups of such a vector bundle and of a twist of it by an automorphism of the ground field are the same. This generalizes to the higher rank case a result of Pink and Roessler.

1. ÉTALE NORI FINITE VECTOR BUNDLES

Let X be a smooth geometrically connected projective variety over a perfect field k with a k-rational point $x \in X(k)$. Let V be a vector bundle on X. The Hodge cohomology groups with coefficients in V are defined as

$$H^i_{\mathrm{Hdg}}(X, V) := \bigoplus_j H^{i-j}(X, V \otimes_{\mathcal{O}_X} \Omega^j_{X/k})$$

and we set $h^i_{\mathrm{Hdg}}(V) := \dim_k H^i_{\mathrm{Hdg}}(X, V)$. In [9], Pink and Roessler showed that

Theorem 1.1 (Pink-Roessler). *Let $k = \overline{\mathbb{Q}}$ and L be a line bundle such that $L^{\otimes n} \simeq \mathcal{O}_X$ for some natural number n. Then $h^i_{\mathrm{Hdg}}(L) = h^i_{\mathrm{Hdg}}(L^{\otimes a})$ for all a prime to n and $i \in \mathbb{Z}$.*

They also posed the following question for the positive characteristic case (see [9, Conjecture 5.1]): Assume that $\mathrm{char}(k) = p > 0$, X is liftable over the ring $W_2(k)$ of 2-Witt vectors, $\dim(X) \leqslant p$ and L is a torsion line bundle of order n on X. Then is it the case that $h^i_{\mathrm{Hdg}}(L) = h^i_{\mathrm{Hdg}}(L^{\otimes a})$ for a prime to n, $i \in \mathbb{Z}$? In fact, when $(n, p) = 1$ and a is congruent to a power of p modulo n, Pink and Roessler showed that the conjecture is true as an easy consequence of a result of Deligne-Illusie [1, Lemme 2.9]. It is also noticed in [3, Remark 10] that a Riemann-Roch calculation implies a positive answer for the conjecture in the case of curves. For the remaining cases, as far as we are aware of, the only result until now is a positive answer of Esnault-Ogus for $n = p$ and X ordinary.

The aim of this note is to generalize the result of Pink-Roessler to the higher rank case with a hope to have a better understand of the above conjecture. Recall that Nori [8] defined a vector bundle V to be finite if V satisfies an equality $f(V) \simeq g(V)$ for some distinct polynomials f, g whose coefficients are non-negative integers. We say a vector bundle V is Nori finite if it is a subquotient of a finite direct sum of finite vector bundles. We denote still by x the fiber functor at the point x from the category of coherent sheaves on X to the category of k-vector spaces. If V is a Nori finite vector bundle, Nori proved that by taking direct sums, tensors,

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duals, subquotients, V generates a k-linear abelian rigid tensor category (see [8, Chapter 1, Proposition 3.7]). Let (V, x) be the pair consisting of this category and the fiber functor restricted on it, then by a well-known theorem of Saavedra, (V, x) is equivalent to the representation category $\mathrm{Rep}_k(G)$ of a finite group scheme G of tensor automorphisms of the fiber functor x, together with the forgetful functor $\mathrm{Rep}_k(G) \to \mathrm{Vect}_k$. This is called the Tannaka duality and G is called the Tannaka group scheme of V. For more on Nori finite vector bundles and Tannaka duality we refer to [8, 2].

Definition 1.2. A vector bundle V on X is called étale Nori finite if it is Nori finite and the Tannaka group scheme is étale.

Torsion line bundles are exactly Nori finite vector bundles of rank 1, and the Tannaka group scheme is μ_n where n is the torsion order. Hence, if the ground field has characteristics 0 then torsion line bundles are always étale. More generally, due to a result of Cartier, any finite group scheme over a field of characteristic zero is always étale, thus every Nori finite vector bundle in characteristic zero is étale and moreover, the category generated by a Nori finite vector bundle is always semi-simple. The situation is more complicated in characteristic $p > 0$. If $\mathrm{char}(k) = p > 0$, a torsion line bundle of torsion order n is étale if and only if n is prime to p. For higher rank Nori finite vector bundles, from Artin-Schreier theory it is known that the Tannaka group scheme G can be a p-group scheme and still étale. In this case the category $\mathrm{Rep}_k(G)$ is no longer semi-simple, this causes a lot of difficulties in studying the corresponding bundle. In the present note we consider only étale Nori finite vector bundles.

Now let V be an étale Nori finite vector bundle with the Tannaka group scheme G and assume k is algebraically closed. The category $\mathrm{Rep}_k(G)$ is equivalent to the representation category $\mathrm{Rep}_k(G(k))$ of the abstract group $G(k)$ which carries a natural action of $\mathrm{Aut}(k)$. Let $\rho : G(k) \to \mathrm{GL}_r(k), r = \mathrm{rank}(V)$, be the representation corresponding to V and $\sigma \in \mathrm{Aut}(k)$. We denote the vector bundle corresponding to $\sigma \circ \rho$ by V_σ. For example, let $\mathrm{char}(k) = 0$ and L be a torsion line bundle of torsion order n. Let $\xi \in \mu_n(k)$ be a primitive n-th root of unity then there corresponds to L via the Tannaka duality a representation $\rho : \mu_n(k) \to k^*, \xi \mapsto \rho(\xi)$. Since $(a, n) = 1$, there always exists an automorphism $\sigma \in \mathrm{Aut}(k)$ such that $\sigma(\xi) = \xi^a$. Then $\sigma \circ \rho(\xi) = \rho \circ \sigma(\xi) = \rho(\xi)^a$. This means $L_\sigma \simeq L^{\otimes a}$. This suggests us the following generalization of Pink-Roessler's conjecture for higher rank vector bundles.

Question 1. When $\mathrm{char}(k) = p > 0$, we assume in addition that X is liftable over the ring $W_2(k)$ of 2-Witt vectors on k and $\dim(X) \leqslant p$. For $\mathrm{char}(k) \geqslant 0$, is it true that $h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(V_\sigma)$ for all $\sigma \in \mathrm{Aut}(k)$ and $i \in \mathbb{Z}$?

In this note we give a positive answer to Question 1 in certain cases. The first case is when $\mathrm{char}(k) = p > 0$ and the associated representation of V is realizable over a finite field, that is, if it is conjugated over k to a representation on $\mathrm{GL}_r(\mathbb{F}_q)$ for some power q of p. Using this we are able to give an algebraic proof for a positive answer for Question 1 in the case of characteristic zero (Theorem 3.3). Note that an answer for Question 1 in characteristic zero has been handled before by Sauter [10, Satz 4.2.1] generalizing the proofs for torsion line bundles of Pink-Roessler and Esnault which rely on the comparison theorem between de Rham cohomology and Betti cohomology.

The main result of this note is a part of our article which appears in the International Mathematics Research Notices 2010.

2. Positive characteristic case

In the whole of this section we assume that the ground field k is a perfect field of characteristic p. We first note that the étale assumption on G in Question 1 is necessary, without this restriction the conclusion in Question 1 should be at the opposite side as the following example shows.

Example 2.1. Let X be a super singular elliptic curve over k. Let F_X be the Frobenius on X, then the induced map $F_X^* : H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X)$ vanishes [5, page 332]. Let $0 \neq \alpha \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X)$ be a cohomology class of an extension V of $\mathcal{O}_X$ by itself. Since $F_X^*(\alpha) = 0$, the sequence $0 \longrightarrow \mathcal{O}_X \longrightarrow F_X^* V \longrightarrow \mathcal{O}_X \longrightarrow 0$ splits. It is proved by Mehta-Subramanian [6, Section 2] that V is Nori finite with a finite local group scheme. It is easy to show that the non-splitting of the sequence $0 \longrightarrow \mathcal{O}_X \longrightarrow V \longrightarrow \mathcal{O}_X \longrightarrow 0$ implies (in fact, is equivalent to) $\dim_k H^0(X, V) = \dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim_k H^0(X, F_X^* V) = 2 \dim_k H^0(X, \mathcal{O}_X) = 2$. Therefore, $h^0_{\mathrm{Hdg}}(V) < h^0_{\mathrm{Hdg}}(F_X^* V)$.

Definition 2.2. Let H be a group and $\rho : H \longrightarrow \mathrm{GL}_r(k)$ be a representation of H. Let $k_0 \subseteq k$ be a subfield. We say that ρ is realizable over k_0 if it is conjugated over k to a representation $\rho_0 : H \longrightarrow \mathrm{GL}_r(k_0)$, that is, $\rho \simeq \rho_0 \otimes_{k_0} k$.

In positive characteristic, the most emphasized automorphism in Question 1 is $\sigma = F^n$ a power of the Frobenius of k. In that case, it is easy to see that $V_\sigma \simeq (F_X^n)^* V$ where F_X is the Frobenius morphism of X. The first attempt to answer Question 1 is the following proposition.

Proposition 2.3. *Let X/k be a smooth geometrically connected projective scheme which is liftable over the ring $W_2(k)$ of 2-Witt vectors over k. Let V be an étale Nori finite vector bundle whose associated representation is ρ. Let $\sigma \in \mathrm{Aut}(k)$. If ρ is realizable over $\overline{\mathbb{F}}_p$ then*

$$h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(V_\sigma), \quad \textit{for all } i \leq p.$$

In particular, the equality holds for all $i \in \mathbb{Z}$ if $\dim X \leq p$.

Proof. Let $F \in \mathrm{Aut}(k)$ be the Frobenius homomorphism. Assume that ρ is conjugated to a representation $\rho_0 : G(k) \longrightarrow \mathrm{GL}_r(\overline{\mathbb{F}}_p)$. Then $\sigma \circ \rho$ is conjugated to $\sigma \circ \rho_0$. Moreover, from the Galois theory $\sigma|_{\overline{\mathbb{F}}_p} = F^n$ is a power of the Frobenius for some $n > 0$. Hence $\sigma \circ \rho_0 \sim F^n \circ \rho_0$ and we can replace σ by F^n. It then suffices to prove

$$h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}((F_X^n)^* V), \quad \text{for all } i \leq p.$$

In fact, ρ is realizable over a finite field $\mathbb{F}_{p^N}$ for some $N > 0$. Thus $F^N \circ \rho = \rho$ and $(F_X^N)^* V \simeq V$. As a consequence, the sequence $\{h^i_{\mathrm{Hdg}}((F_X^n)^* V)\}_{n=0}^\infty$ is periodic. On the other hand, by the proof of [1, Lemme 2.9] (see [4, Lemma 11.1] for a precise statement), for each $i \leqslant p$ the sequence $\{h^i_{\mathrm{Hdg}}((F_X^n)^* V)\}_{n=0}^\infty$ does not decrease. So we get immediately the assertion of the proposition. □

>From Proposition 2.3 we get immediately the following consequence.

Corollary 2.4. *If $k = \overline{\mathbb{F}}_p$ and $\dim(X) \leq p$ then the answer to Question 1 is positive.*

In the proof of Proposition 2.3, the vector bundle V satisfies $(F_X^N)^* V \simeq V$ and we say that V is Frobenius periodic. In fact, this property also characterizes those étale Nori vector bundles whose associated representations are realizable over a finite field.

Proposition 2.5. *Let X/k be a smooth geometrically connected projective scheme. A vector bundle V is Frobenius periodic if and only if V is an étale Nori finite vector bundle and the associated representation is realizable over a finite field. In particular, this is the case if $p \nmid \#G(\bar{k})$ where G is the Tannaka group scheme of V.*

Combining two Propositions 2.3 and 2.5 we get

Corollary 2.6. *Let X/k be a smooth geometrically connected projective scheme which is liftable over the ring $W_2(k)$ of 2-Witt vectors over k. Let V be an étale Nori finite vector bundle whose Tannaka group scheme is denoted by G. Assume in addition that $G(\bar{k})$ has order prime to p. Then $h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(V_\sigma)$, for all $i \leq p$.*

3. Main result

In the case of characteristic zero, Esnault, Pink-Roessler, Sauter gave an affirmative answer to Question 1 by using the comparison theorem between de Rham cohomology and Betti cohomology (see [10, Satz 4.2.1]). Using Proposition 2.3, in the rest of this section we give an algebraic proof for this fact. We first recall Brauer's theory of decomposition map in representation theory of finite groups. Let A be a Dedekind domain with the field of fractions K of characteristic zero and a residue field k of $\mathrm{char}(k) = p > 0$. Assume that G is a finite group scheme over $\mathrm{Spec}\, A$ whose Hopf algebra $A[G]$ is a finitely generated projective A-module. Let G_K and G_k denote respectively the generic fiber and the special fiber of G at k. Assume in addition that G_k is étale. Let $\mathrm{Rep}_K(G_K)$ (resp. $\mathrm{Rep}_k(G_k)$) be the category of finite dimensional representations of G_K (resp. G_k) over K (resp. k) and $\mathcal{R}(G_K)$ (resp. $\mathcal{R}(G_k)$) the corresponding Grothendieck group. By taking finite extensions of the fields if necessary, we can assume that K and k are big enough and G_K and G_k are defined by the abstract groups $G_K(K)$ and $G_k(k)$, that means, $\mathrm{Gal}(K)$ and $\mathrm{Gal}(k)$ act trivially on $G_K(K)$ and $G_k(k)$ respectively (see [12, 6.4]). If we denote by $\mathcal{R}(G_K(K))$ and $\mathcal{R}(G_k(k))$ the Grothendieck groups of the representation categories of $G_K(K)$ and $G_k(k)$ over K and k respectively, we have $\mathcal{R}(G_K) \simeq \mathcal{R}(G_K(K))$ and $\mathcal{R}(G_k) \simeq \mathcal{R}(G_k(k))$. In the theory of representations of finite groups, it is well-known that there is a ring epimorphism $d : \mathcal{R}(G_K(K)) \longrightarrow \mathcal{R}(G_k(k))$ which is called the decomposition map and is constructed as follows (see [11, Theorem 33]). Let $[V] \in \mathcal{R}(G_K(K))$. Let V_0 be an A-lattice of V, that is, V_0 is a finitely generated A-submodule of V and V_0 generates V as a K-vector space. Replacing V_0 by the sum of its image under the actions of the elements of $G_K(K)$, we assume that V_0 is stable under the action of $G_K(K)$. Hence $V_0 \otimes_A k$ is a $G_k(k)$-module. From V one can get different $V_0 \otimes_A k$ (even non-isomorphic) by choosing different V_0, but we get the same equivalent class $[V_0 \otimes_A k] \in \mathcal{R}(G_k(k))$. This defines the map $d : \mathcal{R}(G_K(K)) \longrightarrow \mathcal{R}(G_k(k))$. We record some properties of this map in [11].

Lemma 3.1. [11, Theorem 33 and Proposition 43] *The map $d : \mathcal{R}(G_K) \longrightarrow \mathcal{R}(G_k)$ is a ring epimorphism. Moreover, if the order of $G_K(K)$ is prime to p, d is an isomorphism.*

On $\mathrm{Rep}_K(G_K(K))$ and $\mathrm{Rep}_k(G_k(k))$ there are natural actions of $\mathrm{Aut}(K)$ and $\mathrm{Aut}(k)$ which induce actions on the Grothendieck groups $\mathcal{R}(G_K) \simeq \mathcal{R}(G_K(K))$ and $\mathcal{R}(G_k) \simeq \mathcal{R}(G_k(k))$. The next lemma relates the action of $\mathrm{Aut}(K)$ on $\mathcal{R}(G_k)$ via the decomposition map with the action of the Frobenius homomorphism on $\mathcal{R}(G_k)$.

Lemma 3.2. *Let $n = \dim_K K[G_K]$ and $\xi \in A$ be a primitive n-th root of unity. Let $\sigma \in \mathrm{Aut}(K)$ and write $\sigma(\xi) = \xi^a$. Assume that $p \gg 0$ and $p \equiv a \mod n$. Then the action of σ on $\mathcal{R}(G_K)$ via the decomposition map coincides with the action of the Frobenius homomorphism F of k, that is, $d \circ \sigma = F \circ d$.*

Proof. The assumption implies $G_K(K) \simeq G_k(k)$, we denote this group by G_0. Let $\rho : G_0 \longrightarrow \mathrm{GL}_r(K)$ be a group homomorphism. Following [11, Theorem 24], ρ is realizable over $\mathbb{Q}(\xi)$, that is, there is a representation $\rho_0 : G_0 \longrightarrow \mathrm{GL}_r(\mathbb{Q}(\xi))$ such that $\rho = \rho_0 \otimes_{\mathbb{Q}(\xi)} K$. Fix a basis $e_1, \dots, e_r$ of the vector space $\mathbb{Q}(\xi)^r$ and denote

$$V_0 = \sum_{g\in G_0} \sum_{i=1}^{r} g(e_i)\mathbb{Z}[\xi].$$

V_0 is stable under the action of G_0 and by taking modulo p, one obtains a representation $\bar{\rho}_0 : G_0 \longrightarrow \mathrm{GL}_r(\mathbb{F}_p(\xi))$. Similarly, from the representation $\sigma \circ \rho$, we get a lattice $(V_0)_\sigma$ which induces a representation $\overline{\sigma \circ \rho_0}$. Put $\bar{\rho} = \bar{\rho}_0 \otimes_{\mathbb{F}_p(\xi)} k$. Clearly $d([\rho]) = [\bar{\rho}]$. On the other hand, from the assumption on p, $\overline{\sigma \circ \rho_0} = F \circ \bar{\rho}_0$ where F is the Frobenius homomorphism of k. Therefore, $d \circ \sigma([\rho]) = d([\sigma \circ \rho]) = [F \circ \bar{\rho}_0] = F \circ d([\rho])$. □

Turning back to étale vector bundles. Assume that K is an algebraically closed field of characteristic zero. Let X be a connected smooth projective variety over K and V be a Nori finite vector bundle on X. By Tannaka duality, V corresponds to a representation of a finite group scheme G which is always étale by a result of Cartier (see [12, 11.4]). Since $\mathrm{Rep}_K(G)$ is equivalent to $\mathrm{Rep}_K G(K)$, there is a representation $\rho : G(K) \longrightarrow \mathrm{GL}_r(K)$ corresponding to V through these equivalences. For each automorphism $\sigma \in \mathrm{Aut}(K)$, we denote by V_σ the vector bundle corresponding to the representation $\sigma \circ \rho$. With these assumptions, we have (see also [10, Satz 4.2.1]),

Theorem 3.3. *$h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(V_\sigma)$ for all $i \in \mathbb{Z}, \sigma \in \mathrm{Aut}(K)$.*

Proof. By a standard argument in algebraic geometry, there is a subfield $K_0 \subset K$ which is a finite (possibly transcendental) extension of $\mathbb{Q}$ such that X, V, G have models over K_0 (see [1, Proof of Théorème 2.1]). There is an open subset $\mathrm{Spec}\, A \subset \mathrm{Spec}\, \mathcal{O}_{K_0}$ such that $\mathrm{Spec}\, A/\mathrm{Spec}\, \mathbb{Z}$ is smooth and there are smooth models $\mathcal{X}, \mathcal{G}$ defining over $\mathrm{Spec}\, A$ whose generic fibers are X, G respectively. Moreover, localizing A if necessary, we can assume that the A-modules $H^i_{\mathrm{Hdg}}(\mathcal{X}, \mathcal{V})$ are locally free of constant rank over A, where $\mathcal{V}$ is a model of V on $\mathcal{X}/\mathrm{Spec}(A)$ and we denote this rank by $h^i_{\mathrm{Hdg}}(\mathcal{V})$ too, (see [7, Section II.5] or [5, Section III.12]). Due to Dirichlet's theorem on arithmetic progressions, there are infinitely many prime numbers p such that $p \equiv a \mod n$. Let p be such a prime such that p is not invertible in A. Localizing the ring A at a minimal prime ideal containing pA, we get a 1-dimensional regular local ring A_0. Note that A_0 and A have the same field of fractions K_0 and the residue field k of A_0 is of characteristic p. Further more, by base change one gets $h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(\mathcal{V}) = h^i_{\mathrm{Hdg}}(\mathcal{V} \otimes_A K_0)$ and $h^i_{\mathrm{Hdg}}(\mathcal{V}) = h^i_{\mathrm{Hdg}}(\mathcal{V} \otimes_A A_0) = h^i_{\mathrm{Hdg}}(\mathcal{V} \otimes_A k)$, for all $i \in \mathbb{Z}$ (see [7, Section II.5] or [5, Section III.12]). So we can assume from the beginning that A is a discrete valuation ring with the residue field k of characteristic $p \gg 0$. Denote the fibers of $\mathcal{X}, \mathcal{V}, \mathcal{G}$ at p by X_k, V_k, G_k. Since $p \gg 0$, we can assume also G_k is étale over k.

With the notations before Lemma 3.1, there is a decomposition map $d : \mathcal{R}(G_K) \longrightarrow \mathcal{R}(G_k)$ (we use freely a finite extension of K_0 if necessary) which is in fact an isomorphism. Let $\bar{\rho} : G_k \longrightarrow \mathrm{GL}_r(k)$ be the representation defining V_k, then $d([\rho]) = [\bar{\rho}]$. Let F and F_{X_k} be the Frobenius morphisms on k and X_k respectively. Using Lemma 3.2, we obtain $d([\sigma \circ \rho]) = [F \circ \bar{\rho}]$. By the choice of p, the category $\mathrm{Rep}_k G_k$ is semi-simple. Combining these facts we see that $F^*_{X_k} V_k$ is isomorphic to the fiber at p of $\mathcal{V}_\sigma$. Therefore, from Corollary 2.6 we obtain

$$h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(V_k) = h^i_{\mathrm{Hdg}}((V_\sigma)_k) = h^i_{\mathrm{Hdg}}(V_\sigma),$$

for all $i \in \mathbb{Z}$. □

Remark 3.4. (i) It should be noted that the answer to Question 1 is always positive at the zero level. Indeed, let V be an étale Nori finite vector bundle and $\rho : G(k) \longrightarrow GL_r(k)$ be the associated representation of V. One gets

$$h^0_{\mathrm{Hdg}}(V) = \dim_k \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, V) = \dim_k \mathrm{Hom}_{G(k)}(k, \rho) = \dim_k (k^r)^\rho,$$

where in $\mathrm{Hom}_{G(k)}(k, \rho)$, k is the trivial representation and $(k^r)^\rho$ is the invariant subspace of k^r under ρ. Clearly, $(k^r)^\rho \simeq (k^r)^{\sigma\circ\rho}$ for any $\sigma \in \mathrm{Aut}(k)$. So, $h^0_{\mathrm{Hdg}}(V) = h^0_{\mathrm{Hdg}}(V_\sigma)$.

(ii) Using (i), the Riemann-Roch theorem on curves allows to give a complete answer to Question 1 for this case: If X is a smooth curve over $k = \bar{k}$ of any characteristic and V is an étale Nori finite vector bundle on X, then $h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}(V_\sigma)$ for all $i \in \mathbb{Z}$ and $\sigma \in \mathrm{Aut}(k)$. Let $\check{V}$ be the dual vector bundle of V, one has $h^0_{\mathrm{Hdg}}(V) = h^0_{\mathrm{Hdg}}(V_\sigma)$ and $h^2_{\mathrm{Hdg}}(V) = h^0_{\mathrm{Hdg}}(\check{V}) = h^0_{\mathrm{Hdg}}(\check{V}_\sigma) = h^2_{\mathrm{Hdg}}(V_\sigma)$ by (i) and Serre duality theorem. Moreover, $h^1_{\mathrm{Hdg}}(V) = \chi(V) - \chi(V \otimes_{\mathcal{O}_X} \Omega^1_{X/k}) + h^0(V) + h^0(\check{V}) = \chi(V) + h^0(V) + h^0(\check{V})$. Note that $\deg(V) = 0$ since V is defined by the representation of a finite group (see also [8, Chapter 1, Proposition 3.4]). Then Riemann-Roch theorem implies $h^1_{\mathrm{Hdg}}(V) = h^1_{\mathrm{Hdg}}(V_\sigma)$.

Remark 3.5. One also might think about a converse of Question 1: Let X be a smooth geometrically connected projective varietey over $k = \bar{k}$ of characteristic $p > 0$ and V be a Nori finite bundle on X. If $h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}((F^n_X)^*V)$ for all $i \geqslant 0$ and $n > 0$, is V étale? Note that when the Tannaka group scheme of V is local, Mehta and Subramanian [6, Section 2] showed that $(F^n_X)^*V$ is a trivial bundle for some $n > 0$. Hence, if V is not trivial, $h^0(V) < h^0((F^n_X)^*V) = \mathrm{rank}(V)$. Unfortunately, the answer for the question above is negative in general. For an example, let X be a hyperelliptic curve of genus g such that $1 \leqslant p\mathrm{-rank}(X) \leqslant g-1$. Take an extension V of O_X by it self such that the cohomology class in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq H^1(X, O_X)$ is $\alpha+\beta$, where $\alpha, \beta \neq 0$, $F^*_X(\alpha) = 0$ and $F^*_X(\beta) = \beta$. Then V is Nori finite but its Tannaka group scheme is neither étale nor local (see [6, 2]). We have $h^0(V) = h^0((F^n)^*V) = 1$ for all $n > 0$. By the same argument as in Remark 3.4(ii) we obtain $h^i_{\mathrm{Hdg}}(V) = h^i_{\mathrm{Hdg}}((F^n)^*V)$ for all $n > 0, i \geqslant 0$.

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Irreducibility criterion for algebroid curves

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Let $\mathbb{C}$ be the complex number field. An *algebroid curve* over $\mathbb{C}$ is a Noetherian local ring A satisfying the following properties: (1) A is complete, (2) A is unmixed and of Krull dimension one (i.e. $\dim \mathfrak{p} = 1$ for all $\mathfrak{p} \in \operatorname{Ass} A$), (3) A has a coefficient field $\mathbb{C}$. If A is domain, we say that A is *irreducible*. In this paper, we give criteria for algebroid curve $A = \mathbb{C}[\![x_1, \dots, x_r]\!]/I$ to be irreducible. In case of bivariate formal power series $F(x,y) \in \mathbb{C}[x,y]$ over the field of complex numbers $\mathbb{C}$, there is an irreducibility criterion for $F(x,y)$ by Abhyankar ([1]).

We denote by $\mathbb{N} = \{0,1,2,3,\dots\}$ the set of non-negative integers, and by $\mathbb{N}_+$, $\mathbb{Q}_+$, and $\mathbb{R}_+$ the set of positive integers, positive rational numbers, and positive real numbers, respectively. We set $\widetilde{\mathbb{N}}_+ = \mathbb{N}_+ \cup \{\infty\}$, $\widetilde{\mathbb{Q}}_+ = \mathbb{Q}_+ \cup \{\infty\}$ and $\widetilde{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

Definition 1. Let $w = (w_1, \dots, w_r) \in \widetilde{\mathbb{R}}_+^r$. For $f = \sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} x^{\mathbf{a}} \in \mathbb{C}[\![x]\!]$, $c_{\mathbf{a}} \in k$, we define the *order* of f with respect to w as

$$\operatorname{ord}_w(f) = \min\{w \cdot \mathbf{a} \mid c_{\mathbf{a}} \neq 0\} \in \widetilde{\mathbb{R}}_+,$$

where $w \cdot \mathbf{a} = \sum w_i a_i \in \widetilde{\mathbb{R}}_+$. We set $\operatorname{ord}_w(0) = \infty$. If $\operatorname{ord}_w(f) < \infty$, we define the *initial form* of f as

$$\operatorname{in}_w(f) \coloneqq \sum_{w \cdot \mathbf{a} = \operatorname{ord}_w(f)} c_{\mathbf{a}} x^{\mathbf{a}} \in k[x_i \mid w_i \neq \infty] \cong \mathbb{C}[x]/\langle x_j \mid x_j = \infty \rangle.$$

If $\operatorname{ord}_w(f) = \infty$, we set $\operatorname{in}_w(f) = 0$. For an ideal $I \subset \mathbb{C}[\![x]\!]$, we define the *initial ideal* of I with respect to w as the ideal

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) \mid f \in I \rangle \subset k[x_i \mid w_i \neq \infty].$$

Definition 2. Let $I \subset \mathbb{C}[\![x]\!]$ be an ideal. We call

$$\mathcal{T}_{\mathrm{loc}}(I) = \{w \in \widetilde{\mathbb{R}}_+^r \mid \operatorname{in}_w(I) \text{ contains no monomial }\}.$$

the *local tropical variety* of I. We say that an element $w \in \mathcal{T}_{\mathrm{loc}}(I) \cap \widetilde{\mathbb{N}}_+^r$ is a *tropism* of I if w is primitive.

The local tropical variety $\mathcal{T}_{\mathrm{loc}}(I)$ is closed under the multiplication by $\widetilde{\mathbb{R}}_+$. Note that $\operatorname{in}_w(I)$ is homogeneous with respect to w, and thus $\operatorname{in}_w(I)$ contains no monomial if and only if $\operatorname{in}_w(f)$ is not monomial for any $f \in I$. Similarly to the usual varieties, the following hold.

Lemma 3. *Let $I, J, I_1, \dots, I_l \subset \mathbb{C}[\![x]\!]$ be ideals. Then the following hold.*

(1) *If* $I \subset J$, *then* $\mathcal{T}_{\mathrm{loc}}(I) \supset \mathcal{T}_{\mathrm{loc}}(J)$.

(2) $\sqrt{\mathrm{in}_w(I)} = \sqrt{\mathrm{in}_w(\sqrt{I})}$ *for any* $w \in \widetilde{\mathbb{R}}^r_+$. *In particular,* $\mathcal{T}_{\mathrm{loc}}(I) = \mathcal{T}_{\mathrm{loc}}(\sqrt{I})$.

(3) $\mathcal{T}_{\mathrm{loc}}(\bigcap_{i=1}^{l} I_i) = \bigcup_{i=1}^{l} \mathcal{T}_{\mathrm{loc}}(I_i)$.

Proof. The assertions of (1) is trivial.

(2) Since $I \subset \sqrt{I}$, $\sqrt{\mathrm{in}_w(I)} \subset \sqrt{\mathrm{in}_w(\sqrt{I})}$. To prove the converse, let $f \in \mathrm{in}_w(\sqrt{I})$. Then there exists $g \in \sqrt{I}$ such that $\mathrm{in}_w(g) = f$. Since $g^n \in I$ for some $n \in \mathbb{N}$, $f^n = \mathrm{in}_w(g^n) \in \mathrm{in}_w(I)$. Thus $f \in \sqrt{\mathrm{in}_w(I)}$. This implies $\mathrm{in}_w(\sqrt{I}) \subset \sqrt{\mathrm{in}_w(I)}$ and thus $\sqrt{\mathrm{in}_w(\sqrt{I})} \subset \sqrt{\mathrm{in}_w(I)}$.

(3) Since $\bigcap I_i \subset I_i$ for all i, we have $\mathcal{T}_{\mathrm{loc}}(\bigcap I_i) \supset \bigcup_i \mathcal{T}_{\mathrm{loc}}(I_i)$. Let $w \notin \bigcup_i \mathcal{T}_{\mathrm{loc}}(I_i)$. For each i, there exists $f_i \in I_i$ such that $\mathrm{in}_w(f_i)$ is a monomial. Since $\prod_i f_i \in \prod I_i \subset \bigcap I_i$, we conclude that $\mathrm{in}_w(\bigcap I_i)$ contains a monomial $\mathrm{in}_w(\prod_i f_i) = \prod_i \mathrm{in}_w(f_i)$. □

Corollary 4. $\mathcal{T}_{\mathrm{loc}}(I) = \bigcup_{P \in \mathrm{Ass}\, I} \mathcal{T}_{\mathrm{loc}}(P)$.

Definition 5. Let $I \subset \mathbb{C}[\![x]\!]$ be an unmixed ideal of dimensional one, and $f \in \mathbb{C}[\![x]\!]$. We define the *intersection number* of f and I as

$$\mathrm{int}(f; I) = \ell_{\mathbb{C}[\![x]\!]}(\mathbb{C}[\![x]\!]/(I + \langle f \rangle)) = \dim_k \mathbb{C}[\![x]\!]/(I + \langle f \rangle) \in \widetilde{\mathbb{N}}_+,$$

where $\ell_R(M)$ denotes the length of an R-module M. We set

$$\omega(I) = (\mathrm{int}(x_1; I), \ldots, \mathrm{int}(x_r; I)) \in \widetilde{\mathbb{N}}^r_+.$$

The main theorem of this paper is the following.

Theorem 6. *([2]) Let* $I \subset \mathbb{C}[\![x]\!]$ *be an unmixed ideal of dimension one. Then the following hold.*

(1) *If* I *is a prime ideal, then* $\mathcal{T}_{\mathrm{loc}}(I) = \widetilde{\mathbb{R}}_+ \cdot \omega(I)$.

(2) *If* $\omega(I)$ *is a tropism of* I, *then* I *is a prime ideal.*

(3) *Assume that* $I = \sqrt{I}$. *Then there exists an algorithm for computing* $J \subset \mathbb{C}[\![x_1, \ldots, x_{r'}]\!]$, $r' \geq r$, *such that the residue ring of* I *is isomorphic to the residue ring of* J, *and if* I *is prime then* $\omega(J)$ *is a tropism of* J, *and if* I *is not prime then* $\mathcal{T}_{\mathrm{loc}}(J) \neq \widetilde{\mathbb{R}}_+ \cdot \omega(J)$.

Proof. (1) We may assume that $x_i \notin I$ for all $1 \leq i \leq r$. Let $A = \mathbb{C}[\![x]\!]/I$, and $B := \mathbb{C}[\![t]\!] = \overline{A}$, the integral closure of A. We denote by $\xi_i(t) \in \mathbb{C}[\![t]\!]$ the image of x_i under the natural morphism $\mathbb{C}[\![x]\!] \twoheadrightarrow A \hookrightarrow B$. For $\eta = \beta t^n$ + (higher order terms) $\in \mathbb{C}[\![t]\!]$, $\beta \in \mathbb{C}^\times$, we set $\mathrm{ord}_t(\eta) = n$ and $\mathrm{in}_t(\eta) = \beta t^n$. Let $\mathrm{ord}_t(\xi_1) = N$. By taking $\xi_1^{1/N}$ as a parameter of B, we may assume that $\xi_1 = t^N$. In this way, we obtain a Puiseux expansion $\boldsymbol{\xi} = (t^N, \xi_2, \ldots, \xi_r) \in V_M(I) \cap (t\mathbb{C}[\![t]\!])^{\oplus r}$ of I. Note that for $f \in \mathbb{C}[\![x]\!]$, $\mathrm{int}(f; I) = e_A(f) = e_B(f) = \mathrm{ord}_t(f(\boldsymbol{\xi}))$, and in particular, $\omega(I) = (N, \mathrm{ord}_t(\xi_2), \ldots, \mathrm{ord}_t(\xi_r))$.

We will prove that $\mathcal{T}_{\text{loc}}(I) \supset \widetilde{\mathbb{R}}_+ \cdot \omega(I)$. Let $f \in \mathbb{C}[\![\boldsymbol{x}]\!]$ and $f_0 = \text{in}_{\omega(I)}(f)$. Then the lowest order part appearing in the expansion of $f(\boldsymbol{\xi})$ is $f_0(t^N, \text{in}_t(\xi_2), \ldots, \text{in}_t(\xi_r))$. In particular, if $f \in I$ then $f_0(t^N, \text{in}_t(\xi_2), \ldots, \text{in}_t(\xi_r)) = 0$. Since $t^N, \text{in}_t(\xi_2), \ldots, \text{in}_t(\xi_r)$ are monomials, f_0 is not a monomial. Therefore $\text{in}_{\omega(I)}(I)$ contains no monomial, and thus $\omega(I) \in \mathcal{T}_{\text{loc}}(I)$.

We will prove that $\mathcal{T}_{\text{loc}}(I) \subset \widetilde{\mathbb{R}}_+ \cdot \omega(I)$. Note that $\mathbb{C}[\![\boldsymbol{x}]\!] \subset \mathbb{C}[\![x_1^{1/N}, x_2, \ldots, x_r]\!]$ is a Galois extension of normal domains whose Galois group is generated by $\sigma : x_1^{1/N} \mapsto \zeta_N x_1^{1/N}$ where ζ_N is a primitive N-th root of unity. Let $J = \langle x_2 - \xi_2(x_1^{1/N}), \ldots, x_r - \xi_r(x_1^{1/N})\rangle \subset \mathbb{C}[\![x_1^{1/N}, x_2, \ldots, x_r]\!]$. Then J is a prime ideal lying over I. Take $\boldsymbol{w} \in \widetilde{\mathbb{R}}^r_+$ such that $\boldsymbol{w} \notin \widetilde{\mathbb{R}}_+ \cdot \omega(I)$. Then there exists $1 \leq j \leq r$ such that $\text{in}_{\boldsymbol{w}}(x_j - \xi_r(x_1^{1/N}))$ is a monomial in $\mathbb{C}[x_1^{1/N}, x_2, \ldots, x_r]$. We set $f = \prod_{i=0}^{N-1} \sigma^i(x_j - \xi_r(x_1^{1/N}))$. Then $f \in J \cap \mathbb{C}[\![\boldsymbol{x}]\!] = I$, and $\text{in}_{\boldsymbol{w}}(f) = \prod_{i=0}^{N-1} \sigma^i(\text{in}_{\boldsymbol{w}}(x_j - \xi_r(x_1^{1/N})) \in \text{in}_{\boldsymbol{w}}(I) \subset \mathbb{C}[\boldsymbol{x}]$ is a monomial since $\sigma^i(\text{in}_{\boldsymbol{w}}(x_j - \xi_r(x_1^{1/N}))$ is a monomial for all i. Therefore $\boldsymbol{w} \notin \mathcal{T}_{\text{loc}}(I)$. This proves that $\mathcal{T}_{\text{loc}}(I) \subset \widetilde{\mathbb{R}}_+ \cdot \omega(I)$.

(2) Let $A = \mathbb{C}[\![\boldsymbol{x}]\!]/I$. By the multiplicity formula, $\omega(I) = \sum_{\mathfrak{p} \in \text{Ass}\, I} \ell(A_{\mathfrak{p}})\omega(\mathfrak{p})$. By (1) and Corollary 4, we have $\mathcal{T}_{\text{loc}}(I) = \bigcup_{\mathfrak{p} \in \text{Ass}\, I} \mathcal{T}_{\text{loc}}(\mathfrak{p}) = \bigcup_{\mathfrak{p} \in \text{Ass}\, I} \widetilde{\mathbb{R}}_+ \cdot \omega(\mathfrak{p})$. As $\omega(I) \in \mathcal{T}_{\text{loc}}(I)$ and $\gcd(\omega(I)) = 1$, we have $\omega(I) = \gcd(\omega(\mathfrak{p}_0))^{-1}\omega(\mathfrak{p}_0)$ for some $\mathfrak{p}_0 \in \text{Ass}\, I$. Since $\gcd(\omega(\mathfrak{p}_0))^{-1} \leq 1$ and $\ell(A_{\mathfrak{p}_0}) \geq 1$, the equality $\omega(I) = \sum_{\mathfrak{p} \in \text{Ass}\, I} \ell(A_{\mathfrak{p}})\omega(\mathfrak{p})$ is possible only if $\text{Ass}\, I = \{\mathfrak{p}_0\}$ and $\ell(A_{\mathfrak{p}_0}) = 1$. This proves that I $\mathfrak{p}_0$.

(3) See [2]. □

Example 7 ([1] Kuo's Example). Let $F(x,y) = (y^2 - x^3)^2 - x^7 \in \mathbb{C}[\![x,y]\!]$, $\text{char}(k) = 2$. Let $I = \langle F, z - (y^2 - x^3 - x^2y)\rangle \subset \mathbb{C}[\![x,y,z]\!]$. Then $\omega(I) = (4, 6, 15)$. Note that F is irreducible if and only if I is a prime ideal as $\mathbb{C}[\![x,y]\!]/\langle F\rangle \cong \mathbb{C}[\![x,y,z]\!]/I$. Let $G = 2x^2yz + z^2 + x^6y + x^4z$. Since $G \equiv G(x,y,y^2 - x^3 - x^2y) = (y^2 - x^3)^2 - x^7 \equiv 0 \mod I$, we have $G \in I$. As $\text{in}_{(4,6,15)}(G) = 2x^2yz$ is a monomial, I is not prime ideal. Thus F is a reducible power series.

Example 8. Let $F(x,y) = (y^2 - x^3)^2 - x^5y - x^7 \in \mathbb{C}[\![x,y]\!]$. Then $\omega(F) = (4,6)$. Let $J = \langle F, z - (x^2 - y^3)\rangle \subset \mathbb{C}[\![x,y,z]\!]$. Then $\omega(J) = (4,6,13)$, $\gcd(\omega(J)) = 1$, and $\text{in}_{\omega(J)}(J) = \langle y^2 - x^3, z^2 - x^5y\rangle$ contains no monomial. Hence F is an irreducible power series.

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ON THE BETTI NUMBERS OF EDGE IDEALS OF CHORDAL GRAPHS

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1. Introduction

Let G be a finite graph with no loop and no multiple edge. We denote its vertex set by $V = V(G)$ and its edge set by $E(G)$. Let $S = K[x_v : v \in V]$ be a polynomial ring over a field K with $\deg x_v = 1$ for any $v \in V$. The edge ideal of G is the squarefree monomial ideal $I(G) \subset S$ generated by all products $x_i x_j$ with $\{x_i, x_j\} \in E(G)$. Let us consider a minimal graded free resolution of $S/I(G)$ over S:

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{p,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j S(-j)^{\beta_{1,j}} \longrightarrow S \longrightarrow S/I(G) \longrightarrow 0.$$

The length p of this resolution is called the projective dimension of $S/I(G)$ over S, denoted by $\mathrm{pd}_S\, S/I(G)$. The integers $\beta_{i,j}(S/I(G)) := \beta_{i,j}$ is called the (i,j)th graded Betti numbers of $S/I(G)$. The regularity of $S/I(G)$ is defined by $\mathrm{reg}\, S/I(G) := \max\{j - i : \beta_{i,j}(S/I(G)) \neq 0\}$. We are interested in describing these invariants in terms of combinatorial data of the graph G.

A finite graph G is said to be *chordal* if each cycle of G whose length is more than 3 has a chord. A special class of chordal graphs is the *forest*, which is a graph containing no cycle. Zheng [7] characterized the projective dimension $\mathrm{pd}_S\, S/I(G)$ and the regularity $\mathrm{reg}\, S/I(G)$ when G was a forest. Later, Hà and Van Tuyl [4] extended this characterization of the regularity to that for chordal graphs.

In this report, we extend Zheng's characterization of the projective dimension to that for chordal graphs (Theorem 4.1). Moreover, we characterize the vanishing of the graded Betti numbers for chordal graphs (Theorem 5.2). Especially, we give a characterization of the graded Betti numbers for forests.

This report is organized as follows. First in Section 2, we introduce some notions on graphs needed to our characterization. In Section 3, we recall the recursive formula of graded Betti numbers for chordal graphs due to Hà and Van Tuyl [4, Theorem 5.8], which is indispensable on the proof of our theorems. Then in Sections 4 and 5, we prove main theorems on this report (Theorems 4.1 and 5.2). Finally in Section 6, we note on the non-vanishingness of graded Betti numbers for general graphs (Theorem 6.1).

The author is grateful to Professor Naoki Terai for giving her many useful suggestions after the talk, especially for Corollary 4.4.

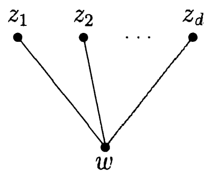


FIGURE 1. Bouquet B

2. Disjointnesses on a set of bouquets

Let B be a graph with $V(B) = \{w, z_1, \ldots, z_d\}$ and $E(B) = \{\{w, z_i\} : i = 1, \ldots, d\}$ $(d \geq 1)$; see Figure 1. Zheng [7, Definition 1.7] called this graph B a *bouquet*. Also she called the vertex w the *root* of B, vertices z_i *flowers* of B, and edges $\{w, z_i\}$ *stems* of B. In this section, we will define two kinds of disjointnesses on a set of bouquets.

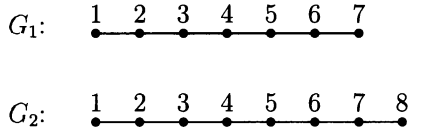
Let G be a finite graph. First we recall the definition of 3-disjointness for two edges of G (Hà and Van Tuyl [4, Definitions 2.2 and 6.3]). Let e, e' be two distinct edges of G. The *distance* of e and e' in G, denoted by $\mathrm{dist}_G(e, e')$, is defined to be the minimum length ℓ of a sequence $e_0 = e, e_1, \ldots, e_\ell = e'$ with $e_{i-1} \cap e_i \neq \emptyset$, where $e_i \in E(G)$. If there is no such a sequence, we define $\mathrm{dist}_G(e, e') = \infty$. We say that e and e' are 3-*disjoint* in G if $\mathrm{dist}_G(e, e') \geq 3$. Let $\mathcal{E} \subset E(G)$ be a subset of the edge set. We say that $\mathcal{E}$ is pairwise 3-disjoint in G if any two distinct edges of $\mathcal{E}$ are 3-disjoint in G. Then the regularity can be characterized as follows.

Theorem 2.1 (Zheng [7], Hà and Van Tuyl [4, Theorems 6.5 and 6.8]). *Let G be a finite graph. We set c as the maximum cardinality of a subset $\mathcal{E} \subset E(G)$ which is pairwise 3-disjoint in G. Then*

$$\operatorname{reg} S/I(G) \geq c.$$

Moreover when G is a chordal graph, the equality holds.

Example 2.2. Let us consider the following line graphs G_1, G_2:



For example, edges $\{1, 2\}, \{4, 5\} \in E(G_1)$ are 3-disjoint in G_1. But we cannot choose 3 edges from G_1 those are pairwise 3-disjoint. Hence, $\operatorname{reg} S/I(G_1) = 2$. On the other hand, edges $\{1, 2\}$, $\{4, 5\}$, $\{7, 8\} \in E(G_2)$ are pairwise 3-disjoint in G_2. Thus $\operatorname{reg} S/I(G_2) = 3$.

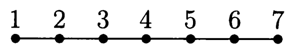
Let $\mathcal{B} = \{B_1, \ldots, B_j\}$ be a set of bouquets those are subgraphs of G. We denote by $R(\mathcal{B})$, the set of roots of the bouquets in $\mathcal{B}$, and by $F(\mathcal{B})$, the union of the sets of flowers of the bouquets in $\mathcal{B}$. We define the *type* of $\mathcal{B}$ as $(\#F(\mathcal{B}), \#R(\mathcal{B}))$.

Definition 2.3. (1) We say that $\mathcal{B}$ is *strongly disjoint* in G if for any $k \neq \ell$, bouquets B_k, B_ℓ contain no common vertex, and there exists the set of edges $\{s_1, \dots, s_j\}$ with $s_k \in E(B_k)$, which is pairwise 3-disjoint in G.

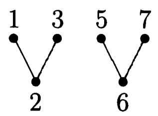
(2) We say that $\mathcal{B}$ is *semi-strongly disjoint* in G if for any $k \neq \ell$, bouquets B_k, B_ℓ contain no common vertex and the roots of B_k, B_ℓ have no common edge in G.

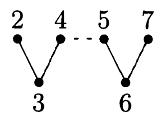
Let B_1, B_2 be two bouquets of G which do not have common vertex. If the roots of B_1, B_2 have common edge, then the distance of any stems of B_1 and B_2 is less than 3. This implies that a set of bouquets of G which is strongly disjoint in G is also semi-strongly disjoint in G.

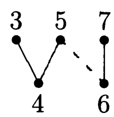
Example 2.4. Let us consider the line graph with 7 vertices:

G:


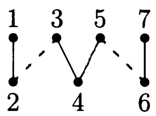
The following sets of bouquets of G are examples of strongly disjoint ones:

$\mathcal{B}_1$:


$\mathcal{B}_2$:


$\mathcal{B}_3$:


Also, there exists a semi-strongly disjoint set of bouquets which is not strongly disjoint:

$\mathcal{B}_4$:


3. The recursive formula due to Hà and Van Tuyl

In this section, we recall the recursive formula of graded Betti numbers for chordal graphs due to Hà and Van Tuyl [4, Theorem 5.8]. It is indispensable on the proofs of our theorems.

Let G be a finite graph on V and v a vertex of G. We say that $u \in V$ is a neighborhood of v in G if $\{u, v\} \in E(G)$. We denote by $N(v)$, the set of neighborhoods of v. Let W be a subset of V. The induced subgraph of G on W, denoted by G_W, is the graph whose edge set consists of all edges $\{u, v\} \in E(G)$ with $u, v \in W$. For an edge $e \in E(G)$, we denote by $G \setminus e$ the subgraph of G obtained from G by delating e.

Dirac [3] proved that when G is chordal, there exists a perfect elimination ordering on $E(G)$. This means that there exists an edge $e = \{u, v\}$ such that $G_{N(v)}$ is a complete graph.

Lemma 3.1 (Hà and Van Tuyl [4, Theorem 5.8]). *Let G be a chordal graph. Suppose that $e = \{u, v\}$ is an edge of G such that $G_{N(v)}$ is a complete graph. Set $t = \#N(u) - 1$. Let G' be the subgraph of G with*

$$E(G') = \{e' \in E(G) \ : \ \mathrm{dist}_G(e, e') \geq 3\}.$$

Then both of $G \setminus e$ and G' are chordal, and

$$\beta_{i,j}(S/I(G)) = \beta_{i,j}(S/I(G \setminus e)) + \sum_{\ell=0}^{i-1} \binom{t}{\ell} \beta_{i-1-\ell, j-2-\ell}(S/I(G')). \tag{3.1}$$

Remark 3.2. The subgraph G' of G in the above theorem coincides with the induced subgraph G_W where

$W = \{z \in V \ : \ \text{there exists } e' \in E(G) \text{ with } z \in e' \text{ such that } \mathrm{dist}_G(e, e') \geq 3\}$.

Set $W_0 := V \setminus (N(u) \cup \{u\})$. Then G' coincides with G_{W_0} up to isolated vertices. That is, the edge sets of G' and G_{W_0} coincide.

4. Characterization of the projective dimension

Let G be a finite graph. We set

$d_G = \max\{\#F(\mathcal{B}) \ : \ \mathcal{B} \text{ is a strongly disjoint set of bouquets of } G\}$,

$d'_G = \max\{\#F(\mathcal{B}) \ : \ \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}$.

In general, the inequality $d_G \leq d'_G$ holds since a strongly disjoint set of bouquets of G is also semi-strongly disjoint in G. In this section, we prove the following theorem, which is a generalization of Zheng's result [7].

Theorem 4.1. *Let G be a chordal graph. Then*

$$\mathrm{pd}_S\, S/I(G) = d_G = d'_G.$$

Remark 4.2. Later, we will prove that the inequality $\mathrm{pd}_S\, S/I(G) \geq d_G$ holds for a general graph G; see Theorem 6.1.

First, we observe the relation between d_G and $d_{G\setminus e}$. Note that a set of bouquets of $G \setminus e$ is also that of G.

Lemma 4.3. *Let G be a chordal graph and $e = \{u, v\}$ an edge of G as in Lemma 3.1. Let $\mathcal{B}$ be a set of bouquets of $G \setminus e$. If $\mathcal{B}$ is strongly disjoint in $G \setminus e$, then $\mathcal{B}$ is also strongly disjoint in G. In particular,*

$$d_G \geq d_{G\setminus e}.$$

Proof. We set $\mathcal{B} = \{B_1, \ldots, B_j\}$ and assume that $\mathcal{B}$ is strongly disjoint in $G\setminus e$. If one of u and v does not belong to $R(\mathcal{B}) \cup F(\mathcal{B})$, then it is clear that $\mathcal{B}$ is strongly disjoint also in G. Thus we assume $u, v \in R(\mathcal{B}) \cup F(\mathcal{B})$. If u and v belong to the same bouquet, then it is easy to see that $\mathcal{B}$ is strongly disjoint in G. Thus we may assume $u \in V(B_1)$ and $v \in V(B_2)$.

We must consider 4 cases.

Case 1: $u, v \in R(\mathcal{B})$. In this case, all flowers of B_2 are neighborhood of u since $G_{N(v)}$ is a complete graph. Then there are no stems s_1, s_2 ($s_1 \in E(B_1)$,

$s_2 \in E(B_2)$) with $\mathrm{dist}_{G\setminus e}(s_1, s_2) \geq 3$. This contradicts the assumption that $\mathcal{B}$ is strongly disjoint in $G \setminus e$.

Case 2: $u \in R(\mathcal{B})$ and $v \in F(\mathcal{B})$. In this case, the root of B_2 is a neighborhood of u. This leads a contradiction as in Case 1.

Case 3: $u \in F(\mathcal{B})$ and $v \in R(\mathcal{B})$. Let w_1 be the root of B_1. Since $G_{N(v)}$ is complete, for all stems $s_2 = \{v, z\}$ of B_2, the vertex u has a common edge with the vertex z. Then it follows that the distance of $\{u, w_1\}$ and s_2 in $G \setminus e$ is equal to 2 and that $\mathcal{B}$ is also strongly disjoint in G.

Case 4: $u, v \in F(\mathcal{B})$. Let w_1, w_2 be roots of B_1, B_2 respectively. Then $\{w_2, u\} \in E(G \setminus e)$ and $\mathrm{dist}_{G\setminus e}(\{u, w_1\}, \{v, w_2\}) = 2$. This implies that $\mathcal{B}$ is also strongly disjoint in G. $\square$

Before proving Theorem 4.1, we rewrite the recursive formula (3.1) as the form for total Betti numbers:

$$\beta_i(S/I(G)) = \beta_i(S/I(G \setminus e)) + \sum_{\ell=0}^{i-1} \binom{t}{\ell} \beta_{i-1-\ell}(S/I(G')). \tag{4.1}$$

Proof of Theorem 4.1. Set $d = d_G$ and $d' = d'_G$. The proof is distinguished with two parts. First we prove $\beta_{d'}(S/I(G)) \neq 0$. Then we have $d' \leq \mathrm{pd}_S\, S/I(G)$. Next we prove $\beta_{d+1}(S/I(G)) = 0$. Then we have $\mathrm{pd}_S\, S/I(G) \leq d$. Since the inequality $d \leq d'$ always holds, these two inequalities imply the desired assertion.

We proceed the proof by induction on the number of edges in G. First note that we can easily see that $\mathrm{pd}_S\, S/I(G) = d = d' = 1$ when $\#E(G) = 1$.

(Step 1) We prove $\beta_{d'}(S/I(G)) \neq 0$. Let $\mathcal{B} = \{B_1, \ldots, B_j\}$ be a semi-strongly disjoint set of bouquets of G with $\#F(\mathcal{B}) = d'$. We write the root of B_k as w_k. We take an edge $e \in E(G)$ as in Lemma 3.1. If e is not a stem of any of $B_1, \ldots, B_j$, then $\mathcal{B}$ is also a set of bouquets of $G \setminus e$ which is semi-strongly disjoint in $G \setminus e$. Then by inductive hypothesis, we have $\beta_{d'}(S/I(G)) \geq \beta_{d'}(S/I(G \setminus e)) \neq 0$.

Next we assume that $e = \{u, v\} \in E(B_j)$. If there exists some $k < j$, say $k = 1$, such that $\{w_1, u\} \in E(G)$ (in this case v is the root of B_j), then $\mathcal{B}' = \{B'_1, B_2, \ldots, B_{j-1}, B'_j\}$ is a semi-strongly disjoint set of bouquets of $G \setminus e$ with $\#F(\mathcal{B}') = \#F(\mathcal{B}) = d'$, where B'_1 is the bouquet obtained by adding u with B_1 as a flower and B'_j is the one obtained by delating u from B_j. Therefore we may assume that none of roots $w_1, \ldots, w_{j-1}$ is a neighborhood of u. Since $G_{N(v)}$ is a complete graph, it follows that none of roots of $B_1, \ldots, B_{j-1}$ is a neighborhood of v. Moreover we may assume that u is the root of B_j. If a flower $z \in E(B_k)$ $(k < j)$ is not a neighborhood of u, then $\mathrm{dist}_G(\{w_k, z\}, e) \geq 3$. Let f be the number of such flowers. Then there exists a semi-strongly disjoint set of bouquets of G' with $\#F(\mathcal{B}') = f$. By inductive hypothesis, we have $\beta_f(S/I(G')) \neq 0$.

Set $t = \#N(u) - 1$. Then $t \geq d' - f - 1$. By the recursive formula (4.1), we have

$$\begin{aligned}\beta_{d'}(S/I(G)) &= \beta_{d'}(S/I(G \setminus e)) + \sum_{\ell=0}^{d'-1} \binom{t}{\ell} \beta_{d'-1-\ell}(S/I(G')) \\ &\geq \binom{t}{d'-f-1} \beta_f(S/I(G')) \neq 0.\end{aligned}$$

(Step 2) Next we prove $\beta_{d+1}(S/I(G)) = 0$. Since $d = d_G \geq d_{G \setminus e}$ by Lemma 4.3, inductive hypothesis implies that $\beta_{d+1}(S/I(G \setminus e)) = 0$.

Let B be the bouquet whose root is u and the set of flowers is $N(u)$. Note that $t+1 = \#N(u) \leq d$. Let $\mathcal{B}'$ be a strongly disjoint set of bouquets of G' with $\#F(\mathcal{B}') = d_{G'} =: d''$. Then $\mathcal{B} = \mathcal{B}' \cup \{B\}$ is a strongly disjoint set of bouquets of G. Thus $d'' + t + 1 = \#F(\mathcal{B}) \leq d$. For $\ell \leq t$, since $d - \ell \geq d - t \geq d'' + 1$, we have $\beta_{d-\ell}(S/I(G')) = 0$.

Combining these result with (4.1) for $i = d + 1$, we have $\beta_{d+1}(S/I(G)) = 0$, as desired. □

Let G be a finite graph on V. A subset $\mathcal{C} \subset V$ is called a vertex cover of G if $\mathcal{C} \cap e \neq \emptyset$ for all $e \in E(G)$. A vertex cover is said to be minimal if it is minimal among vertex covers of G under inclusion. We define big height of $I(G)$, denoted by $\operatorname{bight} I(G)$, as the maximum height of minimal prime divisors of $I(G)$. There exists a one-to-one correspondence between minimal vertex covers of G and minimal prime divisors of $I(G)$. Precisely, $(x_v : v \in \mathcal{C})$ is a minimal prime divisor of $I(G)$ if and only if $\mathcal{C}$ is a minimal vertex cover of G. In particular, $\operatorname{bight} I(G)$ coincides with the maximum cardinality of a minimal vertex cover of G.

As a corollary of Theorem 4.1, we have the following result.

Corollary 4.4. *Let G be a chordal graph. Then*

$$\operatorname{bight} I(G) = \operatorname{pd}_S S/I(G).$$

In particular, if $I(G)$ is unmixed, then $S/I(G)$ is Cohen–Macaulay.

Remark 4.5. The latter part of Corollary 4.4 have proved by Herzog, Hibi, and Zheng [5, Theorem].

Proof. In general, the inequality $\operatorname{bight} I(G) \leq \operatorname{pd}_S S/I(G)$ holds. We prove the opposite inequality. Let $\mathcal{B}$ be a semi-strongly disjoint set of bouquets of G with $\#F(\mathcal{B}) = d'_G$. We claim that $F(\mathcal{B})$ is a minimal vertex cover of G.

Assume that there exists an edge $e = \{u, v\} \in E(G)$ with $u, v \notin F(\mathcal{B})$. If one of u, v belongs to $R(\mathcal{B})$, then we can construct a semi-strongly disjoint set of bouquets $\mathcal{B}'$ with $F(\mathcal{B}') = d'_G + 1$ by adding e as a stem to $\mathcal{B}$. This is a contradiction. Now assume that $u, v \notin R(\mathcal{B}) \cup F(\mathcal{B})$. If one of u, v has a common edge with a root of $\mathcal{B}$, then we can leads a contradiction as above. Otherwise, $\mathcal{B}' = \mathcal{B} \cup \{e\}$ is also a semi-strongly disjoint set of bouquets with $\#F(\mathcal{B}') = d'_G + 1$, a contradiction. Hence $F(\mathcal{B})$ is a vertex cover of G. The minimality is clear. □

5. Non-vanishingness of graded Betti numbers

In this section, we characterize the vanishing of graded Betti numbers of edge ideals of chordal graphs. In particular, we characterize the graded Betti numbers for forests.

First, we give one more definition on a graph.

Definition 5.1. We say that *a graph G contains a strongly disjoint set of bouquets of type (i,j)* if there exists a set $\mathcal{B}$ of bouquets of G which is of type (i,j) and strongly disjoint in G with

$$R(\mathcal{B}) \cup F(\mathcal{B}) = V(G).$$

The following theorem is the main result in this section.

Theorem 5.2. *Let G be a chordal graph on V and $i,j \geq 1$ integers. Then $\beta_{i,i+j}(S/I(G)) \neq 0$ if and only if there exists a subset $W \subset V$ such that G_W contains a strongly disjoint set of bouquets of type (i,j).*

Moreover, when G is a forest, the graded Betti number $\beta_{i,i+j}(S/I(G))$ coincides with the number of such subsets $W \subset V$.

Before proving this theorem, we rewrite the recursive formula (3.1) as the following form:

(5.1)

$$\beta_{i,i+j}(S/I(G)) = \beta_{i,i+j}(S/I(G \setminus e)) + \sum_{\ell=0}^{i-1} \binom{t}{\ell} \beta_{i-1-\ell,(i-1-\ell)+(j-1)}(S/I(G')).$$

First, we prove the following lemma.

Lemma 5.3. *Let G be a finite graph on V. Then*

$$\beta_{i,j}(S/I(G)) = \sum_{\substack{W \subset V \\ \#W=j}} \beta_{i,j}(S/I(G_W)).$$

Proof. Since $I(G)$ is a squarefree monomial ideal, there exists a simplicial complex Δ such that $I(G) = I_\Delta$, where I_Δ stands for the Stanley–Reisner ideal of Δ. By Hochster's formula for Betti numbers (see e.g., [1, Theorem 5.5.1]), we have

$$\beta_{i,j}(S/I(G)) = \beta_{i,j}(K[\Delta]) = \sum_{\substack{W \subset V \\ \#W=j}} \dim_K \tilde{H}_{j-i-1}(\Delta_W; K),$$

where $\tilde{H}_i(\Delta; K)$ stands for the ith reduced homology group of Δ and where Δ_W denotes the restriction of Δ to W. Note that $I(G_W) = I_{\Delta_W}$. Moreover, when $\#W = j$,

$$\begin{aligned}\beta_{i,j}(S/I(G_W)) = \beta_{i,j}(K[\Delta_W]) &= \sum_{\substack{W' \subset W \\ \#W'=j}} \dim_K \tilde{H}_{j-i-1}((\Delta_W)_{W'}; K) \\ &= \dim_K \tilde{H}_{j-i-1}(\Delta_W; K).\end{aligned}$$

Therefore we have the desired formula. □

Proof of Theorem 5.2. By Lemma 5.3, we may assume that $\#V = i + j$ and may prove that $\beta_{i,i+j}(S/I(G)) \neq 0$ if and only if G contains a strongly disjoint set of bouquets of type (i, j). We proceed the proof by induction on $\#E(G)$. When $\#E(G) = 1$, the graded Betti number $\beta_{i,i+j}(S/I(G))$ is 0 except for $(i, j) = (1, 1)$; $\beta_{1,2}(S/I(G)) = 1$. Then the claim is trivially true.

Assume that $\#E(G) \geq 2$. We use the same notation as in Lemma 3.1. Also we set $N(u) = \{v, x_1, \ldots, x_t\}$. Since $G' = G_{W_0}$ where $W_0 = V \setminus \{v, u, x_1, \ldots, x_t\}$ and $\#W_0 = i + j - (t + 2)$, the summands of the second term of the righthand-side of (5.1) vanish except for $\ell = t$. Therefore we can rewrite (5.1) as

$$(5.2) \qquad \beta_{i,i+j}(S/I(G)) = \beta_{i,i+j}(S/I(G \setminus e)) + \beta_{i\ 1\ t,(i\ 1\ t)+(j\ 1)}(S/I(G')).$$

First suppose that G contains a strongly disjoint set of bouquets of type (i, j). Let $\mathcal{B} = \{B_1, \ldots . B_j\}$ be such a set. If e is not a stem of any bouquets $B_k \in \mathcal{B}$, then $\mathcal{B}$ is also a set of bouquets of $G \setminus e$ which is strongly disjoin in $G \setminus e$ and the first term of the righthand-side of (5.2) does not vanish. Now assume that $e = \{u, v\}$ is a stem of B_1. Then either u or v is the root of B_1.

When we can consider u as the root of B_1, a stem which contains x_i is not 3-disjoint with the stems of B_1. Thus we may assume that $V(B_1) = \{u, v, x_1, \ldots, x_t\}$. Then $\mathcal{B}' := \mathcal{B} \setminus B_1$ is a strongly disjoint set of bouquets of G'. Since $\#F(\mathcal{B}') = i - 1 - t$ and $\#R(\mathcal{B}') = j - 1$, the graph G' contains a strongly disjoint set of bouquets of type $(i - 1 - t, j - 1)$. Thus the second term of the righthand-side of (5.2) does not vanish.

When we cannot consider u as the root of B_1, then v is the root of B_1. Let $\{s_1, \ldots, s_j\}$ be the set of stems which guarantees the strongly disjointness of $\mathcal{B}$. Our assumption on u implies that $s_1 = \{v, x_i\}$. Then we can consider x_i as the root of B_1 since $G_{N(v)}$ is a complete graph. When this is the case, $\mathcal{B}$ is a strongly disjoint set of bouquets of $G \setminus e$ and the first term of the righthand-side of (5.2) does not vanish.

Next, we assume that $\beta_{i,i+j}(S/(I(G))) \neq 0$ and prove that G contains a strongly disjoint set of bouquets of type (i, j). Then the equation (5.2) implies that at least one of $\beta_{i,i+j}(S/(I(G \setminus e)))$ and $\beta_{i-1-t,(i-1-t)-(j-1)}(S/(I(G')))$ does not vanish.

First assume that $\beta_{i-1-t,(i-1-t)-(j-1)}(S/(I(G'))) \neq 0$. By the inductive hypothesis, G' contains a strongly disjoint set of bouquets $\mathcal{B}'$ of type $(i - 1 - t, j - 1)$. Let B be the bouquet of G whose root is u and whose flowers are $v, x_1, \ldots, x_t$. Since each edge in $E(G')$ is 3-disjoint with e, the set of bouquets $\mathcal{B}' \cup \{B\}$ is strongly disjoint in G, whose type is (i, j). Next assume that $\beta_{i,i+j}(S/I(G \setminus e)) \neq 0$ and $\mathcal{B}$ is a strongly disjoint set of bouquets of $G \setminus e$ of type (i, j) . Then by Lemma 4.3, we have that $\mathcal{B}$ is also strongly disjoint in G.

The case where G is a forest, we may prove that $\beta_{i,i+j}(S/I(G)) = 1$ when $\#V(G) = i + j$ and G contains a strongly disjoint set of bouquets of type (i, j). In this case, u is the only neighborhood of v. This implies $\#V(G \setminus e) \leq i + j - 1$ and $\beta_{i,i+j}(S/I(G \setminus e)) = 0$. Then we have $\beta_{i,i+j}(S/I(G)) = 1$ by (5.2) and the inductive hypothesis. □

6. On general graphs

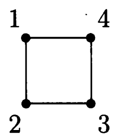
In this section, we consider the non-vanishingness of graded Betti numbers for general finite graphs.

Theorem 6.1. *Let G be a finite graph on a vertex set V. If there exists a subset $W \subset V$ such that G_W contains a strongly disjoint set of bouquets of type (i, j), then $\beta_{i,i+j}(S/I(G)) \neq 0$.*
In particular, $\mathrm{pd}_S\, S/I(G) \geq d_G$.

In general, the converse is not true for a general graph as the following example shows. In other words, the chordal graph is a special class of graphs, which satisfies the converse.

Example 6.2. Let us consider the 4-cycle G:

G:



Then $I(G) = (x_1x_2, x_1x_4, x_2x_3, x_3x_4)$ and the graded Betti number $\beta_{i,i+j}(S/I(G))$ is given by the following diagram:

$j \backslash i$	0	1	2	3
0	1			
1		4	4	1

Then $\beta_{3,3+1}(S/I(G)) \neq 0$ but G does not contain a set of bouquets of type $(3, 1)$.

Remark 6.3. Theorems 5.2 and 6.1 are generalization of Theorem 2.1 due to Hà and Van Tuyl [4].

On the proof of Theorem 6.1, we use a Lyubeznik resolution [6]. Let I be a monomial ideal. We define an ordering on the minimal system of monomial generators of I as $m_1, \ldots, m_\mu$. A symbol $[i_1, \ldots, i_q] := e_{i_1 \cdots i_q}$ $(1 \leq i_1 < \cdots < i_q \leq \mu)$ is said to be *L-admissible* if for all $t > 1$ and for all $s < i_t$, the least common multiple $\mathrm{lcm}(m_{i_t}, m_{i_{t+1}}, \ldots, m_{i_q})$ is not divisible by m_s. A Lyubeznik resolution of I (with respect to this ordering of monomial generators) is a subcomplex of Taylor resolution of I generated by all L-admissible symbols.

An L-admissible symbol $[i_1, \ldots, i_q]$ is called maximal if there are no L-admissible symbol $[j_1, \ldots, j_{q'}]$ with $\{i_1, \ldots, i_q\} \subsetneq \{j_1, \ldots, j_{q'}\}$. Note that if maximal L-admissible symbol $[i_1, \ldots, i_q]$ satisfies the following condition $(*)$:

$$(*) \quad \mathrm{lcm}(m_{i_1}, \ldots, \widehat{m_{i_k}}, \ldots, m_{i_q}) \neq \mathrm{lcm}(m_{i_1}, \ldots, m_{i_q}) \quad \text{for } k = 1, 2, \ldots, q,$$

then the symbol $[i_1, \ldots, i_q]$ must be rest in a minimal free resolution of I (see also Barile [2, Remark 1]).

Now we prove Theorem 6.1.

Proof of Theorem 6.1. By Lemma 5.3, we may prove $\beta_{i,i+j}(S/I(G)) \neq 0$ when $\#V = i + j$ and G contains a strongly disjoint set of bouquets of type (i, j).

Let $\mathcal{B} = \{B_1, \dots, B_j\}$ be such a set of bouquets and $\{s_1, \dots, s_j\}$ the set of stems with $s_i \in E(B_i)$ which guarantees the strongly disjointness of $\mathcal{B}$. Now we define an ordering on the minimal system of monomial generators of $I(G)$. Since it is equivalent to the ordering on the edge set $E(G)$, we define the ordering on $E(G)$:

$$E(B_1) \setminus \{s_1\},\ E(B_2) \setminus \{s_2\}, \dots, E(B_j) \setminus \{s_j\},\ \{\text{other edges}\},\ s_1, s_2, \dots, s_j.$$

First, we consider the symbol σ corresponding to

$$E(B_1) \setminus \{s_1\},\ E(B_2) \setminus \{s_2\}, \dots, E(B_j) \setminus \{s_j\},\ s_1, s_2, \dots, s_j.$$

Then it is easy to see that this is an L-admissible symbol satisfying the condition $(*)$.

Next we consider the symbol τ corresponding to

$$E(B_1) \setminus \{s_1\},\ E(B_2) \setminus \{s_2\}, \dots, E(B_j) \setminus \{s_j\},\ \{\text{one edge } e\},\ s_1, s_2, \dots, s_j.$$

Set $e = \{u, v\}$. By the assumption $V = F(\mathcal{B}) \cup R(\mathcal{B})$ and $\mathcal{B}$ is strongly disjoint, we have $\{u, v\} \cap F(\mathcal{B}) \neq \emptyset$. Moreover, it follows that at least one of $\{u, v\}$ belongs to $F(\mathcal{B}) \setminus \{s_1, \dots, s_j\}$. We assume that u is such a vertex and $u \in V(B_k)$. Then the product of monomials corresponding to s_k and e is divisible by the monomial corresponding to the stem of B_k whose flower is u. Therefore, τ is not L-admissible.

Hence σ is a maximal L-admissible symbol satisfying $(*)$ and we conclude that $\beta_{i,i+j}(S/I(G)) \neq 0$, as desired. □

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THE FIRST HILBERT COEFFICIENTS AND EULER CHARACTERISTICS OF MODULES

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The set of the first Hilbert coefficients of parameter ideals relative to a module over a local Noetherian ring codes for considerable information about its structure–noteworthy properties such as that of Cohen-Macaulayness, Buchsbaumness, and of having finitely generated local cohomology. In the joint work with L. Ghezzi, S. Goto, K. Ozeki, T. Phuong, and W. V. Vasconcelos ([4], [5]), the author has studied the ring cases and extended those results to the cases of modules in a more transparent manner. Another series of integers arise from partial Euler characteristics and are shown to carry similar properties of the module. Given their similar role as predictors of the Cohen-Macaulay property, we consider several questions about the direct comparison of the first Hilbert coefficient and the first partial Euler characteristic of a module.

Let $(R, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ of dimension $d > 0$. Let M be a finitely generated R–module of dimension d. Let I be an $\mathfrak{m}$–primary R–ideal. For sufficiently large n, the Hilbert–Samuel function $\lambda(M/I^{n+1}M)$ is of polynomial type :

$$P_{I,M}(n) = \sum_{i=0}^{d}(-1)^i e_i(I, M)\binom{n+d-i}{d-i}.$$

The integers $e_i(I, M)$'s are called *Hilbert coefficients* of I with respect to M.

Let $\mathbf{x} = x_1, x_2, \ldots, x_d$ be a system of parameters for M. Let $\mathrm{H}_i(\mathbf{x},\ M)$ be ith Koszul homology module of $\mathbf{x}$ with coefficients in M. The *first Euler characteristic* of M with respect to $\mathbf{x}$ is

$$\chi_1(\mathbf{x},\ M) = \sum_{i\geq 1}^{d}(-1)^{i+1}\lambda_R(\mathrm{H}_i(\mathbf{x},\ M)).$$

By a classical result of Serre, (see [1], [11]), we have

$$\chi_1(\mathbf{x},\ M) = \lambda_R(M/QM) - e_0(Q, M),$$

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where $Q = (\mathbf{x}) = (x_1, \ldots, x_d)$.

Let $\mathcal{P}(M)$ be the set of parameter ideals $\mathbf{x} = \{x_1, \ldots, x_d\}$ for M. Set

$$\Omega(M) = \{ -e_1(\mathbf{x}, M) \mid \mathbf{x} \in \mathcal{P}(M)\}$$

$$\Delta(M) = \{ \chi_1(\mathbf{x}, M) \mid \mathbf{x} \in \mathcal{P}(M)\}$$

Identical assertions about the character of $\Omega(M)$ and $\Delta(M)$ with emphasis on the range are expressed in the following table:

M = arbitrary	$\Omega(M) \subseteq [0, \infty)$	$\Delta(M) \subseteq [0, \infty)$	[10]	[11, Appendice II]
M = Cohen-Macaulay	$0 \in \Omega(M)$ ♯	$0 \in \Delta(M)$	[3], [4]	[11, Appendice II]
M = generalized Cohen-Macaulay	$\lvert\Omega(M)\rvert < \infty$ ♯	$\lvert\Delta(M)\rvert < \infty$	[3], [4]	[2]
M = Buchsbaum	$\lvert\Omega(M)\rvert = 1$ ♯	$\lvert\Delta(M)\rvert = 1$	[6], [4]	[12]
$\overline{(\mathbf{x})} = \overline{Q}$ fixed	$\lvert\Omega_Q(M)\rvert < \infty$	$\lvert\Delta_Q(M)\rvert < \infty$	[4]	[4]

TABLE 1. Properties of a finitely generated module M carried by the values of either function. ♯ requires that M is unmixed. The fourth and fifth columns refer to $\Omega(M)$ and $\Delta(M)$ respectively.

1 Vanishing of the first Hilbert coefficients of parameter ideals

A Noetherian local ring $(R, \mathfrak{m})$ of dimension d is said to be *unmixed* if $\dim \widehat{R}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass}(\widehat{R})$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. Similarly, if M is a finitely generated R-module of dimension d, M is *unmixed* if $\dim \widehat{R}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass}(M \otimes \widehat{R})$. W. Vasconcelos, exploring the vanishing of $e_1(Q)$ for parameter ideals Q, posed the following conjecture in his lecture at the conference in Yokohama in March 2008.

Conjecture 1.1 (W. Vasconcelos)([13]) A unmixed Noetherian local ring R is not Cohen-Macaulay if and only if $e_1(Q) < 0$ for a parameter ideal Q.

L. Ghezzi, S. Goto, K. Ozeki, T. Phuong, W. Vasconcelos, and the author settled the conjecture affirmatively for the ring cases ([3, Theorem 2.1]) and then were able to extend the results to the cases of modules in a more transparent manner ([4, Theorem 3.1]).

Proposition 1.2 (Mandal, Singh, and Verma)([10]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M is a finitely generated R-module of positive dimension and Q a parameter ideal for M. Then* $e_1(Q, M) \leq 0$.

Since $e_1(Q) \leq 0$ for every parameter ideal Q, the Conjecture 1.1 can be rephrased as follows.

Conjecture 1.3 (W. Vasconcelos) A unmixed Noetherian local ring R is Cohen-Macaulay if and only if $e_1(Q) = 0$ for a parameter ideal Q.

Theorem 1.4 ([4, Theorem 3.1]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let M be a finitely generated R–module with $\dim_R M \geq 2$. Let Q be a parameter ideal for M. Suppose that M is unmixed. Then M is Cohen–Macaulay if and only if $e_1(Q, M) = 0$.*

Proof. We set $e_1(Q) = e_1(Q, M)$. It is enough to show that if R is unmixed but not Cohen-Macaulay, then $e_1(Q, M) < 0$. We may assume that R is complete with an infinite residue field and $\dim R = d = \dim M$. Then there exists a Gorenstein local ring S with $\dim(S) = d$ such that R is a homomorphic image of S. For a parameter ideal Q of M, there exists a parameter ideal $\mathfrak{q}$ of M as a S–module such that $\mathfrak{q}R = Q$. Then the associated graded module of Q relative to M is isomorphic to the associated graded module of $\mathfrak{q}$ with respect to the S–module M. This implies that

$$e_1(Q, M) = e_1(\mathfrak{q}, M),$$

where $e_1(\mathfrak{q}, M)$ denotes the first Hilbert coefficient of $\mathfrak{q}$ with respect to the S–module M. Since M is unmixed, there exists an embedding of M into a free S-module. Consider the exact sequence of S–modules :

$$0 \longrightarrow M \longrightarrow S^n \longrightarrow C \longrightarrow 0.$$

Let y be a superficial element for $\mathfrak{q}$ with respect to S–module M. By tensoring the exact sequence with S/yS,

$$0 \to T = \operatorname{Tor}_1^S(S/yS, C) \longrightarrow M/yM \xrightarrow{\zeta} S^n/yS^n \to C/yC \to 0.$$

We use induction on $d = \dim M$.

Let $d = 2$ and $\mathfrak{q} = (y, z)$. Then

$$e_1(\mathfrak{q}, M) = e_1(\mathfrak{q}/(y), M/yM) = -\lambda(T) < 0.$$

Now suppose that $d \geq 3$. From the exact sequence

$$0 \to T \to M/yM \to N \to 0,$$

we have

$$e_1(\mathfrak{q}, M) = e_1(\mathfrak{q}/(y), M/yM) = e_1(\mathfrak{q}/(y), N).$$

By an induction argument, it is enough to show that N is not Cohen–Macaulay since $\dim(S/(y)) = d - 1$.

Suppose that N is Cohen–Macaulay. Let $\mathfrak{n}$ be the maximal ideal of S/yS. From the exact sequence

$$0 \to T \to M' = M/yM \to N \to 0,$$

we obtain the long exact sequence:

$$0 \to \mathrm{H}^0_{\mathfrak{n}}(T) \to \mathrm{H}^0_{\mathfrak{n}}(M') \to \mathrm{H}^0_{\mathfrak{n}}(N) \to \mathrm{H}^1_{\mathfrak{n}}(T) \to \mathrm{H}^1_{\mathfrak{n}}(M') \to \mathrm{H}^1_{\mathfrak{n}}(N).$$

By the assumption that N is Cohen–Macaulay of dimension $d-1 \geq 2$ and the fact that T is a torsion module, we get

$$0 \to T \simeq \mathrm{H}^0_{\mathfrak{n}}(M') \to 0 \to 0 \to \mathrm{H}^1_{\mathfrak{n}}(M') \to 0.$$

From the exact sequence

$$0 \to M \xrightarrow{\cdot y} M \to M' = M/yM \to 0,$$

we obtain the following exact sequence:

$$0 \to T \simeq \mathrm{H}^0_{\mathfrak{n}}(M') \to \mathrm{H}^1_{\mathfrak{n}}(M) \xrightarrow{\cdot y} \mathrm{H}^1_{\mathfrak{n}}(M) \to \mathrm{H}^1_{\mathfrak{n}}(M') = 0.$$

Since $\mathrm{H}^1_{\mathfrak{n}}(M)$ is finitely generated and $\mathrm{H}^1_{\mathfrak{n}}(M) = y\mathrm{H}^1_{\mathfrak{n}}(M)$, we have $\mathrm{H}^1_{\mathfrak{n}}(M) = 0$. This means that $T = 0$. Therefore

$$0 \to T = 0 \to M/yM \simeq N \to 0.$$

Since N is Cohen–Macaulay, M/yM is Cohen–Macaulay. Since y is regular on M, M is Cohen–Macaulay, which is a contradiction. □

The assumption of unmixedness in Theorem 1.4 is necessary as shown in the following example.

Example 1.5 Let $R = k[[x, y, z]]/(z(x, y, z))$. Then R is not Cohen–Macaulay. For every parameter ideal Q, from the exact sequence $0 \to H^0_{\mathfrak{m}}(R) \to R \to S \to 0$, we get

$$e_1(Q) = e_1(QS) = 0,$$

because S is Cohen–Macaulay. Notice that R is not unmixed because $\mathfrak{m} \in \mathrm{Ass}(R) \setminus \mathrm{Min}(R)$.

2 Bounding the first Hilbert coefficients of parameter ideals

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. The *homological degree* of an R–module M is the integer

$$\mathrm{hdeg}(M) = e_0(M) + \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_R(M, R)).$$

For an $\mathfrak{m}$–primary ideal I, we define

$$\mathrm{hdeg}_I(M) = e_0(I, M) + \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \mathrm{hdeg}_I(\mathrm{Ext}^i_R(M, R)).$$

Note that $\mathrm{hdeg}_I M$ depends only on the integral closure of I. A *generic hyperplane section used for* $\mathrm{hdeg}(M)$ is a superficial element for M, all $\mathrm{Ext}^i_R(M, R)$, and all the iterated Ext^j_R, $j \geq 1$, of these modules (there are only a finite number of them).

Definition 2.1 Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. The *homological torsion* of an R–module M is the integer

$$\mathbf{T}(M) = \sum_{i=1}^{d-1} \binom{d-2}{i-1} \cdot \operatorname{hdeg}(\operatorname{Ext}_R^i(M, R)).$$

For an $\mathfrak{m}$–primary ideal I, we define

$$\mathbf{T}_I(M) = \sum_{i=1}^{d-1} \binom{d-2}{i-1} \cdot \operatorname{hdeg}_I(\operatorname{Ext}_R^i(M, R)).$$

Lemma 2.2 ([4, Theorem 7.5]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 3$ and M a finitely generated R–module. Let h be a generic hyperplane section used for* $\operatorname{hdeg}(M)$. *Then*

$$\mathbf{T}(M/hM) \leq \mathbf{T}(M).$$

Theorem 2.3 ([4, Theorem 7.10]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Let Q be a parameter ideal for a finitely generated R–module M. Then*

$$-e_1(Q, M) \leq \mathbf{T}_Q(M).$$

Proof. Note that

$$-e_1(Q, M) = -e_1(Q, M/H_{\mathfrak{m}}^0(M)), \text{ and } T_Q(M/H_{\mathfrak{m}}^0(M)) \leq T_Q(M).$$

By replacing M with $M/H_{\mathfrak{m}}^0(M)$, if necessary, we may assume that M has a positive depth. Let $x \in Q$ be a generic hyperplane section used for $\operatorname{hdeg}_Q(M)$. We use induction on d. Let $d = 2$. We may assume that x is a regular element which is superficial for M and $L = \operatorname{Ext}_R^1(M, R)$.

$$\begin{aligned} T_Q(M) &= \operatorname{hdeg}_Q(\operatorname{Ext}_R^1(M, R)) \\ -e_1(Q, M) &= -e_1(M/xM) = \lambda(H_{\mathfrak{m}}^0(M/xM)) = \lambda(\operatorname{Ext}_R^2(M/xM, R)). \end{aligned}$$

From the short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0,$$

we obtain the exact sequence

$$L = \operatorname{Ext}_R^1(M, R) \xrightarrow{x} L = \operatorname{Ext}_R^1(M, R) \longrightarrow \operatorname{Ext}_R^2(M/xM, R) \longrightarrow 0.$$

Note that $\dim(L) \leq 1$. If $\dim(L) = 0$, then

$$T_Q(M) = \operatorname{hdeg}_Q(L) = \lambda(L) \geq \lambda(\operatorname{Ext}_R^2(M/xM, R)) = -e_1(Q, M).$$

On the other hand, if $\dim(L) = 1$, then

$$T_Q(M) = \operatorname{hdeg}_Q(L) \geq \operatorname{hdeg}_Q(L/xL) = \lambda(L/xL) = \lambda(\operatorname{Ext}_R^2(M/xM, R))$$

Now suppose that $d \geq 3$. Then using Lemma 2.2, we obtain

$$-e_1(Q, M) = -e_1(M/xM) \leq T_{Q/(x)}(M/xM) \leq T_Q(M).$$

□

For an ideal I, we denote the integral closure of I by $\overline{I}$.

Corollary 2.4 ([4, Theorem 7.12]) *Let Q be a fixed parameter ideal. Then the set*

$$\Omega_Q(M) = \{e_1(I, M) \mid I \text{ is a parameter ideal with } \overline{I} = \overline{Q}\}$$

is finite.

Proof. This follows from

$$0 \geq e_1(I, M) \geq -\mathbf{T}_I(M) = -\mathbf{T}_Q(M).$$

□

3 Bounding the first Euler characteristics

We prove that the first Euler characteristics can be uniformly bounded by homological degrees. For a finitely generated R–module M and for a system of parameters $\mathbf{x}$ for M, the integer $\operatorname{hdeg}_{(\mathbf{x})}(M) - \deg_{(\mathbf{x})}(M)$ is called the *Cohen Macaulay deficiency* of M with respect to $(\mathbf{x})$.

Theorem 3.1 ([4, Theorem 8.2]) *Let R be a Noetherian local ring of infinite residue field and M a finitely generated R-module of positive dimension. For every system of parameters $\mathbf{x}$ for M,*

$$\chi_1(\mathbf{x},\ M) \leq \operatorname{hdeg}_{(\mathbf{x})}(M) - \deg_{(\mathbf{x})}(M).$$

Proof. We may replace the system of parameters by another generating the same ideal but formed by a superficial sequence $\mathbf{x}$ for $\operatorname{hdeg}_{(\mathbf{x})}$.

If $n = 1$, $\chi_1(\mathbf{x}; M) = \lambda(0 :_M x_1)$ while $\operatorname{hdeg}_{(\mathbf{x})}(M) = \lambda(H^0_{\mathfrak{m}}(M)) + \deg_{(\mathbf{x})}(M)$, and therefore the assertion holds.

If $n \geq 2$, $0 :_M x_1$ has finite length and thus $e_0(\mathbf{x}'; 0 :_M x_1) = 0$ so that by Proposition ??,

$$\chi_1(\mathbf{x}; M) = \chi_1(\mathbf{x}'; M/x_1M) + \chi_0(\mathbf{x}'; 0 :_M x_1) = \chi_1(\mathbf{x}'; M/x_1M).$$

We iterate this $n - 1$ steps so that we have

$$\chi_1(\mathbf{x}; M) = \chi_1(x_n; M/(x_1, \ldots, x_{n-1})M).$$

By the dimension 1 case we have that

$$\chi_1(x_n; M/(x_1, \ldots, x_{n-1})M) \leq \lambda(H^0_{\mathfrak{m}}(M/(x_1, \ldots, x_{n-1})M)) \leq \mathrm{hdeg}_{(\mathbf{x})}(M) - \deg_{(\mathbf{x})}(M),$$

where the last inequality follows by [13, Theorem 7.1]. □

Corollary 3.2 ([4, Corollary 8.3]) *Let R be a Noetherian local ring with infinite residue field. Let $Q = (\mathbf{x})$ be a fixed parameter ideal. Then the set*

$$\Delta_Q(M) = \{\chi_1(\mathbf{y}, M) \mid (\mathbf{y}) \text{ is a parameter ideal with } \overline{(\mathbf{y})} = \overline{Q}\}$$

is finite.

Proof. This follows from

$$0 \leq \chi_1(\mathbf{y}, M) \leq \mathrm{hdeg}_{(\mathbf{y})}(M) - \deg_{(\mathbf{y})}(M) = \mathrm{hdeg}_Q(M) - \deg_Q(M),$$

where both $\mathrm{hdeg}_{(\mathbf{y})}(M)$ and $\deg_{(\mathbf{y})}(M)$ depend only on the integral closure of $(\mathbf{y})$. □

4 $e_1(\mathbf{x})$ vs $\chi_1(\mathbf{x})$

The functions $e_1(\mathbf{x}, M)$ and $\chi_1(\mathbf{x}, M)$ have mirror images consequences as shown in previous sections. Hence we are interested in a comparative study of functions $e_1(\mathbf{x}, M)$ and $\chi_1(\mathbf{x}, M)$. In particular, we would like study when is

$$-e_1(\mathbf{x}, M) \leq \chi_1(\mathbf{x}, M)\ ?$$

Let us recall the notion of d–sequences and proper sequences, which are extensions of regular sequences.

Definition 4.1 Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be a sequence of elements in R.

(a) $\mathbf{x}$ is a d-sequence if

$$(x_1, x_2, \ldots, x_i) : x_{i+1}x_k = (x_1, x_2, \ldots, x_i) : x_k,$$

for $i = 0, \ldots, n-1,\ k \geq i+1$.

(b) $\mathbf{x}$ is a proper sequence if

$$x_{i+1}\mathrm{H}_j(x_1, x_2, \ldots, x_i) = 0,$$

for $i = 0, 1, \ldots, n-1,\ j > 0$, where $\mathrm{H}_j(x_1, x_2, \ldots, x_i)$ is the Koszul homology associated to the subsequence $\{x_1, x_2, \ldots, x_i\}$.

It is known that a d–sequence is a proper sequence ([9, Theorem 5.6], [8, Corollary 12.4]).

Proposition 4.2 ([5]) *Let R be a Noetherian local ring of dimension $d \geq 2$ and let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be a sequence of elements of R. Set $\mathbf{x}' = \{x_1, \ldots, x_{n-1}\}$. Suppose that $\lambda(\mathrm{H}_j(\mathbf{x}')) < \infty$ and $\lambda(\mathrm{H}_j(\mathbf{x})) < \infty$ for all $j \geq 1$. Then*

$$\chi_1(\mathbf{x}) = \lambda_R((0) :_{R/(\mathbf{x}')} x_n).$$

Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters that is a d-sequence. Let $S = R[T_1, \ldots, T_d]$. Let $G = \mathrm{gr}_Q(R)$. Then the approximation complex $\mathcal{M}(\mathbf{x})$ associated to $\mathbf{x}$

$$0 \to \mathrm{H}_d(\mathbf{x}) \otimes S[-d] \to \cdots \to \mathrm{H}_1(\mathbf{x}) \otimes S[-1] \to \mathrm{H}_0(\mathbf{x}) \otimes S \to G \to 0$$

is acyclic. Therefore the Hilbert series of G is

$$H(G,t) = \frac{\sum_{i=0}^{d}(-1)^i h_i(\mathbf{x})t^i}{(1-t)^d} = \frac{f(t)}{(1-t)^d},$$

where $h_i(\mathbf{x}) = \lambda(\mathrm{H}_i(\mathbf{x}))$. Moreover,

$$e_i(\mathbf{x}) = \frac{f^{(i)}(1)}{i!}.$$

Lemma 4.3 [5] *Let $\mathbf{x} = x_1, \ldots, x_n$ be a proper sequence of R. Then we obtain the following set of exact sequences :*

$$0 \to (0) :_{R_i} x_{i+1} \longrightarrow R_i = R/(x_1, \ldots, x_i) \xrightarrow{x_{i+1}} R/(x_1, \ldots, x_i) \longrightarrow R/(x_1, \ldots, x_i, x_{i+1}) \to 0$$

$$0 \to \mathrm{H}_1(x_1, \ldots, x_i) \longrightarrow \mathrm{H}_1(x_1, \ldots, x_i, x_{i+1})$$

$$0 \to \mathrm{H}_j(x_1, \ldots, x_i) \longrightarrow \mathrm{H}_j(x_1, \ldots, x_i, x_{i+1}) \longrightarrow \mathrm{H}_{j-1}(x_1, \ldots, x_i) \to 0$$

for all $j \geq 2$.

Theorem 4.4 [5] *Let R be a Noetherian local ring of dimension $d \geq 2$ and let $\mathbf{x} = x_1, x_2, \ldots, x_d$ be a system of parameters that is a d-sequence in R. Let $Q_i = (x_1, \ldots, x_i)$ with $Q = Q_d = (\mathbf{x})$. Then*

(a) $\chi_1(x_1, \ldots, x_i) = \lambda_R((0) :_{R/Q_{i-1}} x_i)$.
In particular, $\chi_1(x_1) = \lambda_R((0) :_R x_1) = \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(R))$.

(b) $(-1)^i e_i(Q) = \chi_1(x_1, \ldots, x_{d-i}, x_{d-i+1}) - \chi_1(x_1, \ldots, x_{d-i}) \geq 0$ *for all $1 \leq i \leq d$.*
In particular the $e_i(Q)$ alternate.

Proof. (a) follows from Proposition 4.2. For (b), we first consider the case $-e_1(Q)$. For simplicity, set

$$\mathbf{x}' = x_1, \ldots, x_{d-1}, \ \mathrm{h}_i = \lambda(\mathrm{H}_i(\mathbf{x})), \ \mathrm{h}'_i = \lambda(\mathrm{H}_i(\mathbf{x}')).$$

Then

$$
\begin{aligned}
-e_1(Q) &= -h'(1) \\
&= \mathrm{h}_1 - 2\mathrm{h}_2 + 3\mathrm{h}_3 + \cdots + (-1)^{j+1} j\mathrm{h}_j + \cdots + (-1)^{d+1} d\mathrm{h}_d \\
&= \mathrm{h}_1 - \mathrm{h}_2 + \mathrm{h}_3 + \cdots + (-1)^{j+1}\mathrm{h}_j + \cdots + (-1)^{d+1}\mathrm{h}_d \\
&\quad -(\mathrm{h}_2 - 2\mathrm{h}_3 + \cdots + (-1)^j (j-1)\mathrm{h}_j + \cdots + (-1)^d (d-1)\mathrm{h}_d),
\end{aligned}
$$

where $\chi_1(\mathbf{x}) = \mathrm{h}_1 - \mathrm{h}_2 + \mathrm{h}_3 + \cdots + (-1)^{j+1}\mathrm{h}_j + \cdots + (-1)^{d+1}\mathrm{h}_d$. It is enough to show that

$$\chi_1(\mathbf{x}') = \mathrm{h}_2 - 2\mathrm{h}_3 + \cdots + (-1)^j (j-1)\mathrm{h}_j + \cdots + (-1)^d (d-1)\mathrm{h}_d.$$

Since a d–sequence $\mathbf{x}$ is a proper sequence, we can use the short exact sequences given in Lemma 4.3 to obtain the following:

$$
\begin{array}{rcll}
(-1)^d (d-1)\mathrm{h}_d & = & & + \ (-1)^d (d-1)\mathrm{h}'_{d-1} \\
(-1)^{d-1}(d-2)\mathrm{h}_{d-1} & = & (-1)^{d-1}(d-2)\mathrm{h}'_{d-1} & + \ (-1)^{d-1}(d-2)\mathrm{h}'_{d-2} \\
\vdots & & & \\
(-1)^{j+1} j\mathrm{h}_{j+1} & = & (-1)^{j+1} j\mathrm{h}'_{j+1} & + \ (-1)^{j+1} j\mathrm{h}'_j \\
(-1)^j (j-1)\mathrm{h}_j & = & (-1)^j (j-1)\mathrm{h}'_j & + \ (-1)^j (j-1)\mathrm{h}'_{j-1} \\
\vdots & & & \\
-2\mathrm{h}_3 & = & -2\mathrm{h}'_3 & + \ (-2)\mathrm{h}'_2 \\
\mathrm{h}_2 & = & \mathrm{h}'_2 & + \ \mathrm{h}'_1
\end{array}
$$

Therefore by adding these equations, we get

$$
\begin{aligned}
& \mathrm{h}_2 - 2\mathrm{h}_3 + \cdots + (-1)^j (j-1)\mathrm{h}_j + \cdots + (-1)^d (d-1)\mathrm{h}_d \\
= \ & \mathrm{h}'_1 - \mathrm{h}'_2 + \cdots + (-1)^{j+1}\mathrm{h}'_j + \cdots + (-1)^d \mathrm{h}'_{d-1} \\
= \ & \chi_1(\mathbf{x}'),
\end{aligned}
$$

which completes the proof for $-e_1(Q)$. This relation can be generalized as follows. Let $\mathbf{x}_n = x_1, \ldots, x_n$ and $\mathrm{h}_i(\mathbf{x}_n) = \lambda(\mathrm{H}_i(\mathbf{x}_n))$. Then

$$
\begin{aligned}
& \mathrm{h}_2(\mathbf{x}_n) - 2\mathrm{h}_3(\mathbf{x}_n) + \cdots + (-1)^j (j-1)\mathrm{h}_j(\mathbf{x}_n) + \cdots + (-1)^n (n-1)\mathrm{h}_n(\mathbf{x}_n) \\
= \ & \mathrm{h}_1(\mathbf{x}_{n-1}) - \mathrm{h}_2(\mathbf{x}_{n-1}) + \cdots + (-1)^{j+1}\mathrm{h}_j(\mathbf{x}_{n-1}) + \cdots + (-1)^n \mathrm{h}_{n-1}(\mathbf{x}_{n-1}) \\
= \ & \chi_1(\mathbf{x}_{n-1})
\end{aligned}
$$

More generally,

$$
\begin{aligned}
(-1)^i e_i(Q) &= \mathrm{h}_i - (i+1)\mathrm{h}_{i+1} + \binom{i+2}{2}\mathrm{h}_{i+2} + \cdots + (-1)^{i+j}\binom{i+j}{j}\mathrm{h}_{i+j} + \cdots + (-1)^{d+i}\binom{d}{d-i}\mathrm{h}_d \\
&= (\mathrm{h}_i(\mathbf{x}_{d-1}) + \mathrm{h}_{i-1}(\mathbf{x}_{d-1})) - (i+1)(\mathrm{h}_{i+1}(\mathbf{x}_{d-1}) + \mathrm{h}_i(\mathbf{x}_{d-1})) \\
&\quad + \binom{i+2}{2}(\mathrm{h}_{i+2}(\mathbf{x}_{d-1}) + \mathrm{h}_{i+1}(\mathbf{x}_{d-1})) + \cdots \\
&= \mathrm{h}_{i-1}(\mathbf{x}_{d-1}) - i\mathrm{h}_i(\mathbf{x}_{d-1}) + \binom{i+1}{2}\mathrm{h}_{i+1}(\mathbf{x}_{d-1}) + \cdots \\
&\ \vdots \\
&= \mathrm{h}_1(\mathbf{x}_{d-i+1}) - 2\mathrm{h}_2(\mathbf{x}_{d-i+1}) + 3\mathrm{h}_3(\mathbf{x}_{d-i+1}) + \cdots + (-1)^{d-i+2}(d-i+1)\mathrm{h}_{d-i+1} \\
&= \mathrm{h}_1(\mathbf{x}_{d-i+1}) - \mathrm{h}_2(\mathbf{x}_{d-i+1}) + \mathrm{h}_3(\mathbf{x}_{d-i+1}) + \cdots + (-1)^{d-i+2}\mathrm{h}_{d-i+1}(\mathbf{x}_{d-i+1}) \\
&\quad -(\mathrm{h}_2(\mathbf{x}_{d-i+1}) - 2\mathrm{h}_3(\mathbf{x}_{d-i+1}) + \cdots + (-1)^{d-i+1}(d-i)\mathrm{h}_{d-i+1}(\mathbf{x}_{d-i+1})), \\
&= \chi_1(x_1, \ldots, x_{d-i+1}) - \chi_1(x_1, \ldots, x_{d-i}).
\end{aligned}
$$

As for the assertion of positivity, using (a)

$$
\begin{aligned}
(-1)^i e_i(Q) &= \chi_1(x_1, \ldots, x_{d-i+1}) - \chi_1(x_1, \ldots, x_{d-i}) \\
&= \lambda((0) :_{R/Q_{d-i}} x_{d-i+1}) - \lambda((0) :_{R/Q_{d-i-1}} x_{d-i}) \\
&= \lambda(((x_1, \ldots, x_{d-i}) : x_{d-i+1})/(x_1, \ldots, x_{d-i})) - \lambda(((x_1, \ldots, x_{d-i-1}) : x_{d-i})/(x_1, \ldots, x_{d-i-1}))
\end{aligned}
$$

Now observe that since $\mathbf{x}$ is d-sequence,

$$((\ldots, x_{d-i-1}) : x_{d-i}) \subset ((\ldots, x_{d-i-1}) : x_{d-i+1}) \subset ((\ldots, x_{d-i}) : x_{d-i+1}),$$

which induces a natural mapping :

$$\frac{((x_1, \ldots, x_{d-i-1}) : x_{d-i})}{(x_1, \ldots, x_{d-i-1})} \xrightarrow{\varphi} \frac{((x_1, \ldots, x_{d-i}) : x_{d-i+1})}{(x_1, \ldots, x_{d-i})}.$$

The kernel is zero since

$$((x_1, \ldots, x_{d-i-1}) : x_{d-i}) \cap (x_1, \ldots, x_{d-i}) = (x_1, \ldots, x_{d-i-1})$$

because $\mathbf{x}$ is d-sequence. This gives the formula

$$(-1)^i e_i(Q) = \lambda\left(\frac{(x_1, \ldots, x_{d-i}) : x_{d-i+1}}{(x_1, \ldots, x_{d-i}) + (x_1, \ldots, x_{d-i-1}) : x_{d-i}}\right).$$

□

Corollary 4.5 ([5]) *Let R be a Noetherian local ring of dimension $d \geq 2$. Let $\mathbf{x}$ be a system of parameters that is a d-sequence in R and set $Q = (\mathbf{x})$. Then $\chi_1(\mathbf{x}) = -e_1(Q)$ if and only if* $\operatorname{depth} R \geq d - 1$

For a parameter ideal $Q = (\mathbf{x})$, the key to comparing the values of $\chi_1(\mathbf{x})$ and $e_1(Q)$ is the following.

Proposition 4.6 ([5]) *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ with* $\operatorname{depth} R \geq d - 1$ *and infinite residue field. Let Q be a parameter ideal of R. Then there exists a generating system $\mathbf{x} = \{x_1, \ldots, x_d\}$ for Q such that $\{x_1, \ldots, x_{d-1}\}$ is a superficial sequence of R relative to Q. Set $Q_{d-1} = (a_1, \ldots, a_{d-1})$. Then*

(a) $\chi_1(\mathbf{x}) = \lambda_R(0 :_{R/Q_{d-1}} x_d) = \lambda_R([Q_{d-1} : x_d]/Q_{d-1})$.

(b) $e_1(Q) = -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(R/Q_{d-1}))$.

(c) $\chi_1(\mathbf{x}) \leq -e_1(Q)$, *where the equality holds if and only if $\mathbf{x}$ is a d-sequence.*

We are now in position to give the following converse to Theorem 4.4:

Theorem 4.7 ([5]) *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 2$ with* $\operatorname{depth} R \geq d - 1$ *and infinite residue field. Let Q be a parameter ideal of R. Then we may assume that $Q = (\mathbf{x}) = (x_1, \ldots, x_d)$, where $\mathbf{x}' = x_1, x_2, \ldots, x_{d-1}$ forms a superficial sequence for R with respect to Q. Then the followings are equivalent :*

(i) $\chi_1(\mathbf{x}) \geq -e_1(Q)$;

(ii) $\chi_1(\mathbf{x}) = -e_1(Q)$;

(iii) $\mathbf{x}$ *is a d-sequence* ;

(iv) $e_2(Q) = 0$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow from Proposition 4.6.

(iii) $\Rightarrow$ (iv) : By assumption, $\mathbf{x}$ is a system of parameters that is a d-sequence. We use induction on d. If $d = 2$, then by Theorem 4.4,

$$e_2(Q) = \chi_1(x_1) = \lambda_R(H^0_{\mathfrak{m}}(R)) = 0,$$

where the last equality follows from $\operatorname{depth} R \geq 1$. Suppose that $d \geq 3$. By Theorem 4.4,

$$e_2(Q) = \chi_1(x_1, \ldots, x_{d-1}) - \chi_1(x_1, \ldots, x_{d-2}) = 0,$$

since $\operatorname{depth} R \geq d - 1$ implies that the subsequence $\mathbf{x}' = x_1, \ldots, x_{d-1}$ is a regular sequence.

(iv) $\Rightarrow$ (iii) : We use induction on d. If $d = 2$, the assertion follows from [7, Theorem 3.2]. Suppose that $d \geq 3$. Then

$$0 = e_2(Q) = e_2(Q/(x_1)).$$

By induction hypothesis, this means that $x_2, \ldots, x_d$ forms a d–sequence in $R/(x_1)$. Hence $\mathbf{x} = x_1, x_2, \ldots, x_d$ is a d–sequence in R because x_1 is R–regular. □

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On the Limit Closure of a Sequence of Elements in Local Rings [1]

By
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1 Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-modules of dimension d. Let $\underline{x} = x_1, ..., x_r$ be a sequence of r elements in $\mathfrak{m}$. Then the *limit closure* of the sequence $\underline{x}$ in M is a submodule of M defined by

$$(\underline{x})_M^{lim} = \bigcup_{t>0} \left((x_1^{t+1}, ..., x_r^{t+1})M : x_1^t ... x_r^t\right).$$

When $M = R$ we write $(\underline{x})^{lim}$ for short. It should be noticed that the limit closure of a system of parameters of R is closely related to the Monomial Conjecture. In fact, R satisfies the monomial conjecture if and only if $(\underline{x})^{lim} \neq R$ for all systems of parameters $\underline{x}$ of R. Moreover, it was proved by R. Hartshorne [13] that if $\underline{x} = x_1, ..., x_r$ is a M-sequence, then $(\underline{x})M = (\underline{x})_M^{lim}$. The converse is also true (see [4]), and therefore M is a Cohen-Macaulay module if and only if $(\underline{x})M = (\underline{x})_M^{lim}$ for some system of parameters $\underline{x}$ of M. The aim of this paper is to present basic properties of this submodule $(\underline{x})_M^{lim}$ in Section 2, and then some applications of them in commutative algebra. First, we can give in Section 3 a very simple and short proof for the well-known Lichthenbaum-Hartshorne Vanishing Theorem for local cohomology. Next, we prove an intersection formula for limit closures of powers of a system of parameters (Theorem 4.1) in Section 4. This intersection formula plays an important role in the next two sections. Namely, by virtue of this formula we can prove in Sections 5 several results concerning the following question: Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of M and $\underline{y} = y_1, ..., y_d$ a sequence of elements in $\mathfrak{m}$ such that $(\underline{y}) \subseteq (\underline{x})$. When is $\underline{y}$ a system of parameters of M? Then our results in this section generalize theorems of S. Dutta-P. Roberts [10] and of L. Fouli-C. Huneke in [11]. And in Section 6 we show that a local ring $(R, \mathfrak{m})$ is unmixed if and only if the $\mathfrak{m}$-adic topology is equivalent to the topology defined by the chain of ideals $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ for some (and therefore all) system of parameters $\underline{x}$ of R.

2 Basic properties

Throughout this paper, $(R, \mathfrak{m})$ is a Noetherian local ring of dimension t and M is a finitely generated R-modules of dimension $d > 0$. Let $\underline{x} = x_1, ..., x_r$ be a sequence of r elements in $\mathfrak{m}$. For a positive integer n, we set $\underline{x}^{[n]} = x_1^n, ..., x_r^n$.

Definition 2.1 ([16]). The *limit closure* of the sequence $\underline{x}$ in M is a submodule of M defined by

$$(\underline{x})_M^{lim} = \bigcup_{n>0} \left((\underline{x}^{[n+1]})M : (x_1 ... x_r)^n\right),$$

when $M = R$ we write $(\underline{x})^{lim}$ for short.

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Now, let $\underline{y} = y_1, ..., y_r$ be another sequence of elements such that $(\underline{y}) \subseteq (\underline{x})$. Then there exists a matrix $A = (a_{ij})$, $a_{ij} \in R$, $1 \leq i, j \leq r$ such that $y_i = \sum_{j=1}^{r} a_{ij}x_j$, it means $\mathbf{y} = A\mathbf{x}$, where $\mathbf{x}$ (res. $\mathbf{y}$) denotes the column vector with entries $x_1, ..., x_r$ (res. $y_1, ..., y_r$). Following [11], we abbreviate it by writing $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. Put $\delta = \det A$. It easily follows from Crammer's rule that $\delta(\underline{x}) \subseteq (\underline{y})$. Therefore, we obtain a canonical map

$$\delta : M/(\underline{x})M \to M/(\underline{y})M, \quad m + (\underline{x})M \mapsto \delta m + (\underline{y})M.$$

By ([25], 5.1.15) we also have that $\delta(\underline{x})_M^{lim} \subseteq (\underline{y})_M^{lim}$. Hence we obtain a homomorphism

$$\delta : M/(\underline{x})_M^{lim} \to M/(\underline{y})_M^{lim},$$

which is independent of the choice of the matrix A. The map δ is called *determinantal maps.*

Remark 2.2.

(i) For a sequence $\underline{x} = x_1, ..., x_r$ and s a positive integer we denote the sequence $x_1^s, ..., x_r^s$ by $\underline{x}^{[s]}$. We have a direct system $\{M/(\underline{x}^{[n]})M\}_{n \geq 1}$ given by the determinantal maps

$$(x_1 ... x_r)^{k-s} : M/(\underline{x}^{[s]})M \longrightarrow M/(\underline{x}^{[k]})M$$

for all $1 \leq s \leq k$. Then the kernel of the canonical map

$$M/(\underline{x})M \to \varinjlim M/(\underline{x}^{[n]})M \cong H^r_{(\underline{x})}(M)$$

is $(\underline{x})_M^{lim}/(\underline{x})M$, where $H^i_{(\underline{x})}(M)$ is the i-th local cohomology of M with support in $(\underline{x})$. We get that the induced direct system $\{M/(\underline{x}^{[n]})_M^{lim}\}_{n \geq 1}$ with injective maps and

$$\varinjlim M/(\underline{x}^{[n]})_M^{lim} \cong H^r_{(\underline{x})}(M).$$

Therefore we can consider $M/(\underline{x}^{[n]})_M^{lim}$ as a submodule of $H^r_{(\underline{x})}(M)$. Hence

$$\mathrm{Ann}(H^r_{(\underline{x})}(M)) = \cap_{n \geq 1} \mathrm{Ann}(M/(\underline{x}^{[n]})_M^{lim}).$$

In particular, $\mathrm{Ann}(H^r_{(\underline{x})}(R)) = \cap_{n \geq 1}(\underline{x}^{[n]})^{lim}$.

(ii) Let $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ are sequences such that $\sqrt{(\underline{x})} = \sqrt{(\underline{y})}$ i.e. $(\underline{x}^{[n]}) \overset{B}{\subseteq} (\underline{y})$ for some B and n. Then the determinantal map $\delta : M/(\underline{x})_M^{lim} \to M/(\underline{y})_M^{lim}$ is injective (cf. [4], Lemma 3.1). Therefore $(\underline{x})_M^{lim} = (\underline{y})_M^{lim} :_M \delta$. Hence $(\underline{y})_M^{lim} \subseteq (\underline{x})_M^{lim}$.

The following is a slight generalization of [15], Proposition 2, and [4], Theorem 3.3.

Proposition 2.3. *Let M be a finitely generated R-modules of dimension d. Then there exists a positive integer n such that every system of parameters $\underline{y} = y_1, ..., y_d$ contained in $\mathfrak{m}^n$ satisfies the monomial property, i.e. $(\underline{y})_M^{lim} \neq M$.*

Proof. Without any loss of generality we may assume that $\mathrm{Ann}\, M = 0$. Then by 2.2, (i) we can choose a system of parameters $\underline{x} = x_1, ..., x_d$ satisfies the monomial property. Therefore by Remark 2.2(ii), the positive integer n such that $\mathfrak{m}^n \subseteq (\underline{x})$ satisfies the requirement of the proposition. ☐

Lemma 2.4. *Let M be a finitely generated R-modules of dimension d, and $\underline{x} = x_1, ..., x_r$ a sequence of elements in $\mathfrak{m}$. Then the following assertions hold true.*

(i) $(\underline{x})^{lim}_M = M$ *if* $H^r_{(\underline{x})}(M) = 0$. *In particular, if* $r > d$, *then* $(\underline{x})^{lim}_M = M$.

(ii) *If* $r = d$, *then* $M/(\underline{x})^{lim}_M$ *has finite length.*

Proof. Note first that $M/(\underline{x})^{lim}_M$ is a submodule of $H^r_{(\underline{x})}(M)$ by Remark 2.2, (i). So the statement (i) is trivial.
(ii). If $r = d$, $H^d_{(\underline{x})}(M)$ is an Artinian module, and hence $M/(\underline{x})^{lim}_M$ is Noetherian and Artinian. □

Corollary 2.5. *Let $\underline{x} = x_1, ..., x_r$ be a sequence of elements in $\mathfrak{m}$, and N a submodule of M such that* $\dim N < r$. *Then* $N \subseteq (\underline{x})^{lim}_M$.

Proof. By Lemma 2.4 we have $(\underline{x})^{lim}_N = N$. Hence

$$\begin{aligned} (\underline{x})^{lim}_M &= \bigcup_{n>0} ((x_1^{n+1}, ..., x_r^{n+1})M :_M x_1^n ... x_r^n) \\ &\supseteq \bigcup_{n>0} ((x_1^{n+1}, ..., x_r^{n+1})N :_N x_1^n ... x_r^n) \\ &= (\underline{x})^{lim}_N = N. \end{aligned}$$

□

Proposition 2.6. *Let $\underline{x} = x_1, ..., x_r$ be a sequence of elements in $\mathfrak{m}$, and N a submodule of M such that* $N \subseteq \cap_{n>0} (\underline{x}^{[n]})^{lim}_M$. *Set* $\overline{M} = M/N$. *We have* $(\underline{x})^{lim}_{\overline{M}} = (\underline{x})^{lim}_M / N$.

Proof. It is sufficient to prove that

$$(\underline{x})^{lim}_M = \bigcup_{n>0} (((x_1^{n+1}, ..., x_r^{n+1})M + N) :_M x_1^n ... x_r^n).$$

In fact, the set on the left hand is clear contained in the set on the right hand. Conversely, it is easy to check that

$$((x_1^{n+1}, ..., x_r^{n+1})M + N) :_M x_1^n ... x_r^n \subseteq ((x_1^{n'+1}, ..., x_r^{n'+1})M + N) :_M x_1^{n'} ... x_r^{n'},$$

for all $n \leq n'$. Then there exists a positive integer s such that

$$\bigcup_{n>0} (((x_1^{n+1}, ..., x_r^{n+1})M + N) :_M x_1^n ... x_r^n) = ((x_1^{s+1}, ..., x_r^{s+1})M + N) :_M x_1^s ... x_r^s.$$

Since $N \subseteq (\underline{x}^{[s+1]})^{lim}_M$, there exists a positive integer k such that

$$N \subseteq (x_1^{(k+1)(s+1)}, ..., x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)} ... x_r^{k(s+1)}.$$

Therefore

$$(x_1^{s+1}, ..., x_r^{s+1})M + N \subseteq (x_1^{(k+1)(s+1)}, ..., x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)} ... x_r^{k(s+1)}.$$

Thus

$$\begin{aligned}(\underline{x})_M^{lim} &\supseteq (x_1^{(k+1)(s+1)},...,x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)+s}...x_r^{k(s+1)+s}\\ &= ((x_1^{(k+1)(s+1)},...,x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)}...x_r^{k(s+1)}) :_M x_1^s...x_r^s\\ &\supseteq ((x_1^{s+1},...,x_r^{s+1})M + N) :_M x_1^s...x_r^s\\ &= \bigcup_{n>0} (((x_1^{n+1},...,x_r^{n+1})M + N) :_M x_1^n...x_r^n).\end{aligned}$$

□

Proposition 2.7. *Let $\underline{x} = x_1, ..., x_t$ be a sequence of elements of R. Then the following conditions are equivalent*

(i) $H^t_{(\underline{x})}(R) = 0$.

(ii) $\cap_{n\geq 1}(\underline{x}^{[n]})^{lim} = R$.

(iii) $\dim R/(\mathrm{Ann}\,(H^t_{(\underline{x})}(R))) < t$.

Proof. (i) ⇔ (ii) follows from Remark 2.2, (i) that $\mathrm{Ann}\,(H^t_{(\underline{x})}(R)) = \cap_{n\geq 1}(\underline{x}^{[n]})^{lim}$.
(i) ⇒ (iii) is trivial.
(iii) ⇒ (ii). Set $R' = R/(\cap_{n\geq 1}(\underline{x}^{[n]})^{lim})$ and a' is the image of a in R'. By Proposition 2.6 we have $\cap_{n\geq 1}(\underline{x'}^{[n]})^{lim}_{R'} = 0$. On the other hand, since $\dim R/(\mathrm{Ann}\,(H^t_{(\underline{x})}(R))) < t$, $\dim R' < t$. it follows by Lemma 2.4 that $\cap_{n\geq 1}(\underline{x'}^{[n]})^{lim}_{R'} = R'$. Therefore $R' = 0$. □

3 The Lichtenbaum-Hartshorne Vanishing Theorem

There are several proofs for the Lichtenbaum-Hartshorne Vanishing Theorem for local cohomology (cf. [1], [14]). All most of them are based on the Chevalley Theorem (see Lemma 6.2 below). By applying Proposition 2.7 we shall give here a simple and short proof for this theorem based on Matlis' duality.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t, and I an ideal of R. Then the following statements are equivalent*

(i) $H^t_I(R) = 0$.

(ii) *For all minimal primes P of $\widehat{R}$, the completion of R, satisfying* $\dim \widehat{R}/P = \dim R$, *we have* $\dim \widehat{R}/(I\widehat{R} + P) > 0$.

Proof. We only need to prove the difficult part (ii) ⇒ (i) of Theorem 3.1. We may assume without loss of generality that $(R, \mathfrak{m})$ is a complete local ring. Let $\underline{x} = x_1, ..., x_t$ be a I-filter regular sequence in I. By [21, Lemma 3.4] we have $H^t_I(R) \cong H^0_I(H^t_{(\underline{x})}(R))$. Therefore $H^t_I(R) \cong H^t_{(\underline{x})}(R)$, since $H^t_{(\underline{x})}(R)$ is Artinian. So we may assume henceforth that $I = (\underline{x})$. Let $E(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$, and $N = \mathrm{Hom}_R(H^t_I(R), E(R/\mathfrak{m}))$ the Matlis dual of $H^t_I(R)$. By duality (see [24] Theorem 11.57) we have

$$\mathrm{Hom}_R(\mathrm{Hom}_R(R/I, H^t_I(R)), E(R/\mathfrak{m})) \cong R/I \otimes_R N \cong N/IN.$$

Since $\mathrm{Hom}_R(R/I, H_I^t(R))$ is finitely generated by [19] Proposition 5.1, $\mathrm{Hom}_R(R/I, H_I^t(R))$ and N/IN have finite length. Suppose that $H_I^t(R) \neq 0$. Then $\dim R/\mathrm{Ann}(H_I^t(R)) = t$ by Proposition 2.7. Hence N is a Noetherian module of dimension t. Therefore there exists $\mathfrak{p} \in \mathrm{Assh}\, R$ such that $\dim R/(I+\mathfrak{p}) = 0$. This is a contradiction. □

4 An intersection formula

The following theorem is the main result of this section and plays an important role for the study in next sections. Recall that the *unmixed component* $U_M(0)$ of M is a submodule defined by $U_M(0) = \bigcap_{\substack{\mathfrak{p}\in \mathrm{Ass} M\\ \dim R/\mathfrak{p}=d}} N(\mathfrak{p})$, where $0 = \bigcap_{\mathfrak{p}\in \mathrm{Ass}\, M} N(\mathfrak{p})$ is a reduced primary decomposition of the zero module of M (see [8]).

Theorem 4.1. *Let M be a finitely generated R-module of dimension d and $\underline{x} = x_1, ..., x_d$ a system of parameters of $M/U_M(0)$. Then*

$$U_M(0) = \bigcap_{n>0} (\underline{x}^{[n]})_M^{lim}.$$

To prove the theorem we need some auxiliary results from [3], [6], which are collected in the theorem below. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of M. Then we can consider the differences

$$I_{M,\underline{x}}(n) = \ell(M/(\underline{x}^{[n]})M) - e(\underline{x}^{[n]}, M), \text{ and}$$
$$J_{M,\underline{x}}(n) = e(\underline{x}^{[n]}, M) - \ell(M/(\underline{x}^{[n]})_M^{lim})$$

as functions in n, where $e(\underline{x}; M)$ is the Serre multiplicity of M with respect to the sequence $\underline{x}$. In general, these functions are not polynomials in n (see [7]), but they are bounded above by polynomials. Moreover, we have

Theorem 4.2. ([3], [6]) *With the notations as above, the both functions $I_{M,\underline{x}}(n)$ and $J_{M,\underline{x}}(n)$ are non-negative increasing, and the least degrees of polynomials in n bounding above these functions are independent of the choice of $\underline{x}$. Moreover, if we denote by $p(M)$ and $pf(M)$ for these least degrees with respect to $I_{M,\underline{x}}(n)$ and $J_{M,\underline{x}}(n)$ respectively, then $p(M) \leq d-1$ and $pf(M) \leq d-2$.*

We are now able to prove the theorem.

Proof of Theorem 4.1. We set $N = \bigcap_{n>0}(\underline{x}^{[n]})_M^{lim}$. . By Corollary 2.5 we have $U_M(0) \subseteq N$. Put $\overline{M} = M/U_M(0)$ and $M' = M/N$. Then by Theorem 4.2, there are polynomials $f(n)$ of degree at most $d-1$ and $g(n)$ of degree at most $d-2$ such that

$$\ell(M/(\underline{x}^{[n]})_M^{lim}) = \ell(M'/(\underline{x}^{[n]})_{M'}^{lim}) \leq \ell(M'/(\underline{x}^{[n]})M') \leq n^d e(\underline{x}; M') + f(n),$$

and

$$\ell(M/(\underline{x}^{[n]})_M^{lim}) = \ell(\overline{M}/(\underline{x}^{[n]})_{\overline{M}}^{lim}) \geq n^d e(\underline{x}; \overline{M}) - g(n).$$

Therefore

$$f(n) + g(n) \geq n^d (e(\underline{x}, \overline{M}) - e(\underline{x}, M'))$$

for all $n > 0$. It follows that $e(\underline{x}; N/U_M(0)) = e(\underline{x}; \overline{M}) - e(\underline{x}; M') = 0$. Hence $\dim N < d$, and so $N = U_M(0)$, since $U_M(0)$ is the largest submodule of M with the dimension less than d. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.3. *$\bigcap_{\underline{x}}(\underline{x})^{lim}_M = U_M(0)$, where $\underline{x}$ runs through the set of all systems of parameters of M.*

Corollary 4.4. *Let M be a finitely generated R-module of dimension d.*

$$\mathrm{Ann}\,(H^d_{\mathfrak{m}}(M)) = \mathrm{Ann}\,(M/U_M(0)) = \{r \in R : \dim M/(0:_M r) < d\}.$$

In particular, $H^d_{\mathfrak{m}}(M) \neq 0$.

Proof. Let $\underline{x} = x_1, ..., x_d$ be a system of parameter of M. By Remark 2.2, we may consider $M/(\underline{x}^{[n]})^{lim}_M$ as a submodule of $H^d_{\mathfrak{m}}(M) = H^d_{(\underline{x})}(M)$ for all $n \geq 1$.

$$\begin{aligned}\mathrm{Ann}\,(H^d_{\mathfrak{m}}(M)) &= \{r \in R : rM \subseteq M/(\underline{x}^{[n]})^{lim}_M, \forall n \geq 1\}\\ &= \{r \in R : rM \subseteq U_M(0)\}\\ &= \mathrm{Ann}\,(M/U_M(0))\\ &= \{r \in R : \dim(rM) < d\}\\ &= \{r \in R : \dim M/(0:_M r) < d\}.\end{aligned}$$

The last assertion follows from the first. □

It should be noticed that Corollary 4.4 was proved by A. Grothendieck [12, Proposition 6.6] when R is complete.

Corollary 4.5. *Let $(R, \mathfrak{m})$ be a complete local ring of dimension t, $\underline{x} = x_1, ..., x_t$ a sequence of elements. Let $\cap_{\mathfrak{p} \in \mathrm{Ass}\, R} N(\mathfrak{p}) = 0$ be a reduced primary decomposition of (0). Then*

$$\bigcap_{n \geq 1}(\underline{x}^{[n]})^{lim}_M = \bigcap_{\mathfrak{p} \in J} N(\mathfrak{p}),$$

where $J = \{\mathfrak{p} \in \mathrm{Ass} R : \underline{x}$ is a system of parameters of $R/\mathfrak{p}\}$.

Proof. Set $N = \cap_{\mathfrak{p} \in J} N(\mathfrak{p})$. Then $\mathrm{Ass}\, N = \mathrm{Ass}\, R \setminus J$. It is easy to check by Theorem 4.1 that $H^t_{(\underline{x})}(R) = 0$. Therefore $\cap_{n \geq 1}(\underline{x}^{[n]})^{lim} \supseteq \cap_{n \geq 1}(\underline{x}^{[n]})^{lim}_N = N$ by Lemma 2.4. Set $R' = R/N$ and x' the image of x. We have that $\underline{x}$ is a system of parameters of R' and $U_{R'}(0) = 0$. Hence $\cap_{n \geq 1}(\underline{x'}^{[n]})^{lim}_{R'} = 0$ by Theorem 3.1. Thus $\cap_{n \geq 1}(\underline{x}^{[n]})^{lim} = N$ by Proposition 2.6. □

5 Systems of parameters

In the rest of the paper, $\underline{x} = x_1, ..., x_t$ is a system of parameters of R, and $\underline{y} = y_1, ..., y_t$ a sequence of elements such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. S. Dutta and P. Roberts proved in [10] that if R is a Cohen-Macaulay ring, then $\underline{y}$ is a system of parameters if and only if the determinantal map $\det A : R/(\underline{x}) \to R/(\underline{y})$ is injective. Without the assumption that R is Cohen-Macaulay, we can prove the following result.

Theorem 5.1. *Let R be a local ring such that $R/U_R(0)$ is Cohen-Macaulay. Let $\underline{x}$ be a system of parameters and $\underline{y}$ a sequence of elements in R such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. Then $\underline{y}$ is a system of parameters of $R/U_R(0)$ if and only if the determinantal map $\det A : R/(\underline{x})^{lim} \to R/(\underline{y})^{lim}$ is injective.*

Proof. By Corollary 2.5 we have that $U_R(0) \subseteq (\underline{x}^{[s]})^{lim}$ for all $s \geq 1$. Set $\overline{R} = R/U_R(0)$. For an element $a \in R$, we denote by $\overline{a}$ the image of a in $\overline{R}$. Set $\underline{\overline{x}} = \overline{x_1}, ..., \overline{x_t}$ and $\underline{\overline{y}} = \overline{y_1}, ..., \overline{y_t}$. Then it is easy to check by Proposition 2.6 that we also have $(\underline{\overline{y}}) \overset{A}{\subseteq} (\underline{\overline{x}})$. Then the conclusion follows from the mentioned result of Dutta and Roberts above. □

As a consequence of Theorem 5.1 we get again a recently result of L. Fouli and C. Huneke, [11, Theorem 4.4] as follows.

Corollary 5.2. *Let R be a 1-dimensional Noetherian local ring. Let x be a parameter, and let $y = ux$. Then y is a parameter if and only if the map $R/(x)^{lim} \overset{u}{\longrightarrow} R/(y)^{lim}$ is injective.*

Proof. Since $\dim R = 1$ we have that $U_R(0) = H^0_{\mathfrak{m}}(R)$ and $\overline{R} = R/U_R(0)$ is Cohen-Macaulay. Moreover, x is a parameter of R if and only if $\overline{x}$ is also a parameter of $\overline{R}$. Hence the assertion follows from Theorem 5.1. □

The following theorem is a correction of a result of Fouli and Huneke [11, Corollary 5.4].

Theorem 5.3. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring of dimension t. There exists an integer ℓ with the following property: whenever $\underline{x} = x_1, ..., x_t$ is a system of parameters of $R/U_R(0)$ with $(\underline{x}) \subseteq \mathfrak{m}^\ell$ and $\underline{y} = y_1, ..., y_t$ a sequence of elements such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$, then $\underline{y}$ forms a system of parameters of $R/U_R(0)$ if and only if the determinantal map $R/(\underline{x})^{lim} \overset{\det A}{\longrightarrow} R/(\underline{y})^{lim}$ is injective.*

Proof. It suffices to prove the only if part. By Corollary 2.5 and Proposition 2.6 we may assume henceforth that $U_R(0) = 0$. Assume that $\mathrm{Ass} R = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ and $\cap_{\mathfrak{p}_i \in \mathrm{Ass}\, R} N(\mathfrak{p}_i) = 0$ is a reduced primary decomposition of (0). For $1 \leq i \leq n$ we set

$$N_i = \bigcap_{j \neq i, \mathfrak{p}_j \in \mathrm{Ass} R} N(\mathfrak{p}_j).$$

Let $\underline{z} = z_1, ..., z_t$ is a system of parameters of R. By Theorem 3.1 we have $\cap_{n \geq 1} (\underline{z}^{[n]})^{lim} = 0$. Then there are positive integers ℓ_1 and ℓ such that $N_i \nsubseteq (\underline{z}^{[\ell_1]})^{lim}$ and $\mathfrak{m}^\ell \subseteq (\underline{z}^{[\ell_1]})$. Let $\underline{x} = x_1, ..., x_t$ and $\underline{y} = y_1, ..., y_t$ are sequences of elements contained in $\mathfrak{m}^\ell$ such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ and $\underline{x}$ is a system of parameters of R but $\underline{y}$ is not. Assume that $\underline{y} = y_1, ..., y_t$ is not a system of parameters of $R/\mathfrak{p}_1$. By Corollary 4.5 we have $N_1 \subseteq (\underline{y})^{lim}$. On the other hand, it follows from Remark 2.2, (b) that $(\underline{x})^{lim} \subseteq (\underline{z}^{[\ell_1]})^{lim}$. Hence $N_i \nsubseteq (\underline{x})^{lim}$ for all $1 \leq i \leq n$. Thus $u \in (\underline{y})^{lim} \setminus (\underline{x})^{lim} \neq \emptyset$. Therefore the determinantal map $R/(\underline{x})^{lim} \overset{\det A}{\longrightarrow} R/(\underline{y})^{lim}$ is not injective. □

6 A characterization of unmixed local rings

Unmixed local rings were introduced first by M. Nagata [20] as follows.

Definition 6.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t. Then R is *unmixed* if $U_{\widehat{R}}(0) = 0$ i.e. $\mathrm{Assh}\, \widehat{R} = \mathrm{Ass}\, \widehat{R}$, where $\widehat{R}$ denotes the completion of R with respect to the $\mathfrak{m}$-adic topology.

Almost of domains in Commutative Algebra are unmixed. However, in [20] N. Nagata constructed a domain of dimension two which is not unmixed. Unmixed local rings were investigated by several authors (cf. [22], [23], [26]). Let $\underline{x} = x_1, ..., x_t$ be a system of parameters of R. By Krull's intersection theorem we have $\cap_{n\geq 1}(\underline{x}^{[n]}) = 0$. It means that the topology defined by $\{(\underline{x}^{[n]})\}_{n\geq 1}$ is always Hausdorff. However, the topology defined by $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ may be not Hausdorff. In fact, $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ is a Hausdorff topology if and only if $U_R(0) = 0$ by Theorem 3.1. The aim of this section is to give a characterization of unmixed local rings in term of the $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ topology. First, we need the following result proved by Chevalley (cf. [2, Lemma 7]).

Lemma 6.2 (Chevalley). *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, and $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ a chain of ideals of R such that $\cap_{n\geq 1}\mathfrak{a}_n = 0$. Then for each n there exists an integer $v(n)$ such that $\mathfrak{a}_{v(n)} \subseteq \mathfrak{m}^n$. In other words, the linear topology defined by $\{\mathfrak{a}_n\}_{n\geq 1}$ is stronger or equal to the $\mathfrak{m}$-adic topology.*

Theorem 6.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t, $\underline{x} = x_1, ..., x_t$ a system of parameters. Then R is unmixed if and only if the topology defined by $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ is equivalent to the $\mathfrak{m}$-adic topology.*

Proof. We note that the $\mathfrak{m}$-adic topology is always stronger or equal to the topology defined by $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ since $(\underline{x}^{[n]})^{lim}$ is $\mathfrak{m}$-primary for all $n \geq 1$.
($\Rightarrow$). We assume that R is unmixed. Then by Theorem 4.1 the topology defined by $\{(\underline{x}^{[n]})^{lim}_{\widehat{R}}\}_{n\geq 1}$ is Hausdorff. By Chevalley's theorem, for each n there exists an integer $v(n)$ such that $(\underline{x}^{[v(n)]})^{lim}_{\widehat{R}} \subseteq \widehat{\mathfrak{m}}^n$. Thus $(\underline{x}^{[v(n)]})^{lim} \subseteq \mathfrak{m}^n$. Therefore the topology defined by $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ is stronger or equal to the $\mathfrak{m}$-adic topology. So they are equivalence.
($\Leftarrow$). Suppose that R is not unmixed i.e. $U_{\widehat{R}} \neq 0$. By Krull's intersection theorem, there exists n_0 such that $U_{\widehat{R}} \not\subseteq \widehat{\mathfrak{m}}^{n_0}$. On the other hand, we get by Proposition 2.6 that $U_{\widehat{R}} \subseteq (\underline{x}^{[n]})^{lim}_{\widehat{R}}$ for all n. Therefore $(\underline{x}^{[n]})^{lim}_{\widehat{R}} \not\subseteq \widehat{\mathfrak{m}}^{n_0}$ for all n. Thus $(\underline{x}^{[n]})^{lim} \not\subseteq \mathfrak{m}^{n_0}$ for all n so the topology defined by $\{(\underline{x}^{[n]})^{lim}\}_{n\geq 1}$ is not equivalent to the $\mathfrak{m}$-adic topology. □

Corollary 6.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $U_R(0) = 0$. Suppose that the $\mathfrak{m}$-adic topology is minimal among all Hausdorff topologies of R. Then R is unmixed.*

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ASYMPTOTIC BEHAVIOR OF THE GRADE ASSOCIATED TO MULTIGRADED MODULES

FUTOSHI HAYASAKA

ABSTRACT. Let R be a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $\mathbf{d}_1, \ldots, \mathbf{d}_r$ which are linearly independent vectors over $\mathbb{R}$, and let $\mathfrak{a}$ be an ideal in $R_\mathbf{0}$. In this note, we investigate the asymptotic behavior of the grade of the ideal $\mathfrak{a}$ on the homogeneous components $M_\mathbf{n}$ of a finitely generated $\mathbb{Z}^r$-graded R-module M, and show that the periodicity occurs in a cone.

1. INTRODUCTION

Let A be a commutative Noetherian ring and $\mathfrak{a}$ an ideal in A. Let R be a Noetherian $\mathbb{N}^r$-graded ring with $R_\mathbf{0} = A$, and let M be a finitely generated $\mathbb{Z}^r$-graded R-module. In this note, we study the asymptotic behavior of the numerical function grade$(\mathfrak{a}, M_\mathbf{n})$ the grade of the ideal $\mathfrak{a}$ on the homogeneous components $M_\mathbf{n}$ of M.

The first result in this setting is due to McAdam and Eakin [7]. They proved that if R is a standard $\mathbb{N}$-graded ring, then the set of primes $\mathrm{Ass}_A M_n$ is stable for all large n and hence, by using the technique due to Brodmann [3], we have that grade$(\mathfrak{a}, M_n)$ is constant for all large n. They also showed that Brodmann's result [2] about the asymptotic prime divisors of an ideal followed from their result as a direct consequence. Afterwards, a number of authors have extended these results to multigraded cases, especially in connection with the study of the asymptotic prime divisors and the analytic spread of ideals. In particular, West [8] extended McAdam-Eakin's results to multigraded cases. He proved that if R is a standard $\mathbb{N}^r$-graded ring, then the set of primes $\mathrm{Ass}_A M_\mathbf{n}$ is eventually stable and hence grade$(\mathfrak{a}, M_\mathbf{n})$ is constant for all large $\mathbf{n}$. He also considered several interesting non-standard multigraded cases. For more general results in the standard graded cases, see [1, 5].

On the other hand, when R is a standard $\mathbb{N}$-graded ring with a local ring A, Herzog and Hibi [6] gave a direct proof of the stability of $\mathrm{depth}_A M_n$ by using the Hilbert polynomial of Koszul homology modules of M with respect to the maximal ideal of A, instead of the asymptotic stability of $\mathrm{Ass}_A M_n$. More recently, Colomé-Nin and Elias [4] investigated in a similar way the asymptotic behavior of the depth associated to graded modules over certain non-standard multigraded rings.

The purpose of this note is to give a common generalization to all of the results concerning the asymptotic grade stated above with a more direct approach. The result is the following:

Theorem 1.1. *Let R be a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$, where $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r \in \mathbb{N}^r$ are linearly independent vectors over $\mathbb{R}$, with $R_\mathbf{0} = A$. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} M_\mathbf{n}$ be a finitely generated $\mathbb{Z}^r$-graded R-module. Then, for any ideal $\mathfrak{a}$ in A, there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that, in the cone $C_\mathbf{k}$ with vertex $\mathbf{k}$ generated by $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$,* grade$(\mathfrak{a}, M_\mathbf{n})$ *is periodic with respect to $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$. Namely, the equality*

$$\operatorname{grade}(\mathfrak{a}, M_\mathbf{n}) = \operatorname{grade}(\mathfrak{a}, M_{\mathbf{n}+\mathbf{m}})$$

holds true for all $\mathbf{n} \in C_\mathbf{k}$ and all $\mathbf{m} \in \Gamma$, where Γ is the semigroup generated by $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$.

In the next section, we fix notation and give some facts about cones and graded modules. In section 3, we will give a proof of Theorem 1.1.

Throughout this note, A is a commutative Noetherian ring with identity. $\mathbb{N}$ (resp. $\mathbb{R}$) denotes the set of non-negative integers (resp. real numbers), and r is any fixed positive integer. Vectors will be always written by Bold-faced letters, e.g., $\mathbf{a}$, and they will be represented by row vectors, e.g., $\mathbf{a} = (a_1, a_2, \ldots, a_r)$.

2. Preliminaries

Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r \in \mathbb{N}^r$ be any fixed linearly independent vectors over $\mathbb{R}$. We denote by $\Gamma \subseteq \mathbb{N}^r$ the semigroup generated by $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$, i.e.,

$$\Gamma = \left\{ \sum_{i=1}^{r} c_i \mathbf{d}_i \mid c_i \in \mathbb{N} \right\}.$$

For any vector $\mathbf{k} \in \mathbb{N}^r$, let

$$C_\mathbf{k} := \left\{ \mathbf{k} + \sum_{i=1}^{r} c_i \mathbf{d}_i \mid c_i \in \mathbb{R}_{\geq 0} \right\} \cap \mathbb{N}^r$$

be the *cone with vertex* $\mathbf{k}$ *generated by* $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ and

$$\Delta_\mathbf{k} := \left\{ \mathbf{k} + \sum_{i=1}^{r} c_i \mathbf{d}_i \mid 0 \leq c_i < 1,\ c_i \in \mathbb{R} \right\} \cap \mathbb{N}^r$$

the *basic cell of* $C_\mathbf{k}$. Then it is easy to see that (i) $\Delta_\mathbf{k}$ is a finite subset of $C_\mathbf{k}$, (ii) for any $\mathbf{n} \in C_\mathbf{k}$, there is a unique expression $\mathbf{n} = \delta + \mathbf{m}$ with $\delta \in \Delta_\mathbf{k}$ and $\mathbf{m} \in \Gamma$, and hence (iii) $C_\mathbf{k} = \bigcup_{\delta \in \Delta_\mathbf{k}} (\delta + \Gamma)$. Moreover, we have the following:

Lemma 2.1. *For any vectors $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that*

$$\mathbf{k}' + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r) \in C_\mathbf{k} \text{ for all } \ell \geq \ell_0.$$

In particular, $C_\mathbf{k} \cap C_{\mathbf{k}'} \neq \emptyset$ and hence there exists a cone $C_{\mathbf{k}''}$ such that $C_{\mathbf{k}''} \subseteq C_\mathbf{k} \cap C_{\mathbf{k}'}$.

Proof. Consider the system

$$\mathbf{x}D = \mathbf{k} - \mathbf{k}', \tag{1}$$

where D is a square matrix of size r whose i-th row is $\mathbf{d}_i$. Since $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ are linearly independent, there is a unique solution $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{R}^r$ of the system (1). Let

$$\ell_0 := \max\{\lceil |a_i| \rceil \mid i = 1, 2, \ldots, r\},$$

where $\lceil * \rceil$ denotes the least integer $\geq *$. Take $\ell \geq \ell_0$ and put $c_i := \ell - a_i \in \mathbb{R}_{\geq 0}$. Then $\mathbf{a} = (\ell - c_1, \ell - c_2, \ldots, \ell - c_r)$ and we have

$$\mathbf{k}' + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r) = \mathbf{k} + c_1\mathbf{d}_1 + c_2\mathbf{d}_2 + \cdots + c_r\mathbf{d}_r \in C_{\mathbf{k}}.$$

The last assertions follow from the fact that the above vector is in $C_{\mathbf{k}'}$. □

Let $R = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} R_{\mathbf{n}}$ be a Noetherian $\mathbb{N}^r$-graded ring with $R_{\mathbf{0}} = A$. Assume that R is generated in degrees $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$, i.e., $R = A[R_{\mathbf{d}_1}, R_{\mathbf{d}_2}, \ldots, R_{\mathbf{d}_r}]$. Let $R_{++} = (R_{\mathbf{d}_1} R_{\mathbf{d}_2} \cdots R_{\mathbf{d}_r})R$ be the irrelevant ideal of R. For a finitely generated $\mathbb{Z}^r$-graded R-module $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$, we define the *homogeneous support of M* as

$$\operatorname{Supp}_{++} M := \{P \in \operatorname{Spec} R \mid P \text{ is a graded ideal}, M_P \neq (0), \text{and } R_{++} \not\subseteq P\}.$$

For any vector $\delta \in \mathbb{N}^r$, we set $M^{(\delta+\Gamma)} = \bigoplus_{\mathbf{m} \in \Gamma} M_{\delta+\mathbf{m}}$, which is a graded submodule of M.

Lemma 2.2. *Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded R-module. Then the following statements are equivalent:*

(1) $\operatorname{Supp}_{++} M = \emptyset$;
(2) *there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$.*

Proof. Suppose $\operatorname{Supp}_{++} M = \emptyset$. Let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. Then we claim the following:

Claim. For any $\delta \in \Delta_{\mathbf{k}_0}$, there exists $\mathbf{k} = \mathbf{k}(\delta) \in \mathbb{N}^r$ such that $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$.

Let $\delta \in \Delta_{\mathbf{k}_0}$. The assertion is clear if $M^{(\delta+\Gamma)} = (0)$. Assume $M^{(\delta+\Gamma)} \neq (0)$. We write

$$M^{(\delta+\Gamma)} = Rm_1 + Rm_2 + \cdots + Rm_t,$$

where $m_i \in M_{\mathbf{k}_i}$ and $\mathbf{k}_i \in \delta + \Gamma$. Since $\operatorname{Supp}_{++} M = \emptyset$, $\operatorname{Supp}_{++} M^{(\delta+\Gamma)} = \emptyset$ so that $R_{++} \subseteq \sqrt{\operatorname{Ann}_R(M^{(\delta+\Gamma)})}$. Therefore there exists an integer $\ell \geq 0$ such that $R_{++}^{\ell} \cdot m_i = (0)$ for all $i = 1, 2, \ldots, t$. This implies that

$$[Rm_i]_{\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r)} = (0)$$

for all $i = 1, 2, \ldots, t$. Let $\mathbf{l}_i := \mathbf{k}_i + \ell(\mathbf{d}_1 + \cdots + \mathbf{d}_r) \in \delta + \Gamma$. Then $[Rm_i]_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{l}_i} \cap (\delta + \Gamma)$. Hence, by taking a vector $\mathbf{k} = \mathbf{k}(\delta) \in \mathbb{N}^r$ such that

$$C_{\mathbf{k}} \subseteq C_{\mathbf{l}_1} \cap C_{\mathbf{l}_2} \cap \cdots \cap C_{\mathbf{l}_t},$$

we have that $M_{\mathbf{n}} = [M^{(\delta+\Gamma)}]_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$. This completes the proof of Claim.

Now, let $\mathbf{k} \in \mathbb{N}^r$ be a vector such that

$$C_{\mathbf{k}} \subseteq C_{\mathbf{k}_0} \cap \left[\bigcap_{\delta \in \Delta_{\mathbf{k}_0}} C_{\mathbf{k}(\delta)} \right],$$

where $\mathbf{k}(\delta)$ is the vector in the Claim. Then $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$. Indeed, since $\mathbf{n} \in C_{\mathbf{k}_0}$, there is a unique expression $\mathbf{n} = \delta + \mathbf{m}$ for some $\delta \in \Delta_{\mathbf{k}_0}$ and $\mathbf{m} \in \Gamma$. Hence $\mathbf{n} \in C_{\mathbf{k}(\delta)} \cap (\delta + \Gamma)$ so that $M_{\mathbf{n}} = (0)$ by Claim.

We prove the other implication. Suppose that there exists $\mathbf{k} \in \mathbb{N}^r$ such that $M_{\mathbf{n}} = (0)$ for all $\mathbf{n} \in C_{\mathbf{k}}$. We write $M = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M_{\mathbf{k}_i}$. Then it is enough to show that for any $i = 1, 2, \ldots, t$, there exists an integer $\ell \geq 0$ such that $R_{++}^{\ell} \cdot m_i = (0)$. For the vectors $\mathbf{k}, \mathbf{k}_i \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that

$$\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r) \in C_{\mathbf{k}} \text{ for all } \ell \geq \ell_0$$

by Lemma 2.1. Thus

$$[Rm_i]_{\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r)} \subseteq M_{\mathbf{k}_i + \ell(\mathbf{d}_1 + \mathbf{d}_2 + \cdots + \mathbf{d}_r)} = (0)$$

and hence $R_{++}^{\ell} \cdot m_i = (0)$. □

Lemma 2.3. *Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded R-module and let $\mathbf{k}_0 \in \mathbb{N}^r$ and $\delta \in \Delta_{\mathbf{k}_0}$. Then there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that*

$$M_{\mathbf{n}} \neq (0) \textit{ for all } \mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma),$$

if $\operatorname{Supp}_{++} M^{(\delta+\Gamma)} \neq \emptyset$.

Proof. Suppose $\operatorname{Supp}_{++} M^{(\delta+\Gamma)} \neq \emptyset$. Assume the contrary, so that for any vector $\mathbf{k} \in \mathbb{N}^r$, there exists $\mathbf{n} \in C_{\mathbf{k}} \cap (\delta + \Gamma)$ such that $M_{\mathbf{n}} = (0)$. Write

$$M^{(\delta+\Gamma)} = Rm_1 + Rm_2 + \cdots + Rm_t,$$

where $m_i \in M_{\mathbf{k}_i}$ and $\mathbf{k}_i \in \delta + \Gamma$. Then, for each $\mathbf{k}_i$, there exists $\mathbf{n}_i \in C_{\mathbf{k}_i} \cap (\delta + \Gamma)$ such that $M_{\mathbf{n}_i} = (0)$ by the assumption. Therefore $[Rm_i]_{\mathbf{n}_i} = (0)$ and hence

$$[Rm_i]_{\mathbf{n}} = (0) \text{ for all } \mathbf{n} \in C_{\mathbf{n}_i}.$$

By taking a cone $C_{\mathbf{k}}$ such that $C_{\mathbf{k}} \subseteq C_{\mathbf{n}_1} \cap C_{\mathbf{n}_2} \cap \cdots \cap C_{\mathbf{n}_t}$, we have

$$[M^{(\delta+\Gamma)}]_{\mathbf{n}} = (0) \text{ for all } \mathbf{n} \in C_{\mathbf{k}},$$

which implies $\operatorname{Supp}_{++} M^{(\delta+\Gamma)} = \emptyset$ by Lemma 2.2. This is a contradiction. □

3. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. Recall that R is a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$, where $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r \in \mathbb{N}^r$ are linearly independent vectors over $\mathbb{R}$, with $R_\mathbf{0} = A$. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} M_\mathbf{n}$ be a finitely generated $\mathbb{Z}^r$-graded R-module and let $\mathfrak{a}$ be an ideal in A.

Proof of Theorem 1.1. Let $\mathfrak{a} = (a_1, a_2, \ldots, a_p)A$ and let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. Then we claim the following:

Claim. For any $\delta \in \Delta_{\mathbf{k}_0}$, there exist a vector $\mathbf{k} = \mathbf{k}(\delta) \in \mathbb{N}^r$ and a constant $c = c(\delta) \in \mathbb{N} \cup \{\infty\}$ such that $\operatorname{grade}(\mathfrak{a}, M_\mathbf{n}) = c$ for all $\mathbf{n} \in C_\mathbf{k} \cap (\delta + \Gamma)$.

Let $\delta \in \Delta_{\mathbf{k}_0}$ and $L := M/\mathfrak{a}M$. If $\operatorname{Supp}_{++} L^{(\delta+\Gamma)} = \emptyset$, then there exists $\mathbf{k} \in \mathbb{N}^r$ such that $L_\mathbf{n} = (0)$ for all $\mathbf{n} \in C_\mathbf{k} \cap (\delta + \Gamma)$ by Lemma 2.2. Therefore $M_\mathbf{n} = \mathfrak{a}M_\mathbf{n}$ so that $\operatorname{grade}(\mathfrak{a}, M_\mathbf{n}) = \infty$ for all $\mathbf{n} \in C_\mathbf{k} \cap (\delta + \Gamma)$. Suppose $\operatorname{Supp}_{++} L^{(\delta+\Gamma)} \neq \emptyset$. By Lemma 2.3, there exists $\mathbf{k}_1 \in \mathbb{N}^r$ such that for any $\mathbf{n} \in C_{\mathbf{k}_1} \cap (\delta+\Gamma)$, $L_\mathbf{n} \neq (0)$ so that $M_\mathbf{n} \neq \mathfrak{a}M_\mathbf{n}$. Thus, by the grade sensitivity of the Koszul complex, we have that for any $\mathbf{n} \in C_{\mathbf{k}_1} \cap (\delta + \Gamma)$,

$$\operatorname{grade}(\mathfrak{a}, M_\mathbf{n}) = p - \max\{i \mid H_i(\underline{a}; M_\mathbf{n}) \neq (0)\}$$

where $H_i(\underline{a}; *)$ denotes the i-th Koszul homology module of $*$ with respect to the sequence $\underline{a} = a_1, a_2, \ldots, a_p$. Let

$$q := \max\left\{i \mid \operatorname{Supp}_{++}\left(H_i\left(\underline{a}; M^{(\delta+\Gamma)}\right)\right) \neq \emptyset\right\}.$$

For any $i > q$, since $\operatorname{Supp}_{++}\left(H_i\left(\underline{a}; M^{(\delta+\Gamma)}\right)\right) = \emptyset$, there exists $\mathbf{l}_i \in \mathbb{N}^r$ such that

$$H_i(\underline{a}; M_\mathbf{n}) = (0) \text{ for all } \mathbf{n} \in C_{\mathbf{l}_i} \cap (\delta + \Gamma)$$

by Lemma 2.2. On the other hand, since $\operatorname{Supp}_{++}\left(H_q(\underline{a}; M^{(\delta+\Gamma)})\right) \neq \emptyset$, there exists $\mathbf{l}_q \in \mathbb{N}^r$ such that

$$H_q(\underline{a}; M_\mathbf{n}) \neq (0) \text{ for all } \mathbf{n} \in C_{\mathbf{l}_q} \cap (\delta + \Gamma)$$

by Lemma 2.3. Thus, by taking a cone $C_{\mathbf{k}_2}$ such that

$$C_{\mathbf{k}_2} \subseteq C_{\mathbf{l}_q} \cap C_{\mathbf{l}_{q+1}} \cap \cdots \cap C_{\mathbf{l}_p},$$

we have that for all $\mathbf{n} \in C_{\mathbf{k}_2} \cap (\delta + \Gamma)$,

$$H_q(\underline{a}; M_\mathbf{n}) \neq (0), \text{ and } H_i(\underline{a}; M_\mathbf{n}) = (0) \text{ if } i > q.$$

By taking a cone $C_\mathbf{k}$ such that $C_\mathbf{k} \subseteq C_{\mathbf{k}_1} \cap C_{\mathbf{k}_2}$, we get that

$$\operatorname{grade}(\mathfrak{a}, M_\mathbf{n}) = p - q \text{ for all } \mathbf{n} \in C_\mathbf{k} \cap (\delta + \Gamma).$$

This completes the proof of Claim.

Now let us take a cone $C_\mathbf{k}$ such that

$$C_\mathbf{k} \subseteq C_{\mathbf{k}_0} \cap \left[\bigcap_{\delta\in\Delta_{\mathbf{k}_0}} C_{\mathbf{k}(\delta)}\right],$$

where $\mathbf{k}(\delta)$ is the vector in the Claim. Then we have the equality

$$\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}) = \mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}+\mathbf{m}})$$

for all $\mathbf{n} \in C_{\mathbf{k}}$ and all $\mathbf{m} \in \Gamma$. Indeed, since $\mathbf{n} \in C_{\mathbf{k}_0}$, there is a unique $\delta \in \Delta_{\mathbf{k}_0}$ such that $\mathbf{n} \in \delta + \Gamma$. Thus $\mathbf{n}, \mathbf{n} + \mathbf{m} \in C_{\mathbf{k}(\delta)} \cap (\delta + \Gamma)$ and hence $\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}) = \mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}+\mathbf{m}})$ by Claim. □

If we take each $\mathbf{d}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ to be the i-th standard basis element of $\mathbb{N}^r$, then we can readily get the known results in the standard graded cases [6, 7, 8]. Moreover, as a direct consequence, we have the stability of the grade in the special non-standard graded cases considered in [8].

Corollary 3.1. *Let $R, M, \mathfrak{a}$ be the same as in Theorem 1.1. Assume that each vector $\mathbf{d}_i$ has the form $\mathbf{d}_i = (*, \ldots, *, 1, 0, \ldots, 0)$. Then there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that, in the cone $C_{\mathbf{k}}$, $\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}})$ is constant. Namely, the equality*

$$\mathrm{grade}(\mathfrak{a}, M_{\mathbf{n}}) = \mathrm{grade}(\mathfrak{a}, M_{\mathbf{k}})$$

holds true for all $\mathbf{n} \in C_{\mathbf{k}}$.

Proof. Let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. Then $\Delta_{\mathbf{k}_0} = \{\mathbf{k}_0\}$ because of the form of the vectors $\mathbf{d}_i$'s. Thus we have the assertion as a direct consequence of Theorem 1.1. □

Colomé-Nin and Elias studied in [4] the asymptotic behavior of $\mathrm{depth}_A\, M_{\mathbf{n}}$ in the case where each $\mathbf{d}_i$ has the form $(*, \ldots, *, \lambda_i, 0, \ldots, 0)$ with $\lambda_i \neq 0$. This case is also a special case of Theorem 1.1.

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ON GORENSTEIN INJECTIVITY OF TOP LOCAL COHOMOLOGY MODULES

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ABSTRACT. R. Sazeedeh showed that top local cohomology modules are Gorenstein injective in a Gorenstein local ring with at most two dimension. In this paper, it is proved that the condition of dimension in his result cannot be relaxed and the conclusion in his result holds for complete local hypersurface rings with arbitrary dimension.

INTRODUCTION

Through this paper, all rings are commutative noetherian ring and all modules are unitary. All maximal Cohen-Macaulay modules considered here will be finitely generated.

In [8], R. Sazeedeh showed a following result: if R is a Gorenstein local ring with at most two dimension, then the top local cohomology module $H_J^{\dim R}(R)$ is a Gorenstein injective R-module for any ideal J of R. Therefore it is natural to ask whether the conclusion in his result holds for a Gorenstein local rings with arbitrary dimension. But it seems difficult to decide whether modules are Gorenstein injective from the definition immediately. Thus, to study the above problem, we shall try to find a practical way of concluding that top local cohomology modules are Gorenstein injective.

Our strategy is the following. P. Schenzel proved the existence of a monomorphism from the Matlis dual module of the top local cohomology module to the canonical module in [11]. R. Takahashi, Y. Yoshino and the author introduced a notion of generalized local cohomology modules associated to a pair of ideals and showed that the image of above monomorphism is isomorphic to the generalized local cohomology module in [12]. By using these results, we shall give an example of non-Gorenstein injective top local cohomology module over 3-dimensional Gorenstein local ring and show that the conclusion in R. Sazeedeh's result is valid for complete local hypersurface rings with arbitrary dimension.

1. PRELIMINARIES

Let us recall some definitions of modules. J. Xu introduced notions of strongly cotorsion modules and strongly torsion free modules in [13].

E. E. Enochs and O. M. G. Jenda introduced the notion of Gorenstein injective modules in [5].

Definition 1.1. (1) An R-module M is called strongly cotorsion if $\mathrm{Ext}^1_R(X, M) = 0$ for all R-modules X of finite flat dimension.
(2) An R-module M is called strongly torsion free if $\mathrm{Tor}^R_1(X, M) = 0$ for all R-modules X of finite flat dimension.
(3) An R-module M is called Gorenstein injective if there is an exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective modules such that $M = \mathrm{Ker}(E^0 \to E^1)$ and such that $\mathrm{Hom}_R(E, -)$ leaves the above sequence exact whenever E is injective R-module.

Remark 1.2. (1) Any injective module is a strongly cotorsion module and a Gorenstein injective module.
(2) In a Gorenstein local ring, strongly cotorsion modules are precisely Gorenstein injective modules.
(3) In a Cohen-Macaulay local ring, maximal Cohen-Macaulay modules are precisely strongly torsion free finitely generated modules. (See [2, 3, 6].)

The following theorem was proved by R. Sazeedeh. We shall refer to this theorem as R. Sazeedeh's Theorem.

Theorem 1.3. [8, Theorem 2.6], [10, Theorem 2.15] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and M be a maximal Cohen-Macaulay R-module. If $d \leq 2$ and J is a nonzero ideal of R, then $H^d_J(M)$ is strongly cotorsion. In particular, if R is Gorenstein, then $H^d_J(R)$ is Gorenstein injective.*

2. Characterization of Gorenstein ring

In this section, we shall give characterization of Gorenstein ring by Gorenstein injectivity of local cohomology module.

Throughout the rest of this paper, let $(R, \mathfrak{m})$ be a local ring with a maximal ideal $\mathfrak{m}$ and the injective hull of $R/\mathfrak{m}$ is denoted by $E_R(R/\mathfrak{m})$, the functor $\mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$ is denoted by $(-)^\vee$ and the canonical module of R-module M is denoted by K_M.

Let us recall the definition of local cohomology functors associated to a pair of ideals which defined in [12] as the generalization of local cohomology functors with a closed support.

Definition 2.1. Let I and J be ideals of R. We set

$$W(I, J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some integer } n \gg 0\}.$$

For an R-module M, $\Gamma_{I,J}(M)$ denotes a submodule of M consisting of all elements of M with support in $W(I, J)$, that is

$$\Gamma_{I,J}(M) = \{x \in M \mid \operatorname{Supp}(Rx) \subseteq W(I, J)\}.$$

The left exact functor $\Gamma_{I,J}$ is called (I, J)-torsion functor.

Remark 2.2. Let $(R, \mathfrak{m})$ be a local ring and J be an ideal of R. For a prime ideal $\mathfrak{p}$ of R, it holds $\mathfrak{p} \in W(\mathfrak{m}, J)$ if and only if $\mathfrak{p} + J$ is $\mathfrak{m}$-primary ideal.

First of all, we shall show a following proposition which is the base of this paper.

Proposition 2.3. *Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring, J be an ideal of R and M be a finitely generated R-module of dimension t. Then the following conditions are equivalent:*

(1) *$H^t_J(M)$ is a strongly cotorsion R-module;*

(2) *$\Gamma_{\mathfrak{m},J}(K_M)$ is a maximal Cohen-Macaulay R-module or zero.*

Proof. Since $H^t_J(M)$ is an Artinian R-module by [1, 7.1.6 Theorem], we have $H^t_J(M) \cong H^t_J(M)^{\vee\vee}$. By [9, Lemma 2.1], it holds that $H^t_J(M)$ is strongly cotorsion if and only if $H^t_J(M)^\vee$ is strongly torsion free. Here, we note that $H^t_J(M)^\vee \cong \Gamma_{\mathfrak{m},J}(K_M)$ by [12, Theorem 5.11] and this is a finitely generated R-module. Therefore, it holds that $H^t_J(M)^\vee$ is strongly torsion free if and only if $\Gamma_{\mathfrak{m},J}(K_M)$ is a maximal Cohen-Macaulay R-module. (See Remark 1.2 (3).) □

In particular, it holds the following result which has already seen in [10].

Corollary 2.4. *Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring of dimension d. Then $H^d_{\mathfrak{m}}(R)$ is a strongly cotorsion R-module.*

Remark 2.5. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, J be an ideal of R and M be a finitely generated R-module of dimension t. We denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R.

(1) Since $H^t_J(M)$ is Artinian R-module, it also has an $\hat{R}$-module structure. In the proofs of [10, Theorem 2.4, Theorem 2.15], R. Sazeedeh showed that if the top local cohomology module $H^t_J(M)$ is a strongly cotorsion as $\hat{R}$-module then it is a strongly cotorsion as R-module.

(2) If R is complete and $H^t_J(M)$ is a non-zero strongly cotorsion R-module, then it holds

$$\dim R = \dim \Gamma_{\mathfrak{m},J}(K_M) \leq \dim K_M = \dim M$$

by Proposition 2.3. Thus, if $\dim M < \dim R$, then the non-zero top local cohomology module $H^{\dim M}_J(M)$ is not a strongly cotorsion R-module.

If R is a Gorenstein local ring, then strongly cotorsion R-modules are precisely Gorenstein injective R-modules. Therefore it is natural to ask whether $H^d_{\mathfrak{m}}(R)$ is always a Gorenstein injective R-module and the following assertion holds.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring of dimension d. Then the following conditions are equivalent:*

(1) *R is Gorenstein;*

(2) *$H^d_{\mathfrak{m}}(R)$ is an injective R-module;*

(3) *$H^d_{\mathfrak{m}}(R)$ is a Gorenstein injective R-module.*

Proof. The implication (1) $\Leftrightarrow$ (2) is well-known and (2) $\Rightarrow$ (3) are clear. Therefore we have only to prove (3) $\Rightarrow$ (2).

We assume that $H^d_{\mathfrak{m}}(R)$ is a Gorenstein injective R-module. Since $H^d_{\mathfrak{m}}(R)$ is Artinian, $H^d_{\mathfrak{m}}(R)^\vee$ is a finitely generated Gorenstein projective R-module by [4, Theorem 4.8]. We note that injective dimension of $H^d_{\mathfrak{m}}(R)^\vee$ is d. It follows from [7, Theorem 2.2] that $\operatorname{pd} H^d_{\mathfrak{m}}(R)^\vee = \operatorname{Gpd} H^d_{\mathfrak{m}}(R)^\vee = 0$ where pd (resp. Gpd) is projective dimension (resp. Gorenstein projective dimension). Thus $H^d_{\mathfrak{m}}(R)^\vee \cong R^n$ for some integer n, so it holds $H^d_{\mathfrak{m}}(R) \cong E(R/\mathfrak{m})^n$. Hence $H^d_{\mathfrak{m}}(R)$ is injective. □

By Corollary 2.4 and Theorem 2.6, we can see the following assertion holds.

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring. Then the following conditions are equivalent:*

(1) *R is Gorenstein;*

(2) *Strongly cotorsion modules are precisely Gorenstein injective modules.*

3. The Gorenstein injectivity of top local cohomology modules over Gorenstein rings

The main aim of this section is to give an example of non-Gorenstein top local cohomology module over 3-dimension Gorenstein ring.

We begin with the following lemma.

Lemma 3.1. *Let I and J be ideals of R such that $0 \neq \Gamma_{I,J}(R) \subsetneq R$ and $(0) = \cap_{i=1}^m \mathfrak{q}_i$ be an irredundant primary decomposition of zero ideal of R. Then one has*

$$\Gamma_{I,J}(R) = \bigcap_{\sqrt{\mathfrak{q}_i} \notin W(I,J)} \mathfrak{q}_i.$$

Proof. Let $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ for each i. We may assume that $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in W(I, J)$ and $\mathfrak{p}_{n+1}, \ldots, \mathfrak{p}_m \notin W(I, J)$. By the definition of $W(I, J)$, there exists an integer s such that $I^s \subseteq \Pi_{i=1}^n \mathfrak{q}_i + J \subseteq \cap_{i=1}^n \mathfrak{q}_i + J$. We note

$\cap_{i=1}^{n}\mathfrak{q}_i \subseteq \mathrm{Ann}_R(\cap_{i=n+1}^{m}\mathfrak{q}_i)$. Hence we have $I^s \subseteq \mathrm{Ann}_R(\cap_{i=n+1}^{m}\mathfrak{q}_i) + J$, and so $\cap_{i=n+1}^{m}\mathfrak{q}_i \subseteq \Gamma_{I,J}(R)$. Now, we consider a short exact sequence

$$0 \to \cap_{i=n+1}^{m}\mathfrak{q}_i \to R \to R/\cap_{i=n+1}^{m}\mathfrak{q}_i \to 0.$$

Applying the left exact functor $\Gamma_{I,J}$ to this sequence, we get an exact sequence

$$0 \to \Gamma_{I,J}(\cap_{i=n+1}^{m}\mathfrak{q}_i) \to \Gamma_{I,J}(R) \to \Gamma_{I,J}(R/\cap_{i=n+1}^{m}\mathfrak{q}_i).$$

Since $\cap_{i=1}^{m}\mathfrak{q}_i$ is irredundant, we see that

$$\mathrm{Ass}(R/\cap_{i=n+1}^{m}\mathfrak{q}_i) \cap W(I,J) = \{\mathfrak{p}_{n+1},\ldots,\mathfrak{p}_m\} \cap W(I,J) = \emptyset.$$

Hence it holds $\Gamma_{I,J}(R/\cap_{i=n+1}^{m}\mathfrak{q}_i) = 0$ by [12, Proposition 1.10]. Consequently, it holds $\Gamma_{I,J}(R) = \cap_{i=n+1}^{m}\mathfrak{q}_i$. □

Definition 3.2. Let I and J be ideals of R such that $0 \neq \Gamma_{I,J}(R) \subsetneq R$ and $(0) = \cap\mathfrak{q}_i$ be an irredundant primary decomposition of zero ideal of R with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$. Then we set

$$R(I,J) = R/\cap_{\mathfrak{p}_i \notin W(I,J)}\mathfrak{q}_i.$$

In the case $\Gamma_{I,J}(R) = 0$ (resp. $\Gamma_{I,J}(R) = R$), we set $R(I,J) = R$ (resp. $R(I,J) = 0$).

We recall that R is called almost Cohen-Macaulay ring if $\dim R - \operatorname{depth} R$ is at most one. Now we are able to give a practical way of concluding that top local cohomology modules are Gorenstein injective.

Proposition 3.3. *Let $(R,\mathfrak{m})$ be a complete Gorenstein local ring of dimension d and J be an ideal of R. Then the following conditions are equivalent:*

(1) *$H_J^d(R)$ is a Gorenstein injective R-module;*

(2) *$R(\mathfrak{m},J)$ is an almost Cohen-Macaulay ring.*

Proof. By Proposition 2.3, this is clear. □

If ring R has the form $S/(f)$ where S is a regular local ring and f is an element of S, R is called a local hypersurface ring defined by f in S. The following result is an application of Proposition 3.3.

Theorem 3.4. *Let $(R,\mathfrak{m})$ be a complete local hypersurface ring with dimension d. Then $H_J^d(R)$ is Gorenstein injective R-module for any ideal J of R.*

Proof. Let $R = S/(f)$ where S is regular local ring and $f \in S$. We shall show that $R(\mathfrak{m},J)$ is also complete local hypersurface ring.

Since regular local ring is a factorial domain, f can be expressed as a product of prime elements, that is

$$f = p_1^{n_1}p_2^{n_2}\cdots p_s^{n_s},$$

where p_i is a prime element in S and n_i is a positive integer for each i. We may also assume a condition $(*)$ which is $(p_i) \neq (p_j)$ if $i \neq j$. Then it holds

$$(f) = (p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s}) = (p_1^{n_1}) \cap (p_2^{n_2}) \cap \cdots \cap (p_s^{n_s}).$$

In fact, if $g \in \cap_{i=1}^s (p_i^{n_i})$, $p_i^{n_i}$ must appear in unique factorization of g for each i. By assumption $(*)$ and since the prime element of S is irreducible, this means that g have $p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s}$ in unique factorization of itself, so $g \in (p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s})$.

Now we assume $(p_1)R, \ldots, (p_l)R \in W(\mathfrak{m}, J)$ and $(p_{l+1})R, \ldots, (p_s)R \notin W(\mathfrak{m}, J)$. We note $(\cap_{i=l+1}^s (p_i^{n_i})R) \cap S = \cap_{i=l+1}^s (p_i^{n_i})$ and $\mathrm{Ass}(R) = \{(p_i)R\}_{i=1,\cdots,s}$, we have

$$R(\mathfrak{m}, J) = R/ \cap_{i=l+1}^s (p_i^{n_i})R = S/ \cap_{i=l+1}^s (p_i^{n_i}) = S/(p_{l+1}^{n_{l+1}} \cdots p_s^{n_s})$$

by same argument in above. So $R(\mathfrak{m}, J)$ is a local hypersurface ring. In particular, $R(\mathfrak{m}, J)$ is Cohen-Macaulay ring. Thus $H_J^d(R)$ is Gorenstein injective by Proposition 3.3. □

Finally, we give two examples of top local cohomology module: one is a Gorenstein injective but not injective, and the other is non-Gorenstein injective.

Example 3.5. Let k be a field and $\mathfrak{m}$ be a maximal ideal of local ring R.

(1) Let $R = k[[x, y, z, w]]/(xy)$ and $J = (y, z, w)R$. R is hypersurface and 3-dimensional complete Gorenstein local ring. We have a primary decomposition $(xy) = (x) \cap (y)$ and $\mathrm{Ass}(R) = \{(x)R, (y)R\}$. Therefore it holds $\mathrm{Ass}(R) \cap W(\mathfrak{m}, J) = \{(x)R\}$ and $R(\mathfrak{m}, J) = k[[x, y, z, w]]/(y)$. This is a Cohen-Macaulay ring. Consequently, $H_J^3(R)$ is Gorenstein injective R-module.

But this module is not injective. In fact, we suppose that $H_J^3(R)$ is an injective R-module. Since $H_J^3(R)$ is Artinian, $H_J^3(R)^\vee$ is finitely generated free R-module R^n for some integer n. Let an R-isomorphism

$$\varphi : R^n \xrightarrow{\sim} H_J^3(R)^\vee \xrightarrow{\sim} \Gamma_{\mathfrak{m},J}(R) = (y)R.$$

Then it holds that $0 \neq \varphi(x \cdot R^n) = x \cdot \varphi(R^n) = x \cdot (y)R = 0$. This is a contradiction.

Therefore the top local cohomology module

$$H^3_{(y,z,w)}(k[[x, y, z, w]]/(xy))$$

is a Gorenstein injective but not injective R-module.

(2) Let $R = k[[x, y, z, u, v, w]]/(xu, yv, zw)$ and $J = (x + y, y + z, u + v, v + w)R$. R is a 3-dimensional complete intersection local ring but not hypersurface.

A primary decomposition of (xu, yv, zw) is

$$(x, y, z) \cap (x, y, w) \cap (x, z, v) \cap (x, v, w)$$
$$\cap (y, z, u) \cap (y, u, w) \cap (z, u, v) \cap (u, v, w),$$

and

$$\mathrm{Ass}(R) = \begin{Bmatrix} (x,y,z)R, & (x,y,w)R, & (x,z,v)R, & (x,v,w)R, \\ (y,z,u)R, & (y,u,w)R, & (z,u,v)R, & (u,v,w)R \end{Bmatrix}.$$

$(x,y,z)R$ and $(u,v,w)R$ are two associated prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p}+J$ is not $\mathfrak{m}$-primary ideal. Thus it holds $\mathrm{Ass}(R)\cap W(\mathfrak{m},J) = \mathrm{Ass}(R)\setminus\{(x,y,z)R,(u,v,w)R\}$, and so

$$R(\mathfrak{m},J) = k[[x,y,z,u,v,w]]/(x,y,z)\cap(u,v,w).$$

But $\dim R(\mathfrak{m},J) = 3$ and $\operatorname{depth} R(\mathfrak{m},J) = 1$, so $R(\mathfrak{m},J)$ is not almost Cohen-Macaulay ring. Therefore it follows from Proposition 3.3 that the top local cohomology module

$$H^3_{(x+y,y+z,u+v,v+w)}(k[[x,y,z,u,v,w]]/(xu,yv,zw))$$

is not Gorenstein injective R-module.

Thus R. Sazeedeh's Theorem for arbitrary dimension does not hold.

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Binomial edge ideals of complete r-partite graphs

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Abstract

In this paper, we prove two results about the binomial edge ideal J_G of a complete r-partite graph $G = K_{a_1,\dots,a_r}$: (1) the t^{th} symbolic and ordinary powers of J_G coincide for all $t \in \mathbb{N}$, (2) the quotient ring S/J_G is F-pure.

1. Introduction

Binomial edge ideals are introduced as common generalizations of determinantal ideals and ideals generated by the adjacent 2-minors in a $2 \times n$ generic matrix by Herzog et.al. [4] and by the author [12] independently.

Let H be a simple graph (i.e., H is a graph which has neither multiple edges nor loops) on the vertex set $[n] = \{1, 2, \dots, n\}$, $S = k[x_1, \dots, x_n, y_1, \dots, y_n]$ a polynomial ring over a field k with $2n$ variables. The binomial edge ideal J_H of H is defined by

$$J_H := ([i,j] := x_i y_j - y_i x_j \mid \{i,j\} \in E),$$

where E is the edge set of H. We denote by X the matrix $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$, then $[i,j]$ is the determinant of the submatrix which is formed by i^{th} and j^{th} columns of X.

For a general graph H, it is too difficult to determine some properties of the binomial edge ideal J_H. So, in this paper, we deal with only complete r-partite graphs.

Let $G = (V = [n], E)$ be a simple graph. If V is a disjoint union $\coprod_{1 \le l \le r} V_l$ of r subsets, (V_l is called the l^{th} part, and we set $a_\ell = \sharp V_\ell$), and if

$$E = \{\{a,b\} \mid a \text{ and } b \text{ is not contained in the same part}\},$$

then G is called a *complete r-partite graph*, denoted by $K_{a_1,a_2,\dots,a_r}$. Reordering vertices if necessary, we can assume that

$$a_1 \le a_2 \le \ldots \le a_r$$

and

$$V_l = \left\{ 1 + \sum_{m=1}^{l-1} a_m,\ 2 + \sum_{m=1}^{l-1} a_m,\ \dots,\ \sum_{m=1}^{l} a_m \right\}.$$

We investigate two topics in this paper.

One of them is symbolic power, and we discuss in Section 2. For an ideal I of a noetherian ring A, the t^{th} symbolic power $I^{(t)}$ of I is the intersection of the minimal components of I^t. So we have $I^t \subset I^{(t)}$, the converse does not hold in general.

Many researchers gave conditions to characterize when the equations hold. Hochster [5] proved that when I is a prime ideal such that A_I is regular, $I^{(t)} = I^t$ for all $t \in \mathbb{N}$ if and only if the associated graded ring $\mathrm{Gr}_A(I) = \bigoplus_{m \ge 0} I^m / I^{m+1}$ is an integral domain. This

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theorem was generalized by Li–Swanson [9] to radical ideals and to rings with zero divisors. Huckaba–Huneke [8] and Nishida [11] gave conditions to hold the equations $I^{(t)} = I^t$ by the analytic spread of I. Simis–Vasconcelos–Villarreal [13] proved that when I is the edge ideal of a graph G, then $I^{(t)} = I^t$ for all $t \in \mathbb{N}$ if and only if G is bipartite, i.e., G has no odd cycle. For the ideal generated by the maximal minors of a generic matrix, the symbolic and ordinary powers coincides, see [5] and [14]. In particular, if G is a complete graph, then the symbolic and ordinary powers of J_G coincide.

In this paper, for a complete r-partite graph $G = K_{a_1,a_2,\dots,a_r}$, we determine the minimal associated prime ideals of J_G explicitly, and prove that $J_G^{(t)} = J_G^t$ for any integer $t \geq 1$.

The other topic is the F-purity of the quotient ring S/J_G, and we discuss in Section 3. We say that a commutative ring R of prime characteristic $p > 0$ is F-pure if the Frobenius map F_R of R is pure as a ring homomorphism. The notion of F-purity is introduced by Hochster–Roberts in 1970s [6] [7], and it is corresponding to the notion of log-canonical singularity in characteristic zero. The global analogue of F-purity is Frobenius split, and the Kodaira vanishing theorem holds on Frobenius split schemes [10]. The determinantal rings are strongly F-regular, in particular, F-pure. So the quotient ring S/J_G is F-pure for a complete graph G. In this paper, we prove that S/J_G is F-pure for a complete r-partite graph G.

First, we recall the relations which play important roles in the discussion.

Proposition 1.1. *For indexes a, b, c and d, the following equations hold :*

(1, Plücker relation) $$[a,b][c,d] - [a,c][b,d] + [a,d][b,c] = 0$$

(2) $$x_a[b,c] - x_b[a,c] + x_c[a,b] = 0, \quad y_a[b,c] - y_b[a,c] + y_c[a,b] = 0.$$

2. symbolic powers

Let A be a noetherian ring, I an ideal of A. We set $W := A \setminus \bigcup_{P \in \operatorname{Min} I} P$, where $\operatorname{Min} I$ is the set of minimal prime ideals of I. Then we denote the contracted ideal $I^t A_W \cap A$ by $I^{(t)}$, and call it the t^{th} symbolic power of I. We also have

$$I^{(t)} = \{a \in A \mid sa \in I^t \text{ for some } s \in W\}.$$

Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ be the irredundant primary decomposition. We suppose that $\operatorname{Min} I = \{P_1, P_2, \dots, P_s\}$, where $P_i = \sqrt{Q_i}$ for $i = 1, 2, \dots, s$. Then it is well known the following :

Lemma 2.1. *Under the above assumption, we have*

(1) $$I^{(t)} = Q_1^{(t)} \cap Q_2^{(t)} \cap \cdots \cap Q_s^{(t)}$$

for any integer $t \geq 1$. □

In general, binomial edge ideals are radical ideals, and some ways to determine all associated prime ideals of them are known. In this paper, we use the method of Herzog et.al. to determine $\operatorname{Min} J_G$. We recall it below.

Let G be a simple connected graph on V, and J_G the binomial edge ideal of G. For a subset $U \subset V$, we denote by $G_1, G_2, \dots, G_{c(U)}$ the connected components of $G \setminus U$, where $c(U)$ means the number of components of $G \setminus U$. We also denote by $\tilde{G}_i$ the complete graph whose vertex set coincides with that of G_i. Then we put

$$P_U = (x_u, y_u \mid u \in U) + J_{\tilde{G}_1} + J_{\tilde{G}_2} + \cdots + J_{\tilde{G}_{c(U)}}.$$

Proposition 2.2. *By the above notation, the following hold :*
(1) *For any* $U \subset V$, P_U *is a prime ideal of* S.

(2)([4] Theorem 3.2) *We have the equation*

$$J_G = \bigcap_{U \subset V} P_U.$$

(3)([4] Corollary 3.9) *For $U \subset V$, P_U is a minimal prime ideal of J_G if and only if $U = \emptyset$, or $U \neq \emptyset$ and $c(U \setminus \{i\}) < c(U)$ for any $i \in U$.*

We determine the associate prime ideals of the binomial edge ideal J_G of a complete r-partite graph $G = K_{a_1,a_2,\ldots,a_r}$.

Lemma 2.3. *Let $G = K_{a_1,a_2,\ldots,a_r}$ be a complete r-partite graph with $a_1 \leq a_2 \leq \ldots \leq a_r$. We denote by V_l the l^{th} part of G and set $s := \sharp\{l \mid a_l = 1\}$. Then J_G has $r-s+1$ associated prime ideals. Moreover, the associated prime ideals of J_G are the ideal $I_2(X)$ generated the 2-minors of X and $P_l = (x_v, y_v \mid v \notin V_l)$ for $l = s+1, s+2, \ldots, r$.*

Proof. For a non-empty subset $U \subset V$, we have

$$c(U) = \begin{cases} \sharp(V_l \setminus U) & \text{if } V \setminus V_l \subset U \\ 1 & \text{otherwise} \end{cases}.$$

If $V \setminus V_l \not\subset U$ for any $l > s$, $c(U) = 1$ and the condition of Proposition 2.2 (3) does not hold. $c(U \setminus \{i\}) = 1 < c(U) = \sharp V_l$ for any $i \in U = V \setminus V_l$ with $l > s$, then $P_{V \setminus V_l}$ is an associated prime ideal of J_G. If $V \setminus V_l \subsetneq U$, $c(U \setminus \{i\}) > c(U)$ for some $i \in U \cap V_l$. Then J_G has $r - s + 1$ associated prime ideals $I_2(X)$ and $P_{V \setminus V_l}$ for $l = s+1, s+2, \ldots, r$.

If $U = V \setminus V_l$, then the graph $G \setminus U$ has no edges and $P_U = P_l = (x_v, y_v \mid v \notin V_l)$. ☐

Symbolic and ordinary powers of these prime ideals coincide. Moreover, we can characterized when an element of S is contained in their (symbolic) powers by valuations.

S is an algebra with straightening lows on the set of minors of X. In fact, we have the ASL structure by the order defined by

$$\begin{aligned}&[a_1, \ldots, a_u | b_1, \ldots, b_u] \leq [c_1, \ldots, c_v | d_1, \ldots, d_v] \\ &\iff u \geq v \text{ and } a_1 \leq c_1, \ldots, a_v \leq c_v, b_1 \leq d_1, \ldots, b_v \leq d_v,\end{aligned}$$

where $x_i = [1|i]$, $y_i = [2|i]$ and $[i,j] = [1,2|i,j]$. Then any element of S can be expressed as a linear combination of standard monomials uniquely.

The valuation γ is defined by $\gamma(\delta) = s - 1$ for a s-minor δ of X. For a formal monomial $\mu = \delta_1\delta_2\cdots\delta_p$, $\gamma(\mu) \geq t$ if and only if μ has at least t factors which are 2-minors.

Proposition 2.4 ([1], Theorem 10.4). *The t^{th} symbolic powers of $I_2(X)$ is generated by the formal monomials μ such that $\gamma(\mu) \geq t$, i.e., $I_2(X)^t = I_2(X)^{(t)}$*

We also have degrees $\deg^{(l)}$, $l = s+1, s+2, \ldots, r$, on S defined by

$$\deg^{(l)} x_v = \deg^{(l)} y_v = \begin{cases} 0 & \text{if } v \in V_l \\ 1 & \text{if } v \notin V_l \end{cases}.$$

Then, for a (formal) monomial μ, $\mu \in P_l^{(t)}$ if and only if $\mu \in P_l^t$ if and only if $\deg^{(l)} \mu \geq t$.

Theorem 2.5. *Let $G = K_{a_1,a_2,\ldots,a_r}$ be a complete r-partite graph, J_G the binomial edge ideal of G. For an integer $t \geq 1$, we have that $J_G^{(t)} = J_G^t$.*

Proof. In this proof, we say that a 2-minor $[i,j]$ is *bad* if i and j is contained in the same part V_l, i.e., $[i,j]$ is not contained in J_G. At this time, we also say that $[i,j]$ is a bad 2-minor of *type l*.

It is obvious that $J_G^t \subset J_G^{(t)}$. To prove the converse, it is sufficient to prove that any standard monomial μ with $\gamma(\mu) \geq t$ and $\deg^{(l)} \mu \geq t$ for $l = s+1, s+2, \ldots, r$, is contained in J_G^t.

If μ has 2 bad factors $[b_1, b_2]$ and $[c_1, c_2]$ whose types are not the same, using the Plücker relation

$$[b_1, b_2][c_1, c_2] - [b_1, c_1][b_2, c_2] + [b_1, c_2][b_2, c_1] = 0,$$

we have an expression of μ as a linear combination of formal monomials whose bad factors have only one type. So we have to prove the following claim.

Claim. *Let l be an integer with $s < l \leq r$. Assume that a formal monomial*

$$\mu = N \cdot \left(\prod_{i=1}^{\alpha}[a_i, b_i]\right) \cdot \left(\prod_{i=1}^{\beta}[c_i, d_i^{(1)}]\right) \cdot \left(\prod_{i=1}^{\gamma}[d_i^{(2)}, d_i^{(3)}]\right)$$

satisfies the following :
(1) N is an ordinary monomial, i.e., N is a product of some indeterminates,
(2) $d_i^{(j)} \in V_l$ for each i and j, and $a_i, b_i, c_i \notin V_l$ for each i,
(3) a_i and b_i are not contained in the same part for any i,
(4) $\gamma(\mu) \geq t$ and $\deg^{(l)} \mu \geq t$.
Then $\mu \in J_G^t$.

We prove the claim by reverse induction on $\alpha + \beta$. If $\alpha + \beta \geq t$, then $\mu \in J_G^t$ and we have nothing to prove.

We assume that $\alpha + \beta < t$, then $\gamma > 0$ since $\gamma(\mu) = \alpha + \beta + \gamma \geq t$. If $\alpha > 0$, using the Plücker relation

$$[a_1, b_1][d_1^{(2)}, d_1^{(3)}] - [a_1, d_1^{(2)}][b_1, d_1^{(3)}] + [a_1, d_1^{(3)}][b_1, d_1^{(2)}] = 0,$$

μ is a sum of similar formal monomials with α of them $= \alpha - 1$, β of them $= \beta + 2$ and γ of them $= \gamma - 1$.

If $\alpha = 0$, then $\deg^{(l)} N > 0$ since $\deg^{(l)} \mu = \deg^{(l)} N + \beta \geq t$ and $\beta < t$. So there is an indeterminate x_v or y_v with $v \notin V_l$ which divides N. We suppose that x_v divides N. Then we use the relation

$$x_v[d_1^{(2)}, d_1^{(3)}] - x_{d_1^{(2)}}[v, d_1^{(3)}] + x_{d_1^{(3)}}[v, d_1^{(2)}] = 0,$$

μ can be expressed as a sum of similar formal monomials with β of them $= \beta + 1$ and γ of them $= \gamma - 1$.

In both cases, μ is a sum of similar formal monomials with $\alpha + \beta$ of them $= \alpha + \beta + 1$, so we have that $\mu \in J_G^t$ by hypothesis of induction. □

3. F-purity

For a commutative ring A and $\phi : N \to M$ be an A-linear map. We say that ϕ is *pure* if $\phi \otimes_A 1_L : N \otimes_A L \to M \otimes_A L$ is injective for any A-module L. A pure map is injective, so N is called a pure submodule of M in this case. We say that ϕ is a split monomorphism if there is an A-linear map $\psi : M \to N$ such that $\psi\phi = 1_N$. A split monomorphism is pure, but the converse is not true in general. A pure map $\phi : N \to M$ is a split monomorphism if M/N is finitely presented, see [7],(5.2). A ring homomorphism $\phi : A \to A'$ is said to be pure (resp. split) if it is pure (resp. split) as an A-linear map.

For a commutative ring R of prime characteristic $p > 0$, the Frobenius map $F_R : R \to R$ is given by $r \mapsto r^p$, and it is a ring homomorphism. We say that R is *F-pure* (resp. F-finite) if F_R is pure (resp. finite) as a ring homomorphism. F-purity is defined by Hochster–Roberts [6] [7], and is corresponding to the notion of log-canonical singularity in characteristic 0 [15].

Let X be an $\mathbb{F}_p$-scheme. We say that X is Frobenius split if the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-linear map [10]. An F-finite ring R of characteristic $p > 0$, R is F-pure if and only if $\operatorname{Spec} R$ is Frobenius split. For an $\mathbb{N}$-graded F-finite noetherian ring R of prime characteristic $p > 0$, if R is F-pure, then $\operatorname{Proj} R$ is Frobenius split. On every Frobenius split projective non-singular varieties, the Kodaira vanishing theorem holds, see [10] Proposition 2.

From now on, we suppose that the base field k is a perfect field of characteristic $p > 0$. In this paper, we prove the following.

Theorem 3.1. *Let $G = K_{a_1,a_2,\ldots,a_r}$ be a complete r-partite graph, $S = k[x_v, y_v \mid v \in V]$ a polynomial ring over k and J_G the binomial edge ideal of G. Then S/J_G is F-pure.*

For the proof, we use the Fedder's criterion.

Proposition 3.2 (Fedder's criterion, [2] Theorem 1.12). *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$ and $R = S/I$. Then R is F-pure if and only if $(I^{[p]} : I) \not\subset \mathfrak{m}^{[p]}$, where $J^{[p]} = (x^p \mid x \in J)$ for an ideal J of S.*

Since S/J_G is an $\mathbb{N}$-graded k-algebra, it is enough to show that $(S/J_G)_{\mathfrak{m}}$ is F-pure, where $\mathfrak{m}$ is the unique homogeneous maximal ideal of S by Matijevic–Roberts type theorem, see [3] Corollary 5.4. In particular, taking completion, we can assume that $S = K[[x_v, y_v|v \in V]]$ is a formal power series ring. Let $\mathfrak{m}$ be the maximal ideal $(x_v, y_v|v \in V)$ generated by the indeterminates of S.

Proof of Theorem 3.1. It is sufficient to find an element $f \in (J_G^{[p]} : J_G)$ with $f \notin \mathfrak{m}^{[p]}$ by Proposition 3.2.

For a sequence $v_1, v_2, \ldots, v_s$ of integers, we put

$$ {}_y(v_1, v_2, \ldots, v_s)_x := (y_{v_1}[v_1, v_2][v_2, v_3] \cdots [v_{s-1}, v_s] x_{v_s})^{p-1}. $$

We prove the following claim :

Claim. ${}_y(1, 2, \ldots, n)_x \in (J_G^{[p]} : J_G) \setminus \mathfrak{m}^{[p]}$.

First, we think the lexicographic order on S with $x_1 > x_2 > \ldots > x_n > y_1 > y_2 > \ldots > y_n$. One has $\operatorname{in}_< {}_y(1, 2, \ldots, n)_x = (x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n)^{p-1}$ and the coefficient of the monomial is one, so ${}_y(1, 2, \ldots, n)_x \notin \mathfrak{m}^{[p]}$.

To prove ${}_y(1, 2, \ldots, n)_x \in (J_G^{[p]} : J_G)$, we show two formulae.

Formula 1. *If a and b are not contained in the same part, then*

$$ (\heartsuit) \qquad {}_y(v_1, \ldots, c, \underline{a, b}, d, \ldots, v_n)_x \equiv {}_y(v_1, \ldots, c, \underline{b, a}, d, \ldots, v_n)_x $$

modulo $J_G^{[p]}$.

Proof. Using the Plücker relation, we have

$$ \begin{aligned} [a, b]^{p-1}([c, a][b, d])^{p-1} &= [a, b]^{p-1}([c, b][a, d] - [c, d][a, b])^{p-1} \\ &= \sum_{s=0}^{p-1} \binom{p-1}{s} (-1)^s [c, b]^{p-s-1} [a, d]^{p-s-1} [c, d]^s [a, b]^{s+p-1}. \end{aligned} $$

One has $[a, b] \in J_G$ by assumption, then the terms with $s > 0$ are contained in $J_G^{[p]}$. In particular,

$$ \begin{aligned} \text{LHS of } (\heartsuit) &\equiv (y_{v_1}[v_1, v_2] \cdots [c, b][a, b][b, d] \cdots [v_{n-1}, v_n] x_{v_n})^{p-1} \\ &= (y_{v_1}[v_1, v_2] \cdots [c, b][b, a][b, d] \cdots [v_{n-1}, v_n] x_{v_n})^{p-1} \\ &= \text{RHS of } (\heartsuit) \end{aligned} $$

modulo $J_G^{[p]}$, since $[a, b]^{p-1} = [b, a]^{p-1}$ for any prime p. ☐

Formula 2. *If a and b are not contained in the same part, then*

$$(\diamondsuit)\qquad {}_y(\underline{a,b},c,\ldots,v_n)_x \equiv {}_y(\underline{b,a},c,\ldots,v_n)_x,$$
$$ {}_y(v_1,\ldots,c,\underline{a,b})_x \equiv {}_y(v_1,\ldots,c,\underline{b,a})_x$$

modulo $J_G^{[p]}$.

Proof. We prove only $(\diamondsuit)$. Using Proposition 1.1 (2), we have

$$\begin{aligned}[a,b]^{p-1}(y_a[b,c])^{p-1} &= [a,b]^{p-1}(y_b[a,c]-y_c[a,b]) \\ &= \sum_{s=1}^{p-1}\binom{p-1}{s}(-1)^s y_b^{p-s-1}y_c^s[a,c]^{p-s-1}[a,b]^{p+s-1}.\end{aligned}$$

It holds that $[a,b]\in J_G$ by assumption, the terms with $s>0$ are contained in $J_G^{[p]}$. So we have

$$\begin{aligned}\text{LHS of } (\diamondsuit) &\equiv (y_b[a,c][a,b]\cdots[v_{n-1},v_n]x_{v_n})^{p-1} \\ &= (y_b[b,a][a,c]\cdots[v_{n-1},v_n]x_{v_n})^{p-1} \\ &= \text{RHS of } (\diamondsuit)\end{aligned}$$

modulo $J_G^{[p]}$. □

By these formulae, we have the equivalences

$${}_y(1,2,\ldots,n)_x \equiv {}_y(\sigma(1),\sigma(2),\ldots,\sigma(n))_x$$

for any permutation σ of $[n]$ such that

$$\sigma(1+\sum_{m=1}^{l-1}a_m) < \sigma(2+\sum_{m=1}^{l-1}a_m) < \cdots < \sigma(\sum_{m=1}^{l}a_m)$$

for any l. So, for any $[a,b]\in J_G$, ${}_y(1,2,\ldots,n)_x$ is equivalent to a polynomial which can be divided by $[a,b]^{p-1}$. In particular, ${}_y(1,2,\ldots,n)_x\cdot[a,b]\in J_G^{[p]}$, that is, ${}_y(1,2,\ldots,n)_x\in(J_G^{[p]}:J_G)$. □

Example 3.3. Theorem 3.1 does not hold for a general graph H. For example, let H be a pentagon or a hexagon and $\operatorname{char} k=2$. Then the quotient ring S/J_H is not F-pure.

The author computed this example by Macaulay2.

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Sliding functor and polarization functor for multigraded modules and their application

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Most results in §§2,3,5 of this note are taken from [17] (and [11]), and those of §§4,6 are from [18].

1. Introduction

First, we introduce the convention used throughout this note. The i^{th} coordinate of $\mathbf{a} \in \mathbb{Z}^n$ is denote by a_i. However, for a vector represented by several letters, such as $\mathbf{a}_j \in \mathbb{Z}^n$, its i^{th} coordinate is denoted by $(\mathbf{a}_j)_i$. Let $\succeq$ be the order on $\mathbb{Z}^n$ defined by $\mathbf{a} \succeq \mathbf{b} \Leftrightarrow a_i \geq b_i$ for all i. Set $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{Z}^n$.

Let $S := \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. Clearly, S is a $\mathbb{Z}^n$-graded ring with $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} \Bbbk x^{\mathbf{a}}$, where $x^{\mathbf{a}}$ denotes the monomial $\prod_{i=1}^n x^{a_i}$. Let $\operatorname{mod}_{\mathbb{Z}^n} S$ denote the category of $\mathbb{Z}^n$-graded finitely generated S-modules. Recall that if $M, N \in \operatorname{mod}_{\mathbb{Z}^n} S$ then $\operatorname{Ext}_S^i(M, N) \in \operatorname{mod}_{\mathbb{Z}^n} S$ in the natural way.

For $i \in [n] := \{1, \ldots, n\}$ and $j \in \mathbb{Z}$, we define the *sliding functors* $(-)^{\triangleleft\langle i,j\rangle}$ and $(-)^{\triangleright\langle i,j\rangle}$, which are exact endofunctors of $\operatorname{mod}_{\mathbb{Z}^n} S$ with $S(-\mathbf{a})^{\triangleleft\langle i,j\rangle} = S(-\mathbf{a}')$ and $S(-\mathbf{a})^{\triangleright\langle i,j\rangle} = S(-\mathbf{a}'')$. Here, $a_k = a'_k = a''_k$ for all $k \neq i$ and

$$a'_i = \begin{cases} a_i + 1 & \text{if } a_i \geq j, \\ a_i & \text{if } a_i < j, \end{cases} \quad \text{and} \quad a''_i := \begin{cases} a_i & \text{if } a_i > j, \\ a_i - 1 & \text{if } a_i \leq j. \end{cases}$$

Sliding functors preserve the depth and dimension. Moreover, we have

$$\operatorname{Ext}_S^l(M^{\triangleleft\langle i,j\rangle}, S) \cong \operatorname{Ext}_S^l(M, S)^{\triangleright\langle i,-j\rangle} \quad \text{and} \quad \operatorname{Ext}_S^l(M^{\triangleright\langle i,j\rangle}, S) \cong \operatorname{Ext}_S^l(M, S)^{\triangleleft\langle i,-j\rangle}$$

for $M \in \operatorname{mod}_{\mathbb{Z}^n} S$.

The *Stanley depth* of $M \in \operatorname{mod}_{\mathbb{Z}^n} S$, which is denoted by $\operatorname{sdepth}_S M$, is a combinatorial invariant. In [13], it is conjectured that the inequality $\operatorname{sdepth}_S M \geq \operatorname{depth}_S M$ holds for all M (*Stanley's conjecture*). In Theorem 3.2, we will show that

$$\operatorname{sdepth}_S M = \operatorname{sdepth}_S M^{\triangleleft\langle i,j\rangle} = \operatorname{sdepth}_S M^{\triangleright\langle i,j\rangle}.$$

Using this result, the author and Okazaki ([11]) improved a result in [1], see Theorem 3.3 below.

Take $\mathbf{d} \in \mathbb{N}^n$ with $\mathbf{d} \succeq \mathbf{1}$, and set

$$\widetilde{S} := \Bbbk[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i].$$

Note that

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \leq i \leq n,\ 2 \leq j \leq d_i\} \subset \widetilde{S}$$

forms a regular sequence with $\widetilde{S}/(\Theta) \cong S$.

Definition 1.1. For a monomial ideal $I \subset S$, a *polarization* of I is a squarefree monomial ideal $J \subset \widetilde{S}$ satisfying the following conditions:

(i) $\widetilde{S}/J \otimes_{\widetilde{S}} \widetilde{S}/(\Theta) \cong S/I$,

(ii) Θ forms a $\widetilde{S}/J$-regular sequence.

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For a monomial $x^{\mathbf{a}}$ with $\mathbf{a} \preceq \mathbf{d}$, set

$$\mathsf{pol}(x^{\mathbf{a}}) := \prod_{1 \le i \le n} x_{i,1}x_{i,2}\cdots x_{i,a_i} \in \widetilde{S}.$$

Let $I = (x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_r}) \subset S$ be a monomial ideal with $\mathbf{a}_i \preceq \mathbf{d}$ for all i. Then it is well-known that

$$\mathsf{pol}(I) = (\,\mathsf{pol}(x^{\mathbf{a}_1}), \dots, \mathsf{pol}(x^{\mathbf{a}_r})\,)$$

gives a polarization of I, which is called the *standard polarization* of I (the explicit value of $\mathbf{d}$ is not important for the study in this note). While all monomial ideals have the standard polarizations, some have alternative ones.

Let $\mathrm{mod}_{\mathbf{d}} S$ be the category of *positively* $\mathbf{d}$*-determined modules* over S, and $\mathrm{Sq}\, S$ the category of *squarefree modules* over S (see Definition 5.1 below). Both of them are full subcategories of $\mathrm{mod}_{\mathbb{Z}^n} S$. For a monomial ideal I minimally generated by $x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_r}$, we have $I \in \mathrm{mod}_{\mathbf{d}} S$ if and only if $S/I \in \mathrm{mod}_{\mathbf{d}} S$ if and only if $\mathbf{a}_i \preceq \mathbf{d}$ for all i. Recall that squarefree modules are nothing other than positively $\mathbf{1}$-determined modules.

The standard polarization induces the exact functor $\mathsf{pol}_{\mathbf{d}} : \mathrm{mod}_{\mathbf{d}} S \to \mathrm{Sq}\, \widetilde{S}$. Here, for all $M \in \mathrm{mod}_{\mathbf{d}} S$, Θ forms a regular sequence over $\mathsf{pol}_{\mathbf{d}}(M) \in \mathrm{Sq}\, \widetilde{S}$ and $\mathsf{pol}_{\mathbf{d}}(M) \otimes_{\widetilde{S}} \widetilde{S}/(\Theta) \cong M$ holds. This idea has appeared in [4, 12] essentially, however we will give slightly different approach here. More precisely, we introduce the "reversed copy" $\mathsf{pol}^{\mathbf{d}}$ of $\mathsf{pol}_{\mathbf{d}}$. The relation between $\mathsf{pol}_{\mathbf{d}}$ and $\mathsf{pol}^{\mathbf{d}}$ is analogous to the relation between $(-)^{\triangleleft\langle i,j\rangle}$ and $(-)^{\triangleright\langle i,-j\rangle}$. For example, we have

$$\mathrm{Ext}^i_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M), \widetilde{S}(-\mathbf{1})) \cong \mathsf{pol}^{\mathbf{d}}(\mathrm{Ext}^i_S(M, S(-\mathbf{d}))).$$

This result corresponds to a result of Sbarra ([12, Corollary 4.10]). However the pair $\mathsf{pol}_{\mathbf{d}}$ and $\mathsf{pol}^{\mathbf{d}}$ is only implicit in [12], since a different convention is used there.

In the last section, we give an "alternative" polarization $\mathsf{b}\text{-}\mathsf{pol}(I)$ of a Borel fixed monomial ideal $I \subset S$. For example, $\mathsf{b}\text{-}\mathsf{pol}(x^2yz) = x_1x_2y_3z_4$, while $\mathsf{pol}(x^2yz) = x_1x_2y_1z_1$. If I is not Borel fixed, $\mathsf{b}\text{-}\mathsf{pol}(I)$ is not a polarization in general. Using the polarization $\mathsf{b}\text{-}\mathsf{pol}(I)$, we can refine results in [2, 9].

2. Sliding functors

For $M \in \mathrm{mod}_{\mathbb{Z}^n} S$ and $\mathbf{a} \in \mathbb{Z}^n$, $M(\mathbf{a})$ denotes the shifted module with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. We say $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ is $\mathbb{N}^n$-graded, if $M_{\mathbf{a}} = 0$ for all $\mathbf{a} \notin \mathbb{N}^n$. Let $\mathrm{mod}_{\mathbb{N}^n} S$ be the full subcategory of $\mathrm{mod}_{\mathbb{Z}^n} S$ consisting of $\mathbb{N}^n$-graded modules. As usual, for $M \in \mathrm{mod}_{\mathbb{Z}^n} S$ and $\mathbf{a} \in \mathbb{Z}^m$, we call $\beta_{i,\mathbf{a}}(M) := \dim_{\Bbbk}[\mathrm{Tor}^S_i(\Bbbk, M)]_{\mathbf{a}}$ the $(i, \mathbf{a})^{\text{th}}$ *Betti number* of M. We also treat the $\mathbb{Z}$-graded Betti numbers $\beta_{i,j}(M)$.

From an order preserving map $q : \mathbb{Z}^n \to \mathbb{Z}^n$ and $M \in \mathrm{mod}_{\mathbb{Z}^n} S$, Brun and Fløystad [3] constructed the new module $q^*M \in \mathrm{mod}_{\mathbb{Z}^n} S$ so that $(q^*M)_{\mathbf{a}} = M_{q(\mathbf{a})}$ and the multiplication map $(q^*M)_{\mathbf{a}} \ni y \longmapsto x^{\mathbf{b}}y \in (q^*M)_{\mathbf{a}+\mathbf{b}}$ is given by $M_{q(\mathbf{a})} \ni y \longmapsto x^{q(\mathbf{a}+\mathbf{b})-q(\mathbf{a})}y \in M_{q(\mathbf{a}+\mathbf{b})}$ for all $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbf{b} \in \mathbb{N}^n$. Similarly, for a morphism $f : M \to N$ in $\mathrm{mod}_{\mathbb{Z}^n} S$, we can define $q^*(f) : q^*(M) \to q^*(N)$. This construction gives the functor $q^* : \mathrm{mod}_{\mathbb{Z}^n} S \to \mathrm{mod}_{\mathbb{Z}^n} S$, which is clearly exact.

Let $i \in [n]$ and $j \in \mathbb{Z}$. Define the order preserving map $\sigma_{\langle i,j\rangle} : \mathbb{Z}^n \to \mathbb{Z}^n$ by $(\sigma_{\langle i,j\rangle}(\mathbf{a}))_k = a_k$ for all $k \neq i$ and

$$(\sigma_{\langle i,j\rangle}(\mathbf{a}))_i = \begin{cases} a_i - 1 & \text{if } a_i \geq j, \\ a_i & \text{if } a_i < j. \end{cases}$$

The functor $(\sigma_{\langle i,j\rangle})^* : \operatorname{mod}_{\mathbb{Z}^n} S \to \operatorname{mod}_{\mathbb{Z}^n} S$ is denoted by $(-)^{\triangleleft\langle i,j\rangle}$.

We also define the map $\tau_{\langle i,j\rangle} : \mathbb{Z}^n \to \mathbb{Z}^n$ by $\tau_{\langle i,j\rangle}(\mathbf{a})_k = a_k$ for all $k \neq i$ and

$$\tau_{\langle i,j\rangle}(\mathbf{a})_i = \begin{cases} a_i + 1 & \text{if } a_i \geq j, \\ a_i & \text{if } a_i < j. \end{cases}$$

If there is no danger of confusion, we simply denote $\sigma_{\langle i,j\rangle}$ and $\tau_{\langle i,j\rangle}$ by σ and τ respectively. We have $(M^{\triangleleft\langle i,j\rangle})_{\mathbf{a}} = M_{\sigma(\mathbf{a})}$ and $M_{\mathbf{a}} = (M^{\triangleleft\langle i,j\rangle})_{\tau(\mathbf{a})}$ for all $\mathbf{a} \in \mathbb{Z}^n$. However, if $a_i = j-1$ (in this case, $\sigma(\mathbf{a}) = \tau(\mathbf{a}) = \mathbf{a}$), then $M_{\mathbf{a}} = (M^{\triangleleft\langle i,j\rangle})_{\mathbf{a}} = (M^{\triangleleft\langle i,j\rangle})_{\mathbf{a}+\mathbf{e}_i}$ and the multiplication map $(M^{\triangleleft\langle i,j\rangle})_{\mathbf{a}} \ni y \longmapsto x_i y \in (M^{\triangleleft\langle i,j\rangle})_{\mathbf{a}+\mathbf{e}_i}$ is bijective. Here, $\mathbf{e}_i$ denotes the i^{th} unit vector of $\mathbb{Z}^n$. It is easy to see that $S(-\mathbf{a})^{\triangleleft\langle i,j\rangle} \cong S(-\tau(\mathbf{a}))$.

Let $I = (x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_r})$ be a monomial ideal. If $j < 0$, then $(S/I)^{\triangleleft\langle i,j\rangle} \cong (S/I)(-\mathbf{e}_i)$. So, **when we treat $(S/I)^{\triangleleft\langle i,j\rangle}$ or $I^{\triangleleft\langle i,j\rangle}$, we always assume that $j > 0$.** It is easy to see that $(S/I)^{\triangleleft\langle i,j\rangle} \cong S/I^{\triangleleft\langle i,j\rangle}$ and $I^{\triangleleft\langle i,j\rangle} = (x^{\tau(\mathbf{a}_1)}, \dots, x^{\tau(\mathbf{a}_r)})$.

Proposition 2.1 (c.f. [11, Proposition 5.2]). *For $M \in \operatorname{mod}_{\mathbb{Z}^n} S$, we have*

$$\beta_{l,\mathbf{a}}(M) = \beta_{l,\tau(\mathbf{a})}(M^{\triangleleft\langle i,j\rangle}) \text{ for all } l \in \mathbb{N} \text{ and } \mathbf{a} \in \mathbb{Z}^n,$$

$$\operatorname{depth}_S M^{\triangleleft\langle i,j\rangle} = \operatorname{depth}_S M \quad \text{and} \quad \operatorname{Ass} M^{\triangleleft\langle i,j\rangle} = \operatorname{Ass} M.$$

In particular, $\dim_S M^{\triangleleft\langle i,j\rangle} = \dim_S M$.

Proof. Let $F_\bullet$ be a $\mathbb{Z}^n$-graded minimal free resolution of M. Since $(-)^{\triangleleft\langle i,j\rangle}$ is exact and $S(-\mathbf{a})^{\triangleleft\langle i,j\rangle} \cong S(-\tau(\mathbf{a}))$, $(F_\bullet)^{\triangleleft\langle i,j\rangle}$ is a minimal free resolution of $M^{\triangleleft\langle i,j\rangle}$. Hence the equations on the Betti numbers and depths hold. We omit the proofs of the remaining equations here. □

Corollary 2.2. *For $M \in \operatorname{mod}_{\mathbb{Z}^n} S$, M is Cohen-Macaulay, if and only if so is $M^{\triangleleft\langle i,j\rangle}$. For a monomial ideal I, S/I is Gorenstein if and only if so is $S/I^{\triangleleft\langle i,j\rangle}$.*

Let $\lambda_{\langle i,j\rangle} : \mathbb{Z}^n \to \mathbb{Z}^n$ be the order preserving map defined by $(\lambda_{\langle i,j\rangle}(\mathbf{a}))_k = a_k$ for all $k \neq i$ and

$$(\lambda_{\langle i,j\rangle}(\mathbf{a}))_i := \begin{cases} a_i & \text{if } a_i \geq j, \\ a_i + 1 & \text{if } a_i < j. \end{cases}$$

Let $(-)^{\triangleright\langle i,j\rangle}$ denote the functor $(\lambda_{\langle i,j\rangle})^* : \operatorname{mod}_{\mathbb{Z}^n} S \to \operatorname{mod}_{\mathbb{Z}^n} S$.

We define $\rho_{\langle i,j\rangle} : \mathbb{Z}^n \to \mathbb{Z}^n$ by $(\rho_{\langle i,j\rangle}(\mathbf{a}))_k = a_k$ for all $k \neq i$ and

$$(\rho_{\langle i,j\rangle}(\mathbf{a}))_i := \begin{cases} a_i & \text{if } a_i > j, \\ a_i - 1 & \text{if } a_i \leq j, \end{cases}$$

Then we have $S(-\mathbf{a})^{\triangleright\langle i,j\rangle} \cong S(-\rho_{\langle i,j\rangle}(\mathbf{a}))$ for all $\mathbf{a} \in \mathbb{Z}^n$.

Remark 2.3. It is easy to see that $M^{\triangleleft\langle i,j\rangle} \cong (M^{\triangleright\langle i,j\ \ 1\rangle})(-\mathbf{e}_i)$. In this sense, $(-)^{\triangleright\langle i,j\rangle}$ is "parallel" to $(-)^{\triangleleft\langle i,j\rangle}$. Hence Proposition 2.1, the first assertion of Corollary 2.2, and Theorem 3.2 below also hold for $M^{\triangleright\langle i,j\rangle}$.

Let $\mathsf{D}^b(\mathrm{mod}_{\mathbb{Z}^n} S)$ denote the bounded derived category of $\mathrm{mod}_{\mathbb{Z}^n} S$. Since the functor $(-)^{\triangleleft\langle i,j\rangle} : \mathrm{mod}_{\mathbb{Z}^n} S \to \mathrm{mod}_{\mathbb{Z}^n} S$ is exact, it can be extended to $(-)^{\triangleleft\langle i,j\rangle} : \mathsf{D}^b(\mathrm{mod}_{\mathbb{Z}^n} S) \to \mathsf{D}^b(\mathrm{mod}_{\mathbb{Z}^n} S)$ in the natural way. The same is true for $(-)^{\triangleright\langle i,j\rangle}$. We denote the exact functor $\mathbf{R}\mathrm{Hom}_S(-, S) : \mathsf{D}^b(\mathrm{mod}_{\mathbb{Z}^n} S) \to \mathsf{D}^b(\mathrm{mod}_{\mathbb{Z}^n} S)^{\mathsf{op}}$ by $\mathbb{D}$.

Theorem 2.4. *We have natural isomorphisms*

$$\mathbb{D} \circ (-)^{\triangleleft\langle i,j\rangle} \cong (-)^{\triangleright\langle i,\ \ j\rangle} \circ \mathbb{D} \quad \textit{and} \quad \mathbb{D} \circ (-)^{\triangleright\langle i,j\rangle} \cong (-)^{\triangleleft\langle i,\ \ j\rangle} \circ \mathbb{D}.$$

In particular, we have

$$\mathrm{Ext}^l_S(M^{\triangleleft\langle i,j\rangle}, S) \cong \mathrm{Ext}^l_S(M, S)^{\triangleright\langle i,\ \ j\rangle} \quad \textit{and} \quad \mathrm{Ext}^l_S(M^{\triangleright\langle i,j\rangle}, S) \cong \mathrm{Ext}^l_S(M, S)^{\triangleleft\langle i,\ \ j\rangle}.$$

Proof. For $M^\bullet \in \mathsf{D}^b(\mathrm{mod}_{\mathbb{Z}^n} S)$, we can take its minimal free resolution $F^\bullet$. Then $C_1^\bullet := \mathrm{Hom}_S((F^\bullet)^{\triangleleft\langle i,j\rangle}, S)$ (resp. $C_2^\bullet := \mathrm{Hom}_S(F^\bullet, S)^{\triangleright\langle i,-j\rangle}$) is a minimal free resolution of $\mathbb{D}((M^\bullet)^{\triangleleft\langle i,j\rangle})$ (resp. $\mathbb{D}(M^\bullet)^{\triangleright\langle i,-j\rangle}$). If the l^{th} term F^l of $F^\bullet$ is of the form $\bigoplus_{\mathbf{a}\in\mathbb{Z}^n} S(-\mathbf{a})^{b_{l,\mathbf{a}}}$, then both C_1^l and C_2^l are isomorphic to $\bigoplus_{\mathbf{a}\in\mathbb{Z}^n} S(\tau_{\langle i,j\rangle}(\mathbf{a}))^{b\ \ l,\mathbf{a}}$. We can also easily show that $C_1^\bullet$ and $C_2^\bullet$ are isomorphic as cochain complexes. Hence we have $\mathbb{D}((M^\bullet)^{\triangleleft\langle i,j\rangle}) \cong \mathbb{D}(M^\bullet)^{\triangleright\langle i,-j\rangle}$. The remaining statements are clear now. ☐

Let M be a finitely generated S-module. We say M is *sequentially Cohen-Macaulay* if $\mathrm{Ext}^{n\ \ i}_S(M, S)$ is either a Cohen-Macaulay module of dimension i or the 0 module for all i. The original definition is given by the existence of a certain filtration, however it is equivalent to the above one.

Corollary 2.5. *For $M \in \mathrm{mod}_{\mathbb{Z}^n} S$, it is sequentially Cohen-Macaulay, if and only if so is $M^{\triangleleft\langle i,j\rangle}$, if and only if so is $M^{\triangleright\langle i,\ \ j\rangle}$. For a monomial ideal I, S/I satisfies Serre's condition (S_r) if and only if so dose $S/I^{\triangleleft\langle i,j\rangle}$.*

3. Sliding functors and Stanley depth

For a subset $Z \subset \{x_1, \ldots, x_n\}$, $\Bbbk[Z]$ denotes the subalgebra of S generated by all $x_i \in Z$. Clearly, $\Bbbk[Z]$ is a polynomial ring with $\dim \Bbbk[Z] = \#Z$. Let $M \in \mathrm{mod}_{\mathbb{Z}^n} S$. We say the $\Bbbk[Z]$-submodule $m\,\Bbbk[Z]$ of M generated by a homogeneous element $m \in M_{\mathbf{a}}$ is a *Stanley space*, if it is $\Bbbk[Z]$-free. A *Stanley decomposition* $\mathcal{D}$ of M is a presentation of M as a finite direct sum of Stanley spaces. That is,

$$\mathcal{D} : \bigoplus_{l=1}^{s} m_l\,\Bbbk[Z_l] = M$$

as $\mathbb{Z}^n$-graded $\Bbbk$-vector spaces, where each $m_l\,\Bbbk[Z_l]$ is a Stanley space.

Let $\mathrm{sd}(M)$ be the set of Stanley decompositions of M. For all $0 \neq M \in \mathrm{mod}_{\mathbb{Z}^n} S$, we have $\mathrm{sd}(M) \neq \emptyset$. For $\mathcal{D} = \bigoplus_{l=1}^{s} m_l\,\Bbbk[Z_l] \in \mathrm{sd}(M)$, we set

$$\mathrm{sdepth}_S(\mathcal{D}) := \min\{\, \#Z_l \mid l = 1, \ldots, s \,\},$$

and call it the *Stanley depth* of $\mathcal{D}$. The Stanley depth of M is defined by

$$\mathrm{sdepth}_S(M) := \max\{\, \mathrm{sdepth}_S \mathcal{D} \mid \mathcal{D} \in \mathrm{sd}(M) \,\}.$$

This invariant behaves somewhat strangely. For example, contrary to the usual depth, no relation between the Stanley depth of a monomial ideal I and that of the quotient S/I is known.

The following conjecture, which is widely open even for monomial ideals I and the quotients S/I, is a leading problem of this subject.

Conjecture 3.1 (Stanley). *For any $M \in \mathrm{mod}_{\mathbb{Z}^n} S$, we have*

$$\mathrm{sdepth}_S M \geq \mathrm{depth}_S M.$$

The following is the main result of this section.

Theorem 3.2. *For $M \in \mathrm{mod}_{\mathbb{Z}^n} S$, we have $\mathrm{sdepth}_S M = \mathrm{sdepth}_S M^{\triangleleft\langle i,j\rangle}$. Hence Stanley's conjecture holds for M if and only if it holds for $M^{\triangleleft\langle i,j\rangle}$.*

Proof. Let $M = \bigoplus_{l=1}^{s} m_l \, \Bbbk[Z_l]$ be a Stanley decomposition with $m_l \in M_{\mathbf{a}_l}$. We may assume the following.

- If $1 \leq l \leq t$, then $(\mathbf{a}_l)_i = j-1$ and $x_i \notin Z_l$;
- If $t+1 \leq l \leq s$, then either $(\mathbf{a}_l)_i \neq j-1$ or $x_i \in Z_l$.

For $1 \leq l \leq s$, let $\acute{m}_l \in (M^{\triangleleft\langle i,j\rangle})_{\tau(\mathbf{a}_l)}$ be the element corresponding to m_l. Furthermore, for $1 \leq l \leq t$, set $\acute{m}_{s+l} := x_i \, \acute{m}_l \in (M^{\triangleleft\langle i,j\rangle})_{\tau(\mathbf{a}_l)+\mathbf{e}_i}$ (note that $\tau(\mathbf{a}_l) = \mathbf{a}_l$ in this case) and $Z_{l+s} := Z_l$. Now we have homogeneous elements $\acute{m}_1, \dots, \acute{m}_{s+t}$ of $M^{\triangleleft\langle i,j\rangle}$, and it is easy to check that $\bigoplus_{l=1}^{s+t} \acute{m}_l \, \Bbbk[Z_l]$ is a Stanley decomposition of $M^{\triangleleft\langle i,j\rangle}$. Hence we have $\mathrm{sdepth}_S M \leq \mathrm{sdepth}_S M^{\triangleleft\langle i,j\rangle}$.

The converse inequality can be proved by a similar way, but we omit it here. □

Using this result somewhat indirectly, we can get the following result.

Theorem 3.3 (Okazaki-Y.[11, Theorem 6.5]). *If $I \subset S$ is a cogeneric monomial ideal (see* [14] *for the definition), then the Stanley conjecture holds for S/I.*

Remark 3.4. Under the further assumption that S/I is Cohen-Macaulay, Theorem 3.3 has been proved in [1]. Roughly speaking, our proof reduces the assertion to the Cohen-Macaulay case ([1]) using Theorem 3.2 and some other techniques developed in [11]. We also remark that the paper [11] is earlier than [16], in which $(-)^{\triangleleft\langle i,j\rangle}$ is defined in full generality. More precisely, [11] only concerns the case $j = 1$. Since $\mathbb{N}^n$-graded modules are treated there, this restriction makes sense.

4. Faithful and non-faithful polarizations

In the rest of this note, let $\widetilde{S}$ and Θ be as in Introduction. For our definition of polarizations of monomial ideals, which is a bit more general than the usual one, see Definition 1.1 in Introduction. First, we remark the following.

Lemma 4.1 (c.f. [10, Lemma 6.9]). *Let I and J be monomial ideals of S and $\widetilde{S}$ respectively. Assume that the condition (i) of Definition 1.1 is satisfied. Then the condition (ii) is equivalent to the following.*

(ii') $\beta^{\widetilde{S}}_{i,j}(J) = \beta^{S}_{i,j}(I)$ *for all i, j.*

Definition 4.2. We say a polarization J of I is *faithful*, if Θ forms an $\mathrm{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$-regular sequence for all i.

If a polarization J of I is faithful, then we have

$$\operatorname{Ext}^i_S(S/I, S) \cong \operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) \otimes_{\widetilde{S}} \widetilde{S}/(\Theta),$$

in particular, $\deg_S(\operatorname{Ext}^i_S(S/I, S)) = \deg_{\widetilde{S}}(\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}))$. By [12, Corollary 4.10] (see also Theorem 5.3 below), the standard polarization $\mathsf{pol}(I)$ is always faithful.

Example 4.3. Here, we give an example of a non-faithful polarization. Let $I := (x^2y, x^2z, xyz, xz^2, y^3, y^2z, yz^2)$ be an ideal of $S := \Bbbk[x, y, z]$. Then the ideal

$$J = (x_1x_2y_3, x_1x_2z_3, x_1y_2z_3, x_1z_2z_3, y_1y_2y_3, y_1y_2z_3, y_1z_2z_3)$$

of $\widetilde{S} := \Bbbk[x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3]$ is a polarization of I. *Macaulay2* computation shows that $\deg_S(\operatorname{Ext}^3_S(S/I, S)) = 6$ and $\deg_{\widetilde{S}}(\operatorname{Ext}^3_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})) = 5$. Hence this polarization is not faithful.

Proposition 4.4. *Let J be a faithful polarization of I. Then we have*

$$\beta^{\widetilde{S}}_{i,j}(\operatorname{Ext}^l_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})) = \beta^S_{i,j}(\operatorname{Ext}^l_S(S/I, S))$$

for all i, j, l. Moreover, $\widetilde{S}/J$ is sequentially Cohen-Macaulay (satisfies Serre's condition (S_r)) if and only if so is (resp. so does) S/I.

We will use following facts in the last section.

Lemma 4.5. *Let M be a sequentially Cohen-Macaulay S-module, and $y \in S$ a non-zero divisor of M. Then M/yM is a sequentially Cohen-Macaulay module with $\operatorname{Ext}^{i+1}_S(M/yM, S) \cong \operatorname{Ext}^i_S(M, S)/y \cdot \operatorname{Ext}^i_S(M, S)$ for all i. Moreover, we have*

$$\operatorname{Ass}(M/yM) = \{\, \mathfrak{p} \mid \mathfrak{p} \textit{ is a minimal prime of } \mathfrak{p}' + (y) \textit{ for some } \mathfrak{p}' \in \operatorname{Ass}(M) \,\}.$$

The above lemma easily follows from the next property of a finitely generated S-module M, which is well-known to the specialist.

1. (1) $\dim_S \operatorname{Ext}^i_S(M, S) \leq n - i$ for all i.
2. (2) For a prime ideal $\mathfrak{p} \subset S$ of codimension c, $\mathfrak{p} \in \operatorname{Ass}(M)$ if and only if $\mathfrak{p}$ is an associated (equivalently, minimal) prime of $\operatorname{Ext}^c_S(M, S)$.

Proposition 4.6. *Let J be a polarization of I. If $\widetilde{S}/J$ is sequentially Cohen-Macaulay, then so is S/I, and J is faithful.*

Proof. Immediate from Lemma 4.5. □

Remark 4.7. Even if S/I is sequentially Cohen-Macaulay, a polarization J is not necessarily faithful. In fact, S/I of Example 4.3 is sequentially Cohen-Macaulay.

5. Standard polarization as a functor

Definition 5.1 (Miller [8]). Let $\mathbf{d} \in \mathbb{N}^n$ with $\mathbf{d} \succeq \mathbf{1}$. We say $M \in \operatorname{mod}_{\mathbb{N}^n} S$ is *positively* $\mathbf{d}$*-determined*, if the multiplication map $M_\mathbf{a} \ni y \longmapsto x_i y \in M_{\mathbf{a}+\mathbf{e}_i}$ is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and $i \in [n]$ with $a_i \geq d_i$. Let $\operatorname{mod}_\mathbf{d} S$ denote the full subcategory of $\operatorname{mod}_{\mathbb{N}^n} S$ consisting of positively $\mathbf{d}$-determined modules.

A positively $\mathbf{1}$-determined module is called a *squarefree module*, and we denote $\operatorname{mod}_\mathbf{1} S$ by $\operatorname{Sq} S$.

For a free module $S(-\mathbf{a})$, $\mathbf{a} \in \mathbb{N}^n$, $S(-\mathbf{a}) \in \operatorname{mod}_{\mathbf{d}} S$ if and only if $\mathbf{a} \preceq \mathbf{d}$.

Extending the polarization of monomial ideals, we will define the functor $\mathsf{pol}_{\mathbf{d}} : \operatorname{mod}_{\mathbf{d}} S \to \operatorname{Sq} \widetilde{S}$. This idea has already appeared in [4, Theorem 2.1] and [12, §4] in slightly different setting. See Remark 5.5 below.

Set $[\mathbf{d}] := \{ (i, j) \mid 1 \le i \le n, 1 \le j \le d_i \}$. Then $\mathbb{N}^{[\mathbf{d}]}$ is the set of vectors $\mathbf{a} = (a_{i,j} \in \mathbb{N} \mid 1 \le i \le n, 1 \le j \le d_i)$. We can regard $\widetilde{S}$ as an $\mathbb{N}^{[\mathbf{d}]}$-graded ring in the natural way. Define the order preserving map $\eta : \mathbb{N}^{[\mathbf{a}]} \to \mathbb{N}^n$ by

$$\eta(\mathbf{a})_i = \begin{cases} d_i & \text{if } a_{i,j} > 0 \text{ for all } j, \\ \min\{ j - 1 \mid a_{i,j} = 0 \} & \text{otherwise.} \end{cases}$$

For $M \in \operatorname{mod}_{\mathbf{d}} S$, we can construct a new module $\eta^*(M) \in \operatorname{mod}_{\mathbb{N}^{[\mathbf{d}]}} \widetilde{S}$ so that $\eta^*(M)_{\mathbf{a}} \cong M_{\eta(\mathbf{a})}$ and the multiplication map $\eta^*(M)_{\mathbf{a}} \ni y \longmapsto x^{\mathbf{b}} y \in \eta^*(M)_{\mathbf{a}+\mathbf{b}}$ is given by $M_{\eta(\mathbf{a})} \ni y \longmapsto x^{\eta(\mathbf{a}+\mathbf{b}) - \eta(\mathbf{a})} y \in M_{\eta(\mathbf{a}+\mathbf{b})}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{[\mathbf{d}]}$. It is easy to show that $\eta^*(M) \in \operatorname{Sq} \widetilde{S}$. Hence η^* gives an exact functor $\operatorname{mod}_{\mathbf{d}} S \to \operatorname{Sq} \widetilde{S}$.

For $\mathbf{a} \in \mathbb{N}^n$ with $\mathbf{a} \preceq \mathbf{d}$, we define $\widetilde{\mathbf{a}} \in \mathbb{N}^{[\mathbf{d}]}$ by

$$\widetilde{a}_{i,j} = \begin{cases} 1 & \text{if } j \le a_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\eta^*(S(-\mathbf{a})) \cong \widetilde{S}(-\widetilde{\mathbf{a}})$ and $\mathsf{pol}(x^{\mathbf{a}}) = x^{\widetilde{\mathbf{a}}} \in \widetilde{S}$ (see Introduction for the construction of $\mathsf{pol}(x^{\mathbf{a}})$). For a positively $\mathbf{d}$-determined monomial ideal $I = (x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_r}) \subset S$, $\eta^*(I)$ coincides with the standard polarization $\mathsf{pol}(I) = (x^{\widetilde{\mathbf{a}}_1}, \dots, x^{\widetilde{\mathbf{a}}_r}) \subset \widetilde{S}$. So the functor η^* is denoted by $\mathsf{pol}_{\mathbf{d}}$.

Lemma 5.2. *For $M \in \operatorname{mod}_{\mathbf{d}} S$, $\Theta \subset \widetilde{S}$ forms a $\mathsf{pol}_{\mathbf{d}}(M)$-regular sequence, and $\mathsf{pol}_{\mathbf{d}}(M) \otimes_{\widetilde{S}} \widetilde{S}/(\Theta) \cong M$. Hence we have*

$$\beta^{\widetilde{S}}_{i,j}(\mathsf{pol}_{\mathbf{d}}(M)) = \beta^{S}_{i,j}(M),$$

$$\operatorname{depth}_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M)) = \operatorname{depth}_S M + (\#\Theta), \qquad \dim_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M)) = \dim_S M + (\#\Theta),$$

and

$$\deg_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M)) = \deg_S M.$$

We can also construct the "reversed" copy $\mathsf{pol}^{\mathbf{d}}$ of $\mathsf{pol}_{\mathbf{d}}$. Here, for a monomial $x^{\mathbf{a}}$ with $\mathbf{a} \preceq \mathbf{d}$,

$$\mathsf{pol}^{\mathbf{d}}(x^{\mathbf{a}}) := \prod_{a_i > 0} x_{i,d_i} x_{i,d_i - 1} \cdots x_{i, d_i - a_i + 1} \in \widetilde{S}.$$

Clearly, $\mathsf{pol}^{\mathbf{d}}$ is the same thing as $\mathsf{pol}_{\mathbf{d}}$ modulo suitable exchange of the variables of $\widetilde{S}$. Hence Lemma 5.2 also holds for $\mathsf{pol}^{\mathbf{d}}$.

Since $\operatorname{Ext}^i_S(M, S(-\mathbf{d})) \in \operatorname{mod}_{\mathbf{d}} S$ for all $M \in \operatorname{mod}_{\mathbf{d}} S$, $\mathbf{R}\operatorname{Hom}_S(-, S(-\mathbf{d}))$ gives a functor $\mathsf{D}^b(\operatorname{mod}_{\mathbf{d}} S) \to \mathsf{D}^b(\operatorname{mod}_{\mathbf{d}} S)^{\mathsf{op}}$, which is denoted by $\mathbb{D}_S$. Similarly, $\mathbb{D}_{\widetilde{S}}$ denotes the duality functor $\mathbf{R}\operatorname{Hom}_{\widetilde{S}}(-, \widetilde{S}(-\mathbf{1})) : \mathsf{D}^b(\operatorname{Sq} \widetilde{S}) \to \mathsf{D}^b(\operatorname{Sq} \widetilde{S})^{\mathsf{op}}$. Here $\mathbf{1} \in \mathbb{N}^{[\mathbf{d}]}$ is the vector whose coordinates are all 1. Note that the exact functors $\mathsf{pol}_{\mathbf{d}}$ and $\mathsf{pol}^{\mathbf{d}}$ can be extended to the functors $\mathsf{D}^b(\operatorname{mod}_{\mathbf{d}} S) \to \mathsf{D}^b(\operatorname{Sq} \widetilde{S})$.

Theorem 5.3 (c.f. Sbarra [12, Corollary 4.10]). *We have a natural isomorphism* $\mathbb{D}_{\widetilde{S}} \circ \mathsf{pol}_{\mathbf{d}} \cong \mathsf{pol}^{\mathbf{d}} \circ \mathbb{D}_S$. *In particular, for all* $M \in \mathrm{mod}_{\mathbf{d}} S$,

$$\mathrm{Ext}^i_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M), \widetilde{S}(-\mathbf{1})) \cong \mathsf{pol}^{\mathbf{d}}(\mathrm{Ext}^i_S(M, S(-\mathbf{d}))).$$

The proof is somewhat analogous to that of Theorem 2.4, and we omit it here.

The *arithmetic degree* $\mathrm{adeg}_S(M)$ of a finitely generated S-module M is the "degree" reflecting the contribution of all associated primes of M. In [15, Proposition 1.11], it is shown that

$$\mathrm{adeg}_S M = \sum_{i=0}^{n} \deg_S(\mathrm{Ext}^i_S(\mathrm{Ext}^i_S(M, S), S)).$$

Corollary 5.4 (c.f. Frübis-Kruger and Terai [5]). *For* $M \in \mathrm{mod}_{\mathbf{d}} S$, *we have*

$$\mathrm{adeg}_S(M) = \mathrm{adeg}_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M)).$$

Proof. The assertion follows from Lemma 5.2 and Theorem 5.3. □

When $M = S/I$ for a monomial ideal I, Corollary 5.4 has been given by Frübis-Kruger and Terai [5]. Our proof also works for the *homological degree* $\mathrm{hdeg}_S(M)$ of M, which is defined by the iterated use of the operation $\mathrm{Ext}^i_S(-, S)$ (see [15] for detail). Hence the equation $\mathrm{hdeg}_S(M) = \mathrm{hdeg}_{\widetilde{S}}(\mathsf{pol}_{\mathbf{d}}(M))$ holds.

Remark 5.5. Bruns and Herzog [4] gave the same construction as $\mathsf{pol} : \mathrm{mod}_{\mathbf{d}} S \to \mathrm{Sq}\, \widetilde{S}$, but they did not recognize it as a functor. On the other hand, Sbarra [12] treated a polarization as a functor. However he used the convention something like $\mathsf{pol}_{\pm \mathbf{e}_i} : \mathrm{mod}_{\mathbb{Z}^n} S \to \mathrm{mod}_{\mathbb{Z}^n} S'$ with $S' = S[x_i']$. Roughly speaking, his functor is similar to our $\mathsf{pol}_{\mathbf{d}}$ (resp. $\mathsf{pol}^{\mathbf{d}}$) in the positive (resp. negative) degree parts.

6. Alternative polarizations of Borel fixed ideals

We say a monomial ideal I is *Borel fixed*, if $x^{\mathbf{a}} \in I$, $x_i | x^{\mathbf{a}}$ and $j < i$ imply $(x_j/x_i) \cdot x^{\mathbf{a}} \in I$. If $\mathrm{char}(\Bbbk) > 0$, this terminology is unnatural. However, we use it for simplicity. Borel fixed ideals play a role in Gröbner basis theory, since they appear as the "generic initial ideals" of homogeneous ideals (when $\mathrm{char}(\Bbbk) = 0$).

Let d be a positive integer, and set

$$\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \le i \le n,\ 1 \le j \le d\,].$$

For a monomial $x^{\mathbf{a}} \in S$ with $\deg(x^{\mathbf{a}}) \le d$, set $b_i := \sum_{j=1}^{i} a_j$ for each $i \ge 0$ (here $b_0 = 0$), and

$$\mathsf{b\text{-}pol}(x^{\mathbf{a}}) := \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j} \in \widetilde{S}.$$

Let $I \subset S$ be a monomial ideal with $\deg(x^{\mathbf{a}}) \le d$ for all $x^{\mathbf{a}} \in G(I)$, where $G(I)$ denotes the set of minimal (monomial) generators of I. Set

$$\mathsf{b\text{-}pol}(I) := (\, \mathsf{b\text{-}pol}(x^{\mathbf{a}}) \mid x^{\mathbf{a}} \in G(I)\,) \subset \widetilde{S}.$$

Occasionally, this ideal gives a polarization of I. Note that $\mathsf{b\text{-}pol}(I)$ always satisfies the condition (i) of Definition 1.1, and the problem is the condition (ii).

Example 6.1. For $I = (x^2, xy, xz, y^2, yz) \subset \Bbbk[x, y, z]$, we have

$$\mathsf{b\text{-}pol}(I) = (x_1x_2, x_1y_2, x_1z_2, y_1y_2, y_1z_2),$$

and it gives a polarization. In fact, since I is Borel fixed, we can use Theorem 6.2 below. It is an easy exercise to show that $\mathsf{b\text{-}pol}(I)$ is essentially different (i.e., different even after permutations of the variables) to the standard polarization $\mathsf{pol}(I) = (x_1x_2, x_1y_1, x_1z_1, y_1y_2, y_1z_1)$.

The polarization of the form $\mathsf{b\text{-}pol}(I)$ first appeared in Nagel and Reiner [10, Corollary 2.21]. H. Lohne, who studied this idea systematically, called $\mathsf{b\text{-}pol}(I)$ the *box polarization*, since the ideals treated in [10] are constructed from combinatorial data described by several "boxes" (something like Young diagrams).

Theorem 6.2. *If $I \subset S$ is a Borel fixed ideal, then $J := \mathsf{b\text{-}pol}(I)$ gives a polarization of I, which is faithful.*

The idea of the proof. Using the notion of a *pretty clean filtration* ([6]), we can show that $\widetilde{S}/J$ is a sequentially Cohen-Macaulay, and an associated prime of J is of the form $(x_{i,c_i} \mid 1 \leq i \leq l)$ for some l and a non-decreasing sequence $c_1, c_2, \dots, c_l \in \mathbb{N}$. Hence we see that Θ forms an $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$-regular sequence by Lemma 4.5. □

Let $\widehat{S} = \Bbbk[x_1, \dots, x_N]$ be a polynomial ring with $N \gg 0$. A monomial $x^{\mathbf{a}} \in S = \Bbbk[x_1, \dots, x_n]$ of degree d can be expressed as $x^{\mathbf{a}} = \prod_{i=1}^d x_{\alpha_i}$ for some $\alpha_1, \dots, \alpha_d$ with $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d \leq n$. G. Kalai [7] introduced the *squarefree operation*

$$(x^{\mathbf{a}})^\sigma := \prod_{i=1}^{d} x_{\alpha_i+i-1} = x_{\alpha_1} \cdot x_{\alpha_2+1} \cdots x_{\alpha_d+d-1} \in \widehat{S}.$$

Note that $(x^{\mathbf{a}})^\sigma$ is a squarefree monomial. For a monomial ideal I, set

$$I^\sigma := (\, (x^{\mathbf{a}})^\sigma \mid x^{\mathbf{a}} \in G(I) \,) \subset \widehat{S}.$$

The following is a fundamental result on I^σ, which is useful in combinatorics on simplicial complexes (especially, the *shifting theory*).

Theorem 6.3 (Kalai [7], Aramova-Herzog-Hibi [2]). *Let $I \subset S$ be a Borel fixed monomial ideal. Then the following hold.*

(i) $\beta^S_{i,j}(I) = \beta^{\widehat{S}}_{i,j}(I^\sigma)$ *for all* i, j.

(ii) *If I is a generic initial ideal of some squarefree monomial ideal in S, then we can "regard" I^σ as an ideal of S (i.e., we can take $N = n$).*

Using Theorem 6.2, we can give a new proof of Theorem 6.3 (i), which is a bit more conceptual than the original one in [2]. By an argument similar to the proof of Theorem 6.2, we can show that

$$\Theta' := \{x_{i,j} - x_{i+1,j-1} \mid 1 \leq i < n,\ 2 \leq j \leq d\} \subset \widetilde{S}$$

forms a $(\widetilde{S}/J)$-regular sequence, where we set $J := \mathsf{b\text{-}pol}(I)$. Moreover, through the ring homomorphism $\widetilde{S}/(\Theta') \ni x_{i,j} \longmapsto x_{i+j} \in \widehat{S}$, we have $\widetilde{S}/J \otimes_{\widetilde{S}} \widetilde{S}/(\Theta') \cong \widehat{S}/I^\sigma$. Hence we have $\beta^S_{i,j}(I) = \beta^{\widetilde{S}}_{i,j}(J) = \beta^{\widehat{S}}_{i,j}(I^\sigma)$ for all i, j. Here the first equality follows from Theorem 6.2 and the second follows from the above argument.

Remark 6.4. S. Murai [9] defined the generalized squarefree operators $(-)^{\sigma(a)}$ associated with a non-decreasing sequence $\{a_i\}$ of non-negative integers (if $a_i = i$ for all i, $(-)^{\sigma(a)}$ coincides with $(-)^{\sigma}$). In [9, Proposition 1.9], he showed that $\beta^{S}_{i,j}(I) = \beta^{\widehat{S}}_{i,j}(I^{\sigma(a)})$ for all i, j. We can prove this fact also by Theorem 6.2. In fact, the above argument for I^{σ} also works for $I^{\sigma(a)}$, if one replaces Θ' by a "minimal set of generators" Θ_a of $\{x_{i,j} - x_{i',j'} \mid i + a_j = i' + a_{j'}\} \subset \widetilde{S}$. Conversely, by virtue of Lemma 4.1, [9, Proposition 1.9] induces our Theorem 6.2. However our argument has some advantage. We can show that $I^{\sigma(a)}$ satisfies the same properties as Proposition 4.4, since Θ_a forms an $\mathrm{Ext}^{i}_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$-regular sequence.

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On equivariant total ring of fractions and factoriality of rings generated by semiinvariants

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1. Introduction

The purpose of this paper is two fold.

First, we construct an equivariant analogue of the total ring of fractions. If B is a commutative ring on which an abstract group Γ acts, then Γ acts on the total ring of fractions $Q(B)$ via $g(a/b) = ga/gb$, and $B \hookrightarrow Q(B)$ is Γ-linear.

Let R be a commutative ring, F a flat R-group scheme, and S an F-algebra (that is, R-algebra with a rational F-action). In this case, $Q(S)$ need not have an F-algebra structure such that $S \hookrightarrow Q(S)$ is F-linear. This is sometimes inconvenient in invariant theory. If R is an algebraically closed field and F is a linear algebraic group over R, then we may consider the group of rational points $F(R)$, which is an abstract group acting on $Q(S)$. This action behaves well. For example, $Q(S)^{F(R)} = Q(S)^F$. However, if R is not algebraically closed, then $Q(S)^{F(R)}$ may be strictly larger than $Q(S)^F$.

The F-total ring of fractions of S, denoted by $Q_F(S)$, is the largest T such that $S \subset T \subset Q(S)$, T is an S-subalgebra of $Q(S)$, T is also an F-algebra such that $S \hookrightarrow T$ is F-linear. In this paper, we give some basic properties and examples. This machinery is useful when we consider a non-algebraically closed base field.

The second purpose is to study the factoriality (the UFD property) of the ring of (semi-)invariants. This is an application of the F-total ring of fractions.

Throughout this paper, k denotes a field, and G denotes an affine (smooth) algebraic group over k. Let S be a G-algebra.

Popov [Pop2, p. 376] remarked the following:

Theorem 1.1 (Popov). *Let k be algebraically closed. Assume that S is a UFD, the character group $X(G)$ of G is trivial, and one of the following hold:* **(a)** *S is finitely generated and G is connected; or* **(b)** *$S^\times \subset S^G$.*

Then S^G is a UFD.

Some variation of the theorem for the case that **(b)** is assumed is treated in [Hoc]. The case that G is a finite group and S is a polynomial ring is found in [Sm, (1.5.7)].

We give some generalizations of this theorem. Our main theorem is Theorem 3.14.

Note that under the assumption of Theorem 1.1, if P is a height one G-stable prime ideal, then $P = Sf$ for some semiinvariant f of S, and $P \cap A = Af$ is principal, see Lemma 3.3 and Lemma 3.4. Theorem 3.14 is a generalization of Theorem 1.1, **(a)**, in the sense that the assumption of Theorem 1.1, **(a)** is stronger than that of Theorem 3.14, and the conclusion of Theorem 1.1, **(a)** is weaker than that of Theorem 3.14.

There are three directions of generalizations. First, we need not assume that k is algebraically closed. Second, we need not assume that the all G-stable height one prime ideals of S are principal. Third, we treat not only the ring of invariants, but also semiinvariants.

As an example of Theorem 3.14, we give an example of S which is not a UFD, but S^G is a UFD by the theorem (Example 3.15). Another generalization is stated as Theorem 3.7. In the theorem, we do not assume that S is a UFD, either (we do not even assume that S is normal).

We also state and prove some classical results and their variations on factoriality of (semi-)invariant subrings. For example, in Theorem 1.1 for the case that **(a)** is assumed, the finite generation of S is unnecessary, the assumption that k is algebraically closed can be weakened, and we can treat the ring of semiinvariants (Proposition 3.8). Theorem 1.1 for the case that **(b)** is assumed is also stated as a result on the ring of invariants over k which is not necessarily algebraically closed (Lemma 3.9).

As we work over a non-separably closed field k, G may not have sufficiently many rational points. That is, $G(k)$ is not Zariski dense in G. So the action of $G(k)$ on $Q(S)$ is not useful, and thus we utilize $Q_G(S)$.

The detailed version of these notes is available as [Has].

2. Equivariant total ring of fractions

Let R be a commutative ring, F a flat R-group scheme, and S an F-algebra.

The action of F on S may not be extended to an action of F on $Q(S)$, the total ring of fractions.

Example 2.1. Let $R = k$ be a field, $F = \mathbb{G}_m$ $(= GL_1)$, and $S = k[x]$. F acts on S via $\deg x = 1$ (that is, F acts on $\operatorname{Spec} S = \mathbb{A}^1$ via $t \cdot \xi = t^{-1}x$. Then this action cannot be extended to $Q(S) = k(x)$. That is, there is no $\mathbb{Z}$-grading on $Q(S)$ such that k is of degree zero and $\deg x = 1$.

Let $\omega : S \to S \otimes R[F]$ be the coaction, that is, ω is the R-algebra map corresponding to the morphism $\operatorname{Spec} S \times F \to \operatorname{Spec} S$ given by $(x, f) \mapsto f^{-1}x$, where $R[F]$ is the coordinate ring of F. As F is R-flat, ω is flat. So the map $\omega' : Q(S) \to Q(S \otimes R[F])$ between the total ring of fractions is induced. We set

$$\Omega := \{M \subset Q(S) \mid M \text{ is an } R\text{-submodule of } Q(S), \text{ and } \omega'(M) \subset M \otimes R[F]\},$$

and define $Q_F(S) := \sum_{M \in \Omega} M$. We call $Q_F(S)$ the *F-total ring of fractions.*

Here we list some basic properties of $Q_F(S)$.

Lemma 2.2. **(1)** *$Q_F(S)$ is an R-subalgebra of $Q(S)$.* **(2)** *Letting $\omega' : Q_F(S) \to Q_F(S) \otimes R[F]$ be the coaction, $Q_F(S)$ is an F-algebra.* **(3)** *S is an F-subalgebra of $Q_F(S)$.* **(4)** *If $S \subset T \subset Q(S)$, T is an S-submodule of $Q(S)$, and T has an (F, S)-module structure such that $S \hookrightarrow T$ is F-linear, then $T \subset Q_F(S)$.* **(5)** $(\omega')^{-1}(Q(S) \otimes R[F]) = Q_F(S)$.

In Noetherian case, there is another description of $Q_F(S)$, which shows that $Q_F(S)$ is really an equivariant analogue of $Q(S)$.

Lemma 2.3. *Let S be Noetherian. Then* **(1)** $Q_F(S) = \bigcup_I S :_{Q(S)} I$, *where I runs through all the F-ideals of S containing a nonzerodivisor.* **(2)** $Q_F(S) = \varinjlim \Gamma(U, \mathcal{O}_{\operatorname{Spec} S})$, *where U runs through all the F-stable open subsets such that $S \to \Gamma(U, \mathcal{O}_{\operatorname{Spec} S})$ are injective.*

Corollary 2.4. *Let S be Noetherian, and I and J be F-stable ideals of S. If J contains a nonzerodivisor, then $I :_{Q(S)} J$ is an (F, S)-submodule of $Q_F(S)$.*

If F is smooth (of finite type) and S is Noetherian and reduced, then the action of F on S is extended to the normalization S' of S. As $Q_F(S)$ is the largest among S-subalgebra T of $Q(S)$ containing S such that T is an F-algebra and $S \hookrightarrow T$ is F-linear, we have:

Lemma 2.5. *Let F be smooth over R, and S be Noetherian and reduced. Then the integral closure S' of S in $Q(S)$ is an F-subalgebra of $Q_F(S)$.*

In Noetherian normal case, $Q_F(S)$ is a subintersection of S.

Lemma 2.6. *Let S be a Noetherian normal domain. Then*

$$Q_F(S) = \bigcap_{P \in X^1(S),\ P^* = 0} S_P,$$

where $X^1(S)$ is the set of height one prime ideals of S, and P^* is the largest F-ideal of S contained in P. In particular, $Q_F(S)$ is a Krull domain.

We show some examples.

Example 2.7. If S is Noetherian and F is finite over R, then $Q_F(S) = Q(S)$.

Example 2.8. Let $R = \mathbb{Z}$, $F = \mathbb{G}_m^n$, and S be a domain. Then S is $\mathbb{Z}^n$-graded. We have $Q_F(S) = S_\Gamma$, where Γ is the set of nonzero homogeneous elements of S.

Example 2.9. Let $R = k$ be a field, V a finite dimensional k-vector space, $F = GL(V)$, and $S = \operatorname{Sym} V$. If $\dim V \geq 2$, then $Q_F(S) = S$.

Next we study what the invariant subring $Q_F(S)^F$ is.

Let $\iota : S \to S \otimes R[F]$ be the map given by $\iota(s) = s \otimes 1$. As it is flat, it induces $\iota' : Q(S) \to Q(S \otimes R[F])$. We define $Q(S)^F$ to be the kernel of the map $\iota' - \omega' : Q(S) \to Q(S \otimes R[F])$. The notation $Q(S)^F$ does not mean that F acts on $Q(S)$. However, if $R = k$ is a field, F is smooth, and $F(k)$ is dense in k, then $Q(S)^F$ agrees with $Q(S)^{F(k)}$, the invariant under the action of the group $F(k)$ of k-rational points of F. Note that in general $Q(S)^F$ is a subring of $Q(S)$. As $Q(S)^F \cap Q(S)^\times = (Q(S)^F)^\times$, if S is a domain, then $Q(S)^F$ is a subfield of $Q(S)$. The following is useful.

Lemma 2.10. $Q(S)^F = Q_F(S)^F$.

3. Factoriality of the ring of semiinvariants

Let k be a field, and G a (smooth) linear algebraic group over k. Let S be a G-algebra. We study when the ring of invariants S^G is a UFD.

It seems that the following lemma is basic, and is a prototype of the known resluts on factoriality of the rings of invariants. As the author does not know any written proof in the literature, we give a proof for reader's convenience.

Lemma 3.1. *Let B be a UFD on which an abstract group Γ acts. If the first cohomology group $H^1(\Gamma, B^\times)$ vanishes, then B^Γ is a UFD.*

Proof. Let $f \in B^\Gamma \setminus \{0\}$. We prove that f is either a unit of B^Γ or a product of prime elements of B^Γ. As $f \in B \setminus \{0\}$ and B is a UFD, we can write $f = uf_1 \cdots f_r$ for some $r \geq 0$, $u \in B^\times$, and prime elements f_i of B.

We use the induction on r. If $r = 0$, that is, $f \in B^\times$, then $f \in (B^\Gamma)^\times$, and the assertion is true. So let $r > 0$, and assume that the assertion is true for smaller r. Note that Γ acts on the set of prime ideals $\{Bf_1, \ldots, Bf_r\}$. Without loss of generality, we may assume that the Γ-orbit of Bf_1 is $\{Bf_1, \ldots, Bf_n\}$ $(1 \leq n \leq r)$, and $Bf_1, \ldots, Bf_n$ are distinct. Let $h := f_1 \cdots f_n$. Then the ideal Bh is Γ-stable. It is easy to see that $\sigma : \Gamma \to B^\times$ given by $\sigma(\gamma) = (\gamma h)/h$ $(\gamma \in \Gamma)$ is well-defined, and is a 1-cocycle of $B^\times$. As $H^1(\Gamma, B^\times) = 0$, σ is a 1-coboundary. That is, there exists some $v \in B^\times$ such that $\sigma(\gamma) = (\gamma v)/v$ for any $\gamma \in \Gamma$. Then $g := v^{-1}h$ is in B^Γ.

We show that g is a prime element of B^Γ. As $h \notin B^\times$, $g \notin B^\times \cap B^\Gamma = (B^\Gamma)^\times$. Let $a, b \in B^\Gamma$ and $ab = gB^\Gamma$. Then as $ab \in gB$, $ab \in f_1B$. So $a \in f_1B$ or $b \in f_1B$. Let $a \in f_1B$. Then $a \in f_iB$ for $i = 1, \ldots, n$. So $a \in gB \cap B^\Gamma = gB^\Gamma$. Similarly, if $b \in f_1B$, then $b \in gB^\Gamma$. Thus g is a prime element of B^Γ.

Note that $f/g = (uv)f_{n+1} \cdots f_r \in B^\Gamma \setminus \{0\}$. By induction assumption, f/g is either a unit of B^Γ, or a product of prime elements of B^Γ. So $f = g(f/g)$ is a product of prime elements of B^Γ, as desired. □

If $B^\times \subset B^\Gamma$, then a 1-cocycle $\Gamma \to B^\times$ is nothing but a group homomorphism, and a 1-coboundary is nothing but the constant map $\gamma \mapsto 1$. So we have

Corollary 3.2. *Let B be a UFD on which an abstract group Γ acts. If $B^\times \subset B^\Gamma$, and if there is no nontrivial group homomorphism $\Gamma \to B^\times$, then B^Γ is a UFD.*

The algebraic group version of this corollary is Theorem 1.1.

Theorem 1.1 for the case that (b) is assumed is proved similarly to Lemma 3.1. Let $h \in S$ be as in the proof of Lemma 3.1 (where S is B and G is Γ). Note that $\sigma : G \to S$ given by $\sigma(g) = (gh)/h$ is a homomorphism from G to $S^\times$. Let $p : S^\times \to S^\times/k^\times$ be the projection. Then $p\sigma : G \to S^\times/k^\times$ is trivial, as $S^\times/k^\times \cong \mathbb{Z}^s$ for some s by Rosenlicht's theorem [Ros], and there is no homomorphism $G \to \mathbb{Z}^s$ of abstract groups. So $\sigma(G) \subset k^\times$. Now σ is a character of G, and hence is trivial by assumption. So $h \in S^G$, and the rest of the proof is the same as in Lemma 3.1.

The case that (a) is assumed is more complicated. See [Has] for details.

We extend the results on invariants to those on semiinvariants. Let χ be a character (that is, one-dimensional G-module) of G. Let V be a G-module. We define

$$V^\chi := \{v \in V \mid \omega_V(v) = v \otimes \chi\} = \sum_{\phi \in \mathrm{Hom}_G(\chi, V)} \mathrm{Im}\,\phi,$$

where we identify $\chi \in \mathrm{Hom}_{\mathrm{Alggrp}}(G, \mathbb{G}_m) \subset \mathrm{Hom}_{\underline{\mathrm{Sch}}/k}(G, \mathbb{A}^1 \setminus \{0\}) = k[G]^\times$. Note that $S_G := \bigoplus_{\chi \in X(G)} S^\chi$ is a k-subalgebra of S. It is $X(G)$-graded, where $X(G)$ is the character group of G. A homogeneous element of S_G is called a *semiinvariant* of S. The degree zero component S_G is S^G.

Let B be a domain, and $f \in B$. There is a unique largest open subset U of $\mathrm{Spec}\,B$ such that $f \in \Gamma(U, \mathcal{O}_{\mathrm{Spec}\,B})$. We call U the domain of definition of f, and denote it by $U(f)$.

Then $f : U(f) \to \mathbb{A}^1_{\mathbb{Z}}$ is a morphism. Let $(\mathbb{A}^1_{\mathbb{Z}})^* := \mathbb{A}^1_{\mathbb{Z}} \setminus 0$, where $0 \cong \mathrm{Spec}\,\mathbb{Z}$ is the origin. We denote $f^{-1}((\mathbb{A}^1_{\mathbb{Z}})^*)$ by $U^*(f)$.

The following is a generalization of a result of Popov [Pop2, Theorem 1] and Kamke [Kam, (3.11)].

Lemma 3.3. *Let G be connected. Let S be a G-algebra domain of finite type over k. Let K be the integral closure of k in $Q(S)$. Assume that $X(G) \to X(K \otimes_k G)$ is surjective. Then for $f \in Q(S)$, the following are equivalent.* **(1)** *$f \in Q_G(S)$, and f is a semiinvariant of $Q_G(S)$;* **(2)** *$U^*(f)$ is a G-stable open subset of* Spec S*;* **(3)** *$Sf \subset Q_G(S)$ is a G-submodule.*

In particular, any unit of S is a homogeneous unit of S_G.

A similar result on (possibly) disconnected G can also be proved.

Lemma 3.4. *Let S be a domain, and K denote the integral closure of k in $Q(S)$. Assume that* **(1)** *$S^\times \subset S^G$;* **(2)** *$G(K)$ is dense in $K \otimes_k G$;* **(3)** *Sf is a G-submodule of $Q_G(S)$;* **(4)** *$X(G) \to X(K \otimes_k G)$ is surjective.*

Then f is a semiinvariant of $Q_G(S)$. If, moreover, $X(G)$ is trivial, then $f \in Q(S)^G$.

In both lemmas, we do not assume that k is algebraically closed.

Lemma 3.5. *Let S be a domain. Let $G(k)$ be dense in G. Assume that $S^\times = k^\times$. If Sf is a $G(k)$-submodule of $Q(S)$, then $f \in Q_G(S)$, and f is a semiinvariant.*

If $X(G)$ is trivial, then a semiinvariant is an invariant.

Remark 3.6. Let k be algebraically closed. **(1)** If N is a normal subgroup of G and $X(N)$ is trivial, then $X(G/N) \cong X(G)$. **(2)** The canonical map $X(G/[G,G]) \to X(G)$ is an isomorphism. **(3)** If G is unipotent, then $X(G)$ is trivial. **(4)** If G is semisimple, then $G = [G,G]$, and $X(G)$ is trivial.

The following is a generalization of Theorem 1.1.

Theorem 3.7. *Let G be connected. Let S be a finitely generated G-algebra domain over k. Let K be the integral closure of k in $Q(S)$. Assume that $X(G) \to X(K \otimes_k G)$ is surjective. Set $A := S_G$. Assume that if P is a G-stable height one prime ideal of S such that $P \cap A$ is a minimal prime of some nonzero principal ideal, then P is a principal ideal. Then* **(1)** *If P is a G-stable height one prime ideal of S such that $P \cap A$ is a minimal prime of a nonzero principal ideal, then $P = Sf$ for some homogeneous prime element f of A.* **(2)** *A is a UFD.* **(3)** *Any homogeneous prime element of A is a prime element of S.* **(4)** *If, moreover, $X(G)$ is trivial, then $S^G = A$ is a UFD.*

In the following proposition, we need *not* assume that S is finitely generated.

Proposition 3.8. *Let G be connected. Let S be a G-algebra. Assume that S is a UFD. Assume that $X(G) \to X(K \otimes_k G)$ is surjective, where K is the integral closure of k in S. Then $A := S_G$ is a UFD. Any homogeneous prime element of A is a prime element of S. If, moreover, $X(G)$ is trivial, then $S^G = A$ is a UFD.*

Lemma 3.9. *Let S be a G-algebra which is a UFD. Assume that $G(K)$ is dense in $K \otimes_k G$, where K is the integral closure of k in S. Assume that $X(K \otimes_k G)$ is trivial. Assume also that $S^\times \subset A = S^G$. Then A is a UFD.*

Corollary 3.10. *Let S be a G-algebra which is a UFD. Assume that $S^\times = k^\times$. If $G(k)$ is dense in G and $X(G)$ is trivial, then S^G is a UFD.*

In [Muk], Mukai posed the question which asks when $Q(S)^G = Q(S^G)$ holds. This question is called the *Italian problem*.

The following is a refinement of [Pop1, Lemma 1]. See also [Kam, (3.14)].

Proposition 3.11. *Let G be connected. Let S be a G-algebra which is a Krull domain. Assume also that any G-stable height one prime ideal of S is principal (e.g., S is a UFD). Moreover, assume that $X(G) \to X(K \otimes_k G)$ is surjective, where K is the integral closure of k in S. Then $Q_G(S)_G = Q_T(A)$, where $T = \operatorname{Spec} kX(G)$. If, moreover, $X(G)$ is trivial, then $Q(S)^G = Q(S^G)$.*

Let S be a finitely generated G-algebra domain. Set $X := \operatorname{Spec} S$. Let

$$s := \max\{\dim Gx \mid x \in X\} = \dim G - \min\{\dim G_x \mid x \in X\}.$$

Proposition 3.12. *We have*

$$s = \dim S - \operatorname{tdeg}_k Q(S)^G.$$

Let S be a finitely generated G-algebra domain. Set $r := \dim S - \operatorname{tdeg}_k Q(S^G)$.

Lemma 3.13. *If S is normal, then $Q(S^G) = Q(S)^G$ if and only if $r = s$.*

We give an example. Let $G = \mathbb{G}_m$ act on $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ via $\deg x = \deg y = 1$. Then $r = 2$ and $s = 1$. $Q(S)^G = k(x/y)$ and $Q(S^G) = k$.

The following is our main theorem.

Theorem 3.14. *Let S be a finitely generated G-algebra which is a normal domain. Assume that G is connected. Assume that $X(G) \to X(K \otimes_k G)$ is surjective, where K is the integral closure of k in S. Let $X^1_G(S)$ be the set of height one G-stable prime ideals of S. Let $M(G)$ be the subgroup of the class group $\operatorname{Cl}(S)$ of S generated by the image of $X^1_G(S)$. Let Γ be a subset of $X^1_G(S)$ whose image in $M(G)$ generates $M(G)$. Set $A := S_G$. Assume that $Q_G(S)_G \subset Q(A)$. Assume that if $P \in \Gamma$, then either the height of $P \cap A$ is not one or $P \cap A$ is principal. Then A is a UFD. If, moreover, $X(G)$ is trivial, then $S^G = A$ is a UFD.*

Example 3.15. We give an example of Theorem 3.14. Let $n \geq m \geq t \geq 2$ be positive integers, $V := k^m$, $W := k^n$, and $M := V \otimes W$. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be the standard bases of V and W, respectively. Let $S := (\operatorname{Sym} M)/I_t$, where $I_t = I_t(v_i \otimes w_j)$ is the determinantal ideal. Let G be the subgroup of the unipotent upper triangular matrices in $GL_m = GL(V)$. Then $A = S^G$ is a UFD.

We give a sketch of the proof of Example 3.15. Let P be the ideal of S generated by the $(t-1)$-minors of the first $(t-1)$ rows of the matrix $(v_i \otimes w_j)$. P is G-invariant, and generates $\operatorname{Cl}(S) = M(G) \cong \mathbb{Z}$. We set $\Gamma := \{P\}$. It is easy to check that **(1)** $\dim S = (t-1)(m+n-t+1)$; **(2)** S^G is finitely generated, and $\dim S^G = (t-1)(n+1-t/2)$; **(3)** $\dim S^G/P^G = (t-2)(n+1-(t-1)/2)$; **(4)** $\operatorname{ht} P^G = n - t + 2 \geq 2$.

Note that S is normal and $K = k$. As G is unipotent, $X(G)$ is trivial. To apply the theorem, it remains to show that $Q_G(S)_G \subset Q(S_G)$. As $X(G)$ is trivial. This is equivalent to $Q(S)^G = Q(S^G)$. So it suffices to show that $r = s$. Clearly $r = \dim S - \dim S^G = (t-1)(m - t/2)$. On the other hand, the orbit Gx, where

$$x = \begin{pmatrix} E_{t-1} & O \\ O & O \end{pmatrix} \in (\operatorname{Spec} S)(k),$$

is $(t-1)(m-t/2)$-dimensional, as can be seen easily. So $r = s$, a desired.

We give another example.

Example 3.16. A finite group G acting on a UFD S such that there is no nontrivial homomorphism $G \to S^\times$, but S^G is not a UFD.

Construction. Let $G := \mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle$, and k an algebraically closed field of characteristic 3. $S := k[A^{-1}, B^{-1}]$, and G acts on S via $\sigma A = B$ and $\sigma B = (AB)^{-1}$. Then S is a UFD. $\operatorname{Spec} S \to \operatorname{Spec} S^G$ is étale in codimension one. So by Fossum's theorem [Fos, (16.1)], $\mathrm{Cl}(S^G) \cong H^1(G, S^\times) \cong \mathbb{Z}/3\mathbb{Z}$.

Example 3.17. S is a finitely generated UFD over k, G is connected, $X(G)$ is trivial, but S^G is not a UFD.

Construction. Set $k = \mathbb{R}$, and

$$G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\} \subset GL_2(k).$$

Let G act on $S := \mathbb{C}[x, y, s, t]$ by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} x = (a + b\sqrt{-1})x, \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} y = (a + b\sqrt{-1})y,$$
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} s = (a - b\sqrt{-1})s, \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} t = (a - b\sqrt{-1})t$$

(G acts trivially on $\mathbb{C}$). Then S is a finitely generated UFD over $\mathbb{R}$, G is connected, $X(G)$ is trivial, but $S^G = \mathbb{C}[xs, xt, ys, yt]$ is not a UFD.

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SOME PROPERTIES OF MATROIDAL IDEALS

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ABSTRACT. Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K. A matroidal ideal I is a square-free monomial ideal in R whose minimal generators satisfying the following exchange condition that for any $u = x_1^{a_1} \cdots x_n^{a_n}, v = x_1^{b_1} \cdots x_n^{b_n} \in G(I)$, if $a_i > b_i$ for some i, then there exists some j with $a_j < b_j$ such that $x_j u/x_i \in G(I)$. We study relations between the linear quotient index, the degree, the height, and the arithmetical rank of a matroidal ideal in this paper.

1. INTRODUCTION

Let K be a field and $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables ($n \geq 2$) over K. Denote by $G(I)$ the set of minimal monomial generators of a given monomial ideal I. We call this I a *matroidal ideal* if each member of $G(I)$ is square-free (hence I is reduced) and the following condition is satisfied: for any pair $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n} \in G(I)$, if $a_i > b_i$ for some i, then there exists some j with $a_j < b_j$ such that $x_j u/x_i \in G(I)$. We note that if we consider the set $\mathcal{B}(I) = \{\operatorname{supp}(u) \mid u \in G(I)\}$, where $\operatorname{supp}(u) = \{x_i \mid u = x_1^{a_1} \cdots x_n^{a_n},\ a_i \neq 0\}$, then it is not hard to see that $\mathcal{B}(I)$ satisfies the following so-called *exchange condition* (cf. [6] or [8]) in matroid theory.

- If B_1 and B_2 are elements of $\mathcal{B}(I)$ and $x \in B_1 - B_2$, then there is an element $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}(I)$.

Form the basic property [6, Theorem 1.2.3] in matroid theory, we know that there exists a "matroid" having $\mathcal{B}(I)$ as its collection of bases (maximal independent sets). Since each maximal independent set of a matroid has the same cardinality (see [6, Lemma 1.2.4]), we conclude that each monomial $u \in G(I)$ must be of the same degree, say d, and call this number d the degree of the matroidal ideal I.

To generalize the notion of matroid, Herzog and Hibi propose the concept of *discrete polymatroid* in the paper [2] and study some combinatoric and algebraic properties related to it. They define the *polymatroidal ideal*, a monomial ideal having the exchange property, in [2, Remark 6.4] and indicate that this class of ideals have the property of linear quotients (cf. [1, Theorem 5.2]) so that they have linear resolutions and then some holomogical results can be obtained easily.

As a special case of polymatroidal ideal, matroidal ideals also adopt these good properties. In this paper, we investigate the invariants like the linear quotient index, the degree, the height, and the arithmetical rank of a matroidal ideal and aim to find some relations between them. We briefly outline this work as follows.

We first discuss the linear quotient index $q(I)$ of a matroidal ideal in section two and get the following result.

Theorem 2.5. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a matroidal ideal of degree d. If I is full-supported, then $q(I) = n - d$.*

With this result and the fact [4, Corollary 1.6] we obtain that the projective dimension of a matroidal ideal is $pd_R(R/I) = n - d + 1$.

An ideal is said to be *pure* if all its prime divisors are of the same height. It is known that the Cohen-Macaulay monomial ideals hold this property. In section three, we discuss the pure matroidal ideal and find the relation between the height and the degree of a pure matroidal ideal as follows.

Theorem 3.6. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a matroidal ideal of degree d. If I is full-supported and is pure of height h, then $\frac{n}{d} \leq h \leq n - d + 1$. In particular, the extreme cases (i) $h = n - d + 1$ if and only if I is square-free Veronese; and (ii) $h = \frac{n}{d}$ if and only if $I = J_1 J_2 \cdots J_d$, where each J_i is generated by h distinct variables and $\operatorname{supp}(J_i) \cap \operatorname{supp}(J_j) = \varnothing$.*

For an ideal I, the minimal number of elements which generate I up to radical is called the *arithmetical rank* of this ideal and is denoted by $ara\, I$. When this numerical invariant equals to the height of I, we say that I is a set-theoretic complete intersection. We discuss the relation between the arithmetical rank $ara\, I$ and the linear quotient index $q(I)$ of a matroidal ideal in the final section. The main result we obtain is as below.

Theorem 4.4. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a matroidal ideal of degree d. Then $ara\ I = q(I) + 1$. In particular, if I is full-supported, then $ara\ I = n - d + 1$.*

2. Linear quotients and matroidal ideals

Throughout, $R = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables ($n \geq 2$) over a field K. By Cohen-Macaulay for an ideal I, we mean that the quotient ring R/I is Cohen-Macaulay. For a monomial ideal I, the support of I is defined for the set $\operatorname{supp}(I) = \bigcup_{u \in G(I)} \operatorname{supp}(u)$; and we say that I is full-supported if $\operatorname{supp}(I) = \{x_1, \ldots, x_n\}$.

In this section, we discuss the linear quotient index $q(I)$ of a matroidal ideal. We first recall the definition of linear quotients from [4].

Definition 2.1. *We say that a monomial ideal $I \subseteq R$ has linear quotients if there is an ordering $u_1, \ldots, u_s$ of the monomials belonging to $G(I)$ with $deg\, u_1 \leq deg\, u_2 \leq \cdots \leq deg\, u_s$ such that, for each $2 \leq j \leq s$, the colon ideal $\langle u_1, u_2, \ldots, u_{j-1} \rangle : u_j$ is generated by a subset of $\{x_1, \ldots, x_n\}$.*

Let I be a monomial ideal with linear quotients with respect to the ordering $\{u_1, \ldots, u_s\}$ of the monomials belonging to $G(I)$. We write $q_j(I)$ for the number of variables which is required to generate the colon ideal $\langle u_1, u_2, \ldots, u_{j-1} \rangle : u_j$. Let $q(I) = max\{q_j(I) \mid 2 \leq j \leq s\}$. From the fact [4, Corollary 1.6] that the length of the minimal free resolution of R/I over R is equal to $q(I)+1$, we see that the index $q(I)$ is independent of the particular choice of the ordering of the monomials which gives linear quotients. Moreover, by the Auslander-Buchsbaum formula, we have $depth\, R/I = n - q(I) - 1$. It then follows from the equality $\dim R/I = n - ht(I)$ that a monomial ideal I with linear quotients satisfies $ht(I) \leq q(I) + 1$ and is Cohen-Macaulay if and only if $ht(I) = q(I) + 1$. We summarize the above as the following proposition.

Proposition 2.2. *Let I be a monomial ideal of R with linear quotients. Then $ht(I) \leq q(I)+1$; and I is Cohen-Macaulay if and only if $ht(I) = q(I)+1$.*

As stated in the introduction, it is known that the polymatroidal ideals have linear quotients. Therefore all the above discussion also applied to polymatroidal ideals. Next, we introduce two lemmas which are useful later.

Lemma 2.3. *Let $I \subseteq K[x_1, \dots, x_n]$ be a full-supported matroidal ideal of degree d. Assume that x and y are variables in R such that $xy \nmid u$ for any $u \in G(I)$. If $xf \in G(I)$ for some monomial f of degree $d-1$, then $yf \in G(I)$.*

Proof. Write $f = x_{i_1} \cdots x_{i_{d-1}}$. The assertion is clear if $d = 2$ so we may assume that $d \geq 3$. Let $g = x_{j_1} \cdots x_{j_{d-1}}$ be a monomial in R different from f such that $yg \in G(I)$ and $|\operatorname{supp}(f) \cap \operatorname{supp}(g)|$ is maximal. We may assume that $j_r = i_r$ for $r = 1, \dots, k$, with $k \leq d-2$. Since $xy \nmid u$ for any $u \in G(I)$, by the definition of matroidal ideal there are integers $s, t \geq k+1$ such that $\frac{yg}{x_{j_s}} x_{i_t} \in G(I)$, which contradicts to the choice of g. Therefore, $f = g$ and the assertion holds. □

Lemma 2.4. *Let $I \subseteq K[x_1, \dots, x_n]$ be a full-supported matroidal ideal of degree d. If there are $d+1$ distinct variables $\{y, y_1, \dots, y_d\} \subseteq \{x_1, \dots, x_n\}$ such that $f = y_1 \cdots y_d \in G(I)$, then there exists an integer i such that $\frac{f}{y_i} y \in G(I)$.*

Proof. The assertion is clear if d is small. We may assume that $d \geq 3$. Let $g = z_1 \cdots z_d$ be a monomial in I different from f such that $y \in \operatorname{supp}(g)$ and $|\operatorname{supp}(f) \cap \operatorname{supp}(g)|$ is maximal. We may assume that $z_i = y_i$ for $i = 1, \dots, k$ and $z_d = y$. Suppose that $k \leq d-2$. Then by the definition of matroidal ideal there are integers $i, j \geq k+1$ such that $\frac{g}{z_j} x_i \in G(I)$, which contradicts to the choice of g. Therefore, $k = d-1$ and the assertion holds. □

Theorem 2.5. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal of degree d. If I is full-supported, then $q(I) = n-d$.*

Proof. Since I has linear quotients, there is an ordering $u_1, \dots, u_s$ of the monomials belonging to $G(I)$ such that, for each $2 \leq j \leq s$, the colon ideal $\langle u_1, u_2, \dots, u_{j-1} \rangle : u_j$ is generated by a subset of $\{x_1, \dots, x_n\}$.
To show the assertion, it is enough to show that

$$\langle u_1, u_2, \dots, u_{j-1} \rangle : u_j \subseteq \{x_1, \dots, x_n\} - \operatorname{supp}(u_j) \tag{2.1}$$

for each $2 \leq j \leq s$ and

$$\langle u_1, u_2, \dots, u_{s-1} \rangle : u_s = \{x_1, \dots, x_n\} - \operatorname{supp}(u_s).$$

Write $u_j = x_{i_1} \cdots x_{i_d}$. If $x_{i_t} \in \langle u_1, u_2, \dots, u_{j-1} \rangle : u_j$ for some $t \leq d$, then $u_j \in \langle u_1, u_2, \dots, u_{j-1} \rangle$ as $\langle u_1, u_2, \dots, u_{j-1} \rangle$ is a square-free monomial ideal, a contradiction. Thus, (2.1) holds. By (2.1), to finish the proof, it suffices to show that $y \in \langle u_1, u_2, \dots, u_{s-1} \rangle : u_s$ if $y \notin \operatorname{supp}(u_s)$. However, this follows by Lemma 2.4 with $u_s = y_1 \cdots y_d$. □

3. Pure matroidal ideals

An ideal is *pure* if all its prime divisors are of the same height. This property is held for a Cohen-Macaulay monomial ideal. In this section, we study the relations between the height, degree, and the number of variables of an pure matroidal ideal.

We first recall one special kind of matroidal ideal, the square-free Veronese ideals.

Example 3.1. The square-free Veronese ideal of degree d in variables $\{x_1, \ldots, x_n\}$ is the ideal generated by all square-free monomials in $\{x_1, \ldots, x_n\}$ of degree d. It is easy to see that the square-free Veronese ideals are matroidal and pure. In particular, from [3, Theorem 4.2] one see that the square-free Veronese ideals are the only case for matroidal ideals being Cohen-Macaulay.

We now give a characterization of matroidal ideals of degree 2.

Theorem 3.2. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a full-supported matroidal ideal of degree two. Then there are subsets $S_1, \ldots, S_m$ of $\{x_1, \ldots, x_n\}$ such that the following hold:*

(i) $m \geq 2$ and $|S_i| \geq 1$ for each i;

(ii) $S_i \cap S_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^m S_i = \{x_1, \ldots, x_n\}$;

(iii) if $x \in S_i$, $y \in S_j$ for $i \neq j$, then $xy \in G(I)$;

(iv) if $x, y \in S_i$ for some i, then $xy \notin G(I)$.

Moreover, let P_i be the the prime ideals generated by the set $\{x_1, \ldots, x_n\} - S_i$ for all i. Then $P_1 \cap P_2 \cap \cdots \cap P_m$ gives the primary decomposition of I.

Proof. Let $t - 1 = |\{x_i \mid i \neq 1, x_ix_1 \notin G(I)\}|$; then $1 \leq t \leq n-1$. Without loss of generality, we may assume that $x_1x_i \notin G(I)$ if $i = 2, \ldots, t$ and $x_1x_i \in G(I)$ if $i = t+1, \ldots, n$. We first show the following two statements:
(a) $x_ix_j \in G(I)$ if $i \leq t$ and $j \geq t+1$;
(b) $x_ix_j \notin G(I)$ if $i, j \leq t$.

To show (a) holds, suppose on the contrary that $x_ix_j \notin G(I)$ for some $i \leq t$ and $j \geq t+1$. Since $x_i \in \text{supp}(I)$, there is a variable x_k such that $x_ix_k \in G(I)$. Moreover, $x_1x_j, x_ix_k \in G(I)$ and I is matroidal imply that either x_ix_1 or x_ix_j is in $G(I)$, a contradiction. Thus (a) holds. For (b), suppose on the contrary that $x_ix_j \in G(I)$ for some $i, j \leq t$. Since $x_1x_n, x_ix_j \in G(I)$, it follows from the exchange property of matroidal ideals that either x_1x_i or x_1x_j belongs to $G(I)$, a contradiction. Thus (b) holds.

Let $S_1 = \{x_1, \ldots, x_t\}$. Observe that $\{x_ix_j \mid i \leq t, \text{ and } j \geq t+1\}$ is a subset of $G(I)$. If $G(I) = \{x_ix_j \mid i \leq t, \text{ and } j \geq t+1\}$ then by setting $S_2 = \{x_{t+1}, \ldots, x_n\}$ and we are done. Therefore, we may assume that there are $j, k \geq t+1$ such that $x_jx_k \in G(I)$. Let I' be the monomial ideal in $K[x_{t+1} \ldots, x_n]$ generated by the set $G(I) - \{x_ix_j \mid i \leq t, \text{ and } j \geq t+1\}$. Then $\text{supp}(I') \subseteq \{x_{t+1}, \ldots, x_n\}$. In fact, $\text{supp}(I') = \{x_{t+1}, \ldots, x_n\}$. For if not, then there is a variable x_l with $l \geq t+1$ such that $x_l \notin \text{supp}(I')$. Since $x_lx_1, x_jx_k \in G(I)$ and I is matroidal, either x_lx_j or x_lx_k is in $G(I)$. Therefore, either x_lx_j or x_lx_k is in $G(I')$, a contradiction. We note that I' is a matroidal ideal of degree 2 of the polynomial ring $K[x_{t+1}, \ldots, x_n]$. Thus, the assertion follows by induction.

Let P_i be the prime ideals generated by the set $\{x_1, \ldots, x_n\} - S_i$. By the properties of S_i, it is easy to see that $P_i = I : y$ for every $y \in S_i$. Therefore each P_i is an associated prime ideal of I. Let $w \in P_1 \cap \cdots \cap P_m$; then $w \cdot y \in I$ whence $y \in \bigcup_{i=1}^m S_i$. It follows that $(I : w) \supseteq \langle x_1, \ldots, x_n \rangle$. Since I is reduced, I has no embedded prime ideals. Therefore $I : w = R$ and so that $w \in I$. Hence, $P_1 \cap \cdots \cap P_m = I$; and this completes the proof. □

From the above theorem we see that the S_i's are uniquely determined. Moreover, if I is pure then we have that $|S_i| = |S_j| = n - ht(I)$ for all i, j. Therefore we have the following corollary.

Corollary 3.3. *Let $I \subseteq K[x_1, \dots, x_n]$ be a full-supported pure matroidal ideal of degree 2; then one has $\frac{n}{2} \le ht(I) \le n-1$. In particular, $ht(I) = n-1$ if and only if I is square-free Veronese; and $ht(I) = \frac{n}{2}$ if and only if n is even and $I = I_1 \cdot I_2$ such that each I_i is generated by $\frac{n}{2}$ distinct variables.*

Proof. It is obvious that $ht(I) \le n-1$ and the equality holds when $m = n$ and $|S_i| = 1$ for all i, i.e., I is square-free Veronese. On the other hand, since $|S_1| + |S_2| = 2(n - ht(I)) \le \sum_{i=1}^{m} |S_i| = n$, we obtain that $n \le 2\,ht(I)$. This equality holds when $m = 2$ and in this case $I = I_1 \cdot I_2$ such that each I_i is generated by $\frac{n}{2}$ distinct variables. □

Here, we connect matroidal ideals with graphs. Observe that if $I \subseteq K[x_1, \dots, x_n]$ is a square-free monomial ideal of degree 2 then I defines a simple graph G with vertex set $\{x_1, \dots, x_n\}$ and edge set $\{x_i x_j \mid x_i x_j \in I\}$. If this is the case, we also say that I is the defining ideal of G. The following corollary is a consequence of Theorem 3.2.

Corollary 3.4. *Let I be a matroidal ideal of degree 2 of a polynomial ring $R = K[x_1, \dots, x_n]$. If I is the defining ideal of a simple graph Γ, then there are positive integers $t_1, \dots, t_m$ such that $n = t_1 + \cdots + t_m$ and $\Gamma \simeq K_{t_1, t_2, \dots, t_m}$. In particular, if I is pure, then $\Gamma \simeq K_{t,t,\dots,t}$.*

Example 3.5. *Let Γ be a graph defined by a full-supported matroidal ideal of degree 2 of the polynomial ring $R = K[x_1, \dots, x_6]$. If Γ is pure, then by Corollary 3.4, $\Gamma \simeq K_6$ or $K_{3,3}$ or $K_{2,2,2}$.*

Next, we shall give an elementary proof to show the range of the height of a pure full-supported matroidal ideal.

Theorem 3.6. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal of degree $d \ge 2$. If I is full-supported and is pure of height h, then $\frac{n}{d} \le h \le n - d + 1$. In particular, $h = n - d + 1$ if and only if I is square-free Veronese; and $h = \frac{n}{d}$ if and only if $I = J_1 J_2 \cdots J_d$ such that* $\text{supp}(J_i) \cap \text{supp}(J_j) = \varnothing$ *for distinct i, j and each J_i is generated by h variables.*

Proof. It suffices to show the inequalities $h + d - 1 \le n \le h\,d$ and we proceed by induction on d. It is easy to see that if $d = 2$, then it is the content of Corollary 3.3. Thus we assume now that $d \ge 3$. For $i = 1, \dots, n$, we let $S_i = \{\frac{u}{x_i} \mid u \in G(I),\ and\ x_i \mid u\}$ and I_i be the ideal generated by S_i. Then I_i is a matroidal ideal of degree $d-1$ with $\text{supp}(I_i) \subseteq \{x_1, \dots, \hat{x}_i, \dots, x_n\}$ and

$$I = \sum_{i=1}^{n} x_i I_i.$$

We first show that I_i is pure for each i; and by symmetry, we prove I_1 for example. Let $P_1, \dots, P_r$ be the minimal primes of I that contain x_1 and $Q_1, \dots, Q_s$ be the minimal primes of I that do not contain x_1; then

$$I = P_1 \cap \cdots \cap P_r \cap Q_1 \cap \cdots \cap Q_s$$

is a minimal primary decomposition of I. Therefore

$$x_1 I_1 \subseteq \langle x_1 \rangle \cap I \subseteq \langle x_1 \rangle \cap Q_1 \cap \cdots \cap Q_s = \langle x_1 \rangle \cdot (Q_1 \cap \cdots \cap Q_s) \subseteq x_1 I_1$$

as $\langle x_1 \rangle \cdot (Q_1 \cap \cdots \cap Q_s) \subseteq I$. Hence, $I_1 = Q_1 \cap \cdots \cap Q_s$ which is pure as $ht(Q_j) = h$ for all j.

To obtain the inequality $n \geq h + d - 1$, we let $t_i = |\operatorname{supp}(I_i)|$ for each $i \in \{1, \dots, n\}$. Since each I_i is an pure matroidal ideal of degree $d-1$ and of height h, $t_i \geq h+(d-1)-1$ by inductive assumptions. Therefore $n \geq h+d-1$ as $t_i \leq n-1$. We note that if I is square-free Veronese then $n = h+d-1$. Conversely, if $n = h+d-1$, then $t_i = n - 1 = h + (d-1) - 1$, so that I_i is square-free Veronese by inductive assumptions. This implies that I is square-free Veronese as $I = \sum_{i=1}^{n} x_i I_i$.

To obtain $n \leq hd$, we let $T_1 = \{ x_i \mid i \neq 1, x_1 x_i \text{ divides } u \text{ for some } u \in G(I)\}$. Then we have $T_1 \subseteq \operatorname{supp}(I_1) \subseteq \{x_2, \dots, x_n\}$. Choose $f_i \in (Q_1 \cap \dots \cap Q_s) - P_i$ for each $i \in \{1, \dots, r\}$ and let $y \in \{x_2, \dots, x_n\} - T_1$. It is easy to see that $x_1 f_i \in I$, so that $y f_i \in I \subseteq P_i$ by Lemma 2.3. This implies that $y \in P_i$ for each i and hence $h = ht(P_i) \geq 1 + n - 1 - |T_1| = n - t_1$, where $t_1 = |T_1|$. We note that I_1 is an pure matroidal ideal of degree $d-1$ and of height h. Hence, by inductive assumptions we have $h(d-1) \geq |\operatorname{supp}(I_1)| \geq t_1 \geq n - h$. This then implies $hd \geq n$. It is clear that if $I = J_1 J_2 \cdots J_d$ such that each J_i is generated by h distinct variables, then $n = hd$. Conversely, if $n = hd$, then P_i is generated by the set $\{x_1, \dots, x_n\} - T_1$, so that $r = 1$ and $I = P_1 \cdot (Q_1 \cap \dots \cap Q_s)$. The assertion follows by induction. □

4. Arithmetical rank of a matroidal ideal

Let R be a Noetherian ring and I be an ideal of R. We say that the elements $z_1, \dots, z_m \in R$ *generate I up to radical* if $\sqrt{\langle z_1, \dots, z_m \rangle} = \sqrt{I}$. The minimal number m with this property is called the *arithmetical rank of I*, denoted by $ara\ I$. It is known that

$$ht(I) \leq ara\ I \leq \mu(I),$$

where $ht(I)$ and $\mu(I)$ denote the height of I and the minimal number of generators for I respectively. We call I a *set-theoretic complete intersection* when $ht(I) = ara\ I$.

The goal of this section is to study the *arithmetical rank* of a matroidal ideal. Before we start, we first introduce some useful results.

Lemma 4.1. [7] *Let P be a finite subset of a ring R. Let $P_0, \dots, P_r$ be subsets of P such that the following three conditions are satisfied.*
(i) $\bigcup_{i=0}^{r} P_i = P$.
(ii) P_0 has exactly one element.
(iii) If p and p' are different elements of P_i $(0 < i \leq r)$ then there exists $h \in P_j$ for some $j < i$ such that h divides pp'.
Let $q_i = \sum_{p \in P_i} p$. Then $\sqrt{\langle P \rangle} = \sqrt{\langle q_0, \dots, q_r \rangle}$.

Lemma 4.2. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal. Then $ara\ I \geq q(I) + 1$.*

Proof. From [5], we know that for square-free monomial ideal I, the following inequality holds:

$$pd_R\ R/I \leq ara\ I.$$

Moreover, by [4, Corollary 1.6], we have

$$pd_R\ R/I = q(I) + 1.$$

Thus, the assertion holds. □

Proposition 4.3. *Let $I \subseteq K[x_1, \cdots, x_n]$ be a square-free monomial ideal and let $d = \min_i \{ d_i \mid d_i = \deg m_i,$ where $m_i \in G(I) \}$. Then $ara\, I \leq n - d + 1$.*

Proof. For $1 \leq t \leq n$, we denote

$$S_t = \{ n_k \mid n_k \in R \text{ is a monomial such that } deg\, n_k = t \}.$$

For $d \leq r \leq n$, we let

$$P_{n-r} = \{ n_j \in S_r \mid n_j \text{ is divisible by some } m_i \in G(I) \}.$$

Now, we let $\bigcup_{i=0}^{n-d} P_i = P$. It is easy to see that $\langle P \rangle = I$. We shall prove that the these sets P_i satisfy the conditions required in Lemma 4.1. Given two different elements $n_j, n_k \in P_{n-r}$, then there must exist $m_j \in G(I)$ and $m_k \in T$ such that m_j divides n_j and m_k divides n_k. Since n_j and n_k are distinct, there exists one variable x_l such that x_l divides n_j but n_k. We note that $n_j x_l \in S_{r+1}$ and m_j divides $n_j x_l$ as m_j divides n_j. Therefore, $n_j x_l \in P_{n-r-1}$ and $n_j x_l$ divides $n_j n_k$. So the conditions required in Lemma 4.1 is fulfilled. Hence, we obtain that $\sqrt{\langle P \rangle}$ can be generated by $n - d + 1$ elements and conclude that

$$ara\, I = ara \sqrt{\langle P \rangle} \leq n - d + 1.$$

□

We now prove the main theorem in this section.

Theorem 4.4. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a matroidal ideal of degree d. Then $ara\ I = q(I) + 1$. In particular, if I is full-supported, then $ara\ I = n - d + 1$.*

Proof. It suffices to show the case when I is full-supported. We note that

$$q(I) + 1 \leq ara\, I \leq n - d + 1$$

by Lemma 4.1 and Proposition 4.3. Moreover, we have $q(I) = n - d$ by Theorem 2.5. Therefore, $ara\ I = q(I) + 1$. This completes the proof. □

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ON TORIC SCHEMES

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ABSTRACT. Studying toric varieties from a scheme-theoretical point of view leads to toric schemes, i.e. "toric varieties over arbitrary base rings". It is shown how the base ring affects the geometry of a toric scheme. Moreover, generalisations of results by Cox and Mustaţă allow to describe quasicoherent sheaves on toric schemes in terms of graded modules. Finally, a toric version of the Serre-Grothendieck correspondence relates cohomology of quasicoherent sheaves on toric schemes to local cohomology of graded modules.

0. FROM TORIC VARIETIES TO TORIC SCHEMES

During the last forty years a huge amount of work on toric varieties was and still is published. Their theory was generalised in several directions, and this often lead to a better understanding of classical toric varieties. However, the generalisation that seems to be the most natural and the most important – the ***study of toric varieties from a scheme-theoretical point of view*** – was never actually carried out. It is clear that to do this one has to be able to make arbitrary base changes. Hence, instead of considering toric varieties over an algebraically closed field (or, as often done, over the field of complex numbers), one needs to study *toric schemes*, that is "toric varieties over arbitrary base rings". Special cases of this generalisation were mentioned briefly in [3, §4] (for regular fans and mainly over the ring of integers) and [9, IV.3] (over discrete valuation rings). But unfortunately later authors seemed to ignore this, and hence the knowledge of toric schemes is very small compared to the one of toric varieties.

Besides yielding a better understanding of the geometry of toric varieties, there are concrete applications of the above generalisation, as the following remark shows.

(0) Let X be the toric variety over an algebraically closed field K associated with a fan Σ. A fundamental question in algebraic geometry is then if the Hilbert functor $\mathrm{Hilb}_{X/K}$ of X over K is representable, i.e. if the Hilbert scheme of X exists (cf. [7]). If X is projective, then this is indeed the case and follows from Grothendieck's more general result [7, Théorème 3.1]. However, toric varieties are not necessarily projective, and in general it is not known whether their Hilbert schemes exist. Studying $\mathrm{Hilb}_{X/K}$ amounts to studying quasicoherent sheaves on the base change $X \otimes_K R$ for every K-algebra R, and it turns out that $X \otimes_K R$ is the same as the toric scheme over R associated with Σ. Hence, in order to study Hilbert functors of toric varieties ***it is necessary to study toric schemes over more general bases than just over algebraically closed fields.***

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The development of a theory of toric schemes was begun in the PhD Thesis [11], and its contents were refined and extended in [12], [13], [14] and [15]. Here we give an overview of the most important results and refer the reader to the aforementioned sources for a more extensive treatment including proofs.

1. The geometry of toric schemes

We start by briefly describing the construction of toric schemes from fans. In [13], toric schemes are obtained as a special case of the more general construction of schemes from so-called openly immersive projective systems of monoids (also yielding the Cox schemes introduced below).

• *From now on let V be an $\mathbb{R}$-vector space of finite dimension n, let N be a $\mathbb{Z}$-structure on V (i.e. a subgroup of rank n of the additive group underlying V with $\langle N\rangle_{\mathbb{R}} = V$), and let $M := N^*$ denote the dual of N which is a $\mathbb{Z}$-structure on the dual V^* of V.*

An *N-polycone (in V)* is the set of $\mathbb{R}$-linear combinations with coefficients in $\mathbb{R}_{\geq 0}$ of a finite subset of N, and an N-polycone is called ***sharp*** if it does not contain a line. If σ is an N-polycone then a *face of* σ is a set of the form $\sigma \cap \mathrm{Ker}(u)$ for some $u \in \sigma^{\vee} \cap M$ (where $E^{\vee} := \{v \in V^* \mid v(E) \subseteq \mathbb{R}_{\geq 0}\}$ for a subset $E \subseteq V$). The set of faces of a (sharp) N-polycone is a finite set of (sharp) N-polycones. An *N-fan (in V)* is a finite set Σ of sharp N-polycones that is closed under taking faces and such that the intersection of two cones in Σ is a common face of both of them. By means of the relation "τ is a face of σ", denoted by $\tau \preccurlyeq \sigma$, we consider an N-fan as an ordered set.

• *From now on let Σ be an N-fan in V and let R be a ring*[1].

If $\sigma \in \Sigma$ then $\sigma^{\vee} \cap M$ is a torsionfree, cancellable, finitely generated submonoid of M, and if moreover $\tau \preccurlyeq \sigma$ then $\sigma^{\vee} \cap M$ is a submonoid of $\tau^{\vee} \cap M$. Taking spectra of algebras of monoids over R and setting $X_{\sigma}(R) := \mathrm{Spec}(R[\sigma^{\vee} \cap M])$ for $\sigma \in \Sigma$ we get an inductive system $(X_{\sigma}(R))_{\sigma \in \Sigma}$ of R-schemes over Σ. Its inductive limit exists and is an R-scheme denoted by $X_{\Sigma}(R) \to \mathrm{Spec}(R)$ and called *the toric scheme over R associated with Σ (and N)*. It can be understood as obtained by glueing $(X_{\sigma}(R))_{\sigma \in \Sigma}$ along $(X_{\sigma \cap \tau}(R))_{(\sigma,\tau) \in \Sigma^2}$.

The above construction of toric schemes gives rise to a contravariant functor X_{Σ} from the category of rings to the category of schemes together with a morphism $X_{\Sigma} \to \mathrm{Spec}$. Moreover, the functor X_{Σ} is compatible with base change in the following sense.

(1) Proposition ([13, 5.9]) *There is a canonical isomorphism*

$$X_{\Sigma}(\bullet) \cong X_{\Sigma}(R) \otimes_R \bullet$$

of contravariant functors from the category of R-algebras to the category of R-schemes.

In particular, if $\mathfrak{a} \subseteq R$ is an ideal then $X_{\Sigma}(R/\mathfrak{a})$ is canonically identified with a closed subscheme of $X_{\Sigma}(R)$.

The first important question is now of course how the base ring affects the geometry of a toric scheme. It turns out that some basic properties hold for all

[1] By a ring, group or monoid we always mean a commutative ring, group or monoid, respectively, and by an algebra we always mean a commutative, unital and associative algebra.

toric schemes, making them a class of "nice schemes". More precisely, on use of the above base change property we get the following result.

(2) Proposition ([13, 5.8]) *The R-scheme* $X_\Sigma(R) \to \operatorname{Spec}(R)$ *is separated, quasicompact, flat, and of finite presentation; it is faithfully flat if and only if* $\Sigma \neq \emptyset$ *or* $R = 0$.

In contrast, a lot of other basic properties are respected and reflected by X_Σ. The following statements are proved by reducing to the affine case, i.e. X_σ, and then applying corresponding results about algebras of monoids (see e.g. [6]).

(3) Proposition ([13, 5.8]) *a) The scheme* $X_\Sigma(R)$ *is reduced, connected, or normal if and only if* R *is so or* $\Sigma = \emptyset$*; it is irreducible, or integral if and only if* R *is so and* $\Sigma \neq \emptyset$.

b) If $\Sigma \neq \emptyset$ *then there is a bijection* $\mathfrak{p} \mapsto X_\Sigma(R/\mathfrak{p})$ *from the set of minimal prime ideals of* R *to the set of irreducible components of* $X_\Sigma(R)$.

c) The scheme $X_\Sigma(R)$ *is Noetherian if and only if* R *is so or* $\Sigma = \emptyset$*; it is Artinian if and only if* R *is so and* $n = 0$*, or* $R = 0$*, or* $\Sigma = \emptyset$.

d) If $\Sigma \neq \emptyset$ *then*

$$\dim(R) + n \leq \dim(X_\Sigma(R)) \leq (n+1)\dim(R) + n;$$

if R *is moreover Noetherian then* $\dim(R) + n = \dim(X_\Sigma(R))$.

e) If R *is Noetherian, then* $X_\Sigma(R)$ *is equidimensional if and only if* R *is so or* $\Sigma = \emptyset$.

The above shows in particular that on general toric schemes *no satisfying theory of Weil divisors is available*. Since a lot of results about toric varieties were proved by heavy use of Weil divisor techniques (see e.g. [2], [5]), one has to come up with new proofs in order to generalise these results to toric schemes.

Finally, as an example of a property depending on the fan but not on the base ring we consider properness. Its characterisation needs the notion of a *complete* N-fan Σ, i.e. an N-fan Σ with $\bigcup \Sigma = V$. This result is well-known for toric varieties (see e.g. [5, 2.4]), and proved on use of torus operations for toric schemes associated with regular fans in [3, §4 Proposition 4]. Our proof for arbitrary fans avoids speaking of torus operations and relies only on the valuative criterion for properness and on properties of projections of fans proved in [12].

(4) Proposition ([13, 6.12]) *The R-scheme* $X_\Sigma(R) \to \operatorname{Spec}(R)$ *is proper if and only if* Σ *is complete, or* $\Sigma = \emptyset$*, or* $R = 0$.

2. Sheaves on toric schemes

Generalising work by Cox [2] and Mustaţă [10] we introduce a notion of Cox ring (not to be confused with the one introduced in [8]) and describe quasicoherent modules on toric schemes in terms of graded modules over these rings. In order to do so we need to define some objects encoding the combinatorics of the fan Σ.

Let Σ_1 denote the set of 1-dimensional cones in Σ. Every $\rho \in \Sigma_1$ has a unique minimal N-generator (i.e. an $x \in N$ with $\rho = \mathbb{R}_{\geq 0}x$ such that $rx \notin N$ for every $r \in]0,1[$), denoted by ρ_N. There is an exact sequence of groups

$$M \xrightarrow{c} \mathbb{Z}^{\Sigma_1} \xrightarrow{a} A \longrightarrow 0,$$

where $c(u) := (u(\rho_N))_{\rho\in\Sigma_1}$ for $u \in M$ and where a is defined as the cokernel of c. Note that c is a monomorphism if and only if Σ if *full*, i.e. $\langle\bigcup\Sigma\rangle_{\mathbb{R}} = V$. We denote by $(\delta_\rho)_{\rho\in\Sigma_1}$ the canonical basis of $\mathbb{Z}^{\Sigma_1}$ and we set $\alpha_\rho := a(\delta_\rho)$ for $\rho \in \Sigma_1$.

Now, we denote by S the polynomial algebra $R[(Z_\rho)_{\rho\in\Sigma_1}]$ in indeterminates $(Z_\rho)_{\rho\in\Sigma_1}$ over R, furnished with the A-graduation induced by a, i.e. such that $\deg(Z_\rho) = \alpha_\rho$ for $\rho \in \Sigma_1$. For $\sigma \in \Sigma$ we set $\widehat{Z}_\sigma := \prod_{\rho\in\Sigma_1\setminus\sigma_1} Z_\rho \in S$ (where σ_1 denotes the set of 1-dimensional faces of σ) and $\widehat{\alpha}_\sigma := \deg(\widehat{Z}_\sigma) \in A$, and we define a graded ideal $I := \langle\widehat{Z}_\sigma \mid \sigma \in \Sigma\rangle_S$.

- *From now on let $B \subseteq A$ be a subgroup.*

The B-graded R-algebra $S_B := \bigoplus_{\alpha\in B} S_\alpha$ obtained from S by degree restriction to B is called ***the B-restricted Cox ring over R associated with Σ_1 (and N)***, and its graded ideal $I_B := I \cap S_B$ is called ***the B-restricted irrelevant ideal over R associated with Σ (and N)***. One can show that I_B is generated by finitely many monomials.

To proceed we need to "invert the monomials $\widehat{Z}_\sigma$ in the Cox ring", and hence we have to assure that some power of these monomials lies in S_B. This amounts to supposing that B is "big enough" in the following sense: B is called ***big (with respect to Σ)*** if it has finite index in $B + \langle\widehat{\alpha}_\sigma \mid \sigma \in \Sigma\rangle_{\mathbb{Z}}$. Clearly, A is big.

- *From now on suppose that B is big, so that there exists $m \in \mathbb{N}_0$ with $\widehat{Z}_\sigma^m \in S_B$ for every $\sigma \in \Sigma$.*

For $\sigma \in \Sigma$ the B-graded ring of fractions $(S_B)_{\widehat{Z}_\sigma^m}$ is independent of the choice of m. Its component of degree 0 is independent of the choice of B and is denoted by $S_{(\sigma)}$. Moreover, for $\tau \preccurlyeq \sigma$ there is a canonical morphism of rings $S_{(\sigma)} \to S_{(\tau)}$ which is independent of m and B. Taking spectra and setting $Y_{(\sigma)}(R) := \operatorname{Spec}(S_{(\sigma)})$ for $\sigma \in \Sigma$ we obtain an inductive system $(Y_\sigma(R))_{\sigma\in\Sigma}$ of R-schemes over Σ. Its inductive limit exists and is an R-scheme denoted by $Y_\Sigma(R) \to \operatorname{Spec}(R)$ and called ***the Cox scheme over R associated with Σ (and N)***. It can be understood as obtained by glueing $(Y_\sigma(R))_{\sigma\in\Sigma}$ along $(Y_{\sigma\cap\tau}(R))_{(\sigma,\tau)\in\Sigma^2}$.

The above construction of Cox schemes gives rise to a contravariant functor Y_Σ from the category of rings to the category of schemes together with a morphism $Y_\Sigma \to \operatorname{Spec}$, and Y_Σ is compatible with base change in the sense of (1).

Cox schemes are closely related to toric schemes as follows. The morphism of groups $c : M \to \mathbb{Z}^{\Sigma_1}$ induces morphisms of rings $R[\sigma^\vee \cap M] \to S_{(\sigma)}$ for $\sigma \in \Sigma$, and these induce a canonical morphism of contravariant functors $\gamma : Y_\Sigma \to X_\Sigma$. Then, we have the following result.

(5) Proposition ([14, 3.16]) *The canonical morphism of contravariant functors $\gamma : Y_\Sigma \to X_\Sigma$ is an isomorphism if and only if Σ is full.*

Using the (non-canonical) procedure to consider a toric scheme associated with a non-full fan as a toric scheme associated with a full fan ([13, 5.10]) it is sufficient to study from now on Cox schemes instead of toric schemes. (Note that this reduction demands a base change and is in general *not available for toric varieties.*)

Now we are ready to explain how B-graded S_B-modules give rise to quasicoherent sheaves on $Y_\Sigma(R)$. We denote by $\mathsf{GrMod}^B(S_B)$ and $\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma(R)})$ the categories of B-graded S_B-modules and of quasicoherent $\mathscr{O}_{Y_\Sigma(R)}$-modules. Moreover, for a B-graded S_B-module F we denote by $F_{(\sigma)}$ the component of degree 0 of the B-graded module of fractions $F_{\widehat{Z}_\sigma^m} = F \otimes_{S_B} (S_B)_{\widehat{Z}_\sigma^m}$, and for an $S_{(\sigma)}$-module G we denote by $\widetilde{G}$ the $\mathscr{O}_{Y_\sigma(R)}$-module associated with G.

(6) Proposition ([14, 4.2]) *There exists a unique functor*

$$\mathscr{S}_B : \mathsf{GrMod}^B(S_B) \to \mathsf{QCMod}(\mathscr{O}_{Y_\Sigma(R)})$$

with $\mathscr{S}_B(F) \upharpoonright_{Y_\sigma(R)} = \widetilde{F_{(\sigma)}}$ *for every* $\sigma \in \Sigma$ *and every B-graded S_B-module F.*

Since $\mathscr{S}_B$ coincides locally with the canonical equivalence between modules and quasicoherent sheaves on affine schemes, and hence it is exact and commutes with inductive limits. Furthermore, we can construct a right quasiinverse

$$\Gamma_*^B(\bullet) := \bigoplus_{\alpha \in B} (\bullet \otimes_{\mathscr{O}_{Y_\Sigma(R)}} \mathscr{S}_B(S(\alpha)))$$

for $\mathscr{S}_B$, called *the first total functor of sections associated with Σ and B over R*. Thus, we get the following generalisation of [10, Theorem 1.1], itself a generalisation of [2, Theorem 3.2].

(7) Theorem ([14, 4.22]) *The functor* $\mathscr{S}_B : \mathsf{GrMod}^B(S_B) \to \mathsf{QCMod}(\mathscr{O}_{Y_\Sigma(R)})$ *is essentially surjective.*

Next, we restrict our attention to ideals. A graded ideal $\mathfrak{a} \subseteq S_B$ is called *I_B-saturated* if $\mathfrak{a} = \bigcup_{k \in \mathbb{N}_0} (\mathfrak{a} :_{S_B} I_B^k)$. Let $\mathbb{J}_B^{\mathrm{sat}}$ and $\widetilde{\mathbb{J}}$ denote the sets of I_B-saturated graded ideals of S_B and of quasicoherent ideals of $\mathscr{O}_{Y_\Sigma(R)}$, respectively. Then, $\mathscr{S}_B$ induces by exactness a map $\Xi_B : \mathbb{J}_B^{\mathrm{sat}} \to \widetilde{\mathbb{J}}$. The next result treats the question whether this map is surjective or injective. To get injectivity, besides being "big enough" the subgroup B must not be "too big". More precisely, B is called *small (with respect to Σ)* if it is contained in $\bigcap_{\sigma \in \Sigma} \langle \{\alpha_\rho \mid \rho \in \Sigma_1 \setminus \sigma_1\} \rangle_{\mathbb{Z}}$.

(8) Theorem ([14, 4.28]) *The map* $\Xi_B : \mathbb{J}_B^{\mathrm{sat}} \to \widetilde{\mathbb{J}}$ *is surjective, and if B is small then it is bijective.*

An example of a subgroup that is big and small (and moreover well understood) is given in the following remark (cf. [4, V.5]).

Consider a family $(U_\sigma)_{\sigma \in \Sigma}$ of subsets of V^* such that for every $\sigma \in \Sigma$ there exists a (not necessarily unique) $m_\sigma \in M$ with $U_\sigma = m_\sigma + \sigma^\vee$. Such a family is called a *virtual polytope over* Σ if $\tau \subseteq \mathrm{Ker}(m_\sigma - m_\tau)$ for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$, and this condition is independent of the choice of the family $(m_\sigma)_{\sigma \in \Sigma}$. There is a canonical structure of group on the set of virtual polytopes over Σ, and the set of virtual polytopes of the form $(m + \sigma^\vee)_{\sigma \in \Sigma}$ is a subgroup. The corresponding quotient group is denoted by $\mathrm{Pic}(\Sigma)$ and called *the Picard group of* Σ. It can be considered as the group of virtual polytopes over Σ modulo M-rational translations.

The map

$$(m_\sigma + \sigma^\vee)_{\sigma \in \Sigma} \mapsto (m_\rho(\rho_N))_{\rho \in \Sigma_1}$$

yields a monomorphism from the group of virtual polytopes over Σ to $\mathbb{Z}^{\Sigma_1}$, and this induces a monomorphism $\mathrm{Pic}(\Sigma) \rightarrowtail A$ by means of which we consider $\mathrm{Pic}(\Sigma)$ as a subgroup of A. Then, $\mathrm{Pic}(\Sigma)$ is small, and if Σ is simplicial then $\mathrm{Pic}(\Sigma)$ is big. Hence, it provides an example of a subgroup of A to which (8) can be applied.

Finally, since $\mathrm{Pic}(\Sigma) \cong \mathrm{Pic}(X_\Sigma(\mathbb{C}))$ by [4, Theorem VII.2.15] we get back [2, Corollary 3.9] as a special case.

Our results about quasicoherent sheaves in the last section reveals that toric schemes (or more precisely, Cox schemes) are very similar to projective schemes. Hence, we ask if there is a toric version of the Serre-Grothendieck correspondence (cf. [1, 20.4.4]), relating cohomology of quasicoherent sheaves on a Cox scheme to graded local cohomology of B-graded S_B-modules with respect to the irrelevant ideal I_B. This is indeed the case.

First, we have to explain what we mean by graded local cohomology. We denote by

$$ {}^B\Gamma_{I_B} : \mathsf{GrMod}^B(S_B) \to \mathsf{GrMod}^B(S_B) $$

the B-graded I_B-torsion functor. Its right derived cohomological functor is denoted by $({}^B H^i_{I_B})_{i\in\mathbb{Z}}$ and called *B-graded local cohomology with respect to I_B*. The reason for this clumsy notation is that the ungraded module underlying a graded local cohomology module of a graded module F might not be the same as the local cohomology module of the ungraded module underlying F. (A sufficient condition for this to hold is coherence of the graded ring S_B.)

Next, we introduce a variant of sheaf cohomology that is useful for our purpose. We define a functor

$$ \Gamma^B_{**}(\bullet) : \mathsf{GrMod}^B(S_B) \to \mathsf{GrMod}^B(S_B), $$

called *the second total functor of sections associated with Σ and B over R*, by setting

$$ \Gamma^B_{**}(\bullet) := \bigoplus_{\alpha\in B} \Gamma(Y_\Sigma(R), \mathscr{S}_B(\bullet(\alpha))), $$

where $\bullet(\alpha)$ denotes the functor of shifting degrees by α. Note that despite its name it is defined on the category $\mathsf{GrMod}^B(S_B)$. However, by (7) this is merely a technical point. The reason for two (in general different) total functors of sections is that the canonical morphism

$$ \mathscr{S}_B(\bullet) \otimes_{\mathscr{O}_{Y_\Sigma(R)}} \mathscr{S}_B(S_B(\alpha)) \to \mathscr{S}_B(\bullet(\alpha)) $$

is not necessarily an isomorphism. The right derived cohomological functor of $\Gamma^B_{**}(\bullet)$ is denoted by $(H^i_{**,B})_{i\in\mathbb{Z}}$ and contains the usual sheaf cohomology as a direct summand.

To go on we need a certain behaviour of injectives in the category $\mathsf{GrMod}^B(S_B)$. Namely, the B-graded ring S_B is said to have *the ITR-property with respect to I_B* if every B-graded I_B-torsion S_B-module has an injective resolution whose components are B-graded I_B-torsion S_B-modules. This is fulfilled for example if S_B is Noetherian (as a graded ring), and in particular if R is Noetherian. Using this notion and imitating the corresponding proof in the projective case we arrive at the Toric Serre-Grothendieck Correspondence.

(9) Theorem ([15, 5.10]) *If S_B has the ITR-property with respect to I_B, then there exist an exact sequence of functors*

$$ 0 \longrightarrow {}^B\Gamma_{I_B} \longrightarrow \mathrm{Id}_{\mathsf{GrMod}^B(S_B)} \longrightarrow \Gamma^B_{**} \xrightarrow{\zeta_B} {}^B H^1_{I_B} \longrightarrow 0 $$

and a unique morphism of δ-functors

$$ (\zeta^i_B)_{i\in\mathbb{Z}} : (H^i_{**,B})_{i\in\mathbb{Z}} \longrightarrow ({}^B H^{i+1}_{I_B})_{i\in\mathbb{Z}} $$

with $\zeta^0_B = \zeta_B$, and ζ^i_B is an isomorphism for every $i \in \mathbb{N}$.

As an application we can prove a toric version of Serre's Finiteness Theorem.

(10) Proposition ([15, 5.12]) *Let F be a finitely generated B-graded S_B-module, and suppose that Σ is complete and that R is Noetherian. Then, the R-modules $H^i_{**,B}(F)_\alpha$ and ${}^B H^i_{I_B}(F)_\alpha$ are finitely generated for every $i \in \mathbb{Z}$ and every $\alpha \in B$.*

Considering the fibres of a toric scheme, this allows us to define and investigate Hilbert functions of toric schemes, a task we would like to address in future research. Note that the above hypothesis of a complete fan Σ can be achieved by the Completion Theorem ([12], [13, 6.13]).

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REES ALGEBRAS AND SINGULARITIES

BERND ULRICH

1. INTRODUCTION

This is a report on joint work with A. Kustin and C. Polini [5], [6], [7] and with D. Cox, A. Kustin, and C. Polini [2]. Our basic setting is as follows: Let k be an algebraically closed field, $R = k[x,y]$ a polynomial ring in two variables, and I an ideal of R minimally generated by forms f_1, f_2, f_3 of the same degree $d > 0$. Extracting a common divisor we may harmlessly assume that I has height two. We will keep these assumptions throughout, though some of our results hold in greater generality.

On the one hand, the forms f_1, f_2, f_3 define a morphism

$$F: \mathbb{P}^1_k \xrightarrow{[f_1:f_2:f_3]} \mathbb{P}^2_k$$

whose image is a curve C. After reparametrizing we may assume that the map F is birational onto its image or, equivalently, that the curve C has degree d.

On the other hand, associated to f_1, f_2, f_3 is a syzygy matrix φ that gives rise to a homogeneous free resolution of the ideal I,

$$0 \longrightarrow R(-d-d_1) \oplus R(-d-d_2) \xrightarrow{\varphi} R(-d)^3 \longrightarrow I \longrightarrow 0\,.$$

Here φ is a 3 by 2 matrix with homogeneous entries in R, of degree d_1 in the first column and of degree d_2 in the second column. We may assume that $d_1 \leq d_2$. Notice that $d = d_1 + d_2$ by the Hilbert-Burch Theorem.

The two aspects, the curve C parametrized by the forms f_1, f_2, f_3 and the syzygy matrix φ of these forms, are mediated by the Rees algebra $\mathcal{R}$ of I. The Rees algebra is defined as the subalgebra $R[It] = R[f_1t, f_2t, f_3t]$ of the polynomial ring $R[t]$. It becomes a standard bigraded k-algebra if one sets deg $x =$ deg $y = (1,0)$ and deg $t = (-d, 1)$, which gives deg$f_it = (0,1)$. The biprojective spectrum of $\mathcal{R}$ is the graph $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ of the morphism $F = [f_1 : f_2 : f_3]$. Projecting to the second factor of $\mathbb{P}^1_k \times \mathbb{P}^2_k$ one obtains a surjection $\Gamma \twoheadrightarrow C$, which corresponds to an inclusion of coordinate rings

$$\mathcal{R} \hookleftarrow k[f_1t, f_2t, f_3t]\,.$$

Thus the coordinate ring $A(C)$ of the curve C can be recovered as a direct summand of the Rees algebra $\mathcal{R}$, namely

$$A(C) = \bigoplus_i \mathcal{R}_{(0,i)}\,.$$

The same holds for the ideal I,

$$I \simeq It = \bigoplus_i \mathcal{R}_{(i,1)}\,.$$

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Finally, the inclusion $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ corresponds to a homogeneous epimorphism $\mathcal{R} \twoheadleftarrow B$, where $B = k[x,y,T_1,T_2,T_3]$ is a bigraded polynomial ring with $\deg x = \deg y = (1,0)$ and $\deg T_i = (0,1)$, and the variables T_i are mapped to $f_i t$. The kernel of this epimorphism is a bihomogeneous ideal $\mathcal{J}$ of B, the 'defining ideal' of the Rees algebra $\mathcal{R}$. Now the syzygy module of I can be recovered as well,

$$\operatorname{syz}(I) \simeq \bigoplus_i \mathcal{J}_{(i,1)} \, .$$

Thus, the philosophy underlying this work can be summarized as follows: One wishes to study local properties of the rational plane curve $\mathcal{C}$, such as the types of its singularities, by means of the syzygies of I, since linear relations among polynomials are easier to handle than polynomial relations. The mediator is the Rees algebra, which in turn carries more information than the coordinate ring $A(\mathcal{C})$ of the curve, just like the graph of a map reveals more than the image of the map. One may therefore hope that even relatively simple numerical data associated to this algebra, such as the (first) bigraded Betti numbers, say a great deal about the curve. The syzygies of I appear in the defining ideal $\mathcal{J}$, which leads one to study defining ideals of Rees algebras. Finding such ideals or, equivalently, describing Rees rings explicitly in terms of generators and relations, is a fundamental problem in elimination theory, that has occupied commutative algebraists, algebraic geometers, and, more recently, applied mathematicians. The problem is wide open, even for ideals of polynomial rings in two variables. In the next section we will report on some recent progress in this direction.

2. Defining ideals of Rees rings: explicit formulas

In the setting described at the beginning of the Introduction, we have an explicit description of the defining ideal $\mathcal{J}$ of the Rees algebra in either of these cases:

- $d_1 = 1$ or, more generally, for ideals in $k[x,y]$ generated by any number of forms of the same degree as long as all but one generating syzygies are linear;
- $d_1 = 2$;
- $d_1 = d_2$.

In each case, we have explicit formulas for generators of the ideal $\mathcal{J}$, which tend to be quite complicating. Rather than listing them here, we refer to [6], [7].

We briefly outline our approach in the case where all but one syzygies are linear. The crucial observation is that the existence of 'many' linear syzygies forces the Rees algebra $\mathcal{R}$ to be a codimension one factor ring of the homogeneous coordinate ring A of a rational normal scroll. In order to describe $\mathcal{J}$ it suffices to know $A\mathcal{J}$. But the latter ideal is a height one prime ideal of A and hence gives rise to an element of the divisor class group $C\ell(A)$. This group is cyclic with a known generator. What remained to be done though was to exhibit, explicitly, divisorial A-ideals K_i, $i \in \mathbb{Z}$, so that $C\ell(A) = \{[K_i] \mid i \in \mathbb{Z}\}$. This amounted to computing the symbolic powers of a specific divisorial A-ideal. This is carried out in [5]. One then needed to determine which of the ideals K_i is isomorphic to $A\mathcal{J}$, and one had to make this isomorphism explicit in order to obtain formulas for generators of $A\mathcal{J}$, hence of $\mathcal{J}$. This is done in [6].

The cases $d_1 = 2$ and $d_1 = d_2$ are treated in [7]. We describe the former in more detail. Thus write $\mathfrak{m} = (x,y)$ for the homogeneous maximal ideal of R and S for the polynomial ring $k[T_1,T_2,T_3]$.

Recall that $B = k[x,y,T_1,T_2,T_3] = R \otimes_k S$ and that $\mathcal{R} = B/\mathcal{J}$. To study the Rees algebra of an ideal one customarily maps the symmetric algebra onto it,

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathrm{Sym}(I) \longrightarrow \mathcal{R} \longrightarrow 0\,.$$

One readily sees that $\mathcal{A} = H^0_{\mathfrak{m}}(\mathrm{Sym}(I)) = 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^\infty$. A presentation of the symmetric algebra is well understood, $\mathrm{Sym}(I) \simeq B/(g_1,g_2)$, where

$$[g_1,g_2] = [T_1,T_2,T_3] \cdot \varphi\,.$$

The polynomials g_i are homogeneous of bidegree $(d_i,1)$ and together they form a B-regular sequence. Thus the Koszul complex provides a homogeneous B-resolution of the symmetric algebra,

$$K_\bullet(g_1,g_2;B) \longrightarrow \mathrm{Sym}(I) \longrightarrow 0\,.$$

One computes local cohomology with support in $\mathfrak{m}$ along this resolution, uses the symmetry of the Koszul complex, and the isomorphism $H^2_{\mathfrak{m}}(R) \simeq \underline{\mathrm{Hom}}_k(R,k)(2)$, where $\underline{\mathrm{Hom}}$ denotes the graded dual. Thus one sees:

Proposition 2.1. *One has* $\mathcal{A} = H^0_{\mathfrak{m}}(\mathrm{Sym}(I)) \simeq \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I),S)(2-d,-2)$.

The goal is to compute the graded components $[\mathcal{A}]_{(i,\star)}$ of $\mathcal{A}$. Notice that $[\mathcal{A}]_{(0,\star)}$ is the defining ideal of the coordinate ring $A(C)$, which is generated by the resultant $\mathrm{Res}_{\{x,y\}}(g_1,g_2)$. Thus we may assume that $i > 0$. Proposition 2.1 shows in particular that there is an isomorphism of graded S-modules

$$[\mathcal{A}]_{(i,\star)} \simeq \mathrm{Hom}_S([\mathrm{Sym}(I)]_{(d-2-i,\star)},S)(-2)\,. \tag{2.1}$$

We now make use of the assumption $d_1 = 2$. In this case $d-2-i = d_1+d_2-2-i = d_2-i < d_2$. Thus, writing j for $d-2-i$ in (2.1), the task becomes to investigate

$$\mathrm{Hom}_S([\mathrm{Sym}(I)]_{(j,\star)},S)$$

for $j < d_2$. In this range,

$$[\mathrm{Sym}(I)]_{(j,\star)} \simeq [B/(g_1,g_2)]_{(j,\star)} = [B/(g_1)]_{(j,\star)}\,.$$

The last equality obtains because g_2 has degree $(d_2,1)$, whereas $j < d_2$.

We compute $\mathrm{Hom}_S([B/(g_1)]_{(j,\star)},S)$ for every j. To do so we notice that after elementary row operations and a linear change of variables we may assume that

$$\varphi = \begin{pmatrix} x^2 & \star \\ xy & \star \\ y^2 & \star \end{pmatrix} \quad \text{or} \quad \varphi = \begin{pmatrix} x^2+y^2 & \star \\ xy & \star \\ 0 & \star \end{pmatrix}.$$

The first case, where the entries of the first column of φ are k-linearly independent, has been treated by Busé, see [1]. Thus we focus on the second case.

Notice that $g_1 = x^2T_1 + y^2T_1 + xyT_2$. We consider a free presentation of graded S-modules,

$$S(-1)^{j-1} \xrightarrow{\Psi_j} S^{j+1} \longrightarrow [B/(g_1)]_{(j,\star)} \longrightarrow 0\,,$$

where Ψ_j is a matrix of linear forms in S that can be described explicitly. We then resolve the map $\mathrm{Hom}_S(\Psi_j,S)$. This gives, in particular, the S-module $\ker(\mathrm{Hom}_S(\Psi_j,S)) \simeq \mathrm{Hom}_S([B/(g_1)]_{(j,\star)},S)$.

Applying (2.1) we thus describe the modules $\mathcal{A}_i$ up to graded S-isomorphisms. One can make these isomorphisms explicit by using Joanolou's work on Morley forms, see [4]. Thus we obtain generators of the modules $\mathcal{A}_i$ and, eventually, a homogenous minimal generating set of the entire ideal $\mathcal{A}$.

Application. The results outlined in this section suffice to provide explicit defining equations for $\mathcal{R}$ if $d = d_1 + d_2 \leq 6$, since then $d_1 \leq 2$ or $d_1 = d_2$. We focus on the case $d = 6$, the case of a sextic curve.

As it turn out, there is, essentially, a one-to-one correspondence between the bidegrees of the defining equations of $\mathcal{R}$ on the one hand and the types of the singularities on or infinitely near the curve C on the other hand. Here one says that a singularity is infinitely near C if it is obtained from a singularity on C by a sequence of quadratic transformations. This one-to-one correspondence can be justified using the results of the next section. It is summarized in the following chart; the first column gives the possible values of d_1, d_2, namely 1,5 or 2,4 or 3,3; the second column lists the corresponding bidegrees of minimal generators of $\mathcal{J}$ together with the multiplicities by which they appear, suppressing the obvious bidegrees $(d_1, 1), (d_2, 1)$ (of the equations defining $\mathrm{Sym}(I)$) and $(0,6)$ (of the implicit equation of C); this second column reflects the results of the present section; the third column, finally, gives the multiplicities of the singularities on or infinitely near C.

d_1	d_2	equations of $\mathcal{R}$	singularities of C
1	5	(4,2) (3,3) (2,4) (1,5)	1 of multiplicity 5 on C
2	4	(2,2) 2(1,3)	1 of multiplicity 4 on C 4 double points on or near C
		(3,2) 3(2,3) 4(1,4)	10 double points on or near C
3	3	3(2,2) 4(1,4)	10 double points on or near C
		3(2,2) (1,3) 2(1,4)	1 of multiplicity 3 on C 7 double points on or near C
		3(2,2) 2(1,3)	2 of multiplicity 3 and 4 double points on or near C
		(2,2) (1,2) (1,4)	3 of multiplicity 3 and 1 double point on or near C

Notice that in the above chart, the constellation of 10 double points on or infinitely near the curve corresponds to two distinct numerical types of Rees algebras. Thus, the Rees algebra provides a finer distinction.

3. Relation to the syzygy matrix

The results described is this section are from [2]. Again, we use the notation introduced at the beginning of the Introduction. We are going to explain the interplay between the syzygy matrix φ of I and the types of singularities of the curve C. It is this connection that is also used to justify the chart at the end of Section 2.

Let $p \in \mathbb{P}^2_k$. Abusing notation, we also think of p as a nonzero row vector, and hence of $p \cdot \varphi$ as the row of a matrix obtained from φ by row operations; such a row is called a 'generalized row' of φ. Write $p \cdot \varphi = [a, b]$. One can show that $F^{-1}(p) = V(Ra + Rb) \subset \mathbb{P}^1_k$, see [3]. Is follows, in particular, that $p \in C$, equivalently $F^{-1}(p) \neq \emptyset$, if and only if $\mathrm{ht}(Ra + Rb) = 1$.

Notice that the morphism $F : \mathbb{P}^1_k \twoheadrightarrow C$ corresponds to the normalization map $\overline{A(C)} \hookleftarrow A(C)$ and, locally at every point $p \in C$, to the map $\overline{O_{C,p}} \hookleftarrow O_{C,p}$. Combining these facts with classical properties of integral closures of one-dimensional analytically unramified local rings one obtains the next result that relates generalized rows of φ and singularities of C; here $m_p = e(O_{C,p})$ denotes the multiplicity of C at the point p.

Theorem 3.1. *Let p be a point on C.*

(a) *Write $\gcd(a,b) = \prod \ell_i^{e_i}$ with ℓ_i pairwise non-associated linear forms in R and $e_i > 0$. There is a one-to-one correspondence between the ℓ_i and the branches of C through p, and the e_i are the multiplicities if these branches.*

(b) *In particular, $m_p = \deg(\gcd(a,b))$.*

Corollary 3.2. *One has:*

(a) *If p is a point on C, then $m_p \leq d_1$ or $m_p = d_2$.*

(b) *If q is a point infinitely near C, then $m_q \leq d_1$.*

Part (a) of the corollary is an immediate consequence of Theorem 3.1(b). Part (b) follows from an analogue of that theorem for points that are infinitely near C.

¿From now on we will always assume that $d_1 = d_2$, and we will write c for this number. Note that $c = d/2$. We recall that there is an explicit description of the defining ideal of the Rees algebra in this case, due to the results of Section 2. Furthermore, according to Corollary 3.2, every singularity on or infinitely near C has multiplicity at most c. For the remainder of this section we will focus on singularities having maximal multiplicity c. From Max Noether's formula for the genus,

$$\binom{d-1}{2} = \sum \binom{m_q}{2}$$

with q any point on or infinitely near C, it follows that there are at most three q with $m_q = c$. Since, moreover, $m_q \leq c$ and since quadratic transformations can only decrease the multiplicity, one obtains:

Theorem 3.3. *There are at most 7 configurations of multiplicity c singularities on or infinitely near the curve:* $\emptyset$ $\{c\}$ $\{c,c\}$ $\{c,c,c\}$ $\{c:c\}$ $\{c:c,c\}$ $\{c:c:c\}$.

Here a comma is meant to separate multiplicity c singularities on the curve, whereas a colon separates points of multiplicity c obtained from each other by blowing up. For instance, $\{c:c,c\}$ means that there are two multiplicity c singularities p_1, p_2 on the curve, one multiplicity c singularity obtained from p_1 by a quadratic transformation, and no other points of multiplicity c on or infinitely near C.

To understand how the 7 possible constellations of multiplicity c singularities correspond to certain normal forms of the syzygy matrix φ, we recall that a point $p \in C$ has multiplicity c if and only if $p \cdot \varphi = [a,b]$ with $\deg(\gcd(a,b)) = c$, see Theorem 3.1(b). As $\deg a = \deg b = c$, the latter condition means that after an elementary column operation the generalized row $p \cdot \varphi$ has a zero entry, i.e., $p \cdot \varphi \cdot q^t = 0$ for some $q \in \mathbb{P}^1_k$. There is a similar characterization for infinitely near singularities of multiplicity c. Thus one establishes the correspondence expressed in the next chart; here the matrices in the right column are obtained from φ by elementary row and column operations; by $Q_1,\ldots,Q_{c+1}$ we denote suitable k-linearly independent forms of degree c in R.

multiplicity c singularities	φ after row and column operations
none	$\begin{pmatrix} Q_1 & Q_4 \\ Q_2 & Q_5 \\ Q_3 & Q_6 \end{pmatrix}$ or $\begin{pmatrix} Q_1 & Q_4 \\ Q_2 & Q_5 \\ Q_3 & Q_1 \end{pmatrix}$ or $\begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \\ Q_3 & Q_4 \end{pmatrix}$
$\{c\}$	$\begin{pmatrix} Q_1 & Q_3 \\ Q_2 & Q_4 \\ 0 & Q_5 \end{pmatrix}$ or $\begin{pmatrix} Q_1 & Q_3 \\ Q_2 & Q_1 \\ 0 & Q_4 \end{pmatrix}$
$\{c,c\}$	$\begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_4 \end{pmatrix}$
$\{c,c,c\}$	$\begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_2 \\ 0 & Q_3 \end{pmatrix}$
$\{c:c\}$	$\begin{pmatrix} Q_1 & Q_3 \\ Q_2 & Q_4 \\ 0 & Q_2 \end{pmatrix}$
$\{c:c,c\}$	$\begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix}$
$\{c:c:c\}$	$\begin{pmatrix} Q_1 & Q_3 \\ Q_2 & Q_1 \\ 0 & Q_2 \end{pmatrix}$

This chart does not answer the following questions though: Given a syzygy matrix φ, is there a direct way to find its normal form? What is the constellation of multiplicity c singularities on or infinitely near the curve, and how can one locate these singularities?

To address these problems, we recall that singularities of multiplicity c on C are related to 'generalized zeros' of φ. A generalized zero is a zero entry obtained after row and column operations, $p \cdot \varphi \cdot q^t = 0$. In fact, we have just seen that the 'row operations' p are the multiplicity c singularities on C. Thus, to identify these singularities, we consider the subset of $\mathbb{P}^2_k \times \mathbb{P}^1_k$ consisting of all pairs (p,q) with $p \cdot \varphi \cdot q^t = 0$, and project to $\mathbb{P}^2_k$.

More specifically, let T_1,T_2,T_3 and u_1,u_2 be two sets of variables, so that

$$\mathbb{P}^2_k \times \mathbb{P}^1_k = \mathrm{BiProj}(k[T_1,T_2,T_3] \otimes_k k[u_1,u_2]) \ .$$

We define a $c+1$ by 2 matrix C with linear entries in $k[T_1,T_2,T_3]$ and a $c+1$ by 3 matrix A with linear entries in $k[u_1,u_2]$ via the matrix equations

$$[T_1,T_2,T_3] \cdot \varphi \cdot [u_1,u_2]^t = [x^c, x^{c-1}y, \ldots, y^c] \cdot C \cdot [u_1,u_2]^t = [x^c, x^{c-1}y, \ldots, y^c] \cdot A \cdot [T_1,T_2,T_3]^t \ .$$

These matrices exist because the entries of φ are forms of degree c in $k[x,y]$. We consider the subscheme

$$Z = V(I_1(C \cdot [u_1,u_2]^t)) \subset \mathbb{P}^2_k \times \mathbb{P}^1_k$$

and its images $Z_1 \subset \mathbb{P}^2_k$ and $Z_2 \subset \mathbb{P}^1_k$ under the projections onto the first and second factor, respectively. Here I_i denotes the ideal generated by the i by i minors of a given matrix. One obtains the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ \mathbb{P}^2_k \supset Z_1 & & & & Z_2 \subset \mathbb{P}^1_k \end{array}$$

The sets supporting these schemes are

$$\begin{aligned} Z_{\text{red}} &= \{(p,q) \mid p \cdot \varphi \cdot q^t = 0\} \\ (Z_1)_{\text{red}} &= \{p \mid p \cdot \varphi \cdot q^t = 0 \text{ for some } q\} \\ &= \{p \mid p \text{ is a multiplicity } c \text{ singularity on } \mathcal{C}\} \\ (Z_2)_{\text{red}} &= \{q \mid p \cdot \varphi \cdot q^t = 0 \text{ for some } p\}\ . \end{aligned}$$

It will be important though to keep in mind that Z, Z_1, and Z_2 also have a scheme structure.

Theorem 3.4. *For the schemes Z, Z_1, Z_2 one has:*

(a) *The projection maps π_1 and π_2 are isomorphisms, and hence induce an isomorphism between Z_1 and Z_2.*
(b) $Z_1 = V(I_2(C))$ *and* $Z_2 = V(I_3(A))$.

Part (a) of the theorem says that information about Z_1, the primary object of interest, can be obtained from Z_2, a subscheme of $\mathbb{P}^1_k$. Part (b) provides a description of both subschemes, Z_1 and Z_2, in terms of the matrices C and A that can be easily extracted from φ.

Theorem 3.5. *Let $\gcd(I_3(A))$ be a greatest common divisor of polynomials generating $I_3(A)$ and write $\gcd(I_3(A)) = \prod \lambda_i^{f_i}$, were λ_i are pairwise non-associated linear forms in $k[u_1,u_2]$ and f_i are positive integers.*

(a) *There is a one-to-one correspondence between the linear forms λ_i and the singularities p_i of multiplicity c on $\mathcal{C}$, induced by the isomorphism $\pi_1\pi_2^{-1}$.*
(b) *The integer $f_i - 1$ is the number of singularities of multiplicity c that are infinitely near p_i.*

The theorem provides a great deal of information about the multiplicity c singularities on or infinitely near the curve, based on data that are readily available from the syzygy matrix. Part (a) is an immediate consequence of Theorem 3.4 and the description of Z_1 as a scheme of multiplicity c singularities on $\mathcal{C}$. The proof of (b) is more involved – it uses the correspondence between constellations of singularities and normal forms of the syzygy matrix that is described in the chart of Section 3.

Corollary 3.6. *The number of multiplicity c singularities on or infinitely near $\mathcal{C}$ equals the multiplicities*

$$e(k[T_1,T_2,T_3]/I_2(C)) = e(k[u_1,u_2]/I_3(A)),$$

and is equal to $6 - \mu(I_2(C))$, where μ denotes minimal number of generators.

The corollary follows from Theorem 3.4(a) and Theorem 3.5, except for the last statement involving minimal numbers of generators. This number of generators, $\mu(I_2(C))$, also plays a role in our work [7], where it is shown to characterize the numerical type of the Rees algebra, meaning the bidegrees of its defining equations. The reader may want to verify this for $d_1 = d_2 = 3$ by examining the chart of Section 2 – indeed, the chart shows that distinct numerical types of Rees rings correspond to distinct total numbers of triple points on or infinitely near C.

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Local slice construction and wild automorphisms

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Let k be a field of characteristic zero, and $k[\mathbf{x}] = k[x_1, \dots, x_n]$ the polynomial ring in n variables over k. An automorphism $\phi \in \operatorname{Aut}_k k[\mathbf{x}]$ is said to be *elementary* if there exist $c \in k^\times$, $i \in \{1, \dots, n\}$ and $f \in k[\{x_j \mid j \neq i\}]$ such that $\phi(x_i) = cx_i + f$, and $\phi(x_j) = x_j$ for $j \neq i$. The subgroup $\mathrm{T}(k, \mathbf{x})$ of $\operatorname{Aut}_k k[\mathbf{x}]$ generated by all the elementary automorphisms of $k[\mathbf{x}]$ is called the *tame subgroup*. An element of $\operatorname{Aut}_k k[\mathbf{x}]$ is said to be *tame* if it belongs to $\mathrm{T}(k, \mathbf{x})$, and *wild* otherwise. Due to Jung [3], it holds that $\operatorname{Aut}_k k[\mathbf{x}] = \mathrm{T}(k, \mathbf{x})$ when $n \leq 2$. When $n = 3$, it was a longstanding open question whether $\operatorname{Aut}_k k[\mathbf{x}] = \mathrm{T}(k, \mathbf{x})$ (cf. [10]). Shestakov-Umirbaev [11] recently answered this question by showing that $\operatorname{Aut}_k k[\mathbf{x}] \neq \mathrm{T}(k, \mathbf{x})$. They not only settled the question, but also gave a criterion for deciding tameness of elements of $\operatorname{Aut}_k k[\mathbf{x}]$ for $n = 3$. The theory of Shestakov-Umirbaev was generalized and improved by us in [4] and [5]. This makes it possible to show the wildness of elements of $\operatorname{Aut}_k k[\mathbf{x}]$ more easily and efficiently. Using this, we showed that various elements of $\operatorname{Aut}_k k[\mathbf{x}]$ are wild. These results are announced in the series of papers [7], [8] and [9] with a total of nearly hundred pages (see also [6] for a summary of these three papers). Part of the results were reported in the fifth Japan-Vietnam Joint Seminar in Hanoi. In this talk, we report the results in [9] most of which are newly obtained after the seminar.

Let D be a *locally nilpotent derivation* of $k[\mathbf{x}]$, i.e., an element of $\operatorname{Der}_k k[\mathbf{x}]$ such that $D^l(f) = 0$ holds for some $l \in \mathbf{N}$ for each $f \in k[\mathbf{x}]$. Then, an element $\exp D$ of $\operatorname{Aut}_k k[\mathbf{x}]$ is defined by

$$(\exp D)(f) = \sum_{i \geq 0} \frac{D^i(f)}{i!}$$

for each $f \in k[\mathbf{x}]$. The *rank* of D is by definition the minimal number $r \geq 0$ for which $D(\sigma(x_i)) \neq 0$ holds for $i = 1, \dots, r$ for some $\sigma \in \operatorname{Aut}_k k[\mathbf{x}]$. When $n = 3$, locally nilpotent derivations of rank at most two are rather easy and well-understood. On the other hand, locally nilpotent derivations of $k[\mathbf{x}]$ of rank three are difficult to study. Actually, until Freudenburg [2] gave some examples in 1998, it was even unknown whether there exists such a locally nilpotent derivation. It is not easy to construct a locally nilpotent derivation of $k[\mathbf{x}]$ of rank three, for which it is previously not known whether $\exp D$ is tame. In this talk, we give a large family of locally nilpotent derivations of

$k[\mathbf{x}]$ by means of *local slice construction* due to Freudenburg [1], and determine tameness of $\exp D$ for each D. The family includes the locally nilpotent derivations of Freudenburg, and many other locally nilpotent derivations of rank three. The result is that $\exp D$ is wild whenever D is of rank three for the locally nilpotent derivations we give.

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On a characterization of cofinite complexes

～余有限複体の特徴付けについて～

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Abstract

Let R be a regular ring of dimension d, J an ideal of R, and $N^\bullet \in \mathcal{D}^+(R)$ a complex bounded below. Our purpose of this report is to give equivalent conditions for $N^\bullet$ to be J-cofinite, which is a characterization of cofinite complexes in the derived category.

1 Introduction

We assume that all rings are commutative and noetherian with identity throughout this paper.

In this report, we shall prove the following result.

Claim 1 *Let R be a ring. Let $N^\bullet \in \mathcal{D}^+(R)$ be a complex, J an ideal of R. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}^j(R/J, N^\bullet)$ *is of finite type over R for all j;*

(ii) $\mathrm{Ext}^j(R/\sqrt{J}, N^\bullet)$ *is of finite type over R for all j;*

(iii) $\mathrm{Ext}^j(R/P, N^\bullet)$ *is of finite type over R for all j and for each $P \in \mathrm{Min}(R/J)$;*

(iv) $\mathrm{Ext}^j(W, N^\bullet)$ *is of finite type over R for all j and for each finitely generated R-module W such that* $\mathrm{Supp}\, W \subseteq V(J)$;

(v) $\mathrm{Ext}^j(W^\bullet, N^\bullet)$ *is of finite type over R for all j and for each $W^\bullet \in \mathcal{D}^b_{ft}(R)$ such that* $\mathrm{Supp}\, H^l(W^\bullet) \subseteq V(J)$ *for all l;*

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(vi) $\mathrm{Ext}^j(W^\bullet, N^\bullet)$ *is of finite type over* R *for all* j *and for each* $W^\bullet \in \mathcal{D}^-_{ft}(R)$ *such that* $\mathrm{Supp}\, H^l(W^\bullet) \subseteq V(J)$ *for all* l.

The statement of this claim is included in [4, Lemma 8] as a lemma without proofs. So we prove Claim 1 in this report.

Next we recall the following lemma (cf. [3, lemma 1]), which is motivated to prove Claim 1:

Lemma 2 *Let* A *be a ring,* I *an ideal, and* T *an* A*-module. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}^i(A/I, T)$ *is of finite type for all* $i \geq 0$;

(ii) $\mathrm{Ext}^i(N, T)$ *is of finite type for all* $i \geq 0$ *and for all* A*-modules* N *of finite type with support in* $V(I)$,

(iii) $\mathrm{Ext}^i(A/P, T)$ *is of finite type for all* $i \geq 0$ *and for each minimal prime ideal* P *in* $V(I)$.

Further we recall the result due to Hartshorne in [1, Theorem 5.1]:

Theorem 3 (A characterization of cofinite complexes) *Let* $N^\bullet$ *be in* $\mathcal{D}^+(R)$. *Suppose that* R *is complete with respect to the* J*-adic topology. Then* $N^\bullet$ *is* J*-cofinite if and only if*

(a) $\mathrm{Supp}\, H^i(N^\bullet) \subseteq V(J)$ *for each* i, *and*

(b) $\mathrm{Ext}^j(R/J, N^\bullet)$ *is of finite type over* R, *for each* j *(See [1, Definition, p. 149] for the definition on* J*-cofiniteness).*

In [7], Lemma 2 was extended. In this report, we shall extend Lemma 2 as Claim 1 to a different direction in terms of the hyperexts.

2 Proofs of the lemmas and the claim

Lemma 4 *Let* A *be a ring,* I *an ideal of* A, $N^\bullet \in \mathcal{D}^+(A)$ *a complex. And let* $W^\bullet$ *be in* $\mathcal{D}^-(A/I)$ *(we also see this complex over* A *via the natural map* $A \to A/I$*). Then there is a spectral sequence between the hyperexts:*

$$E_2^{p,q} = \mathrm{Ext}^{\ p}(W^\bullet, \mathrm{Ext}^q(A/I, N^\bullet)) \Longrightarrow H^{p+q} = \mathrm{Ext}^{(\ p)+q}(W^\bullet, N^\bullet).$$

Proof. Let $P^\bullet$ be a projective resolution of $W^\bullet$ and $E^\bullet$ an injective resolution of $N^\bullet$. Consider a double complex $\mathrm{Hom}(P^\bullet, \mathrm{Hom}(A/I, E^\bullet))$, so the assertion follows from the Hom-Tensor adjunction (cf. [5, p. 53]. See also [6, Theorem 11.66 p. 365] for the version on modules). □

Proof of Claim 1

Proof. The following implications are clear: (iv) $\Rightarrow$ (i), (iv) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (iii), (v) $\Rightarrow$ (iv) and (vi) $\Rightarrow$ (v).

First we prove the implication (i) $\Rightarrow$ (iv). Considering Serre's composition series, it is enough to prove that for the case $W = R/P$, where $P \in \operatorname{Spec} R$. Now there is a spectral sequence in the first quadrant:

$$E_2^{p,q} = \operatorname{Ext}^p(R/P, \operatorname{Ext}^q(R/J, N^\bullet)) \Longrightarrow H^{p+q} = \operatorname{Ext}^{p+q}(R/P, N^\bullet)$$

by Lemma 4. Since $\operatorname{Ext}^q(R/J, N^\bullet)$ is of finite type for all q by assumption, the initial terms $E_2^{p,q}$ of the spectral sequence are of finite type for all p and all q. Therefore the abutment terms $H^n = \operatorname{Ext}^n(R/P, N^\bullet)$ are of finite type for all n, as required.

Secondly we prove the implication (ii) $\Rightarrow$ (i). It is straightforward from the above paragraph, applying the module W for $R/\sqrt{J}$, here we note that $\operatorname{Supp}(R/\sqrt{J}) = V(\sqrt{J}) = V(J)$.

Further we prove the implication (iii) $\Rightarrow$ (ii). We may assume that J is radical ideal of R. Let $J = P_1 \cap P_2 \cap \cdots \cap P_r$ be the primary decomposition of J.

We proceed by induction on r. For the case $r = 1$, we have the claim by assumption. Suppose that $r > 1$, and set $J_l = P_1 \cap P_2 \cap \cdots \cap P_l$ for a positive integer l with $1 \leq l \leq r$. Note that $J_r = J$ and $J_{r-1} \cap P_r = J$. There is a short exact sequence:

$$0 \longrightarrow J_{r-1}/J_{r-1} \cap P_r \longrightarrow R/J_{r-1} \cap P_r \longrightarrow R/J_{r-1} \longrightarrow 0. \qquad (*)$$

Since $J_{r-1}/J_{r-1} \cap P_r \simeq J_{r-1} + P_r/P_r$ and $\operatorname{Supp}(J_{r-1}/J_{r-1} \cap P_r) \subseteq V(P_r)$, $\operatorname{Ext}^j(J_{r-1}/J_{r-1} \cap P_r, N^\bullet))$ is of finite type over R for all j by assumption and the implication (i) $\Rightarrow$ (iv), which was just proven above. Form the short exact sequence $(*)$, we get the triangle in $\mathcal{D}(R)$ to which we apply the contravariant cohomology functor $\operatorname{Hom}_{D(R)}(\,-\,, T^j(N^\bullet))$. So we obtain the long exact sequence:

$$\cdots \longrightarrow \operatorname{Hom}_{D(R)}(R/I_{r-1}, T^j(N^\bullet)) \longrightarrow \operatorname{Hom}_{D(R)}(R/I, T^j(N^\bullet))$$
$$\longrightarrow \operatorname{Hom}_{D(R)}(I_{r-1}/I_{r-1} \cap P_r, T^j(N^\bullet)) \longrightarrow \cdots.$$

From the inductive hypothesis, it follows that $\operatorname{Ext}^j(R/I_{r-1}, N^\bullet) = \operatorname{Hom}_{D(R)}(R/I_{r-1}, T^j(N^\bullet))$ is of finite type over R for all j and consequently $\operatorname{Ext}^j(R/I, N^\bullet)$ is of finite type over R for all j.

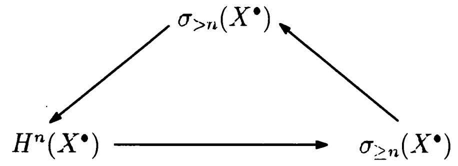
Next we shall prove the implication (iv) $\Rightarrow$ (v). Before proving the implication, we shall introduce the notations in [2, Definition, p. 69]. For a complex $X^\bullet \in \mathcal{D}(R)$, the truncated complexes are defined as follows:

$$\begin{aligned}
\sigma_{>n}(X^\bullet): &\quad \cdots \to 0 \longrightarrow 0 \longrightarrow \operatorname{Im} d^n \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \cdots, \\
\sigma_{\leq n}(X^\bullet): &\quad \cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow \ker d^n \longrightarrow 0 \longrightarrow 0 \to \cdots.
\end{aligned}$$

Then there is an exact sequence in $C(R)$:

$$0 \longrightarrow K^\bullet \longrightarrow \sigma_{\geq n}(X^\bullet) \longrightarrow \sigma_{>n}(X^\bullet) \longrightarrow 0,$$

where $K^\bullet$ is a complex: $0 \longrightarrow \operatorname{Im} d^{n-1} \longrightarrow \ker d^n \longrightarrow 0$ and $C(R)$ is a category of complexes consisting of R-modules. Notice that the complex $K^\bullet$ is quasi-isomorphic to the complex consisting of a single module: $\cdots \to 0 \to H^n(X^\bullet) \to 0 \to \cdots$. So we have a triangle:



in the derived category $\mathcal{D}(R)$.

Now we prove the implication (iv) $\Rightarrow$ (v). Let $W^\bullet$ be a bounded complex in $\mathcal{D}^b_{ft}(R)$ with $\operatorname{Supp} H^l(W^\bullet) \subset V(J)$ for all l. Then we have $\sigma_{>n}(W^\bullet) \in \mathcal{D}^b_{ft}(R)$. We proceed by descending induction on n. Since $W^\bullet \in \mathcal{D}^b_{ft}(R)$, we have $\operatorname{Ext}^j(\sigma_{>n}(W^\bullet), N^\bullet) = \operatorname{Ext}^j(0, N^\bullet) = 0$ for $n \gg 0$. By assumption, it holds that $H^n(W^\bullet)$ is of finite type over R and $\operatorname{Supp}(H^n(W^\bullet)) \subseteq V(J)$. So $\operatorname{Ext}^j(H^n(W^\bullet), N^\bullet)$ is of finite type over R for all j, by the assumption (iv). Further $\operatorname{Ext}^j(H^n(\sigma_{>n}(W^\bullet)), N^\bullet)$ is of finite type over R for all j, by the inductive hypothesis. Applying the cohomology functor $\operatorname{Hom}_{D(R)}(\ -\ , T^j(N^\bullet))$ to the triangle, we have the long exact sequence in the category of R-modules:

$$\cdots \longrightarrow \operatorname{Hom}_{D(R)}(\sigma_{>n}(W^\bullet), T^j(N^\bullet)) \longrightarrow \operatorname{Hom}_{D(R)}(\sigma_{\geq n}(W^\bullet), T^j(N^\bullet))$$
$$\longrightarrow \operatorname{Hom}_{D(R)}(H^n(W^\bullet), T^j(N^\bullet)) \longrightarrow \cdots.$$

So $\operatorname{Ext}^j(\sigma_{\geq n}(W^\bullet), N^\bullet) = \operatorname{Hom}_{D(R)}(\sigma_{\geq n}(W^\bullet), T^j(N^\bullet))$ is of finite type over R for all j. Therefore $\operatorname{Ext}^j(W^\bullet, N^\bullet) = \operatorname{Hom}_{D(R)}(W^\bullet, T^j(N^\bullet))$ is of finite type over R for all j. We have done the proof of the implication.

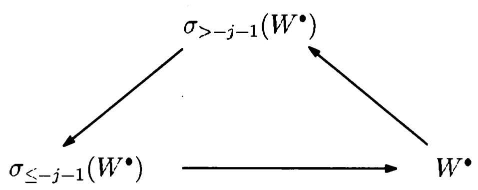
Finally we prove the implication (v) $\Rightarrow$ (vi). Since $N^\bullet \in \mathcal{D}^+_{fl}(R)$, we may assume that $N^i = 0$ for all $i < 0$. Now let n be an integer and suppose that $W^i = 0$ for all $i > n$, from the assumption that $W^\bullet \in \mathcal{D}_{ft}(R)$. Let j be any integer, and fixed. If $n < -j$, then we have

$$\operatorname{Ext}^j(W^\bullet, N^\bullet) = \operatorname{Hom}_{D(R)}(W^\bullet, T^j(N^\bullet)) = 0,$$

since $W^i = 0$ for $i > n$. If $n \geq -j$, then there is a short exact sequence:

$$0 \longrightarrow \sigma_{\leq -j-1}(W^\bullet) \longrightarrow W^\bullet \longrightarrow \sigma_{>-j-1}(W^\bullet) \longrightarrow 0,$$

in $C(R)$ for all $l \geq 0$. Then we have a triangle:



in the derived category $\mathcal{D}(R)$. Since $W^\bullet \in \mathcal{D}^-_{ft}(R)$, we have $\sigma_{>-j-1}(W^\bullet) \in \mathcal{D}^b_{fl}(R)$. From the assumption (iv), it follows that $\mathrm{Ext}^j(\sigma_{>-j-1}(W^\bullet), N^\bullet)$ is of finite type over R. On the other hand, we have $\mathrm{Ext}^j(\sigma_{\leq -j-1}(W^\bullet), N^\bullet) = \mathrm{Hom}_{D(R)}(\sigma_{\leq -j-1}(W^\bullet), T^j(N^\bullet)) = 0$. So we have an exact sequence:

$$\cdots \longrightarrow \mathrm{Ext}^j(\sigma_{>-j-1}(W^\bullet), N^\bullet) \longrightarrow \mathrm{Ext}^j(W^\bullet, N^\bullet) \longrightarrow \mathrm{Ext}^j(\sigma_{\leq -j-1}(W^\bullet), N^\bullet) = 0,$$

which prove that $\mathrm{Ext}^j(W^\bullet, N^\bullet)$ is of finite type over R for the j. Now an integer j was assumed to be any, so we have the assertion. □

Finally we introduce Hartshorne's Theorem (cf. [1, Theorem 5.1]). We rewrite the theorem as follows, combining it with Claim 1 together:

Theorem 5 (cf. [1]) *Let R be a regular ring of dimension d, J an ideal of R, and $N^\bullet \in \mathcal{D}^+(R)$ a complex bounded below. Suppose that R is complete with respect to the J-adic topology. Then the following conditions are equivalent:*

(i) *$N^\bullet$ is J-cofinite;*

(ii) (a) $\mathrm{Supp}(H^i(N^\bullet)) \subseteq V(J)$ *for all i, and*

(b) *The equivalent conditions in the above claim.*

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SECOND POWERS OF STANLEY-REISNER IDEALS

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(*joint work with Giancarlo Rinaldo*)

1. Introduction

Throughout this talk, let Δ be a *simplicial complex* on the vertex set $V = [n]$. Namely, Δ is a nonempty subset of the power set 2^V which satisfies the following two conditions: (i) $F \in \Delta$, $F' \subseteq F$ imply $F' \in \Delta$ and (ii) $\{v\} \in \Delta$ for all $v \in V$. An element $F \in \Delta$ is called a *face* of Δ. The *dimension* of F is defined by $\dim F = \sharp(F) - 1$, where $\sharp(F)$ denotes the cardinality of a set F. The dimension of Δ, denoted by $\dim \Delta$, is the maximum of the dimensions of all faces. A maximal face of Δ is called a *facet* of Δ, and $\mathcal{F}(\Delta)$ denotes the set of all facets of Δ. For any subset σ of V, x_σ denotes the squarefree monomial in $K[x_1, \ldots, x_n]$ with support σ.

In the following, suppose that $\dim \Delta = d - 1$, and let K be a field. The *Stanley–Reisner ideal* of Δ, denoted by I_Δ, is the squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$ generated by

$$\{x_{i_1}x_{i_2}\cdots x_{i_p} \,:\, 1 \le i_1 < \cdots < i_p \le n,\ \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta\},$$

and $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$ is called the *Stanley–Reisner ring* of Δ. Note that the Krull dimension of $K[\Delta]$ is equal to d.

Let G be a graph, which means a finite graph without loops and multiple edges. Let $V(G)$ (resp. $E(G)$) denote the set of vertices (resp. edges) of G. Put $V(G) = [n]$. Then the *edge ideal* of G, denoted by $I(G)$, is a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$ defined by

$$I(G) = (x_i x_j \,:\, \{i, j\} \in E(G)).$$

If Δ (or I_Δ) is a complete intersection, then all powers I_Δ^ℓ are Cohen-Macaulay. Moreover, the converse is also true. So it is natural to ask the following question.

Question 1.1. Fix an integer $\ell \ge 2$. When is I_Δ^ℓ Cohen-Macaulay?

It is proved in [18] that Δ is a complete intersection if the third power I_Δ^3 (or I_Δ^ℓ for some $\ell \ge 3$) is Cohen-Macaulay, using a result in [11, 21]. On the other hand, there is a simplicial complex Δ which is not a complete intersection such that I_Δ^2 is Cohen-Macaulay (for example, the simplicial complex associated with a pentagon).

So the main purpose of this talk is to give an answer to the following question:

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Question 1.2. What constraints does Cohen-Macaulayness (or Buchsbaumness) of I_Δ^2 impose upon a simplicial complex Δ?

Moreover, in Sections 3,4, we give two families of examples of simplicial complexes Δ whose second powers are Cohen-Macaulay. One is a simplicial join of pentagons; the other is a stellar subdivision of a complete intersection complex.

We refer [1, 12, 15, 16] for several definitions and notations.

2. Cohen-Macaulayness of second powers and Gorensteinness

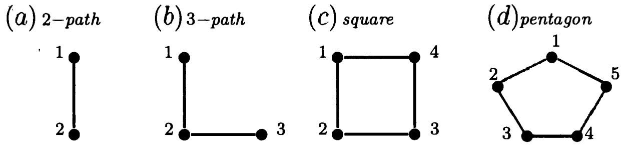
In this section, we discuss Question 1.2. In what follows, let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and put $\mathfrak{m} = (x_1, \dots, x_n)S$. Let I be a homogeneous ideal of S. Then S/I is called *quasi-Buchsbaum* if $\mathfrak{m}H_\mathfrak{m}^i(S/I) = 0$ for each $i = 0, 1, \dots, \dim S/I - 1$. A simplicial complex Δ is called *Cohen–Macaulay* (resp. Gorenstein, (FLC) etc.) if so is $K[\Delta] - S/I_\Delta$ over any field K.

We recall the following two classfication theorems.

Theorem 2.1 (Minh-Trung [10]). *Let Δ be a simplicial complex on $V = [n]$. Suppose that* $\dim\Delta - 1$. *Then the following conditions are equivalent:*

(1) *S/I_Δ^2 is Cohen-Macaulay.*

(2) *Δ is one of the following complexes up to a permutation of the verices,*



When this is the case, Δ is Gorenstein.

Theorem 2.2 (Trung-Tuan [20, Theorem 3.8]). *Let Δ be a simplicial complex on $V = [n]$. Suppose that* $\dim\Delta - 2$. *Then the following conditions are equivalent:*

(1) *S/I_Δ^2 is Cohen-Macaulay.*

(2) *Δ is one of the following complexes up to a permutation of the vertices,*

(a) $n = 3$ *and* $\mathcal{F}(\Delta) - \{\{1,2,3\}\}$.

(b) $n = 4$ *and* $\mathcal{F}(\Delta) = \{\{1,2,3\}, \{2,3,4\}\}$.

(c) $n = 4$ *and* $\mathcal{F}(\Delta) = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$.

(d) $n = 5$ *and* $\mathcal{F}(\Delta) = \{\{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}\}$.

(e) $n = 5$ *and* $\mathcal{F}(\Delta) - \{\{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}\}$.

(f) $n = 6$ *and* $\mathcal{F}(\Delta) = \{\{1,3,6\}, \{1,4,6\}, \{2,4,6\}, \{2,5,6\}, \{3,5,6\}\}$.

(g) $n = 6$ *and*

$$\mathcal{F}(\Delta) = \{\{1,2,3\}, \{1,2,6\}, \{1,3,5\}, \{1,5,6\}, \{2,3,4\}, \{2,4,6\}, \{3,4,5\}, \{4,5,6\}, \}.$$

(h) $n = 6$ *and*

$$\mathcal{F}(\Delta) = \{\{1,2,4\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,5,6\}, \{2,3,4\}, \{2,3,6\}, \{3,5,6\}\}.$$

(i) $n = 7$ *and*

$$\begin{aligned}\mathcal{F}(\Delta) &= \{\{1,3,6\},\{3,5,6\},\{2,5,6\},\{2,4,6\},\{1,4,6\},\\ &\quad \{1,3,7\},\{3,5,7\},\{2,5,7\},\{2,4,7\},\{1,4,7\}\}.\end{aligned}$$

(j) $n = 7$ *and*

$$\begin{aligned}\mathcal{F}(\Delta) &= \{\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,6\},\{1,5,6\},\\ &\quad \{2,3,4\},\{2,3,7\},\{2,5,7\},\{3,6,7\},\{5,6,7\}\}.\end{aligned}$$

(k) $n = 8$ *and*

$$\begin{aligned}\mathcal{F}(\Delta) &= \{\{1,3,6\},\{1,3,8\},\{1,4,7\},\{1,4,8\},\{1,6,7\},\{2,4,7\},\\ &\quad \{2,4,8\},\{2,5,6\},\{2,5,8\},\{2,6,7\},\{3,5,6\},\{3,5,8\}\}.\end{aligned}$$

When this is the case, Δ is Gorenstein.

From the above theorems, we want to pose the following question.

Question 2.3. If S/I_Δ^2 is Cohen-Macaulay over a fixed field K, then is Δ Gorenstein over K?

The main purpose of this section is to prove the following theorem, which gives a partial affirmative answer to the above question (including the case where $\operatorname{char} K = 2$).

Theorem 2.4. *Let Δ be a simplicial complex on $V = [n]$. Suppose that $d = \dim S/I_\Delta \geq 3$. If S/I_Δ^2 is quasi-Buchsbaum for any field K, then Δ is Gorenstein.*

We first prove the following lemma, which is closely related to the conjecture by Vasconcelos (see also [17, Conjecture 3.12]): Let R be a regular local ring and I Cohen-Macaulay. If I is syzygetic and I/I^2 is Cohen-Macaulay, then I is Gorenstein.

Lemma 2.5. *If S/I_Δ^2 is Cohen-Macaulay for any field K, then Δ is Gorenstein.*

Proof. We may assume that any variable appears in I_Δ. Let K be a field and fix it. Let F be a face of Δ and put $\Gamma = \operatorname{link}_\Delta F := \{H \in \Delta \mid H \cup F \in \Delta,\ H \cap F = \emptyset\}$.

First note that S/I_Γ^2 and S/I_Δ are Cohen-Macaulay if so is S/I_Δ^2. Indeed, since S/I_Δ^2 is Cohen-Macaulay and $I_\Delta = \sqrt{I_\Delta^2}$, we have that S/I_Δ is Cohen-Macaulay; see e.g. [6]. On the other hand, by localizing at $x_F = \prod_{i \in F} x_i$, we get

$$I_\Delta S[x_F^{-1}] = (I_\Gamma, x_{i_1}, \ldots, x_{i_k})S[x_F^{-1}]$$

for some variables $x_{i_1}, \ldots, x_{i_k}$. Hence the assumption implies that $(I_\Gamma, x_{i_1}, \ldots, x_{i_k})^2$ is Cohen-Macaulay. This yields that I_Γ^2 is also Cohen-Macaulay.

Suppose that $\dim \Gamma = 0$. Then one can take a complete graph G such that $I(G) = I_\Gamma$. Since $S/I(G)^2$ is Cohen-Macaulay, we have $I(G)^{(2)} = I(G)^2$. Hence G does not contain any triangle (e.g. see Corollary 3.3). Thus $\sharp(V(\Gamma)) = \sharp(V(G)) \leq 2$.

By the above argument, $\Lambda = \operatorname{link}_\Delta F$ is a locally complete intersection complex whenever $\dim \Lambda = 1$. Moreover, since S/I_Λ is Cohen-Macaulay and thus Λ is connected, Λ is an n-cycle or an n-pointed path; see [19, Proposition 1.11]. On the other hand, since $\operatorname{diam}\Lambda \leq 2$ by [10, Theorem 2.3], we get $n \leq 3$ if Λ is an n-pointed path. Hence $\Lambda - \operatorname{link}_\Delta F$ is Gorenstein.

Now suppose that $K = \mathbb{Z}/2\mathbb{Z}$. By [15, Chapter II, Theorem 5.1], $K[\Delta]$ is Gorenstein. Then we obtain that the reduced Euler characteristic of Δ, $\widetilde{\chi}(\Delta)$ is equal to $(-1)^{d-1}$.

Let K be any field. Then $\widetilde{\chi}(\Delta) = (-1)^{d-1}$ because $\widetilde{\chi}(\Delta)$ does not depend on K. Therefore we conclude that Δ is Gorenstein over K by [15, Chapter II, Theorem 5.1] again. □

A complex Δ is called a *locally Gorenstein* complex if $\operatorname{link}_\Delta\{x\}$ is Gorenstein for every vertex $x \in V$. Then the following corollary immediately follows from Lemma 2.5.

Corollary 2.6. *If S/I_Δ^2 has (FLC) for any field K, then Δ is a locally Gorenstein complex.*

Lemma 2.7. *Suppose $d \geq 2$. If S/I_Δ^2 is quasi-Buchsbaum, then S/I_Δ is Cohen-Macaulay.*

Proof. By assumption that S/I_Δ^2 has (FLC). Then S/I_Δ has (FLC) by [6, Theorem 2.6] and thus it is Buchsbaum.

Now suppose that S/I_Δ is *not* Cohen-Macaulay. Then there exists an i with $0 \leq i \leq d-2$ such that $H_\mathfrak{m}^{i+1}(S/I_\Delta)_0 \cong \widetilde{H}_i(\Delta; K) \neq 0$. Then we get the following commutative diagram (see [9]):

$$\begin{array}{ccc} H_\mathfrak{m}^{i+1}(S/I_\Delta^2)_0 & \xrightarrow{\ x_1\ } & H_\mathfrak{m}^{i+1}(S/I_\Delta^2)_{\mathbf{e}_1} \\ \downarrow & & \downarrow \\ \widetilde{H}^i(\Delta; K) & \longrightarrow & \widetilde{H}^i(\Delta; K), \end{array}$$

where the bottom map is identity and the vertical maps are isomorphism (see [18] for details). This yields $x_1 H_\mathfrak{m}^{i+1}(S/I_\Delta^2) \neq 0$. But this contradicts the assumption. □

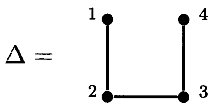
We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. By assumption and Corollary 2.6, we have that Δ is locally Gorenstein. Moreover, Δ is Cohen-Macaulay by Lemma 2.7. Take any face F of Δ with $\dim \operatorname{link}_\Delta F = 1$. As $d \geq 3$, $\operatorname{link}_\Delta F$ is given by some link of $\operatorname{link}_\Delta\{x\}$ for $x \in F$. Hence such a $\operatorname{link}_\Delta F$ is also Gorenstein. By a similar argument as in the proof of Lemma 2.5, we get the required assertion. □

Remark 2.8. The Gorensteinness of S/I_Δ does not necessarily imply the quasi-Buchsbaumness of S/I_Δ^2.

In Theorem 2.4, we cannot remove the assumption that $\dim S/I_\Delta \geq 3$ as the next example shows.

Example 2.9. Put $I_\Delta = (x_1x_3, x_1x_4, x_2x_4)$, the Stanley-Reisner ideal of the 4-pointed path Δ. Then S/I_Δ^2 is Buchsbaum by [19, Example 2.9] and S/I_Δ is Cohen-Macaulay but not Gorenstein of dimension 2.



Remark 2.10. We can classify all simplicial complexes Δ such that S/I_Δ^2 is quasi-Buchsbaum.

3. When does $I^{(2)} = I^2$ hold

Let $I = I_\Delta$ be any squarefree monomial ideal. We first recall the definition of symbolic powers. For any facet F of Δ, we set $P_F = (x \in [n] \setminus F)$. Given an integer $\ell \geq 1$, the *ℓth symbolic power* of I_Δ is defined to be the ideal

$$I_\Delta^{(\ell)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^\ell S_{P_F} \cap S = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^\ell.$$

The aim of this section is to give a criterion of $I^{(2)} = I^2$ for any squarefree monomial ideal I; see Theorem 3.2. As an application of the theorem, we give an example of edge ideals $I(G)$ (in other words, Stanley-Reisner ideals generated by degree 2 monomials) whose second powers $I(G)^2$ are Cohen-Macaulay.

Next we introduce the notion of special triangles.

Definition 3.1 (cf. [5]). Let I be a squarefree monomial ideal of $S = K[x_1, \dots, x_n]$. Let $G(I) = \{x^{F_1}, \dots, x^{F_\mu}\}$ be the minimal set of monomial generators, where $x^F = x_{i_1} \cdots x_{i_r}$ for $F = \{i_1, \dots, i_r\}$. Then $\mathcal{H}(I)$ is called the hypergraph associated to I if the vertex set of $\mathcal{H}(I)$ is V and the edge set is $\{F_1, \dots, F_\mu\}$.

Then $\{i, j, k\}$ is called a *special triangle* of $\mathcal{H}(I)$ if there exist $F_1, F_2, F_3 \in \mathcal{H}(I)$ such that

$$F_1 \cap \{i,j,k\} = \{j,k\}, \qquad F_2 \cap \{i,j,k\} = \{i,k\}, \qquad F_3 \cap \{i,j,k\} = \{i,j\}.$$

Then we say that "*F_1, F_2, F_3 make a special triangle $\{i,j,k\}$*".

The following is the main theorem in this section.

Theorem 3.2. *Let I be a squarefree monomial ideal. Then the following conditions are equivalent:*

1. $I^{(2)} = I^2$ *holds.*
2. *If there exist $\{F_1, F_2, F_3\} \subseteq \mathcal{H}(I)$ such that F_1, F_2, F_3 make a special triangle, then $x^{F_1 \cap F_2 \cap F_3} x^{F_1 \cup F_2 \cup F_3} \in I^2$.*

The following criterion is well known; see [13].

Corollary 3.3. *Let $I(G)$ denote the edge ideal of a graph G. Then $I(G)^{(2)} = I(G)^2$ holds if and only if G has no triangles (the cycles of length 3).*

In what follows, we give a sketch of the proof of the above theorem. First we need the following key lemma.

Lemma 3.4. *Suppose that the condition* (2) *in Theorem* 3.2 *holds. Then* $xI \cap (I^2 \colon x) \subseteq I^2$ *holds for every* $x \in V$.

Proof. Suppose that there exist a variable x_1 and a monomial M such that $M \in x_1 I \cap (I^2 \colon x) \setminus I^2$. As $x_1 M \in I^2$, we can take N_2, $N_3 \in G(I)$ and a monomial L such that

$$(3.1) \qquad x_1 M = N_2 N_3 L.$$

On the other hand, as $M \in x_1 I$, we can choose $N_1 \in G(I)$ and a monomial L' such that

$$(3.2) \qquad M = N_1 L' \quad \text{and} \quad x_1 \,|\, L'.$$

Then we can easily see that $x_1 \,|\, N_2$, $x_1 \,|\, N_3$ but $x_1 \not| N_1$ and that $N_2 \neq N_3$ and $\gcd(N_2, N_3) \,|\, L'$. Furthermore, one can find variables x_2, x_3 such that

$$x_2 \,\Big|\, \frac{N_3}{\gcd(N_2, N_3)}, \quad x_3 \,\Big|\, \frac{N_2}{\gcd(N_2, N_3)}, \quad x_2, x_3 \,|\, N_1.$$

Take $F_i \in \mathcal{H}(I)$ such that $x^{F_i} = N_i$ for each $i = 1, 2, 3$. Then F_1, F_2, F_3 make a special triangle $\{1, 2, 3\}$. Hence, by assumption, we get

$$\gcd(N_1, N_2, N_3)\sqrt{N_1 N_2 N_3} = x^{F_1 \cap F_2 \cap F_3} \cdot x^{F_1 \cup F_2 \cup F_3} \in I^2.$$

Since N_1 divides $N_2 N_3 L$ and $x_1 \,|\, N_2$, N_3, we have

$$(3.3) \qquad \sqrt{N_1 N_2 N_3} \,\Big|\, \frac{N_2 N_3 L}{x_1} = M.$$

On the other hand, since $x_1 \not| \gcd(N_1, N_2, N_3)$, we have

$$(3.4) \qquad \gcd(N_1, N_2, N_3)^2 \,\Big|\, \frac{N_2 N_3}{x_1} \,\Big|\, M.$$

Hence Eqs. (3.3), (3.4) imply

$$\gcd(N_1, N_2, N_3)\sqrt{N_1 N_2 N_3} \,\big|\, M.$$

Therefore $M \in I^2$, which contradicts the choice of M. □

The following lemma is useful when we use an induction.

Lemma 3.5 (See the proof of [17, Theorem 5.9]). *Let I be a squarefree monomial ideal of S with* $\dim S/I \geq 1$. *Now suppose that* $xI \cap (I^2 \colon x) \subseteq I^2$ *for every variable* x. *Then* $\mathfrak{m} \notin \mathrm{Ass}_S(S/I^2)$.

Proof. We omit the proof. □

Proof of Theorem 3.2. First we show (2) $\Longrightarrow$ (1). Suppose (2). Since this condition preserves under localization, we may assume that $(I^{(2)})_x = (I^2)_x$ for any variable x by an induction on $\dim S/I$. By the above two lemmata, we have $\mathfrak{m} \notin \mathrm{Ass}_S(S/I^2)$. Hence $I^{(2)} = I^2$, as required.

Next we show (1) $\Longrightarrow$ (2). Suppose that there exists a subset $\{F_1, F_2, F_3\} \subseteq \mathcal{H}(I)$ such that F_1, F_2, F_3 make a special triangle and $x^{F_1\cap F_2\cap F_3}x^{F_1\cup F_2\cup F_3} \notin I^2$. Then it suffices to show $I^2 \subsetneq I^{(2)}$.

Put $F = F_1\cup F_2\cup F_3$. Let I_F be the squarefree monomial ideal of $K[x : x \in V\setminus F]$ such that $I_F S + (x \in V \setminus F) = I + (x \in V \setminus F)$. Let P be any minimal prime ideal of I_F. If height $P = 1$, then there exists a vertex $j \in F_1 \cap F_2 \cap F_3$ such that $P = (x_j)$. Then $M := x^{F_1\cap F_2\cap F_3}\, x^F \in (x_j^2) = P^2$. If height $P \geq 2$, then P contains two variables x_i, x_j with $i, j \in F$. Then $x^F \in P^2$ and hence $M \in P^2$. Therefore $M \in I_F^{(2)}$ but $M \notin I_F^2$ by the assumption that $M \notin I^2$. □

Suppose $U \cap V = \emptyset$. Let Γ (resp. Λ) be a simplicial complex on U (resp. V). Then the *simplicial join* of Γ and Λ, denoted by $\Gamma * \Lambda$, is defined by $\Gamma * \Lambda = \{F \cup G : F \in \Delta,\ G \in \Lambda\}$. It is a simplicial complex on $U \cup V$.

The following corollary is probably well-known, but we give a proof as an application of Theorem 3.2.

Corollary 3.6. *Let Γ be a simplicial complex on U and Λ a simplicial complex on V. Let $\Delta = \Gamma * \Lambda$ denote the simplicial join of Γ and Λ. Then Δ is a simplicial complex on $W = U \coprod V$. Put $R = K[U]$, $S = K[V]$ and $T = R \otimes_K S \cong K[W]$. Then:*

(1) $I_\Delta^{(2)} = I_\Delta^2$ *if and only if* $I_\Gamma^{(2)} = I_\Gamma^2$ *and* $I_\Lambda^{(2)} = I_\Lambda^2$.

(2) T/I_Δ^2 *is Cohen-Macaulay if and only if so do* R/I_Γ^2 *and* S/I_Λ^2.

Proof. (1) Note that $I_\Delta = I_\Gamma T + I_\Lambda T$ and $G(I_\Delta)$ is a disjoint union of $G(I_\Gamma)$ and $G(I_\Lambda)$. Thus it immediately follows from Theorem 3.2.

(2) It immediately follows from (1) and [11, Theorem 2.7]. □

A disjoint union of two graphs G_1 and G_2, denoted by $G_1 \coprod G_2$, is the graph G which satisfies $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Let $G = G_1 \coprod \cdots \coprod G_r$ be a disjoint union of graphs $G_1, \ldots, G_r$, and let Δ_i (resp. Δ) be the complementary simplicial complex of G_i for each $i = 1, \ldots, r$ (resp. G). Then Δ is equal to the simplicial join $\Delta_1 * \cdots * \Delta_r$.

The second symbolic power of the edge ideal of the pentagon is Cohen–Macaulay. Thus we get the following corollary.

Example 3.7. Let $G = G_1 \coprod \cdots \coprod G_r$ be a disjoint union of the pentagons G_i for $i = 1, \ldots, r$. Then $I(G)^2$ is Cohen–Macaulay.

4. Examples of Stanley-Reisner ideals whose square is Cohen-Macaulay

We give another examples of simplicial complexes whose second powers are Cohen-Macaulay, using liaison theory. The following key proposition is due to Buchweitz [3]; see also Kustin and Miller [8]. Note that it gives a partial converse of Theorem 2.4.

Proposition 4.1 (cf. [3, 6.2.11], [8, Proposition 7.1]). *Let I be a Gorenstein homogeneous ideal in a polynomial ring S. Assume that there exist a homogeneous polynomial ring $T = S[z_1, \dots, z_r]$ $(\deg z_i = 1)$ and a homogeneous radical ideal L such that*

(a) $S/I \cong T/(z_1, \dots, z_r, L)$.
(b) $z_1, \dots, z_r$ *is a regular sequence on* T/L.
(c) L *is in the linkage class of a complete intersection in* T.

Then S/I^2 is Cohen-Macaulay.

It is well-known that any Gorenstein ideal of codimension 3 lies in the linkage class of a complete intersection; see [2, 24] or [22, Theorem 4.15]. Thus we can obtain the following corollary.

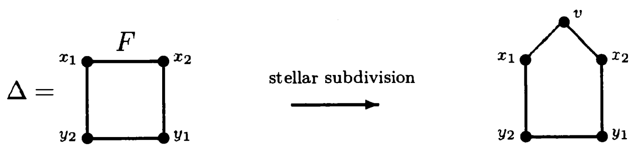
Corollary 4.2. *Let $I_\Delta \subseteq S$ be a Gorenstein Stanley-Reisner ideal of codimension 3. Then S/I_Δ^2 is Cohen-Macaulay.*

In the rest of this section we prove the second power of the Stanley-Reisner ideal of a stellar subdivision of any non-acyclic complete intersection complex is Cohen-Macaulay.

We recall the definition of the stellar subdivision of a complex. For a given face F of Δ with $\dim F \geq 1$ and a new vertex v, the *stellar subdivision* of Δ on F is the simplicial complex Δ_F on the vertex set $V \cup \{v\}$ defined by

$$\Delta_F = (\Delta \setminus \{H \mid F \subseteq H \in \Delta\}) \cup \{H \cup \{v\} \mid H \in \Delta,\ F \not\subseteq H,\ F \cup H \in \Delta\}.$$

Notice that Δ_F is homeomorphic to Δ.



In what follows, as vertices of simplicial complexes we use indeterminates instead of natural numbers for convenience. Let Γ be a non-acyclic complete intersection simplicial complex whose Stanley-Reisner ideal is

$$I_\Gamma = (x_{11}x_{12}\cdots x_{1i_1}, x_{21}x_{22}\cdots x_{2i_2}, \dots, x_{\mu 1}x_{\mu 2}\cdots x_{\mu i_\mu}).$$

Let $\mathcal{F}(\Gamma)$ be the set of all facets of Γ. Then

$$\begin{aligned}\mathcal{F}(\Gamma) = \{ & \{x_{11}, \dots, \widehat{x_{1k_1}}, \dots, x_{1i_1}, x_{21}, \dots, \widehat{x_{2k_2}}, \dots, x_{2i_2}, \dots, \\ & x_{\mu 1}, \dots, \widehat{x_{\mu k_\mu}}, \dots, x_{\mu i_\mu}\} \\ & \mid 1 \leq k_1 \leq i_1, 1 \leq k_2 \leq i_2, \dots, 1 \leq k_\mu \leq i_\mu\}.\end{aligned}$$

Let Δ be the *stellar subdivision* of Γ on

$$F = \{x_{11}, \dots, x_{1j_1}, x_{21}, \dots, x_{2j_2}, \dots, x_{p1}, \dots, x_{pj_p}\},$$

where $1 \leq p \leq \mu$ and $1 \leq j_1 < i_1, \dots, 1 \leq j_p < i_p$ and $j_1 + \cdots + j_p \geq 2$.

Let v be the new added vertex. Then

$$I_\Delta = (I_\Gamma, x_F, vx_{1j_1+1}\cdots x_{1i_1}, vx_{2j_2+1}\cdots x_{2i_2}, \dots, vx_{pj_p+1}\cdots x_{pi_p})$$

is an ideal of a polynomial ring

$$S = k[x_{11}, \dots, x_{1i_1}, x_{21}, \dots, x_{2i_2}, \dots, x_{\mu 1}, \dots, x_{\mu i_\mu}, v].$$

Applying Proposition 4.1 to this ideal $I = I_\Delta$, we obtain the following theorem. It is proved the two-dimensional case in [20].

Theorem 4.3. *Let $\Delta = \Gamma_F$ be the stellar subdivision of the non-acyclic complete intersection complex Γ as above. Then S/I_Δ^2 is Cohen–Macaulay.*

Example 4.4 (Cross Polytope). Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the fundamental vectors of the Euclidean space $\mathbb{R}^d$. Then the convex hull $\mathcal{P} = \mathrm{CONV}(\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \dots, \pm\mathbf{e}_d\})$ is called the *cross d-polytope*. Let Γ be the boundary complex of the cross d-polytope $\mathcal{P}$. Then Γ can be regarded as a simplicial complex on $W = \{x_1, \dots, x_d, y_1, \dots, y_d\}$ with $I_\Gamma = (x_1y_1, x_2y_2, \dots, x_dy_d)$.

Let v be a new vertex, and choose a facet $F = \{x_1, \dots, x_d\}$ of Γ. Let Δ be the stellar subdivision of Γ on F. Then Δ is a $(d-1)$-dimensional Gorenstein complex on $V = W \cup \{v\}$ and its geometric realization of Δ is homeomorphic to $\mathbb{S}^{d-1}$. The above theorem says that the second power of

$$I = (x_1y_1, x_2y_2, \dots, x_dy_d, vy_1, \dots, vy_d, x_1x_2\cdots x_d)$$

is Cohen-Macaulay, but the third power is not if $d \geq 2$ because the third power of the Stanley-Reisner ideal $(x_1y_1, x_2y_2, vy_1, vy_2, x_1x_2)$ of a pentagon is not.

In the last of the paper, we give candidates of edge ideals $I(G)$ for which $S/I(G)^2$ is Cohen–Macaulay (but $S/I(G)^3$ is not by [13]). For the case that $n = 2$ it is mentioned in [20, Theorem 3.7 (iv)].

Conjecture 4.5. *Let G be a graph on the vertex set $V = \{x_1, x_2, \dots, x_{3n+2}\}$ with*

$$I(G) = \big(x_1x_2,\ \{x_{3k-1}x_{3k}, x_{3k}x_{3k+1}, x_{3k+1}x_{3k+2}, x_{3k+2}x_{3k-2}\}_{k=1,2,\dots,n},\ \{x_{3\ell-3}x_{3\ell}\}_{\ell=2,3,\dots,n}\big).$$

Then $S/I(G)^2$ is Cohen–Macaulay but $S/I(G)^3$ is not.

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COHEN-MACAULAYFICATION OF CERTAIN LOCAL RINGS

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Abstract [1]. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Denote by $Q(A)$ the total quotient ring of A. An intermediate Cohen-Macaulay ring B between A and $Q(A)$ is said to be a *Cohen-Macaulayfication* of A if B is a finitely generated A-module and $\dim_A(B/A) \leqslant \dim A - 2$. In this paper, we modify the construction of Cohen-Macaulayfication for Buchsbaum rings given by S. Goto [Go] in conjunction with the notion of generalized regular elements introduced in [Nh] to construct a Cohen-Macaulayfication of A in case where the non Cohen-Macaulay locus of A is of dimension at most 1.

1 Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. Denote by $Q(A)$ the total quotient ring of A. An intermediate Cohen-Macaulay ring B between A and $Q(A)$ is said to be a *Cohen-Macaulayfication* of A if B is a finitely generated A-module and $\dim_A(B/A) \leqslant \dim A - 2$. Note that if such a Cohen-Macaulayfication B of A exists, it is unique determined (up to an isomorphism). In this case, A must be unmixed (i.e. $\dim A/\mathfrak{p} = \dim A$ for all $\mathfrak{p} \in \operatorname{Ass} A$), and $\dim B_{\mathfrak{n}} = \dim A$ for any maximal ideal $\mathfrak{n}$ of B. If in addition that A is universally catenary and all its formal fibres are Cohen-Macaulay then the non Cohen-Macaulay locus of A is closed and its dimension is equal to $\dim_A(B/A)$. This notion of Cohen-Macaulayfication was introduced by Goto [Go] for Buchsbaum rings. He showed that A has a Cohen-Macaulayfication B with $\mathfrak{m}B \subseteq A$ if and only if A is Buchsbaum with $H^i_{\mathfrak{m}}(A) = 0$ for all $i \neq 1, \dim A$. He also gave an implicit construction of the Cohen-Macaulayfication of A in this case.

The purpose of this paper is to give a construction for the Cohen-Macaulayfication of A in case where the non Cohen-Macaulay locus of A is of dimension less than or equal to 1.

Assume that $\operatorname{depth} A > 0$ and $H^1_{\mathfrak{m}}(A)$ is of finite length. For each A-regular element $a \in \operatorname{Ann}_A(H^1_{\mathfrak{m}}(A))$, we set $U(aA) = \bigcup_{n \in \mathbb{N}}(aA :_A \mathfrak{m}^n)$ and $S_0 = a^{-1}U(aA)$. We will prove that S_0 is an intermediate ring between A and $Q(A)$, isomorphic to $U(aA)$ as A-modules, and independent of the choice of a. Moreover, $\operatorname{depth}(S_0) \geq 2$ and $\dim(S_0/A) \leqslant 0$ (see Lemmas 2.3 and 2.4).

Recall that A is said to be *generalized Cohen-Macaulay* if $H^i_{\mathfrak{m}}(A)$ is of finite length for all $i < \dim A$. The following theorem, which is the first main result of this paper, gives a characterization for a generalized Cohen-Macaulay ring to have a Cohen-Macaulayfication.

Theorem 1.1. *Let* $\dim A = d \geq 2$. *The following statements are equivalent:*

[1] *Key words and phrases:* Cohen-Macaulayfication, Cohen-Macaulay locus, generalized regular sequence.
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(i) A is generalized Cohen-Macaulay with $H^i_{\mathfrak{m}}(A) = 0$ *for all* $i \neq 1, d$.

(ii) A has a Cohen-Macaulayfication B such that $\dim_A(B/A) \leqslant 0$.

If (i) and (ii) are satisfied then there is an A-regular element $a \in \mathrm{Ann}_R(H^1_{\mathfrak{m}}(A))$ *such that* $B = S_0 := a^{-1}U(aA)$ *and* $B/A \cong H^1_{\mathfrak{m}}(A)$.

Next, assume that A is universally catenary and all its formal fibres are Cohen-Macaulay. Let A be unmixed (i.e. $\dim(A/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \mathrm{Ass}\, A$) and $d \geq 3$. Then A satisfies the condition Serre (S_1). Therefore $\mathrm{depth}\, A > 0$, $\ell(H^1_{\mathfrak{m}}(A)) < \infty$, cf. [CNN]. So, the ring S_0 in the above is well defined. Set $I = \mathrm{Ann}_A(H^2_{\mathfrak{m}}(S_0))$. As S_0 is unmixed, $\dim(A/I) \leqslant 1$. For an A-regular element $a \in I$, set $U_1(aS_0) = \bigcup_{n \in \mathbb{N}}(aS_0 :_{S_0} I^n)$ and $S_1 = a^{-1}U_1(aS_0)$. We will show that there exists an integer r such that S_1 is an intermediate ring between A and $Q(A)$, isomorphic to $U_1(aS_0)$ as A-modules, and does not depend on the choice of A-regular element $a \in I^r$ (see Lemma 3.4).

The theory of secondary representation for Artinian modules was introduced by I. G. Macdonald [Mac] which is in some sense dual to the theory of primary decomposition for Noetherian modules. Note that each Artinian A-module L has a minimal secondary representation $L = L_1 + \ldots + L_n$, where L_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of minimal secondary representation of L. It is called the set of attached primes of L and denoted by $\mathrm{Att}_A\, L$.

Theorem 1.2. *Let* $\dim A = d \geq 3$. *Assume that A is universally catenary and all its formal fibres are Cohen-Macaulay. The following statements are equivalent:*

(i) A is unmixed (i.e. $\dim(A/\mathfrak{p}) = d$ *for all* $\mathfrak{p} \in \mathrm{Ass}\, A$*),* $\dim \mathrm{nCM}(A) = 1$, $H^i_{\mathfrak{m}}(A) = 0$ *for all* $i \neq 1, 2, d$, *and* $\mathfrak{m} \notin \mathrm{Att}_R\, H^2_{\mathfrak{m}}(A)$.

(ii) A has a Cohen-Macaulayfication B such that $\dim_A(B/A) = 1$.

If (i) and (ii) are satisfied then $B = S_1 := a^{-1}U_1(aS_0)$ *for some A-regular element* $a \in \mathrm{Ann}_R(H^2_{\mathfrak{m}}(S_0))$, $\dim_A(B/A) = 1$ *and* $H^1_{\mathfrak{m}}(B/A) \cong H^2_{\mathfrak{m}}(A)$.

This paper is divided into 3 sections. In the next section we present the proof of Theorem 1.1. The proof of Theorem 1.2 will be given in Section 3.

2 Proof of Theorem 1.1

In this section we always assume that $\dim A = d \geq 2$. Denote by $Q(A)$ the total quotient ring of A. Let M be a finitely generated A-module. For each element $a \in A$ we set

$$U(aM) = \bigcup_{n \in \mathbb{N}} (aM :_M \mathfrak{m}^n).$$

It is clear that if $a \in \mathfrak{m}$ is an M-regular element then $\dim(U(aM)) = \dim M$.

Following Cuong-Schenzel-Trung 1978 [CST], an element $b \in \mathfrak{m}$ is said to be an *M-filter regular element* if $b \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A\, M \setminus \{\mathfrak{m}\}$. It is clear that $b \in \mathfrak{m}$ is M-filter regular if and only if $0 :_M b$ is of finite length. Therefore if $\mathfrak{a}$ is an ideal of A such that $\dim(M/\mathfrak{a}M) \leqslant 0$ then there always exists an M-filter regular element in $\mathfrak{a}$.

Lemma 2.1. *Assume that* $\operatorname{depth} A > 0$ *and* $H^1_{\mathfrak{m}}(A)$ *is of finite length. Set* $I = \operatorname{Ann}(H^1_{\mathfrak{m}}(A))$. *Let* $a \in I$ *be an* A*-regular element. Set* $M = U(aA)$. *Then we have*

(i) $U(aA) = aA :_A I = aA :_A b$ *for any* A/aA*-filter regular element* $b \in I$.

(ii) $U(aM) = (aM :_M I) = (aM :_M b)$ *for any* M/aM*-filter regular element* $b \in I$.

Proof. (i). Since b is A/aA-filter regular, $(aA :_A b)/aA = 0 :_{A/aA} b$ is of finite length. So $(aA :_A b)/aA \subseteq H^0_{\mathfrak{m}}(A/aA) = U(aA)/aA$. Therefore $aA :_A b \subseteq U(aA)$. Consider the exact sequence $0 \longrightarrow A \xrightarrow{.a} A \longrightarrow A/aA \longrightarrow 0$. Since $H^0_{\mathfrak{m}}(A) = 0$, we have $H^0_{\mathfrak{m}}(A/aA) \cong 0 :_{H^1_{\mathfrak{m}}(A)} a$. As $IH^1_{\mathfrak{m}}(A) = 0$, we have $IH^0_{\mathfrak{m}}(A/aA) = 0$. Therefore $IU(aA) \subseteq aA$. Because $b \in I$, we have

$$U(aA) \subseteq aA :_A I \subseteq aA :_A b \subseteq U(aA).$$

(ii) Since b is M/aM-filter regular, $aM :_M b \subseteq U(aM)$. Consider the exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow A/M \longrightarrow 0.$$

Note that $A/M = A/U(aA) \cong A/aA \big/ H^0_{\mathfrak{m}}(A/aA)$. Therefore $H^0_{\mathfrak{m}}(A/M) = 0$. So, we get the exact sequence $0 \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow H^1_{\mathfrak{m}}(A)$. Since $IH^1_{\mathfrak{m}}(A) = 0$, we have $IH^1_{\mathfrak{m}}(M) = 0$. As $\operatorname{Ass}_A M \subseteq \operatorname{Ass} A$, it follows that a is M-regular. Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{.a} M \longrightarrow M/aM \longrightarrow 0.$$

Since $H^0_{\mathfrak{m}}(M) = 0$, we have an exact sequence $0 \longrightarrow H^0_{\mathfrak{m}}(M/aM) \longrightarrow H^1_{\mathfrak{m}}(M)$. Since $IH^1_{\mathfrak{m}}(M) = 0$ as above, we have $IH^0_{\mathfrak{m}}(M/aM) = 0$. So $IU(aM) \subseteq aM$. Since $b \in I$, we have

$$U(aM) \subseteq aM :_M I \subseteq aM :_M b \subseteq U(aM).$$

□

Lemma 2.2. *With the assumptions as in Lemma 2.1, we have* $(U(aA))^2 = aU(aA)$.

Proof. It is clear that $(U(aA))^2 \supseteq aU(aA)$. We need only to show that if $g, f \in U(aA)$ then $fg \in aU(aA)$. Since $\dim A/I \leqslant 0$, there exists $b \in I \cap \mathfrak{m}$ such that b is an A/aA-filter regular element. Since a is A-regular, $\operatorname{Ass}(A/aA) = \operatorname{Ass}(A/a^2A)$. Hence b, b^2 are A/a^2A-filter regular elements.

As $U(aA) = aA :_A b$ by Lemma 2.1(i), $bf = ax, bg = ay$ for some $x, y \in A$. Therefore $b^2 fg = a^2 xy$. So, by Lemma 2.1(i), $fg \in a^2A :_A b^2 = U(a^2A)$ as b^2 is A/a^2A-filter regular. As $U(a^2A) = a^2A :_A I$ by Lemma 2.1(i), we have $a(fg) \in IU(a^2A) \subseteq a^2A$. Therefore $afg = a^2z$ for some $z \in A$. Hence $fg = az$ since a is A-regular. By Lemma 2.1(i), $U(a^2A) = a^2A :_A b$ since b is A/a^2A-filter regular. Since $fg \in U(a^2A)$, we have $bfg \in a^2A$. Hence $bfg = a^2w$ for some $w \in A$. Therefore $bfg = baz = a^2w$. So, $bz = aw$. Hence $z \in aA :_A b = U(aA)$ by Lemma 2.1(i). Thus, $fg = az \in aU(aA)$. □ □

Lemma 2.3. *With the assumptions as in Lemma 2.1, let*

$$S_0 = \{x/a \in Q(A) \mid x \in U(aA)\}.$$

Then we have

(i) S_0 *is an intermediate ring between* A *and* $Q(A)$.

(ii) $S_0 \cong U(aA)$ *as* A*-modules, and* $S_0 \cong \mathrm{End}_R(U(aA))$ *as* A*-algebras.*

(iii) $IS_0 \subseteq A$. *In particular,* $\ell_A(S_0/A) < \infty$.

(iv) S_0 *does not depend on the choice of an* A*-regular element* $a \in I$ *and it is uniquely determined by* A.

Proof. The proof follows by Lemmas 2.1, 2.2 and by the similar arguments as in the proof of Goto [G, (2.6)]. □

Lemma 2.4. *With the assumptions as in Lemma 2.1, we have* $\mathrm{depth}(U(aA)) \geq 2$. *In particular,* $\mathrm{depth}(S_0) \geq 2$, *where* S_0 *is defined in Lemma 2.3.*

Proof. Set $M = U(aA)$. Then $\dim M = d \geq 2$ and a is M-regular since $\mathrm{Ass}_A M \subseteq \mathrm{Ass}\, A$. So, it is enough to show that $H^0_{\mathfrak{m}}(M/aM) = 0$. Since $\dim A/I \leqslant 0$, there exists an element $b \in I \cap \mathfrak{m}$ such that $b \notin \mathfrak{p}$ for all prime ideals

$$\mathfrak{p} \in \Big(\mathrm{Ass}(A/aA) \cup \mathrm{Ass}(M/aM))\Big) \setminus \{\mathfrak{m}\}.$$

Note that $\mathrm{Ass}(A/aA) = \mathrm{Ass}(A/a^2A)$ since a is A-regular. Therefore b and b^2 are filter regular elements with respect to all A/aA, A/a^2A, and M/aM. Let $\overline{f} \in H^0_{\mathfrak{m}}(M/aM)$. Since

$$H^0_{\mathfrak{m}}(M/aM) = U(aM)/aM = (aM :_M b)/aM$$

by Lemma 2.3(ii), we can express $\overline{f} = f + aM$ for some $f \in aM :_M b$. Hence $fb = ax$ for some $x \in M = U(aA)$. As $U(aA) = aA :_A b$ by Lemma 2.1(i), $bx = ay$ for some $y \in A$. Hence $fb^2 = axb = a^2y$. Hence $f \in a^2A :_A b^2 = U(a^2A)$ by Lemma 2.1(i). As $U(a^2A) = a^2A :_A I$ by Lemma 2.1(i), we have $IU(a^2A) \subseteq a^2A$. Because $a \in I$ and $f \in U(a^2A)$, we have $af \in a^2A$. Hence $af = a^2z$ for some $z \in A$. So, $f = az$ as a is A-regular. As $U(a^2A) = a^2A :_A b$ by Lemma 2.1(i) and $f \in U(a^2A)$, we have $bf \in a^2A$. So, $bf = a^2w$ for some $w \in A$. Hence $a^2w = bf = baz$. Hence $bz = aw$. So, $z \in aA :_A b = U(aA)$ by Lemma 2.1(i). Thus, $f = az \in aU(aA) = aM$ and hence $\overline{f} = 0$. □

Proof of Theorem 1.1. (i)⇒ (ii). Since $d \geq 2$ and A is generalized Cohen-Macaulay, $\ell(H^1_{\mathfrak{m}}(A)) < \infty$. As $H^0_{\mathfrak{m}}(M) = 0$, we have $\mathrm{depth}\, A > 0$. Therefore there exists an A-regular element $a \in I = \mathrm{Ann}_A H^1_{\mathfrak{m}}(A)$. Set $S_0 = a^{-1}U(aA)$. By Lemmas 2.3 and 2.4, S_0 is an intermediate ring between A and $Q(A)$, S_0 is a finitely generated A-module which is isomorphic to $U(aA)$, $\mathrm{depth}_A(S_0) \geq 2$ and $\dim_A(S_0/A) \leqslant 0$. As $H^i_{\mathfrak{m}}(A) = 0$ for all $i \neq 1, d$, we get the exact sequence

$$0 \longrightarrow A \longrightarrow S_0 \longrightarrow S_0/A \longrightarrow 0$$

that $H^0_{\mathfrak{m}}(S_0/A) = S_0/A \cong H^1_{\mathfrak{m}}(A)$ and $H^i_{\mathfrak{m}}(S_0) = 0$ for all $i \neq d$. Hence S_0 is Cohen-Macaulay required, and hence $B = S_0$ satisfies all our requirements.

(ii)⇒ (i). Assume that B is an intermediate ring between A and $Q(A)$, B is a finitely generated A-module, B is Cohen-Macaulay and $\dim_A(B/A) \leqslant 0$. From the exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

we have $B/A = H^0_{\mathfrak{m}}(B/A) \cong H^1_{\mathfrak{m}}(A)$ and $H^i_{\mathfrak{m}}(A) = 0$ for all $i \neq 1, d$. Therefore $H^1_{\mathfrak{m}}(A)$ is of finite length, and hence A is generalized Cohen-Macaulay ring.

Now assume that (i), (ii) are satisfied. As $B/A \cong H^1_{\mathfrak{m}}(A)$ by the above proof of (ii)⇒(i), we have $IB \subseteq A$, where $I = \operatorname{Ann}_A H^1_{\mathfrak{m}}(A)$. By (i), $\dim(A/I) \leqslant 0$. Let $a \in I$ be an A-regular element. We will show that $B = a^{-1}U(aA)$. As $IB \subseteq A$, we have $IaB \subseteq aA$. Hence

$$aB \subseteq aA :_A I = U(aA)$$

by Lemma 2.1(i). So, $B \subseteq a^{-1}U(aA) := S_0$. As $\dim_A(S_0/A) \leqslant 0$ by the proof (i)⇒(ii), we have $\dim_A(S_0/B) \leqslant 0$. From the exact sequence $0 \longrightarrow B \longrightarrow S_0 \longrightarrow S_0/B \longrightarrow 0$ with notice that B, S_0 are Cohen-Macaulay of dimension d, we have $S_0/B = H^0_{\mathfrak{m}}(S_0/B) = 0$.

Remark 2.5. *Assume that A is generalized Cohen-Macaulay. It follows by Theorem 1.1 and by the result of Schenzel [Sc1] (see also Trung [Tr, Corollary 6.4]) that A has a Cohen-Macaulayfication if and only if there exists a parameter ideal $\mathfrak{q}$ of A such that the Rees algebra $R_{\mathfrak{q}}(A)$ is Cohen-Macaulay. In this case, $R_{\mathfrak{q}}(A)$ is Cohen-Macaulay for any parameter ideal $\mathfrak{q}$ of A contained in* $\operatorname{Ann} H^1_{\mathfrak{m}}(A)$.

3 Proof of Theorem 1.2

Let M be a finitely generated R-module. Following [Nh], an element $b \in \mathfrak{m}$ is called an *M-generalized regular element* if $b \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$ satisfying $\dim(A/\mathfrak{p}) > 1$. Note that b is an M-generalized regular element if and only if $\dim(0 :_M b) \leqslant 1$. Therefore if $\mathfrak{a}$ is an ideal of A such that $\dim(M/\mathfrak{a}M) \leqslant 1$ then there always exists an M-generalized regular element in $\mathfrak{a}$.

In this section, assume that A is universally catenary and all its formal fibres are Cohen-Macaulay. Suppose that $\dim A = d \geq 3$ and A is unmixed (i.e. $\dim(A/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \operatorname{Ass} A$). Then A satisfies the condition Serre (S_1), i.e. $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{1, \dim A_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. So, $\dim\left(A/\operatorname{Ann}_A H^i_{\mathfrak{m}}(A)\right) \leqslant i-1$ for all $i < d$, see [CNN]. It follows that $H^0_{\mathfrak{m}}(A) = 0$ and $H^1_{\mathfrak{m}}(A)$ is of finite length. Therefore the ring S_0 in Lemma 2.3 is well defined.

Note that $\dim_A(S_0) = d \geq 3$, $\operatorname{depth}_A(S_0) \geq 2$ and S_0 is unmixed, i.e. $\dim(A/\mathfrak{p}) = \dim S_0$ for all $\mathfrak{p} \in \operatorname{Ass}_A S_0$. Therefore S_0 satisfies the Serre condition (S_1). Set $I = \operatorname{Ann}_A H^2_{\mathfrak{m}}(S_0)$. Then $\dim(A/I) \leqslant 1$. For an element $a \in A$ we set

$$U_1(aS_0) = \bigcup_{n \in \mathbb{N}} (aS_0 :_{S_0} I^n).$$

For a subset T of $\operatorname{Spec}(A)$ and an integer $i \geq 0$, we set

$$(T)_i = \{\mathfrak{p} \in T \mid \dim A/\mathfrak{p} = i\}.$$

Lemma 3.1. *Let $I = \operatorname{Ann}_A H^2_{\mathfrak{m}}(S_0)$. Then $U_1(aS_0)/aS_0$ is the largest submodule of S_0/aS_0 of dimension at most 1 for any S_0-regular element $a \in I$.*

Proof. Let $\mathfrak{a}$ denote the intersection of all $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$ such that $\dim(A/\mathfrak{p}) \leqslant 1$. Then $H^0_{\mathfrak{a}}(S_0/aS_0)$ is the largest submodule of S_0/aS_0 of dimension at most 1 (cf. [Sc2]). So, it is enough to show that $U_1(aS_0)/aS_0 = H^0_{\mathfrak{a}}(S_0/aS_0)$. Note that $\big(\operatorname{Ass}_A(S_0/aS_0)\big)_0 = \emptyset$ since $\operatorname{depth}(S_0) \geq 2$. From the exact sequence

$$0 \longrightarrow S_0 \xrightarrow{.a} S_0 \longrightarrow S_0/aS_0 \longrightarrow 0$$

with notice that $a \in I$, we have $H^1_{\mathfrak{m}}(S_0/aS_0) \cong 0 :_{H^2_{\mathfrak{m}}(S_0)} a = H^2_{\mathfrak{m}}(S_0)$. As $\big(\operatorname{Ass} S_0/aS_0\big)_0 = \emptyset$, it follows that

$$\big(\operatorname{Ass} S_0/aS_0\big)_{\leqslant 1} = \big(\operatorname{Ass} S_0/aS_0\big)_1 = \big(\operatorname{Att} H^1_{\mathfrak{m}}(S_0/aS_0)\big)_1 = \big(\operatorname{Att} H^2_{\mathfrak{m}}(S_0)\big)_1$$

by [BS, 11.3.3]. Note that the set of minimal prime ideals containing I is equal to the set of minimal attached primes of $H^2_{\mathfrak{m}}(S_0)$, cf [Mac]. Therefore $\sqrt{I}$ is the intersection of all attached primes of $H^2_{\mathfrak{m}}(S_0)$. So, $\mathfrak{a} \supseteq \sqrt{I}$. Hence $H^0_I(S_0/aS_0) \supseteq H^0_{\mathfrak{a}}(S_0/aS_0)$. As $\dim(A/I) \leqslant 1$, it follows that $H^0_I(S_0/aS_0)$ is of dimension at most 1. Therefore $H^0_I(S_0/aS_0) \subseteq H^0_{\mathfrak{a}}(S_0/aS_0)$. Hence $U_1(aS_0)/aS_0 = H^0_I(S_0/aS_0) = H^0_{\mathfrak{a}}(S_0/aS_0)$. □

Lemma 3.2. *Let $I = \operatorname{Ann}_A H^2_{\mathfrak{m}}(S_0)$. Then*

(i) There exists $r \in \mathbb{N}$ such that $I^r\big(H^1_{IA_{\mathfrak{p}}}(S_0)_{\mathfrak{p}}\big) = 0$ for all $\mathfrak{p} \in \big(\operatorname{Att}_A H^2_{\mathfrak{m}}(S_0)\big) \setminus \{\mathfrak{m}\}$.

(ii) Let r denote an integer satisfying (i). Set $J = I^r$. Let $a \in I$ be an A-regular element. Then for any S_0/aS_0-generalized regular element $b \in J$ we have

$$U_1(aS_0) = (aS_0 :_{S_0} J) = (aS_0 :_{S_0} b).$$

Proof. (i). Let $\mathfrak{p} \in \big(\operatorname{Att}_A H^2_{\mathfrak{m}}(S_0)\big) \setminus \{\mathfrak{m}\}$. Then $\mathfrak{p} \supseteq I$. As $\dim(A/I) \leqslant 1$, we have $\dim(A/\mathfrak{p}) = 1$. Therefore $IA_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$-primary. Since $(S_0)_{\mathfrak{p}}$ is unmixed, $H^1_{IA_{\mathfrak{p}}}(S_0)_{\mathfrak{p}}$ is of finite length. So there exists an integer $r_{\mathfrak{p}} > 0$ such that $I^{r_{\mathfrak{p}}}\big(H^1_{IA_{\mathfrak{p}}}(S_0)_{\mathfrak{p}}\big) = 0$. Set $r = \max r_{\mathfrak{p}}$, where $\mathfrak{p}$ takes over the set $\big(\operatorname{Att}_A H^2_{\mathfrak{m}}(S_0)\big) \setminus \{\mathfrak{m}\}$. Then r satisfies the requirement (i).

(ii). Firstly we claim that $\bigcup_n (a(S_0)_{\mathfrak{p}} : I^n A_{\mathfrak{p}}) = (a(S_0)_{\mathfrak{p}} : JA_{\mathfrak{p}})$ for any $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$. In fact, if $\mathfrak{p} \not\supseteq J$ then $\mathfrak{p} \not\supseteq I$, therefore both sides are equal to $a(S_0)_{\mathfrak{p}}$. So, we assume that $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$ and $\mathfrak{p} \supseteq J$. As $\operatorname{depth}(S_0) \geq 2$, we have $\operatorname{depth}(S_0/aS_0) > 0$. Hence $\mathfrak{p} \neq \mathfrak{m}$. Since $\dim(A/J) = \dim(A/I) \leqslant 1$, it follows that $\dim(A/\mathfrak{p}) = 1$. So, $\mathfrak{p}$ is a minimal prime ideal containing I. Hence $\mathfrak{p} \in \big(\operatorname{Att}_A H^2_{\mathfrak{m}}(S_0)\big) \setminus \{\mathfrak{m}\}$. So, we get by (i) that $JA_{\mathfrak{p}} \subseteq \mathfrak{b}_{\mathfrak{p}}$, where $\mathfrak{b}_{\mathfrak{p}} = \operatorname{Ann}_{R_{\mathfrak{p}}}\big(H^1_{IA_{\mathfrak{p}}}(S_0)_{\mathfrak{p}}\big)$. Since $IA_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$-primary, we have $H^1_{IA_{\mathfrak{p}}}(S_0)_{\mathfrak{p}} = H^1_{\mathfrak{p}A_{\mathfrak{p}}}(S_0)_{\mathfrak{p}}$, and by Lemma 2.1(i) we have

$$\bigcup_{n \in \mathbb{N}} (a(S_0)_{\mathfrak{p}} : I^n A_{\mathfrak{p}}) = \bigcup_{n \in \mathbb{N}} (a(S_0)_{\mathfrak{p}} : \mathfrak{p}^n A_{\mathfrak{p}}) = (a(S_0)_{\mathfrak{p}} : \mathfrak{b}_{\mathfrak{p}}) \subseteq (a(S_0)_{\mathfrak{p}} : JA_{\mathfrak{p}}).$$

Since $J = I^r$, it is clear that $(a(S_0)_{\mathfrak{p}} : JA_{\mathfrak{p}}) \subseteq \bigcup_{n \in \mathbb{N}} (a(S_0)_{\mathfrak{p}} : I^n A_{\mathfrak{p}})$. Therefore the claim is proved.

Now we prove the original equalities. Let $x \in U_1(aS_0)$. Then $x \in (aS_0 : I^n)$ for some n. Hence $x/1 \in (a(S_0)_{\mathfrak{p}} : I^n A_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$, where $x/1$ is the image of x in $A_{\mathfrak{p}}$. So, we have by the claim that $x/1 \in (a(S_0)_{\mathfrak{p}} : JA_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Ass}(S_0/aS_0)$. Let $c \in J$. For each $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$, we have $cx/1 \in a(S_0)_{\mathfrak{p}}$, and hence $cu_{\mathfrak{p}}x \in aS_0$ for some element $u_{\mathfrak{p}} \notin \mathfrak{p}$. Let $\mathfrak{u}$ be the ideal of A generated by all elements $u_{\mathfrak{p}}$ with $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$. Then $\mathfrak{u}cx \subseteq aS_0$. Note that $\mathfrak{u} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$. So, there exists by Prime Avoidance an element $u \in \mathfrak{u}$ such that $u \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0)$. It follows that u is not a zero divisor of S_0/aS_0 and $ucx \in aS_0$. So, $cx \in (aS_0 : u) = aS_0$ for all $c \in J$. Hence $Jx \subseteq aS_0$ and hence $x \in (aS_0 : J)$. Thus $U_1(aS_0) = (aS_0 : J)$.

Let $b \in J$ be (S_0/aS_0)-generalized regular. Then $\dim((aS_0 : b)/aS_0) \leqslant 1$. Since $U_1(aS_0)/aS_0$ is the largest submodule of S_0/aS_0 of dimension at most 1 by Lemma 3.1, it follows that $(aS_0 : b)/aS_0 \subseteq U_1(aS_0)/aS_0$. As $b \in J$ and $U_1(aS_0) = (aS_0 : J)$, we have

$$(aS_0 : b)/aS_0 \subseteq U_1(aS_0)/aS_0 = (aS_0 : J)/aS_0 \subseteq (aS_0 : b)/aS_0.$$

Thus, $U_1(aS_0) = (aS_0 : b)$. □

Lemma 3.3. *Let $I = \operatorname{Ann}_A H^2_{\mathfrak{m}}(S_0)$. Let $J = I^r$ be defined as in Lemma 3.2(ii). Let $a \in J$ be an S_0-regular element. Then $(U_1(aS_0))^2 = aU_1(aS_0)$.*

Proof. We have $(U_1(aS_0))^2 \supseteq aU_1(aS_0)$. Let $g, f \in U_1(aS_0)$. As $\dim(A/J) \leqslant 1$, there exists an element $b \in J \cap \mathfrak{m}$ which is an (S_0/aS_0)-generalized regular element. Since $\operatorname{Ass}_A(S_0/aS_0) = \operatorname{Ass}_A(S_0/a^2S_0)$, it follows that b, b^2 are (S_0/a^2S_0)-generalized regular elements. By Lemma 3.2(ii) we have $U_1(aS_0) = (aS_0 :_{S_0} b)$. Therefore $bf = ax, bg = ay$ for some $x, y \in S_0$. Hence $b^2fg = a^2xy$. So, by Lemma 3.2(ii), $fg \in (a^2S_0 :_{S_0} b^2) = U_1(a^2S_0)$ as b^2 is (S_0/a^2S_0)-generalized regular. As $U_1(a^2S_0) = (a^2S_0 :_{S_0} J)$ by Lemma 3.2(ii), we have $a(fg) \in JU_1(a^2S_0) \subseteq a^2S_0$. Therefore $afg = a^2z$ for some $z \in S_0$. Hence $fg = az$ since a is S_0-regular. By Lemma 3.2(ii) we have $U_1(a^2S_0) = (a^2S_0 :_{S_0} b)$ since b is S_0/a^2S_0-generalized regular. Since $fg \in U(a^2S_0)$, we have $bfg \in a^2S_0$. Hence $bfg = a^2w$ for some $w \in S_0$. Therefore $bfg = baz = a^2w$. So, $bz = aw$. Hence $z \in (aS_0 :_{S_0} b) = U_1(aS_0)$ by Lemma 3.2(ii). Thus, $fg = az \in aU_1(aS_0)$. □

The following lemma follows immediately by Lemma 3.3 and by the same arguments as in the proof of Goto [G, (2.6)].

Lemma 3.4. *Let $I = \operatorname{Ann}_A H^2_{\mathfrak{m}}(S_0)$ and let $J = I^r$ be defined as in Lemma 3.2(ii). Let $a \in J$ be an S_0-regular element. Set*

$$S_1 = \{x/a \in Q(A) \mid x \in U_1(aS_0)\}.$$

Then we have

(i) S_1 is an intermediate ring between S_0 and $Q(A)$.

(ii) $S_1 \cong U_1(aS_0)$ as A-modules, and $S_1 \cong \operatorname{End}_R(U_1(aS_0))$ as A-algebras.

(iii) $JS_1 \subseteq S_0$. In particular, $\dim_A(S_1/A) \leqslant 1$.

(iv) S_1 does not depend on the choice of A-regular element $a \in J$ and it is uniquely determined by A.

Now we need to recall the concept of multiplicity for Artinian modules. Let $L \neq 0$ be an Artinian A-module. Let $\mathfrak{q}$ be an ideal of A such that $\ell(0 :_L \mathfrak{q}) < \infty$. Denote by $\text{N-dim}_A L$ the Noetherian dimension of L defined as in [Ro], [K]. Then $\ell(0 :_L \mathfrak{q}^n)$ is a polynomial for $n \gg 0$, and

$$\text{N-dim}_A(L) = \deg\left(\ell(0 :_L \mathfrak{q}^n)\right) = \inf\{t \mid \exists x_1, \ldots, x_t \in \mathfrak{m} : \ell(0 :_L (x_1, \ldots, x_t)A) < \infty\},$$

cf. [Ro], [K]. By [CN], $\text{N-dim}_A L = \dim(\widehat{A}/\operatorname{Ann}_{\widehat{A}} L)$, where $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A. Assume that $\dim(\widehat{A}/\operatorname{Ann}_{\widehat{A}} L) = t$. Let a_t be the leading coefficient of the polynomial $\ell(0 :_L \mathfrak{q}^n)$ for $n \gg 0$. Following Brodmann and Sharp [BS1], the *multiplicity* of L with respect to $\mathfrak{q}$, denoted by $e'(\mathfrak{q}, L)$, is defined by the formula $e'(\mathfrak{q}, L) := a_t t!$. Note that $e'(\mathfrak{q}, L)$ is a positive integer. Moreover, if $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$ is an exact sequence of Artinian modules such that $\dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} L') = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} L) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} L'')$ then we have

$$e'(\mathfrak{q}, L) = e'(\mathfrak{q}, L') + e'(\mathfrak{q}, L'').$$

The following property on dimension of local cohomology modules was proved by Brodmann and Sharp [BS1].

Lemma 3.5. *Suppose that R is universally catenary and all formal fibers of R are Cohen-Macaulay. Then*

$$\dim(R/\operatorname{Ann}_R H^i_{\mathfrak{m}}(M)) = \dim(\widehat{\mathrm{R}}/\operatorname{Ann}_{\widehat{\mathrm{R}}} H^i_{\mathfrak{m}}(M)) = \text{N-dim}_R(H^i_{\mathfrak{m}}(M))$$

for any finitely generated R-module M.

Proof of Theorem 1.2. (ii)$\Rightarrow$(i). As B is unmixed, A must be unmixed. From the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ of finitely generated A-modules with notice that B is Cohen-Macaulay of dimension $d \geq 3$ and $\dim(B/A) = 1$, we have $H^2_{\mathfrak{m}}(A) \cong H^1_{\mathfrak{m}}(B/A)$ and $H^i_{\mathfrak{m}}(A) = 0$ for all $i \neq 1, 2, d$. As $\dim(B/A) = 1$, it follows that

$$\operatorname{Att}_A H^2_{\mathfrak{m}}(A) = \operatorname{Att}_A H^1_{\mathfrak{m}}(B/A) = \operatorname{Ass}(B/A)_1 \neq \emptyset.$$

Therefore $\mathfrak{m} \notin \operatorname{Att}_A H^2_{\mathfrak{m}}(A)$ and $\dim(A/\operatorname{Ann}_A H^2_{\mathfrak{m}}(A)) = 1$. As A is unmixed, $\operatorname{nCM}(A) = \bigcup_{i=0}^{d-1} \operatorname{Var}(\mathfrak{a}_i(A))$, where $\mathfrak{a}_i(A) = \operatorname{Ann}_R H^i_{\mathfrak{m}}(A)$, cf. [CNN]. Therefore $\dim \operatorname{nCM}(A) = 1$.

(i)$\Rightarrow$(ii). Since A is unmixed, we have $\operatorname{depth} A > 0$ and $\ell(H^1_{\mathfrak{m}}(A)) < \infty$. So, the ring S_0 in Lemma 2.3 is well defined. From the exact sequence

$$0 \longrightarrow A \longrightarrow S_0 \longrightarrow S_0/A \longrightarrow 0$$

with notice that $\ell(S_0/A) < \infty$ by Lemma 2.3 and $\operatorname{depth}(S_0) \geq 2$ by Lemma 2.4, we have by the assumption (i) that $H^i_{\mathfrak{m}}(S_0) = 0$ for all $i \neq 2, d$ and $H^2_{\mathfrak{m}}(S_0) \cong H^2_{\mathfrak{m}}(A)$. As

$$\dim \operatorname{nCM}(A) = 1 = \max_{i=0,\dots,d-1} \dim(A/\operatorname{Ann}_A H^i_{\mathfrak{m}}(A))$$

and $\ell_A(H^1_{\mathfrak{m}}(A)) < \infty$, we have by the assumption (i) that $\operatorname{Att}_A H^2_{\mathfrak{m}}(A) \neq \emptyset$ and $\dim(A/\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \operatorname{Att}_A H^2_{\mathfrak{m}}(A)$. Therefore $\operatorname{Att}_A H^2_{\mathfrak{m}}(S_0) \neq \emptyset$ and $\dim(A/\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \operatorname{Att}_A H^2_{\mathfrak{m}}(S_0)$.

Let $I = \operatorname{Ann}(H^2_{\mathfrak{m}}(M))$. Let $J = I^r$ be defined as in Lemma 3.2(ii). Let $a \in J$ be an S_0-regular element and $S_1 = a^{-1}U_1(aS_0)$ defined in Lemma 3.3. Then S_1 is an intermediate ring between S_0 and $Q(A)$, $S_1 \cong U_1(aS_0)$, and $\dim_A(S_1/S_0) \leqslant 1$. So, S_1 is an intermediate ring between A and $Q(A)$, S_1 is a finitely generated A-module,

$$\dim_A(S_1/A) = \max\{\dim_A(S_0/A), \dim_A(S_1/S_0)\} \leqslant 1.$$

Consider two exact sequences of A-modules

$$0 \longrightarrow U_1(aS_0) \longrightarrow S_0 \longrightarrow S_0/U_1(aS_0) \longrightarrow 0 \tag{3}$$

$$0 \longrightarrow S_0 \longrightarrow S_1 \longrightarrow S_1/S_0 \longrightarrow 0. \tag{4}$$

Since $S_0/U_1(aS_0) \cong (S_0/aS_0)/H_I^0(S_0/aS_0)$, we have

$$H_{\mathfrak{m}}^0(S_0/U_1(aS_0)) \subseteq H_I^0(S_0/U_1(aS_0)) = 0.$$

Since $H_{\mathfrak{m}}^1(S_0) = 0$, we have by (3) that $H_{\mathfrak{m}}^1(U_1(aS_0)) = 0$. Hence $H_{\mathfrak{m}}^1(S_1) = 0$ since $S_1 \cong U_1(aS_0)$ as A-modules. Because $\dim_A(S_1/S_0) \leqslant 1$, we get by (4) that $H_{\mathfrak{m}}^i(S_1) \cong H_{\mathfrak{m}}^i(S_0) = 0$ for all $i = 3, \ldots, d-1$. As $\operatorname{Ass}_A U_1(aS_0) \subseteq \operatorname{Ass}_A S_0$ and $\operatorname{depth}_A(S_0) > 0$, it follows that $H_{\mathfrak{m}}^0(S_1) \cong H_{\mathfrak{m}}^0(U_1(aS_0)) = 0$. So, to prove S_1 is Cohen-Macaulay, it is enough to show that $H_{\mathfrak{m}}^2(S_1) = 0$.

Assume that $H_{\mathfrak{m}}^2(S_1) \neq 0$. Since $\dim_A(S_1/S_0) \leqslant 1$, we get by (4) the exact sequence $H_{\mathfrak{m}}^2(S_0) \longrightarrow H_{\mathfrak{m}}^2(S_1) \longrightarrow 0$. Hence $\operatorname{Att}_A(H_{\mathfrak{m}}^2(S_1)) \subseteq \operatorname{Att}_A(H_{\mathfrak{m}}^2(S_0))$. Therefore $\dim(A/\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \operatorname{Att}_A H_{\mathfrak{m}}^2(S_1)$. Since $U_1(aS_0)/aS_0$ is the largest submodule of S_0/aS_0 of dimension at most 1 by Lemma 3.1, we have by [Sc2] that

$$\operatorname{Ass}_A(S_0/U_1(aS_0)) = \{\mathfrak{p} \in \operatorname{Ass}_A(S_0/aS_0) \mid \dim A/\mathfrak{p} > 1\}.$$

Therefore $\left(\operatorname{Ass}_A(S_0/U_1(aS_0))\right)_1 = \emptyset$. So, $\left(\operatorname{Att}_A H_{\mathfrak{m}}^1(S_0/U_1(aS_0))\right)_1 = \emptyset$, cf. [CNN]. Hence $\ell_A(H_{\mathfrak{m}}^1(S_0/U_1(aS_0))) < \infty$. As $H_{\mathfrak{m}}^3(U_1(aS_0)) \cong H_{\mathfrak{m}}^3(S_1) = 0$ and $H_{\mathfrak{m}}^1(S_0) = 0$, it deduces from (3) the exact sequence

$$0 \to H_{\mathfrak{m}}^1(S_0/U_1(aS_0)) \to H_{\mathfrak{m}}^2(U_1(aS_0)) \to H_{\mathfrak{m}}^2(S_0) \to H_{\mathfrak{m}}^2(S_0/U_1(aS_0)) \to 0. \tag{5}$$

Since $\dim(U_1(aS_0)/aS_0) \leqslant 1$ by Lemma 3.2(ii), from the exact sequence

$$0 \longrightarrow U_1(aS_0)/aS_0 \longrightarrow S_0/aS_0 \longrightarrow S_0/U_1(aS_0) \longrightarrow 0$$

we have $H_{\mathfrak{m}}^2(S_0/U_1(aS_0)) \cong H_{\mathfrak{m}}^2(S_0/aS_0)$. Because $H_{\mathfrak{m}}^3(S_0) = 0$ and $a \in J \subseteq \operatorname{Ann}_A H_{\mathfrak{m}}^2(S_0)$, it follows by the exact sequence $0 \longrightarrow S_0 \xrightarrow{.a} S_0 \longrightarrow S_0/aS_0 \longrightarrow 0$ that $H_{\mathfrak{m}}^2(S_0/aS_0) \cong H_{\mathfrak{m}}^2(S_0)$. Therefore $H_{\mathfrak{m}}^2(S_0/U_1(aS_0)) \cong H_{\mathfrak{m}}^2(S_0)$. Thus, the exact sequence (5) can be reformed as follows

$$0 \longrightarrow C/D \longrightarrow L \longrightarrow L \longrightarrow 0,$$

where $D = H_{\mathfrak{m}}^1(S_0/U_1(aS_0))$, $C = H_{\mathfrak{m}}^2(U_1(aS_0))$, $L = H_{\mathfrak{m}}^2(S_0)$. We have proved that $\ell_A(D) < \infty$. Moreover, $\dim(A/\operatorname{Ann}_A L) = 1$ and $\dim(A/\operatorname{Ann}_A H_{\mathfrak{m}}^2(S_1)) = 1$. Since $C \cong H_{\mathfrak{m}}^2(S_1)$, we have $\dim(A/\operatorname{Ann}_A C) = 1$. Therefore we have by Lemma 3.5 that

$$e'(\mathfrak{m}, L) = e'(\mathfrak{m}, C/D) + e'(\mathfrak{m}, L),$$

where $e'(\mathfrak{m}, -)$ denotes for the multiplicity. Hence $e'(\mathfrak{m}, C/D) = 0$. This gives a contradiction since $\dim(A/\operatorname{Ann}_A(C/D)) = 1$. Hence $H_{\mathfrak{m}}^2(S_1) = 0$. Therefore S_1 is Cohen-Macaulay.

From the exact sequence $0 \longrightarrow A \longrightarrow S_1 \longrightarrow S_1/A$ with notice that $\dim A \geq 3$ and S_1 is Cohen-Macaulay, we have $H_{\mathfrak{m}}^1(S_1/A) \cong H_{\mathfrak{m}}^2(A)$. Since $H_{\mathfrak{m}}^2(A) \neq 0$ and $\dim_A(S_1/A) \leqslant 1$, we have $\dim_A(S_1/A) = 1$. Thus, $B = S_1$ satisfies the requirement (ii).

Finally, suppose (i) and (ii) are satisfied. Set $\mathfrak{a} = \operatorname{Ann}_A(B/A)$. As $\dim(B/A) = 1$, we have $\emptyset \neq \operatorname{Att}_A H_{\mathfrak{m}}^1(B/A) = \operatorname{Ass}(B/A) \setminus \{\mathfrak{m}\}$. Therefore $\sqrt{\mathfrak{a}} = \sqrt{I}$, where $I = \operatorname{Ann}_A H_{\mathfrak{m}}^2(A)$.Note that $\dim(A/\mathfrak{a}) = 1$. Let $J = I^r$ be defined as in Lemma 3.2(ii) for the case $M = A$. Set $\mathfrak{b} = \mathfrak{a} \cap J$. Then $\sqrt{\mathfrak{b}} = \sqrt{I}$ and $\mathfrak{b}B \subseteq A$. Let S_0 be the ring defined as in Lemma 2.3. As $\dim(A/\mathfrak{b}) = 1$ and S_0 is unmixed of dimension $d \geq 3$, there exists $a \in \mathfrak{b}$ which is an S_0-regular element. We will show that $B = a^{-1}U_1(aA)$. As $\mathfrak{b}B \subseteq A$, we have $\mathfrak{b}aB \subseteq aA$. Hence

$$aB \subseteq aA :_A \mathfrak{b} \subseteq aS_0 :_{S_0} \mathfrak{b}.$$

As $\sqrt{\mathfrak{b}} = \sqrt{I}$, we have $aS_0 :_{S_0} \mathfrak{b} \subseteq \bigcup_n (aS_0 :_{S_0} I^n) = U_1(aS_0)$. Therefore $aB \subseteq U_1(aS_0)$. Hence $B \subseteq a^{-1}U_1(aS_0) := S_1$. From the exact sequence

$$0 \longrightarrow B \longrightarrow S_1 \longrightarrow S_1/B \longrightarrow 0$$

with notice that B, S_1 are Cohen-Macaulay of dimension $d \geq 3$, we have $H^1_{\mathfrak{m}}(S_1/B) = 0$ for all $i = 0, 1$. As $\dim_A(S_1/B) \leqslant \dim(S_1/A) \leqslant 1$ by the proof (i)$\Rightarrow$(ii), we have $S_1/B = 0$.

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SYMMETRIC AUSLANDER AND BASS CATEGORIES

KIRIKO KATO

ABSTRACT. We define the symmetric Auslander category $\mathsf{A}^{\mathrm{s}}(R)$ to consist of complexes of projective modules whose left- and right-tails are equal to the left- and right-tails of totally acyclic complexes of projective modules.

The symmetric Auslander category contains $\mathsf{A}(R)$, the ordinary Auslander category. It is well known that $\mathsf{A}(R)$ is intimately related to Gorenstein projective modules, and our main result is that $\mathsf{A}^{\mathrm{s}}(R)$ is similarly related to what can reasonably be called Gorenstein projective homomorphisms. Namely, there is an equivalence of triangulated categories

$$\underline{\mathsf{GMor}}(R) \xrightarrow{\simeq} \mathsf{A}^{\mathrm{s}}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$$

where $\underline{\mathsf{GMor}}(R)$ is the stable category of Gorenstein projective objects in the abelian category $\mathsf{Mor}(R)$ of homomorphisms of R-modules.

This result is set in the wider context of a theory for $\mathsf{A}^{\mathrm{s}}(R)$ and $\mathsf{B}^{\mathrm{s}}(R)$, the symmetric Bass category which is defined dually.

0. Introduction

This is a report based on joint work [8] with Peter Jørgensen.

We aim at investigating symmetry caused by a dualizing complex. Since a ring with a dualizing complex is a homomorphic image of a Gorenstein ring [9], we expect such a ring has symmetries analogous to Gorenstein case. Among all, the featured property is "a triangle of recollements" which results in high symmetry.

Definition 0.1 ([6]). Let T be a triangulated category. A stable t-structure on T is a pair of full subcategories (U, V) such that

- (i) $\Sigma\mathsf{U} = \mathsf{U}$, $\Sigma\mathsf{V} = \mathsf{V}$.
- (ii) $\mathrm{Hom}_{\mathsf{T}}(\mathsf{U}, \mathsf{V}) = 0$.
- (iii) For each T in T there exist U in U and V in V and a distinguished triangle $U \to T \to V \to$.

A triangle of recollements in T is a triple $(\mathsf{U}, \mathsf{V}, \mathsf{W})$ such that each of (U, V), (V, W), (W, U) is a stable t-structure.

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Let T' be another triangulated category with a triangle of recollements $(\mathsf{U}',\mathsf{V}',\mathsf{W}')$ and let $F:\mathsf{T}\to\mathsf{T}'$ be a triangulated functor. We say that F sends $(\mathsf{U},\mathsf{V},\mathsf{W})$ to $(\mathsf{U}',\mathsf{V}',\mathsf{W}')$ if $F(\mathsf{U})\subseteq\mathsf{U}'$, $F(\mathsf{V})\subseteq\mathsf{V}'$, $F(\mathsf{W})\subseteq\mathsf{W}'$.

Proposition 0.2 ([6, Prop. 1.18]). (i) *If a triangulated category* T *has a triangle of recollements* $(\mathsf{U},\mathsf{V},\mathsf{W})$, *then the relevant sub and quotient categories* $\mathsf{U},\mathsf{V},\mathsf{W},\mathsf{T}/\mathsf{U},\mathsf{T}/\mathsf{V}$ *and* T/W *are all equivalent.*

(ii) *Let* $(\mathsf{U},\mathsf{V},\mathsf{W})$ *and* $(\mathsf{U}',\mathsf{V}',\mathsf{W}')$ *be triangles of recollements in* T *and* T' *respectively. Suppose the triangulated functor* $F:\mathsf{T}\to\mathsf{T}'$ *sends* $(\mathsf{U},\mathsf{V},\mathsf{W})$ *to* $(\mathsf{U}',\mathsf{V}',\mathsf{W}')$. *If the restriction* $F\mid\mathsf{U}$ *is an equivalence of triangulated categories, then so is* F.

A triangle of recollements is first reported in some homotopy category of complexes over a Gorenstein ring [6].

From now on, let R be a commutative noetherian ring with a dualizing complex D. Such complexes were introduced in [5, chp. V] where it was also shown that the functor $\mathrm{RHom}_R(-,D)$ is a contravariant autoequivalence of $\mathsf{D}^{\mathrm{f}}(R)$, the finite derived category of R.

Some time later, it was shown in [2, sec. 3] that by restricting to certain subcategories $\mathsf{A}(R)$ and $\mathsf{B}(R)$ of the derived category $\mathsf{D}(R)$, the functors $D\overset{\mathrm{L}}{\otimes}_R-$ and $\mathrm{RHom}_R(D,-)$ become quasi-inverse covariant equivalences

$$\mathsf{A}(R)\underset{\mathrm{RHom}_R(D,-)}{\overset{D\overset{\mathrm{L}}{\otimes}_R-}{\rightleftarrows}}\mathsf{B}(R).$$

The categories $\mathsf{A}(R)$ and $\mathsf{B}(R)$ are known as the Auslander and Bass categories of R. The precise definition is given in Remark 1.5 below, but note that $\mathsf{A}(R)$ and $\mathsf{B}(R)$ contain the bounded complexes of projective, respectively injective, modules.

This paper introduces the symmetric Auslander category $\mathsf{A}^{\mathrm{s}}(R)$ and the symmetric Bass category $\mathsf{B}^{\mathrm{s}}(R)$ which contain $\mathsf{A}(R)$, respectively $\mathsf{B}(R)$, as full subcategories. While $\mathsf{A}(R)$ enjoys a strong relation to Gorenstein projective modules, our result is that $\mathsf{A}^{\mathrm{s}}(R)$ has a similarly close relation to *homomorphisms* of Gorenstein projective modules.

We develop a theory for the symmetric Auslander and Bass categories. One of the highlights is that $\mathsf{A}^{\mathrm{s}}(R)$ is, indeed, a highly symmetric object.

Our main result is the following.

Theorem A. *The quotient category* $\mathsf{A}^{\mathrm{s}}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$ *admits a triangle of recollements. Namely,*

$$(\mathsf{A}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R),\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R),S(\mathsf{B}(R))/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R))$$

forms a triangle of recollements, where $\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)$ *is the full subcategory of* $\mathsf{K}(\mathrm{Prj}\ R)$ *consisting of totally acyclic complexes and* S *is a certain functor introduced in* [7, sec. 4].

This leads us to the following equivalence.

Theorem B. *There is an equivalence of triangulated categories*

$$\underline{\mathsf{GMor}}(R) \xrightarrow{\simeq} \mathsf{A}^{\mathrm{s}}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R).$$

Here $\underline{\mathsf{GMor}}(R)$ is the stable category of Gorenstein projective objects in $\mathsf{Mor}(R)$, the abelian category of homomorphisms of R-modules. Note that there is an equivalence of categories between $\mathsf{Mor}(R)$ and $\mathsf{Mod}\,T_2(R)^{\mathrm{op}}$, the category of right-modules over the upper triangular matrix ring $T_2(R)$; cf. [1]. This implies that $\underline{\mathsf{GMor}}(R)$ is equivalent to the stable category of Gorenstein projective right-modules over $T_2(R)$.

On the other hand, we will show that the objects in $\underline{\mathsf{GMor}}(R)$ are precisely the injective homomorphisms between Gorenstein projective R-modules which have Gorenstein projective cokernels. Hence, whereas the Auslander category $\mathsf{A}(R)$ is related to Gorenstein projective modules, the symmetric Auslander category $\mathsf{A}^{\mathrm{s}}(R)$ is similarly related to *homomorphisms* of Gorenstein projective modules via Theorem B.

There are also several other results, among them the following.

Theorem C. *There are quasi-inverse equivalences of triangulated categories*

$$\mathsf{A}^{\mathrm{s}}(R) \rightleftarrows \mathsf{B}^{\mathrm{s}}(R).$$

Let $\mathsf{K}^{(\mathrm{b})}(\mathrm{Prj}\ R)$ denote the full subcateogry of $\mathsf{K}(\mathrm{Prj}\ R)$ consisting of complexes with bounded homology.

Theorem D. *There are inclusions*

$$\mathsf{A}(R) \subseteq \mathsf{A}^{\mathrm{s}}(R) \subseteq \mathsf{K}^{(\mathrm{b})}(\mathrm{Prj}\ R).$$

The first inclusion is an equality if and only if each Gorenstein projective R-module is projective.

The second inclusion is an equality if and only if R is a Gorenstein ring.

Thus, the property that $\mathsf{A}^{\mathrm{s}}(R)$ is minimal, respectively maximal, characterises two interesting classes of rings.

Let us remark on two important sources of ideas for this paper. First, [6] originated the notion of a triangle of recollements and used it to get a version of Theorems A and B for finitely generated modules when R is a Gorenstein ring. The present paper can be viewed as extending these ideas. Secondly, while it is not obvious from the description above, we make extensive use of the machinery developed in [7] for homotopy categories of complexes of projective, respectively, injective modules and their relation to Auslander and Bass categories.

1. Background

This section recalls the tools we will use; most of them come from [7].

Setup 1.1. Throughout, R is a commutative noetherian ring with a dualizing complex D which is assumed to be a bounded complex of injective modules.

Dualizing complexes were introduced in [5], but see e.g. [3, sec. 1] for a contemporary introduction.

Remark 1.2. There are homotopy categories $\mathsf{K}(\mathrm{Prj}\ R)$ and $\mathsf{K}(\mathrm{Inj}\ R)$ of complexes of projective, respectively, injective modules. They have several important triangulated subcategories:

The subcategories of bounded complexes are denoted by $\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$ and $\mathsf{K}^{\mathrm{b}}(\mathrm{Inj}\ R)$. The subcategories of complexes with bounded homology are denoted by $\mathsf{K}^{(\mathrm{b})}(\mathrm{Prj}\ R)$ and $\mathsf{K}^{(\mathrm{b})}(\mathrm{Inj}\ R)$.

The subcategories of K-projective, respectively, K-injective complexes are denoted by $\mathsf{K}_{\mathrm{prj}}(R)$ and $\mathsf{K}_{\mathrm{inj}}(R)$; see [11].

The subcategories of totally acyclic complexes are denoted $\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)$ and $\mathsf{K}_{\mathrm{tac}}(\mathrm{Inj}\ R)$. Complexes X in $\mathsf{K}(\mathrm{Prj}\ R)$ and Y in $\mathsf{K}(\mathrm{Inj}\ R)$ are called totally acyclic if they are exact and $\mathrm{Hom}_R(X,P)$ and $\mathrm{Hom}_R(I,Y)$ are exact for each projective module P and each injective module I.

Remark 1.3. Consider the subcategories $\mathsf{K}_{\mathrm{prj}}(R) \subseteq \mathsf{K}(\mathrm{Prj}\ R)$ and $\mathsf{K}_{\mathrm{inj}}(R) \subseteq \mathsf{K}(\mathrm{Inj}\ R)$. By [7, sec. 7], the inclusion functors, which we will denote by inc, are parts of adjoint pairs of functors,

$$\mathsf{K}_{\mathrm{prj}}(R) \underset{p}{\overset{\mathrm{inc}}{\rightleftarrows}} \mathsf{K}(\mathrm{Prj}\ R) \quad \text{and} \quad \mathsf{K}_{\mathrm{inj}}(R) \underset{\mathrm{inc}}{\overset{i}{\leftrightarrows}} \mathsf{K}(\mathrm{Inj}\ R).$$

In the terminology of [10, chp. 9], the existence of the right adjoint p places us in a situation of Bousfield localization, and accordingly, the counit morphism of the adjoint pair (inc, p) can be completed to a distinguished triangle

$$pX \xrightarrow{\epsilon_P} X \longrightarrow aX \longrightarrow$$

which depends functorially on X. Both p and a are triangulated functors. Dually, the unit morphism of the adjoint pair (i, inc) can be completed to a distinguished triangle

$$bY \longrightarrow Y \xrightarrow{\eta_Y} iY \longrightarrow$$

which depends functorially on Y.

Remark 1.4. By [7, thm. 4.2] there are quasi-inverse equivalences of categories

$$\mathsf{K}(\mathrm{Prj}\ R) \underset{S}{\overset{T}{\rightleftarrows}} \mathsf{K}(\mathrm{Inj}\ R)$$

where $T(-) = D \otimes_R -$ and $S = q \circ \mathrm{Hom}_R(D, -)$. The functor q is right-adjoint to the inclusion $\mathsf{K}(\mathrm{Prj}\ R) \to \mathsf{K}(\mathrm{Flat}\ R)$ where $\mathsf{K}(\mathrm{Flat}\ R)$ is the homotopy category of complexes of flat modules.

Remark 1.5. Let us recall the following from [2]. The derived category $\mathsf{D}(R)$ supports an adjoint pair of functors

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(D,-)}{\overset{D \overset{\mathrm{L}}{\otimes}_R}{\rightleftarrows}} \mathsf{D}(R).$$

The Auslander category of R is the triangulated subcategory defined in terms of the unit η by

$$\mathsf{A}(R) = \left\{ X \in \mathsf{D}(R) \,\middle|\, \begin{array}{l} X \text{ and } D \overset{\mathrm{L}}{\otimes}_R X \text{ have bounded homology;} \\ X \xrightarrow{\eta_X} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X) \text{ is an isomorphism} \end{array} \right\}$$

and the Bass category of R is the triangulated subcategory defined in terms of the counit ϵ by

$$\mathsf{B}(R) = \left\{ Y \in \mathsf{D}(R) \,\middle|\, \begin{array}{l} Y \text{ and } \mathrm{RHom}_R(D, Y) \text{ have bounded homology;} \\ D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D, Y) \xrightarrow{\epsilon_Y} Y \text{ is an isomorphism} \end{array} \right\}.$$

The functors $D \overset{\mathrm{L}}{\otimes}_R -$ and $\mathrm{RHom}_R(D, -)$ restrict to quasi-inverse equivalences between $\mathsf{A}(R)$ and $\mathsf{B}(R)$.

The canonical functors $\mathsf{K}_{\mathrm{prj}}(R) \to \mathsf{D}(R)$ and $\mathsf{K}_{\mathrm{inj}}(R) \to \mathsf{D}(R)$ are equivalences, and this permits us to view $\mathsf{A}(R)$ as a full subcategory of $\mathsf{K}_{\mathrm{prj}}(R)$ and hence of $\mathsf{K}(\mathrm{Prj}\ R)$, and $\mathsf{B}(R)$ as a full subcategory of $\mathsf{K}_{\mathrm{inj}}(R)$ and hence of $\mathsf{K}(\mathrm{Inj}\ R)$. As such, the adjoint functors

$$\mathsf{K}_{\mathrm{prj}}(R) \underset{pS}{\overset{iT}{\rightleftarrows}} \mathsf{K}_{\mathrm{inj}}(R)$$

restrict to a pair of quasi-inverse equivalences between $\mathsf{A}(R)$ and $\mathsf{B}(R)$ by [7, prop. 7.2].

See [3, sec. 1] for an alternative review of Auslander and Bass categories.

2. SYMMETRIC AUSLANDER AND BASS CATEGORIES

This section develops a theory of symmetric Auslander and Bass categories. It proves Theorems A,C and D from the Introduction.

For the rest of the paper, an unadorned K stands for $\mathsf{K}(\mathrm{Prj}\ R)$. We combine this in an obvious way with various embellishments to form K^{b}, $\mathsf{K}^{(\mathrm{b})}$, $\mathsf{K}_{\mathrm{prj}}$, and $\mathsf{K}_{\mathrm{tac}}$. Likewise, unadorned categories such as A, B, and D stand for $\mathsf{A}(R)$, $\mathsf{B}(R)$, and $\mathsf{D}(R)$.

In the following definition, $\mathsf{X} * \mathsf{Y}$ denotes the full subcategory of objects C which sit in distinguished triangles $X \to C \to Y \to$ with X in X and Y in Y.

Definition 2.1. The *symmetric Auslander category* A^{s} and the *symmetric Bass category* B^{s} of R are the full subcategories of $\mathsf{K}(\mathrm{Prj}\ R)$ and $\mathsf{K}(\mathrm{Inj}\ R)$ defined by

$$\mathsf{A}^{\mathrm{s}} = S(\mathsf{B}) * \mathsf{A} \quad\text{and}\quad \mathsf{B}^{\mathrm{s}} = \mathsf{B} * T(\mathsf{A})$$

where S and T are the functors from [7] described in Remark 1.4.

Remark 2.2. By [3, thm. 4.1], the subcategory A of K consists of complexes isomorphic to right-bounded complexes of projective modules whose left-tail is equal to the left-tail of a complete projective resolution.

Using the theory of [7], one can show that similarly, $S(\mathsf{B})$ consists of complexes isomorphic to left-bounded complexes of projective modules whose right-tail is equal to the right-tail of a complete projective resolution.

From this it follows that A^{s} consists of complexes isomorphic to complexes of projective modules both of whose tails are equal to the tails of complete projective resolutions.

Similar remarks apply to B^{s}, and this is one of the reasons for the terminology "symmetric Auslander and Bass categories".

Remark 2.3. The following lemma and most of the other results in this section will only be given for A^{s}, but there are dual versions for B^{s} with similar proofs.

Lemma 2.4. *Let C be in K. Then C is in A^{s} if and only if the following conditions are satisfied.*

(i) *C and TC have bounded homology.*
(ii) *The mapping cone of $pC \xrightarrow{\epsilon_C} C$ is totally acyclic.*
(iii) *The mapping cone of $TC \xrightarrow{\eta_{TC}} iTC$ is totally acyclic.*

Proposition 2.5. *The category A^{s} is a triangulated subcategory of K, and there are inclusions of triangulated subcategories*

$$\mathsf{K}_{\mathrm{tac}} \subseteq \mathsf{A}^{\mathrm{s}} \subseteq \mathsf{K}^{(\mathrm{b})}.$$

Remark 2.6. We owe the following observations based on Lemma 2.4 to Srikanth Iyengar.

The Auslander and Bass categories A and B also exist in versions $\widehat{\mathsf{A}}$ and $\widehat{\mathsf{B}}$ without boundedness conditions [7, 7.1]. With small modifications, the proof of Lemma 2.4 shows that membership of $S(\widehat{\mathsf{B}}) * \widehat{\mathsf{A}}$ is characterised by conditions (ii) and (iii) of the Lemma.

It is immediate from Lemma 2.4 that $\mathsf{A}*S(\mathsf{B})$ is contained in $\mathsf{A}^{\mathsf{s}} = S(\mathsf{B})*\mathsf{A}$. This is a bit surprising since one would not normally expect any inclusion between categories of the form $\mathsf{X} * \mathsf{Y}$ and $\mathsf{Y} * \mathsf{X}$.

We do not know if $\mathsf{A} * S(\mathsf{B})$ is triangulated, but it will often be considerably smaller than $S(\mathsf{B}) * \mathsf{A}$ since $\mathsf{K}_{\mathsf{tac}}$ is contained in $S(\mathsf{B}) * \mathsf{A}$ by Proposition 2.5 while it is easy to show that the intersection of $\mathsf{A} * S(\mathsf{B})$ with $\mathsf{K}_{\mathsf{tac}}$ is zero.

Theorem 2.7. *The functors T and S restrict to quasi-inverse equivalences of triangulated categories*

$$\mathsf{A}^{\mathsf{s}} \underset{S}{\overset{T}{\rightleftarrows}} \mathsf{B}^{\mathsf{s}}.$$

Theorem 2.8. *There are inclusions*

$$\mathsf{A} \subseteq \mathsf{A}^{\mathsf{s}} \subseteq \mathsf{K}^{(\mathsf{b})}.$$

The first inclusion is an equality if and only if each Gorenstein projective R-module is projective.

The second inclusion is an equality if and only if R is a Gorenstein ring.

In the following theorem, note that $\mathsf{K}_{\mathsf{tac}}$ is a triangulated subcategory of A^{s} which can also be viewed as a triangulated subcategory of the Verdier quotient $\mathsf{A}^{\mathsf{s}}/\mathsf{K}^{\mathsf{b}}$ since there are only zero morphisms from K^{b} to $\mathsf{K}_{\mathsf{tac}}$.

Theorem 2.9. *The category $\mathsf{A}^{\mathsf{s}}/\mathsf{K}^{\mathsf{b}}$ has a triangle of recollements*

$$(\mathsf{A}/\mathsf{K}^{\mathsf{b}}\,,\, \mathsf{K}_{\mathsf{tac}}\,,\, S(\mathsf{B})/\mathsf{K}^{\mathsf{b}}).$$

That is, it has stable t-structures

$$(\mathsf{A}/\mathsf{K}^{\mathsf{b}}\,,\, \mathsf{K}_{\mathsf{tac}})\,,\, (\mathsf{K}_{\mathsf{tac}}\,,\, S(\mathsf{B})/\mathsf{K}^{\mathsf{b}})\,,\, (S(\mathsf{B})/\mathsf{K}^{\mathsf{b}}\,,\, \mathsf{A}/\mathsf{K}^{\mathsf{b}}).$$

3. The category of homomorphisms

This section proves our main result, Theorem A from the Introduction (Theorem 3.7).

Definition 3.1. We let Mor denote the category of homomorphisms of R-modules.

The objects of Mor are the homomorphisms of R-modules.

The morphisms of Mor are defined as follows: A morphism f from $X_\alpha \xrightarrow{\alpha} T_\alpha$ to $X_\beta \xrightarrow{\beta} T_\beta$ is a pair (f_X, f_T) of homomorphisms of R-modules $X_\alpha \xrightarrow{f_X} X_\beta$ and $T_\alpha \xrightarrow{f_T} T_\beta$

such that there is a commutative square

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_X} & X_\beta \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ T_\alpha & \xrightarrow[f_T]{} & T_\beta . \end{array}$$

It is not hard to check that the projective objects of Mor are precisely the split injections between projective R-modules.

Corollary 3.2. *The Gorenstein projective objects of* Mor *are the injective homomorphisms between Gorenstein projective R-modules which have Gorenstein projective cokernels.*

Definition 3.3. We denote the full subcategory of Gorenstein projective objects in Mor by GMor. Inside GMor, we consider the following full subcategories GMor^p, GMor^0, and GMor^1.

(i) GMor^p consists of injective homomorphisms $X \xrightarrow{\iota_X} P$ where X is Gorenstein projective and P is projective.

(ii) GMor^0 consists of zero homomorphisms $0 \xrightarrow{0^T} T$ where T is Gorenstein projective.

(iii) GMor^1 consists of identity homomorphisms $X \xrightarrow{1_M} X$ where X is Gorenstein projective.

There are corresponding stable categories which are defined by dividing out the morphisms which factor through a projective object. The stable categories are denoted by underlining. The category $\underline{\mathsf{GMor}}$ is triangulated, and $\underline{\mathsf{GMor}^p}$, $\underline{\mathsf{GMor}^0}$, and $\underline{\mathsf{GMor}^1}$ are triangulated subcategories.

Theorem 3.4. *The category $\underline{\mathsf{GMor}}$ has a triangle of recollements*

$$(\underline{\mathsf{GMor}^p}\,,\ \underline{\mathsf{GMor}^1}\,,\ \underline{\mathsf{GMor}^0}).$$

That is, it has stable t-structures

$$(\underline{\mathsf{GMor}^p}\,,\ \underline{\mathsf{GMor}^1})\ ,\ (\underline{\mathsf{GMor}^1}\,,\ \underline{\mathsf{GMor}^0})\ ,\ (\underline{\mathsf{GMor}^0}\,,\ \underline{\mathsf{GMor}^p}).$$

Let $X_\alpha \xrightarrow{\alpha} T_\alpha$ be an object of GMor and consider complete projective resolutions P of X_α and $\widetilde{P}$ of T_α. In particular, there is a surjection $P^0 \xrightarrow{\rho} X_\alpha$ and an injection $T_\alpha \xrightarrow{\iota} \widetilde{P}^1$. Let P_α denote the complex

$$\cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{\iota\alpha\rho} \widetilde{P}^1 \longrightarrow \widetilde{P}^2 \longrightarrow \cdots .$$

Proposition 3.5 ([6, lemmas 4.2 and 4.3 and prop. 4.4]). *The operation $\alpha \mapsto P_\alpha$ gives a functor* $\mathsf{GMor} \to \mathsf{A}^s$ *which induces a triangulated functor*

$$\underline{P} : \underline{\mathsf{GMor}} \to \mathsf{A}^s/\mathsf{K}^b.$$

Lemma 3.6 ([6, lemmas 4.6 and 4.7]). (i) *$\underline{P}$ sends the triangle of recollements*

$$(\underline{\mathsf{GMor}}^{p}, \underline{\mathsf{GMor}}^{1}, \underline{\mathsf{GMor}}^{0})$$

to the triangle of recollements

$$(\mathsf{A}/\mathsf{K}^{b}, \mathsf{K}_{\mathrm{tac}}, S(\mathsf{B})/\mathsf{K}^{b}).$$

(ii) *The restriction of $\underline{P}$ to $\underline{\mathsf{GMor}}^{1}$ is an equivalence of triangulated categories $\underline{\mathsf{GMor}}^{1} \to \mathsf{K}_{\mathrm{tac}}$.*

The following main theorem follows immediately by combining Lemma 3.6 and Proposition 0.2; compare with [6, lem. 4.7 and thm. 4.8].

Theorem 3.7. *The functor $\underline{P}$ is an equivalence of triangulated categories*

$$\underline{\mathsf{GMor}} \to \mathsf{A}^{s}/\mathsf{K}^{b}.$$

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ON MODULES OF FINITE PROJECTIVE DIMENSION WITH RESPECT TO A SEMIDUALIZING MODULE

KEI-ICHIRO IIMA

This proceeding is based on joint work with Tokuji Araya and Ryo Takahashi. Throughout this proceeding, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. All modules considered in this proceeding are assumed to be finitely generated.

An R-module C is said to be *semidualizing* if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. A free module of rank one and a canonical module of a Cohen-Macaulay local ring are semidualizing modules. Various homological dimensions with respect to a fixed semidualizing R-module C are invented and investigated (cf. [2, 4, 8]). Among them, the *C-projective dimension* of a nonzero R-module M, denoted by $C\text{-proj.dim}_R\, M$, is defined as the infimum of integers n such that there exists an exact sequence of the form

$$0 \to C^{b_n} \to C^{b_{n-1}} \to \cdots \to C^{b_1} \to C^{b_0} \to M \to 0,$$

where each b_i is a positive integer.

An R-module M is called *totally C-reflexive*, where C is a semidualizing R-module, if the natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism and $\mathrm{Ext}^i_R(M, C) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$.

We denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules. Let $\mathcal{G}_C(R)$ and $\mathcal{F}_C(R)$ denote the full subcategories of $\mathrm{mod}\, R$ consisting of all totally C-reflexive R-modules and consisting of all direct summands of finite direct sums of copies of C, respectively. Let $\mathcal{X}_C(R)$ be the *left perpendicular category* of the category of R-modules of finite C-projective dimension, that is, the subcategory of $\mathrm{mod}\, R$ consisting of all R-modules X satisfying $\mathrm{Ext}^1_R(X, M) = 0$ for each R-module M of finite C-projective dimension. There are inclusion relations of subcategories of $\mathrm{mod}\, R$:

$$\mathcal{X}_C(R) \supset \mathcal{G}_C(R) \supset \mathcal{F}_C(R),\ \mathcal{F}_R(R).$$

The main purpose of this proceeding is to find out what property is characterized by the equalities of $\mathcal{X}_C(R)$ and each of $\mathcal{G}_C(R)$, $\mathcal{F}_C(R)$, $\mathcal{F}_R(R)$. The main result of this proceeding is the following theorem.

Theorem 1. *Let R be a commutative noetherian local ring.*

(1) *The following are equivalent for a semidualizing R-module C.*
   (a) *$\mathcal{X}_C(R) = \mathcal{G}_C(R)$ holds.*
   (b) *C has finite injective dimension.*
   If this is the case, then R is Cohen-Macaulay and C is a canonical module.

Key words and phrases. perpendicular category, projective dimension, semidualizing module, totally reflexive module, strong test module for projectivity.

(2) *The following are equivalent.*
   (a) $\mathcal{X}_R(R) = \mathcal{G}_R(R)$ *holds.*
   (b) R *is Gorenstein.*
(3) *The following are equivalent.*
   (a) $\mathcal{X}_C(R) = \mathcal{F}_R(R)$ *holds for some semidualizing* R*-module* C.
   (b) $\mathcal{X}_C(R) = \mathcal{F}_C(R)$ *holds for some semidualizing* R*-module* C.
   (c) $\mathcal{X}_R(R) = \mathcal{F}_R(R)$ *holds.*
   (d) R *is regular.*

To prove our theorem, we establish two lemmas.

Lemma 2. *For every* R*-module* M *one has an isomorphism* $\operatorname{Hom}_R(\operatorname{Tr} M, R) \cong \Omega^2 M$ *of* R*-modules.*

Lemma 3. *Let* C *be a semidualizing* R*-module, and let* $t = \operatorname{depth} R$.
(1) *If* $\operatorname{Ext}^i_R(X, C) = 0$ *for* $1 \leq i \leq t+1$, *then* X *belongs to* $\mathcal{X}_C(R)$.
(2) *The module* $C \otimes_R \operatorname{Tr} \Omega^{t+1} k$ *belongs to* $\mathcal{X}_C(R)$.

Now we can prove the theorem.

Proof. (1) (a) $\Rightarrow$ (b): Lemma 3(2) shows that $C \otimes_R \operatorname{Tr} \Omega^{t+1} k$ belongs to $\mathcal{X}_C(R) = \mathcal{G}_C(R)$. By definition, the C-dual module $\operatorname{Hom}_R(C \otimes_R \operatorname{Tr} \Omega^{t+1} k, C)$ is also in $\mathcal{G}_C(R)$. There are isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(C \otimes_R \operatorname{Tr} \Omega^{t+1} k, C) &\cong \operatorname{Hom}_R(\operatorname{Tr} \Omega^{t+1} k, \operatorname{Hom}_R(C, C)) \\ &\cong \operatorname{Hom}_R(\operatorname{Tr} \Omega^{t+1} k, R) \\ &\cong \Omega^2 \Omega^{t+1} k = \Omega^{t+3} k\end{aligned}$$

by Lemma 2. Hence $\Omega^{t+3} k$ belongs to $\mathcal{G}_C(R)$, and $\operatorname{Ext}^{i+t+3}_R(k, C) \cong \operatorname{Ext}^i_R(\Omega^{t+3} k, C) = 0$ for all $i > 0$, which shows that C has finite injective dimension.

(b) $\Rightarrow$ (a): Let C be a semidualizing R-module of finite injective dimension. By [7, Corollary (3.9)], R is a Cohen-Macaulay local ring with canonical module C. Denote by $\operatorname{CM}(R)$ the full subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay R-modules. Note here that $\operatorname{CM}(R)$ coincides with $\mathcal{G}_C(R)$. By virtue of the Cohen-Macaulay approximation theorem [3, Theorem 1.1], for each R-module M there exists an exact sequence

$$0 \to Y_M \to X_M \to M \to 0$$

of R-modules such that X_M is maximal Cohen-Macaulay and Y_M has finite C-projective dimension. Now suppose that M belongs to $\mathcal{X}_C(R)$. Then the above exact sequence splits because it corresponds to an element of $\operatorname{Ext}^1_R(M, Y_M)$, which vanishes as $M \in \mathcal{X}_C(R)$ and $C\text{-proj.dim}_R Y_M < \infty$. Hence M is isomorphic to a direct summand of X_M. In particular, M is a maximal Cohen-Macaulay R-module, and thus $\mathcal{X}_C(R)$ is contained in $\operatorname{CM}(R) = \mathcal{G}_C(R)$. Therefore $\mathcal{X}_C(R)$ coincides with $\mathcal{G}_C(R)$.

(2) Letting $C = R$ in (1), we observe that the assertion holds.

(3) (a) $\Rightarrow$ (c): The module C belongs to $\mathcal{X}_C(R) = \mathcal{F}_R(R)$ by Lemma 3(1), so it is isomorphic to R^r for some $r \geq 0$. Since C is semidualizing, we have $R \cong \operatorname{Hom}_R(C, C) \cong R^{r^2}$. Hence we have $r = 1$ and $C \cong R$. Therefore it holds that $\mathcal{X}_R(R) = \mathcal{F}_R(R)$.

(c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b): Let $C = R$.

(b) $\Rightarrow$ (d): Lemma 3(2) says that $C \otimes_R \operatorname{Tr} \Omega^{t+1} k$ is in $\mathcal{X}_C(R) = \mathcal{F}_C(R)$. Similarly to the proof of (a) $\Rightarrow$ (b) in (1), we observe that $\Omega^{t+3} k$ is in $\mathcal{F}_R(R)$. Thus the R-module k has finite projective dimension, and hence R is regular.

(d) $\Rightarrow$ (c): If R is regular, then it is Gorenstein and every maximal Cohen-Macaulay R-module is free. By (2) we have $\mathcal{X}_R(R) = \mathcal{G}_R(R) = \mathrm{CM}(R) = \mathcal{F}_R(R)$. □

On the other hand, the notion of a strong test module for projectivity has been introduced and studied by Ramras [6]. An R-module M is called a *strong test module for projectivity* if every R-module N with $\operatorname{Ext}^1_R(N, M) = 0$ is projective, or equivalently, free. The residue field k is a typical example of a strong test module for projectivity. Ramras shows that the maximal ideal $\mathfrak{m}$ is a strong test module for projectivity. He also proves that every strong test module for projectivity has depth at most one. Using the rigidity theorem for Tor modules, Jothilingam [5] proves that when R is a regular local ring, every R-module of depth at most one is a strong test module for projectivity. Our Theorem 1 yields that the converse of this Jothilingam's result also holds true.

Theorem 4. *The following four conditions are equivalent.*

(1) *R is regular.*

(2) *Every R-module of depth at most one is a strong test module for projectivity.*

(3) *There exists a strong test R-module for projectivity of finite projective dimension.*

(4) *There exist a semidualizing R-module C and a strong test R-module for projectivity of finite C-projective dimension.*

Proof. (1) $\Rightarrow$ (2): This implication follows from [5, Theorem 1].

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): These implications are easy.

(4) $\Rightarrow$ (1): Let M be a strong test R-module for projectivity with $C\text{-proj.d}_R\, M < \infty$. Let X be a module in $\mathcal{X}_C(R)$. Then we have $\operatorname{Ext}^1_R(X, M) = 0$. Since M is a strong test module for projectivity, X is a free R-module. Thus $\mathcal{X}_C(R)$ is contained in $\mathcal{F}_R(R)$. Therefore $\mathcal{X}_C(R) = \mathcal{F}_R(R)$, and R is regular by Theorem 1(3). □

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A HOMOLOGICAL DIMENSION OVER AB RINGS

TOKUJI ARAYA

1. Introduction

Throughout this talk, let $(R, \mathfrak{m}, k)$ be a noetherian local ring. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. All modules considered in this talk are assumed to be finitely generated.

In the commutative ring theory, there are important ring theoretic properties $\mathbb{P}$ with implications below:

$$\mathbb{P} : \text{regular} \Rightarrow \text{complete intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay}$$

Related to each properties, there are homological dimensions $\mathbb{P}\text{-}\dim_R M$ of R-module M with following inequalities:

$$\mathbb{P}\text{-}\dim_R M : \operatorname{pd}_R M \geq \text{CI-}\dim_R M \geq \text{G-}\dim_R M \geq \text{CM-}\dim_R M.$$

The relationship between a ring theoretic property $\mathbb{P}$ and a related homological dimension $\mathbb{P}\text{-}\dim_R$ is following (c.f. [4, 5, 17] for regular, [8] for complete intersection, [3] for Gorenstein and [11] for Cohen-Macaulay, and see also [7]):

Proposition 1.1. (1) *The following conditions are equivalent.*
  (i) *R has property $\mathbb{P}$.*
  (ii) *$\mathbb{P}\text{-}\dim_R M < \infty$ for every R-module M.*
  (iii) *$\mathbb{P}\text{-}\dim_R k < \infty$.*
(2) *Let M be a nonzero R-module. If $\mathbb{P}\text{-}\dim_R M < \infty$ then $\mathbb{P}\text{-}\dim_R M = \operatorname{depth} R - \operatorname{depth} M$.*

The notion of the AB ring is introduced by Huneke and Jorgensen [13]. It has nice homological properties. They prove that complete intersection implies AB.

The aim of this talk is to define AB-dimension which satisfies following theorem.

Theorem 1.2. (1) *The following conditions are equivalent.*
  (i) *R is an AB ring.*
  (ii) *$\text{AB-}\dim_R M < \infty$ for every R-module M.*
  (iii) *$\text{AB-}\dim_R L < \infty$ for every R-module L of finite length.*
  (iv) *R is Gorenstein and the class $\mathcal{AB}$ of $\operatorname{mod} R$ consisting of all modules of finite AB-dimension is closed under extension.*
(2) *Let M be a nonzero R module.*
  (i) *If $\text{AB-}\dim_R M < \infty$ then $\text{AB-}\dim_R M = \operatorname{depth} R - \operatorname{depth} M$.*
  (ii) *$\text{CI-}\dim_R M \geq \text{AB-}\dim_R M \geq \text{G-}\dim_R M$.*

2. AB-dimension

In this section, we shall define AB-dimension and investigate some properties.

We give some notations to define an AB-dimension.

Definition 2.1. Let M be an R-modules.
(1) For an R-module N, we set $\mathrm{P}_R(M,N) := \sup\{ n \mid \mathrm{Ext}_R^n(M,N) \neq 0 \}$.
(2) We denote by $M^\perp$ the full sub category of $\operatorname{mod} R$ consisting of all R-modules N with $\mathrm{Ext}_R^{\gg 0}(M,N) = 0$.
(3) We set $\mathrm{P}_R(M) := \sup\{ \mathrm{P}_R(M,N) \mid N \in M^\perp \}$.
(4) We say that M satisfies an *Auslander condition (AC)* if $(-\infty \leq)\, \mathrm{P}_R(M) < \infty$.
(5) R is called an *AC ring* if every R-module satisfies (AC).
(6) R is called an *AB ring* if R is a Gorenstein AC ring.
(7) We define $\text{AB-dim}_R M := \sup\{ \text{G-dim}_R M, \mathrm{P}_R(M) \}$.

There are some remarks.

Remark 2.2. (1) If $\mathrm{pd}_R M < \infty$, then $M^\perp = \operatorname{mod} R$. For any nonzero R-module N, one can check that $\mathrm{P}_R(M,N) = \mathrm{pd}_R M = \operatorname{depth} R - \operatorname{depth} M$ by Nakayama's lemma (c.f. [1, Theorem 4.2]). Therefore we have $\mathrm{P}_R(M) = \operatorname{depth} R - \operatorname{depth} M$.
(2) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence and let (i,j,l) be a permutation of $(1,2,3)$. Then one can easily check $M_i^\perp \cap M_j^\perp \subset M_l^\perp$. Furthermore, if $\mathrm{pd}_R M_i < \infty$, then $M_j^\perp = M_l^\perp$.
(3) If either M or N is zero, then $\{ n \mid \mathrm{Ext}_R^n(M,N) \neq 0 \}$ is empty. In this case, we define $\mathrm{P}_R(M,N) = -\infty$. In particular, $\mathrm{P}_R(0)$ is $-\infty$. If $M^\perp = \{0\}$, then $\mathrm{P}_R(M)$ is also $-\infty$.
(4) If there exists a nonzero R-module N in $k^\perp$, then N has finite injective dimension and therefore R must be Cohen-Macaulay by Bass' conjecture [15, 12, 16]. Thus if R is not Cohen-Macaulay, then $\mathrm{P}_R(k) = -\infty$. On the other hand, if R is Cohen-Macaulay, then there exists a nonzero R-module I of finite injective dimension. Since $\mathrm{P}_R(M,I) = \operatorname{depth} R - \operatorname{depth} M$ for any nonzero R-module M (c.f. [9, Exercises 3.1.24]), we have $\mathrm{P}_R(M) \geq \operatorname{depth} R - \operatorname{depth} M \geq 0$ and $\mathrm{P}_R(k) = \operatorname{depth} R$. Furthermore we can check that $\text{AB-dim}_R k < \infty$ if and only if R is Gorenstein.

We prepare two lemmas to prove Theorem 1.2.(1).

Lemma 2.3. *Let M be an R-module of finite G-dimension. Let $0 \to M \to Y \to X \to 0$ be a finite projective hull of M. Then* $\text{AB-dim}_R X < \infty$ *if and only if* $\text{AB-dim}_R M < \infty$.

Proof. It comes from $\mathrm{pd}_R Y < \infty$, we obtain $Y^\perp = \operatorname{mod} R$, $\mathrm{P}_R(Y) = \mathrm{pd}_R Y$ and $M^\perp = X^\perp$ by Remark 2.2.(1) and (2). Assume $\text{AB-dim}_R X < \infty$ to prove "only if" part. Let $N \in M^\perp$. Taking $\mathrm{Hom}_R(-,N)$ to $0 \to M \to Y \to X \to 0$, we have $\mathrm{P}_R(M,N) \leq \max\{ \mathrm{P}_R(Y,N),\ \mathrm{P}_R(X,N)+1 \} \leq \max\{ \mathrm{pd}_R Y,\ \mathrm{P}_R(X)+1 \} < \infty$. It yields $\mathrm{P}_R(M) \leq \max\{ \mathrm{pd}_R Y,\ \mathrm{P}_R(X)+1 \} < \infty$ and we get $\text{AB-dim}_R M < \infty$.
Similarly to this argument, we can prove "if" part and we ommit. □

Lemma 2.4. *Let M be an R-module and let $x \in \mathfrak{m}$ be an M-regular element. Then* $\text{AB-dim}_R M < \infty$ *if and only if* $\text{AB-dim}_R M/xM < \infty$.

Proof. Since Auslander and Bridger [3] prove $\text{G-dim}_R M/xM = \text{G-dim}_R M + 1$, we prove that $\mathrm{P}_R(M) < \infty$ if and only if $\mathrm{P}_R(M/xM) < \infty$. Let N be an R-module. Taking $\mathrm{Hom}_R(-,N)$ to the exact sequence $0 \to M \to M \to M/xM \to 0$, we get the long exact sequence

$$\begin{array}{cccccccc} \cdots & \to & \mathrm{Ext}_R^i(M/xM,N) & \to & \mathrm{Ext}_R^i(M,N) & \to & \mathrm{Ext}_R^i(M,N) & \\ & \to & \mathrm{Ext}_R^{i+1}(M/xM,N) & \to & \mathrm{Ext}_R^{i+1}(M,N) & \to & \mathrm{Ext}_R^{i+1}(M,N) & \to \cdots . \end{array}$$

Then one can check $\mathrm{P}_R(M/xM, N) = \mathrm{P}_R(M, N) + 1$ by Nakayama's lemma. Thus we have $\mathrm{P}_R(M/xM) = \mathrm{P}_R(M) + 1$. □

We give a proof of Theorem 1.2.(1).

Proof of Theorem 1.2.(1). (i) $\iff$ (ii): This equivalence is obvious by definition.

(i)(,(ii)) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (iii): Since R is Gorenstein, we have $k \in \mathcal{AB}$ by Remark 2.2.(4). Therefore $\mathcal{AB}$ contains all R-modules of finite length by the assumption.

(iii) $\Rightarrow$ (ii): Since $\text{G-dim}_R\, k < \infty$, R must be Gorenstein. Note that every maximal Cohen-Macaulay R-module has finite AB-dimension. Namely, let X be a maximal Cohen-Macaulay R-module and let $x_1, x_2, \dots, x_t \in \mathfrak{m}$ be a maximal X-regular sequence. Since $\text{AB-dim}_R\, X/(x_1, x_2, \dots, x_t)X < \infty$ by the assumption, $\text{AB-dim}_R\, X < \infty$ by Lemma 2.4.

Let M be an R-module which is not maximal Cohen-Macaulay, and let $0 \to M \to Y \to X \to 0$ be a finite projective hull of M. Since $\text{AB-dim}_R\, X < \infty$, we have $\text{AB-dim}_R\, M < \infty$ by Lemma 2.3. □

The next lemma gives a more strong statement than Theorem 1.2.(2).(i).

Lemma 2.5. *Let M and N be nonzero R-modules. Assume* $\text{AB-dim}_R\, M < \infty$. *If* $\mathrm{P}_R(M, N) < \infty$, *then* $\mathrm{P}_R(M, N) = \operatorname{depth} R - \operatorname{depth} M$.

Proof. Put $t = \operatorname{depth} R - \operatorname{depth} M$. Since $\text{G-dim}_R\, M = t$, we have $\mathrm{P}_R(M, R) = t$. If $t < \mathrm{P}_R(M, N)$, then $\mathrm{P}_R(M, \Omega^n N) = \mathrm{P}_R(M, N) + n < \infty$ for every $n \geq 0$. This contradicts to $\mathrm{P}_R(M) < \infty$. Thus $\mathrm{P}_R(M, N) \leq t$.

If $t = 0$, then $0 \leq \mathrm{P}_R(M, N) \leq t = 0$. Assume $t > 0$. Let $0 \to M \to Y \to X \to 0$ be a finite projective hull of M. Using depth lemma, we see $\operatorname{depth} Y = \operatorname{depth} M$. Since $\operatorname{pd}_R Y < \infty$, we have $\mathrm{P}_R(Y, N) = \operatorname{pd}_R Y = t < \infty$. It follows from Lemma 2.3, $\text{AB-dim}_R\, X < \infty$. Since $\operatorname{depth} X = \operatorname{depth} R$, we have $\mathrm{P}_R(X, N) = 0$. Applying $\operatorname{Hom}_R(-, N)$ to $0 \to M \to Y \to X \to 0$, we get $\mathrm{P}_R(M, N) = t$. □

Now we can prove Theorem 1.2.(2).

Proof of Theorem 1.2.(2). (i) is clear by Lemma 2.5. To prove (ii), we assume $\text{CI-dim}_R\, M < \infty$. Then we have $\text{G-dim}_R\, M < \infty$ by [8, Theorem (1.4)]. For any nonzero R-module N with $\mathrm{P}_R(M, N) < \infty$, we have $\mathrm{P}_R(M, N) = \operatorname{depth} R - \operatorname{depth} M$ by [1, Theorem 4.2]. This yields $\mathrm{P}_R(M) < \infty$ and we have $\text{AB-dim}_R\, M < \infty$. □

We give an example which does not satisfy the equality of Theorem 1.2.(2).(ii). This example is given by Jorgensen and Şega [14].

Example 2.6. Let k be a field and let $\alpha \in k$ be a nonzero element.

Put $R = k[x_1, x_2, x_3, x_4, x_5]/I$, where I is a ideal of R generated by

$$\alpha x_1 x_3 + x_2 x_3,\ x_1 x_4 + x_2 x_4,\ x_3^2 + \alpha x_1 x_5 - x_2 x_5,$$
$$x_4^2 + x_1 x_5 - x_2 x_5,\ x_1^2,\ x_2^2,\ x_3 x_4,\ x_3 x_5,\ x_4 x_5,\ x_5^2.$$

Consider the sequence

$$\mathbf{C} : \cdots \xrightarrow{d_{i+1}} R^2 \xrightarrow{d_i} R^2 \xrightarrow{d_{i-1}} R^2 \xrightarrow{d_{i-2}} \cdots,$$

where d_i be a matrix $\begin{pmatrix} x_1 & \alpha^i x_3 \\ x_4 & x_2 \end{pmatrix}$ over R. We put $M = \operatorname{Coker} d_1$.

Jorgensen and Şega [14, Lemma 2.2] prove that **C** is a complete resolution of M and therefore M is totally reflexive. Furthermore, if α is not a root of unity, then they [14, Proposition 3.1.(b)] give an R-module T_q for every positive integer q such that $\mathrm{Ext}^i_R(M, T_q) \neq 0$ if and only if $i = 0, q-1, q$. This yields that M does not satisfy (AC). In particular, we have $\text{AB-dim}_R\, M = \infty > 0 = \text{G-dim}_R\, M$.

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CLASSIFYING RESOLVING SUBCATEGORIES OVER A COHEN-MACAULAY LOCAL RING

RYO TAKAHASHI

INTRODUCTION

The classification problem of subcategories having a specific property has been studied actively in a lot of areas of mathematics. The first classification theorem is a result of Gabriel [3] in the early 1960s. He classified the localizing subcategories of the category of modules over a commutative Noetherian ring in terms of specialization-closed subsets of the prime ideal spectrum of the ring. After more than two decades passed, Hopkins [6] and Neeman [8] classified the thick subcategories of the derived category of perfect complexes over a commutative Noetherian ring in terms of specialization-closed subsets of the prime ideal spectrum. Later on, Benson, Carlson and Rickard [2] classified the thick subcategories of the stable category of finitely generated representations of a finite p-group in terms of closed homogeneous subvarieties of the maximal ideal spectrum of the group cohomology ring.

In recent years, the studies have been getting more active. Garkusha and Prest [4, 5] classified the Serre subcategories of the category of finitely presented modules over a commutative coherent ring and the torsion classes of finite type in the category of modules over a commutative ring, in terms of unions of closed subsets of the prime ideal spectrum whose complements are quasi-compact. Takahashi [9, 11] classified the full subcategories closed under direct summands and extensions of the category of finitely generated modules over a commutative Noetherian ring in terms of subsets of the prime ideal spectrum, and the thick subcategories of the stable category of Cohen-Macaulay modules over an abstract hypersurface local ring in terms of specialization-closed subsets of the prime ideal spectrum. Krause [7] classified the thick subcategories closed under direct sums of the category of modules over a commutative Noetherian ring in terms of coherent subsets of the prime ideal spectrum.

Note that all the classification theorems stated above are given by using sets of prime ideals. Many other analogous classification theorems of subcategories of a category and related results have been obtained so far.

On the other hand, in the late 1960s, Auslander and Bridger [1] introduced the notion of a resolving subcategory of an abelian category with enough projectives. A resolving subcategory is by definition a full subcategory containing the projective objects which is closed under direct summands, extensions and kernels of epimorphisms. The projective objects form a resolving subcategory. The maximal Cohen-Macaulay modules over a Cohen-Macaulay local ring form a resolving subcategory of the category of finitely generated modules. Auslander and Bridger proved that over a left and right noetherian ring the modules of Gorenstein dimension zero, which are also called totally reflexive modules, form a resolving subcategory of the category of finitely generated modules. There are a lot of other examples of a resolving subcategory.

In the present article, we are interested in classifying, by using sets of prime ideals, resolving subcategories of the category of finitely generated modules over a Cohen-Macaulay local ring.

Convention

1. Throughout the rest of this article, we assume that all rings are commutative Noetherian rings, and all modules are finitely generated. Let R be a commutative Noetherian local ring of (Krull) dimension d. We denote by $\mathfrak{m}$ the maximal ideal of R, by k the residue field of R and by $\mathsf{mod}\, R$ the category of finitely generated R-modules.

2. Let $\mathcal{C}$ be a category. In this article, by a *subcategory* of $\mathcal{C}$, we always mean a strict full subcategory of $\mathcal{C}$. (Recall that a subcategory $\mathcal{X}$ of $\mathcal{C}$ is said to be *strict* if every object of $\mathcal{C}$ that is isomorphic in $\mathcal{C}$ to some object of $\mathcal{X}$ belongs to $\mathcal{X}$.) By the *subcategory* of $\mathcal{C}$ consisting of objects $\{M_\lambda\}_{\lambda\in\Lambda}$, we always mean the smallest strict full subcategory of $\mathcal{C}$ to which M_λ belongs for all $\lambda \in \Lambda$. Note that this coincides with the full subcategory of $\mathcal{C}$ consisting of all objects $X \in \mathcal{C}$ such that $X \cong M_\lambda$ for some $\lambda \in \Lambda$.

3. When R is a Cohen-Macaulay local ring, we say that an R-module M is *Cohen-Macaulay* if $\operatorname{depth}_R M = d$. Such a module is usually called a *maximal* Cohen-Macaulay R-module, but in this article, we call it just Cohen-Macaulay. We denote by $\mathsf{CM}(R)$ the subcategory of $\mathsf{mod}\, R$ consisting of all Cohen-Macaulay R-modules. Note that a subcategory of $\mathsf{CM}(R)$ can naturally be regarded as a subcategory of $\mathsf{mod}\, R$.

1. Preliminaries

In this section, we recall and make several definitions of the notions which we will deal with in this article, and state their basic properties for later use.

We make a list of several closed properties of a subcategory.

Definition 1.1. Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\, R$. We say that:

(1) $\mathcal{X}$ is *closed under (finite) direct sums* provided that if $X_1, \dots, X_n$ are a finite number of R-modules in $\mathcal{X}$, then the direct sum $X_1 \oplus \cdots \oplus X_n$ belongs to $\mathcal{X}$.

(2) $\mathcal{X}$ is *closed under direct summands* provided that if X is an R-module in $\mathcal{X}$ and M is a direct summand of X, then M belongs to $\mathcal{X}$.

(3) $\mathcal{X}$ is *closed under extensions* provided that for each exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N belong to $\mathcal{X}$, then so does M.

(4) $\mathcal{X}$ is *closed under kernels of epimorphisms* provided that for each exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M and N belong to $\mathcal{X}$, then so does L.

(5) $\mathcal{X}$ is *closed under syzygies* provided that if M is an R-module in $\mathcal{X}$, then $\Omega^i M$ belongs to $\mathcal{X}$ for all $i \geq 0$.

Now we recall the definition of a resolving subcategory, which is a main object we study in this article.

Definition 1.2. A subcategory of $\mathsf{mod}\, R$ is called *resolving* if it contains R, and if it is closed under direct summands, extensions and kernels of epimorphisms.

The nonfree loci of a module and a subcategory are defined as follows.

Definition 1.3. (1) For an R-module M, we denote by $\mathbb{V}_R(M)$ the *nonfree locus* of M, namely, the set of prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module.

(2) For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, we denote by $\mathbb{V}_R(\mathcal{X})$ the *nonfree locus* of $\mathcal{X}$, that is, the union of all $\mathbb{V}_R(X)$ where X runs over all modules in $\mathcal{X}$.

Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\, R$. We denote by $\mathsf{add}_R\, \mathcal{X}$ the *additive closure* of $\mathcal{X}$, that is, the subcategory of $\mathsf{mod}\, R$ consisting of all direct summands of finite direct sums of modules in $\mathcal{X}$. By definition, when $\mathcal{X}$ is closed under direct sums, each R-module in $\mathsf{add}_R\, \mathcal{X}$ is isomorphic to a direct summand of some single R-module in $\mathcal{X}$.

For a prime ideal $\mathfrak{p}$ of R, we denote by $\mathcal{X}_\mathfrak{p}$ the subcategory of $\mathsf{mod}\, R_\mathfrak{p}$ consisting of all $R_\mathfrak{p}$-modules of the form $X_\mathfrak{p}$ with $X \in \mathcal{X}$.

The following proposition will play an important role in the proofs of the main results of this article.

Proposition 1.4. *Let $\mathcal{X}$ be a resolving subcategory of* $\mathsf{mod}\, R$*, and let M be an R-module. Let Γ be a nonempty finite subset of* $\mathsf{Spec}\, R$*. Assume that $M_\mathfrak{p}$ belongs to $\mathsf{add}_{R_\mathfrak{p}}\, \mathcal{X}_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ in Γ. Then*

(1) *There exists an exact sequence*

$$0 \to L \to N \to X \to 0 \tag{1.4.1}$$

of R-modules satisfying the following four conditions:
- (i) *X belongs to $\mathcal{X}$,*
- (ii) *M is a direct summand of N,*
- (iii) *$\mathbb{V}_R(L)$ is contained in $\mathbb{V}_R(M)$,*
- (iv) *$\mathbb{V}_R(L)$ does not intersect Γ.*

(2) *Suppose that R is a Cohen-Macaulay local ring, that $\mathcal{X}$ is contained in* $\mathsf{CM}(R)$ *and that M is a Cohen-Macaulay R-module. Then the modules L and N in* (1.4.1) *can be chosen as Cohen-Macaulay R-modules.*

Let Φ be a subset of $\mathsf{Spec}\, R$. Recall that the *dimension* $\dim \Phi$ of Φ is defined as the supremum of $\dim R/\mathfrak{p}$ where $\mathfrak{p}$ runs through all prime ideals in Φ. By definition, one has $\dim \Phi = -\infty$ if and only if Φ is empty. We denote by $\min \Phi$ the set of minimal elements of Φ with respect to inclusion relation. Note that $\dim \Phi$ is equal to the supremum of $\dim R/\mathfrak{p}$ where $\mathfrak{p}$ runs through all prime ideals in $\min \Phi$.

For a subset Φ of $\mathsf{Spec}\, R$, the subcategory of $\mathsf{mod}\, R$ consisting of all R-modules whose nonfree loci are contained in Φ is denoted by $\mathbb{V}^{-1}(\Phi)$. When R is Cohen-Macaulay, $\mathbb{V}^{-1}_{\mathsf{CM}}(\Phi)$ denotes the restriction of $\mathbb{V}^{-1}(\Phi)$ to $\mathsf{CM}(R)$, that is, the subcategory of $\mathsf{CM}(R)$ consisting of all Cohen-Macaulay R-modules whose nonfree loci are contained in Φ.

We denote by $\mathsf{S}(R)$ the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is not a field. The *singular locus* $\mathsf{Sing}\, R$ is defined as the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is singular, i.e., nonregular. Recall that a subset Φ of $\mathsf{Spec}\, R$ is called *specialization-closed* if every prime ideal of R containing some prime ideal in Φ belongs to Φ. The following two lemmas will be used as basic tools in the proofs of our results. The second one is a Cohen-Macaulay module version of the first one; Lemma 1.6(i) corresponds to Lemma 1.5(i) for each $1 \le i \le 10$.

Lemma 1.5. *Let R be a Cohen-Macaulay local ring. Let M be an R-module. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$. *Let Φ be a specialization-closed subset of* $\operatorname{Spec} R$ *contained in* $\mathsf{S}(R)$.

(1) *If* $\dim \mathbb{V}(M) = -\infty$, *then* $M \in \mathcal{X}$.
(2) *If* $\dim \mathbb{V}(M) = 0$ *and* $k \in \mathcal{X}$, *then* $M \in \mathcal{X}$.
(3) *If* $\mathfrak{p} \in \min \mathbb{V}_R(M)$, *then* $\mathbb{V}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$, *and* $\dim \mathbb{V}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$.
(4) $\operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ *is a resolving subcategory of* $\operatorname{mod} R_\mathfrak{p}$.
(5) *If* $M_\mathfrak{p} \in \operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ *for all* $\mathfrak{p} \in \min \mathbb{V}(M)$, *then there is an exact sequence* $0 \to L \to N \to X \to 0$ *of R-modules with* $X \in \mathcal{X}$, $\mathbb{V}(L) \subseteq \mathbb{V}(M)$ *and* $\dim \mathbb{V}(L) < \dim \mathbb{V}(M)$ *such that M is a direct summand of N.*
(6) $\mathbb{V}(\mathcal{X})$ *is a specialization-closed subset of* $\operatorname{Spec} R$ *contained in* $\mathsf{S}(R)$.
(7) $R/\mathfrak{p}$ *belongs to* $\mathbb{V}^{-1}(\Phi)$ *for all* $\mathfrak{p} \in \Phi$.
(8) $\mathbb{V}^{-1}(\Phi)$ *is a resolving subcategory of* $\operatorname{mod} R$.
(9) $\mathcal{X}$ *is contained in* $\mathbb{V}^{-1}(\mathbb{V}(\mathcal{X}))$.
(10) *One has* $\Phi = \mathbb{V}(\mathbb{V}^{-1}(\Phi))$.

Lemma 1.6. *Let R be a Cohen-Macaulay local ring. Let M be a Cohen-Macaulay R-module. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *contained in* $\mathsf{CM}(R)$. *Let Φ be a specialization-closed subset of* $\operatorname{Spec} R$ *contained in* $\operatorname{Sing} R$.

(1) *If* $\dim \mathbb{V}(M) = -\infty$, *then* $M \in \mathcal{X}$.
(2) *If* $\dim \mathbb{V}(M) = 0$ *and* $\Omega^d k \in \mathcal{X}$, *then* $M \in \mathcal{X}$.
(3) *If* $\mathfrak{p} \in \min \mathbb{V}_R(M)$, *then* $\mathbb{V}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$, *and* $\dim \mathbb{V}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$.
(4) $\operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ *is a resolving subcategory of* $\operatorname{mod} R_\mathfrak{p}$ *contained in* $\mathsf{CM}(R_\mathfrak{p})$.
(5) *If* $M_\mathfrak{p} \in \operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ *for all* $\mathfrak{p} \in \min \mathbb{V}(M)$, *then there is an exact sequence* $0 \to L \to N \to X \to 0$ *of Cohen-Macaulay R-modules with* $X \in \mathcal{X}$, $\mathbb{V}(L) \subseteq \mathbb{V}(M)$ *and* $\dim \mathbb{V}(L) < \dim \mathbb{V}(M)$ *such that M is a direct summand of N.*
(6) $\mathbb{V}(\mathcal{X})$ *is a specialization-closed subset of* $\operatorname{Spec} R$ *contained in* $\operatorname{Sing} R$.
(7) $\Omega^d(R/\mathfrak{p})$ *belongs to* $\mathbb{V}_{\mathsf{CM}}^{-1}(\Phi)$ *for all* $\mathfrak{p} \in \Phi$.
(8) $\mathbb{V}_{\mathsf{CM}}^{-1}(\Phi)$ *is a resolving subcategory of* $\operatorname{mod} R$ *contained in* $\mathsf{CM}(R)$.
(9) $\mathcal{X}$ *is contained in* $\mathbb{V}_{\mathsf{CM}}^{-1}(\mathbb{V}(\mathcal{X}))$.
(10) *One has* $\Phi = \mathbb{V}(\mathbb{V}_{\mathsf{CM}}^{-1}(\Phi))$.

2. Main results

We start by the following two propositions. The latter one, which is a consequence of the former one, plays a key role in the proof of our first main theorem.

Proposition 2.1. *Let R be a Cohen-Macaulay local ring with minimal multiplicity. Suppose that the residue field k is infinite. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *contained in* $\mathsf{CM}(R)$ *containing a nonfree R-module. Then $\Omega^d k$ belongs to $\mathcal{X}$.*

Proposition 2.2. *Let R be a Cohen-Macaulay local ring locally with minimal multiplicity on the punctured spectrum. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *contained in* $\mathsf{CM}(R)$ *containing $\Omega^d k$. If M is a Cohen-Macaulay R-module such that $\mathbb{V}(M)$ is contained in $\mathbb{V}(\mathcal{X})$, then M belongs to $\mathcal{X}$.*

The theorem below is one of our main results, which yields a classification of resolving subcategories of Cohen-Macaulay modules.

Theorem 2.3. *Let R be a Cohen-Macaulay singular local ring which locally has minimal multiplicity on the punctured spectrum. Then one has the following one-to-one correspondence:*

$$\left\{\begin{array}{c}\textit{Resolving subcategories of } \mathsf{mod}\, R\\ \textit{contained in } \mathsf{CM}(R) \textit{ containing } \Omega^d k\end{array}\right\}$$

$$\mathbb{V}\big\downarrow \quad \big\uparrow \mathbb{V}_{\mathsf{CM}}^{-1}$$

$$\left\{\begin{array}{c}\textit{Nonempty specialization-closed subsets of } \mathsf{Spec}\, R\\ \textit{contained in } \mathsf{Sing}\, R\end{array}\right\}.$$

Proof. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\, R$ containing $\Omega^d k$. If $\mathbb{V}(\mathcal{X})$ is empty, then $\Omega^d k \in \mathcal{X}$ is free, and hence the R-module k has finite projective dimension, which contradicts the assumption that R is singular. Therefore $\mathbb{V}(\mathcal{X})$ is nonempty. If Φ is a nonempty specialization-closed subset of $\mathsf{Spec}\, R$, then it must contain the maximal ideal $\mathfrak{m}$, and $\Omega^d k$ belongs to $\mathbb{V}_{\mathsf{CM}}^{-1}(\Phi)$ because $\mathbb{V}(\Omega^d k)$ consists of $\mathfrak{m}$. Applying Lemma 1.6(6)(8), we see that the maps $\mathbb{V}$ and $\mathbb{V}^{-1}$ are well-defined. It follows Lemma 1.6(9)(10) and Proposition 2.2 that the two maps form mutually inverse bijections. ■

Next, we define the left and right perpendicular subcategories of a given subcategory, and introduce the notions of a right approximation and a contravariantly finite subcategory.

Definition 2.4. (1) Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\, R$. We denote by ${}^{\perp}\mathcal{X}$ (respectively, $\mathcal{X}^{\perp}$) the subcategory of $\mathsf{mod}\, R$ consisting of all R-modules M satisfying $\mathrm{Ext}_R^i(M, X) = 0$ (respectively, $\mathrm{Ext}_R^i(X, M) = 0$) for all $X \in \mathcal{X}$ and all $i > 0$.

(2) Let R be a Cohen-Macaulay local ring. Let $\mathcal{X}$ be a subcategory of $\mathsf{CM}(R)$. We denote by ${}^{\perp}_{\mathsf{CM}}\mathcal{X}$ (respectively, $\mathcal{X}^{\perp}_{\mathsf{CM}}$) the subcategory of $\mathsf{CM}(R)$ consisting of all Cohen-Macaulay R-modules M satisfying $\mathrm{Ext}_R^i(M, X) = 0$ (respectively, $\mathrm{Ext}_R^i(X, M) = 0$) for all $X \in \mathcal{X}$ and all $i > 0$.

Definition 2.5. Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\, R$.

(1) Let $\phi : X \to M$ be a homomorphism of R-modules with $X \in \mathcal{X}$. We say that ϕ is a *right $\mathcal{X}$-approximation* (of M) if the homomorphism $\mathrm{Hom}_R(X', \phi) : \mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M)$ is surjective for all $X' \in \mathcal{X}$, in other words, every homomorphism $X' \to M$ with $X' \in \mathcal{X}$ factors through ϕ.

(2) We say that $\mathcal{X}$ is *contravariantly finite* if all modules $M \in \mathsf{mod}\, R$ admit right $\mathcal{X}$-approximations.

We make here several statements for later use.

Lemma 2.6. (1) *For each subcategory $\mathcal{X}$ of* $\mathsf{mod}\, R$*, one has that ${}^{\perp}\mathcal{X}$ is a resolving subcategory of* $\mathsf{mod}\, R$.

(2) *Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of* $\mathsf{mod}\, R$*, and assume that $\mathcal{Y}$ is contained in $\mathcal{X}$. Let $Y \xrightarrow{g} X \xrightarrow{f} M$ be homomorphisms of R-modules. If f is a right $\mathcal{X}$-approximation and g is a right $\mathcal{Y}$-approximation, then fg is a right $\mathcal{Y}$-approximation.*

(3) *Let R be a Henselian local ring, and let $\mathcal{X}$ be a resolving subcategory of* $\mathsf{mod}\, R$*. Suppose that M admits a right $\mathcal{X}$-approximation. Then M admits a surjective right*

$\mathcal{X}$-approximation whose kernel belongs to $\mathcal{X}^{\perp}$. Hence there exists an exact sequence

$$0 \to Y \to X \to M \to 0$$

of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$.

(4) *Let R be a Cohen-Macaulay local ring with a canonical module. Then $\mathsf{CM}(R)$ is a contravariantly finite resolving subcategory of* $\mathsf{mod}\, R$.

Let R be a Cohen-Macaulay local ring with a canonical module ω. Let M be a Cohen-Macaulay R-module and $\mathcal{X}$ a subcategory of $\mathsf{CM}(R)$. Then we set $M^{\dagger} = \mathrm{Hom}_R(M, \omega)$, and denote by $\mathcal{X}^{\dagger}$ the subcategory of $\mathsf{mod}\, R$ consisting of all modules of the form $X^{\dagger}$ with $X \in \mathcal{X}$. Note that $M^{\dagger}$ is also a Cohen-Macaulay R-module, and hence $\mathcal{X}^{\dagger}$ is contained in $\mathsf{CM}(R)$. Note also that $M^{\dagger}$ is uniquely determined up to isomorphism since so is a canonical module, and hence $\mathcal{X}^{\dagger}$ is uniquely determined.

Lemma 2.7. *Let R be a Cohen-Macaulay local ring with a canonical module ω. Let $\mathcal{X}$ be a subcategory of $\mathsf{CM}(R)$. Let M be a module in $\mathcal{X}^{\perp}_{\mathsf{CM}}$. Then the following hold.*

(1) *The module $M^{\dagger}$ belongs to ${}^{\perp}_{\mathsf{CM}}(\mathcal{X}^{\dagger})$.*

(2) *If $\mathcal{X}$ contains ω, then $\mathrm{Ext}^i_R(M^{\dagger}, R) = 0$ for every $i > 0$.*

Now we can prove a result on determination of contravariantly finite resolving subcategories over a Cohen-Macaulay Henselian local ring.

Theorem 2.8. *Let R be a Cohen-Macaulay Henselian local ring with a canonical module ω. Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{CM}(R)$ containing ω. Then $\mathcal{X}$ coincides with either $\mathsf{add}\, R$ or $\mathsf{CM}(R)$.*

Proof. (1) First, we prove that ${}^{\perp}_{\mathsf{CM}}(\mathcal{X}^{\dagger})$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\, R$. The resolving property of ${}^{\perp}_{\mathsf{CM}}(\mathcal{X}^{\dagger})$ is easily verified by using Lemma 2.6(1). By Lemma 2.6(2)(4), it is enough to show that each Cohen-Macaulay R-module M admits a right ${}^{\perp}_{\mathsf{CM}}(\mathcal{X}^{\dagger})$-approximation. Since $\mathcal{X}$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\, R$, we see from Lemma 2.6(3) that there exists an exact sequence

$$0 \to Y \to X \to (\Omega M)^{\dagger} \to 0$$

of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. As X and $(\Omega M)^{\dagger}$ are Cohen-Macaulay, so is Y, and hence Y is in $\mathcal{X}^{\perp}_{\mathsf{CM}}$. Lemma 2.7(1) implies that $Y^{\dagger}$ belongs to ${}^{\perp}_{\mathsf{CM}}(\mathcal{X}^{\dagger})$. Dualizing the above exact sequence by ω, we get an exact sequence $0 \to \Omega M \to X^{\dagger} \to Y^{\dagger} \to 0$. Also, there is an exact sequence $0 \to \Omega M \to R^{\oplus n} \to M \to 0$. From these two exact sequences, we make the following pushout diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & X^\dagger & \longrightarrow & Y^\dagger & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & R^{\oplus n} & \longrightarrow & Z & \longrightarrow & Y^\dagger & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Lemma 2.7(2) gives $\mathrm{Ext}_R^i(Y^\dagger, R) = 0$ for all $i > 0$. Hence the middle row in the above diagram splits, and Z is isomorphic to $Y^\dagger \oplus R^{\oplus n} \in {}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$. Thus we obtain an exact sequence

$$0 \to X^\dagger \to Y^\dagger \oplus R^{\oplus n} \xrightarrow{\phi} M \to 0.$$

Let W be any R-module in ${}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$. Then there is an exact sequence

$$\mathrm{Hom}_R(W, Y^\dagger \oplus R^{\oplus n}) \xrightarrow{\mathrm{Hom}_R(W,\phi)} \mathrm{Hom}_R(W, M) \to \mathrm{Ext}_R^1(W, X^\dagger).$$

Since $X^\dagger \in \mathcal{X}^\dagger$ and $W \in {}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$, we have $\mathrm{Ext}_R^1(W, X^\dagger) = 0$, and the map $\mathrm{Hom}_R(W, \phi)$ is surjective. Therefore the homomorphism ϕ is a right ${}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$-approximation. Consequently, ${}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$ is a contravariantly finite resolving subcategory of $\mathsf{mod}\, R$.

(2) Now, let us prove that either $\mathcal{X} = \mathsf{add}\, R$ or $\mathcal{X} = \mathsf{CM}(R)$ holds true. Assume that $\mathcal{X}$ is different from $\mathsf{add}\, R$. We want to show that $\mathcal{X}$ coincides with $\mathsf{CM}(R)$. Let M be a Cohen-Macaulay R-module. Then by Lemma 2.6(3) there exists an exact sequence

$$0 \to Y \to X \to M \to 0 \qquad (2.8.1)$$

of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$. As X and M are Cohen-Macaulay, so is Y, and we have $Y \in \mathcal{X}_{\mathsf{CM}}^\perp$. Lemma 2.7 implies that $Y^\dagger$ belongs to ${}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$ and that $\mathrm{Ext}_R^i(Y^\dagger, R) = 0$ for every $i > 0$. Thus, by virtue of [12, Theorem 1.4], one of the following two statements holds.

(i) As a subcategory of $\mathsf{mod}\, R$, ${}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$ coincides with either $\mathsf{mod}\, R$ or $\mathsf{CM}(R)$.

(ii) The R-module $Y^\dagger$ has finite projective dimension.

Suppose that the statement (i) holds. Then $\Omega^d k$ belongs to ${}_{\mathsf{CM}}^{\perp}(\mathcal{X}^\dagger)$. Let Z be a module in $\mathcal{X}$. Then we have $0 = \mathrm{Ext}_R^i(\Omega^d k, Z^\dagger) \cong \mathrm{Ext}_R^{i+d}(k, Z^\dagger)$ for all $i > 0$, which implies that the Cohen-Macaulay R-module $Z^\dagger$ has finite injective dimension. Hence $Z^\dagger$ is isomorphic to a direct sum of copies of ω, and therefore Z is free. Thus we have $\mathcal{X} = \mathsf{add}\, R$, which contradicts our assumption.

Consequently, the statement (i) cannot hold true, and the statement (ii) must hold. Since $Y^\dagger$ is a Cohen-Macaulay R-module, it is free. Hence Y is isomorphic to a direct sum of copies of ω. Then the exact sequence (2.8.1) splits, and M is isomorphic to a direct summand of X. As $\mathcal{X}$ is closed under direct summands, M belongs to $\mathcal{X}$. Now we conclude that $\mathcal{X}$ coincides with $\mathsf{CM}(R)$. ∎

Next we want to exclude the Henselian assumption on R from Theorem 2.8. For this, we need to lift the contravariant finite and resolving properties of a subcategory of $\mathsf{mod}\, R$ to the completion of R. We consider this in the following two results.

Let $R \to S$ be a flat (not necessarily local) homomorphism of local rings. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, we denote by $\mathcal{X} \otimes_R S$ the subcategory of $\mathsf{mod}\, S$ consisting of all S-modules of the form $X \otimes_R S$ with $X \in \mathcal{X}$. Note that if $\mathcal{X}$ is closed under direct sums, then so is $\mathcal{X} \otimes_R S$.

Lemma 2.9. (1) *Let $\mathcal{X}$ be a subcategory of* $\mathsf{mod}\, R$. *Let $\phi : X \to M$ be a right $\mathcal{X}$-approximation, and let $\pi : M \to N$ be a split epimorphism of R-modules. Then $\pi\phi : X \to N$ is a right $\mathcal{X}$-approximation.*

(2) *Let $R \to S$ be a flat (not necessarily local) homomorphism of local rings. Let $\mathcal{X}$ be a subcategory of* $\mathsf{mod}\, R$ *closed under direct sums. If $\phi : X \to M$ is a right $\mathcal{X}$-approximation, then $\phi \otimes_R S : X \otimes_R S \to M \otimes_R S$ is a right* $\mathsf{add}_S(\mathcal{X} \otimes_R S)$*-approximation.*

Let $\widehat{R}$ be the completion of R in the $\mathfrak{m}$-adic topology. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, we denote by $\widehat{\mathcal{X}}$ the subcategory of $\mathsf{mod}\, \widehat{R}$ consisting of all $\widehat{R}$-modules of the form $\widehat{X}$ with $X \in \mathcal{X}$. This can be identified as $\mathcal{X} \otimes_R \widehat{R}$.

When R is a Cohen-Macaulay local ring, we denote by $\mathsf{CM}_0(R)$ the subcategory of $\mathsf{CM}(R)$ consisting of all Cohen-Macaulay R-modules that are locally free on the punctured spectrum of R.

Proposition 2.10. (1) *Let $\mathcal{X}$ be a resolving subcategory of* $\mathsf{mod}\, R$. *Assume that all R-modules in $\mathcal{X}$ are locally free on the punctured spectrum of R. Then* $\mathsf{add}_{\widehat{R}}\, \widehat{\mathcal{X}}$ *is a resolving subcategory of* $\mathsf{mod}\, \widehat{R}$.

(2) *Let R be a Cohen-Macaulay local ring, and let $\mathcal{X}$ be a resolving subcategory of* $\mathsf{mod}\, R$ *contained in* $\mathsf{CM}_0(R)$. *Then the following hold.*

  (i) *The subcategory* $\mathsf{add}_{\widehat{R}}\, \widehat{\mathcal{X}}$ *of* $\mathsf{mod}\, \widehat{R}$ *is contained in* $\mathsf{CM}_0(\widehat{R})$.

  (ii) *If every module in* $\mathsf{CM}_0(R)$ *admits a right $\mathcal{X}$-approximation, then every module in* $\mathsf{CM}_0(\widehat{R})$ *admits a right* $\mathsf{add}_{\widehat{R}}\, \widehat{\mathcal{X}}$*-approximation*

Now we can prove the following theorem. We can actually exclude from Theorem 2.8 the assumption that R is Henselian, if instead we assume that the completion of R has an isolated singularity.

Theorem 2.11. *Let R be a Cohen-Macaulay local ring with a canonical module ω. Suppose that the completion of R has an isolated singularity. Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\mathsf{mod}\, R$ *contained in* $\mathsf{CM}(R)$ *containing ω. Then $\mathcal{X}$ is either* $\mathsf{add}\, R$ *or* $\mathsf{CM}(R)$.

Proof. Since $\widehat{R}$ has an isolated singularity, so does R (cf. [11, Proposition 3.4]). Hence we have $\mathsf{CM}(\widehat{R}) = \mathsf{CM}_0(\widehat{R})$ and $\mathsf{CM}(R) = \mathsf{CM}_0(R)$. Proposition 2.10(2) and Lemma 2.6(2)(4) show that $\mathsf{add}_{\widehat{R}}\, \widehat{\mathcal{X}}$ is a contravariantly finite subcategory of $\mathsf{mod}\, \widehat{R}$. Proposition 2.10(1) says that $\mathsf{add}_{\widehat{R}}\, \widehat{\mathcal{X}}$ is also resolving. Since $\widehat{R}$ is Henselian, it follows from Theorem 2.8 that $\mathsf{add}_{\widehat{R}}\, \widehat{\mathcal{X}}$ coincides with either $\mathsf{add}_{\widehat{R}}\, \widehat{R}$ or $\mathsf{CM}(\widehat{R})$. In the former case we have $\mathcal{X} = \mathsf{add}\, R$, so let us consider the latter case, i.e., the case where $\mathsf{add}_{\widehat{R}}\widehat{\mathcal{X}} = \mathsf{CM}(\widehat{R})$ holds. We want to prove that $\mathcal{X} = \mathsf{CM}(R)$. According to [11, Corollary 2.7], it is enough to show that the R-module $\Omega^d k$ belongs to $\mathcal{X}$. By the contravariant finiteness of $\mathcal{X}$, there is a right $\mathcal{X}$-approximation $\phi : X \to \Omega^d k$. There is a surjective homomorphism from a free R-module to $\Omega^d k$, and it factors through ϕ. We see from this that ϕ is surjective, and have an exact sequence

$$\sigma : 0 \to L \to X \xrightarrow{\phi} \Omega^d k \to 0$$

of R-modules. Taking the completion, we get an exact sequence

$$\widehat{\sigma} : 0 \to \widehat{L} \to \widehat{X} \xrightarrow{\widehat{\phi}} \widehat{\Omega^d k} \to 0$$

of $\widehat{R}$-modules. Lemma 2.9(2) says that $\widehat{\phi}$ is a right $\mathsf{add}_{\widehat{R}}\widehat{\mathcal{X}}$-approximation. Note that $\widehat{\Omega^d k}$ is in $\mathsf{CM}(\widehat{R}) = \mathsf{add}_{\widehat{R}}\widehat{\mathcal{X}}$. Hence the identity map $\mathrm{id}_{\widehat{\Omega^d k}} : \widehat{\Omega^d k} \to \widehat{\Omega^d k}$ factors through $\widehat{\phi}$, which implies that $\widehat{\phi}$ is a split epimorphism. Therefore we have $\widehat{\sigma} = 0$ in $\mathrm{Ext}^1_{\widehat{R}}(\widehat{\Omega^d k}, \widehat{L})$. Since the R-module $\mathrm{Ext}^1_R(\Omega^d k, L)$ has finite length, the natural map $\mathrm{Ext}^1_R(\Omega^d k, L) \to \mathrm{Ext}^1_{\widehat{R}}(\widehat{\Omega^d k}, \widehat{L})$ is an isomorphism. Hence $\sigma = 0$ in $\mathrm{Ext}^1_R(\Omega^d k, L)$. Thus $\Omega^d k$ is isomorphic to a direct summand of $X \in \mathcal{X}$, and $\Omega^d k$ belongs to $\mathcal{X}$, as desired. ■

Here we state a lemma to get a corollary of the above theorem.

Lemma 2.12. *Let X be an R-module. Then $\mathsf{add}\,X$ is a contravariantly finite subcategory of $\mathsf{mod}\,R$.*

Corollary 2.13. *Let R be a Cohen-Macaulay local ring admitting a canonical module ω. Assume that R is of finite Cohen-Macaulay representation type and that the completion $\widehat{R}$ has an isolated singularity. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{CM}(R)$ containing ω. Then one has either $\mathcal{X} = \mathsf{add}\,R$ or $\mathcal{X} = \mathsf{CM}(R)$.*

Taking advantage of the above corollary, we prove the following proposition, which is the essential part of the classification theorem.

Proposition 2.14. *Let R be a Cohen-Macaulay local ring with a canonical module ω. Suppose that for every nonmaximal prime ideal $\mathfrak{p}$ the local ring $R_\mathfrak{p}$ has finite Cohen-Macaulay representation type whose completion has an isolated singularity. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{CM}(R)$ containing ω and $\Omega^d k$. Let M be a Cohen-Macaulay R-module. If $\mathrm{V}(M)$ is contained in $\mathrm{V}(\mathcal{X})$, then M belongs to $\mathcal{X}$.*

Now we obtain a classification theorem of resolving subcategories.

Theorem 2.15. *Let R be a Cohen-Macaulay singular local ring with a canonical module ω. Suppose that for each nonmaximal prime ideal $\mathfrak{p}$ the local ring $R_\mathfrak{p}$ has finite Cohen-Macaulay representation type whose completion has an isolated singularity. Then one has the following one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{Resolving subcategories of } \mathsf{mod}\,R \\ \textit{contained in } \mathsf{CM}(R) \textit{ containing } \omega, \Omega^d k\end{matrix}\right\}$$

$$\mathrm{V}\Big\downarrow \quad \Big\uparrow \mathrm{V}^{-1}_{\mathsf{CM}}$$

$$\left\{\begin{matrix}\textit{Nonempty specialization-closed subsets of } \mathsf{Spec}\,R \\ \textit{contained in } \mathsf{Sing}\,R \textit{ containing } \mathsf{NonGor}(R)\end{matrix}\right\}.$$

Proof. Note that $\mathsf{NonGor}(R) = \mathbb{V}(\omega)$ holds. The assertion is proved by the same argument as the proof of Theorem 2.3 except that Proposition 2.2 is replaced with Proposition 2.14. ∎

Corollary 2.16. *Let R be a Cohen-Macaulay excellent singular local ring with a canonical module ω locally of finite Cohen-Macaulay representation type on the punctured spectrum. Then one has the following one-to-one correspondence:*

$$\left\{\begin{array}{c}\textit{Resolving subcategories of } \mathsf{mod}\, R \\ \textit{contained in } \mathsf{CM}(R) \textit{ containing } \omega, \Omega^d k\end{array}\right\}$$

$$\mathbb{V}\Big\downarrow \quad \Big\uparrow \mathbb{V}_{\mathsf{CM}}^{-1}$$

$$\left\{\begin{array}{c}\textit{Nonempty specialization-closed subsets of } \mathsf{Spec}\, R \\ \textit{contained in } \mathsf{Sing}\, R \textit{ containing } \mathsf{NonGor}(R)\end{array}\right\}$$

Proof. Let $\mathfrak{p}$ be a nonmaximal prime ideal of R. Since R is excellent, so is $R_\mathfrak{p}$. The assumption that $R_\mathfrak{p}$ is of finite Cohen-Macaulay representation type implies that $R_\mathfrak{p}$ has an isolated singularity, hence so does the completion of the local ring $R_\mathfrak{p}$ (cf. [11, Proposition 3.4]). Thus, the assertion follows from Theorem 2.15 ∎

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CANONICAL MODULES OF MULTI-SECTION RINGS

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1. Introduction

We shall describe the divisor class groups and the graded canonical modules of Noetherian multi-section rings associated with a normal projective variety.

Let X be a d-dimensional normal projective variety over a field k. We always assume $d > 0$. Let $D_1, \ldots, D_s$ be Weil divisors on X. We define the multi-section rings $R(X; D_1, \ldots, D_s)$ and $R'(X; D_1, \ldots, D_s)$ as follows:

$$\begin{aligned} & R(X; D_1, \ldots, D_s) \\ = & \bigoplus_{(n_1,\ldots,n_s)\in\mathbb{N}_0{}^s} H^0(X, \mathcal{O}_X(\sum_i n_i D_i)) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1, \ldots, t_s] \\ & R'(X; D_1, \ldots, D_s) \\ = & \bigoplus_{(n_1,\ldots,n_s)\in\mathbb{Z}^s} H^0(X, \mathcal{O}_X(\sum_i n_i D_i)) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}] \end{aligned}$$

Here, $\mathbb{Z}$ is the set of integers, and $\mathbb{N}_0$ is the set of non-negative integers.

We want to describe the divisor class groups and the graded canonical modules of the above rings.

For a Weil divisor F on X, we set

$$M_F = \bigoplus_{(n_1,\ldots,n_s)\in\mathbb{Z}^s} H^0(X, \mathcal{O}_X(\sum_i n_i D_i + F)) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}],$$

that is, M_F is a $\mathbb{Z}^s$-graded $R'(X; D_1, \ldots, D_s)$-module such that

$$[M_F]_{(n_1,\ldots,n_s)} = H^0(X, \mathcal{O}_X(\sum_i n_i D_i + F)) t_1^{n_1} \cdots t_s^{n_s}.$$

We denote by $\overline{M_F}$ the isomorphism class of M_F in $\mathrm{Cl}(R'(X; D_1, \ldots, D_s))$.

For a normal variety X, we denote by $\mathrm{Cl}(X)$ the class group of X, and for a Weil divisor F on X, we denote by $\overline{F}$ the class of F in $\mathrm{Cl}(X)$.

We denote by $C^1(X)$ the set of closed subvarieties of codimension 1.

In the case where $\mathrm{Cl}(X)$ is freely generated by $\overline{D_1}, \ldots, \overline{D_s}$, the ring $R'(X; D_1, \ldots, D_s)$ is usually called the *Cox ring* of X and denoted by $\mathrm{Cox}(X)$.

Remark 1. Assume that D is an ample divisor on X. In this case, $R(X; D)$ coincides with $R'(X; D)$, and it is a Noetherian normal domain by a famous result of Zariski. It is well-known that $\mathrm{Cl}(R(X; D))$ is isomorphic to $\mathrm{Cl}(X)/\mathbb{Z}\overline{D}$. Mori ([6]) constructed a lot of examples of non-Cohen Macaulay factorial domains using this isomorphism.

The canonical module of $R(X; D)$ is isomorphic to M_{K_X}, and ω_X is $M_{K_X}^{\sim}$.

We want to generalize the results in the above remark for rings as $R(X; D_1, \dots, D_s)$ or $R'(X; D_1, \dots, D_s)$.

For rings as $R'(X; D_1, \dots, D_s)$, we had already proved the following:

Theorem 2 (Elizondo-Kurano-Watanabe [1], Hashimoto-Kurano [3]). *Let X be a normal projective variety over a field such that $\dim X > 0$. Assume that $D_1, \dots, D_s$ are Weil divisors on X such that $\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$ contains an ample Cartier divisor. Then, we have the following:*

(1) $R'(X; D_1, \dots, D_s)$ is a Krull domain.

(2) The set $\{P_V \mid V \in C^1(X)\}$ coincides with the set of homogeneous prime ideals of R' of height 1, where $P_V = M_{-V}$.

(3) We have an exact sequence

$$0 \longrightarrow \sum_i \mathbb{Z}\overline{D_i} \longrightarrow \mathrm{Cl}(X) \xrightarrow{p} \mathrm{Cl}(R'(X; D_1, \dots, D_s)) \longrightarrow 0$$

such that $p(\overline{F}) = \overline{M_F}$.

(4) Assume that $R'(X; D_1, \dots, D_s)$ is Noetherian. Then $\omega_{R'(X; D_1, \dots, D_s)}$ is isomorphic to M_{K_X} as a $\mathbb{Z}^s$-graded module. Therefore, $\omega_{R'(X; D_1, \dots, D_s)}$ is $R'(X; D_1, \dots, D_s)$-free if and only if $\overline{K_X} \in \sum_i \mathbb{Z}\overline{D_i}$ in $\mathrm{Cl}(X)$.

Suppose that $\mathrm{Cl}(X)$ is freely generated by $\overline{D_1}, \dots, \overline{D_s}$. By the above theorem, the Cox ring $\mathrm{Cox}(X)$ is factorial and

$$\omega_{\mathrm{Cox}(X)} \simeq M_{K_X} \simeq \mathrm{Cox}(X)(\overline{K_X}),$$

where we regard $\mathrm{Cox}(X)$ as a $\mathrm{Cl}(X)$-graded ring.

We have obtained the following results:

Theorem 3 ([4]). *Let X be a normal projective variety over a field k such that $d = \dim X > 0$. Assume that $D_1, \dots, D_s$ are Weil divisors on X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor. Set*

$$U = \{i \mid \mathrm{tr.deg}_k R(X; D_1, \dots, D_{i-1}, D_{i+1}, \dots, D_s) = d + s - 1\}.$$

Then, we have the following:

(1) $R(X; D_1, \dots, D_s)$ is a Krull domain.

(2) The set $\{Q_V \mid V \in C^1(X)\} \cup \{Q_i \mid i \in U\}$ coincides with the set of homogeneous prime ideals of R of height 1, where $Q_V = P_V \cap B$ and

$$Q_i = \bigoplus_{\substack{n_1, \dots, n_s \in \mathbb{Z} \\ n_i > 0}} R_{(n_1, \dots, n_s)}.$$

(3) We have an exact sequence

$$0 \longrightarrow \sum_{i \notin U} \mathbb{Z}\overline{D_i} \longrightarrow \mathrm{Cl}(X) \xrightarrow{q} \mathrm{Cl}(R(X; D_1, \dots, D_s)) \longrightarrow 0$$

such that $q(\overline{F}) = \overline{M_F \cap k(X)[t_1, \dots, t_s, \{t_i^{-1} \mid i \notin U\}]}$.

(4) Assume that $R(X; D_1, \dots, D_s)$ is Noetherian. Then $\omega_{R(X; D_1, \dots, D_s)}$ is isomorphic to

$$M_{K_X} \cap (t_i \mid i \in U) k(X)[t_1, \dots, t_s, \{t_i^{-1} \mid i \notin U\}]$$

as a $\mathbb{Z}^s$*-graded module. Further, we have*

$$q(\overline{K_X + \sum_i D_i}) = \overline{\omega_{R(X;D_1,\dots,D_s)}}.$$

Therefore, $\omega_{R(X;D_1,\dots,D_s)}$ *is* $R(X; D_1, \dots, D_s)$*-free if and only if*

$$\overline{K_X + \sum_i D_i} \in \sum_{i \notin U} \mathbb{Z}\overline{D_i}$$

in $\mathrm{Cl}(X)$.

2. Examples

Example 4. Assume that both D_1 and D_2 are ample divisors. Then, $R(X; D_1, D_2)$ is Noetherian by a famous result of Zariski. In this case, $U = \{1, 2\}$. By Theorem 3 (3), $\mathrm{Cl}(X)$ is isomorphic to $\mathrm{Cl}(R(X; D_1, D_2))$. By Theorem 3 (4), $\omega_{R(X;D_1,D_2)}$ is $R(X; D_1, D_2)$-free module if and only if $\overline{K_X} = \overline{-D_1 - D_2}$ in $\mathrm{Cl}(X)$. This is the case, $-K_X$ is ample, that is, X is a Fano variety.

Example 5. Set $X = \mathbb{P}^m \times \mathbb{P}^n$. Let p_1 (resp. p_2) be the first (resp. second) projection.
Let H_1 be a hyperplane of $\mathbb{P}^m$, and H_2 a hyperplane of $\mathbb{P}^n$. Put $A_i = p_i^{-1}(H_i)$ for $i = 1, 2$. In this case, $\mathrm{Cl}(X) = \mathbb{Z}\overline{A_1} + \mathbb{Z}\overline{A_2} \simeq \mathbb{Z}^2$, and $K_X = -(m+1)A_1 - (n+1)A_2$.
We have

$$\mathrm{Cox}(X) = R'(X; A_1, A_2) = k[x_0, x_1, \dots, x_m, y_0, y_1, \dots, y_n].$$

$\mathrm{Cox}(X)$ is a $\mathbb{Z}^2$-graded ring such that x_i's (resp. y_j's) are of degree $(1, 0)$ (resp. $(0, 1)$).
Let a, b, c, d be positive integers such that $ad - bc \neq 0$. Put $D_1 = aA_1 + bA_2$ and $D_2 = cA_1 + dA_2$. Then, both D_1 and D_2 are ample divisors. Consider the section rings:

$$\begin{aligned} R'(X; D_1, D_2) &= \oplus_{p,q \in \mathbb{Z}} \mathrm{Cox}(X)_{p(a,b)+q(c,d)} \\ R(X; D_1, D_2) &= \oplus_{p,q \geq 0} \mathrm{Cox}(X)_{p(a,b)+q(c,d)} \end{aligned}$$

Here, both $R'(X; D_1, D_2)$ and $R(X; D_1, D_2)$ are Cohen-Macaulay rings.
By Theorem 2, we know

$$\begin{aligned} R'(X; D_1, D_2) \text{ is a Gorenstein ring} &\iff \overline{K_X} \in \mathbb{Z}\overline{D_1} + \mathbb{Z}\overline{D_2} \text{ in } \mathrm{Cl}(X) \\ &\iff (m+1, n+1) \in \mathbb{Z}(a, b) + \mathbb{Z}(c, d). \end{aligned}$$

In this case, we have $U = \{1, 2\}$. By Theorem 3, we have

$$\begin{aligned} R(X; D_1, D_2) \text{ is a Gorenstein ring} &\iff \overline{K_X + D_1 + D_2} = 0 \text{ in } \mathrm{Cl}(X) \\ &\iff m + 1 = a + c \text{ and } n + 1 = b + d. \end{aligned}$$

Example 6. Let a, b, c be pairwise coprime positive integers. Let $\mathfrak{p}$ be the kernel of the k-algebra map $S = k[x, y, z] \to k[T]$ given by $x \mapsto T^a$, $y \mapsto T^b$, $z \mapsto T^c$.
Let $\pi : X \to \mathbb{P} = \mathrm{Proj}(k[x, y, z])$ be the blow-up at $V_+(\mathfrak{p})$, where $a = \deg(x)$, $b = \deg(y)$, $c = \deg(z)$. Put $E = \pi^{-1}(V_+(\mathfrak{p}))$. Let A be a Weil divisor on X satisfying $\pi^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_X(A)$. In this case, we have $\mathrm{Cl}(X) = \mathbb{Z}\overline{E} + \mathbb{Z}\overline{A} \simeq \mathbb{Z}^2$, and $K_X = E - (a + b + c)A$.
Then, we have

$$\begin{aligned} \mathrm{Cox}(X) = R'(X; -E, A) &= R'_s(\mathfrak{p}) := S[t^{-1}, \mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \dots] \subset S[t^{-1}], \\ R(X; -E, A) &= R_s(\mathfrak{p}) := S[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \dots] \subset S[t]. \end{aligned}$$

By Theorem 2, we have

$$\omega_{R'_s(\mathfrak{p})} = R'_s(\mathfrak{p})(\overline{K_X}) = R'_s(\mathfrak{p})(-1, -a-b-c).$$

In this case, $U = \{1\}$. By Theorem 3,

$$\begin{aligned}\omega_{R_s(\mathfrak{p})} &= M_{K_X} \cap t_1 k(X)[t_1, t_2^{-1}] \\ &= \omega_{R'_s(\mathfrak{p})} \cap t_1 k(X)[t_1, t_2^{\pm 1}] \\ &= R'_s(\mathfrak{p})(-1, -a-b-c) \cap t_1 k(X)[t_1, t_2^{-1}] \\ &= R_s(\mathfrak{p})(-1, -a-b-c).\end{aligned}$$

Therefore, both of $R'_s(\mathfrak{p})$ and $R_s(\mathfrak{p})$ are quasi-Gorenstein rings.

3. Outline of the proof of Theorem 3

In this section, we shall prove Theorem 3.

We use Theorem 2 here without proof.

Put $A = k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ and $B = k(X)[t_1, \ldots, t_s]$. Assume that $D_1, \ldots, D_s$ are Weil divisors on a normal projective variety X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor.

We denote $R(X; D_1, \ldots, D_s)$ and $R'(X; D_1, \ldots, D_s)$ simply by R and R', respectively.
Since

$$R = R' \cap B,$$

R is a Krull domain.

By Theorem 2 (2), we have

$$\begin{aligned} R' &= A \cap \left(\bigcap_{V \in C^1(X)} R'_{P_V} \right) \\ A &= \bigcap_{P \in NHC^1(R')} R'_P, \end{aligned}$$

where $NHC^1(R')$ is the set of non-homogeneous prime ideals of R' of height 1.

It is easy to see that, for each non-homogeneous prime ideal P of R' of height 1, $R'_P = R_{P \cap R}$. Therefore, we have

$$A = \bigcap_{P \in NHC^1(R')} R_{P \cap R}.$$

Since $R_{P\cap R}$ is a discrete valuation ring, $P \cap R$ is a non-homogeneous prime ideal of R of height 1.

For $V \in C^1(X)$, remark that $R'_{P_V} = R_{Q_V}$, where $Q_V = P_V \cap R$.

Then, we have

$$\begin{aligned} R &= R' \cap B \\ &= \left(\bigcap_{V \in C^1(X)} R'_{P_V} \right) \cap A \cap B \\ &= \left(\bigcap_{V \in C^1(X)} R_{Q_V} \right) \cap \left(\bigcap_{P \in NHC^1(R')} R_{P \cap R} \right) \cap \left(\bigcap_{i=1}^{s} B_{(t_i)} \right). \qquad (1) \end{aligned}$$

Here, we have $Q_i = R \cap t_i B_{(t_i)}$ and $R_{Q_i} \subset B_{(t_i)}$. Remark that $B = A \cap (\cap_{i=1}^{s} B_{(t_i)})$.

Put

$$R_i = \bigoplus_{(n_1,\dots,n_{i-1},n_{i+1},\dots,n_s)\in \mathbb{N}_0{}^{s-1}} H^0(X, \mathcal{O}_X(\sum_i n_i D_i)) t_1^{n_1} \cdots t_{i-1}^{n_{i-1}} t_{i+1}^{n_{i+1}} \cdots t_s^{n_s}.$$

We need the following Lemma.

Lemma 7. *With notation as above, the following conditions are equivalent:*

(1) $R_{Q_i} = B_{(t_i)}$.

(2) The height of Q_i *is* 1.

(3) $i \in U$, *that is,* $\mathrm{tr.deg}_k R_i = d + s - 1$.

We omit the proof here.

By the equality (1), Lemma 7 and Theorem 12.3 in [5], we obtain

$$R = \left(\bigcap_{V \in C^1(X)} R_{Q_V} \right) \cap \left(\bigcap_{P \in NHC^1(R')} R_{P\cap R} \right) \cap \left(\bigcap_{i \in U} R_{Q_i} \right).$$

Then,

$$\{Q_V \mid V \in C^1(X)\} \cup \{Q_i \mid i \in U\}$$

is the set of homogeneous prime ideals of R of height 1, and

$$\{P \cap R \mid P \in NHC^1(R')\}$$

is the set of non-homogeneous prime ideals of R of height 1.

Let

$$\mathrm{Div}(X) = \bigoplus_{V \in C^1(X)} \mathbb{Z}V$$

be the set of Weil divisors of X. Let

$$\mathrm{HDiv}(R) = \left(\bigoplus_{V \in C^1(X)} \mathbb{Z}[\mathrm{Spec}(R/Q_V)] \right) \oplus \left(\bigoplus_{i \in U} \mathbb{Z}[\mathrm{Spec}(R/Q_i)] \right)$$

be the set of homogeneous Weil divisors of $\mathrm{Spec}(R)$.

Here, we define

$$\phi : \mathrm{Div}(X) \longrightarrow \mathrm{HDiv}(R)$$

by $\phi(V) = [\mathrm{Spec}(R/Q_V)]$ for each $V \in C^1(X)$. For each $a \in k(X)^{\times}$, we have

$$\phi(\mathrm{div}_X(a)) = \mathrm{div}_R(a) \in \bigoplus_{V \in C^1(X)} \mathbb{Z}[\mathrm{Spec}(R/Q_V)].$$

If $i \in U$, then

$$\phi(\mathrm{div}_X(t_i)) = [\mathrm{Spec}(R/Q_i)] + D_i.$$

If $i \notin U$, then

$$\phi(\mathrm{div}_X(t_i)) = D_i.$$

Then, we have an exact sequence

$$0 \longrightarrow \sum_{i \notin U} \mathbb{Z}\overline{D_i} \longrightarrow \mathrm{Cl}(X) \xrightarrow{q} \mathrm{Cl}(R) \longrightarrow 0$$

such that $q(\overline{F}) = \overline{\phi(F)}$. It is easy to see that the class of the Weil divisor $q(\overline{F})$ corresponds to the isomorphism class of the reflexive module

$$\begin{aligned} M_F \cap \left(\bigcap_{i\in U} R_{Q_i}\right) &= M_F \cap A \cap \left(\bigcap_{i\in U} R_{Q_i}\right) \\ &= M_F \cap k(X)[t_1, \ldots, t_s, \{t_i^{-1} \mid i \notin U\}]. \end{aligned}$$

Here, it is easy to see that the class of the Weil divisor $q(\overline{F + \sum_i a_i D_i})$ corresponds to the isomorphism class of

$$M_{F+\sum_i a_i D_i} = M_F \cap (t_1^{a_1}, \ldots, t_s^{a_s})k(X)[t_1, \ldots, t_s, \{t_i^{-1} \mid i \notin U\}]. \tag{2}$$

In the rest, We assume that R is Noetherian. We shall prove that ω_R is isomorphic to

$$\begin{aligned} & M_{K_X} \cap (t_1, \ldots, t_s)k(X)[t_1, \ldots, t_s, \{t_i^{-1} \mid i \notin U\}] \\ = \; & M_{K_X} \cap (t_i \mid i \in U)k(X)[t_1, \ldots, t_s, \{t_i^{-1} \mid i \notin U\}] \end{aligned}$$

as a $\mathbb{Z}^s$-graded module. If we forget the grading, it is isomorphic to $M_{K_X + \sum_i D_i}$ by the equality (2), that is corresponding to the divisor class $q(\overline{K_X + \sum_i D_i})$.

Put $X' = X \setminus \mathrm{Sing}(X)$. We choose positive integers $a_1, \ldots, a_s$ and sections $f_1, \ldots, f_t \in H^0(X, \sum_i a_i D_i)$ such that

- $\sum_i a_i D_i$ is an ample divisor,
- $X' = \cup_j D_+(f_j)$, and
- all the D_i's are principal Cartier divisors on $D_+(f_j)$ for $j = 1, \ldots, t$.

Put $W = \{\underline{n} \in \mathbb{Z}^s \mid n_i \geq 0 \text{ if } i \in U\}$. Put $D_i' = D_i|_{X'}$ for $i = 1, \ldots, s$. Consider the morphism

$$\pi : Y = \mathrm{Spec}_{X'}\left(\bigoplus_{\underline{n}\in W} \mathcal{O}_{X'}(\sum_i n_i D_i')\right) \longrightarrow X'.$$

We have a natural map

$$Y = \mathrm{Spec}_{X'}\left(\bigoplus_{\underline{n}\in W} \mathcal{O}_{X'}(\sum_i n_i D_i')\right) \xrightarrow{\xi} \mathrm{Spec}(R).$$

The group $\mathbb{G}_m^s$ naturally acts on $\mathrm{Spec}(R)$ and Y, and ξ is an equivariant morphism. $\mathbb{G}_m^s$ trivially acts on X'.

Claim 8. *There exist equivariant open subsets $Y' \subset Y$ and $Z \subset \mathrm{Spec}(R)$ such that*

- *the codimension of $Y \setminus Y'$ in Y is bigger than or equal to 2,*
- *the codimension of $\mathrm{Spec}(R) \setminus Z$ in $\mathrm{Spec}(R)$ is bigger than or equal to 2,*
- *$Y' \simeq Z$ as a $\mathbb{G}_m^s$-scheme.*

We omit the proof of Claim 8 here.

We can define the graded canonical module as in Definition 3.1 in [3] using the theory of the equivariant twisted inverse functors [2].

By Claim 8, we have $\omega_R = H^0(Y, \omega_Y)$. On the other hand, we have

$$\begin{aligned} \omega_Y &= \bigwedge^s \Omega_{Y/X'} \otimes \pi^* \mathcal{O}_{X'}(K_{X'}) \\ &= \pi^* \mathcal{O}_{X'}(\sum_i D'_i + K_{X'})(-1, \ldots, -1), \end{aligned}$$

where $(-1, \ldots, -1)$ denotes the shift of degree.

Then, we have

$$H^0(Y, \omega_Y) = H^0(X', \pi_* \pi^* \mathcal{O}_{X'}(\sum_i D'_i + K_{X'})(-1, \ldots, -1)).$$

By the projection formula,

$$\begin{aligned} & \pi_* \pi^* \mathcal{O}_{X'}(\sum_i D'_i + K_{X'})(-1, \ldots, -1) \\ = & \mathcal{O}_{X'}(\sum_i D'_i + K_{X'}) \otimes \pi_* \mathcal{O}_Y(-1, \ldots, -1) \\ = & \mathcal{O}_{X'}(\sum_i D'_i + K_{X'}) \otimes \left(\bigoplus_{\underline{n} \in W} \mathcal{O}_{X'}(\sum_i n_i D'_i) \right) (-1, \ldots, -1) \\ = & \left(\bigoplus_{\underline{n} \in W} \mathcal{O}_{X'}(\sum_i (n_i + 1) D'_i + K_{X'}) \right) (-1, \ldots, -1) \\ = & \bigoplus_{\underline{n} \in W + (1, \ldots, 1)} \mathcal{O}_{X'}(\sum_i n_i D'_i + K_{X'}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} H^0(Y, \omega_Y) &= H^0(X', \bigoplus_{\underline{n} \in W + (1, \ldots, 1)} \mathcal{O}_{X'}(\sum_i n_i D'_i + K_{X'})) \\ &= \bigoplus_{\underline{n} \in W + (1, \ldots, 1)} H^0(X', \mathcal{O}_{X'}(\sum_i n_i D'_i + K_{X'})) \\ &= \bigoplus_{\underline{n} \in W + (1, \ldots, 1)} H^0(X, \mathcal{O}_X(\sum_i n_i D_i + K_X)) \\ &= M_{K_X} \cap (t_i \mid i \in U) k(X)[t_1, \ldots, t_s, \{t_i^{-1} \mid i \notin U\}]. \end{aligned}$$

We have completed the proof.

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GALOIS EXTENSIONS, PLUS CLOSURE, AND MAPS ON LOCAL COHOMOLOGY

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1. Introduction

Let R be a commutative Noetherian integral domain. We use R^+ to denote the integral closure of R in an algebraic closure of its fraction field. Hochster and Huneke proved the following:

Theorem 1.1. [HH2, Theorem 1.1] *If R is an excellent local domain of prime characteristic, then each system of parameters for R is a regular sequence on R^+, i.e., R^+ is a balanced big Cohen-Macaulay algebra for R.*

It follows that for a ring R as above and $i < \dim R$, the local cohomology modules $H^i_{\mathfrak{m}}(R^+)$ vanish. Hence, for each element $[\eta] \in H^i_{\mathfrak{m}}(R)$, there exists a finite integral extension S of R such that $[\eta] \longmapsto 0$ under the induced map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$. This was strengthened by Huneke and Lyubeznik, albeit under mildly different hypotheses:

Theorem 1.2. [HL, Theorem 2.1] *Let $(R, \mathfrak{m})$ be a local domain of prime characteristic which is a homomorphic image of a Gorenstein ring. Then there exists a finitely generated integral extension S of R such that the induced map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$ is zero for each $i < \dim R$.*

We prove here that the extension S may be chosen to be generically Galois, and analyze the Galois groups that arise; we say *"S is a generically Galois extension of R"* if S is an extension domain that is integral over R and the extension of fractions fields is Galois. In this case, $\operatorname{Gal}(S/R)$ will denote the Galois group of the corresponding extension of fraction fields.

Theorem 1.3. *Let R be a domain of prime characteristic.*

(1) *Let $\mathfrak{a}$ be an ideal of R and $[\eta]$ an element of $H^i_{\mathfrak{a}}(R)_{\mathrm{nil}}$ (see Section 2.3). Then there exists a finitely generated generically Galois extension S of R, with $\operatorname{Gal}(S/R)$ a solvable group, such that $[\eta]$ maps to zero under the induced map $H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(S)$.*
(2) *Suppose $(R, \mathfrak{m})$ is a local ring that is a homomorphic image of a Gorenstein ring. Then there exists a finitely generated generically Galois extension S of R such that the induced map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$ is zero for each $i < \dim R$.*

We next record the analogous results for closure operations; the relevant definitions may be found in 2.1. Theorem 1.4 (2) was essentially proved by the second author in [Si2].

Theorem 1.4. *Let R be an integral domain of positive characteristic, and let $\mathfrak{a}$ be an ideal of R.*

(1) *For each element z of $\mathfrak{a}^F$, there exists a finitely generated generically Galois extension S of R, with $\operatorname{Gal}(S/R)$ solvable, such that $z \in \mathfrak{a}S$.*
(2) *For each element z of $\mathfrak{a}^+$, there exists a finitely generated generically Galois extension S of R such that $z \in \mathfrak{a}S$.*

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Remark 1.5. The assertion of Theorem 1.2 does not hold for rings of characteristic zero: Let $(R, \mathfrak{m})$ be a normal domain of characteristic zero. If S is a domain that is finitely generated as an R-module, then the field trace map $\mathrm{tr}\colon \mathrm{frac}(S) \longrightarrow \mathrm{frac}(R)$ provides an R-linear splitting of $R \subseteq S$, namely

$$\frac{1}{[\mathrm{frac}(S) : \mathrm{frac}(R)]}\,\mathrm{tr}\colon S \longrightarrow R\,.$$

It follows that the induced maps on local cohomology $H^i_\mathfrak{m}(R) \longrightarrow H^i_\mathfrak{m}(S)$ are R-split. A variation is explored in [RSS], where the authors investigate whether the image of $H^i_\mathfrak{m}(R)$ in $H^i_\mathfrak{m}(R^+)$ is killed by elements of R^+ having arbitrarily small positive valuation. This is motivated by Heitmann's proof of the direct summand conjecture for rings $(R, \mathfrak{m})$ of dimension 3 and mixed characteristic $p > 0$, [He], which involves showing that the image of

$$H^2_\mathfrak{m}(R) \longrightarrow H^2_\mathfrak{m}(R^+)$$

is killed by $p^{1/n}$ for each positive integer n.

A detailed account of the Galois theory of rings may be found in [DI]; see also [Gr], which is closer to our context. Standard notions from commutative algebra that are used here may be found in [BH]; for more on local cohomology, consult [ILL]. For the original proof of the existence of big Cohen-Macaulay modules for local rings of prime characteristic, see [Ho].

2. Preliminary Remarks

2.1. **Closure operations.** Let R be an integral domain. The *plus closure* of an ideal $\mathfrak{a}$ of R is the ideal $\mathfrak{a}^+ = \mathfrak{a}R^+ \cap R$.

When R is a domain of prime characteristic $p > 0$, we set

$$R^\infty = \bigcup_{e \geqslant 0} R^{1/p^e}\,,$$

which is a subring of R^+. The *Frobenius closure* of an ideal $\mathfrak{a}$ is the ideal $\mathfrak{a}^F = \mathfrak{a}R^\infty \cap R$. Alternatively, set

$$\mathfrak{a}^{[p^e]} = \left(a^{p^e} \mid a \in \mathfrak{a}\right).$$

Then $\mathfrak{a}^F = (r \in R \mid r^{p^e} \in \mathfrak{a}^{[p^e]}$ for some $e \in \mathbb{N})$.

2.2. **Solvable extensions.** Let L/K be a finite separable field extension; recall that L/K is *solvable* if $\mathrm{Gal}(M/K)$ is a solvable group for some Galois extension M of K that contains L. Solvable extensions form a *distinguished class*, i.e.,

(1) for finite extensions $K \subseteq L \subseteq M$, the extension M/K is solvable if and only if M/L and L/K are solvable;
(2) for finite extensions L/K and M/K contained in a common field, if L/K is solvable, then so is LM/M.

A finite separable extension L/K of fields of characteristic $p > 0$ is solvable precisely if it is obtained by successively adjoining

(1) roots of unity;
(2) roots of polynomials $T^n - a$ for n coprime to p;
(3) roots of *Artin-Schreier polynomials*, $T^p - T - a$.

2.3. **Frobenius-nilpotent submodules.** Let R be a ring of prime characteristic p. A *Frobenius action* on an R-module M is an additive map $F\colon M \longrightarrow M$ with $F(rm) = r^p F(m)$ for each $r \in R$ and $m \in M$. In this case, $\ker F$ is a submodule of M, and we have an ascending sequence

$$\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \dots .$$

The union of these is the F-*nilpotent* submodule of M, denoted M_{nil}. When the ring R is local and the module M is Artinian, there exists an integer e such that $F^e(M_{\mathrm{nil}}) = 0$; see [Ly, Proposition 4.4] or [HS, Theorem 1.12].

3. Proofs

We first record an elementary lemma:

Lemma 3.1. *Let K be a field of characteristic $p > 0$. Let a and b elements of K where a is nonzero. Then the Galois group of the polynomial*

$$T^p + aT - b$$

is a solvable group.

Proof. Form an extension of K by adjoining a primitive $p-1$ root of unity and an element c that is a root of $T^{p-1} - a$. The polynomial $T^p + aT - b$ has the same roots as

$$\left(\frac{T}{c}\right)^p - \left(\frac{T}{c}\right) - \frac{b}{c^p},$$

which is an Artin-Schreier polynomial in T/c. □

Proof of Theorem 1.3. Since solvable extensions form a distinguished class, (1) reduces by induction to the case where $F([\eta]) = 0$. Compute $H^i_{\mathfrak{a}}(R)$ using a Čech complex $C^\bullet(\boldsymbol{x}; R)$, where $\boldsymbol{x} = x_0, \dots, x_n$ are nonzero elements generating the ideal $\mathfrak{a}$; recall that $C^\bullet(\boldsymbol{x}; R)$ is the complex

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=0}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_0 \cdots x_n} \longrightarrow 0 .$$

Consider a cycle η in $C^i(\boldsymbol{x}; R)$ that maps to $[\eta]$ in $H^i_{\mathfrak{a}}(R)$. As $F([\eta]) = 0$, it follows that $F(\eta)$ must be a boundary; say $F(\eta) = \partial(\alpha)$ for $\alpha \in C^{i-1}(\boldsymbol{x}; R)$.

There exist a power q of the characteristic p of R, and elements $a_{k_0, \dots, k_{i-1}}, b_{k_0, \dots, k_{i-2}}$ in R, such that α and η can be written in the above direct sum as

$$\alpha = \left(\frac{b_{k_0, \dots, k_{i-2}}}{(x_{k_0} \cdots x_{k_{i-2}})^q}\right), \eta = \left(\frac{a_{k_0, \dots, k_{i-1}}}{(x_{k_0} \cdots x_{k_{i-1}})^{q/p}}\right)$$

Since $F(\eta) = \partial(\alpha)$,

$$\frac{a^p_{k_0, \dots, k_{i-1}}}{(x_{k_0} \cdots x_{k_{i-1}})^q} = \frac{\sum_{j=0}^{i-1} (-1)^j b_{k_0, \dots \check{k_j} \dots, k_{i-1}} x^q_{k_j}}{(x_{k_0} \cdots x_{k_{i-1}})^q}$$

This implies

$$a^p_{k_0, \dots, k_{i-1}} = \sum_{j=0}^{i-1} (-1)^j b_{k_0, \dots \check{k_j} \dots, k_{i-1}} x^q_{k_j} \tag{1}$$

For $0 \le k_0 < \cdots < k_{i-2} \le n$, define $t_{k_0, \dots, k_{i-2}}$ as follows.
For $k_0 = 0$, let $t_{0, k_1 \dots, k_{i-2}}$ be a root of the polynomial of $T^p + x_0^q T + b_{0, k_1 \dots, k_{i-2}}$.
And for $k_0 \neq 0$, put

$$t_{k_0 \dots, k_{i-2}} = \frac{a_{0, k_0, \dots, k_{i-2}} - \sum_{j=0}^{i-2} (-1)^{j+1} t_{0, k_0 \dots \check{k_j} \dots, k_{i-2}} x^{q/p}_{k_j}}{x_0^{q/p}} \tag{2}$$

and let L be a finite extension field where these have roots $t_{0,k_1\dots,k_{i-2}}$ respectively. By Lemma 3.1, we may assume L is Galois over frac(R) with the Galois group being solvable. Let S be a finitely generated integral extension of R that contains $t_1,\dots,t_m$, and has L as its fraction field; if R is excellent, we may take S to be the integral closure of R in L.
Then for $k_0 \neq 0$,

$$
\begin{aligned}
x_0^q t^p_{k_0,\dots,k_{i-2}} &= a^p_{0,k_0,\dots,k_{i-2}} - \sum_{j=0}^{i-2}(-1)^{j+1} t^p_{0,k_0\dots\check{k_j}\dots,k_{i-2}} x^q_{k_j} \\
&= b_{k_0\dots,k_{i-2}} x_0^q + \sum_{j=0}^{i-2}(-1)^{j+1} b_{0,k_0\dots\check{k_j}\dots,k_{i-2}} x^q_{k_j} \\
&- \left(x_0^q \sum_{j=0}^{i-2}(-1)^{j+1} t_{0,k_0\dots\check{k_j}\dots,k_{i-2}} x^q_{k_j} + \sum_{j=0}^{i-2}(-1)^{j+1} b_{0,k_0\dots\check{k_j}\dots,k_{i-2}} x^q_{k_j} \right) \\
&= x_0^q \left(b_{k_0\dots,k_{i-2}} + \sum_{j=0}^{i-2}(-1)^{j+1} t_{0,k_0\dots\check{k_j}\dots,k_{i-2}} x^q_{k_j} \right),
\end{aligned}
$$

We used (1) and the defining polynomial equations of $t_{0,k_0\dots\check{k_j}\dots,k_{i-2}}$ for second equality.
This implies that $t_{k_0,\dots,k_{i-2}}$'s are integral over R,and by definition of $t_{k_0,\dots,k_{i-2}}$, they are in L. Therefore we can add $t_{k_0,\dots,k_{i-2}}$'s to S without changing the conditions.
Put

$$\beta = \left(\frac{t_{k_0,\dots,k_{i-2}}}{(x_{k_0}\cdots x_{k_{i-2}})^{q/p}} \right).$$

We claim $\eta = \partial(\beta)$.
From (2),we can get the equation for $k_0 = 0$.

$$\frac{a_{k_0,\dots,k_{i-1}}}{(x_{k_0}\cdots x_{k_{i-1}})^{q/p}} = \frac{\sum_{j=0}^{i-1}(-1)^j t_{k_0,\dots\check{k_j}\dots,k_{i-1}} x^{q/p}_{k_j}}{(x_{k_0}\cdots x_{k_{i-1}})^{q/p}}$$

This means

$$\eta_{x_0,x_{k_1},\cdots,x_{k_{i-1}}} = \partial(\beta)_{x_0,x_{k_1},\cdots,x_{k_{i-1}}}.$$

And for $k_0 \neq 0$,

$$
\begin{aligned}
&\partial(\partial(\beta) - \eta)_{x_0,x_{k_0},\cdots,x_{k_{i-1}}} \\
&= \sum(-1)^j \partial(\beta)_{x_0,x_{k_0},\dots\check{x_{k_j}}\dots,x_{k_{i-1}}} - \sum(-1)^j \eta_{x_0,x_{k_0},\dots\check{x_{k_j}}\dots,x_{k_{i-1}}} \\
&= \eta_{x_{k_0},\cdots,x_{k_{i-1}}} - \partial(\beta)_{x_{k_0},\cdots,x_{k_{i-1}}}
\end{aligned}
$$

Since $\partial(\beta)$ and η are in ker∂,this is zero.
Therefore $\eta = \partial(\beta)$. this completes the proof of (1).
For (2), we need a lemma.

Lemma 3.2. *Let R be a domain and $\mathfrak{p}$ be a its prime ideal.Let S be a finite extension of $R_\mathfrak{p}$. Then there is a ring T which is finite over R such that $T_\mathfrak{p} = S$.*

proof
Let $a_1,\dots,a_n$ be generater of S over R.Then there are monic polynomials $f_1,\dots,f_n$ such that all coefficients are in $R_\mathfrak{p}$ and $f_i(a_i) = 0$ for all i.By multipling $r_i^{deg f_i} \in R - \mathfrak{p}$ to f_i,we get the monic polynomials $g_1,\dots,g_n$ such that all coefficients are in R and $g_i(r_i a_i) = 0$ for all i.Put

$T = R[r_1a_1, \dots, r_na_n]$,then $T_{\mathfrak{p}} = S$ and this is finite over R.

To prove (2),We first use induction on $d = \dim R$, as in [HL]. If $d = 0$, there is nothing to be proved; if $d = 1$, the inductive hypothesis is again trivially satisfied since $H^0_{\mathfrak{m}}(R) = 0$.

Fix $i < \dim R$. Let $(A, \mathfrak{M})$ be a Gorenstein local ring that has R as a homomorphic image, and set

$$M = \mathrm{Ext}_A^{\dim A - i}(R, A).$$

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the elements of the set $\mathrm{Ass}_A M \smallsetminus \{\mathfrak{M}\}$.

Let $\mathfrak{q}$ be a prime ideal of R that is not maximal. Since R is catenary, one has $\dim R = \dim R_{\mathfrak{q}} + \dim R/\mathfrak{q}$. Thus, the condition $i < \dim R$ may be rewritten as

$$i - \dim R/\mathfrak{q} < \dim R_{\mathfrak{q}}.$$

By the lemma,the inductive hypothesis implies that there exists a finite Galois field extension K' of $\mathrm{frac}(R_{\mathfrak{q}}) = \mathrm{frac}(R)$ and R' which is finite over R such that the induced map

$$H^{i-\dim R/\mathfrak{q}}_{\mathfrak{q}R_{\mathfrak{q}}}(R_{\mathfrak{q}}) \longrightarrow H^{i-\dim R/\mathfrak{q}}_{\mathfrak{q}R_{\mathfrak{q}}}(R'_{\mathfrak{q}}) \tag{3.2.1}$$

is zero. Taking the compositum of finitely many extensions inside a fixed algebraic closure of $\mathrm{frac}(R)$, we may assume that K' and R' are such that the map (3.2.1) is zero when $\mathfrak{q}$ is any of the primes $\mathfrak{p}_1R, \dots, \mathfrak{p}_sR$. We claim that the image of the induced map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R')$ has finite length.

Using local duality over A, it suffices to show that

$$M' = \mathrm{Ext}_A^{\dim A - i}(R', A) \longrightarrow \mathrm{Ext}_A^{\dim A - i}(R, A) = M$$

has finite length. This, in turn, would follow if

$$M'_{\mathfrak{p}} = \mathrm{Ext}_{A_{\mathfrak{p}}}^{\dim A - i}(R'_{\mathfrak{p}}, A_{\mathfrak{p}}) \longrightarrow \mathrm{Ext}_{A_{\mathfrak{p}}}^{\dim A - i}(R_{\mathfrak{p}}, A_{\mathfrak{p}}) = M_{\mathfrak{p}}$$

is zero for each prime ideal $\mathfrak{p}$ in $\mathrm{Ass}_A M \smallsetminus \{\mathfrak{M}\}$. Using local duality over $A_{\mathfrak{p}}$, it suffices to verify the vanishing of

$$H^{\dim A_{\mathfrak{p}} \ \dim A + i}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \longrightarrow H^{\dim A_{\mathfrak{p}} \ \dim A + i}_{\mathfrak{p}R_{\mathfrak{p}}}(R'_{\mathfrak{p}})$$

for each $\mathfrak{p}$ in $\mathrm{Ass}_A M \smallsetminus \{\mathfrak{M}\}$. This, however, follows from our choice of K' and R' since

$$\dim A_{\mathfrak{p}} - \dim A + i \ = \ i - \dim A/\mathfrak{p} \ = \ i - \dim R/\mathfrak{p}R.$$

What we have arrived at thus far is a finitely generated generically Galois extension R' of R such that the image of $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R')$ has finite length; in particular, this image is finitely generated.

By using theorem1.2,there is a ring R_1 which is finite over R such that the induced map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R_1)$ is zero.Taking the separable closure in R_1,we get the ring R_2 which is finite separable over R and the image of the map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R_2)$ is in $H^i_{\mathfrak{m}}(R_2)_{nil}$.Taking the Galois closure of the fraction field of R_2 and using the lemma3.2.,there is a ring R_3 which is generically Galois over R. And by using the claim above,we can asuume the image of the map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R_3)$ is finitely generated and in $H^i_{\mathfrak{m}}(R_3)_{nil}$. Working with one generator at a time and taking the compositum of extensions, given $[\eta]$ in $H^i_{\mathfrak{m}}(R_3)$, it suffices to construct a finitely generated generically Galois extension S of R_3 such that $[\eta]$ maps to 0 under $H^i_{\mathfrak{m}}(R_3) \longrightarrow H^i_{\mathfrak{m}}(S)$. But this is guaranteed by (1). □

Let R be an integral domain with fraction field K. Fix an algebraic closure $\overline{K}$ of K. Let $R^{+\mathrm{sep}}$ denote the set of all elements of R^+ that are separable over K; this is indeed a ring.Then we get a corollary.

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a local domain of prime characteristic which is a homomorphic image of a Gorenstein ring. Then each system of parameters for R is a regular sequence on $R^{+\mathrm{sep}}$. Hence $H^i_{\mathfrak{m}}(R^{+\mathrm{sep}}) = 0$ for each $i < \dim R$.*

Proof. Same proof of [HL,Corollary2.3.] works. □

As observed in [HH2, page 77], the algebra R^∞, i.e., the purely inseparable part of R^+, is not a Cohen-Macaulay R-algebra in general; for example, take R to be an F-pure domain that is not Cohen-Macaulay.

Proof of Theorem 1.4. Let p be the characteristic of R. Suppose $z \in \mathfrak{a}^F$. Then there exists a prime power $q = p^e$ with $z^q \in \mathfrak{a}^{[q]}$. In this case, $z^{q/p}$ belongs to the Frobenius closure of $\mathfrak{a}^{[q/p]}$, and

$$(z^{q/p})^p \in (\mathfrak{a}^{[q/p]})^{[p]}.$$

Since solvable extensions form a distinguished class, we reduce to the case $e = 1$, i.e., $q = p$.

There exist nonzero elements, $a_0, \ldots, a_m \in \mathfrak{a}$ and $b_0, \ldots, b_m \in R$ with

$$z^p = \sum_{i=0}^{m} b_i a_i^p.$$

Consider the polynomials

$$T^p + a_0^p T - b_i \qquad \text{for } i = 1, \ldots, m,$$

and let L be a finite extension field where these have roots $t_1, \ldots, t_m$ respectively. By Lemma 3.1, we may assume L is Galois over frac(R) with the Galois group being solvable. Set

$$t_0 = \frac{1}{a_0}\Big(z - \sum_{i=1}^{m} t_i a_i\Big). \tag{3.3.1}$$

Taking p-th powers, we have

$$t_0^p = \frac{1}{a_0^p}\Big(\sum_{i=0}^{m} b_i a_i^p - \sum_{i=1}^{m} t_i^p a_i^p\Big) = b_0 + \frac{1}{a_0^p}\sum_{i=1}^{m}(b_i - t_i^p)a_i^p$$

$$= b_0 + \sum_{i=1}^{m} t_i a_i^p.$$

Thus, t_0 belongs to the integral closure of $R[t_1, \ldots, t_m]$ in its field of fractions. Let S be a finitely generated integral extension of R that contains $t_0, \ldots, t_m$, and has L as its fraction field; if R is excellent, we may take S to be the integral closure of R in L. Since (3.3.1) may be rewritten as

$$z = \sum_{i=0}^{m} t_i a_i,$$

it follows that $z \in \mathfrak{a}S$, completing the proof of (1).

(2) follows from [Si2, Corollary 3.4], though we provide a proof using (1). There exists an extension domain T, finitely generated as an R-module, such that $z \in \mathfrak{a}T$. Decompose the field extension frac$(R) \subseteq$ frac(T) as a separable extension frac$(R) \subseteq$ frac$(T)^{+\text{sep}}$ followed by a purely inseparable extension frac$(T)^{+\text{sep}} \subseteq$ frac(T). Let T_0 be the integral closure of R in frac$(T)^{+\text{sep}}$.

Since T is a purely inseparable extension of T_0, and $z \in \mathfrak{a}T$, it follows that z belongs to the Frobenius closure of the ideal $\mathfrak{a}T_0$. By (2) there exists a generically separable extension S_0 of T_0 with $z \in \mathfrak{a}S_0$. Enlarge S_0 to a generically Galois extension S of R. This concludes the argument in the case R is excellent; in the event that S is not a finitely generated R-module, one may replace it by a subring, finitely generated over R, satisfying $z \in \mathfrak{a}S$ and having the same fraction field. □

The equational construction used in the proof of Theorem 1.4 arose from the study of symplectic invariants in [Si1].

4. Some Galois groups that are not solvable

Let R be an integral domain of positive characteristic, and let $\mathfrak{a}$ be an ideal of R. If z is an element of $\mathfrak{a}^F$, then Theorem 1.4 states that there exists an extension S of R, with $\mathrm{Gal}(S/R)$ solvable, such that $z \in \mathfrak{a}S$. In the following example one has $z \in \mathfrak{a}^+$, but we conjecture $z \notin \mathfrak{a}S$ for any generically Galois extension S of R with $\mathrm{Gal}(S/R)$ solvable.

Example 4.1. Take algebraically independent elements a, b, c_1, c_2 over $\mathbb{F}_p$, and set R be the hypersurface

$$\frac{\mathbb{F}_p(a,b,c_1,c_2)[x,y,z]}{\left(z^{p^2} + c_1(xy)^{p^2-p}z^p + c_2(xy)^{p^2-1}z + ax^{p^2} + by^{p^2}\right)}\,.$$

We claim $z \in (x,y)^+$. Let u and v be elements of R^+ that are, respectively, roots of the monic polynomials

$$T^{p^2} + c_1y^{p^2-p}T^p + c_2y^{p^2-1}T + a\,, \tag{4.1.1}$$

and

$$T^{p^2} + c_1x^{p^2-p}T^p + c_2x^{p^2-1}T + b\,.$$

Set S to be the integral closure of R is the Galois closure of $\mathrm{frac}(R)(u,v)$ over $\mathrm{frac}(R)$. Then $(z - ux - vy)/xy$ is an element of S satisfying

$$T^{p^2} + c_1T^p + c_2T = 0\,,$$

and hence $(z - ux - vy)/xy$ belongs to S. It follows that $z \in (x,y)S$.

We next show that $\mathrm{Gal}(S/R)$ is not solvable for the extension S constructed above. Since u is a root of (4.1.1), u/y is a root of

$$T^{p^2} + c_1T^p + c_2T + \frac{a}{y^{p^2}}\,. \tag{4.1.2}$$

The polynomial (4.1.2) is irreducible over $\mathbb{F}_q(c_1, c_2, a/y^{p^2})$, and hence over the purely transcendental extension $\mathbb{F}_q(c_1,c_2,a,x,y,z) = \mathrm{frac}(R)$. Since $\mathrm{frac}(S)$ is a Galois extension of $\mathrm{frac}(R)$ containing a root of (4.1.2), it contains all roots of (4.1.2). As (4.1.2) is separable, its roots are distinct; taking differences of roots, it follows that $\mathrm{frac}(S)$ contains the p^2 distinct roots of

$$T^{p^2} + c_1T^p + c_2T\,. \tag{4.1.3}$$

We next verify that the Galois group of (4.1.3) over $\mathrm{frac}(R)$ is $\mathrm{GL}_2(\mathbb{F}_q)$.

Quite generally, let L be a field of characteristic p. Consider the standard linear action of $\mathrm{GL}_2(\mathbb{F}_p)$ on the polynomial ring $L[x_1,x_2]$. The ring of invariants for this action is generated over L by the *Dickson invariants* c_1, c_2, which occur as the coefficients in the polynomial

$$\prod_{\alpha,\beta\in\mathbb{F}_p} (T - \alpha x_1 - \beta x_2) = T^{p^2} + c_1T^p + c_2T\,,$$

see [Di] or [Be, Chapter 8]. Hence the extension $L(x_1,x_2)/L(c_1,c_2)$ has Galois group $\mathrm{GL}_2(\mathbb{F}_p)$.

It follows from the above that if c_1, c_2 are algebraically independent elements over a field L of characteristic p, then the polynomial

$$T^{p^2} + c_1T^p + c_2T \;\in\; L(c_1,c_2)[T]$$

has Galois group $\mathrm{GL}_2(\mathbb{F}_p)$.

The group $\mathrm{PSL}_2(\mathbb{F}_p)$ is a subquotient of $\mathrm{GL}_2(\mathbb{F}_p)$, and, we conjecture, a subquotient of *any* generically Galois extension S of R for which $z \in \mathfrak{a}S$. For $p \geqslant 5$, the group $\mathrm{PSL}_2(\mathbb{F}_p)$ is a nonabelian simple group and thus, conjecturally, $\mathrm{Gal}(S/R)$ is not solvable for any generically Galois extension S of R with $z \in \mathfrak{a}S$.

Example 4.2. Extending the example above, take algebraically independent elements $a, b, c_1, \ldots, c_n$ over $\mathbb{F}_q$, and set R be the homomorphic image of the polynomial ring $\mathbb{F}_q(a, b, c_1, \ldots, c_n)[x, y, z]$ modulo the principal ideal generated by

$$z^{q^n} + c_1(xy)^{q^n - q^{n-1}} z^{q^{n-1}} + c_2(xy)^{q^n - q^{n-2}} z^{q^{n-2}} + \cdots$$
$$+ c_n(xy)^{q^n - 1} z + ax^{q^n} + by^{q^n}.$$

Then $z \in (x, y)^+$; imitate the previous example with u, v being roots of

$$T^{q^n} + c_1 y^{q^n - q^{n-1}} T^{q^{n-1}} + c_2 y^{q^n - q^{n-2}} T^{q^{n-2}} + \cdots + c_n y^{q^n - 1} T + a,$$

and

$$T^{q^n} + c_1 x^{q^n - q^{n-1}} T^{q^{n-1}} + c_2 x^{q^n - q^{n-2}} T^{q^{n-2}} + \cdots + c_n x^{q^n - 1} T + b.$$

If S is any generically Galois extension of R with $z \in \mathfrak{a}S$, we conjecture that frac(S) contains the splitting field of

$$T^{q^n} + c_1 T^{q^{n-1}} + c_2 T^{q^{n-2}} + \cdots + c_n T. \tag{4.2.1}$$

Using a similar argument with Dickson invariants, the Galois group of (4.2.1) over frac(R) is $\mathrm{GL}_n(\mathbb{F}_q)$. Its subquotient $\mathrm{PSL}_n(\mathbb{F}_q)$ is a nonabelian simple group for $n \geqslant 3$ and $n = 2$, $q \geqslant 4$.

Likewise, we record conjectural examples R where $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$ is nonzero for each generically Galois extension S with $\mathrm{Gal}(S/R)$ solvable:

Example 4.3. Take algebraically independent elements a, b, c_1, c_2 over $\mathbb{F}_p$, and consider the hypersurface

$$A = \frac{\mathbb{F}_p(a, b, c_1, c_2)[x, y, z]}{\left(z^{2p^2} + c_1(xy)^{p^2 - p} z^{2p} + c_2(xy)^{p^2 - 1} z^2 + ax^{p^2} + by^{p^2}\right)}.$$

Let $(R, \mathfrak{m})$ be the Rees ring $A[xt, yt, zt]$ localized at the maximal ideal x, y, z, xt, yt, zt. The elements $x, yt, y + xt$ form a system of parameters for R, and the relation

$$z^2 t \cdot (y + xt) = z^2 t^2 \cdot x + z^2 \cdot yt,$$

defines an element $[\eta]$ of $H^2_{\mathfrak{m}}(R)$. We conjecture that if S is any generically Galois extension such that $[\eta]$ maps to 0 under $H^2_{\mathfrak{m}}(R) \longrightarrow H^2_{\mathfrak{m}}(S)$, then frac($S$) contains the splitting field of

$$T^{p^2} + c_1 T^p + c_2 T,$$

and hence that $\mathrm{Gal}(S/R)$ is not solvable in the case of characteristic $p \geqslant 5$.

5. Graded rings and extensions

Remark 5.1. If R is an $\mathbb{N}$-graded domain that is finitely generated over a field R_0, set $R^{+\mathrm{GR}}$ to be the $\mathbb{Q}_{\geqslant 0}$-graded ring generated by elements of R^+ that can be assigned a degree such that they then satisfy a homogeneous equation of integral dependence over R. Note that $[R^{+\mathrm{GR}}]_0$ is the algebraic closure of the field R_0. One has the following:

Theorem 5.2. [HH2, Theorem 6.1] *Let R be an $\mathbb{N}$-graded domain that is finitely generated over a field R_0 of positive characteristic. Then every homogeneous system of parameters for R is a regular sequence on $R^{+\mathrm{GR}}$.*

Let R be as in the above theorem. Since $R^{+\mathrm{GR}}$ and $R^{+\mathrm{sep}}$ are Cohen-Macaulay R-algebras, it is natural to ask whether the algebra consisting of separable elements of $R^{+\mathrm{GR}}$ is also Cohen-Macaulay. The answer is negative:

Example 5.3. Let R be the Rees ring

$$\frac{\overline{\mathbb{F}}_2[x,y,z]}{(x^3+y^3+z^3)}[xt,yt,zt]$$

with the $\mathbb{N}$-grading where the generators x, y, z, xt, yt, zt have degree 1. We prove that the ring B consisting of separable elements of $R^{+\mathrm{GR}}$ is not a balanced Cohen-Macaulay R-module.

The elements x, yt, $y+xt$ are a system of parameters for R. Suppose, to the contrary, that they form a regular sequence on B. Since

$$z^2t\cdot(y+xt) \;=\; z^2t^2\cdot x + z^2\cdot yt\,,$$

it follows that $z^2t \in (x,yt)B$. Thus, there exist elements $u, v \in B_1$ with

$$z^2t \;=\; u\cdot x \;+\; v\cdot yt\,. \tag{5.3.1}$$

Since $z^3 = x^3+y^3$, we also have $z^2 \;=\; x\sqrt{xz}+y\sqrt{yz}$ in $R^{+\mathrm{GR}}$, and hence

$$z^2t \;=\; t\sqrt{xz}\cdot x \;+\; \sqrt{yz}\cdot yt\,. \tag{5.3.2}$$

Comparing (5.3.1) and (5.3.2), we see that

$$(u+t\sqrt{xz})\cdot x \;=\; (v+\sqrt{yz})\cdot yt$$

in $R^{+\mathrm{GR}}$. But x, yt is a regular sequence on $R^{+\mathrm{GR}}$, so there exists an element c in $[R^{+\mathrm{GR}}]_0$ with $u+t\sqrt{xz} = cyt$ and $v+\sqrt{yz} = cx$. Since $[R^{+\mathrm{GR}}]_0 = \overline{\mathbb{F}}_2$, it follows that $c \in R$, and hence that $\sqrt{yz} \in B$. This contradicts the hypothesis that elements of B are separable over R.

The above argument shows that any graded Cohen-Macaulay R-algebra must contain the elements $\sqrt{yz}$ and $t\sqrt{xz}$.

We next show that the ring R in Example 5.3 has no finitely generated $\mathbb{Q}$-graded extension in $R^{+\mathrm{GR}}$ that is Cohen-Macaulay.

Example 5.4. Let R be the Rees ring from Example 5.3, and let S be a graded Cohen-Macaulay ring with $R \subseteq S \subseteq R^{+\mathrm{GR}}$. We prove that S is not finitely generated over R.

By the previous example, S contains $\sqrt{yz}$ and $t\sqrt{xz}$. Using the symmetry between x, y, z, it follows that $\sqrt{xy}$, $\sqrt{xz}$, $t\sqrt{xy}$, $t\sqrt{yz}$ are all elements of S. In fact, as we next prove inductively, S contains

$$\begin{array}{lll} x^{1-2/q}(yz)^{1/q}, & y^{1-2/q}(xz)^{1/q}, & z^{1-2/q}(xy)^{1/q}, \\ tx^{1-2/q}(yz)^{1/q}, & ty^{1-2/q}(xz)^{1/q}, & tz^{1-2/q}(xy)^{1/q}, \end{array} \tag{5.4.1}$$

for each $q = 2^e$ with $e \geqslant 1$. The case $e = 1$ has been settled.

Suppose S contains the elements (5.4.1) for some $q = 2^e$. Then, one has

$$\begin{aligned} x^{1-2/q}(yz)^{1/q}\cdot ty^{1-2/q}(xz)^{1/q}\cdot(y+xt) \\ = tx^{1-2/q}(yz)^{1/q}\cdot ty^{1-2/q}(xz)^{1/q}\cdot x \\ + x^{1-2/q}(yz)^{1/q}\cdot y^{1-2/q}(xz)^{1/q}\cdot yt\,. \end{aligned}$$

Using as before that x, yt, $y+xt$ is a regular sequence on S, we conclude

$$x^{1-2/q}(yz)^{1/q}\cdot ty^{1-2/q}(xz)^{1/q} \;=\; u\cdot x \;+\; v\cdot yt$$

for some $u, v \in S_1$, i.e., that

$$t(xy)^{1-1/q}z^{2/q} \;=\; u\cdot x \;+\; v\cdot yt\,. \tag{5.4.2}$$

Taking q-th roots in

$$z^2 \;=\; x\sqrt{xz}+y\sqrt{yz}$$

and multiplying by $t(xy)^{1-1/q}$ yields

$$(5.4.3) \qquad t(xy)^{1-1/q}z^{2/q} \;=\; ty^{1-1/q}(xz)^{1/2q}\cdot x \;+\; x^{1-1/q}(yz)^{1/2q}\cdot yt\,.$$

Comparing (5.4.2) and (5.4.3), we see that

$$\big(u+ty^{1-1/q}(xz)^{1/2q}\big)\cdot x \;=\; \big(v+x^{1-1/q}(yz)^{1/2q}\big)\cdot yt\,,$$

so there exists c in $[R^{+\mathrm{GR}}]_0=\overline{\mathbb{F}}_2$ with

$$u+ty^{1-1/q}(xz)^{1/2q}=cy \qquad\text{and}\qquad v+x^{1-1/q}(yz)^{1/2q}=cx\,.$$

It follows that $ty^{1-1/q}(xz)^{1/2q}$ and $x^{1-1/q}(yz)^{1/2q}$ are elements of S. In view of the symmetry between x,y,z, this completes the inductive step. Setting

$$\theta=\frac{xy}{z^2}\,,$$

we have proved that

$$\theta^{1/q}\;\in\;\mathrm{frac}(S)\qquad\text{for each } q=2^e\,.$$

We claim $\theta^{1/2}$ does not belong to frac(R). Indeed if it does, then $(xy)^{1/2}$ belongs to frac(R), and hence to R, as R is normal; this is readily seen to be false. The extension

$$\mathrm{frac}(R)\;\subsetneq\;\mathrm{frac}(R)(\theta^{1/q})$$

is purely inseparable, so the minimal polynomial of $\theta^{1/q}$ over frac(R) has the form $T^Q-\theta^{Q/q}$ for some $Q=p^E$. Since $\theta^{1/2}\notin\mathrm{frac}(R)$, we conclude that the minimal polynomial is $T^q-\theta$. Hence

$$\Big[\mathrm{frac}(R)(\theta^{1/q}):\mathrm{frac}(R)\Big]=q\qquad\text{for each } q=2^e\,.$$

It follows that $[\mathrm{frac}(S):\mathrm{frac}(R)]$ is not finite.

Theorem 1.2 and Theorem 1.3 (2) discuss the vanishing of the image of $H^i_{\mathfrak{m}}(R)$ for $i<\dim R$. In the case of graded rings, one also has the following result for $H^d_{\mathfrak{m}}(R)$.

Proposition 5.5. *Let R be an $\mathbb{N}$-graded domain that is finitely generated over a field R_0 of positive characteristic. Set $d=\dim R$. Then the submodule $[H^d_{\mathfrak{m}}(R)]_{\geqslant 0}$ maps to zero under the induced map*

$$H^d_{\mathfrak{m}}(R)\longrightarrow H^d_{\mathfrak{m}}(R^{+\mathrm{GR}})\,.$$

Hence, there exists a finitely generated graded extension S of R, contained in $R^{+\mathrm{GR}}$, such that $[H^d_{\mathfrak{m}}(R)]_{\geqslant 0}\longrightarrow H^d_{\mathfrak{m}}(S)$ is zero.

Proof. Let $F\colon H^d_{\mathfrak{m}}(R)\longrightarrow H^d_{\mathfrak{m}}(R)$ denote the Frobenius map, and F^e its e-th iteration. Suppose $[\eta]\in[H^d_{\mathfrak{m}}(R)]_n$ for some $n\geqslant 0$. Then $F^e([\eta])$ belongs to $[H^d_{\mathfrak{m}}(R)]_{np^e}$ for each e. As $[H^d_{\mathfrak{m}}(R)]_{\geqslant 0}$ has finite length, there exists a positive integer e and homogeneous elements $r_1,\dots,r_e\in R$ such that

$$(5.5.1)\qquad F^e([\eta])+r_1F^{e-1}([\eta])+\cdots+r_e[\eta]=0\,.$$

We imitate the equational construction from [HL]: Consider a homogeneous system of parameters $\boldsymbol{x}=x_1,\dots,x_d$, and compute $H^i_{\mathfrak{m}}(R)$ as the cohomology of the Čech complex $C^\bullet(\boldsymbol{x};R)$ displayed below:

$$0\longrightarrow R\longrightarrow\bigoplus_{i=1}^{d}R_{x_i}\longrightarrow\bigoplus_{i<j}R_{x_ix_j}\longrightarrow\cdots\longrightarrow R_{x_1\cdots x_d}\longrightarrow 0\,.$$

This complex is $\mathbb{Q}$-graded; let η be a homogeneous element of $C^d(\boldsymbol{x};R)$ that maps to $[\eta]$ in $H^d_{\mathfrak{m}}(R)$. By (5.5.1);

$$F^e(\eta)+r_1F^{e-1}(\eta)+\cdots+r_e\eta$$

is a boundary in $C^d(\boldsymbol{x}; R)$, say it equals $\partial(\alpha)$ for a homogeneous element α of $C^{d-1}(\boldsymbol{x}; R)$. Solving integral equations in each coordinate of $C^{d-1}(\boldsymbol{x}; R)$, there exists a finite normal extension S of R and β in $C^{d-1}(\boldsymbol{x}; S)$ with

$$F^e(\beta) + r_1 F^{e-1}(\beta) + \cdots + r_e\beta = \alpha\,.$$

Since $\eta - \partial(\beta)$ is an element on frac(S) satisfying

$$T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T = 0\,,$$

it belongs to S. But then $\eta - \partial(\beta)$ maps to zero in $H^d_{\mathfrak{m}}(S)$.

Thus, each homogeneous element of $[H^d_{\mathfrak{m}}(R)]_{\geqslant 0}$ maps to 0 in $H^d_{\mathfrak{m}}(R^{+\mathrm{GR}})$. To complete the proof, note that $[H^d_{\mathfrak{m}}(R)]_{\geqslant 0}$ has finite length. □

The next example justifies why Proposition 5.5 is limited to $[H^d_{\mathfrak{m}}(R)]_{\geqslant 0}$.

Example 5.6. Let K be a field of characteristic $p > 0$ and take R to be the affine semigroup ring

$$R = K[x_1 \cdots x_d,\ x_1^d,\ \ldots,\ x_d^d]\,.$$

It is easily seen that R is normal, and that $[H^d_{\mathfrak{m}}(R)]_n$ is nonzero for each integer $n < 0$. We claim that the induced map

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(S)$$

is injective for each extension ring S that is finitely generated as an R-module. For this, it suffices to check that R is a *splinter* ring, i.e., that R is a direct summand of each module-finite extension ring; the splitting of $R \subseteq S$ then induces an R-splitting of $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(S)$. But weakly F-regular rings are splinter by [HH3, Theorem 5.25], and normal affine semigroup rings are weakly F-regular by [HH1, Proposition 4.12].

For more on splinters, we refer the reader to [Ma, HH3, Si3].

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A POSITIVE CHARACTERISTIC APPROACH TO WANG'S THEOREM

KEI-ICHI WATANABE AND KEN-ICHI YOSHIDA

Introduction

Throughout this paper, let R be a commutative Noetherian ring with unit element. Our study of this paper was motivated by the following Wang's theorem.

Theorem 0.1 (**Wang** [Wa, Theorems 1.1 and 1.2]). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$, with $d \geq 3$ if R is regular. Let $s \geq 2$ be an integer. If J is a parameter ideal with $J \subseteq \mathfrak{m}^s$, then $J \colon \mathfrak{m}^s \subseteq \mathfrak{m}^s$ and $L^2 = JL$, where $L = J \colon \mathfrak{m}^s$. In particular, if $1 \leq t \leq s$, then $J \colon \mathfrak{m}^t$ is integral over J.*

It is natural to ask the following question from Wang's theorem.

Question 0.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and let s, t be positive integers. Let $J \subseteq R$ be a parameter ideal. What can we say about the *Goto number*

$$g(J) = \max\{t \in \mathbb{N} \mid J \colon \mathfrak{m}^t \text{ is integral over } J\}?$$

The Goto number was introduced by Heinzer and Swanson [HS]. Recently, they have been studied by many researchers. For instance, Heinzer and Swanson [HS] computed or gave bounds for Goto numbers of parameter ideals in the following cases: (a) local rings of dimension 1 (b) regular local rings of dimension 2 (c) numerical semigroup rings. Moreover, Goto et.al. [GKPT] determined $g(J)$ for many parameter ideals J under the assumption that the associated graded ring $G(\mathfrak{m})$ is Gorenstein. See also [GKM, GTM]. Their approaches need a high standard technique in ideal theory.

But, our method is a little bit different from the others. Namely, we use the so-called characteristic p method involving the theory of tight closure. Indeed, in this talk, we first recall the description of the integral closure $\overline{J}$ for a parameter ideal J in terms of the ideal-adic tight closures introduced by [HY], and we establish a new formula for $J \colon \overline{J}$ for any parameter ideal J in a (complete) Gorenstein local ring R (see Section 1). Secondly, we extend Skoda-type theorem to compare the powers of the maximal ideal and their test ideals (see Section 2). Using these results, we give the following theorem, which is the main theorem in this talk (see Section 3).

Theorem 0.3 (**Wang-type**). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of characteristic $p > 0$. For any parameter ideal J of R with $J \subseteq \mathfrak{m}^s$, $J \colon \mathfrak{m}^{(d-1)(s-1)}$ is*

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integral over J. *If, in addition,* R *is not regular, then* $J\colon \mathfrak{m}^{(d-1)(s-1)+1}$ *is integral over* J.

In other words,

$$g(J) \geq (d-1)(s-1) \quad (\textit{resp.} \geq (d-1)(s-1)+1)$$

if R *is Cohen-Macaulay (resp. and not regular).*

This bound is the best possible one if R is regular.

Example 0.4. Let $s \geq 1$, $d \geq 2$ be integers. Let $R = k[[x_1, \ldots, x_d]]$ be a formal power series ring over a field k of characteristic $p > 0$. Let $J = (x_1^s, \ldots, x_d^s)$ be a parameter ideal in $\mathfrak{m}^s$. Then $J\colon \mathfrak{m}^{(d-1)(s-1)} = \mathfrak{m}^s = \overline{J}$ is integral over J, but $J\colon \mathfrak{m}^{(d-1)(s-1)+1}$ is *not*.

1. A new formula for $J\colon \overline{J}$

Throughout this section, let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic $p > 0$ with $d = \dim R \geq 1$. The main purpose of this section is to provide a formula of $J\colon \overline{J}$ for any parameter ideal J in a complete Gorenstein local ring R in terms of test ideals. Namely, we prove the following theorem.

Theorem 1.1 (Formula). *Assume that* R *is a complete Gorenstein local ring. Then for any parameter ideal* J *of* R, *we have*

$$J : \overline{J} = J + \tau(J^{d-1}),$$

where $\overline{J}$ *denotes the integral closure of* J *and* $\tau(\mathfrak{a})$ *denotes the* $\mathfrak{a}$*-test ideal of* R.

In order prove the above theorem, we need two propositions. The first proposition enables us to describe the integral closure of a parameter ideal in terms of ideal-adic tight closures. The key idea can be seen in [HY, Section 2].

We first recall the definition of tight closures.

Definition 1.2 (tight closure (cf. [HY, Section 1])). Put $R^\circ = R \setminus \cup_{P \in \mathrm{Min}(R)} P$, where $\mathrm{Min}(R)$ denotes the set of all minimal prime divisors of R. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Let $N \subseteq M$ be an R-modules. The $\mathfrak{a}^t$*-tight closure of* N *in* M, denoted by $N_M^{*\mathfrak{a}^t}$, is the submodule generated by all elements $z \in M$ for which there exists an element $c \in R^\circ$ such that $cz^{p^e}\mathfrak{a}^{\lceil tp^e \rceil} \subseteq N_M^{[p^e]}$ for all sufficiently large $e \gg 0$.

The *finitistic* $\mathfrak{a}^t$*-tight closure* $N_M^{*\mathrm{fg},\mathfrak{a}^t}$ is defined by the union of $N_{M'}^{*\mathfrak{a}^t}$ for finitely generated R-submodule M' of M. For an ideal I of R, we define $I^{*\mathfrak{a}^t} = I_R^{*\mathfrak{a}^t}$.

When $\mathfrak{a} = R$ and $t = 1$, $N_M^* = N_M^{*R}$ denotes the original tight closure introduced by Hochster and Huneke [HH2]. An R-submodule N is called *tightly closed* in M if $N_M^* = N$.

If $\mathfrak{b} \subseteq \mathfrak{a}$ are ideals of R such that $\mathfrak{b} \cap R^\circ \neq \emptyset$, then $N_M^{*\mathfrak{b}^t} \supseteq N_M^{*\mathfrak{a}^t}$ and equality holds if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$.

Proposition 1.3. *Assume that* R *is quasi-unmixed with* $d = \dim R \geq 1$. *Let* $a_1, \ldots, a_d$ *be a system of parameters of* R, *and put* $J = (a_1, \ldots, a_d)$. *Then we have*

$$\overline{J} = J^{*J^{d-1}}.$$

Proof. We first prove $\subseteq$. Suppose $z \in \overline{J}$. Then there exists an element $c \in R^{\circ}$ such that $cz^q \in J^q$ for all sufficiently large $q = p^e$ by the valuative criterion. This implies

$$cz^q(J^{d-1})^q \subseteq J^q J^{(d-1)q} = J^{dq} \subseteq J^{[q]}.$$

Hence $z \in J^{*J^{d-1}}$ by definition.

Next, we show the converse. Since $\overline{J} = \overline{J\widehat{R}} \cap R$ and $\overline{JR_{\mathrm{red}}} = \overline{J}R_{\mathrm{red}}$, we may assume that R is complete and reduced.

Now suppose that $z \in J^{*J^{d-1}}$. Then

$$cz^q \in J^{[q]} \colon J^{(d-1)q}.$$

If $a_1, \ldots, a_d$ were an R-sequence, then $J^{[q]} \colon J^{(d-1)q} = J^{[q]} + J^{q-d+1}$. Since R is quasi-unmixed and a homomorphic image of a Cohen-Macaulay local ring, by Colon-capturing property for tight closures, we have

$$J^{[q]} \colon J^{(d-1)q} \subseteq (J^{[q]} + J^{q-d+1})^* \subset \overline{J^{q-d+1}}.$$

Hence $cz^q \in \overline{J^{q-d+1}}$ and by valuative criterion, $z \in \overline{J}$, as required. □

Before stating the next proposition, we recall the definition of test ideals.

Definition 1.4 (test ideal (cf. [HY, HT])). Let $\mathfrak{a}$ be an ideal of R with $\mathfrak{a} \cap R^{\circ} \neq \emptyset$. Let $t \geq 0$ be a real number. Let $E := \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$, the direct sum, taken over all maximal ideal $\mathfrak{m}$ of R, of the injective envelope of the residue fields $R/\mathfrak{m}$. Then the $\mathfrak{a}^t$*-test ideal* (resp., *CS* $\mathfrak{a}^t$*-test ideal*) is defined by

$$\tau(\mathfrak{a}^t) = \bigcap_{M \subseteq E} \mathrm{Ann}_R((0)^{*\mathfrak{a}^t}_M), \qquad (\text{resp., } \widetilde{\tau}(\mathfrak{a}^t) = \mathrm{Ann}_R((0)^{*\mathfrak{a}^t}_E))$$

where M runs through all finitely generated R-submodules of E.

The following properties are easy to check; see also [HY, Proposition 1.11].

Lemma 1.5. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R, and let $t \geq 0$ be a real number. Then*

(1) $\tau(\mathfrak{a})\mathfrak{b} \subseteq \tau(\mathfrak{a}\mathfrak{b})$.
(2) *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\tau(\mathfrak{b}^t) \subseteq \tau(\mathfrak{a}^t)$. Equality holds if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$.*
(3) *$\widetilde{\tau}(\mathfrak{a}^t) \subseteq \tau(\mathfrak{a}^t)$. If R is Gorenstein, then equalty holds.*

The next proposition plays a key role in the proof of Theorem 1.1.

Proposition 1.6. *Assume that R is a complete Gorenstein local ring. Let $t \geq 0$ be a real number. Let $J \subseteq R$ be a parameter ideal and $\mathfrak{a} \subseteq R$ an ideal with $\mathfrak{a} \cap R^{\circ} \neq \emptyset$. Then we have*

$$J \colon J^{*\mathfrak{a}^t} = J + \tau(\mathfrak{a}^t).$$

Proof. We show the case of $t = 1$ only. Put $E = E_R(R/\mathfrak{m})$ and $J = (a_1, \ldots, a_d)$. Since R is Gorenstein, there exists an injection

$$R/J \hookrightarrow E = H^d_{\mathfrak{m}}(R) = \varinjlim R/(a_1^{\ell}, \ldots, a_d^{\ell}).$$

Then

$$J^{*\mathfrak{a}}/J = (0)^{*\mathfrak{a}}_{R/J} = [0\colon J]_E \cap (0)^{*\mathfrak{a}}_E.$$

As R is complete, $(0)^{*\mathfrak{a}}_E = [0\colon \tau(\mathfrak{a})]_E$ by Matlis duality. Hence

$$J^{*\mathfrak{a}}/J = [0 : J]_E \cap [0 : \tau(\mathfrak{a})]_E = \operatorname{Hom}_R(R/J + \tau(\mathfrak{a}), E).$$

By taking annihilators of both sides, we get

$$J\colon J^{*\mathfrak{a}} = \operatorname{Ann}_R \operatorname{Hom}_R(R/J + \tau(\mathfrak{a}), E) = J + \tau(\mathfrak{a}),$$

as required. □

Proof of Theorem 1.1. By Proposition 1.3, we have

$$J\colon \overline{J} = J\colon J^{*J^{d-1}}.$$

On the other hand, by Proposition 1.6, we get

$$J\colon J^{*J^{d-1}} = J + \tau(J^{d-1}).$$

Combining two equalities, we obtain the required formula. □

We can replace the completeness with the F-finite normality in the hypothesis of Theorem 1.1 because test ideals commutes with completion in that case.

Corollary 1.7. *Assume that $(R, \mathfrak{m})$ is an F-finite normal Gorenstein local domain. Let $J \subseteq R$ be a parameter ideal. Then*

$$J\colon \overline{J} = J + \tau(J^{d-1}).$$

Proof. First note that $J + \tau(J^{d-1}) \subseteq J\colon \overline{J}$ holds true in general. Indeed, we can see that $\tau(J^{d-1})\overline{J} = \tau(\overline{J}^{d-1})\overline{J} \subseteq \tau(\overline{J}^{d}) = \tau(J^d) \subseteq J$ by [HY, Proposition 1.11, Theorem 2.1]; see also the next section.

So it is enough to show the converse. The assumption implies that $\tau(J^{d-1})\widehat{R} = \tau((J\widehat{R})^{d-1})$ by [HY, Theorem 1.13] and [HT, Proposition 3.2]. On the other hand, since J is an $\mathfrak{m}$-primary ideal, we have that $\overline{J} = \overline{J\widehat{R}} \cap R$ and thus $\overline{J}\widehat{R} = \overline{J\widehat{R}}$. Then the assertion immediately follows from Theorem 1.1. □

Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k, and let I be a monomial ideal. For each monomial $M = x_1^{n_1} \cdots x_d^{n_d}$, its exponent vector is $(n_1, \dots, n_d) \in \mathbb{N}^d$. The convex hull in $\mathbb{R}^d$ of the set of all exponent vectors of monomials in I is called the *Newton Polyhedron* of I, and denoted by $P(I)$.

The integral closure $\overline{I}$ of a monomial ideal I is also a monomial ideal and it can be represented using the Newton Polyhedron; see e.g. [SH, Proposition 1.4.6]. That is,

$$x_1^{m_1} \cdots x_d^{m_d} \in \overline{I} \Longleftrightarrow (m_1, \dots, m_d) \in P(I).$$

Also, the test ideal $\tau(I^t)$ is a monomial ideal, and

$$x_1^{m_1} \cdots x_d^{m_d} \in \tau(\mathfrak{a}^t) \iff (m_1, \dots, m_d) + (1, \dots, 1) \in \operatorname{Int}(t \cdot P(I)),$$

where $\operatorname{Int}(t \cdot P(I)$ denotes the relative interior of $t \cdot P(I)$ in $\mathbb{R}^d$. This formula is well-known as *Howald-type theorem*; see [Ho] and [HY, Theorem 4.8].

In the case of monomial ideals, let us confirm our formula using Howald-type theorem.

Example 1.8. Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field of characteristic $p > 0$. Let $J = (x_1^{a_1}, \dots, x_d^{a_d})$. Take a monomial $M = x_1^{m_1} \cdots x_d^{m_d} \in J : \overline{J} \setminus J$. Then $m_i \le a_i - 1$ for each i. Set $N = x_1^{a_1 - m_1 - 1} \cdots x_d^{a_d - m_d - 1}$. Then N is the "largest" monomial among monomials which are *not* contained in $J : M$. As $\overline{J} \subseteq J : M$, we have $N \notin \overline{J}$. This yields $(a_1 - m_1 - 1, \dots, a_d - m_d - 1) \notin P(J)$, which means

$$\frac{a_1 - m_1 - 1}{a_1} + \cdots + \frac{a_d - m_d - 1}{a_d} < 1.$$

Hence

$$\frac{m_1 + 1}{a_1} + \cdots + \frac{m_d + 1}{a_d} > d - 1.$$

Therefore Howald-type theorem implies that $M \in \tau(J^{d-1})$, as required.

This argument implies that $J : \overline{J} \subseteq J + \tau(J^{d-1})$. The converse inclusion essentially follows from Skoda-type theorem in the next section.

2. Skoda-type theorem

Let R be a Noetherian ring of characteristic $p > 0$. If $\mathfrak{a} \subseteq R$ is an ideal generated by r elements, then $\tau(\mathfrak{a}^{n+r-1}) \subseteq \mathfrak{a}^n$ for every $n \ge 0$; see e.g. [HY, Theorem 2.1]. This is known as Skoda-type theorem.

In order to prove our main theorem, we need a refinement of this theorem. . Namely, we prove the following theorem.

Theorem 2.1 (Skoda-type). *Let $(R, \mathfrak{m}.k)$ be a complete local ring of characteristic $p > 0$ with $d = \dim R \ge 1$. Then the following statements hold:*

1. *If R is regular, then $\tau(\mathfrak{m}^t) = \mathfrak{m}^{t-d+1}$ holds for every $t \ge d - 1$.*
2. *If R is not regular, then $\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t-d+2}$ holds for every $t \ge d - 1$.*
3. *If R is Gorenstein with $e(R) \ge 3$, then $\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t-d+3}$.*

Without extra assumptions, the second statement is the best possible result as the next example shows.

Example 2.2. If $R = k[[x_0, x_1, \dots, x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2)$, where $d \ge 1$ and k is a field of characteristic $p > 2$, then $\tau(\mathfrak{m}^t) = \mathfrak{m}^{t-d+2}$ for every integer $t \ge d - 1$.

Proof of Theorem 2.1. Put $E = E_R(R/\mathfrak{m}) \cong H^d_{\mathfrak{m}}(R)$. By virtue of [HT, Theorem 4.1], it sufficces to show $\tau(\mathfrak{m}^{d-1}) \subseteq \mathfrak{m}^2$. Namely, it is enough to show

$$[0 : \mathfrak{m}^2]_E \subseteq (0)^{*\mathfrak{m}^{d-1}}_E.$$

We first show the following claim.

> **Claim:** For any $\xi \in [0 : \mathfrak{m}^2]_E$, we can take a minimal reduction J of $\mathfrak{m}$ such that $\xi \in [0 : J]_E$.

To show the claim, we may assume that $\xi \notin [0 : \mathfrak{m}]_E$. Then the image of ξ in

$$[0 : \mathfrak{m}^2]_E / [0 : \mathfrak{m}]_E \cong \mathrm{Hom}_R(\mathfrak{m}/\mathfrak{m}^2, E)$$

is nonzero. Notice that the right-hand side is a k-vector space of dimension $v \ge d+1$. A one-dimensional subspace $k\overline{\xi}$ corresponds to a homomorphic image $\mathfrak{m}/L$ of $\mathfrak{m}/\mathfrak{m}^2$, where L is an ideal with $\mathfrak{m}^2 \subseteq L \subseteq \mathfrak{m}$ and $\dim_k \mathfrak{m}/L = 1$. Then we can regard

$\xi \in [0\colon L]_E$. As $\dim_k L/\mathfrak{m}^2 = v - 1 \geq d$, we can choose a minimal reduction J of $\mathfrak{m}$ such that $J + \mathfrak{m}^2 \subseteq L$. Then J is the required ideal.

We now return to the proof. Fix a minimal reduction $J = (a_1, \ldots, a_d)$ such that $\xi \in [0\colon J]_E$. Recall that

$$E = H^d_{\mathfrak{m}}(R) = \varinjlim_e R/J^{[p^e]} = \{[b + J^{[q]}] \mid b \in R,\ q = p^e\}.$$

Moreover, we can regard the Frobenius map $F\colon E \to \mathbb{F}(E) \cong E$ as follows:

$$F([b + J^{[q]}]) = [b^p + J^{[qp]}].$$

Under this notation, we have $[0\colon J]_E = \{[b+J] \mid b \in R\}$. Write $\xi = [b + J]$ for some $b \in R$. Since $\xi \in [0\colon \mathfrak{m}^2]_E$, we get $\mathfrak{m}^2 b \subseteq J$. By the assumption that $e(R) \geq 3$, we have $\mathfrak{m}^2 \not\subseteq J$. Hence $b \in \mathfrak{m}$. If we put $\eta = [1 + J] \in [0\colon J]_E$, then $\xi = b\eta$.

Fix an integer $r \geq 0$ such that $\mathfrak{m}^{r+1} = J\mathfrak{m}^r$. Then for any power $q = p^e$, we have

$$\begin{aligned} J^r F^e(\xi) J^{(d-1)q} &\subseteq \mathfrak{m}^q J^{(d-1)q} J^r F^e(\eta) \\ &\subseteq \mathfrak{m}^r J^{dq} F^e(\eta) \\ &\subseteq \mathfrak{m}^r J^{[q]} F^e(\eta) = 0. \end{aligned}$$

This yields $\xi \in (0)^{*J^{d-1}}_E = (0)^{*\mathfrak{m}^{d-1}}_E$, as required. □

Let $\mathrm{gr}_{\mathfrak{m}}(R)$ denote the *associated graded ring* of R with respect to $\mathfrak{m}$. Moreover, let $R(\mathfrak{m}) = R[\mathfrak{m}t]$ denote the *Rees algebra* of R with respect to $\mathfrak{m}$. Theorem 2.1 tempts us to pose the following question.

Question 2.3. Let $G = \mathrm{gr}_{\mathfrak{m}}(R)$ denote the associated graded ring of R with respect to the maximal ideal $\mathfrak{m}$, and let $a(G)$ denote the *a-invariant* of G. Then does $\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t+a(G)+1}$ hold?

Example 2.4 ([HY, Proposition 5.8]). Assume that $G = \mathrm{gr}_{\mathfrak{m}}(R)$ is Gorenstein and $R(\mathfrak{m})$ is F-rational. Then $\tau(\mathfrak{m}^t) = \mathfrak{m}^{t+a(G)+1}$ for every $t \geq d-1$.

3. A variant of Wang's theorem in positive characteristic

Throughout this section, let $(R, \mathfrak{m}, k)$ be a d-dimensional Noetherian local ring of characteristic $p > 0$. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. In this section, we prove the main theorem.

Theorem 3.1. *Assume that R is a Cohen-Macaulay local ring with $d = \dim R \geq 2$. Let $s \geq 2$ be an integer, and let J be a parameter ideal with $J \subseteq \mathfrak{m}^s$. Then:*

(1) *If R is regular, then $J\colon \mathfrak{m}^{(d-1)(s-1)}$ is integral over J.*
(2) *If R is not regular, then $J\colon \mathfrak{m}^{(d-1)(s-1)+1}$ is integral over J.*

Corollary 3.2. *Under the same notations as in Theorem 3.1, we further assume that $R(\mathfrak{m})$ is normal and $d \geq 3$. Then for any parameter ideal J with $J \subseteq \mathfrak{m}^s$, we have*

$$J\colon \mathfrak{m}^s \subseteq J\colon \mathfrak{m}^{(d-1)(s-1)} \subseteq \mathfrak{m}^s.$$

In what follows, we prove Theorem 3.1. Notice the following remark.

Remark 3.3. In the proof of the theorem, we may assume that R is complete with infinite residue field. Indeed, if we put $S = \widehat{R}[X]_{\mathfrak{m}\widehat{R}[X]}$, then $R \to S$ is a faithfully flat extension, and $\overline{J} = \overline{JS} \cap R$ by [SH, Proposition 1.6.2]. If $JS\colon \mathfrak{m}^t S$ is integral over JS, then we have

$$J\colon \mathfrak{m}^t = (J\colon \mathfrak{m}^t)S \cap R = (JS\colon \mathfrak{m}^t S) \cap R \subseteq \overline{JS} \cap R = \overline{J}.$$

Proof of Theorem 3.1. By the above remark, we may assume that R is complete with infinite residue field.

Case 1: R is regular.

By Skoda-type theorem (Theorem 2.1) (1) and our formula (Theorem 1.1) we have

$$J : \mathfrak{m}^{(d-1)(s-1)} = J\colon \tau(\mathfrak{m}^{s(d-1)}) \subseteq J\colon [J + \tau(J^{d-1})] = J\colon [J\colon \overline{J}] = \overline{J},$$

where the last equality follows from the local duality theorem.

Case 2: R is Gorenstein but *not* regular.

By a similar argument as above, we have

$$J : \mathfrak{m}^{(d-1)(s-1)+1} \subseteq J\colon \tau(\mathfrak{m}^{s(d-1)}) \subseteq J\colon [J + \tau(J^{d-1})] = J\colon [J\colon \overline{J}] = \overline{J}.$$

Case 3: R is Cohen-Macaulay but *not* Gorenstein.

Let $S = R \ltimes \omega_R$ be the trivial extension of R, where ω_R denotes the canonical module of R. Put $\mathfrak{n} = \mathfrak{m} \ominus \omega_R$. Then $(S, \mathfrak{n})$ is a d-dimensional Gorenstein local ring with $e(S) \geq 3$. Moreover, $\mathfrak{m}^\ell S \subseteq \mathfrak{n}^\ell \subseteq \mathfrak{m}^{\ell-1}S$ for every $\ell \geq 1$. As J is a parameter ideal of R which is contained in $\mathfrak{m}^s$, JS is also a parameter ideal of S which is contained in $\mathfrak{n}^s$. Therefore, by Theorem 2.1(3), we have

$$(J\colon \mathfrak{m}^{(d-1)(s-1)+1})S \subseteq JS\colon \mathfrak{m}^{(d-1)(s-1)+1}S \subseteq JS\colon \mathfrak{n}^{(d-1)(s-1)+2} \subseteq \overline{JS}.$$

On the other hand, one can easily see that $\overline{JS} = \overline{J} \ominus \omega_R$. It follows that

$$J\colon \mathfrak{m}^{(d-1)(s-1)+1} \subseteq \overline{J},$$

as required. □

4. The case of equicharacteristic zero

In this section, we discuss a formula and a variant of Wang's theorem in the equicharacteristic zero case using so-called modulo p reduction.

We first recall the notion of $\mathbb{Q}$-Gorenstein rings. Let R be a Cohen-Macaulay normal local domain. The ring R is called *$\mathbb{Q}$-Gorenstein* if there exists a height one ideal ω that is isomorphic to a canonical module ω_R of R and such that $\omega^{(r)}$ is principal for some integer $r \geq 1$. Then the minimum integer $r \geq 1$ such that $\omega^{(r)}$ is principal is said to be the *index* of R.

We next recall the notion of multiplier ideals. Assume that R is a normal $\mathbb{Q}$-Gorenstein domain essentially of finite type over a field k of characteristic zero. Put $Y = \operatorname{Spec} R$. Let $\mathfrak{a} \subseteq \mathcal{O}_Y = R$ be a nonzero ideal sheaf. Let $f\colon X \to Y$ be a *log resolution* of the ideal $\mathfrak{a}$, that is, a resolution of singularities of Y such that the ideal sheaf $\mathfrak{a}\mathcal{O}_X$ is invertible (say, $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-D)$) for an effective divisor D on X, and that the union $\mathrm{Exc}(f) \cup \mathrm{Supp}(D)$ of the f-exceptional locus and the support of D is a simple normal cross divisor. Then

$$\mathcal{J}(\mathfrak{a}) = H^0(X, \mathcal{O}_X(\lceil K_X - f^*K_Y - D \rceil))$$

is, independent on the choice of a log resolution $f\colon X \to Y$ of $\mathfrak{a}$, is called the *multiplier ideal* of $\mathfrak{a}$. Notice that for any real number $t \geq 0$, $\mathcal{J}(t \cdot \mathfrak{a})$ can be defined.

We recall the following theorem.

Theorem 4.1 (See [HY, Theorems 3.4, 6.8]). *Let R be a normal $\mathbb{Q}$-Gorenstein local ring essentially of finite type over a field of characteristic zero, and let $\mathfrak{a}$ be a nonzero ideal of R. Then, after reduction to characteristic $p > 0$, we have $\tau(\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$.*

Note that $\mathcal{J}(J^{d-1}) \subseteq J\colon \overline{J}$ always holds true by Skoda's theorem , which follows from [HY, Theorem 2.14]. Thus, the following formula is obtained from Corollary 1.7 and Theorem 4.1.

Theorem 4.2 (Formula-zero). *Let $(R, \mathfrak{m})$ be a normal Gorenstein local domain essentially of finite type over a field of characteristic zero. Set $d = \dim R \geq 2$. For any parameter ideal J of R, we have*

$$J\colon \overline{J} = J + \mathcal{J}(J^{d-1}).$$

We want to prove an analogous result of Theorem 3.1. In the case of Gorenstein local rings, we can use Theorem 4.2 instead of Theorem 1.1. But, since the trivial extension is *not* reduced, we must use another method in the case of Cohen-Macaulay local rings. Namely, we use the so-called "canonical cover trick". Notice that the canonical cover of R is quasi-Gorenstein normal domain and it is module-finite over R (but not necessarily Cohen-Macaulay!).

Theorem 4.3 (Wang-type-zero). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay $\mathbb{Q}$-Gorenstein normal local domain essentially of finite type over a field of characteristic zero. Moreover, suppose that a canonical cover $S = \oplus_{i=0}^{r-1} \omega_R^{(i)}$ is Cohen-Macaulay (and thus Gorenstein). Put $d = \dim R \geq 2$. Let J be a parameter ideal of R with $J \subseteq \mathfrak{m}^s$. Then $J\colon \mathfrak{m}^{(d-1)(s-1)}$ is integral over J.*

5. A conjecture

Conjecture 5.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension $d \geq 2$. Let $s \geq 2$ be an integer. Let $J \subseteq R$ be a parameter ideal with $J \subseteq \mathfrak{m}^s$. Then $J\colon \mathfrak{m}^{(d-1)(s-1)}$ is integral over J.*

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