研究集会

第33回可換環論シンポジウム

The 33rd Symposium on Commutative Algebra in Japan

2011年11月7日 –11月10日

於 浜名湖カリアック

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序 (Preface)

この報告集に収録されている原稿は、第33回可換環論シンポジウムの講演の記録です。本研究集会は2011年11月7日（月）から11月10日（木）にかけて、浜名湖カリアック（静岡県浜松市）において開催されました。この研究集会には、国内(約70名)の研究者・大学院生の他、海外からも3名の研究者が参加し、合計22もの興味深い講演が行われました。特に、Gregor Kemper 氏には震災後の不安定な状況にも関わらず快く招待講演を引き受けて下さり, 大変感謝致しております。また、限られた時間の中で素晴らしい講演をしていただいた講演者の皆様と、研究集会の運営に協力していただいた大学院生の皆様には、この場を借りて感謝したいと思っております。

シンポジウム開催にあたり、下記の援助を受けました。吉野雄二先生には感謝しております。事務手続きに携わった方々にもこの場を借りて感謝の気持ちを表したいと思います。

- 平成23年度科学研究費基盤研究（B） 研究課題番号 21340008 (研究代表者：吉野雄二)
  「三角圏の研究とその Cohen-Macaulay 加群への応用」
- 平成23年度科学研究費基盤研究（C）研究課題番号 22540046 (研究代表者：橋本光靖)
  「同変層を用いた不変式論」
- 平成23年度科学研究費基盤研究（C）研究課題番号 22540047 (研究代表者：吉田健一)
  「イデアルのべきに付随する環論的不変量の研究」

2012年1月

名古屋大学　橋本光靖
吉田健一

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  http://www.curreac.co.jp/
- 世話人：橋本光靖・吉田健一　（名大・多元数理）

11月7日(月)

14:55 ~ 15:00　あいさつと諸注意

15:00 ~ 15:50　西田 康二 (千葉大学)
"On a method for computing symbolic powers of ideals"

16:05 ~ 16:45　成 博勝 (日本大学)
"Symmetries on almost symmetric numerical semigroups"

17:00 ~ 17:50　木村 杏子 (静岡大学), 寺井 直樹 (佐賀大学), 吉田 健一 (名古屋大学)
"Licci monomial ideals"

11月8日(火)

9:10 ~ 10:00　後藤 四郎 (明治大学), 高橋 亮 (信州大学), 大関 一秀 (明治大学)
"Ulrich 加群の一般化"

10:15 ~ 10:55　早坂 太 (鹿児島高専)
"Asymptotic periodicity of primes associated to multigraded modules"

11:10 ~ 11:50　渡辺 敬一 (日本大学), 吉田 健一 (名古屋大学)
"トーリック環上の F-thresholds"

- 昼食 -

13:30 ~ 14:30　Gregor Kemper (Technische Universität München)
"Depth of invariant rings and wild ramification"

14:45 ~ 15:15　衛藤 和文 (日本工業大学)
"Monomial curves in affine four space"

15:30 ~ 16:00　松田 一徳 (名古屋大学)
"Weakly closed graph"

16:20 ~ 17:10　大関 一秀 (明治大学)
"The equality of Elias and Valla and Buchsbaumness of associated graded rings"

17:25 ~ 17:55　堀内 淳 (明治大学)
"Stability of quasi-socle ideals"

11 月 9 日 (水)

9:10 ~ 9:50 渡辺 敬一 (日本大学)
"Upper bound of multiplicity of F-rational rings and F-pure rings"

10:05 ~ 10:55 後藤 四郎 (明治大学), 松岡 直之 (明治大学), Tran Thi Phuong (Ton Duc Thang Univ.)
"Almost Gorenstein rings"

11:10 ~ 11:50 Pham Hung Quy (FPT Univ.)
"Some results on the finiteness of associated primes of local cohomology"

- 昼食 -

13:30 ~ 14:30 Gregor Kemper (Technische Universität München)
"The transcendence degree over a ring"

14:45 ~ 15:15 渋田 敬史 (立教大学)
"An algorithm for computing the value-semigroup of an irreducible algebroid curve"

15:30 ~ 16:10 東谷 章弘 (大阪大学)
"Roots of Ehrhart polynomials and symmetric δ-vectors"

16:30 ~ 17:10 荒谷 督司 (徳山高専), 飯間 圭一郎 (奈良高専)
"On the Auslander-Bridger type approximation of modules"

17:25 ~ 17:55 吉澤毅 (岡山大学)
"Subcategories of extension modules by Serre subcategories"

19:00 ~ 21:00 懇親会

11 月 10 日 (木)

9:00 ~ 9:30 宮崎 充弘 (京都教育大学)
"A note on Cohen-Macaulay algebras with straightening law"

9:45 ~ 10:25 岡崎 亮太 (大阪大学), 柳川 浩二 (関西大学)
"Alternative polarizations of Borel fixed ideals and Eliahou-Kervaire type resolution"

10:40 ~ 11:20 藤野 修 (京都大学), 高木 俊輔 (東京大学)
"F-purity of isolated log canonical singularities"

The 33rd Symposium on Commutative Algebra in Japan

Monday, November 7

14:55 ∼ 15:00	Opening
15:00 ∼ 15:50	Koji Nishida (Chiba Univ.) *"On a method for computing symbolic powers of ideals"*
16:05 ∼ 16:45	Hirokatsu Nari (Nihon Univ.) *"Symmetries on almost symmetric numerical semigroups"*
17:00 ∼ 17:50	Kyouko Kimura (Shizuoka Univ.), Naoki Terai (Saga Univ.) and Ken-ichi Yoshida (Nagoya Univ.) *"Licci monomial ideals"*

Tuesday, November 8

9:10 ∼ 10:00	Shiro Goto (Meiji Univ.), Ryo Takahashi (Shinsyu Univ.) and Kazuho Ozeki (Meiji Univ.) *"Ulrich modules - a generalization"*
10:15 ∼ 10:55	Futoshi Hayasaka (Kagoshima National College of Technology) *"Asymptotic periodicity of primes associated to multigraded modules"*
11:10 ∼ 11:50	Kei-ichi Watanabe (Nihon Univ.) and Ken-ichi Yoshida (Nagoya Univ.) *"F-thresholds on toric rings"*
	- Lunch time -
13:30 ∼ 14:30	Gregor Kemper (Technische Universität München) *"Depth of invariant rings and wild ramification"*
14:45 ∼ 15:15	Kazufumi Eto (Nippon Inst. Tech.) *"Monomial curves in affine four space"*
15:30 ∼ 16:00	Kazunori Matsuda (Nagoya Univ.) *"Weakly closed graph"*
16:20 ∼ 17:10	Kazuho Ozeki (Meiji Univ.) *"The equality of Elias and Valla and Buchsbaumness of associated graded rings"*
17:25 ∼ 17:55	Jun Horiuchi (Meiji Univ.) *"Stability of quasi-socle ideals"*

Wednesday, November 9

9:10 ∼ 9:50	Kei-ichi Watanabe (Nihon Univ.) "*Upper bound of multiplicity of F-rational rings and F-pure rings*"
10:05 ∼ 10:55	Shiro Goto (Meiji Univ.), Naoyuki Matsuoka (Meiji Univ.) and Tran Thi Phuong (Ton Duc Thang Univ.) "*Almost Gorenstein rings*"
11:10 ∼ 11:50	Pham Hung Quy (FPT Univ.) "*Some results on the finiteness of associated primes of local cohomology*"
	- Lunch time -
13:30 ∼ 14:30	Gregor Kemper (Technische Universität München) "*The transcendence degree over a ring*"
14:45 ∼ 15:15	Takafumi Shibuta (Rikkyo Univ.) "*An algorithm for computing the value-semigroup of an irreducible algebroid curve*"
15:30 ∼ 16:10	Akihiro Higashitani (Osaka Univ.) "*Roots of Ehrhart polynomials and symmetric δ-vectors*"
16:30 ∼ 17:10	Tokuji Araya (Tokuyama College of Technology) and Kei-ichiro Iima (Nara National College of Technology) "*On the Auslander-Bridger type approximation of modules*"
17:25 ∼ 17:55	Takeshi Yoshizawa (Okayama Univ.) "*Subcategories of extension modules by Serre subcategories*"
19:00 ∼ 21:00	Banquet

Thursday, November 10

9:00 ∼ 9:30	Mitsuhiro Miyazaki (Kyoto Univ. of Education) "*A note on Cohen-Macaulay algebras with straightening law*"
9:45 ∼ 10:25	Ryota Okazaki (Osaka Univ.) and Kohji Yanagawa (Kansai Univ.) "*Alternative polarizations of Borel fixed ideals and Eliahou-Kervaire type resolution*"
10:40 ∼ 11:20	Osamu Fujino (Kyoto Univ.) and Shunsuke Takagi (Tokyo Univ.) "*F-purity of isolated log canonical singularities*"

On a method for computing symbolic powers of ideals

Koji Nishida (Chiba University)

1 Introduction

This is a joint work with K. Fukumuro and T. Inagawa. The purpose of this report is to introduce a new method for computing symbolic powers. Let us assume one of the following two cases:

Local case : R is a 3-dimensional regular local ring with the maximal ideal $\mathfrak{m} = (x, y, z)R$ and $\mathfrak{p}$ is a prime ideal of $\mathrm{ht}_R\,\mathfrak{p} = 2$.

Graded case : $R = K[x, y, z]$ is a polynomial ring over a field K. Putting suitable weights on each variables, we regard R as a graded ring. In this case $\mathfrak{p}$ is a homogeneous prime ideal of $\mathrm{ht}_R\,\mathfrak{p} = 2$.

The typical example of such prime ideal is the defining ideal of a space monomial curve, which is described as follows: Let $R = K[x, y, z]$ and $K[t]$ be polynomial rings over a field K. Let us consider the homomorphism $R \to K[t]$ of K-algebras such that $x \mapsto t^k$, $y \mapsto t^\ell$, $z \mapsto t^m$, where k, ℓ, m are positive integers with $\mathrm{GCD}\{k, \ell, m\} = 1$. We denote by $\mathrm{P}(k, \ell, m)$ the kernel of this homomorphism. Putting $\deg x = k$, $\deg y = \ell$ and $\deg z = m$, we regard R as a graded ring. Then $\mathrm{P}(k, \ell, m)$ is a homogeneous prime ideal generated by the maximal minors of the matrix of the form

$$\begin{pmatrix} x^\alpha & y^{\beta'} & z^{\gamma'} \\ y^\beta & z^\gamma & x^{\alpha'} \end{pmatrix},$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers.

Let us recall the definition of symbolic power of a prime ideal $\mathfrak{p}$.

$$\begin{aligned} \mathfrak{p}^{(n)} &:= \mathfrak{p}^n R_\mathfrak{p} \cap R \\ &= \{ a \in R \mid \exists s \in R \setminus \mathfrak{p} \text{ such that } sa \in I^n \}. \end{aligned}$$

We would like to find an effective method to compute $\mathfrak{p}^{(n)}$.

2 The usual method

Let us recall the usual (traditional) method used by Huneke, Schenzel, Morales, Shimoda, Goto, $\cdots$. As $\mathrm{Ass}_R(R/\mathfrak{p}^{(n)}) = \{ \mathfrak{p} \}$, $R/\mathfrak{p}^{(n)}$ is a 1-dimensional Cohen-Macaulay R-module, and for any $w \in R \setminus \mathfrak{p}$, we have

$$\begin{aligned} \ell_R(R/wR + \mathfrak{p}^{(n)}) &= \mathrm{e}_{wR}(R/\mathfrak{p}^{(n)}) \\ &= \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) \cdot \mathrm{e}_{wR}(R/\mathfrak{p}) \\ &= \binom{n+1}{2} \cdot \ell_R(R/wR + \mathfrak{p}). \end{aligned}$$

Theorem 2.1 *Let $0 < n \in \mathbb{Z}$. Let $\mathfrak{a}$ be an ideal such that $\mathfrak{a} \subseteq \mathfrak{p}^{(n)}$ and $\ell_R(R/wR + \mathfrak{a}) = \ell_R(R/wR + \mathfrak{p}^{(n)})$ for some $w \in R \setminus \mathfrak{p}$. Then $\mathfrak{a} = \mathfrak{p}^{(n)}$.*

Proof. The assumption means $wR + \mathfrak{a} = wR + \mathfrak{p}^{(n)}$, and so

$$\mathfrak{p}^{(n)} = (wR + \mathfrak{a}) \cap \mathfrak{p}^{(n)} = [wR \cap \mathfrak{p}^{(n)}] + \mathfrak{a} = w\mathfrak{p}^{(n)} + \mathfrak{a}.$$

Hence we get $\mathfrak{p}^{(n)} = \mathfrak{a}$ by Nakayama's lemma.

Let us apply Theorem 2.1 to a concrete example.

Example 2.2 *Let $R = K[x, y, z]$, where K is a field. Let $\mathfrak{p} = \mathrm{P}(9, 10, 13)$, which is generated by the maximal minors of the matrix*

$$\begin{pmatrix} x^3 & y^3 & z^2 \\ y & z & x \end{pmatrix}.$$

Then the following assertions hold.

(1) *$\exists d_2 \in \mathfrak{p}^{(2)}$ such that $d_2 \equiv y^7 z$ mod xR and $\mathfrak{p}^{(2)} = \mathfrak{p}^2 + d_2 R$.*

(2) *If $\mathrm{ch}\, K \neq 2$, then $\exists d_3, \exists d_3', \exists d_3'' \in \mathfrak{p}^{(3)}$ such that $d_3 \equiv 2y^7 z^3$ mod xR, $d_3' \equiv y^{11}$ mod xR, $d_3'' \equiv y^{10} z$ mod xR, and $\mathfrak{p}^{(3)} = \mathfrak{p}\mathfrak{p}^{(2)} + (d_3, d_3', d_3'')R$.*

(3) *If $\mathrm{ch}\, K = 2$, then $\exists e \in \mathfrak{p}^{(3)}$ such that $e \equiv y^{10}$ mod xR and $\mathfrak{p}^{(3)} = \mathfrak{p}\mathfrak{p}^{(2)} + eR$.*

Proof of Example 2.2. We put $a := z^3 - xy^3, b := x^4 - yz^2, c := y^4 - x^3 z$. Then $\mathfrak{p} = (a, b, c)R$ and $xR + \mathfrak{p} = xR + (y^4, yz^2, z^3)R$. Hence $\ell_R(R/xR + \mathfrak{p}) = 9$, and so

$$\ell_R(R/xR + \mathfrak{p}^{(n)}) = 9\binom{n+1}{2}.$$

In particular, $\ell_R(R/xR + \mathfrak{p}^{(2)}) = 27$. On the other hand, as

$$xR + \mathfrak{p}^2 = xR + (y^8, y^5 z^2, y^4 z^3, y^2 z^4, yz^5, z^6)R,$$

we have $\ell_R(R/xR + \mathfrak{p}^2) = 28$. Therefore $\mathfrak{p}^2 \subsetneq \mathfrak{p}^{(2)}$ and we need a "new" element in $\mathfrak{p}^{(2)}$. We have the following two relations:

$$x^3 a + y^3 b + z^2 c = 0 \cdots (1) \quad \text{and} \quad ya + zb + xc = 0 \cdots (2).$$

By (1) $\times\, b$ we get $x^3ab + y^3b^2 + z^2bc = 0$. On the other hand, by (2) $\times\, zc$ we get $yzac + z^2bc + xzc^2 = 0$. Hence $x^3ab + y^3b^2 = yzac + xzc^2$, and so $y(y^2b^2 - zac) = x(zc^2 - x^2ab)$. This means

$$\exists d_2 \in R \text{ such that } \begin{cases} yd_2 = zc^2 - x^2ab \cdots (3) \\ xd_2 = y^2b^2 - zac \cdots (4). \end{cases}$$

By (3), we have $d_2 \in \mathfrak{p}^{(2)}$ and $d_2 \equiv y^7z \bmod xR$. Then

$$xR + \mathfrak{p}^2 + d_2R = xR + (y^8, y^7z, y^5z^2, y^4z^3, y^2z^4, yz^5, z^6)R.$$

Therefore $\ell_R(R/xR + \mathfrak{p}^2 + d_2R) = 27$. Thus we get $\mathfrak{p}^{(2)} = \mathfrak{p}^2 + d_2R$.

Let us compute $\mathfrak{p}^{(3)}$ in the case where $\operatorname{ch} K \neq 2$. Notice that $\mathfrak{p}\mathfrak{p}^{(2)} \subseteq \mathfrak{p}^{(3)}$ and

$$xR + \mathfrak{p}\mathfrak{p}^{(2)} = xR + \begin{pmatrix} y^{12}, & y^{11}z, & y^9z^2, & y^8z^3, & y^6z^4, \\ y^5z^5, & y^3z^6, & y^2z^7, & yz^8, & z^9 \end{pmatrix}.$$

Hence we get $\ell_R(R/xR + \mathfrak{p}\mathfrak{p}^{(2)}) = 69$. On the other hand, as

$$\ell_R(R/xR + \mathfrak{p}^{(3)}) = 9\binom{4}{2} = 54,$$

we see $xR + \mathfrak{p}\mathfrak{p}^{(2)} \subsetneq \mathfrak{p}^{(3)}$, and so we need "new" elements in $\mathfrak{p}^{(3)}$. By (2) $\times\, d_2$ we get $yad_2 + zbd_2 + xcd_2 = 0$. Moreover, by (4) $\times\, c$ we get $xcd_2 = y^2b^2c - zac^2$. Hence $yad_2 + zbd_2 = zac^2 - y^2b^2c$, and so $y(ad_2 + yb^2c) = z(ac^2 - bd_2)$. This means that there exists $d_3 \in R$ such that $yd_3 = ac^2 - bd_2$, which implies $d_3 \in \mathfrak{p}^{(3)}$ and $d_3 \equiv 2y^7z^3 \bmod xR$. Similarly, from (2) $\times\, d_2$ and (4) $\times\, a$, we get $d_3' \in R$ such that $zd_3' = cd_2 - xa^2b$. Then $d_3' \in \mathfrak{p}^{(3)}$ and $d_3' \equiv y^{11} \bmod xR$. Furthermore we can find $d_3'' \in \mathfrak{p}^{(3)}$ such that $d_3'' \equiv y^{10}z \bmod xR$. Put $\mathfrak{a} := \mathfrak{p}\mathfrak{p}^{(2)} + (d_3, d_3', d_3'')R \subseteq \mathfrak{p}^{(3)}$. Then

$$xR + \mathfrak{a} = xR + \begin{pmatrix} y^{11}, & y^{10}z, & y^9z^2, & y^7z^3, & y^6z^4, \\ y^5z^5, & y^3z^6, & y^2z^7, & yz^8, & z^9 \end{pmatrix} R,$$

and so $\ell_R(R/xR + \mathfrak{a}) = 54 = \ell_R(R/xR + \mathfrak{p}^{(3)})$. Thus we get $\mathfrak{a} = \mathfrak{p}^{(3)}$.

Finally we compute $\mathfrak{p}^{(3)}$ in the case where $\operatorname{ch} K = 2$. Taking the square of (2), we have

$$y^2a^2 + z^2b^2 + x^2c^2 = 0 \cdots (2').$$

By (2′) $\times\, c$, we get $y^2a^2c + z^2b^2c + x^2c^3 = 0$. Moreover, by (1) $\times\, b^2$, we get $x^3ab^2 + y^3b^3 + z^2b^2c = 0$. Hence $x^3ab^2 + y^3b^3 = y^2a^2c + x^2c^3$, and so $y^2(yb^3 - a^2c) = x^2(c^3 - xab^2)$. This means that there exists $e_3 \in R$ such that $y^2e_3 = c^3 - xab^2$. Then $e_3 \in \mathfrak{p}^{(3)}$ and $e_3 \equiv y^{10} \bmod xR$. Hence

$$xR + \mathfrak{p}^3 + e_3R = xR + \begin{pmatrix} y^{10}, & y^9z^2, & y^8z^3, & y^6z^4, & y^5z^5, \\ y^3z^6, & y^2z^7, & yz^8, & z^9 & \end{pmatrix}.$$

Therefore $\ell_R(R/xR + \mathfrak{p}^3 + e_3R) = 54 = \ell_R(R/xR + \mathfrak{p}^{(3)})$, and we get $\mathfrak{p}^{(3)} = \mathfrak{p}^3 + e_3R$.

Here, we summarize some weak points of the usual method.

- Finding "new" elements is difficult. It is not clear which relation should be used. Furthermore we may miss very important relations. Some systematic way is desired.
- "New" elements may not be congruent to monomials modulo any variables. In such case, computing the length is quite difficult.
- We can not see the exact value of $\ell_R(\mathfrak{p}^{(n)}/\mathfrak{p}^n)$ directly.

So, let us look for another method !

3 $*$-transform of acyclic complex

In the rest of this report, we work in the local case. However please keep in mind that a parallel theory holds in the graded case. This section is devoted to preparing a tool that plays a key role in the new method for computing symbolic powers.

Let x_1, x_2, x_3 be an sop for R. We put $Q = (x_1, x_2, x_3)R$. Suppose that an acyclic complex

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = R$$

of finitely generated R-free modules are given. We assume $\operatorname{Im}\varphi_3 \subseteq QF_2$ and put $\mathfrak{a} = \operatorname{Im}\varphi_1$. Transforming $F_\bullet$ suitably, we are going to construct an acyclic complex

$$0 \longrightarrow F_3^* \xrightarrow{\varphi_3^*} F_2^* \xrightarrow{\varphi_2^*} F_1^* \xrightarrow{\varphi_1^*} F_0^* = R$$

of finitely generated R-free modules such that $\operatorname{Im}\varphi_3^* \subseteq \mathfrak{m}F_2^*$ and $\operatorname{Im}\varphi_1^* = \mathfrak{a} :_R Q$.

We transform $F_\bullet$ using the Koszul complex

$$0 \longrightarrow K_3 \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 = R$$

of x_1, x_2, x_3. Let e_1, e_2, e_3 be an R-free basis of K_1. We set $\check{e}_1 = e_2 \wedge e_3$, $\check{e}_2 = e_1 \wedge e_3$ and $\check{e}_3 = e_1 \wedge e_2$. Then $\check{e}_1, \check{e}_2, \check{e}_3$ is an R-free basis of K_2. Furthermore $e_1 \wedge e_2 \wedge e_3$ is an R-free basis of K_3. The boundary maps satisfy the following:

$$\begin{aligned} \partial_3(e_1 \wedge e_2 \wedge e_3) &= x_1\check{e}_1 - x_2\check{e}_2 + x_3\check{e}_3\,, \\ \partial_2(e_i \wedge e_j) &= x_ie_j - x_je_i \quad \text{if } 1 \le i < j \le 3\,, \\ \partial_1(e_i) &= x_i \quad \text{for any } i = 1, 2, 3\,. \end{aligned}$$

At first, let us notice that we immediately get the following fact from the given complex $F_\bullet$.

Lemma 3.1 $[\,\mathfrak{a} :_R Q\,]/\mathfrak{a} \cong F_3/QF_3$

Let $\{w_\lambda\}_{\lambda\in\Lambda}$ be an R-free basis of F_3. Let $\{v_\lambda^i \mid 1\le i\le 3,\ \lambda\in\Lambda\}$ be a family of elements in F_2 such that

$$\varphi_3(w_\lambda)=\sum_{i=1}^{3} x_i\cdot v_\lambda^i$$

for any $\lambda\in\Lambda$. We set $\tilde{\Lambda}=\{1,2,3\}\times\Lambda$. The next result is the essential part among the construction of the desired complex $F_\bullet^*$.

Lemma 3.2 *There exists a chain map $\sigma_\bullet : K_\bullet\otimes_R F_3 \longrightarrow F_\bullet$*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_3\otimes_R F_3 & \xrightarrow{\partial_3\otimes\mathrm{id}} & K_2\otimes_R F_3 & \xrightarrow{\partial_2\otimes\mathrm{id}} & K_1\otimes_R F_3 & \xrightarrow{\partial_1\otimes\mathrm{id}} & K_0\otimes_R F_3\\
 & & \downarrow\sigma_3 & & \downarrow\sigma_2 & & \downarrow\sigma_1 & & \downarrow\sigma_0\\
0 & \longrightarrow & F_3 & \xrightarrow{\varphi_3} & F_2 & \xrightarrow{\varphi_2} & F_1 & \xrightarrow{\varphi_1} & F_0
\end{array}$$

satisfying the following conditions.

(1) ${\sigma_0}^{-1}(\mathrm{Im}\,\varphi_1)\subseteq \mathrm{Im}\,(\partial_1\otimes\mathrm{id}_{F_3})$.

(2) $\mathrm{Im}\,\sigma_0+\mathrm{Im}\,\varphi_1=\mathfrak{a}:_R Q$.

(3) $\sigma_2(\check{e}_i\otimes w_\lambda)=(-1)^i v_\lambda^i$ *for any* $(i,\lambda)\in\tilde{\Lambda}$.

(4) $\sigma_3((e_1\wedge e_2\wedge e_3)\otimes w_\lambda)=-w_\lambda$ *for any* $\lambda\in\Lambda$.

In the rest, let $\sigma_\bullet : K_\bullet\otimes_R F_3\longrightarrow F_\bullet$ be the chain map stated in Lemma 3.2. Then, taking the mapping cone of $\sigma_\bullet$, we get the following acyclic complex

$$0\to K_3\otimes F_3\xrightarrow{\psi_4}\begin{array}{c}K_2\otimes F_3\\ \oplus\\ F_3\end{array}\xrightarrow{\psi_3}\begin{array}{c}K_1\otimes F_3\\ \oplus\\ F_2\end{array}\xrightarrow{\varphi_2'}\begin{array}{c}K_0\otimes F_3\\ \oplus\\ F_1\end{array}\xrightarrow{\varphi_1^*}F_0=R,$$

where

$$\psi_4=\begin{pmatrix}\partial_3\otimes\mathrm{id}\\ -\sigma_3\end{pmatrix},\ \psi_3=\begin{pmatrix}\partial_2\otimes\mathrm{id} & 0\\ \sigma_2 & \varphi_3\end{pmatrix},\ \varphi_2'=\begin{pmatrix}\partial_1\otimes\mathrm{id} & 0\\ -\sigma_1 & \varphi_2\end{pmatrix},\ \varphi_1^*=(\sigma_0\ \ \varphi_1).$$

Notice that $\mathrm{Im}\,\varphi_1^*=\mathfrak{a}:_R Q$. On the other hand, as σ_3 is an isomorphism, ψ_4 splits. Hence we get the next acyclic complex

$$0\longrightarrow F_3'\xrightarrow{\varphi_3'}F_2'\xrightarrow{\varphi_2'}F_1^*\xrightarrow{\varphi_1^*}F_0^*=R,$$

where

$$F_3'=K_2\otimes F_3,\ F_2'=\begin{array}{c}K_1\otimes F_3\\ \oplus\\ F_2\end{array},\ F_1^*=\begin{array}{c}K_0\otimes F_3\\ \oplus\\ F_1\end{array},\ \varphi_3'=\begin{pmatrix}\partial_2\otimes\mathrm{id}\\ \sigma_2\end{pmatrix}.$$

Although $\mathrm{Im}\,\varphi_3'\not\subseteq\mathfrak{m}F_2'$ may happen, but removing unnecessary free components from F_3' and F_2', we can get suitable R-free submodules F_3^* and F_2^*, for which $\varphi_3'(F_3^*)\subseteq\mathfrak{m}F_2^*$ holds. In the rest of this section, we describe a concrete procedure to get F_3^* and F_2^*. Here we need to define some notation.

- For any $\xi \in K_1 \otimes F_3$, $[\xi] := \begin{pmatrix} \xi \\ 0 \end{pmatrix} \in F_2' = \begin{matrix} K_1 \otimes F_3 \\ \oplus \\ F_2 \end{matrix}$.

  In particular, for any $(i, \lambda) \in \tilde{\Lambda}$, we set $[i, \lambda] = [e_i \otimes w_\lambda]$.

- For any $\eta \in F_2$, $\langle \eta \rangle := \begin{pmatrix} 0 \\ \eta \end{pmatrix} \in F_2'$.

- Let T be a subset of an R-module. We denote by $R \cdot T$ the R-submodule generated by T .

Let us choose a subset Λ' of Λ as big as possible so that $\{ v_\lambda^i \mid (i, \lambda) \in \Lambda' \}$ form a part of an R-free basis of F_2 . The following fact is convenient when we find Λ' .

Lemma 3.3 *Let V be an R-free basis of F_2 . If a subset Λ' of Λ and a subset V^* of V satisfy*

- $|\Lambda'| + |V^*| \leq \operatorname{rank} F_2$,
- $V \subseteq R \cdot \{ v_\lambda^i \mid (i, \lambda) \in \Lambda' \} + R \cdot V^* + \mathfrak{m} F_2$,

then $\{ v_\lambda^i \mid (i, \lambda) \in \Lambda' \} \cup V^*$ is an R-free basis of F_2 .

Let V^* be a subset of F_2 such that $\{ v_\lambda^i \mid (i, \lambda) \in \Lambda' \} \cup V^*$ is an R-free basis of F_2 . Then

$$\{ [i, \lambda] \mid (i, \lambda) \in \tilde{\Lambda} \} \cup \{ \langle v_\lambda^i \rangle \mid (i, \lambda) \in \Lambda' \} \cup \{ \langle v \rangle \mid v \in V^* \}$$

is an R-free basis of F_2' . We set $F_2^* = R \cdot \{ [i, \lambda] \mid (i, \lambda) \in \tilde{\Lambda} \} + R \cdot \{ \langle v \rangle \mid v \in V^* \}$ which is a free summand of F_2' . Furthermore we set $\varphi_2^* = \varphi_2'|_{F_2^*}$. We need the next result at the final step in the process of computing symbolic powers.

Theorem 3.4 *If we can take $\tilde{\Lambda}$ itself as Λ', then*

$$0 \longrightarrow F_2^* \xrightarrow{\varphi_2^*} F_1^* \xrightarrow{\varphi_1^*} F_0^* = R$$

is acyclic. In particular, $\operatorname{depth} R/\mathfrak{a} :_R Q > 0$.

So, in the rest, we consider the case where $\Lambda' \subsetneq \tilde{\Lambda}$. We set $\Lambda^* = \tilde{\Lambda} \setminus \Lambda'$. If Λ' is big enough, then, for any $(j, \mu) \in \Lambda^*$, v_μ^j can be expressed as follows:

$$v_\mu^j = \sum_{(i,\lambda) \in \tilde{\Lambda}} a_{(i,\lambda)}^{(j,\mu)} \cdot v_\lambda^i + \sum_{v \in V^*} b_v^{(j,\mu)} \cdot v \, ,$$

where

$$a_{(i,\lambda)}^{(j,\mu)} \in \begin{cases} R & \text{if } (i, \lambda) \in \Lambda' \\ \mathfrak{m} & \text{if } (i, \lambda) \in \Lambda^* \end{cases} \quad \text{and} \quad b_v^{(j,\mu)} \in \mathfrak{m} \text{ for any } v \in V^* .$$

For any $(j, \mu) \in \Lambda^*$, we set

$$w_{(j,\mu)}^* = (-1)^j \cdot \check{e}_j \otimes w_\mu + \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} \cdot a_{(i,\lambda)}^{(j,\mu)} \cdot \check{e}_i \otimes w_\lambda \in F_3' .$$

Then we have the next result.

Lemma 3.5 *The following assertions hold.*

(1) $\{ \check{e}_i \otimes w_\lambda \mid (i,\lambda) \in \Lambda' \} \cup \{ w^*_{(j,\mu)} \mid (j,\mu) \in \Lambda^* \}$ *is an R-free basis of F'_3.*

(2) *For $\forall (j,\mu) \in \Lambda^*$, we have*

$$\begin{aligned} \varphi'_3(w^*_{(j,\mu)}) &= (-1)^j \cdot [(\partial_2 \check{e}_j) \otimes w_\mu] + \sum_{(i,\lambda)\in\tilde{\Lambda}} (-1)^{i+1} \cdot a^{(j,\mu)}_{(i,\lambda)} \cdot [(\partial_2 \check{e}_i) \otimes w_\lambda] \\ &\quad + \sum_{v \in V^*} b^{(j,\mu)}_v \cdot \langle v \rangle , \end{aligned}$$

and so $\varphi'_3(w^*_{(j,\mu)}) \in \mathfrak{m} F^*_2$.

We set $F^*_3 = R \cdot \{ w^*_{(j,\mu)} \mid (j,\mu) \in \Lambda^* \}$ which is a free summand of F'_3. Furthermore we set $\varphi^*_3 = \varphi'_3|_{F^*_3}$. Then we get the next result.

Theorem 3.6 $0 \longrightarrow F^*_3 \xrightarrow{\varphi^*_3} F^*_2 \xrightarrow{\varphi^*_2} F^*_1 \xrightarrow{\varphi^*_1} F^*_0 = R$ *is an acyclic complex of finitely generated R-free modules such that* $\operatorname{Im} \varphi^*_3 \subseteq \mathfrak{m} F^*_2$ *and* $\operatorname{Im} \varphi^*_1 = \mathfrak{a} :_R Q$.

Let us call the procedure to construct $F^*_\bullet$ from $F_\bullet$ the $*$-transform on x_1, x_2, x_3.

4 Applications of $*$-transform

Let $\mathfrak{p}$ be the prime ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} x^3 & y^3 & z^2 \\ y & z & x \end{pmatrix}.$$

We would like to compute $\mathfrak{p}^{(3)}$ using $*$-transform. Recall $a = z^3 - xy^3$, $b = x^4 - yz^2$, $c = y^4 - x^3 z$. Here we take new variables A, B, C and put $S = R[A,B,C]$. For any $0 \leq n \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{m}^n_{A,B,C} &:= \{ A^\alpha B^\beta C^\gamma \mid 0 \leq \alpha, \beta, \gamma \in \mathbb{Z} \text{ and } \alpha + \beta + \gamma = n \} \\ S_n &:= R \cdot \mathrm{m}^n_{A,B,C} \subseteq S . \end{aligned}$$

Furthermore, let $\epsilon_n : S_n \longrightarrow \mathfrak{p}^n$ be the R-linear map such that

$$\epsilon_n(A^\alpha B^\beta C^\gamma) = a^\alpha b^\beta c^\gamma$$

for any $A^\alpha B^\beta C^\gamma \in \mathrm{m}^n_{A,B,C}$. Let $f := x^3 A + y^3 B + z^2 C$ and $g := yA + zB + xC$. Because a, b, c is a d-sequence, the following fact holds.

Lemma 4.1 *As a free resolution of $\mathfrak{p}^3$, we have*

$$0 \longrightarrow S_1 \xrightarrow{\varphi_3} S_2 \oplus S_2 \xrightarrow{\varphi_2} S_3 \xrightarrow{\epsilon_3} \mathfrak{p}^3 \longrightarrow 0 .$$

where $\varphi_3 := \begin{pmatrix} -g \\ f \end{pmatrix}$ *and* $\varphi_2 := (f \;\; g)$.

So we put $F_3 = S_1$, $F_2 = S_2 \oplus S_2$, $F_1 = S_3$, $F_0 = R$, and consider the acyclic complex

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_2 \xrightarrow{\varphi_1} F_0 = R \quad (\varphi_1 = \epsilon_3)\,.$$

As a free basis of F_3, we take $\mathrm{m}^1_{A,B,C} = \{ A, B, C \}$. For any $\xi \in S_2$, we set

$$[\xi] := \begin{pmatrix} \xi \\ 0 \end{pmatrix} \in F_2 \quad \text{and} \quad \langle \xi \rangle := \begin{pmatrix} 0 \\ \xi \end{pmatrix} \in F_2\,.$$

Then $V := \{ [L], \langle L \rangle \mid L \in \mathrm{m}^2_{A,B,C} \}$ is a free basis of F_2 . Moreover we have

$$\begin{aligned} \varphi_3(A) &= [-yA^2 - zAB - xAC] + \langle x^3A^2 + y^3AB + z^2AC \rangle \\ &= -y \cdot [A^2] - z \cdot [AB] - x \cdot [AC] + x^3 \cdot \langle A^2 \rangle + y^3 \cdot \langle AB \rangle + z^2 \cdot \langle AC \rangle \\ &= x \cdot v^1_A + y \cdot v^2_A + z \cdot v^3_A\,, \end{aligned}$$

where

$$v^1_A := x^2 \cdot \langle A^2 \rangle - [AC]\,,\ v^2_A := y^2 \cdot \langle AB \rangle - [A^2]\,,\ v^3_A := z \cdot \langle AC \rangle - [AB]\,.$$

Similarly we get

$$\varphi_3(B) = x \cdot v^1_B + y \cdot v^2_B + z \cdot v^3_B \quad \text{and} \quad \varphi_3(C) = x \cdot v^1_C + y \cdot v^2_C + z \cdot v^3_C\,,$$

where

$$v^1_B := x^2 \cdot \langle AB \rangle - [BC]\,, \quad v^2_B := y^2 \cdot \langle B^2 \rangle - [AB]\,, \quad v^3_B := z \cdot \langle BC \rangle - [B^2]\,,$$

$$v^1_C := x^2 \cdot \langle AC \rangle - [C^2]\,, \quad v^2_C := y^2 \cdot \langle BC \rangle - [AC]\,, \quad v^3_C := z \cdot \langle C^2 \rangle - [BC]\,.$$

We set $\tilde{\Lambda} = \{ 1, 2, 3 \} \times \{ A, B, C \}$. As $\varphi_3(F_3) \subseteq \mathfrak{m}F_2$. we get the next result.

Proposition 4.2 $[\mathfrak{p}^3 :_R \mathfrak{m}]/\mathfrak{p}^3 \cong S_1/\mathfrak{m}S_1 \cong (R/\mathfrak{m})^{\oplus 3}$.

Let us apply the $*$-transform on x, y, z to $F_\bullet$. Let $K_\bullet$ be the Koszul complex of x, y, z . At first, we get

$$0 \longrightarrow F'_3 \xrightarrow{\varphi'_3} F'_2 \xrightarrow{\varphi'_2} F^*_1 \xrightarrow{\varphi^*_1} F^*_0 = R\,,$$

where

$$F'_3 = K_2 \otimes_R F_3, F'_2 = (K_1 \otimes_R F_3) \oplus F_2, \varphi'_3 = \begin{pmatrix} \partial_2 \otimes \mathrm{id} \\ \sigma_2 \end{pmatrix}$$

and $\sigma_2 : K_2 \otimes_R F_3 \longrightarrow F_2$ is the R-linear map such that

$$\sigma_2(\check{e}_i \otimes M) = (-1)^i \cdot v^i_M$$

for any $(i, M) \in \tilde{\Lambda}$. We set

- $\Lambda' = \{ (1, A), (2, A), (3, A), (1, B), (3, B), (1, C) \}$,

- $V^* = \{ \langle L \rangle \mid L \in \mathrm{m}^2_{A,B,C} \}$.

Then we have

- $|\Lambda'| + |V^*| = 12 = \operatorname{rank} F_2$,
- $V \subseteq R \cdot \{ v^i_M \mid (i, M) \in \Lambda' \} + R \cdot V^* + \mathfrak{m}F_2$.

Hence $\{ v^i_M \mid (i, M) \in \Lambda' \}$ can be a part of a free basis of F_2, and so we set

$$\Lambda^* := \tilde{\Lambda} \setminus \Lambda' = \{ (2, B), (2, C), (3, C) \}.$$

We have

$$\begin{aligned} v^2_B &= v^3_A - z \cdot \langle AC \rangle + y^2 \cdot \langle B^2 \rangle , \\ v^2_C &= v^1_A - x^2 \cdot \langle A^2 \rangle + y^2 \cdot \langle BC \rangle , \\ v^3_C &= v^1_B - x^2 \cdot \langle AB \rangle + z \cdot \langle C^2 \rangle . \end{aligned}$$

So, we define $w^*_{(2,B)}, w^*_{(2,C)}, w^*_{(3,C)} \in F'_3 = K_2 \otimes_R S_1$ as follows.

$$\begin{aligned} w^*_{(2,B)} &= \check{e}_2 \otimes B + \check{e}_3 \otimes A , \\ w^*_{(2,C)} &= \check{e}_2 \otimes C + \check{e}_1 \otimes A , \\ w^*_{(3,C)} &= -\check{e}_3 \otimes C + \check{e}_1 \otimes B . \end{aligned}$$

These elements form a part of an R-free basis of F'_3. So we set

$$F^*_3 := R \cdot \{ w^*_{(2,B)}, w^*_{(2,C)}, w^*_{(3,C)} \} ,$$

which is a direct summand of F'_3.

We use the following Notation on elements in $F'_2 = (K_1 \otimes_R F_3) \oplus F_2$.

- For any $\xi \in K_1 \otimes_R F_3$, $[\xi] := \binom{\xi}{0} \in F'_2$.
  In particular, for any $(i, M) \in \tilde{\Lambda}$, $[i, M] := [e_i \otimes M]$.
- For any $\eta \in F_2$, $\langle \eta \rangle := \binom{0}{\eta} \in F'_2$.

Then we have the next result.

Lemma 4.3 *The following equalities hold.*

$$\begin{aligned} \varphi'_3(w^*_{(2,B)}) &= x \cdot [3, B] - z \cdot [1, B] + x \cdot [2, A] - y \cdot [1, A] - z \cdot \langle\langle AC \rangle\rangle + y^2 \cdot \langle\langle B^2 \rangle\rangle , \\ \varphi'_3(w^*_{(2,C)}) &= x \cdot [3, C] - z \cdot [1, C] + y \cdot [3, A] - z \cdot [2, A] - x^2 \cdot \langle\langle A^2 \rangle\rangle + y^2 \cdot \langle\langle BC \rangle\rangle , \\ \varphi'_3(w^*_{(3,C)}) &= -x \cdot [2, C] + y \cdot [1, C] + y \cdot [3, B] - z \cdot [2, B] - x^2 \cdot \langle\langle AB \rangle\rangle + z \cdot \langle\langle C^2 \rangle\rangle . \end{aligned}$$

In fact, as

$$\begin{aligned}\varphi_3'(w^*_{(2,B)}) &= [\partial_2\check{e}_2 \otimes B] + [\partial_2\check{e}_3 \otimes A] + \langle v^2_B - v^3_A\rangle \\ &= [(xe_3 - ze_1)\otimes B] + [(xe_2 - ye_1)\otimes A] + \langle -z\cdot\langle AC\rangle + y^2\cdot\langle B^2\rangle\rangle\,,\end{aligned}$$

we get the first equality in Lemma 4.3. The other equalities follow similarly.

Now we define

$$F_2^* := R\cdot\{\,[i,M] \mid (i,M)\in\tilde\Lambda\,\} + R\cdot\{\,\langle\langle L\rangle\rangle \mid L\in \mathrm{m}^2_{A,B,C}\,\}\,,$$

which is a free summand of F_2'. By Lemma 4.3 we have $\varphi_3'(F_3^*) \subseteq \mathfrak{m}F_2^*$. So, we set $\varphi_3^* := \varphi_3'|_{F_3^*}$ and $\varphi_2^* := \varphi_2'|_{F_2^*}$. Then $0 \longrightarrow F_3^* \xrightarrow{\varphi_3^*} F_2^* \xrightarrow{\varphi_2^*} F_1^* \xrightarrow{\varphi_1^*} F_0^* = R$ is an acyclic complex of finitely generated R-free modules such that $\operatorname{Im}\varphi_3^* \subseteq \mathfrak{m}F_2^*$ and $\operatorname{Im}\varphi_1^* = \mathfrak{p}^3 :_R \mathfrak{m}$. Because $[\mathfrak{p}^3 :_R \mathfrak{m}] :_R \mathfrak{m} = \mathfrak{p}^3 :_R \mathfrak{m}^2$, we get the following assertion.

Proposition 4.4 $[\mathfrak{p}^3 :_R \mathfrak{m}^2]/[\mathfrak{p}^3 :_R \mathfrak{m}] \cong F_3^*/\mathfrak{m}F_3^* \cong (R/\mathfrak{m})^{\oplus 3}$.

In the rest, we denote $F_\bullet^*$ by $F_\bullet$ (remove "$*$"). And again we apply $*$-transform on x, y, z to $F_\bullet$. By Lemma 4.3, we have

$$\begin{aligned}\varphi_3(w_{(2,B)}) &= x\cdot v^1_{(2,B)} + y\cdot v^2_{(2,B)} + z\cdot v^3_{(2,B)}\,,\\ \varphi_3(w_{(2,C)}) &= x\cdot v^1_{(2,C)} + y\cdot v^2_{(2,C)} + z\cdot v^3_{(2,C)}\,,\\ \varphi_3(w_{(3,C)}) &= x\cdot v^1_{(3,C)} + y\cdot v^2_{(3,C)} + z\cdot v^3_{(3,C)}\,,\end{aligned}$$

where

$v^1_{(2,B)} := [3,B] + [2,A]$, $\quad v^2_{(2,B)} := -[1,A] + y\cdot\langle\langle B^2\rangle\rangle$, $\quad v^3_{(2,B)} := -[1,B] - \langle\langle AC\rangle\rangle$,

$v^1_{(2,C)} := [3,C] - x\cdot\langle\langle A^2\rangle\rangle$, $\quad v^2_{(2,C)} := [3,A] + y\cdot\langle\langle BC\rangle\rangle$, $\quad v^3_{(2,C)} := -[1,C] - [2,A]$,

$v^1_{(3,C)} := -[2,C] - x\cdot\langle\langle AB\rangle\rangle$, $\quad v^2_{(3,C)} := [1,C] + [3,B]$, $\quad v^3_{(3,C)} := -[2,B] + \langle\langle C^2\rangle\rangle$.

We set $\Lambda := \{\,(2,B),(2,C),(3,C)\,\}$ and $\tilde\Lambda := \{\,1,2,3\,\}\times\Lambda$. Let us recall that

$$V := \{\,[i,M] \mid (i,M)\in\{1,2,3\}\times\{A,B,C\}\,\}\cup\{\,\langle\langle L\rangle\rangle \mid L\in\mathrm{m}^2_{A,B,C}\,\}$$

is a free basis of F_2. We set $V^* := \{\,\langle\langle L\rangle\rangle \mid L\in\mathrm{m}^2_{A,B,C}\,\}$.

Lemma 4.5 *We have* $|\Lambda'| + |V^*| = \operatorname{rank} F_2$. *Moreover, if 2 is a unit in R, we have* $V \subseteq R\cdot\{\,v^i_\lambda \mid (i,\lambda)\in\tilde\Lambda\,\} + R\cdot V^*$.

Therefore we can take $\tilde\Lambda$ as Λ'. Hence by Theorem 3.4 we get the following.

Theorem 4.6 *If 2 is a unit in R, then* $\operatorname{depth} R/[\mathfrak{p}^3 :_R \mathfrak{m}^2] > 0$, *which means*

$$\mathfrak{p}^{(3)} = \mathfrak{p}^3 :_R \mathfrak{m}^2\,,$$

and so we have $\ell_R(\,\mathfrak{p}^{(3)}/\mathfrak{p}^3\,) = 6$.

If $\operatorname{ch} R = 2$, then we can go one step more.

SYMMETRIES ON ALMOST SYMMETRIC NUMERICAL SEMIGROUPS

HIROKATSU NARI

ABSTRACT. The notion of almost symmetric numerical semigroup was given by V. Barucci and R. Fröberg in [BF]. We characterize almost symmetric numerical semigroups by symmetry of pseudo-Frobenius numbers. We give a criterion for H^* (the dual of M) to be almost symmetric numerical semigroup.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* H is a subset of $\mathbb{N}$ which is closed under addition, contains the zero element and whose complement in $\mathbb{N}$ is finite.

Every numerical semigroup H admits a finite system of generators, that is, there exist $a_1, ..., a_n \in H$ such that $H = \langle a_1, ..., a_n \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, ..., \lambda_n \in \mathbb{N}\}$.

Let H be a numerical semigroup and let $\{a_1 < a_2 < \cdots < a_n\}$ be its minimal generators. We call a_1 *the multiplicity* of H and denote it by $\mathrm{m}(H)$, and we call n *the embedding dimension* of H and denote it by $\mathrm{e}(H)$. In general, $\mathrm{e}(H) \leq \mathrm{m}(H)$. We say that H has *maximal embedding dimension* if $\mathrm{e}(H) = \mathrm{m}(H)$. The set $\mathrm{G}(H) := \mathbb{N} \setminus H$ is called *the set of gaps* of H. Its cardinality is said to be *the genus* of H and we denote it by $\mathrm{g}(H)$.

If H is a numerical semigroup, the largest integer in $\mathrm{G}(H)$ is called *Frobenius number* of H and we denote it by $\mathrm{F}(H)$. It is known that $2\,\mathrm{g}(H) \geq \mathrm{F}(H) + 1$. We say that H is *symmetric* if for every $z \in \mathbb{Z}$, either $z \in H$ or $\mathrm{F}(H) - z \in H$, or equivalently, $2\,\mathrm{g}(H) = \mathrm{F}(H) + 1$. We say that H is *pseudo-symmetric* if for every $z \in \mathbb{Z}$, $z \neq \mathrm{F}(H)/2$, either $z \in H$ or $\mathrm{F}(H) - z \in H$, or equivalently, $2\,\mathrm{g}(H) = \mathrm{F}(H) + 2$.

We say that an integer x is a *pseudo-Frobenius number* of H if $x \notin H$ and $x + h \in H$ for all $h \in H, h \neq 0$. We denote by $\mathrm{PF}(H)$ the set of pseudo-Frobenius numbers of H. The cardinality in $\mathrm{PF}(H)$ is called the *type* of H, denoted by $\mathrm{t}(H)$. Since $\mathrm{F}(H) \in \mathrm{PF}(H)$, H is symmetric if and only if $\mathrm{t}(H) = 1$.

This paper studies almost symmetric numerical semigroups. The concept of almost symmetric numerical semigroup was introduced by V. Barucci and R. Fröberg [BF]. They developed a theory of almost symmetric numerical semigroups and gave many results (see [Ba], [BF]). This paper aims at an alternative characterization of almost symmetric numerical semigroups. (see Theorem 2.4).

In [BF] the authors proved that H is almost symmetric and has maximal embedding dimension if and only if $H^* = M - M$ (the dual of M) is symmetric, where

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M denotes the maximal ideal of H. In Section 3 we will study the problem of when H^* is an almost symmetric numerical semigroup.

2. Almost symmetric numerical semigroups

Let H be a numerical semigroup and let n be one of its nonzero elements. We define

$$\mathrm{Ap}(H,n) = \{h \in H \mid h - n \notin H\}.$$

This set is called the Apéry set of h in H. By definition, $\mathrm{Ap}(H,n) = \{0 = w(0), w(1), \dots, w(n-1)\}$, where $w(i)$ is the least element of H congruent with i modulo n, for all $i \in \{0, \dots, n-1\}$. We can get pseudo-Frobenius numbers of H from the Apéry set by the following way: Over the set of integers we define the relation $\leq_H$, that is, $a \leq_H b$ implies that $b - a \in H$. Then we have the following result (see [RG] Proposition 2.20).

Proposition 2.1. *Let H be a numerical semigroup and let n be a nonzero element of H. Then*

$$\mathrm{PF}(H) = \{\omega - n \mid \omega \text{ is maximal with respect to } \leq_H \text{ in } \mathrm{Ap}(H,n)\}.$$

It is easy to check that $\mathrm{F}(H) = \max \mathrm{Ap}(H,n) - n$ and $\mathrm{g}(H) = \frac{1}{n}\sum_{h \in \mathrm{Ap}(H,n)} h - \frac{n-1}{2}$ (see [RG] Proposition 2.12).

Let H be a numerical semigroup. A *relative ideal* I of H is a subset of $\mathbb{Z}$ such that $I + H \subseteq I$ and $h + I = \{h + i \mid i \in I\} \subseteq H$ for some $h \in H$. An *ideal* of H is a relative ideal of H with $I \subseteq H$. It is straightforward to show that if I and J are relative ideals of H, then $I - J := \{z \in \mathbb{Z} \mid z + J \subseteq I\}$ is a also relative ideal of H. The ideal $M := H \setminus \{0\}$ is called *the maximal ideal* of H. We easily deduce that $M - M = H \cup \mathrm{PF}(H)$. We define

$$K = K_H := \{\mathrm{F}(H) - z \mid z \notin H\}.$$

It is clear that $H \subseteq K$ and K is a relative ideal of H. This ideal is called *the canonical ideal* of H.

We define $\mathrm{N}(H) := \{h \in H \mid h < \mathrm{F}(H)\}$. We already know that if $h \in \mathrm{N}(H)$, then $\mathrm{F}(H) - h \notin H$, and if $f \in \mathrm{PF}(H)$, $\neq \mathrm{F}(H)$, then $\mathrm{F}(H) - f \notin H$. Then the map

$$\begin{array}{ccc} \mathrm{N}(H) \cup [\mathrm{PF}(H) \setminus \{\mathrm{F}(H)\}] & \longrightarrow & \mathrm{G}(H) \\ \Psi & & \Psi \\ h & \longmapsto & \mathrm{F}(H) - h \end{array}$$

is injective, which proves the following.

Proposition 2.2. *Let H be a numerical semigroup. Then*

$$2\,\mathrm{g}(H) \geq \mathrm{F}(H) + \mathrm{t}(H).$$

Clearly, if a numerical semigroup is symmetric or pseudo-symmetric, then the equality of Proposition 2.2 holds. In general, a numerical semigroup is called almost symmetric if the equality holds.

Proposition-Definition 2.3. [Ba], [BF] *Let H be a numerical semigroup. Then the following conditions are equivalent.*

(1) $K_H \subset M - M$.
(2) $z \notin H$ *implies that either* $\mathrm{F}(H) - z \in H$ *or* $z \in \mathrm{PF}(H)$.
(3) $2\,\mathrm{g}(H) = \mathrm{F}(H) + \mathrm{t}(H)$.
(4) $K_{M-M} = M - \mathrm{m}(H)$.

A numerical semigroup H *satisfying either of these equivalent conditions is said to be almost symmetric.*

It is easy to show that if H is symmetric or pseudo-symmetric, then H is almost symmetric. Conversely, an almost symmetric numerical semigroups with type two is pseudo-symmetric (see Corollary 2.7).

We now give a characterization of almost symmetric numerical semigroups by symmetry of pseudo Frobenius numbers.

Theorem 2.4. *Let* H *be a numerical semigroup and let* n *be one of its nonzero elements. Set* $\mathrm{Ap}(H,n) = \{0 < \alpha_1 < \cdots < \alpha_m\} \cup \{\beta_1 < \beta_2 < \cdots < \beta_{\mathrm{t}(H)-1}\}$ *with* $m = n - \mathrm{t}(H)$ *and* $\mathrm{PF}(H) = \{\beta_i - n, \alpha_m - n = \mathrm{F}(H) \mid 1 \leq i \leq \mathrm{t}(H) - 1\}$. *We put* $f_i = \beta_i - n$ *and* $f_{\mathrm{t}(H)} = \alpha_m - n = \mathrm{F}(H)$. *Then the following conditions are equivalent.*

(1) H *is almost symmetric.*
(2) $\alpha_i + \alpha_{m-i} = \alpha_m$ *for all* $i \in \{1, 2, \ldots, m-1\}$ *and* $\beta_j + \beta_{\mathrm{t}(H)-j} = \alpha_m + n$ *for all* $j \in \{1, 2, \ldots, \mathrm{t}(H) - 1\}$.
(3) $f_i + f_{\mathrm{t}(H)-i} = \mathrm{F}(H)$ *for all* $i \in \{1, 2, \ldots, \mathrm{t}(H) - 1\}$.

Proof. For simplicity, we put $t = \mathrm{t}(H)$.

(1) $\Longrightarrow$ (2). Since $\alpha_i - n \notin H$, $\mathrm{F}(H) - (\alpha_i - n) = \alpha_m - \alpha_i \in H$ and $\alpha_m - (\alpha_i - n) \notin H$, by 2.3 (2). Hence $\alpha_m - \alpha_i \in \mathrm{Ap}(H,n)$. If $\alpha_m - \alpha_i = \beta_j$ for some j, then $\mathrm{F}(H) = \alpha_i + f_j \in H$. Hence we have that $\alpha_i + \alpha_{m-i} = \alpha_m$ for all $i \in \{1, 2, \ldots, m-1\}$. Next, we see that $\beta_j + \beta_{t-j} = \alpha_m + \mathrm{m}(H)$ for all $j \in \{1, 2, \ldots, t-1\}$. Since $\alpha_m - \beta_j = \mathrm{F}(H) - f_j \notin H$, by 2.3 (2) we get $\alpha_m - \beta_j \in \mathrm{PF}(H)$, that is, $\alpha_m - \beta_j = \beta_{\mathrm{t}(H)-j} - n$ for all $j \in \{1, 2, \ldots, t-1\}$.

(2) $\Longrightarrow$ (3). By hypothesis, $(\beta_j - n) + (\beta_{t-j} - n) = \alpha_m - n$ implies $f_j + f_{t-j} = \mathrm{F}(H)$.

(3) $\Longrightarrow$ (1). In view of Proposition-Definition 2.3, it suffices to prove that $K \subset M - M$. Let $x \in K$ and $x = \mathrm{F}(H) - z$ for some $Z \notin H$. If $z \in \mathrm{PF}(H)$, then $x \in \mathrm{PF}(H)$ by condition (3). If $z \notin \mathrm{PF}(H)$, then $z + h \in \mathrm{PF}(H)$ for some $h \in M$. Then $x = \mathrm{F}(H) - (z+h) + h \in H$, since $\mathrm{F}(H) - (z+h) \in \mathrm{PF}(H)$. Hence we have that H is almost symmetric. □

Remark 2.5. When H is symmetric or pseudo-symmetric, the equivalence of (1) and (2) is shown Proposition 4.10 and 4.15 of [RG]

Example 2.6. (1) Let $H = \langle 5, 6, 9, 13 \rangle$. Then $\mathrm{Ap}(H, 5) = \{0, 6, 9, 12, 13\}$ and $\mathrm{PF}(H) = \{4, 7, 8\}$, we see from Theorem 2.4 (3) that H is not almost symmetric.
(2) Let a be an odd integer greater than or equal to three and let $H = \langle a, a+2, a+4, \ldots, 3a-2 \rangle$. H has maximal embedding dimension, so that $\mathrm{PF}(H) = \{2, 4, \ldots, 2(a-1)\}$. Hence we get H is almost symmetric.

We obtain the following corollary from Theorem 2.4 (3).

Corollary 2.7. *Let* H *be a numerical semigroup. Then* H *is almost symmetric with* $\mathrm{t}(H) = 2$ *if and only if* H *is pseudo-symmetric.*

3. When is H^* almost symmetric ?

Let H be a numerical semigroup with maximal ideal M. If I is a relative ideal of H, then relative ideal $H - I$ is called *the dual of I with respect to H*. In particular, the dual of M is denoted by H^*.

For every relative ideal I of H, $I - I$ is a numerical semigroup. Since $H^* = H - M = M - M$, H^* is numerical semigroup. By definition, it is clear that $\mathrm{g}(H^*) = \mathrm{g}(H) - \mathrm{t}(H)$.

In [BF] the authors solved the problem of when the dual of M is a symmetric.

Theorem 3.1. [BF] *Let H be a numerical semigroup. Then H is almost symmetric and maximal embedding dimension if and only if H^* is symmetric.*

Example 3.2. On the Example 2.6 (2), $H = \langle a, a+2, a+4, \ldots; 3a-2 \rangle$ has maximal embedding dimension and almost symmetric. Hence we have that $H^* = H \cup \{2, 4, \ldots, 2(a-1)\} = \langle 2, a \rangle$ is symmetric.

In this section we will ask when is H^* almost symmetric in general case (see Theorem 3.7). Surprisingly, using our criterion for H^* to be almost symmetric Theorem 3.1 can be easily seen.

Let H be a numerical semigroup. Then we set

$$\mathrm{L}(H) := \{a \in H \mid a - \mathrm{m}(H) \notin H^*\}.$$

By definition we have that $\mathrm{Card}\,\mathrm{L}(H) = \mathrm{m}(H) - \mathrm{t}(H)$ and $\mathrm{Ap}(H, \mathrm{m}(H)) = \{f + \mathrm{m}(H) \mid f \in \mathrm{PF}(H)\} \cup \mathrm{L}(H)$. We describe $\mathrm{Ap}(H^*, \mathrm{m}(H))$ in terms of $\mathrm{PF}(H)$ and $\mathrm{L}(H)$.

Lemma 3.3. *Let H be a numerical semigroup. Then*

$$\mathrm{Ap}(H^*, \mathrm{m}(H)) = \mathrm{PF}(H) \cup \mathrm{L}(H).$$

Proof. Since $H^* = H \cup \mathrm{PF}(H)$, clearly $\mathrm{Ap}(H^*, \mathrm{m}(H)) \supseteq \mathrm{PF}(H) \cup \mathrm{L}(H)$.

Conversely we take $a \in \mathrm{Ap}(H^*, \mathrm{m}(H))$ and $a \notin \mathrm{PF}(H)$. Then $a \in H$ and $a - \mathrm{m}(H) \notin H^*$. Hence we have that $a \in \mathrm{PF}(H) \cup \mathrm{L}(H)$. □

By Lemma 3.3, the Frobenius number of H^* is easy to compute.

Proposition 3.4. [BDF] *Let H be a numerical semigroup. Then*

$$\mathrm{F}(H^*) = \mathrm{F}(H) - \mathrm{m}(H).$$

Proof. Clearly $\mathrm{F}(H) - \mathrm{m}(H) \notin H^*$, by Lemma 3.3. Let $x > \mathrm{F}(H) - \mathrm{m}(H)$ and $h \in M$. Then $x + h > \mathrm{F}(H) - \mathrm{m}(H) + h \geq \mathrm{F}(H)$, thus we get $\mathrm{F}(H^*) = \mathrm{F}(H) - \mathrm{m}(H)$. □

Every numerical semigroup is dual of maximal ideal for some numerical semigroup.

Proposition 3.5. *Let H be a numerical semigroup. Then there exists a numerical semigroup $T \subset H$ such that $T^* = H$.*

Proof. Let $\mathrm{Ap}(H, h) = \{0 < \alpha_1 < \cdots < \alpha_{h-1}\}$ for some $h \in H$. We put $T = \langle h, h + \alpha_1, \ldots, h + \alpha_{h-1} \rangle$. Since T has maximal embedding dimension, $\mathrm{PF}(T) = \{\alpha_1 < \cdots < \alpha_{h-1}\}$. Hence we get $T^* = T \cup \mathrm{PF}(T) = H$. □

Remark 3.6. In Proposition 3.5, such numerical semigroup T is not determined uniquely. Indeed, we put $H_1 = \langle 5,6,8,9 \rangle$ and $H_2 = \langle 3,7,8 \rangle$. Then $\mathrm{PF}(H_1) = \{3,4,7\}$ and $\mathrm{PF}(H_2) = \{4,5\}$. Therefore we have $H_1^* = H_2^* = \langle 3,4,5 \rangle$.

The following is the main Theorem of this section.

Theorem 3.7. *Let H (resp. H^*) be an almost symmetric numerical semigroup. Then H^* (resp. H) is an almost symmetric if and only if* $\mathrm{m}(H) = \mathrm{t}(H) + t(H^*)$.

Proof. If H is almost symmetric, then

$$\begin{aligned} 2\,\mathrm{g}(H^*) &= 2\,\mathrm{g}(H) - 2\,\mathrm{t}(H) \\ &= \mathrm{F}(H) - \mathrm{t}(H) \\ &= \mathrm{F}(H^*) + \mathrm{m}(H) - \mathrm{t}(H). \quad \text{(by Proposition 3.4)} \end{aligned}$$

If H^* is almost symmetric, then

$$\begin{aligned} 2\,\mathrm{g}(H) &= 2\,\mathrm{g}(H^*) + 2\,\mathrm{t}(H) \\ &= \mathrm{F}(H^*) + \mathrm{t}(H^*) + 2\,\mathrm{t}(H) \\ &= \mathrm{F}(H) + 2\,\mathrm{t}(H) + \mathrm{t}(H^*) - \mathrm{m}(H). \quad \text{(by Proposition 3.4)} \end{aligned}$$

Observing these inequalities, we deduce the assertion. □

Using Theorem 3.7 we prove Theorem 3.1.

Proof of Theorem 3.1. We assume that H is almost symmetric and maximal embedding dimension. Then $\mathrm{m}(H) = \mathrm{t}(H) + 1$. Hence we have

$$\begin{aligned} \mathrm{t}(H^*) &\leq 2\,\mathrm{g}(H^*) - \mathrm{F}(H^*) \quad \text{(by Proposition 2.2)} \\ &= 2\,\mathrm{g}(H) - 2\,\mathrm{t}(H) - (\mathrm{F}(H) - \mathrm{m}(H)) \quad \text{(by Proposition 3.4)} \\ &= \mathrm{m}(H) - \mathrm{t}(H) \\ &= 1. \end{aligned}$$

This implies H^* is symmetric.

Conversely, let H^* be symmetric. By Theorem 3.7, it is enough to show that $\mathrm{m}(H) = \mathrm{t}(H) + 1$. We assume $\mathrm{m}(H) > \mathrm{t}(H) + 1$. Then

$$\begin{aligned} 2\,\mathrm{g}(H^*) - \mathrm{F}(H^*) &= 2\,\mathrm{g}(H) - 2\,\mathrm{t}(H) - (\mathrm{F}(H) - \mathrm{m}(H)) \quad \text{(by Proposition 3.4)} \\ &\geq \mathrm{m}(H) - \mathrm{t}(H) \\ &> 1. \end{aligned}$$

Since H^* is symmetric, this is a contradiction. Thus we get H is almost symmetric and maximal embedding dimension. □

Let $H = \langle a_1, a_2, ..., a_n \rangle$ be an almost symmetric numerical semigroup with $a_1 < a_2 < \ldots < a_n$. If $\mathrm{e}(H) = n = a_1$ (that is, H has maximal embedding dimension), then the maximal element of $\mathrm{Ap}(H, a_1)$ is equal to a_n. If $n < a_1$, then the maximal element of $\mathrm{Ap}(H, a_1)$ is greater than a_n.

Lemma 3.8. *Let $H = \langle a_1, a_2, ..., a_n \rangle$ be a numerical semigroup and let $n < a_1$. If H is almost symmetric, then* $\max \mathrm{Ap}(H, a_1) \neq a_n$.

Proof. We assume $\max \mathrm{Ap}(H, a_1) = a_n$. Since H is almost symmetric, by Theorem 2.4 we have that

$$\mathrm{Ap}(H, a_1) = \{0 < \alpha_1 < \cdots < \alpha_m < a_n\} \cup \{\beta_1 < \cdots < \beta_{a_1-m-2}\},$$

where $\alpha_i + \alpha_{m-i+1} = a_n$ for all $i \in \{1, 2, \ldots, m\}$ and $\mathrm{PF}(H) = \{\beta_1 - a_1 < \cdots < \beta_{a_1-m-2} - a_1 < a_n - a_1\}$. Since $\mathrm{e}(H) < \mathrm{m}(H)$, there exist i such that $a_i = \alpha_j$ for some j. Hence we get $a_n = a_i + \alpha_k$ for some k. But this is a contradiction, because a_n is a minimal generator of H. □

Proposition 3.9. *Let H be an almost symmetric numerical semigroup with $\mathrm{e}(H) < \mathrm{m}(H)$. Then the following conditions hold:*

(1) $\mathrm{e}(H) + 1 \leq \mathrm{t}(H) + \mathrm{t}(H^*) \leq \mathrm{m}(H)$,
(2) $\mathrm{t}(H^*) \leq \mathrm{e}(H)$.

Proof. (1) First, we show that $\mathrm{t}(H) + \mathrm{t}(H^*) \leq \mathrm{m}(H)$. Since H is almost symmetric, we get

$$\begin{aligned} 2\,\mathrm{g}(H^*) &= \mathrm{F}(H^*) + \mathrm{m}(H) - \mathrm{t}(H) \\ &\geq \mathrm{F}(H^*) + \mathrm{t}(H^*) \quad \text{(by Proposition 2.2)}. \end{aligned}$$

This inequality means $\mathrm{t}(H) + \mathrm{t}(H^*) \leq \mathrm{m}(H)$. Next, we prove $\mathrm{e}(H) + 1 \leq \mathrm{t}(H) + \mathrm{t}(H^*)$. Assume that $H = \langle a_1, \ldots, a_n \rangle$ and $\mathrm{m}(H) = a_1$. Put $\mathrm{PF}(H) = \{f_1 < \cdots < f_{\mathrm{t}(H)-1} < \mathrm{F}(H)\}$. By Lemma 3.8, $\mathrm{F}(H) + a_1 \neq a_i$ for all $i \in \{2, \cdots, a_1 - 1\}$. Also we have that for any $j \in \{1, \ldots, \mathrm{t}(H) - 1\}$, $f_j \notin \mathrm{PF}(H^*)$ by the symmetries of the pseudo-Frobenius numbers of H. This means

$$0 \leq k := \mathrm{Card}\{a_i \mid a_i - a_1 \in \mathrm{PF}(H)\} \leq \mathrm{t}(H) - 1.$$

Hence we have the inequality

$$\mathrm{e}(H) - (\mathrm{t}(H) - 1) \leq \mathrm{e}(H) - k \leq \mathrm{t}(H^*).$$

(2) Let $H = \langle a_1, \ldots, a_n \rangle$. It is enough to show that $\mathrm{PF}(H^*) \subseteq \{\mathrm{F}(H) - a_i \mid 1 \leq i \leq n\}$. Take $x \in \mathrm{PF}(H^*)$. Since $x \notin H^*$, we get $\mathrm{F}(H) - x \in H$ by 2.3 (2). We assume $\mathrm{F}(H) - x \in 2M$, where M denotes the maximal ideal of H. Then there exist $h \in M$ such that $\mathrm{F}(H) - x = a_i + h$ for some a_i, this means $\mathrm{F}(H) \in H$, a contradiction. Hence we have $\mathrm{F}(H) - x \notin 2M$, that is, $\mathrm{F}(H) - x = a_i$ for some i. Thus we obtain that $\mathrm{PF}(H^*) \subseteq \{\mathrm{F}(H) - a_i \mid 1 \leq i \leq n\}$. □

Corollary 3.10. *Let H be an almost symmetric numerical semigroup. If $\mathrm{e}(H) = \mathrm{m}(H) - 1$, then H^* is an almost symmetric with $\mathrm{t}(H^*) \geq 2$.*

Proof. Assume that H is almost symmetric. By Proposition 3.9 (2), if $e(H) = \mathrm{m}(H) - 1$, then $\mathrm{t}(H) + \mathrm{t}(H^*) = \mathrm{m}(H)$. We see from Theorem 3.7 that H^* is almost symmetric. □

The converse of Corollary 3.10 is not known. But if we assume that H is symmetric, then that is true.

Corollary 3.11. *Let H be a symmetric numerical semigroup with $\mathrm{e}(H) < \mathrm{m}(H)$. Then $\mathrm{e}(H) = \mathrm{m}(H) - 1$ if and only if H^* is an almost symmetric with $\mathrm{t}(H^*) \geq 2$.*

Proof. From Corollary 3.10, it is enough to show that H^* is an almost symmetric with $\mathrm{t}(H^*) \geq 2$, then $\mathrm{e}(H) = \mathrm{m}(H) - 1$. We assume that H is symmetric and H^* is almost symmetric with $\mathrm{t}(H^*) \geq 2$. Then by Proposition 3.9, we get $\mathrm{t}(H^*) = \mathrm{e}(H)$. On the other hand, using Theorem 3.7, we have $\mathrm{t}(H) + \mathrm{t}(H^*) = 1 + \mathrm{t}(H^*) = \mathrm{m}(H)$. Hence $\mathrm{e}(H) = \mathrm{m}(H) - 1$. □

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Licci monomial ideals [1]

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1. Introduction

Throughout this talk, let S be a standard graded polynomial ring over a field K, and $\mathfrak{m}$ denotes the unique graded maximal ideal of S.

Definition 1.1. Let R be a regular local ring, and let I, I' be proper ideals of R. Then I and I' are said to be direct linked, denoted by $I \sim I'$, if there exists a regular sequence $\underline{z} = z_1, \dots, z_g$ in $I \cap I'$ such that $I' = (\underline{z}) \colon I$ and $I = (\underline{z}) \colon I'$. Moreover, I is said to be linked to I' if there exists a sequence of direct links

$$I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_m = I'.$$

Then I is in the linkage class of I'. In particular, if one can choose I' as a complete intersection ideal (i.e. an ideal generated by a regular sequence), then I is called *licci*.

It is known that many good ideals are licci. For example, any Cohen-Macaulay ideal of height 2 and any Gorenstein ideal of height 3 in a regular local ring are licci. On the other hand, licci ideals enjoy nice properties. For instance, any licci ideal is Cohen-Macaulay and the second power of any Gorenstein licci ideal is Cohen-Macaulay. This fact provides many examples of squarefree monomial ideals whose second power is Cohen-Macaulay; see also [13].

The following question (see also [14]) is natural.

Question 1.2. *Suppose that $I \subseteq S$ is a squarefree monomial ideal. When is $I_{\mathfrak{m}}$ licci in $R = S_{\mathfrak{m}}$?*

In this talk, we give two partial answers to the question above. We first consider the question in the case of edge ideals of graphs. An edge ideal can be considered as a squarefree monomial ideal generated in degree 2.

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[1] This is an extended abstract. The final version will be published elsewhere.

Secondly, we consider the case of squarefree monomial ideals with small deviation, that is, the number of the minimal set of generators of I is at most height $I+2$. We know that if the deviation of I is zero then I is complete intersection and thus licci. In this talk, we prove that any Cohen-Macaulay almost complete intersection ideals (i.e, the deviation is one) are always licci. We also classify licci squarefree monomial ideals of deviation 2 completely.

2. Huneke-Ulrich theorem

The following result gives a necessary condition for a homogeneous ideal I in a polynomial ring S to be licci in $R = S_{\mathfrak{m}}$.

Theorem 2.1 (Huneke-Ulrich [5, Corollary 5.13]**).** *Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K. Put $\mathfrak{m} = (x_1, \dots, x_n)S$. Let $I \subseteq S$ be a homogeneous ideal of height g. Suppose that S/I is Cohen-Macaulay with the following graded minimal free resolution:*

$$0 \to \bigoplus_{j=1}^{b_g} S(-n_{gj}) \to \cdots \to \bigoplus_{j=1}^{b_1} S(-n_{1j}) \to S \to S/I \to 0.$$

If $\max\{n_{gj}\} \le (g-1)\min\{n_{1j}\}$ holds true, then $I_{\mathfrak{m}} \subseteq R = S_{\mathfrak{m}}$ is not licci.

Remark 2.2. Under the notation as in Theorem 2.1,

$$\max\{n_{gj}\} = a(S/I) + n = \operatorname{reg}(S/I) + g,$$

where $\operatorname{reg}(S/I)$ (resp. $a(S/I)$) denotes the regularity (resp. the a-invariant) of S/I.

The following lemma is well-known.

Lemma 2.3. *If $I \subseteq S$ is generated by squarefree monomials, then $a(S/I) \le 0$.*

3. Licci edge ideals

Throughout this section, let G be a graph, which means a simple finite graph without loops and multiple edges. Let $V(G)$ (resp. $E(G)$) denote the set of all vertices (resp. edges) of G. Put $V(G) = \{x_1, x_2, \dots, x_n\}$, and let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K. Then the *edge ideal*, denoted by $I(G)$, is defined by

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G))S.$$

For a subset $W \subseteq V(G)$, W is said to be a *vertex cover* of G if $e \cap W \neq \emptyset$ for any $e \in E(G)$. A vertex cover W is called *minimal* if it has no proper subset that is a vertex cover of G. Then an irredundant primary decomposition of $I(G)$ is given by

$$I(G) = \bigcap_{W \subseteq V \text{is a minimal vertex cover of } G} (x_i \mid x_i \in W).$$

In particular,

$$\operatorname{height} I(G) = \min\{\sharp W \mid W \text{ is a minimal vertex cover of } G\}.$$

A graph G is called *unmixed* (or well-covered) if its edge ideal $I(G)$ is *unmixed*, that is, all minimal covers have the same cardinality.

Lemma 3.1 (Gitler-Valencia [2, Corollary 3.4]**).** *Assume that a graph G has no isolated vertices. If G is unmixed, then $2 \cdot \operatorname{height} I(G) \geq \sharp(V(G))$.*

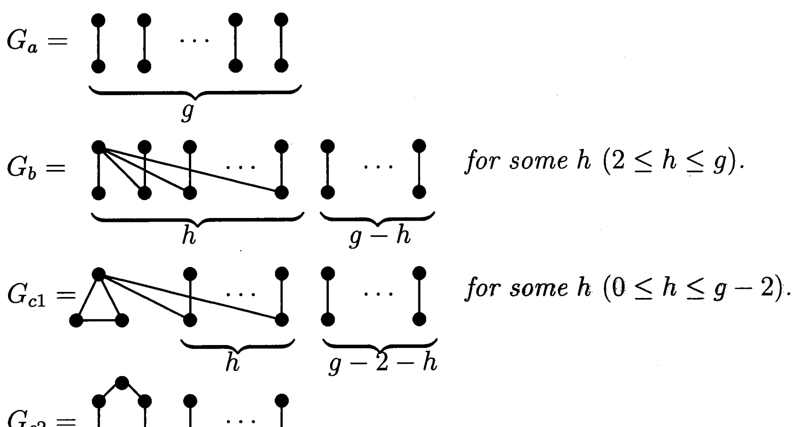
The main result of this section is the following theorem, which classifies all the graphs whose edge ideals are licci.

Theorem 3.2 (Classification of licci edge ideals). *Let G be a graph on V without isolated vertices. Let S be a standard graded polynomial ring with n variables over a field K, where $n = \sharp(V)$. Let $\mathfrak{m}$ denote the unique graded maximal ideal of S and put $R = S_{\mathfrak{m}}$. Let $I(G) \subseteq S$ denote the edge ideal of G. Put $g = \operatorname{height} I(G)$. Then the following conditions are equivalent:*

(1) *$I(G)_{\mathfrak{m}}$ is licci.*

(2) *$I(G)$ is Cohen-Macaulay and one of the following conditions are satisfied:*
   (a) $(\sharp(V), \operatorname{reg} S/I(G)) = (2g, g)$.
   (b) $(\sharp(V), \operatorname{reg} S/I(G)) = (2g, g-1)$.
   (c) $(\sharp(V), \operatorname{reg} S/I(G)) = (2g-1, g-1)$.

(3) *G is isomorphic to one of the following graphs:*



When this is the case,

$$
\begin{aligned}
I(G_a) &= (\{x_i y_i\}_{1 \leq i \leq g}),\\
I(G_b) &= (\{x_i y_i\}_{1 \leq i \leq g},\ \{x_j y_1\}_{2 \leq j \leq h}),\\
I(G_{c1}) &= (\{x_i y_i\}_{1 \leq i \leq g-2},\ \{x_j z_1\}_{1 \leq j \leq h},\ z_1 z_2, z_2 z_3, z_3 z_1),\\
I(G_{c2}) &= (\{x_i y_i\}_{1 \leq i \leq g-3},\ z_1 z_2, z_2 z_3, z_3 z_4, z_4 z_5, z_5 z_1).
\end{aligned}
$$

In what follows, we give a sketch of the proof of this theorem.

Proof of (1) $\Longrightarrow$ (2). Assume that $I(G)_{\mathfrak{m}}$ is licci. Then $I(G)$ is a Cohen-Macaulay ideal. Take a graded minimal free resolution of $S/I(G)$ over S as follows:

$$0 \to \bigoplus_{j=1}^{b_g} S(-n_{gj}) \to \cdots \to \bigoplus_{j=1}^{b_1} S(-n_{1j})(= S(-2)^{\oplus b_1}) \to S \to S/I(G) \to 0.$$

It follows from Huneke-Ulrich theorem (Theorem 2.1) that

$$2g-1 = (g-1)\min\{n_{1j}\} + 1 \le \max\{n_{gj}\} = a(S/I(G)) + n \le n. \tag{3.1}$$

On the other hand, since $I(G)$ is an unmixed edge ideal, we have $n \le 2g$ by Gitler-Valencia theorem (Lemma 3.1). Thus $n = 2g$ or $2g-1$. Moreover, the Cohen-Macaulayness of $S/I(G)$ implies that

$$\operatorname{reg} S/I(G) = a(S/I(G)) + \dim S/I(G) = a(S/I(G)) + n - g. \tag{3.2}$$

First suppose $n = 2g$. Then Equations (3.1) and (3.2) imply that $\operatorname{reg}(S/I(G)) = g-1$ or g. Next suppose $n = 2g-1$. Then we have that $\operatorname{reg}(S/I(G)) = g-1$ similarly. □

Proof of (2) $\Longrightarrow$ (3). First we consider the case of $n = 2g$. Such an unmixed graph G (without isolated vertices) is called *very well-covered.*

An arbitrary very well-covered graph G has a perfect matching, that is, there exists $E' \subseteq E(G)$ such that $\sharp(E') = \frac{\sharp V(G)}{2}$ and that $e \cap e' = \emptyset$ whenever $e, e' \in E'$ with $e \ne e'$. Thus if G is very well-covered, then we may assume that $V(G) = X \cup Y$, where $X = \{x_1, \ldots, x_g\}$ ($g = \operatorname{height} I(G)$) is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_g\}$ is an independent set (that is, $\{y_i, y_j\} \notin E(G)$ for all $1 \le i < j \le g$) and $\{x_i, y_i\} \in E(G)$ for all $i = 1, \ldots, g$. See [1, 10] for more details.

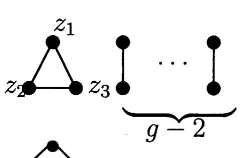
We consider the regularity of $S/I(G)$ for any very well-covered graph G. Suppose $e = \{x_1, x_2\}$, $e' = \{y_1, y_2\} \in E$. Then e and e' are called *pairwise* 3*-disjoint* if $\{x_i, y_j\} \notin E$ holds for all $1 \le i, j \le 2$. Then we have

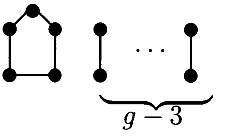
$$\operatorname{reg} S/I(G) = \text{the maximal number of pairwise 3-disjoint edges of } G$$

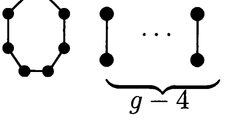
by [10, Theorem 3.2].

Using this, we can conclude that G is isomorphic to G_a (resp. G_b) if $\operatorname{reg} S/I(G) = g$ (resp. $g-1$).

Next we consider the case where $n = \sharp(V) = 2g-1$. By assumption, we have that $a(S/I(G)) = 0$. Since G is an unmixed graph with $\sharp(V) = 2g-1$, it follows from [12, Lemma 14] that G contains one of the following graphs as a spanning subgraph:

$H_1 =$ 

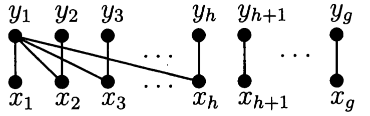
$H_2 =$ 

$H_3 =$ 

By combinatorial argument, we can determine all the graphs whose edge ideals are licci. □

Proof of (3) $\Longrightarrow$ (1). First we consider $I(G_a) = (x_1y_1, x_2y_2, \ldots, x_gy_g)$. Then $I(G_a)_{\mathfrak{m}} \subseteq R$ is a complete intersection ideal, a fortiori, is licci.

Secondly, we consider $I(G_b) = (\{x_iy_i\}_{1\le i\le g},\ \{x_jy_1\}_{2\le j\le h})$.

$G_b =$ 

If we put $J_1 = (\{x_iy_i\}_{1\le i\le g})$ and $I_1 = J_1 \colon I(G_b)$, then

$$I_1 = (x_1, \{x_iy_i\}_{2\le i\le g}, y_2y_3\cdots y_h).$$

Moreover, if we put $J_2 = (x_1, \{x_iy_i\}_{2\le i\le g}) \subseteq I_1$, then

$$J_2 \colon I_1 = (\{x_i\}_{1\le i\le h},\ \{x_jy_j\}_{h+1\le j\le g})$$

is a complete intersection ideal (see e.g. Example 5.2). Therefore $I(G_b)_{\mathfrak{m}} \subseteq R$ is licci. Note that it also follows from [14, Corollary 2.3].

We omit the proof of licciness for $I(G_{c1})$ and $I(G_{c2})$. □

4. Squarefree monomial ideals and hypergraphs

In the latter half of this talk, we discuss licciness of squarefree monomial ideals of small deviation. In order to do that, we need a classification theorem of those ideals in terms of hypergraphs, which were introduced in [8]; see [8, 9] for more details.

For an arbitrary squarefree monomial ideal $I \subseteq S = K[x_1, \ldots, x_n]$, let $\mathcal{G}(I) = \{m_1, \ldots, m_\mu\}$ denote the minimal set of monomial generators of I. Then the *hypergraph* $\mathcal{H}(I)$ *associated to* I on a vertex set $V = [\mu]$ is defined by

$$\mathcal{H}(I) := \Big\{\{j \in V \ :\ m_j \text{ is divisible by } x_i\}\ :\ i = 1, 2, \ldots, n\Big\}.$$

On the other hand, for a hypergraph $\mathcal{H}$, when n is large enough, if we assign a variable x_F to each $F \in \mathcal{H}$, then

$$I_{\mathcal{H}} = \Bigg(\prod_{j\in F\in\mathcal{H}} x_F\ :\ j = 1, 2, \ldots, \mu\Bigg)$$

gives a squarefree monomial ideal of $K[x_F : F \in \mathcal{H}]$. Thus we can construct a squarefree monomial ideal from a given hypergraph. Note that $\mathcal{H}(I_{\mathcal{H}}) = \mathcal{H}$, and that there exist many ideals I such that $\mathcal{H}(I) = \mathcal{H}$.

Let I be a squarefree monomial ideal of S. Then I has a prime component of height h if and only if $\mathcal{H}(I)$ has a minimal cover of cardinality h. Moreover, $\dim \mathcal{H}(I) \le d(I) := \mu(I) - \operatorname{height} I$. In particular, $d(I) = 0$ if and only if $\dim \mathcal{H}(I) = 0$. Namely, I is complete intersection if and only if $\mathcal{H}(I)$ consists of isolated vertices.

On the other hand, $d(I) = 1$ if and only if $\dim \mathcal{H}(I) = 1$ and there are no two disjoint edges. Those ideals have been classified in [8].

In order to classify all licci squarefree monomial ideals with $d(I) = 2$, we use a classification theorem in [9]. Moreover, we use the following notion for simplicity.

Definition 4.1 (Extension). Let $\mathcal{H}$, $\mathcal{H}'$ be hypergraphs on the vertex set V. We say that $\mathcal{H}'$ is an *extension* of $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{H}'$ and the set of facets of $\mathcal{H}$ coincides with that of $\mathcal{H}'$. In particular, for each integer $i \geq 0$, an extension $\mathcal{H}'$ is called an *i-extension* if it is obtained from $\mathcal{H}$ by adding i-faces only.

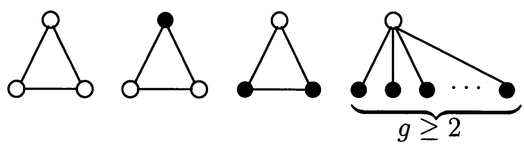
5. Licci almost complete intersection squarefree monomial ideals

Throughout this section, let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and set $\mathfrak{m} = (x_1, \ldots, x_n)$. Then we can regard S an $\mathbb{N}$-graded ring with $\deg x_i = 1$ for all i.

For a homogeneous ideal $I \subseteq S$, I is called an *almost complete intersection* ideal if I is minimally generated by height $I + 1$ elements. We know that if $I_\mathfrak{m}$ is licci then I is Cohen-Macaulay. When I is almost complete intersection, the converse is also true.

Theorem 5.1. *Let $I \subseteq S$ be an almost complete intersection squarefree monomial ideal. Then the following conditions are equivalent.*

(1) *$I_\mathfrak{m}$ is licci.*
(2) *I is Cohen-Macaulay.*
(3) *The associated hypergraph of I is isomorphic to one of the following hypergraphs with isolated vertices:*



Example 5.2. Let $S = K[x_1, \ldots, x_g, y_1, \ldots, y_g]$ be a polynomial ring over a field K. Put $\mathfrak{m} = (x_1, \ldots, x_g, y_1, \ldots, y_g)$ and $R = S_\mathfrak{m}$. For any integers $2 \leq h \leq g$, an ideal

$$I = (x_1y_1, x_2y_2 \ldots, x_gy_g, y_1y_2 \cdots y_h)R$$

is licci because

$$\begin{aligned} (x_1y_1, \ldots, x_gy_g)R \colon I &= (x_1y_1, \ldots, x_gy_g)R \colon y_1y_2 \cdots y_h \\ &= (x_1, \ldots, x_h, x_{h+1}y_{h+1}, \ldots, x_gy_g)R \end{aligned}$$

is complete intersection.

6. Licci squarefree monomial ideals of deviation two

The main purpose of this section is to characterize any squarefree monomial ideal of deviation 2 to be licci. In order to classify those ideals, it is enough to classify Cohen-Macaulay hypergraphs of deviation 2.

The next theorem is the main result in this paper.

Theorem 6.1. *Let $\mathcal{H}$ be a Cohen-Macaulay hypergraph of deviation 2 without isolated vertices, and put $I_{\mathfrak{m}} = (I_{\mathcal{H}})_{\mathfrak{m}} \subseteq R = S_{\mathfrak{m}}$, where S is a standard graded polynomial ring over a field K, and $\mathfrak{m}$ denotes the unique graded maximal ideal of S. Then the following conditions are equivalent:*

(1) *$I_{\mathfrak{m}}$ is not licci.*
(2) *The hypergraph $\mathcal{H}'$ given by removing all 2-faces from $\mathcal{H}$ is a disjoint union of two Cohen-Macaulay hypergraphs of deviation 1.*
(3) *$I_{\mathfrak{m}}$ is in a linkage class in the sum of two Cohen-Macaulay almost complete intersection squarefree monomial ideals L_1 and L_2 such that $\gcd(m_1, m_2) = 1$ for any minimal monomial generators $m_1 \in L_1$ and $m_2 \in L_2$, respectively.*

Corollary 6.2. *Let $I \subseteq S$ be a squarefree monomial ideal generated by at most 5 elements. Then $I_{\mathfrak{m}}$ is licci if and only if $I_{\mathfrak{m}}$ is Cohen-Macaulay.*

Instead of proving the theorem, we give several key lemmata.

Lemma 6.3. *If a Cohen-Macaulay hypergraph $\mathcal{H}_1$ is an extension of a hypergraph $\mathcal{H}_2$ and $I_{\mathcal{H}_1}$ is licci, then so is $I_{\mathcal{H}_2}$.*

Remark 6.4. If K is infinite, then the assertion of Lemma 6.3 follows from [6, Theorem 2.12].

Lemma 6.5. *Let S be a standard graded polynomial ring, and let $\mathfrak{m}$ denote the unique graded maximal ideal of S. Let I, I'', J, $J' \subseteq \mathfrak{m}$ be ideals of the same height. Suppose that the following conditions are satisfied:*

(i) *I and I'' are Cohen-Macaulay ideals.*
(ii) *J and J' are complete intersection ideals.*
(iii) *There exist elements a, b, $c \in S$ such that*

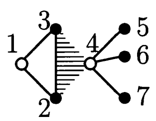
$$I = J + a(b, c), \quad J' = J : a, \quad I'' = J' + (b, c).$$

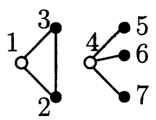
Then I and I'' are doubly linked, that is, we can find an ideal I' such that $I \sim I' \sim I''$.

If, in addition, I'' is almost complete intersection, then I is licci.

Lemma 6.6. *Let $\mathcal{H}$ be a Cohen-Macaulay hypergraph of deviation 2. Let $\mathcal{H}'$ be the hypergraph given by removing all 2-faces from $\mathcal{H}$. Then $I_{\mathcal{H}}$ and $I_{\mathcal{H}'}$ are linked.*

For instance,

$\mathcal{H} =$ 

$\mathcal{H}' =$ 

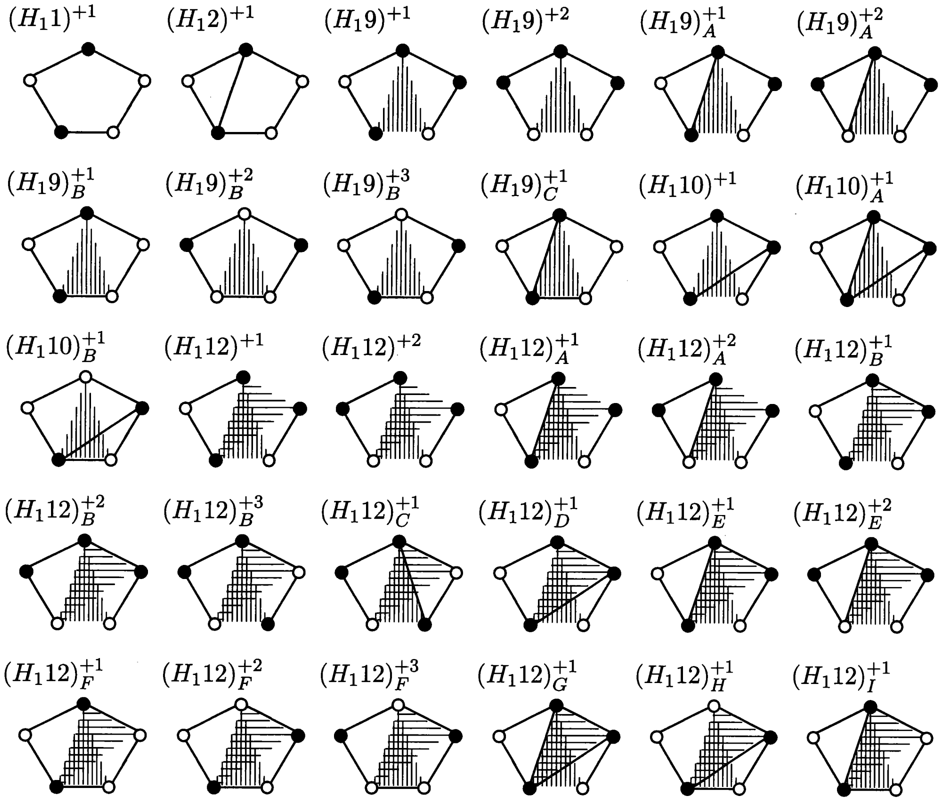
7. Classification of Cohen-Macaulay hypergraphs of deviation 2

For the convenience of the readers, we describe the list of Cohen-Macaulay hypergraphs of deviation 2 in [9] here.

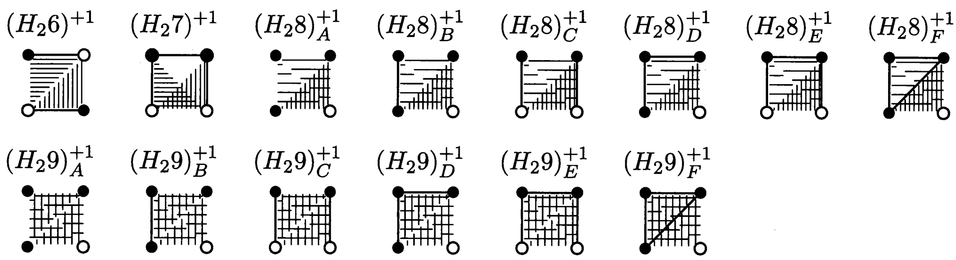
Theorem 7.1 ([9, Theorem 4.9]). *Let $\mathcal{H}$ be a hypergraph without isolated vertices associated with a squarefree monomial ideal of deviation 2. Then it is Cohen–Macaulay if and only if it is a hypergraph of either type (A) or type (B) :*

(A) *a disjoint union of two Cohen–Macaulay hypergraphs of deviation* 1;
(B) *some* 0*-extension of* $\mathcal{H}$ *is isomorphic to one of the following hypergraphs*:

Case 1:

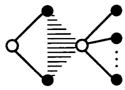


Case 2:

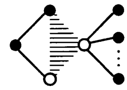


Case 3:

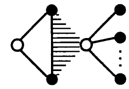
$(H_33)^{+1}_A$



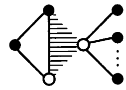
$(H_33)^{+2}_A$



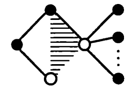
$(H_33)^{+1}_B$



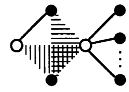
$(H_33)^{+2}_B$



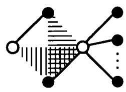
$(H_33)^{+1}_C$



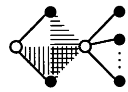
$(H_35)^{+1}_A$



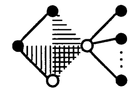
$(H_35)^{+1}_B$



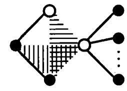
$(H_35)^{+1}_C$



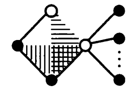
$(H_35)^{+2}_C$



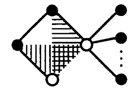
$(H_35)^{+3}_C$



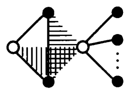
$(H_35)^{+1}_D$



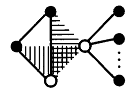
$(H_35)^{+1}_E$



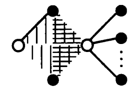
$(H_35)^{+1}_F$



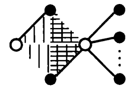
$(H_35)^{+2}_F$



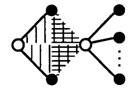
$(H_36)^{+1}_A$



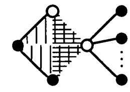
$(H_36)^{+1}_B$



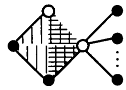
$(H_36)^{+1}_C$



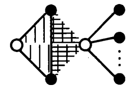
$(H_36)^{+2}_C$



$(H_36)^{+1}_D$

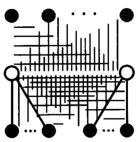


$(H_36)^{+1}_E$

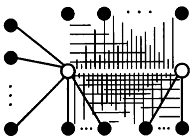


Case 4:

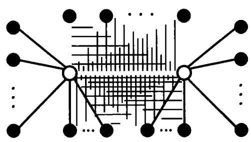
$(H_43)_A$



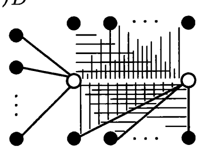
$(H_43)_B$



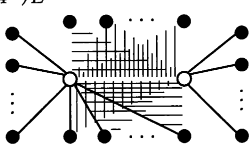
$(H_43)_C$



$(H_43)_D$

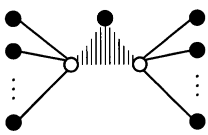


$(H_43)_E$

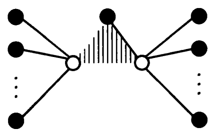


Case 5:

$(H_72)_A$



$(H_72)_B$



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ULRICH MODULES
— A GENERALIZATION —

S. GOTO, R. TAKAHASHI, AND K. OZEKI

1. Introduction

Throughout this note, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 0$. In [BHU] B. Ulrich and other authors explored the structure of **MGMCM** (maximally generated maximal Cohen–Macaulay) A–modules, that is maximal Cohen–Macaulay A–modules M with $e^0_{\mathfrak{m}}(M) = \mu_A(M)$, where $e^0_{\mathfrak{m}}(M)$ (resp. $\mu_A(M)$) denotes the multiplicity of M with respect to $\mathfrak{m}$ (resp. the number of elements in a minimal system of generators of M). In [HK] these modules are simply called *Ulrich* modules.

The purpose of our note is to study Ulrich modules and ideals in a slightly generalized form. Let I be an $\mathfrak{m}$–primary ideal in A and assume that I contains a parameter ideal Q of A as a reduction.

Definition 1.1. Our ideal I is called a Ulrich ideal of A, if

(1) $I \supsetneq Q$,
(2) $I^2 = QI$, and
(3) I/I^2 is A/I–free.

Condition (2) together with condition (1) is equivalent to saying that the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is a Cohen–Macaulay ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - d$, so that Definition 1.1 is independent of the choice of minimal reductions Q of I. If I is a Ulrich ideal, then I/Q is a free A/I–module with $\mathrm{rank}_{A/I}\, I/Q = \mu_A(I) - d$. Therefore, when A is a Gorenstein ring, Ulrich ideals are *good* ideals in the sense of [GIW].

Definition 1.2. Let M ($\neq (0)$) be a finitely generated A–module. Then we say that M is a Ulrich A–module with respect to I, if

(1) M is a Cohen-Macaulay A–module with $\dim_A M = d$,
(2) $e^0_I(M) = \ell_A(M/IM)$, and

AMS 2000 *Mathematics Subject Classi cation:* 13H10, 13H15, 13A30.

(3) M/IM is A/I–free,

where $\mathrm{e}_I^0(M)$ denotes the multiplicity of M with respect to I and $\ell_A(*)$ denotes the length.

Ulrich modules with respect to the maximal ideal $\mathfrak{m}$ of A are **MGMCM** modules in the original sense of [BHU]. Notice that condition (2) in Definition 1.2 is equivalent to saying that $IM = QM$, since M is a Cohen–Macaulay A–module with $\dim_A M = \dim A$ and the parameter ideal Q of A is a reduction of I.

In this note we shall discuss some basic properties of Ulrich modules and ideals, the relation between them, and the structure of minimal free resolutions of Ulrich ideals with some applications.

2. Examples

Let us note the following example.

Example 2.1. Let R be a Cohen–Macaulay local ring with maximal ideal $\mathfrak{n}$ and $\dim R = d$. Let $F = R^n$ for $n > 0$ and

$$A = R \ltimes F$$

the idealization of F over R. Let $\mathfrak{q}$ be a parameter ideal of R and put $I = \mathfrak{q} \times F$ and $Q = \mathfrak{q}A$ $(= \mathfrak{q} \times \mathfrak{q}F)$. Then A is a d-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} = \mathfrak{n} \times F$ and I is an $\mathfrak{m}$–primary ideal of A which contains the parameter ideal Q of A as a reduction. It is standard to check that I is a Ulrich ideal of A.

Hence this ring A contains *infinitely* many Ulrich ideals.

We begin with the following.

Theorem 2.2. *Suppose that I is a Ulrich ideal in A. Then for all $i \geq d$,*

$$\mathrm{Syz}_A^i(A/I)$$

is a Ulrich A–module with respect to I, where $\mathrm{Syz}_A^i(A/I)$ *denotes the i th syzygy of A/I in a minimal free resolution.*

Theorem 2.2 is proven by induction on d. Here we shall explain the basic technique of induction. For the moment, assume that $d > 0$ and let $a \in Q \setminus \mathfrak{m}Q$. Then $a \notin \mathfrak{m}I$. Let $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, and $\overline{Q} = Q/(a)$. We then have the following.

Fact 2.3. The ideal $\overline{I}$ is a Ulrich ideal of $\overline{A}$, if I is a Ulrich ideal of A.

Proof. The exact sequence $0 \to [I^2 + (a)]/I^2 \xrightarrow{\varphi} I/I^2 \to \overline{I}/\overline{I}^2 \to 0$ is split, since

$$[I^2 + (a)]/I^2 \cong (a)/[(a) \cap I^2] \cong (a)/aI \cong A/I$$

and since the homomorphism φ sends 1 to $\overline{a} = a + I^2$ which is a part of an A/I–free basis of I/I^2. Thus $\overline{I}/\overline{I}^2$ is $\overline{A}/\overline{I}$–free. □

Fact 2.4 (Vasconcelos [V]). Suppose that I/I^2 is A/I–free. Then

$$\mathrm{Syz}^i_A(A/I)/a{\cdot}\mathrm{Syz}^i_A(A/I) \cong \mathrm{Syz}^{i-1}_{\overline{A}}(\overline{A}/\overline{I}) \oplus \mathrm{Syz}^i_{\overline{A}}(\overline{A}/\overline{I})$$

for all $i \geq 1$.

Proof. It is enough to show that the exact sequence

$$0 \to A/I \xrightarrow{\widehat{a}} I/aI \to \overline{I} \to 0$$

is split. We write $I = (a) + (x_1, x_2, \ldots, x_\ell)$ with $\ell = \mu_A(I) - 1$. Then

$$I/aI = A\overline{a} + \sum_{i=1}^{n} A\overline{x_i}$$

where $\overline{a}$ and $\overline{x_i}$ denote the images of a and x_i in I/aI, respectively. We claim that this sum is direct. Assume that $c\overline{a} + \sum_{i=1}^{n} c_i\overline{x_i} = 0$ in I/aI with $c, c_i \in A$. Then, because I/I^2 is a homomorphic image of I/aI, we still have that

$$c\overline{a} + \sum_{i=1}^{n} c_i\overline{x_i} = 0$$

in I/I^2 (here $\overline{a}$ and $\overline{x_i}$ denote the images of a and x_i in I/I^2, respectively). Since $\{\overline{a}, \overline{x_i} \in I/I^2\ (1 \leq i \leq n)\}$ forms a free A/I-basis of I/I^2, we get $c \in I$. Thus $c\overline{a} = \sum_{i=1}^{n} c_i\overline{x_i} = 0$ in I/aI, so that $I/aI \cong A/I \oplus \overline{I}$. □

Theorem 2.2 now readily follows by induction on d. Remember that when $d = 0$, we get $I^2 = (0)$ and $I \cong (A/I)^n$ ($n = \mu_A(I) > 0$). Hence

$$\mathrm{Syz}^i_A(A/I) \cong (A/I)^{n^i}$$

for all $i \geq 0$.

Remark 2.5. Fact 2.4 was known by W. V. Vasconcelos [V]. Using this, he proved the famous result that an ideal I ($\neq A$) in a Noetherian local ring $(A, \mathfrak{m})$ is generated by an A–regular sequence, if I has finite projective dimension and if the A/I–module I/I^2 is free. Hence A is a RLR, once $\mathfrak{m}$ has finite projective dimension.

3. Relation between Ulrich modules and Ulrich ideals

The converse of Theorem 2.2 is also true. Namely we have the following.

Theorem 3.1. *The following conditions are equivalent.*

(1) *I is a Ulrich ideal of A.*

(2) $\mathrm{Syz}_A^i(A/I)$ *is a Ulrich A–module with respect to I for all $i \geq d$.*

(3) *There exists an exact sequence*

$$0 \to X \to F \to Y \to 0$$

of finitely generated A–modules such that

(a) *F is free,*

(b) *$X \subseteq \mathfrak{m}F$, and*

(c) *both X and Y are Ulrich A–modules with respect to I.*

When $d > 0$, one can add the following.

(4) *$\mu_A(I) > d$, I/I^2 is A/I–free, and* $\mathrm{Syz}_A^i(A/I)$ *is a Ulrich A–module with respect to I for some $i \geq d$.*

The implication (3) $\Rightarrow$ (1) of Theorem 3.1 is based on the following.

Proposition 3.2. *Let*

$$0 \to X \to F \to Y \to 0$$

be an exact sequence of finitely generated A-modules and assume that

(i) *F is a free A-module,*

(ii) *$X \subseteq \mathfrak{m}F$, and*

(iii) *Y is a Ulrich A-module with respect to I.*

Then X is a Ulrich A-module with respect to I if and only if I is a Ulrich ideal of A.

Proof. Let us note the proof of the *only if* part. We may assume that the field $A/\mathfrak{m}$ is infinite. Suppose that X is a Ulrich A–module with respect to I and look at the exact sequence

$$0 \to X/QX \to F/QF \to Y/QY \to 0;$$

hence $X/QX = \mathrm{Syz}_{A/Q}^1(Y/QY)$. Then, because

$$Y/QY = Y/IY \cong (A/I)^r$$

($r = \mathrm{rank}_A F > 0$), $X/QX \cong (I/Q)^r$, so that $I \supsetneq Q$ and $I^2 \subseteq Q$, because $X \neq (0)$ and $QX = IX$. Besides, I/Q is a free A/I–module, since $X/IX \cong (I/Q)^r$ and X/IX

is a free A/I–module. With the condition that for *every* minimal reduction Q of I, (1) $I^2 \subseteq Q$ and (2) I/Q is a free A/I–module, one can deduce that I/I^2 is A/I–free and that $I^2 = QI$ as well, which we leave to the reader. $\square$

We are now in a position to finish the proof of Theorem 3.1.

Proof of Theorem 3.1. (4) $\Rightarrow$ (1) Let $a \in Q \setminus \mathfrak{m}Q$ and put $\overline{A} = A/(a)$, $\overline{I} = I/(a)$, $\overline{Q} = Q/(a)$. Then by Lemma 2.4

$$\mathrm{Syz}_A^i(A/I)/a{\cdot}\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \oplus \mathrm{Syz}_{\overline{A}}^i(\overline{A}/\overline{I}).$$

Therefore, because both $\mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I})$ and $\mathrm{Syz}_{\overline{A}}^i(\overline{A}/\overline{I})$ are Ulrich modules with respect to $\overline{I}$, thanks to the implication (3) $\Rightarrow$ (1) $\overline{I}$ is a Ulrich ideal of $\overline{A}$. Hence $I^2 \subseteq Q$, which yields $I^2 = QI$, because I/I^2 is A/I–free. $\square$

Remark 3.3. Let $k[[X]]$ be the formal power series ring over a filed k and put $A = k[[X]]/(X^3)$. We look at the exact sequence

$$0 \to \mathfrak{m}^2 \to A \xrightarrow{x} A \to A/\mathfrak{m} \to 0$$

of A–modules, where x denotes the image of X in A and $\mathfrak{m} = (x)$ the maximal ideal in A. Then, since $\mathfrak{m}^3 = (0)$, the A–module

$$\mathfrak{m}^2 = \mathrm{Syz}_A^2(A/\mathfrak{m})$$

is a Ulrich module with respect to $\mathfrak{m}$, but $\mathfrak{m}$ not a Ulrich ideal of A, since $\mathfrak{m}^2 \neq (0)$. This example shows that the implication (4) $\Rightarrow$ (1) is not true in general, unless $d > 0$.

Question 3.4. It seems interesting to explore how many Ulrich ideals are contained in a given Cohen–Macaulay local ring. For example, let $k[[t]]$ be the formal power series ring over a field k and let

$$A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq k[[t]]$$

be a numerical semigroup ring, where $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ such that $\mathrm{GCD}(a_1, a_2, \ldots, a_\ell) = 1$. Let $\mathcal{X}_A^g$ be the set of Ulrich ideals in A which are generated by monomials in t. It is then not difficult to check that $\mathcal{X}_A^g$ is finite and for example, we have the following.

(1) $\mathcal{X}_{k[[t^3,t^4,t^5]]}^g = \{\mathfrak{m}\}$.
(2) $\mathcal{X}_{k[[t^4,t^5,t^6]]}^g = \{(t^4, t^6)\}$.
(3) $\mathcal{X}_{k[[t^a,t^{a+1},\ldots,t^{2a-2}]]}^g = \emptyset$, if $a \geq 5$.

(4) Let $1 < a < b$ be integers such that $\mathrm{GCD}(a,b) = 1$. Then $\mathcal{X}^g_{k[[t^a,t^b]]} \neq \emptyset$ if and only if a or b is even.

(5) Let $A = k[[t^4, t^6, t^{4\ell-1}]]$ $(\ell \geq 2)$. Then $\sharp \mathcal{X}^g_A = 2\ell - 2$.

4. Minimal free resolutions of Ulrich ideals

We now explore minimal free resolutions of Ulrich ideals. Let I be a Ulrich ideal of A which contains a parameter ideal Q of A as a reduction.

Let

$$\cdots \to F_i \overset{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{\partial_1}{\to} F_0 \to A/I \to 0$$

be a minimal free resolution of A/I and let $\beta_i = \beta_i^A(A/I)$ $(i \geq 0)$ be the i-th Betti number of A/I. We put $n = \mu_A(I) = \beta_1$.

We then have the following, which is proven by induction on d.

Theorem 4.1. *The following assertions hold true.*

(1) $A/I \otimes_A \partial_i = 0$ *for all* $i \geq 1$.

(2)

$$\beta_i = \begin{cases} (n-d)^{i-d} \cdot (n-d+1)^d & (i \geq d), \\ \binom{d}{i} + (n-d)\beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0) \end{cases}$$

for $i \geq 0$. *Hence* $\beta_i = \binom{d}{i} + (n-d)\beta_{i-1}$ *for all* $i \geq 1$.

What Theorem 4.1 (3) says is the following.

Corollary 4.2. *The minimal free resolution of I is obtained by the direct sum of those of Q and $(A/I)^{n-d}$.*

Corollary 4.3. $\mathrm{Syz}_A^{i+1}(A/I) \cong [\mathrm{Syz}_A^i(A/I)]^{n-d}$ *for all* $i \geq d$. *Hence*

$$\mathrm{Syz}_A^{i+1}(A/I) \cong \mathrm{Syz}_A^i(A/I)$$

for all $i \geq d$, if A is a Gorenstein ring.

This result shows we can expect, in some sense, only one Ulrich module arising from syzygies. We furthermore have the following. Let $\mathrm{I}_1(\partial_i)$ $(i \geq 1)$ be the ideal of A generated by the entries of the matrix ∂_i.

Theorem 4.4. $\mathrm{I}_1(\partial_i) = I$ *for all* $i \geq 1$.

Proof. By induction on d, we have $\mathrm{I}_1(\partial_i) + Q = I$ for all $i \geq 1$, while by Corollary 4.2 $\mathrm{I}_1(\partial_i) \supseteq Q$ for $1 \leq i \leq d$. Therefore, since by Corollary 4.2 $\mathrm{I}_1(\partial_{i+1}) = \mathrm{I}_1(\partial_i)$ if $i \geq d$, the result readily follows. ☐

Corollary 4.5. *Let I and J be Ulrich ideals of A. Then $I = J$ if and only if*

$$\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_A^i(A/J)$$

for some $i \geq 0$.

Let $\mathcal{X}_A = \{I \mid I \text{ is a Ulrich ideal of } A\}$. We have the following answer to Question 3.4.

Theorem 4.6. *Suppose that A is of finite C-M representation type. Then $\mathcal{X}_A$ is a finite set.*

Proof. Let $\mathcal{Y}_A = \{[\mathrm{Syz}_A^d(A/I)] \mid I \in \mathcal{X}_A\}$, where $[\mathrm{Syz}_A^d(A/I)]$ denotes the isomorphic class of $\mathrm{Syz}_A^d(A/I)$. Let $I \in \mathcal{X}_A$ and $n = \mu_A(I)$. Then, because $I/Q \cong (A/I)^{n-d}$, we have

$$n - d \leq (n-d)\cdot \mathrm{r}_A(A/I) = \mathrm{r}_A(I/Q) \leq \mathrm{r}(A),$$

where $\mathrm{r}_A(*)$ denotes the Cohen–Macaulay type. Hence

$$\mu_A(\mathrm{Syz}_A^d(A/I)) = \beta_d^A(A/I) = (n-d+1)^d \leq (\mathrm{r}(A)+1)^d \ll \infty$$

by Theorem 4.1. Therefore, since A is of finite C-M representation type, the set $\mathcal{Y}_A$ is finite, so that $\mathcal{X}_A$ is also finite, because $\mathcal{X}_A \subseteq \mathcal{Y}_A$ by Proposition 4.5. ☐

Let us explore one example.

Example 4.7. Let $A = k[[X,Y,Z]]/(Z^2 - XY)$, where $k[[X,Y,Z]]$ is the formal power series ring over a field k. Then $\mathcal{X}_A = \{\mathfrak{m}\}$.

Proof. Let x, y, and z be the images of X, Y, and Z in A, respectively. Then the indecomposable maximal Cohen-Macaulay A–modules (up to isomorphisms) are A and $\mathfrak{p} = (x, z)$. Since $\mathfrak{m}^2 = (x,y)\mathfrak{m}$, we get $\mathfrak{m} \in \mathcal{X}_A$. Let $I \in \mathcal{X}_A$. Then $\mu_A(I) = 3$. We put $X = \mathrm{Syz}_A^2(I)$. Then, because $\mu_A(X) = 4$ and $\mathrm{rank}_A X = 2$, we see

$$X \cong \mathfrak{p} \oplus \mathfrak{p} \cong \mathrm{Syz}_A^2(A/\mathfrak{m}),$$

so that $I = \mathfrak{m}$ by Corollary 4.5. ☐

For one-dimensional Cohen-Macaulay local rings possessing finite C-M representation type, we have the following, where $k[[X,Y]]$ and $k[[t]]$ are the formal power series rings over a field k and x, y denote the images of X, Y in the corresponding ring, respectively.

Example 4.8. The following assertions hold true.

(1) $\mathcal{X}_{k[[t^3,t^4]]} = \{(t^4, t^6)\}$.
(2) $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.
(3) $\mathcal{X}_{k[[X,Y]]/(Y(X^2-Y^{2\ell+1}))} = \{(x, y^{2\ell+1}), (x^2, y)\}$, where $\ell \geq 1$.
(4) $\mathcal{X}_{k[[X,Y]]/(Y(Y^2-X^3))} = \{(x^3, y)\}$.
(5) $\mathcal{X}_{k[[X,Y]]/(X^2-Y^{2\ell})} = \{(x^2, y), (x - y^\ell, y(x + y^\ell)), (x + y^\ell, y(x - y^\ell))\}$, where $\ell \geq 1$.

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ASYMPTOTIC PERIODICITY OF PRIMES ASSOCIATED TO MULTIGRADED MODULES

FUTOSHI HAYASAKA

1. Introduction

Let $A \subseteq B$ be a graded ring extension of Noetherian $\mathbb{N}^r$-graded rings generated in degrees $\mathbf{d}_1, \ldots, \mathbf{d}_r$, which are linearly independent vectors over $\mathbb{R}$, with $A_{\mathbf{0}} = B_{\mathbf{0}} = R$. Let $N = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} N_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded A-module and let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded B-module. Assume that N is a graded A-submodule of M. Then what we want to study in this note is the following:

Problem 1.1. *How does the set* $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}})$ *behave asymptotically?*

The problem of this type was originated in a question of Ratliff on the asymptotic behavior of $\mathrm{Ass}_R(R/I^n)$ for a given ideal I in R. This is a special case in our setting where $A = \mathcal{R}(I)$ is the Rees algebra of I and $B = R[t]$ is a polynomial ring over R. In 1979, Brodmann [2] gave an answer to the question of Ratliff showing that the set $\mathrm{Ass}_R(R/I^n)$ is stable for all large $n \gg 0$. Since then, the study of asymptotic prime divisors began and many results have been obtained in the ideal cases. See [10] for the details. One of the direction of the study is to extend Brodmann's original result to more general cases. Problem 1.1 in the standard multigraded cases were investigated by many authors and we know nowadays the corresponding stability results in this cases. Then it is natural to ask the following: What happens in the nonstandard multigraded cases? In this note, we investigate the asymptotic behavior of $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}})$ in the general setting and prove that asymptotic periodicity occurs in a cone. Our main result is the following:

Theorem 1.2. *With our setting as above, there exists a vector* $\mathbf{k} \in \mathbb{N}^r$ *such that, in the cone* $C_{\mathbf{k}}$ *with vertex* $\mathbf{k}$ *generated by* $\mathbf{d}_1, \ldots, \mathbf{d}_r$, *the set* $\mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}})$ *is periodic with respect to* $\mathbf{d}_1, \ldots, \mathbf{d}_r$. *Namely, the equality* $\mathrm{Ass}_R(M_{\mathbf{n}+\mathbf{m}}/N_{\mathbf{n}+\mathbf{m}}) = \mathrm{Ass}_R(M_{\mathbf{n}}/N_{\mathbf{n}})$ *holds true for all* $\mathbf{n} \in C_{\mathbf{k}}$ *and all* $\mathbf{m} \in \Gamma$, *where* Γ *is the semigroup generated by* $\mathbf{d}_1, \ldots, \mathbf{d}_r$.

This recovers all the known results [1, 2, 5, 7, 8, 9, 11, 12] on the asymptotic prime divisors in the standard multigraded cases including the ideal cases. Moreover, by using the technique due to Brodmann [3], we have the same asymptotic behavior of $\operatorname{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}})$ for a given ideal $\mathfrak{a}$ in R as a direct consequence of Theorem 1.2.

Theorem 1.3. *Under the same situation as in Theorem 1.2, for any ideal $\mathfrak{a}$ in R, there exists a vector $\mathbf{k} \in \mathbb{N}^r$ such that, in the cone $C_{\mathbf{k}}$ with vertex $\mathbf{k}$ generated by $\mathbf{d}_1, \dots, \mathbf{d}_r$, $\operatorname{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}})$ is periodic with respect to $\mathbf{d}_1, \dots, \mathbf{d}_r$. Namely, the equality $\operatorname{grade}(\mathfrak{a}, M_{\mathbf{n}+\mathbf{m}}/N_{\mathbf{n}+\mathbf{m}}) = \operatorname{grade}(\mathfrak{a}, M_{\mathbf{n}}/N_{\mathbf{n}})$ holds true for all $\mathbf{n} \in C_{\mathbf{k}}$ and all $\mathbf{m} \in \Gamma$.*

This also recovers all the known results [1, 3, 5, 12] on the asymptotic grade/depth in the standard multigraded cases and generalizes the recent result due to Colomé-Nin and Elias [4] and our previous work [6].

In the next section, we fix our notation and recall some basic results about cones and graded modules that we need in the proof of Theorem 1.2. In section 3, we will give a proof of Theorem 1.2.

Throughout this note, $\mathbb{N}$ (resp. $\mathbb{R}$) denotes the set of non-negative integers (resp. real numbers), and r is any fixed positive integer. Vectors will be always written by Bold-faced letters, e.g., $\mathbf{a}$ or Greek alphabet, e.g., α, β, δ.

2. Preliminaries

Let $\mathbf{d}_1, \dots, \mathbf{d}_r \in \mathbb{N}^r$ be any fixed linearly independent vectors over $\mathbb{R}$. We denote $\Gamma = \{\sum_{i=1}^r c_i\mathbf{d}_i \mid c_i \in \mathbb{N}\} \subseteq \mathbb{N}^r$ the semigroup generated by the fixed vectors $\mathbf{d}_1, \dots, \mathbf{d}_r$. For any vector $\mathbf{k} \in \mathbb{N}^r$, let

$$C_{\mathbf{k}} := \left\{ \mathbf{k} + \sum_{i=1}^r c_i\mathbf{d}_i \mid c_i \in \mathbb{R}_{\geq 0} \right\} \cap \mathbb{N}^r$$

be the *cone with vertex* $\mathbf{k}$ *generated by* $\mathbf{d}_1, \dots, \mathbf{d}_r$ and let

$$\Delta_{\mathbf{k}} := \left\{ \mathbf{k} + \sum_{i=1}^r c_i\mathbf{d}_i \mid 0 \leq c_i < 1,\ c_i \in \mathbb{R} \right\} \cap \mathbb{N}^r$$

be the *basic cell* of the cone $C_{\mathbf{k}}$. Then it is easy to see that

(i) $\Delta_{\mathbf{k}}$ is a finite subset of $C_{\mathbf{k}}$,
(ii) for any $\mathbf{n} \in C_{\mathbf{k}}$, there is a unique expression $\mathbf{n} = \delta + \mathbf{m}$ with $\delta \in \Delta_{\mathbf{k}}$ and $\mathbf{m} \in \Gamma$, and
(iii) $C_{\mathbf{k}} = \bigcup_{\delta \in \Delta_{\mathbf{k}}} (\delta + \Gamma)$.

Moreover, by [6, Lemma 2.1], one can easily see that

(iv) for any vectors $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^r$, there exists an integer $\ell_0 \geq 0$ such that $\mathbf{k}' + \ell(\mathbf{d}_1 + \cdots + \mathbf{d}_r) \in C_{\mathbf{k}}$ for all $\ell \geq \ell_0$. In particular, $C_{\mathbf{k}} \cap C_{\mathbf{k}'} \neq \emptyset$ and hence there exists a cone $C_{\mathbf{k}''}$ such that $C_{\mathbf{k}''} \subseteq C_{\mathbf{k}} \cap C_{\mathbf{k}'}$.

Let $B = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} B_{\mathbf{n}}$ be a Noetherian $\mathbb{N}^r$-graded ring generated in degrees $\mathbf{d}_1, \ldots, \mathbf{d}_r$ with $B_{\mathbf{0}} = R$. Let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^r$-graded B-module. For any vector $\delta \in \mathbb{N}^r$, we set $M^{(\delta+\Gamma)} = \bigoplus_{\mathbf{m} \in \Gamma} M_{\delta+\mathbf{m}}$, which is a graded B-submodule of M. Let $\mathbf{k}_0 \in \mathbb{N}^r$ be any fixed vector. With this notation, we have the following two elementary lemmas that we need in the proof of Theorem 1.2.

Lemma 2.1. *Let $\delta \in \Delta_{\mathbf{k}_0}$. Then there exists a vector $\mathbf{k} = \mathbf{k}(\delta) \in \delta + \Gamma$, depending on the choice of δ, such that $[(0) :_{M^{(\delta+\Gamma)}} B_{\mathbf{d}_i}]_{\mathbf{m}+\mathbf{k}} = (0)$ for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$.*

Proof. Let $\delta \in \Delta_{\mathbf{k}_0}$. Fix i and write $[(0) :_{M^{(\delta+\Gamma)}} B_{\mathbf{d}_i}] = By_1 + \cdots + By_s$ where $y_j \in M_{\delta+\mathbf{m}_j}$ and $\mathbf{m}_j \in \Gamma$. Write each $\mathbf{m}_j = \sum_{k=1}^{r} \alpha_{kj} \mathbf{d}_k$ where $\alpha_{kj} \in \mathbb{N}$. Let $\beta_i := 1 + \max\{\alpha_{ij} \mid j = 1, \ldots, s\}$ and put $\mathbf{k}_i := \beta_i \mathbf{d}_i$. Then $[(0) :_{M^{(\delta+\Gamma)}} B_{\mathbf{d}_i}]_{\mathbf{m}+\delta+\mathbf{k}_i} = (0)$ for all $\mathbf{m} \in \Gamma$. Indeed, suppose $y \in [(0) :_{M^{(\delta+\Gamma)}} B_{\mathbf{d}_i}]_{\mathbf{m}+\delta+\mathbf{k}_i}$ and write $y = b_1 y_1 + \cdots + b_s y_s$ where $b_j \in B$. We may assume that each $b_j \in B_{\mathbf{m}+\delta+\mathbf{k}_i-(\delta+\mathbf{m}_j)} = B_{\mathbf{m}+\mathbf{k}_i-\mathbf{m}_j}$. If we write the vector $\mathbf{k}_i - \mathbf{m}_j$ as the linear combination of $\mathbf{d}_1, \ldots, \mathbf{d}_r$, then the integer coefficient of $\mathbf{d}_i$ must be positive. Hence each $b_j y_j \in B_{\mathbf{d}_i} B \cdot y_j = (0)$. Thus $y = 0$. Here, by putting $\mathbf{k} := \delta + \mathbf{k}_1 + \cdots + \mathbf{k}_r \in \delta + \Gamma$, we have the assertion. □

Lemma 2.2. *Let $\delta \in \Delta_{\mathbf{k}_0}$. Then there exists a vector $\mathbf{k} = \mathbf{k}(\delta) \in \delta + \Gamma$, depending on the choice of δ, such that for any graded B-submodule H of $M^{(\delta+\Gamma)}$, $\mathrm{Ass}_R(H_{\mathbf{m}+\mathbf{k}}) \subseteq \mathrm{Ass}_R(H_{\mathbf{m}+\mathbf{k}+\mathbf{d}_i})$ for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$.*

Proof. Let $\delta \in \Delta_{\mathbf{k}_0}$. By Lemma 2.1, there exists $\mathbf{k} \in \delta + \Gamma$ such that $[(0) :_{M^{(\delta+\Gamma)}} B_{\mathbf{d}_i}]_{\mathbf{m}+\mathbf{k}} = (0)$ for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$. This vector $\mathbf{k}$ is our desired one. Let H be a graded B-submodule of $M^{(\delta+\Gamma)}$. Fix i and take any $P \in \mathrm{Ass}_R(H_{\mathbf{m}+\mathbf{k}})$. Write $P = [(0) :_R h]$ for some nonzero $h \in H_{\mathbf{m}+\mathbf{k}}$. Since $[(0) :_H B_{\mathbf{d}_i}]_{\mathbf{m}+\mathbf{k}} \subseteq [(0) :_{M^{(\delta+\Gamma)}} B_{\mathbf{d}_i}]_{\mathbf{m}+\mathbf{k}} = (0)$, we have $P = [(0) :_R h] = [(0) :_R B_{\mathbf{d}_i} h]$. Let $w_1, \ldots, w_s \in B_{\mathbf{d}_i} h$ generate an R-module $B_{\mathbf{d}_i} h$. Then $P = [(0) :_R B_{\mathbf{d}_i} h] = \bigcap_{j=1}^{s} [(0) :_R w_j]$ and hence $P = [(0) :_R w_j]$ for some j. Since $w_j \in B_{\mathbf{d}_i} h \subseteq H_{\mathbf{m}+\mathbf{k}+\mathbf{d}_i}$, we have $P \in \mathrm{Ass}_R(H_{\mathbf{m}+\mathbf{k}+\mathbf{d}_i})$. □

Before closing this preliminary section, we recall one more lemma we need in the proof of Theorem 1.2.

Lemma 2.3. *Let B be a Noetherian ring which is not necessarily graded and let $J_1, \ldots, J_r$ be ideals in B. Let M be a finitely generated B-module and F a submodule of M. Then there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ such that*

(1) $J_1^{\beta_1} \cdots J_i^{\beta_i+1} \cdots J_r^{\beta_r} F :_M J_i \subseteq J_1^{\beta_1} \cdots J_r^{\beta_r} F + \mathrm{H}^0_{J_i}(M)$, *and*
(2) $J_1^{\beta_1} \cdots J_r^{\beta_r} F \cap \mathrm{H}^0_{J_i}(M) = (0)$

for all $i = 1, \ldots, r$ and all $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ satisfying $\beta \geq \alpha$, that is, each $\beta_j \geq \alpha_j$. Here $\mathrm{H}^0_{J_i}(M)$ denotes the 0-th local cohomology module of M with respect to the ideal J_i.

Proof. See [5, Lemma 2.2]. □

3. Proof of Theorem 1.2

Let me give a proof of Theorem 1.2. Let $A \subseteq B$ be a graded ring extension of Noetherian $\mathbb{N}^r$-graded rings generated in degrees $\mathbf{d}_1, \ldots, \mathbf{d}_r$, where $\mathbf{d}_1, \ldots, \mathbf{d}_r$ are linearly independent vectors over $\mathbb{R}$, with $A_\mathbf{0} = B_\mathbf{0} = R$. Let $N = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} N_\mathbf{n}$ be a finitely generated $\mathbb{Z}^r$-graded A-module and let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_\mathbf{n}$ be a finitely generated $\mathbb{Z}^r$-graded B-module. Assume that N is a graded A-submodule of M. For any vector $\mathbf{n} \in \mathbb{N}^r$, we set $\mathrm{A}(\mathbf{n}) := \mathrm{Ass}_R(M_\mathbf{n}/N_\mathbf{n})$ for short. We begin with the following.

Proposition 3.1. *Let $\mathbf{k}_0 \in \mathbb{N}^r$. Then for any $\delta \in \Delta_{\mathbf{k}_0}$, there exists a vector $\mathbf{k} = \mathbf{k}(\delta) \in \delta + \Gamma$, depending on the choice of δ, such that*

$$\mathrm{A}(\mathbf{m} + \mathbf{k}) \subseteq \mathrm{A}(\mathbf{m} + \mathbf{k} + \mathbf{d}_i)$$

for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$.

Proof. Fix $\delta \in \Delta_{\mathbf{k}_0}$. Since N is a finitely generated A-module, so is $N^{(\delta+\Gamma)}$. Hence there exists $\mathbf{k}_1 \in \delta + \Gamma$ such that $N_{\mathbf{m}+\mathbf{k}_1} = A_\mathbf{m} N_{\mathbf{k}_1}$ for all $\mathbf{m} \in \Gamma$. Let $F := BN_{\mathbf{k}_1}$ be a graded B-submodule of $M^{(\delta+\Gamma)}$ generated by the elements of $N_{\mathbf{k}_1}$. For $i = 1, \ldots, r$, let $J_i := A_{\mathbf{d}_i} B$ be an ideal in B generated by the elements of $A_{\mathbf{d}_i}$ and let $H_i := H^0_{J_i}(M^{(\delta+\Gamma)})$ be the 0-th local cohomology module of $M^{(\delta+\Gamma)}$ with respect to the ideal J_i. Then, by Lemma 2.3, there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ such that

$$(1) \qquad J_1^{\beta_1} \cdots J_i^{\beta_i+1} \cdots J_r^{\beta_r} F :_{M^{(\delta+\Gamma)}} J_i \subseteq J_1^{\beta_1} \cdots J_r^{\beta_r} F + H_i$$

$$(2) \qquad J_1^{\beta_1} \cdots J_r^{\beta_r} F \cap H_i = (0)$$

for all $i = 1, \ldots, r$ and all $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ satisfying $\beta \geq \alpha$. Let $\mathbf{k}_2 := \alpha_1\mathbf{d}_1 + \cdots + \alpha_r\mathbf{d}_r \in \Gamma$. Then the assertions (1) and (2) hold true for all $i = 1, \ldots, r$ and all $\beta \in \mathbb{N}^r$ satisfying $\beta_1\mathbf{d}_1 + \cdots + \beta_r\mathbf{d}_r \in \mathbf{k}_2 + \Gamma$. By Lemma 2.2, there exists a vector $\mathbf{k}_3 \in \Gamma$ such that

$$(3) \qquad \operatorname{Ass}_R([H_i]_{\mathbf{m}+\delta+\mathbf{k}_3}) \subseteq \operatorname{Ass}_R([H_i]_{\mathbf{m}+\delta+\mathbf{k}_3+\mathbf{d}_i})$$

for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$. Let $\mathbf{k} := \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 \in \delta + \Gamma$. Then we claim that

Claim. $\mathrm{A}(\mathbf{m}+\mathbf{k}) \subseteq \mathrm{A}(\mathbf{m}+\mathbf{k}+\mathbf{d}_i)$ for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$.

Fix i and take any $\mathbf{m} \in \Gamma$, and put $H := H_i$. Since $\mathbf{m}+\mathbf{k}_2+\mathbf{k}_3 \in \Gamma$, we can write $\mathbf{m}+\mathbf{k}_2+\mathbf{k}_3 = \beta_1\mathbf{d}_1+\cdots+\beta_r\mathbf{d}_r$ for some $\beta_j \geq 0$. Let $a_1, \ldots, a_p$ generate the ideal J_i. Consider the graded B-linear map $\varphi = {}^t(a_1, \ldots, a_p)$

$$\varphi : M^{(\delta+\Gamma)}(-\mathbf{d}_i) \to \left(M^{(\delta+\Gamma)}/[J_1^{\beta_1} \cdots J_i^{\beta_i+1} \cdots J_r^{\beta_r}F]\right)^{\oplus p}.$$

Then $\operatorname{Ker}\varphi = J_1^{\beta_1} \cdots J_i^{\beta_i+1} \cdots J_r^{\beta_r}F :_{M^{(\delta+\Gamma)}} J_i$ and we have the inclusion

$$\psi : \left(M^{(\delta+\Gamma)}/\operatorname{Ker}\varphi\right)(-\mathbf{d}_i) \hookrightarrow \left(M^{(\delta+\Gamma)}/[J_1^{\beta_1} \cdots J_i^{\beta_i+1} \cdots J_r^{\beta_r}F]\right)^{\oplus p}.$$

By taking a degree $(\mathbf{m}+\mathbf{k}+\mathbf{d}_i)$-part of ψ, we have that

$$\operatorname{Ass}_R(M_{\mathbf{m}+\mathbf{k}}/[\operatorname{Ker}\varphi]_{\mathbf{m}+\mathbf{k}}) \subseteq \mathrm{A}(\mathbf{m}+\mathbf{k}+\mathbf{d}_i).$$

Consider the exact sequence

$$0 \to K \to M^{(\delta+\Gamma)}/[J_1^{\beta_1} \cdots J_r^{\beta_r}F] \to M^{(\delta+\Gamma)}/\operatorname{Ker}\varphi \to 0,$$

where $K := \operatorname{Ker}\varphi/[J_1^{\beta_1} \cdots J_r^{\beta_r}F]$. By taking a degree $(\mathbf{m}+\mathbf{k})$-part of this exact sequence, we have that

$$\mathrm{A}(\mathbf{m}+\mathbf{k}) \subseteq \operatorname{Ass}_R(K_{\mathbf{m}+\mathbf{k}}) \cup \mathrm{A}(\mathbf{m}+\mathbf{k}+\mathbf{d}_i).$$

Therefore it is enough to show that $\operatorname{Ass}_R(K_{\mathbf{m}+\mathbf{k}}) \subseteq \mathrm{A}(\mathbf{m}+\mathbf{k}+\mathbf{d}_i)$. Note that $\operatorname{Ker}\varphi = J_1^{\beta_1} \cdots J_r^{\beta_r}F + [\operatorname{Ker}\varphi \cap H]$ by (1). Let $W := \operatorname{Ker}\varphi \cap H$. Then

$$\begin{aligned} K &= \operatorname{Ker}\varphi/[J_1^{\beta_1} \cdots J_r^{\beta_r}F] \\ &= [J_1^{\beta_1} \cdots J_r^{\beta_r}F + W]/[J_1^{\beta_1} \cdots J_r^{\beta_r}F] \\ &\cong W/[W \cap J_1^{\beta_1} \cdots J_r^{\beta_r}F] \\ &= W \qquad (\text{since } [W \cap J_1^{\beta_1} \cdots J_r^{\beta_r}F] = (0) \text{ by (2)}) \\ &\subseteq H. \end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Ass}_R(K_{\mathbf{m}+\mathbf{k}}) &\subseteq \mathrm{Ass}_R(H_{\mathbf{m}+\mathbf{k}}) \\ &\subseteq \mathrm{Ass}_R(H_{\mathbf{m}+\mathbf{k}+\mathbf{d}_i}) \qquad \text{(by (3))} \\ &\subseteq \mathrm{A}(\mathbf{m}+\mathbf{k}+\mathbf{d}_i).\end{aligned}$$

□

Proof of Theorem 1.2. Take any $\mathbf{k}_0 \in \mathbb{N}^r$ and fix $\delta \in \Delta_{\mathbf{k}_0}$. Put $\mathbf{d} := \mathbf{d}_1 + \cdots + \mathbf{d}_r$. We divide the proof into three steps:

Step 1. For the fixed vector $\delta \in \Delta_{\mathbf{k}_0}$, there exists a vector $\mathbf{k}_1 \in \delta + \Gamma$ such that $\mathrm{A}(\mathbf{m}+\mathbf{k}_1) \subseteq \mathrm{A}(\mathbf{m}+\mathbf{k}_1+\mathbf{d}_i)$ for all $i = 1, \ldots, r$ and all $\mathbf{m} \in \Gamma$.

This follows from Proposition 3.1.

Step 2. $\bigcup_{\ell \geq 0} \mathrm{A}(\mathbf{k}_1 + \ell\mathbf{d})$ is a finite set.

Since $N^{(\delta+\Gamma)}$ is a finitely generated A-module, there exists a vector $\mathbf{k}' \in \delta + \Gamma$ such that $N_{\mathbf{m}+\mathbf{k}'} = A_{\mathbf{m}} N_{\mathbf{k}'}$ for all $\mathbf{m} \in \Gamma$. Take an integer $\ell_0 \geq 0$ satisfying $\delta + \ell_0\mathbf{d} \in \mathbf{k}' + \Gamma$. Then the equalities

$$N_{\mathbf{k}_1+\ell\mathbf{d}} = A_{(\ell-\ell_0)\mathbf{d}} N_{\mathbf{k}_1+\ell_0\mathbf{d}} = [J^{\ell-\ell_0}L]_{\mathbf{k}_1+\ell\mathbf{d}}$$

hold true for all $\ell \geq \ell_0$, where $J := J_1 \cdots J_r$ and $L := BN_{\mathbf{k}_1+\ell_0\mathbf{d}}$. Thus $M_{\mathbf{k}_1+\ell\mathbf{d}}/N_{\mathbf{k}_1+\ell\mathbf{d}} = \left[M/J^{\ell-\ell_0}L\right]_{\mathbf{k}_1+\ell\mathbf{d}}$ for all $\ell \geq \ell_0$. Hence if $\ell \geq \ell_0$, then

$$\begin{aligned}\mathrm{A}(\mathbf{k}_1 + \ell\mathbf{d}) &= \mathrm{Ass}_R\left(\left[M/J^{\ell-\ell_0}L\right]_{\mathbf{k}_1+\ell\mathbf{d}}\right) \\ &\subseteq \mathrm{Ass}_R\left(M/J^{\ell-\ell_0}L\right) \\ &= \left\{Q \cap R \mid Q \in \mathrm{Ass}_B\left(M/J^{\ell-\ell_0}L\right)\right\}.\end{aligned}$$

Since it is easy to see that $\bigcup_{n\geq 0} \mathrm{Ass}_B\left(M/J^nL\right)$ is a finite set, we have $\bigcup_{\ell\geq\ell_0} \mathrm{A}(\mathbf{k}_1 + \ell\mathbf{d})$ is a finite set so that $\bigcup_{\ell\geq 0} \mathrm{A}(\mathbf{k}_1 + \ell\mathbf{d})$ is a finite set.

Step 3. There exists a vector $\mathbf{k}_2 \in \delta + \Gamma$ such that $\mathrm{A}(\mathbf{m} + \mathbf{k}_2) = \mathrm{A}(\mathbf{k}_2)$ for all $\mathbf{m} \in \Gamma$.

Step 1 and Step 2 imply that there exists an integer $\ell_1 \geq 0$ such that

$$\mathrm{A}(\mathbf{k}_1 + \ell\mathbf{d}) = \mathrm{A}(\mathbf{k}_1 + \ell_1\mathbf{d}) \tag{4}$$

for all $\ell \geq \ell_1$. Let $\mathbf{k}_2 := \mathbf{k}_1 + \ell_1\mathbf{d}$. This vector is our desired one. Indeed, take $\mathbf{m} \in \Gamma$ and write $\mathbf{m} = \beta_1\mathbf{d}_1 + \cdots + \beta_r\mathbf{d}_r$. Let $\beta := \max\{\beta_1, \ldots, \beta_r\}$. Then, by Step 1, $\mathrm{A}(\mathbf{k}_2) \subseteq \mathrm{A}(\mathbf{m} + \mathbf{k}_2) \subseteq \mathrm{A}(\mathbf{k}_2 + \beta\mathbf{d})$. Since $\mathrm{A}(\mathbf{k}_2) = \mathrm{A}(\mathbf{k}_2 + \beta\mathbf{d})$ by (4), we have $\mathrm{A}(\mathbf{k}_2) = \mathrm{A}(\mathbf{m} + \mathbf{k}_2)$.

Now, we note here that the above vector $\mathbf{k}_2 = \mathbf{k}_2(\delta)$ depends on the choice of $\delta \in \Delta_{\mathbf{k}_0}$. After these three steps for every point $\delta \in \Delta_{\mathbf{k}_0}$, by taking a cone

$$C_{\mathbf{k}} \subseteq C_{\mathbf{k}_0} \cap \left[\bigcap_{\delta \in \Delta_{\mathbf{k}_0}} C_{\mathbf{k}_2(\delta)} \right],$$

we have that $\mathrm{A}(\mathbf{n} + \mathbf{m}) = \mathrm{A}(\mathbf{n})$ for all $\mathbf{n} \in C_{\mathbf{k}}$ and all $\mathbf{m} \in \Gamma$. Indeed, since $\mathbf{n} \in C_{\mathbf{k}_0}$, there is a unique $\delta \in \Delta_{\mathbf{k}_0}$ such that $\mathbf{n} \in \delta + \Gamma$. Thus $\mathbf{n}, \mathbf{n} + \mathbf{m} \in C_{\mathbf{k}_2(\delta)} \cap (\delta + \Gamma) = \mathbf{k}_2(\delta) + \Gamma$ and hence $\mathrm{A}(\mathbf{n} + \mathbf{m}) = \mathrm{A}(\mathbf{n})$ by Step 3. This completes the proof. □

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F-thresholds on toric rings [1]

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1. Introduction

This is a joint work with Daisuke Hirose.

Throughout this talk, let R be a commutative Noetherian ring containing $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and suppose that R is an F-finite domain, that is, the Frobenius map $F\colon R \to R$, defined by $F(a) = a^p$, is a module-finite map. The ring R is called *F-pure* (resp. *strongly F-regular*) if $R \hookrightarrow R^{1/p}$ splits (resp. for any $c \in R^\circ = R \setminus \{0\}$, there exists a power $q = p^e$ such that $R \to R^{1/q} (a \mapsto c^{1/q}a)$ splits). Any strongly F-regular local ring is F-pure.

Takagi and the first author [TW] defined the notion of F-pure thresholds for (strongly) F-regular rings in terms of the F-purity of pairs. The F-pure threshold $\mathrm{fpt}(\mathfrak{a})$ corresponds to the log canonical threshold, denoted by $\mathrm{lct}(\mathfrak{a})$, which is an important invariant in higher dimensional birational geometry. See e.g. [HMTW, MTW, TW].

Definition 1.1 (**F-pure threshold**). Let $t \geq 0$ be a real number. For any nonzero ideal $\mathfrak{a} \subseteq R$, the pair $(R, \mathfrak{a}^t)$ is said to be *F-pure* if for all large $q = p^e \gg 0$, there exists an element $d \in \mathfrak{a}^{\lfloor t(q-1) \rfloor}$ such that $R \to R^{1/q}$ $(1 \mapsto d^{1/q})$ splits as an R-linear map. Then the *F-pure threshold*, denoted by $\mathrm{fpt}(\mathfrak{a})$, is defined by

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid \text{the pair } (R, \mathfrak{a}^t) \text{ is } F\text{-pure}\}.$$

In order to study the F-purity (of pairs), the following Fedder type criterion is very useful.

Lemma 1.2 (**Fedder type criterion** cf. [Ta, Lemma 3.9]). *Assume that $(S, \mathfrak{n})$ is an F-finite regular local ring (resp. a homogeneous polynomial ring over an F-finite field k with $\mathfrak{n} = S_+$ and $k = S_0$). Let $I \subseteq S$ be a radical (resp. a homogeneous radical) ideal of S. Let $t \geq 0$ be any real number, and let $\mathfrak{b} \subseteq S$ be an ideal. Put $R = S/I$, $\mathfrak{m} = \mathfrak{n}R$ and $\mathfrak{a} = \mathfrak{b}R$. Then the pair $(R, \mathfrak{a}^t)$ is F-pure if and only if for all large $q = p^e \gg 0$, $\mathfrak{b}^{\lfloor t(q-1) \rfloor}(I^{[q]} : I) \not\subseteq \mathfrak{n}^{[q]}$.*

Remark 1.3. Let R be strongly F-regular local ring, and let $\mathfrak{a}, J \subseteq R$ ideals with $\mathfrak{a} \subseteq \sqrt{J}$. Then the *$F$-jumping exponent* of $\mathfrak{a}$ with respect to J is defined by

$$\xi^J(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid \tau(\mathfrak{a}^t) \not\subseteq J\}, \tag{1.1}$$

[1] This is an extended abstract. The final version will be published elsewhere.

where $\tau(\mathfrak{a}^t)$ denotes the $\mathfrak{a}^t$-test ideal defined by Hara and the second author in [HY]. If R is $\mathbb{Q}$-Gorenstein or R and $\mathfrak{a}$ are homogeneous, then the smallest F-jumping exponent $\xi^{\mathfrak{m}}(\mathfrak{a})$ coincides with the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$; see e.g. [Ta, Corollary 3.5].

Let us recall a variant of F-pure thresholds, so-called F-thresholds. Assume that R is an F-finite F-pure domain of characteristic $p > 0$. Let $J^{[q]}$ denote the ideal generated by the $q(=p^e)$th powers of the elements of J.

Definition 1.4 (*F*-**thresholds** [MTW, HMTW])**.** Let $\mathfrak{a} \subseteq R$ be an ideal. Then for every positive integer e, we define $\nu_{\mathfrak{a}}^{\mathfrak{m}}(p^e) = \max\{r \in \mathbb{Z} \mid \mathfrak{a}^r \not\subseteq \mathfrak{m}^{[p^e]}\}$. Then the *F-threshold* of $\mathfrak{a}$ with respect to $\mathfrak{m}$, denoted by $c^{\mathfrak{m}}(\mathfrak{a})$, is defined by

$$c^{\mathfrak{m}}(\mathfrak{a}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{\mathfrak{m}}(p^e)}{p^e}.$$

In particular, $c^{\mathfrak{a}}(\mathfrak{a})$ is said to be the *diagonal F-threshold* of $\mathfrak{a}$ (see [MOY]).

Remark 1.5. When R is F-pure, $c^J(\mathfrak{a})$ always exists (see [HMTW, Lemma 2.3]).

The following proposition is well-known, which is a starting point of our study.

Proposition 1.6 (See [HMTW, Remark 4.5])**.** *Assume $(R, \mathfrak{m})$ is a strongly F-regular $\mathbb{Q}$-Gorenstein local domain (resp. a (not necessarily $\mathbb{Q}$-Gorenstein) homogeneous k-algebra with $R_0 = k$). Let $\mathfrak{a}$ be a nonzero ideal (resp. a nonzero homogeneous ideal). Then*

$$0 < \mathrm{fpt}(\mathfrak{a}) \le c^{\mathfrak{m}}(\mathfrak{a}) \le \dim R.$$

holds true.

If R is regular, then $\mathrm{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$ *holds.*

Remark 1.7. We use the same notation as in Proposition 1.6. If R is regular, then $\mathrm{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m}) = \dim R$. Conversely, if R is not regular, then $c^{\mathfrak{m}}(\mathfrak{m}) < \dim R$; see [HMTW, Corollary 3.2].

So the main aim of this talk is to consider the following problem.

Problem 1.8. Find a relationship between F-pure threshods $\mathrm{fpt}(R)$ and a-invariants $a(R)$ for homogeneous rings R, where

$$a(R) = \max\{n \in \mathbb{Z} \mid [H_{\mathfrak{m}}^{\dim R}(R)]_n \neq 0\}; \quad \text{see [GW]).}$$

In this talk, we give an answer to this problem for affice toric rings.

2. Main result

Before stating our result, we fix some notation.

Let $d \geq 2$ be an integer. Let $N \cong \mathbb{Z}^d$ be a lattice of rank d and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice of N. Put $N_K = N \otimes_{\mathbb{Z}} K$ for any ring K. Then there exists a perfect pairing $\langle\ ,\ \rangle \colon M_{\mathbb{R}} \otimes N_{\mathbb{R}} \to \mathbb{R}$. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone, that is, there exists a minimal system of generators $v_1, \ldots, v_s \in N$ such that $\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s$ with $\{\sigma\} \cap \{-\sigma\} = \{0\}$. Furthermore, we may assume that $v_1, \dots, v_s$ is primitive, that is, for every i, we cannot replace v_i with av_i for any $a \in \mathbb{R}$ with $0 < a < 1$. Let $\sigma^\vee \subseteq M_\mathbb{R}$ be the dual cone of σ defined by

$$\sigma^\vee = \{u \in M_\mathbb{R} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Then

$$R = k[\sigma^\vee \cap M] := k[X^u \mid u \in \sigma^\vee \cap M] \subseteq k[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$$

is called the *affine toric ring defined by* σ over a field k. Note that such a ring is strongly F-regular.

In this talk, we also assume that R is homogeneous (standard graded), that is, there exists a vector

$$v_{\mathrm{nor}} = {}^t[a_1, \dots, a_d] \in N_\mathbb{Q} \quad \text{s.t.} \quad H\colon a_1X_1 + \cdots + a_dX_d = 1$$

contains all the vectors $m_1, \dots, m_\mu$, where $\mathfrak{m} = (X^{m_1}, \dots, X^{m_\mu})$ denotes the unique graded maximal ideal of $k[\sigma^\vee \cap M]$. Under the notation as above, the following theorem is the main result in this talk.

Theorem 2.1. *Let k be a perfect field of characteristic $p > 0$. Let $R = k[\sigma^\vee \cap M]$ be a homogeneous affine toric ring over k, and let $\mathfrak{m}$ denote the unique graded maximal ideal of R. Then the following statements hold:*

(1) $0 < \mathrm{fpt}(\mathfrak{m}) \leq -a(R)$
(2) *Equality holds true if and only if R is Gorenstein.*

If R is $\mathbb{Q}$-Gorenstein of index of $r \geq 2$, then

$$\frac{1}{r} \leq \mathrm{fpt}(\mathfrak{m}) \leq -a(R) - \frac{1}{r}.$$

Chiba and Matsuda [CM] proved Theorem 2.1 holds true for Hibi rings. On the other hand, in characteristic 0, we obtain the following corollary.

Corollary 2.2. *Let $R = k[\sigma^\vee \cap M]$ be a homogeneous affine toric ring over a field k of characteristic 0. Let $\mathfrak{m}$ denote the unique graded maximal ideal of R. Assume that R is $\mathbb{Q}$-Gorenstein. Then the following statements hold:*

(1) $\mathrm{lct}(\mathfrak{m}) \leq -a(R) (\leq \dim R)$.
(2) *Equality holds true if and only if R is Gorenstein.*

In what follows, we give a sketch of the proof of Theorem 2.1.

2.1. **Gorenstein case.** We first consider the Gorenstein case. The following proposition, which was given by Takagi and the first author in [TW] without proof, gives a proof of Theorem 2.1 in the Gorenstein case.

Proposition 2.3 ([TW, Example 2.4 (iv)]). *Assume that R is a homogeneous Gorenstein strongly F-regular domain. Let $\mathfrak{m} = \bigoplus_{n \geq 1} R_n$ denote the unique graded maximal ideal of R. Then $\mathrm{fpt}(\mathfrak{m}) = -a(R)$ and $(R, \mathfrak{m}^{-a(R)})$ is F-pure.*

Proof. Write $R = S/I$, where $S = k[X_1, \ldots, X_n]$ is a polynomial ring and $I \subseteq S$ is an ideal with $h = \operatorname{height} I$. Set $\mathfrak{n} = (X_1, \ldots, X_n)S$. Let

$$\mathbf{F}.: \quad 0 \to F_h \to F_{h-1} \to \cdots \to F_1 \to F_0 = S \to S/I \to 0$$

be a graded minimal free resolution of R over S. As R is Gorenstein, we can write $F_h = S(-c_h)$ for some $c_h \in \mathbb{Z}$. Then since

$$\begin{aligned} K_R = R(a(R)) &= \operatorname{Ext}_S^h(R, S(-n)) \\ &= \operatorname{Cok}(\operatorname{Hom}_S(F_{h-1}, S(-n)) \to \operatorname{Hom}_S(F_h, S(-n))) \\ &\cong R(-n + c_h)), \end{aligned}$$

we have $a(R) = -n + c_h$. Applying the Peskine-Szpiro functor to $\mathbb{F}.$ yields the graded minimal free resolution of $S/I^{[q]}$ as follows:

$$\mathbb{F}^e(\mathbf{F}.): \quad 0 \to F_h = S(-c_h q) \to F_{h-1} \to \cdots \to F_1 \to F_0 = S \to S/I^{[q]} \to 0.$$

In particular, $S/I^{[q]}$ is a Gorenstein ring with $a(S/I^{[q]}) = -n + c_h q$ for all $q = p^e$. Hence

$$\begin{aligned} \frac{I^{[q]} : I}{I^{[q]}} &\cong \operatorname{Hom}_S(S/I, S/I^{[q]}) \\ &\cong \operatorname{Hom}_S(S/I, K_{S/I^{[q]}}(-a(S/I^{[q]}))) \\ &\cong K_{S/I}(-a(S/I^{[q]})) \\ &\cong R(a(S/I) - a(S/I^{[q]})) \\ &= R(-c_h(q-1)) = R(-(a(R)+n)(q-1)). \end{aligned}$$

Therefore, for each $q = p^e$, there exists an element $\alpha \in I^{[q]} \colon I$ of degree $(a(R) + n)(q-1)$ so that

$$I^{[q]} \colon I = I^{[q]} + R\alpha \subseteq I^{[q]} + \mathfrak{n}^{(a(R)+n)(q-1)}.$$

Now suppose that $\mathfrak{n}^{-a(R)(q-1)}(I^{[q]} \colon I) \subseteq \mathfrak{n}^{[q]}$. Since $\mathfrak{n}$ is generated by an S-sequence, we have

$$\alpha \in I^{[q]} \colon I \subseteq \mathfrak{n}^{[q]} \colon \mathfrak{n}^{-a(R)(q-1)} = \mathfrak{n}^{[q]} + \mathfrak{n}^{(n+a(R))(q-1)+1}.$$

As $\deg \alpha = (n + a(R))(q-1)$, we have that $\alpha \in \mathfrak{n}^{[q]}$. But this implies that $I^{[q]} \colon I \subseteq \mathfrak{n}^{[q]}$, which contradicts the assumption that $R = S/I$ is F-pure ([Fe]). Hence $\mathfrak{n}^{-a(R)(q-1)}(I^{[q]} \colon I) \not\subseteq \mathfrak{n}^{[q]}$. Namely, $(R, \mathfrak{m}^{-a(R)})$ is F-pure.

Similarly, we can show that if $t > -a(R)$ then $(R, \mathfrak{m}^t)$ is not F-pure. □

2.2. **General case.** Recall that R is $\mathbb{Q}$-*Gorenstein* if and only if there exists an integer $r \geq 1$ such that $K_R^{(r)}$ is principal. The minimum one among such integers $r \geq 1$ is called the *index* of R, denoted by $\operatorname{Index}(R)$. In particular, a Gorenstein ring is just a $\mathbb{Q}$-Gorenstein ring of index 1.

In what follows, let $R = k[\sigma^\vee \cap M]$ be an affine toric ring. We use the same notation as in Section 2. The following criterion for $\mathbb{Q}$-Gorensteinness of toric rings is known.

Lemma 2.4. *Then the following statements are equivalent:*

(1) *R is $\mathbb{Q}$-Gorenstein.*

(2) *There exists a vector $\omega \in M_{\mathbb{R}}$ such that $\langle \omega, v_j \rangle = 1$ for all $j = 1, \ldots, s$.*

When this is the case, if $r = \mathrm{Index}(R)$, then $m_0 = r \cdot \omega \in M$ and $K_R^{(r)} \cong X^{m_0}R$.

Put

$$P(\mathfrak{m}) = \{\lambda \cdot u \,|\, \lambda \geq 1,\ u \in \mathrm{CONV}(\{m_1, \ldots, m_\mu\}).$$

We define a real function $\lambda_{\mathfrak{m}} \colon \sigma^\vee \to \mathbb{R}$ by

$$\lambda_{\mathfrak{m}}(u) = \sup\{\lambda \in \mathbb{R}_{\geq 0} \,|\, u \in \lambda \cdot P(\mathfrak{m})\}.$$

Since $\lambda_{\mathfrak{m}}(u) = a_1x_1 + \cdots + a_dx_d$ for all $u = {}^t[x_1, \ldots, x_d] \in \sigma^\vee \subseteq \mathbb{R}^d$, $\lambda_{\mathfrak{m}}$ is a linear function which satisfies $\lambda_{\mathfrak{m}}(u) = \deg X^u$ for all $u \in \sigma^\vee \cap M$. Using this function, we can desribe $\mathrm{fpt}(\mathfrak{m})$ and a-invariant. The following lemma is a reformulation of Blickle's theorem ([Bl, Theorem 3]).

Lemma 2.5 (See [Hi, Corollary 4.3]). *Let $R = k[\sigma^\vee \cap M]$ be as above. Then we have*

$$\mathrm{fpt}(\mathfrak{m}) = \sup_{\omega \in \mathbf{X}} \lambda_{\mathfrak{m}}(\omega),$$

where $\mathbf{X} = \{\omega \in M_{\mathbb{R}} \,|\, 0 \leq \langle \omega, v_j \rangle \leq 1$ for all $j = 1, 2, \ldots, s\}$.

If R is $\mathbb{Q}$-Gorenstein, then

$$\mathrm{fpt}(\mathfrak{m}) = \lambda_{\mathfrak{m}}(\omega),$$

where $\omega \in M_{\mathbb{R}}$ with $\langle \omega, v_j \rangle = 1$ for all j.

For a graded R-module M, the *initial degree* of M is defined by $\mathrm{indeg}(M) = \min\{n \in \mathbb{Z} \,|\, [M]_n \neq 0\}$. Also, $\mathrm{relint}(P)$ denotes the relative interior of $P \subseteq M_{\mathbb{R}} = \mathbb{R}^d$.

Lemma 2.6 (Stanley). *Let $R = k[\sigma^\vee \cap M]$ be as above. Then there exists a vector $u_a \in M$ such that*

$$-a(R) = \lambda_{\mathfrak{m}}(u_a) = \deg X^{u_a}.$$

Furthermore, if $u \in \mathrm{relint}(\sigma^\vee) \cap M$ then $\lambda_{\mathfrak{m}}(u) \geq -a(R)$.

Proposition 2.7 (See also [TW, Theorem 2.7(2)]). *Assume that $R = k[\sigma^\vee \cap M]$ is a homogeneous toric ring, which is not Gorenstein. Then*

$$\mathrm{fpt}(\mathfrak{m}) < -a(R).$$

Furthermore, if, in addition, R is $\mathbb{Q}$-Gorenstein ring of index $r \geq 2$, then

$$\mathrm{fpt}(\mathfrak{m}) \in \frac{1}{r}\mathbb{Z}_{\geq 0} \quad \textit{and} \quad \frac{1}{r} \leq \mathrm{fpt}(\mathfrak{m}) \leq -a(R) - \frac{1}{r}.$$

Proof. We divide the proof into two cases.

Case 1: Suppose that R is $\mathbb{Q}$-Gorenstein.

Then we have

$$\mathrm{fpt}(\mathfrak{m}) = \lambda_{\mathfrak{m}}(\omega) = \langle \omega, v_{\mathrm{nor}} \rangle = \frac{1}{r}\langle m_0, v_{\mathrm{nor}} \rangle \in \frac{1}{r}\mathbb{Z}.$$

We first prove the left-hand side. Since R is strongly F-regular, we have $\mathrm{fpt}(\mathfrak{m}) > 0$. Hence it follows that $\mathrm{fpt}(\mathfrak{m}) \geq \frac{1}{r}$ because $\mathrm{fpt}(\mathfrak{m}) > 0$ and $\mathrm{fpt}(\mathfrak{m}) \in \frac{1}{r}\mathbb{Z}$.

Next, we prove the right-hand side. Since R is not Gorenstein, K_R^r is not principal. On the other hand, since $K_R^{(r)}$ is principal and it contains K_R^r, we have $K_R^r \subsetneq K_R^{(r)}$. Therefore

$$r \cdot \mathrm{fpt}(\mathfrak{m}) = r\langle \omega, v_{\mathrm{nor}} \rangle = \langle m_0, v_{\mathrm{nor}} \rangle = \deg X^{m_0} \leq \mathrm{indeg}(K_R^r) - 1 = -r \cdot a(R) - 1.$$

This yields the required inequality.

Case 2: Suppose that R is not $\mathbb{Q}$-Gorenstein.

Let $\{v_1, \ldots, v_s\}$ be the set of primitive vectors of σ. Take a vector $u_a \in \mathrm{relint}(\sigma^\vee) \cap M$ such that $-a(R) = \lambda_{\mathfrak{m}}(u_a) = \deg X^{u_a}$ as in Lemma 2.6. Then $\langle u_a, v_j \rangle \geq 1$ for all $j = 1, 2, \ldots, s$ because $\langle u_a, v_j \rangle > 0$ and $\langle u_a, v_j \rangle \in \mathbb{Z}$. But since R is not $\mathbb{Q}$-Gorenstein, we have that $\langle u_a, v_j \rangle \geq 2$ for some j with $1 \leq j \leq s$. We may assume that $\langle u_a, v_1 \rangle \geq 2$ without loss of generality.

Consider $\mathbf{X} = \{\omega \in M_{\mathbb{R}} \mid 0 \leq \langle \omega, v_j \rangle \leq 1 \text{ for all } j = 1, 2, \ldots, s\}$ as in Lemma 2.5. Then one can easily see that $0 \neq u_a - \omega \in \sigma^\vee$ for all $\omega \in \mathbf{X}$. Indeed,

$$\langle u_a - \omega, v_j \rangle = \langle u_a, v_j \rangle - \langle \omega, v_j \rangle \geq 1 - 1 = 0;$$

and

$$\langle u_a - \omega, v_1 \rangle = \langle u_a, v_1 \rangle - \langle \omega, v_1 \rangle \geq 2 - 1 = 1.$$

Since $\lambda_{\mathfrak{m}}(u_a) - \lambda_{\mathfrak{m}}(\omega) = \lambda_{\mathfrak{m}}(u_a - \omega) > 0$, we have $\lambda_{\mathfrak{m}}(\omega) < \lambda_{\mathfrak{m}}(u_a)$ for all $\omega \in \mathbf{X}$.

Since $\lambda_{\mathfrak{m}}|_{\mathbf{X}}$ is a continuous function on the compact set X, it takes the maximal value on $\mathbf{X}$; say $\lambda_{\mathfrak{m}}(\omega')$ for some $\omega' \in \mathbf{X}$. Summarizing the above argument, we get

$$\lambda_{\mathfrak{m}}(\omega) \leq \lambda_{\mathfrak{m}}(\omega') < \lambda_{\mathfrak{m}}(u_a) = -a(R).$$

Hence Lemma 2.5 yields $\mathrm{fpt}(\mathfrak{m}) \leq \lambda_{\mathfrak{m}}(\omega') < -a(R)$, as required. □

3. F-thresholds vs. a-invariants

The following theorem gives a relationship between F-thresholds and a-invariant.

Theorem 3.1. *Suppose that $R = k[\sigma^\vee \cap M]$ is homogeneous. Let $\mathfrak{m}$ denote the unique graded maximal ideal of R. Then $c^{\mathfrak{m}}(\mathfrak{m}) \geq -a(R)$.*

4. Examples and Questions

In this section, we give typical two examples.

Example 4.1 (Veronese). Let $r, d \geq 2$ be integers. Let $R = k[X_1, \ldots, X_d]^{(r)}$, the rth Veronese subring of a polynomial ring over a field k, and let $\mathfrak{m}$ denote the unique graded maximal ideal. Then R is a $\mathbb{Q}$-Gorenstein homogeneous toric ring with $\dim R = d$. Then we have

$$\mathrm{fpt}(\mathfrak{m}) = \frac{d}{r}, \qquad -a(R) = \left\lceil \frac{d}{r} \right\rceil, \qquad c^{\mathfrak{m}}(\mathfrak{m}) = \frac{d+r-1}{r}.$$

In particular,

(1) $\mathrm{fpt}(\mathfrak{m}) = -a(R)$ if and only if $\frac{d}{r} \in \mathbb{Z}$ that is, R is Gorenstein (see e.g. [GW]).

(2) $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$ if and only if $d \equiv 1 \pmod{r}$.

The Segre product of two polynomial rings can be regarded as a Hibi ring. Thus both the F-pure thresholds and the diagonal F-thresholds of those rings were calculated by Chiba and Matsuda [CM].

Example 4.2 (Segre product; see [CM]). Let $m, n \geq 2$ be integers. Let

$$R = k[X_1, \dots, X_m]\#k[Y_1, \dots, Y_n]$$

be the Segre product of two polynomial rings. Then

(1) R is a homogeneous strongly F-regular ring with $\dim R = m + n - 1$.
(2) R is Gorenstein if and only if $m = n$. Otherwise, R is *not* $\mathbb{Q}$-Gorenstein.
(3) $\mathrm{fpt}(\mathfrak{m}) = \min\{m, n\}$ due to Masahiro Ohtani.
(4) $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \max\{m, n\}$.

In the last of this talk, we pose a conjecture and two questions.

Conjecture 4.3. *Let R be a homogeneous strongly F-regular domain and $\mathfrak{m}$ denotes the unique graded maximal ideal of R. Then*

(1) $\mathrm{fpt}(\mathfrak{m}) \leq -a(R) \leq c^{\mathfrak{m}}(\mathfrak{m})$.
(2) $\mathrm{fpt}(\mathfrak{m}) = -a(R)$ *if and only if R is Gorenstein.*

Question 4.4. When does $\mathrm{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ hold?

Question 4.5. When does $\mathrm{fpt}(\mathfrak{m}) = \dim R - 1$ hold?

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The Cohen–Macaulay Property and Depth in Invariant Theory

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Abstract

This article gives a survey of results about the Cohen–Macaulay property and the depth of invariant rings.

Introduction

The main object of study in invariant theory is the invariant ring of a given group action. Typical questions about such an invariant ring are: Can it be finitely generated? How can generators be obtained algorithmically? To what extent can group orbits be separated by invariants? What ring-theoretic properties does the invariant ring have, and how do they relate to properties of the group action? This paper deals with the last question and focuses on the Cohen–Macaulay property and the depth. We will usually (but not always) restrict to the case of a finite group acting linearly on a finite-dimensional vector space. This paper will present methods using group cohomology to prove results about the Cohen–Macaulay property and depth of invariant rings. These methods were for the most part developed around the last turn of the century.

It is well known that in the nonmodular case (i.e., when the group order is not divisible by the characteristic p of the ground field), the invariant ring is always Cohen–Macaulay. So the interesting case is the modular case. Before the development of the above-mentioned cohomological methods, only sporadic results and examples in the modular case were known. The best of these is a result by Ellingsrud and Skjelbred [7], who considered the case of cyclic groups and obtained an explicit formula for the depth of the invariant ring. Another result by Campbell et al. [4] tells us that if G is a p-group, then vector invariants of three copies are never Cohen–Macaulay.

The first section of this paper is devoted to the nonmodular case. We generalize the above-mentioned result that invariant rings in this case are always

Cohen–Macaulay. The second section introduces the cohomological methods for the study of the Cohen–Macaulay property and depth. The main results in the case of finite groups are presented in Section 3. These include a result on vector invariants and a result saying that the group is generated by certain types of elements, which both go back to joint work with Nikolai Gordeev. Finally in the last section some results where the depth is determined precisely are discussed.

We will use the following notation. For a Noetherian ring R we consider the **Cohen–Macaulay defect**

$$\operatorname{def}(R) := \sup\{\dim(R_P) - \operatorname{depth}(R_P) \mid P \in \operatorname{Spec}(R)\} \in \mathbb{N}_0 \cup \{\infty\},$$

which measures the deviation of R from being Cohen–Macaulay. (In fact, for this definition it suffices to assume that R_P is always finite-dimensional.) This number is particularly interesting in the case that $R = \bigoplus_{i \geq 0} R_i$ is a graded ring with R_0 a field, which occurs for example when R is the invariant ring of a group acting linearly on a vector space. In this case we can use Noether normalization to obtain a graded subalgebra $A \subseteq R$, isomorphic to a polynomial ring, such that R is finitely generated as an A-module. The Auslander–Buchsbaum formula then tells us that $\operatorname{def}(R)$ is equal to the length of a minimal free resolution of R as an A-module. So $\operatorname{def}(R)$ measures the homological complexity of R. In particular, R is free as an A-module if and only if it is Cohen–Macaulay.

We will consider the standard situation of invariant theory, so $V = K^n$ will be a finite-dimensional vector space over a field K, and $G \subseteq \operatorname{GL}(V)$ will be a subgroup of the general linear group. We will often make the restriction that G is finite or, more generally, algebraic. If not stated otherwise,

$$R := K[V] = K[x_1, \ldots, x_n]$$

will denote the polynomial ring on V, on which G acts by $\sigma(f) = f \circ \sigma^{-1}$. (If K is finite, the action is first defined on the dual V^* of V as above and then on $K[V]$ by homomorphic extension.) Moreover,

$$R^G := \{f \in R \mid \sigma(f) = f \text{ for all } \sigma \in G\}$$

will denote the **invariant ring**. This is the main object of interest in invariant theory. In this article, our interest focuses on the Cohen–Macaulay defect $\operatorname{def}(R^G)$. With this notation, the formula by Ellingsrud and Skjelbred [7] mentioned above can be stated as follows. If G is a cyclic group with Sylow p-subgroup P (where $p = \operatorname{char}(K)$), then

$$\operatorname{def}(R^G) = \max\{\operatorname{codim}(V^P) - 2, 0\}. \tag{0.1}$$

1 The nonmodular case

The nonmodular case in invariant theory of finite groups is the case where the group order $|G|$ is finite and not divisible by the characteristic of K. This is the case where the results are nicest. For example, R^G is always Cohen–Macaulay in this case. We will present a proof of this by giving a more general result which, to the best of the author's knowledge, has not yet appeared in the literature in this generality.

Theorem 1.1. *Let R be a commutative ring with unity and let $G \subseteq \operatorname{Aut}(R)$ be a group of ring-automorphisms of R. Furthermore, let $H \subseteq G$ be a subgroup such that the index $(G : H)$ is finite and invertible in R, and assume that R^H is Noetherian. Then*

$$\operatorname{def}(R^G) \leq \operatorname{def}(R^H).$$

Proof. We may assume $\operatorname{def}(R^H) < \infty$. Choose a system $\sigma_1, \ldots, \sigma_n$ of left coset representatives of H in G. Since every $a \in R^H$ satisfies the equation $\prod_{i=1}^n (x - \sigma_i(a)) \in R^G[x]$, R^H is an integral over R^G. Let $Q, Q' \in \operatorname{Spec}(R^H)$ such that

$$R^G \cap Q = R^G \cap Q'.$$

Then for $a \in Q$ we have $\prod_{i=1}^n \sigma_i(a) \in R^G \cap Q \subseteq Q'$, so there exists i with $a \in \sigma_i^{-1}(Q')$. Using the prime avoidance lemma, we conclude that there exists i such that $Q \subseteq \sigma_i^{-1}(Q')$. Since Q and Q' lie over the same prime ideal in R^G, this implies

$$Q' = \sigma_i(Q).$$

We claim that going down holds for the extension $R^G \subseteq R^H$. Let $P \in \operatorname{Spec}(R^G)$ and let $Q' \in \operatorname{Spec}(R^H)$ such that $P \subseteq Q'$. There exist $Q, \widetilde{Q} \in \operatorname{Spec}(R^H)$ such that

$$R^G \cap Q = P, \quad R^G \cap \widetilde{Q} = R^G \cap Q', \quad \text{and} \quad Q \subseteq \widetilde{Q}.$$

By the above, there exists i with $Q' = \sigma_i(\widetilde{Q})$, so $\sigma_i(Q) \subseteq Q'$ and $R^G \cap \sigma_i(Q) = P$. So indeed going down holds.

Now let $P \in \operatorname{Spec}(R^G)$. We need to show that

$$\operatorname{depth}\left(R_P^G\right) \geq \dim(R_P^G) - \operatorname{def}(R^H).$$

Let $a_1, \ldots, a_m$ be a maximal R^H-regular sequence in P. Then every element of P is contained in an associated prime ideal of $R^H/(a_1, \ldots, a_m)R^H$, so by prime avoidance, P itself is contained in an associated prime ideal Q of $R^H/(a_1, \ldots, a_m)R^H$. It is easy to see that as elements of R_Q^H, the a_i form a maximal R_Q^H-regular sequence, so

$$m = \operatorname{depth}(R_Q^H) \geq \dim(R_Q^H) - \operatorname{def}(R^H) \geq \dim(R_P^G) - \operatorname{def}(R^H), \qquad (1.1)$$

where the last inequality follows from going down (see Kemper [17, Corollary 8.14]). We claim that $a_1, \ldots, a_m$ is R^G-regular. So suppose that

$$b \cdot a_k = \sum_{j=1}^{k-1} b_j a_j$$

with $1 \leq k \leq m$ and $b, b_j \in R^G$. The R^H-regularity yields

$$b = \sum_{j=1}^{k-1} c_j a_j$$

with $c_i \in R^H$, so

$$b = \frac{1}{n}\sum_{i=1}^{n} \sigma_i(b) = \sum_{j=1}^{k-1} \left(\frac{1}{n}\sum_{i=1}^{n} \sigma_i(c_j)\right) a_j \in (a_1, \ldots, a_m)R^G.$$

This proves the claim, so

$$m \leq \operatorname{grade}(P, R^G) \leq \operatorname{depth}(R^G_P),$$

where the second inequality follows from Bruns and Herzog [2, Proposition 1.2.10(a)]. Together with (1.1), this completes the proof. □

Returning to our standard situation where $R = K[V]$ and G acts linearly on V, we state the following consequence of Theorem 1.1.

Corollary 1.2 (Hochster and Eagon [12]). *Assume that G is finite such that $|G|$ is not divisible by* $\operatorname{char}(K)$. *Then R^G is Cohen–Macaulay.*

A further consequence of Theorem 1.1 is the result by Campbell et al. [3] that if R^P is Cohen–Macaulay with $P \subseteq G$ a Sylow p-subgroup, $p = \operatorname{char}(K)$, then R^G is also Cohen–Macaulay.

Recall that a linear algebraic group G over an algebraically closed field K is called **linearly reductive** if every G-**module** V (i.e., every finite-dimensional K-vector space V with a linear action given by a morphism $G \times V \to V$) is completely reducible. So a finite group G is linearly reductive if and only if $|G|$ is not divisible by $\operatorname{char}(K)$. The following celebrated result is a generalization of Corollary 1.2.

Theorem 1.3 (Hochster and Roberts [13]). *Let G be a linearly reductive algebraic group over an algebraically closed field K and let V be a G-module. Then $K[V]^G$ is Cohen–Macaulay.*

2 A cohomological obstruction

We will now consider the more difficult nonmodular case of invariant theory and start by considering an example.

Example 2.1. Let K be a field of positive characteristic p. The cyclic group $G = \langle \sigma \rangle \cong C_p$ of order p acts on the polynomial ring $R := K[x_1, x_2, x_3, y_1, y_2, y_3]$ by

$$\sigma(x_i) = x_i \quad \text{and} \quad \sigma(y_i) = y_i + x_i.$$

We find invariants

$$x_i \quad (i = 1, 2, 3) \quad \text{and} \quad u_{i,j} := x_i y_j - x_j y_i \quad (1 \le i < j \le 3)$$

and the relation

$$x_1 u_{2,3} - x_2 u_{1,3} + x_3 u_{1,2} = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = 0.$$

Since $u_{1,2}$ does not lie in the ideal $(x_1, x_2)R^G$, this shows that x_1, x_2, x_3 do not form a R^G-regular sequence. On the other hand, the x_i can be completed to a homogeneous system of parameters (by the invariants $y_i^p - x_i^{p-1} y_i$, for example), so it follows that R^G is not Cohen–Macaulay. This is probably the most accessible example of a non-Cohen–Macaulay invariant ring. Of course we know from (0.1) that $\operatorname{def}(R^G) = 1$. ◁

In the above example, the invariants x_i form an R-regular sequence, but fail to be R^G-regular. From a general investigation of this phenomenon, the cohomological criterion given in the following lemma emerged. We consider group cohomology $H^*(G, R)$ with values in R and write

$$m := \inf \{ i > 0 \mid H^i(G, R) \neq 0 \} \in \mathbb{N} \cup \{\infty\}, \tag{2.1}$$

which we call the **cohomological connectivity** (see Fleischmann et al. [10]). This number is not always easily accessible, but in many cases it is.

Lemma 2.2 (Kemper [14]). *Let $a_1, \ldots, a_r \in R^G$ be an R-regular sequence. If $r < m+2$ (with m defined in (2.1)), then $a_1, \ldots, a_r$ is also R^G-regular. If $r = m+2$, then $a_1, \ldots, a_r$ is R^G-regular if and only if the map*

$$H^m(G, R) \to H^m(G, R^r)$$

induced by the multiplication by $a_1, \ldots, a_r$ is injective.

The last statement may be rephrased as follows: $a_1, \ldots, a_r$ fail to be R^G-regular if and only if there exists a nonzero $\alpha \in H^m(G, R)$ such that $a_i \alpha = 0$ for all i.

The case $m = 1$ of the lemma can be proved by elementary calculations. The general case can be proved by using the long exact sequence of cohomology and a Koszul complex.

Example 2.3. We reconsider Example 2.1 in the light of the above criterion. A nonzero cohomology class in $\alpha \in H^1(G, R)$ is given by the cocycle $G \to R$, $\sigma^j \mapsto j \in K$. So the cohomological connectivity is $m = 1$. For $i = 1, 2, 3$, the class $x_i \alpha \in H^1(G, R)$ is given by

$$G \to R, \ \sigma^j \mapsto j x_i = \sigma^j(y_i) - y_i,$$

which is a coboundary. So $x_i \alpha = 0$, and Lemma 2.2 tells us that x_1, x_2, x_3 are not R^G-regular. ◁

Lemma 2.2 is crucial for proving the general result presented in the following theorem. Before we state it, we recall that a linear algebraic group G over an algebraically closed field K is called **reductive** if it has no infinite normal, unipotent subgroup. Examples of reductive groups include all classical groups (such as $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$ and the symplectic and orthogonal groups) and all finite groups. Every linearly reductive group is reductive, and in characteristic 0 the converse holds. But in positive characteristic there is a wide gap between reductive groups and linearly reductive groups. The following result may be regarded as a converse to Theorem 1.3.

Theorem 2.4 (Kemper [15]). *Let G be a reductive algebraic group. If G is not linearly reductive, there exists a G-module V such that $K[V]^G$ is not Cohen-Macaulay.*

Proof (sketch). It is not hard to see that a linear algebraic group G is linearly reductive if and only if $H^1(G,U) = 0$ for every G-module U. So under our hypotheses there exists a G-module U with $H^1(G,U) \neq 0$. Choose a nonzero $\alpha \in H^1(G,U)$. Such a class α defines an exact sequence

$$0 \to U \to W \to K \to 0$$

of G-modules. (This can be seen by elementary considerations or, more abstractly, by interpreting $H^1(G,U) = \mathrm{Ext}^1_{KG}(K,U)$ as Yoneda Ext.) Dualizing the above sequence yields

$$0 \to K \to W^* \to U^* \to 0.$$

If $w \in (W^*)^G$ is the image of $1 \in K$, it turns out that $w \otimes \alpha$ is 0 as an element of $H^1(G, W^* \otimes U)$. Forming $V := W \oplus W \oplus W \oplus U^*$, we find three copies a_1, a_2, a_3 of w in $R := K[V]$. As an element of $H^1(G,R)$, α remains nonzero, but $a_i\alpha = 0$. So by Lemma 2.2, the a_i do not form an R^G-regular sequence. On the other hand, the a_i can clearly be extended to a homogeneous parameter system of R. Since G is reductive, it can be shown that they can also be extended to a homogeneous parameter system of R^G. Therefore R^G is not Cohen–Macaulay. □

The reductivity hypothesis in Theorem 2.4 cannot be omitted. For example, every invariant ring of the additive group over $K = \mathbb{C}$ is Cohen–Macaulay. It may be worthwhile to mention in this context that, to the best of the author's knowledge, no example of a non-Cohen–Macaulay invariant ring $K[V]^G$ with $\mathrm{char}(K) = 0$ is known to date. An explicit version of Theorem 2.4 can be found in Kohls [18] (see [19] for results on the Cohen-Macaulay defect).

3 Traces and wild ramification

In Theorem 2.4 (and its proof) we have produced a tailor-made representation V for a given group G such that Lemma 2.2 could be used to prove that $K[V]^G$ is not Cohen–Macaulay. The goal of this section is to use Lemma 2.2 to deduce results on a given representation V of a group G.

We will restrict to the case that G is finite and its order is divisible by $p := \mathrm{char}(K)$. The question is which elements of R^G annihilate cohomology classes from $H^i(G,R)$ with $i > 0$. One answer is the following: For a polynomial $f \in R$, define the **trace** as

$$\mathrm{Tr}(f) := \sum_{\sigma \in G} \sigma(f) \in R^G.$$

Then for $\alpha \in H^i(G,R)$ with $i > 0$ and $f \in R$ we have

$$\mathrm{Tr}(f) \cdot \alpha = \mathrm{cores}\left(f \cdot \mathrm{res}_{G,\{\mathrm{id}\}}(\alpha)\right) = 0, \tag{3.1}$$

where cores denotes the corestriction (see Evens [8, Proposition 4.2.4]).

Clearly the image $I := \mathrm{Tr}(R) \subseteq R^G$ of the trace map is an ideal. It is quite easy to determine its height. Since going down holds for $R^G \subseteq R$, the height of I equals the height of the ideal IR in R. So we consider the variety in V determined by I. It is easy to see that for $x \in V$ the equivalence

$$\mathrm{Tr}(f)(x) = 0 \quad \text{for all } f \in R \quad \Longleftrightarrow \quad p \mid |G_x|$$

holds, where

$$G_x := \{\sigma \in G \mid \sigma(x) = x\}$$

denotes the point-stabilizer. So the variety determined by I is the union of all V^σ with $\sigma \in G$ of order p. We obtain

$$\mathrm{ht}\left(\mathrm{Tr}(R)\right) = \min\left\{\mathrm{codim}(V^\sigma) \mid \sigma \in G,\ \mathrm{ord}(\sigma) = p\right\} =: c. \tag{3.2}$$

So the height of the trace ideal is completely accessible. We can now prove the following result.

Theorem 3.1. *Let G be finite. Then*

$$\mathrm{def}\left(R^G\right) \geq c - m - 1$$

with c and m defined by (3.2) *and* (2.1).

Proof. We may assume $m < \infty$. Let P be an associated prime ideal of $H^m(G,R)$ as an R^G-module. We claim that

$$\mathrm{grade}\left(P, R^G\right) \leq m + 1. \tag{3.3}$$

Indeed, if there existed a regular sequence $a_1, \ldots, a_{m+2} \in P$, then the ideal in R generated by the a_i would have height $m+2$ (since going down holds for $R^G \subseteq R$), so the a_i would form an R-regular sequence by the Cohen–Macaulay property of R. Applying Lemma 2.2 then shows that $a_1, \ldots, a_{m+2}$ is not R^G-regular after all. By Bruns and Herzog [2, Proposition 1.2.10(a)], there exists $Q \in \mathrm{Spec}(R^G)$ with $P \subseteq Q$ such that

$$\mathrm{depth}(R^G_Q) = \mathrm{grade}\left(P, R^G\right). \tag{3.4}$$

By (3.1), the trace ideal $\mathrm{Tr}(R)$ is contained in P and therefore also in Q, so $\mathrm{ht}(Q) \geq c$ by (3.2). We obtain

$$\mathrm{def}(R^G) \geq \mathrm{ht}(Q) - \mathrm{depth}(R^G_Q) \geq c - m - 1,$$

where (3.3) and (3.4) were used for the last inequality. □

Now we can give a lower bound for the less accessible quantity m, the cohomological connectivity. In fact, if $|G| = p^a m$ with $(p, m) = 1$ and $a > 0$, then it can be shown that there exists a positive integer $r \leq 2p^{a-1}(p-1)$ such that $H^r(G, \mathbb{F}_p)$ is nonzero. The argument uses the Evens norm in cohomology and can be found in the proof of Theorem 4.1.3 in Benson [1]. It follows that

$$m \leq 2p^{a-1}(p-1) < 2|G|. \tag{3.5}$$

We obtain the following result on vector invariants, i.e., invariants of several copies of the same representation V.

Corollary 3.2 (Gordeev and Kemper [11]). *Assume that G is finite of order divisible by p. Then*

$$\lim_{k\to\infty} \operatorname{def}\left(K[\underbrace{V \oplus \cdots \oplus V}_{k \text{ copies}}]^G \right) = \infty.$$

This follows from Theorem 3.1 and (3.5) since the number c from (3.2) tends (linearly) to infinity when V is replaced be the direct sum of k copies of V. Corollary 3.2 tells us that the vector invariants in the modular case are getting worse and worse, in terms of homological complexity, as the number of copies increases.

We also obtain results that link the Cohen–Macaulay defect to the question by what type of elements G can be generated. An element $\sigma \in \mathrm{GL}(V)$ is called a k-**reflection** if $\operatorname{codim}(V^\sigma) \leq k$. So the 1-reflections are the identity and the pseudo reflections in the classical sense. In this context, a well-known result, attributed to Shephard, Todd, Chevalley, and Serre, says that if R^G is isomorphic to a polynomial ring, then G is generated by 1-reflections. (In the nonmodular case, the converse holds.) Concerning the Cohen–Macaulay defect, we can use our methods to deduce the following result.

Theorem 3.3 (Gordeev and Kemper [11]). *Set $k := \operatorname{def}(R^G) + 2$. Then G is generated by k-reflections and p'-elements (i.e., elements of order not divisible by p).*

Proof (sketch). Let $N \subseteq G$ be the (normal) subgroup generated by the k-reflections and p'-elements in G, and assume, by way of contradiction, that $N \subsetneqq G$. Then $H^1(G/N, K) \neq 0$, so the image of the inflation map $H^1(G/N, R) \to H^1(G, R)$ is a nonzero submodule $M \subseteq H^1(G, R)$. As in the proof of Theorem 3.1, we choose an associated prime ideal P of M and find $Q \in \operatorname{Spec}(R^G)$ with $P \subseteq Q$ such that

$$\operatorname{depth}(R_Q^G) = \operatorname{grade}(P, R^G) \leq 2.$$

On the other hand, it follows as (3.1) that every *relative trace*

$$\operatorname{Tr}_{N,G}(f) := \sum_{\sigma \in G/N} \sigma(f)$$

with $f \in R^N$ annihilates every element from M. So

$$\mathrm{Tr}_{N,G}(R^N) \subseteq P \subseteq Q.$$

As after (3.1), one can determine the variety in V defined by $\mathrm{Tr}_{N,G}(R^N)$ and finds that this is the union of all V^σ with $\sigma \in G$ such that $\sigma N \in G/N$ has order p (see Fleischmann [9] or Campbell et al. [5, Theorem 7]). So

$$\mathrm{ht}(Q) \geq \mathrm{ht}(P) \geq \mathrm{ht}\left(\mathrm{Tr}_{N,G}(R^N)\right) = \min\left\{\mathrm{codim}(V^\sigma) \mid \sigma \in G,\ \mathrm{ord}(\sigma N) = p\right\} > k,$$

since $G \setminus N$ contains no k-reflections by the definition of N. Therefore

$$\mathrm{ht}(Q) - \mathrm{depth}(R_Q^G) > k - 2 = \mathrm{def}(R^G),$$

a contradiction. □

A special case of Theorem 3.3 says that if G is a p-group and R^G is Cohen–Macaulay, then G is generated by 2-reflections (see Kemper [14]). This generalizes the result by Campbell et al. [4] mentioned in the Introduction. Unfortunately, the converse of Theorem 3.3 does not hold. In fact, there are examples of groups generated by 1-reflections such that the Cohen–Macaulay defect of R^G is arbitrarily large.

4 Determining the Cohen–Macaulay defect

So far we have only achieved to establish lower bounds for the Cohen–Macaulay defect. But can anything be said about the exact value? For a given group $G \subseteq \mathrm{GL}(V)$, the invariant ring R^G can be computed algorithmically (given enough time and memory space), and from this $\mathrm{def}(R^G)$ can be determined (see Derksen and Kemper [6, Chapter 3]. Apart from this, theoretical results on the precise value of the Cohen-Macaulay defect are rather sporadic. One is the formula (0.1) by Ellingsrud and Skjelbred [7]. Fleischmann et al. [10] proved the upper bound

$$\mathrm{def}\left(R^G\right) \leq \max\left\{\mathrm{codim}(V^P) - m - 1, 0\right\}, \tag{4.1}$$

where $P \subseteq G$ is a Sylow p-subgroup ($p = \mathrm{char}(K)$) and m is the cohomological connectivity. In all instances of theoretical results where the Cohen–Macaulay defect of R^G was determined, this bound turns out to be an equality. In fact,

$$\mathrm{def}\left(R^G\right) = \max\left\{\mathrm{codim}(V^P) - m - 1, 0\right\}$$

holds if

- $|G|$ is divisible by p but not by p^2 (see Kemper [16, Theorem 3.1]; the determination of m is hard in general),
- $|G|$ is divisible by p but not by p^2 and acts by permutations of a basis of V (see [16, Theorem 3.3], which gives a formula for m is this case),

- P is cyclic and G is p-nilpotent, i.e., there exists a normal subgroup $N \subseteq G$ with $G/N \cong P$ (see Fleischmann et al. [10]; in this case $m = 1$),
- $G = \mathrm{SL}_2(\mathbb{F}_p)$ and V is a symmetric power of the natural representation (see Shank and Wehlau [20]; here $m = 1$),
- more generally, (G, V) is one of the cases dealt with in [16, Section 4] (then $m = 1$).

Notice that the result of Ellingsrud and Skjelbred is included in the one on p-nilpotent groups with a cyclic Sylow p-subgroup.

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Monomial curves in affine four space

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We will give a theorem for a question whether every affine monomial curve is a set-theoretic complete intersection. Let k be a field, $N > 2$ and $n_1, \dots, n_N$ natural numbers whose gcd is one. The curve

$$C = \{(t^{n_1}, t^{n_2}, \dots, t^{n_N}) : t \in k\}$$

is called a **monomial curve** in affine N-space defined by $n_1, n_2, \dots, n_N$. And the kernel of the ring homomorphism

$$k[X_1, \dots, X_N] \longrightarrow k[t], \qquad X_i \longmapsto t^{n_i} \quad \text{for each } i$$

is called the defining ideal of the monomial curve C.

In general, for an ideal I in a ring R, if

$$\exists f_1, \dots, f_r \in I \text{ s.t. } \sqrt{I} = \sqrt{(f_1, \dots, f_r)}, \quad r = \operatorname{ht} I,$$

then I is called a **set-theoretic complete intersection**.

We consider a question if every monomial curve C in affine N-space is a set-theoretic complete intersection. Namely, if we put I be the defining ideal of C, it asks whether there exist

$$f_1, \dots, f_{N-1} \in I \text{ s.t. } \sqrt{I} = \sqrt{(f_1, \dots, f_{N-1})}.$$

There are partrial answers for it. If $N = 3$, it is affirmatively answered (e.g. [6]). If $N = 4$ and if its defining ideal is Gorenstein or an almost complete intersection, then the monomial curve is a set-theoretic complete intersection ([1], [4]). Further, if the characteristic of the field k is positive, Cowsik and Nori prove that any affine algebraic curve is a set-theoretic complete intersection ([2]). Thus we may assume that its characterisitic is zero.

Here is the main theorem:

Theorem 1 ([5, Theorem 3.11]). *If $N = 4$ and if $\min\{n_1, n_2, n_3, n_4\} \leq 13$ then the associated monomial curve with n_1, n_2, n_3 and n_4 is a set-theoretic complete intersection.*

This follows from the two theorems:

Theorem 2 ([5, Theorem 1.6]). *If $n_1 + n_4$ is contained in the semigroup generated by n_2 and n_3, then the associated monomial curve with n_1, n_2, n_3 and n_4 is a set-theoretic complete intersection.*

Theorem 3 ([5, Theorem 1.5]). *Let I be the defining ideal of a monomial curve defined by n_1, n_2, n_3 and n_4. Assume that there are $v_j \in \operatorname{Ker}(n_1, n_2, n_3, n_4)$ and $d_j > 0$ for $j = 1, 2, 3, 4$ with $\operatorname{supp} v_j^+ = \{j\}$, $\sum_j d_j v_j = 0$ and*

$$\operatorname{Ker}(n_1, n_2, n_3, n_4) = \sum_j \mathbb{Z} v_j \subset \mathbb{Z}^4.$$

If two of d_j are one, then I is a set-theoretic complete intersection.

We give some notes. We regard (n_1, n_2, n_3, n_4) as a map $\mathbb{Z}^4 = \bigoplus_{i=1}^4 \mathbb{Z} e_i \to \mathbb{Z}$ which sends $\sum \sigma_i(v) e_i$ to $\sum \sigma_i(v) n_i$. Then $\operatorname{Ker}(n_1, n_2, n_3, n_4)$ is a submodule in $\mathbb{Z}^4$ of rank 3. For $v \in \mathbb{Z}^4$, we denote the support of v by $\operatorname{supp} v$.

Eliahou (1984) proved the following:

Proposition 4 ([3]). *If*

(1) $n_1 = 4,\ n_2, n_3, n_4 \geq 4$,

(2) $n_2 \equiv 1,\ n_3 \equiv 2,\ n_4 \equiv 3 \mod 4$,

(3) $2n_3 \geq n_2 + n_4$,

then I is a set-theoretic complete intersection.

Applying the above, Eliahou also gives an example ([3]) in which I is a set-theoretic complete intersection if $(n_1, n_2, n_3, n_4) = (4, 5, 6, 7)$. But, if $(n_1, n_2, n_3, n_4) = (4, 6, 7, 9)$, then we cannot apply the above.

We observe the case $n_1 = 4$ again. If $n_2 \equiv n_3 \mod 4$, then I is a set-theoretic complete intersection, since it is reduced to the case $N = 3$. If $n_2 \equiv 1,\ n_3 \equiv 2,\ n_4 \equiv 3 \mod 4$, then

$$v = {}^t(a, -1, -1, 1) \in \operatorname{Ker}(4, n_2, n_3, n_4).$$

If $a \leq 0$, then n_4 is contained in the semigroup generated by $4, n_2$ and n_3. If $a > 0$, then I is a set-theoretic complete intersection by Theorem 2. Hence, if $n_1 = 4$, then I is a set-theoretic complete intersection. To generalize this argument, I is a set-theoretic complete intersection in the following cases:

(1) $n_i \equiv n_j \mod n_1$ for any $1 < i < j$,

(2) $n_i + n_j \equiv 0 \mod n_1$ for any $1 < i < j$,

(3) $n_i + n_j \equiv n_k \mod n_1$ for any $1 < i < j$ and $k > 0$.

We observe another case. Assume $n_1 = 7$. If $n_2 \equiv 1$, $n_3 \equiv 2$, $n_4 \equiv 4 \mod 7$, then

$$v_1 = \begin{pmatrix} a_1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} a_2 \\ 2 \\ -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} a_3 \\ 0 \\ 2 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} a_4 \\ -1 \\ 0 \\ 2 \end{pmatrix} \in V,$$

where $V = \mathrm{Ker}(7, n_2, n_3, n_4)$. If one of a_2, a_3 or a_4 is non negative, then I is a set-theoretic complete intersection. If a_2, a_3 and a_4 are negative, then v_1, v_2, v_3 and v_4 satisfy the conditions of Theorem 3 (note $V = \sum \mathbb{Z} v_j$ and $v_1 + v_2 + v_3 + v_4 = 0$). In any case, I is a set-theoretic complete intersection.

Definition 1. Let $V = \mathrm{Ker}(n_1, n_2, n_3, n_4)$. If there are $v_1, v_2, v_3, v_4 \in V$ with $\mathrm{supp}\, v_j^+ \ni j$ such that the existence of them implies that $I(V)$ is a set-theoretic complete intersection, then we denote $M(V) = (\sigma_i(v_j))_{i>1, j\geq 1}$. We also denote it by $M(n_1; n_2, n_3, n_4)$. For example,

$$M(7; 1, 2, 4) = \begin{pmatrix} -1 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Note 1. The following are valid:

(1) $M(n_1; n_2, n_3, n_4) = M(n_1; n_2', n_3', n_4')$, if $(n_2, n_3, n_4) \equiv (n_2', n_3', n_4') \mod n_1$,

(2) $M(n_1; n_2, n_3, n_4)$ exists, if $n_i \equiv n_j \mod n_1$ for any $1 < i < j$,

(3) $M(n_1; n_2, n_3, n_4)$ exists, if $n_i + n_j \equiv 0 \mod n_1$ for any $1 < i < j$,

(4) $M(n_1; n_2, n_3, n_4)$ exists, if $n_i + n_j \equiv n_k \mod n_1$ for any $1 < i < j$ and $k > 0$,

(5) $M(n_1; n_2, n_3, n_4)$ exists, if $M(n_1; n_1 - n_2, n_1 - n_3, n_1 - n_4)$ exists.

For example, $M(7; 3, 5, 6)$ exists, since $M(7; 4, 2, 1)$ exists and

$$(3, 5, 6) \equiv 6(4, 2, 1) \mod 7.$$

We investigate whether $M(n_1; n_2, n_3, n_4)$ exists under the following assumption:

(1) $0 < n_2 < n_3 < n_4 < n_1 - n_2$,

(2) $n_i + n_j \not\equiv 0 \mod n_1$ for any $1 < i < j$,

(3) $n_i + n_j \not\equiv n_k \mod n_1$ for any $1 < i < j$ and $k > 0$,

(4) $\gcd(n_1, n_3, n_4) = \gcd(n_1, n_2, n_4) = \gcd(n_1, n_2, n_3) = 1$,

(5) there are not a known triple (n_2', n_3', n_4') and $d > 0$ satisfying $(n_2, n_3, n_4) \equiv d(n_2', n_3', n_4') \mod n_1$.

For example, we have $(1,2,5) \equiv 5(2,4,1) \mod 9$, hence we have $M(9;1,2,5) = M(9;2,4,1)$. If n_1 is odd, then $M(n_1;1,2,4)$ exists, since

$$M(4\mu_1 + 2\mu_2 + 1; 1,2,4) = \begin{pmatrix} -1 & 2 & 0 & -1 \\ -\mu_2 & -1 & 2 & \mu_2 - 1 \\ -\mu_1 & 0 & -1 & \mu_1 + 1 \end{pmatrix},$$

where $\mu_1 > 0$ and μ_2 is 0 or 1. Note that we need not give $M(n_1;1,2,4)$ if n_1 is even, since the condition (4) above. By Mathematica,

(1) if $n_1 \le 7$, the only case is $M(n_1;1,2,4)$,

(2) if $11 \le n_1 \le 14$, the following cases of (n_2, n_3, n_4) are the rest:

$$\begin{array}{ll} (n_1 = 8) & (1,2,5), \\ (n_1 = 9) & (1,2,4),(1,2,6),(1,4,7), \\ (n_1 = 10) & (1,2,5),(1,2,7),(1,3,5),(1,5,8), \\ (n_1 = 11) & (1,2,4),(1,2,5),(1,2,7),(1,2,8),(1,3,5), \\ (n_1 = 12) & (1,2,5),(1,2,7),(1,2,9),(1,3,5),(1,3,7),(1,3,8), \\ & (1,4,7),(1,5,9),(1,7,10), \\ (n_1 = 13) & (1,2,4),(1,2,5),(1,2,6),(1,2,8),(1,2,9),(1,2,10), \\ & (1,3,5),(1,3,9),(1,4,6),(1,4,11), \\ (n_1 = 14) & (1,2,5),(1,2,7),(1,2,9),(1,2,11),(1,3,5),(1,3,7), \\ & (1,3,8),(1,3,10),(1,4,7),(1,6,11),(1,7,9),(1,7,10), \\ & (1,7,12),(1,9,11). \end{array}$$

If $n_1 \le 13$, the matrix $M(n_1;n_2,n_3,n_4)$ exists! Here is the list of the matrices $M(n_1;n_2,n_3,n_4)$ ([5]):

$$M(8;1,2,5) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(9;1,2,6) = \begin{pmatrix} -1 & 2 & 0 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(9;1,4,7) = \begin{pmatrix} -1 & 2 & -1 & -1 \\ -2 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \; 0v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(10;1,2,5) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ -2 & -1 & 3 & 0 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(10;1,2,7) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ -1 & -1 & 4 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0, \text{ or}$$

$$\begin{pmatrix} -1 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & -2 & 3 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0.$$

Note that, if $\pi(v) = {}^t(0,2,-2)$ for $v \in V$, then $\operatorname{supp} v^+ = \{3\}$ or $\operatorname{supp} v^- = \{4\}$. The above means that there exists $M(V)$ in each case.

$$M(10;1,3,5) = \begin{pmatrix} -2 & 3 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(10;1,5,8) = \begin{pmatrix} -2 & 3 & 0 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(11;1,2,5) = \begin{pmatrix} -1 & 2 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ -2 & 0 & -1 & 5 \end{pmatrix}, \; 2v_1 + 2v_2 + v_3 + v_4 = 0,$$

$$M(11;1,2,7) = \begin{pmatrix} 0 & 2 & -1 & -1 \\ -2 & -1 & 4 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(11;1,2,8) = \begin{pmatrix} -1 & 2 & 0 & 0 \\ -1 & -1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix}, \; 2v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(11;1,3,5) = \begin{pmatrix} -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -2 & 0 & -1 & 3 \end{pmatrix}, \; v_1 + v_2 + v_3 + v_4 = 0,$$

$$M(12;1,2,5)=\begin{pmatrix}0&2&-1&-1\\-1&-1&3&-1\\-2&0&-1&3\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

$$M(12;1,2,7)=\begin{pmatrix}-1&2&-1&0\\-2&-1&4&-1\\-1&0&-1&2\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

$$M(12;1,2,9)=\begin{pmatrix}-1&2&0&-1\\-1&-1&3&-1\\-1&0&-2&3\end{pmatrix},\ v_1+v_2+v_3+v_4=0,\text{ or}$$

$$\begin{pmatrix}-1&2&-1&0\\-1&-1&5&-3\\-1&0&-1&2\end{pmatrix},\ v_1+v_2+v_3+v_4=0.$$

$$M(12;1,3,5)=\begin{pmatrix}-2&3&-1&0\\0&-1&2&-1\\-2&0&-1&3\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

$$M(12;1,3,7)=\begin{pmatrix}0&2&0&-1\\-1&0&4&-1\\-3&-2&0&4\end{pmatrix},\ 2v_1+v_2+v_3+2v_4=0,\text{ or}$$

$$\begin{pmatrix}-2&3&0&-2\\-1&-1&4&0\\-1&0&0&2\end{pmatrix},\ 2v_1+2v_2+v_3+v_4=0.$$

$$M(12;1,3,8)=\begin{pmatrix}-1&3&-1&0\\-1&-1&3&0\\-1&0&-1&3\end{pmatrix},\ 2v_1+v_2+v_3+v_4=0,$$

$$M(12;1,4,7)=\begin{pmatrix}-1&4&-1&0\\-1&-1&2&-1\\-1&0&-1&4\end{pmatrix},\ 2v_1+v_2+2v_3+v_4=0,$$

$$M(12;1,5,9)=\begin{pmatrix}-2&2&-1&-1\\-2&-1&2&-1\\0&-1&-1&2\end{pmatrix},\ v_2+v_3+v_4=0,$$

$$M(12;1,7,10)=\begin{pmatrix}-2&5&-2&-1\\0&-1&2&-1\\-1&-1&0&2\end{pmatrix},\ v_1+v_2+v_3+v_4=0,\text{ or}$$

$$\begin{pmatrix}0&2&-1&-1\\-2&-2&5&-1\\-1&0&-1&2\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

$$M(13;1,2,5)=\begin{pmatrix}-1&2&-1&0\\-1&-1&3&-1\\-2&0&-1&3\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

$$M(13;1,2,6)=\begin{pmatrix}-1&2&0&-1\\0&-1&3&-1\\-2&0&-1&7\end{pmatrix},\ 3v_1+2v_2+v_3+v_4=0,$$

$$M(13;1,2,8)=\begin{pmatrix}-1&2&0&-1\\-2&-1&4&-1\\-1&0&-1&2\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

$$M(13;1,2,9)=\begin{pmatrix}0&2&-1&-1\\-2&-1&5&0\\-1&0&-1&3\end{pmatrix},\ 2v_1+v_2+v_3+v_4=0,$$

$$M(13;1,2,10)=\begin{pmatrix}-1&2&0&-1\\-1&-1&5&0\\-1&0&-1&4\end{pmatrix},\ 3v_1+2v_2+v_3+v_4=0,$$

$$M(13;1,3,5)=\begin{pmatrix}0&3&-1&-1\\-1&-1&2&-1\\-2&0&-1&6\end{pmatrix},\ 2v_1+v_2+2v_3+v_4=0,$$

$$M(13;1,3,9)=\begin{pmatrix}-1&3&0&-1\\-1&-1&3&0\\-1&0&-1&3\end{pmatrix},\ 2v_1+v_2+v_3+v_4=0,$$

$$M(13;1,4,6)=\begin{pmatrix}-1&6&0&-1\\0&0&3&-1\\-2&-1&-2&3\end{pmatrix},\ 3v_1+v_2+v_3+3v_4=0 \text{ or}$$

$$\begin{pmatrix}-1&4&-1&0\\-3&-1&5&-3\\0&0&-1&2\end{pmatrix},\ 2v_1+v_2+2v_3+v_4=0,$$

$$M(13;1,4,11)=\begin{pmatrix}0&2&-1&-1\\-1&-1&3&-1\\-2&-1&-1&4\end{pmatrix},\ v_1+v_2+v_3+v_4=0,$$

Note 2. If $n_1 = 14$, the exsitence of $M(14,1,9,11)$ is not proved, while the others exists.

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WEAKLY CLOSED GRAPH

KAZUNORI MATSUDA

1. Introduction

Throughout this article, let k be an F-finite field of positive characteristic. Let G be a graph on the vertex set $V(G) = [n]$ with edge set $E(G)$. We assume that a graph G is always connected and simple, that is, G is connected and has no loops and multiple edges. Moreover, we note that labeling means numbering of $V(G)$ from 1 to n.

For each graph G, we call $J_G := ([i,j] = X_iY_j - X_jY_i \mid \{i,j\} \in E(G))$ the *binomial edge ideal* of G (see [HeHiHrKR], [O2]). J_G is an ideal of $S := k[X_1, \dots, X_n, Y_1, \dots, Y_n]$.

2. Weakly closed graph

Until we define the notion of weak closedness, we fix a graph G and a labeling of $V(G)$.

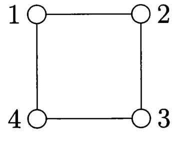
Let $(a_1, \dots, a_n)$ be a sequence such that $1 \leq a_i \leq n$ and $a_i \neq a_j$ if $i \neq j$.

Definition 2.1. We say that a_i is *interchangeable with* a_{i+1} if $\{a_i, a_{i+1}\} \in E(G)$. And we call the following operation $\{a_i, a_{i+1}\}$*-interchanging* :

$$(a_1, \dots, a_{i-1}, \underline{a_i}, \underline{a_{i+1}}, a_{i+2}, \dots, a_n) \rightarrow (a_1, \dots, a_{i-1}, \underline{a_{i+1}}, \underline{a_i}, a_{i+2}, \dots, a_n)$$

Definition 2.2. Let $\{i,j\} \in E(G)$. We say that i is *adjacentable with* j if the following assertion holds: for a sequence $(1, 2, \dots, n)$, by repeating interchanging, one can make a sequence $(a_1, \dots, a_n)$ such that $a_k = i$ and $a_{k+1} = j$ for some k.

Example 2.3. About the following graph G, 1 is adjacentable with 4:



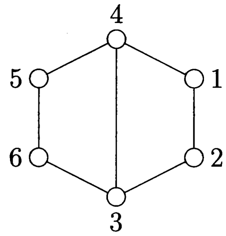
Indeed,

$$(\underline{1}, 2, 3, \underline{4}) \xrightarrow{\{1,2\}} (2, \underline{1}, 3, \underline{4}) \xrightarrow{\{3,4\}} (2, \underline{1}, \underline{4}, 3).$$

Now, we can define the notion of weakly closed graph.

Definition 2.4. Let G be a graph. G is said to be *weakly closed* if there exists a labeling which satisfies the following condition: for all i, j such that $\{i,j\} \in E(G)$, i is adjacentable with j.

Example 2.5. The following graph G is weakly closed:



Indeed,

$$(\underline{1}, 2, 3, \underline{4}, 5, 6) \xrightarrow{\{1,2\}} (2, \underline{1}, 3, \underline{4}, 5, 6) \xrightarrow{\{3,4\}} (2, \underline{1}, \underline{4}, 3, 5, 6),$$
$$(1, 2, \underline{3}, 4, 5, \underline{6}) \xrightarrow{\{3,4\}} (1, 2, 4, \underline{3}, 5, \underline{6}) \xrightarrow{\{5,6\}} (1, 2, 4, \underline{3}, \underline{6}, 5).$$

Hence 1 is adjacentable with 4 and 3 is adjacentable with 6.

Before stating the first main theorem of this article, we recall that the definition of closed graphs.

Definition 2.6 (see [HeHiHrKR]). G is *closed with respect to the given labeling* if the following condition is satisfied: for all $\{i, j\}, \{k, l\} \in E(G)$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$ but $j \neq l$, and $\{i, k\} \in E(G)$ if $j = l$ but $i \neq k$.
In particular, G is *closed* if there exists a labeling for which it is closed.

Remark 2.7. (1) [HeHiHrKR, Theorem 1.1] G is closed if and only if J_G has a quadratic Gröbner basis. Hence if G is closed then S/J_G is Koszul algebra.
(2) [EHeHi, Theorem 2.2] Let G be a graph. Then the following assertion are equivalent:
(a) G is closed.
(b) There exists a labeling of $V(G)$ such that all facets of $\Delta(G)$ are intervals $[a, b] \subset [n]$, where $\Delta(G)$ is the clique complex of G.

The following characterization of closed graphs is a reinterpretation of Crupi and Rinaldo's one. This is relevant to the first main theorem deeply.

Proposition 2.8 (See [CR, Proposition 2.6]). Let G be a graph. Then the following conditions are equivalent:
(1) G is closed.
(2) There exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$ *and* $\{k, j\} \in E(G)$.

Proof. (1) $\Rightarrow$ (2): Let $\{i, j\} \in E(G)$. Since G is closed, there exists a labeling satisfying $\{i, i+1\}, \{i+1, i+2\}, \dots, \{j-1, j\} \in E(G)$ by [HeHiHrKR, Proposition 1.4]. Then we have that $\{i, j-1\}, \{i, j-2\}, \dots, \{i, i+2\} \in E(G)$ by the definition of closedness. Similarly, we also have that $\{k, j\} \in E(G)$ for all $i < k < j$.
(2) $\Rightarrow$ (1): Assume that $i < k < j$. If $\{i, k\}, \{i, j\} \in E(G)$, then $\{k, j\} \in E(G)$ by assumption. Similarly, if $\{i, j\}, \{k, j\} \in E(G)$, then $\{i, k\} \in E(G)$. Therefore G is closed. □

The first main theorem is as follows:

Theorem 2.9. Let G be a graph. Then the following conditions are equivalent:

(1) G is weakly closed.

(2) There exists a labeling which satisfies the following condition: for all i, j such that $\{i,j\} \in E(G)$ and $j > i+1$, the following assertion holds: for all $i < k < j$, $\{i,k\} \in E(G)$ *or* $\{k,j\} \in E(G)$.

Proof. (1) $\Rightarrow$ (2): Assume that $\{i,j\} \in E(G)$, $\{i,k\} \notin E(G)$ and $\{k,j\} \notin E(G)$ for some $i < k < j$. Then i is not adjacentable with j, which is in contradiction with weak closedness of G.

(2) $\Rightarrow$ (1): Let $\{i,j\}E(G)$. By repeating interchanging along the following algorithm, we can see that i is adjacentable with j:

(a): Let $A := \{k \mid \{k,j\} \in E(G), i < k < j\}$ and $C := \emptyset$.

(b): If $A = \emptyset$ then go to (g), otherwise let $s := \max\{A\}$.

(c): Let $B := \{t \mid \{s,t\} \in E(G), s < t \leq j\} \setminus C = \{t_1, \ldots, t_m = j\}$, where $t_1 < \ldots < t_m = j$.

(d): Take $\{s, t_1\}$-interchanging, $\{s, t_2\}$-interchanging, $\ldots$, $\{s, t_m = j\}$-interchanging in turn.

(e): Let $A := A \setminus \{s\}$ and $C := C \cup \{s\}$.

(f): Go to (b).

(g): Let $U := \{u \mid i < u < j, \{i,u\} \in E(G) \text{ and } \{u,j\} \notin E(G)\}$ and $W := \emptyset$.

(h): If $U = \emptyset$ then go to (m), otherwise let $u := \min\{U\}$.

(i): Let $V := \{v \mid \{v,u\} \in E(G), i \leq v < u\} \setminus W = \{v_1 = i, \ldots, v_l\}$, where $v_1 = i < \ldots < v_l$.

(j): Take $\{v_1 = i, u\}$-interchanging, $\{v_2, u\}$-interchanging, $\ldots$, $\{v_l, u\}$-interchanging in turn.

(k): Let $U := U \setminus \{u\}$ and $W := W \cup \{u\}$.

(l): Go to (h).

(m): Finished. □

By comparing this theorem and Proposition 2.8, we get

Corollary 2.10. Closed graphs and complete r-partite graphs are weakly closed.

Proof. Assume that G is complete r-partite and $V(G) = \coprod_{i=1}^{r} V_i$. Let $\{i,j\} \in E(G)$ with $i \in V_a$ and $j \in V_b$. Then $a \neq b$. Hence for all $i < k < j$, $k \notin V_a$ or $k \notin V_b$. This implies that $\{i,k\} \in E(G)$ or $\{k,j\} \in E(G)$. □

3. F-purity of binomial edge ideals

Firstly, we recall that the definition of F-purity of a ring R.

Definition 3.1 (See [HoR]). Let R be an F-finite reduced Noetherian ring of characteristic $p > 0$. R is said to be *F-pure* if the Frobenius map $R \to R$, $x \mapsto x^p$ is pure, equivalently, the natural inclusion $\tau : R \hookrightarrow R^{1/p}$, $(x \mapsto (x^p)^{1/p})$ is pure, that is, $M \to M \otimes_R R^{1/p}$, $m \mapsto m \otimes 1$ is injective for every R-module M.

The following proposition, which is called the Fedder's criterion, is useful to determine the F-purity of a ring R.

Proposition 3.2 (See [Fe]). Let $(S, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Let I be an ideal of S. Put $R = S/I$. Then R is F-pure if and only if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$, where $J^{[p]} = (x^p \mid x \in J)$ for an ideal J of S.

In this section, we consider the following question:

Question 3.3. When is S/J_G F-pure ?

In [O2], Ohtani proved that if G is complete r-partite graph then S/J_G is F-pure. Moreover, it is easy to show that if G is closed then S/J_G is F-pure. However, there are many examples of G such that G is neither complete r-partite nor closed but S/J_G is F-pure. Namely, there is room for improvement about the above studies.

The second main theorem of this article is as follows:

Theorem 3.4. If G is weakly closed, then S/J_G is F-pure.

Proof. For a sequence $v_1, v_2, \dots, v_s$, we put

$$Y_{v_1}(v_1, v_2, \dots, v_s)X_{v_s} := (Y_{v_1}[v_1, v_2][v_2, v_3] \cdots [v_{s-1}, v_s]X_{v_s})^{p-1}.$$

Let $\mathfrak{m} = (X_1, \dots, X_n, Y_1, \dots, Y_n)S$. By taking completion and using Proposition 2.2, it is enough to show that $Y_1(1, 2, \dots, n)X_n \in (J_G^{[p]} : J_G) \setminus \mathfrak{m}^{[p]}$. It is easy to show that $Y_1(1, 2, \dots, n)X_n \notin \mathfrak{m}^{[p]}$ by considering its initial monomial.

Next, we use the following lemmas (see [O2]):

Lemma 3.5 ([O2, Formula 1]). If $\{a, b\} \in E(G)$, then

$$Y_{v_1}(v_1, \dots, c, \underline{a}, \underline{b}, d, \dots, v_n)X_{v_n} \equiv Y_{v_1}(v_1, \dots, c, \underline{b}, \underline{a}, d, \dots, v_n)X_{v_n}$$

modulo $J_G^{[p]}$.

Lemma 3.6 ([O2, Formula 2]). If $\{a, b\} \in E(G)$, then

$$Y_a(\underline{a}, \underline{b}, c, \dots, v_n)X_{v_n} \equiv Y_b(\underline{b}, \underline{a}, c, \dots, v_n)X_{v_n},$$
$$Y_{v_1}(v_1, \dots, c, \underline{a}, \underline{b})X_b \equiv Y_{v_1}(v_1, \dots, c, \underline{b}, \underline{a})X_a$$

modulo $J_G^{[p]}$.

Let $\{i, j\} \in E(G)$. Since G is weakly closed, i is adjacentable with j. Hence there exists a polynomial $g \in S$ such that

$$Y_1(1, 2, \dots, n)X_n \equiv g \cdot [i, j]^{p-1}$$

modulo $J_G^{[p]}$ from the above lemmas. This implies $Y_1(1, 2, \dots, n)X_n \in (J_G^{[p]} : J_G)$. □

4. Difference between closedness and weak closedness and some examples

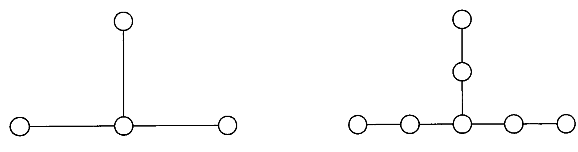
In this section, we state the difference between closedness and weak closedness and give some examples.

Proposition 4.1. Let G be a graph.

(1) [HeHiHrKR, Proposition 1.2] If G is closed, then G is chordal, that is, every cycle of G with length $t > 3$ has a chord.
(2) If G is weakly closed, then every cycle of G with length $t > 4$ has a chord.

Proof. (2) It is enough to show that the pentagon graph G with edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, e\}$ and $\{a, e\}$ is not weakly closed. Suppose that G is weakly closed. We may assume that $a = \min\{a, b, c, d, e\}$ without loss of generality. Then $b \neq \max\{a, b, c, d, e\}$. Indeed, if $b = \max\{a, b, c, d, e\}$, then c, d, e are connected with a or b by the definition of weak closedness, but this is a contradiction. Similarly, $e \neq \max\{a, b, c, d, e\}$. Hence we may assume that $c = \max\{a, b, c, d, e\}$ by symmetry. If $b = \min\{b, c, d\}$, then d, e are connected with b or c, a contradiction. Therefore, $b \neq \min\{b, c, d\}$. Similarly, $b \neq \max\{b, c, d\}$. Hence we may assume that $d = \min\{b, c, d\}$ and $e = \max\{b, c, d\}$ by symmetry. Then $\{a, b\}$ and $a < d < b$, but $\{a, d\}, \{d, b\} \notin E(G)$. This is a contradiction. □

Next, we give a characterization of closed (resp. weakly closed) tree graphs in terms of claw (resp. bigclaw). We consider the following graphs (a) and (b). We call the graph (a) a *claw* and the graph (b) a *bigclaw.*



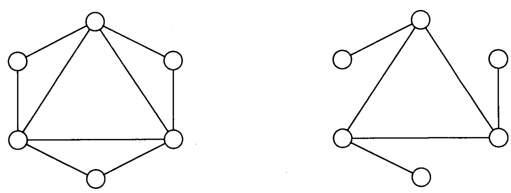
(a) (b)

One can check to a bigclaw graph is not weakly closed, hence we have the following proposition:

Proposition 4.2. Let G be a tree.

(1) [HeHiHrKR, Corollary 1.3] The following conditions are equivalent:
  (a) G is closed.
  (b) G is a path.
  (c) G is a claw-free graph.
(2) The following conditions are equivalent:
  (a) G is weakly closed.
  (b) G is a caterpillar, that is, a tree for which removing the leaves and incident edges produces a path graph.
  (c) G is a bigclaw-free graph.

Remark 4.3. From Proposition 3.2(2), we have that chordal graphs are not always weakly closed. As other examples, the following graphs are chordal, but not weakly closed:



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THE EQUALITY OF ELIAS AND VALLA AND BUCHSBAUMNESS OF ASSOCIATED GRADED RINGS

KAZUHO OZEKI

1. Introduction and the statement of main results

The purpose of this paper is to study the Buchsbaumness of the associated graded ring of ideals in a Buchsbaum local ring satisfying the equality of Elias and Valla [3], without assuming that the base local ring is Cohen-Macaulay.

Throughout this paper let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and suppose that I contains a parameter ideal $Q = (a_1, a_2, \cdots, a_d)$ of A as a reduction, that is $Q \subseteq I$ and the equality $I^{n+1} = QI^n$ holds true for some (and hence any) integer $n \gg 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then we have integers $\{\mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$, which we call the Hilbert coefficients of A with respect to I.

Let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \quad \subseteq A[t]$$

denote, respectively, the Rees algebras of I and Q. Let $F = T/IT$,

$$R' = \mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}] \quad \text{and} \quad G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

In the case where A is a Cohen-Macaulay local ring, we have the inequality

$$2\mathrm{e}_0(I) - \mathrm{e}_1(I) \le 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$$

which is given in [3] and [7], and they showed that the equality $2\mathrm{e}_0(I) - \mathrm{e}_1(I) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$ holds true if and only if $I^3 = QI^2$ and $Q \cap I^2 = QI$. When this is the case, the associated graded ring G of I is Cohen-Macaulay. Thus the ideal I with $2\mathrm{e}_0(I) - \mathrm{e}_1(I) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$ enjoys nice properties.

Key words and phrases: Buchsbaum local ring, associated graded ring, Hilbert function, Hilbert coefficient

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The purpose of this paper is to extend their results without assuming that A is a Cohen-Macaulay ring.

In an arbitrary Noetherian local ring A the inequality

$$2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) \leq 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$$

holds true ([8, Theorem 2.4], [1, Theorem 3.1]). It seems natural to ask, what happens on the ideals I satisfying the equality

$$2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q).$$

To sate the results of the present paper, let us consider the following three conditions:

(C_1) The sequence $a_1, a_2, \cdots, a_d$ is a d^+-sequence in A, that is for all integers $n_1, n_2, \cdots, n_d \geq 1$ the sequence $a_1^{n_1}, a_2^{n_2}, \cdots, a_d^{n_d}$ forms a d-sequence in any order.
(C_2) $(a_1, a_2, \cdots, \check{a}_i, \cdots, a_d) :_A a_i \subseteq I$ for all $1 \leq i \leq d$.
(C_3) $\operatorname{depth} A > 0$.

These conditions $(\mathrm{C}_1), (\mathrm{C}_2)$, and (C_3) are naturally satisfied, when A is a Cohen-Macaulay local ring. Condition (C_1) (resp. condition (C_2)) is always satisfied, if A is a Buchsbaum local ring (resp. $I = \mathfrak{m}$). Here we notice that condition (C_1) is equivalent to saying that our local ring A is a generalized Cohen-Macaulay ring, that is all the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(A)$ $(i \neq d)$ of A with respect to the maximal ideal $\mathfrak{m}$ are finitely generated and the parameter ideal Q is standard, that is the equality

$$\ell_A(A/Q) - \mathrm{e}_0(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$$

holds true. Hence condition (C_1) is independent of the choice of a minimal system $\{a_i\}_{1 \leq i \leq d}$ of generators of the parameter ideal Q. We note here that condition (C_2) is also independent of the choice of a minimal system $\{a_i\}_{1 \leq i \leq d}$ of generators of Q.

Let us now state our own result. The main result of this paper is the following Theorem 1.1, which generalizes the result of [3] and [7] given in the case where A is a Cohen-Macaulay local ring, because $\mathrm{e}_i(Q) = 0$ for all $1 \leq i \leq d$. We notice that, thanks to condition (C_1), the Hilbert coefficients $\mathrm{e}_i(Q)$ of Q are given by the formula

$$(-1)^i \mathrm{e}_i(Q) = \begin{cases} \mathrm{e}_0(Q) & \text{if } i = 0, \\ \ell_A(\mathrm{H}^0_{\mathfrak{m}}(A)) & \text{if } i = d, \\ \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \ell_A(\mathrm{H}^j_{\mathfrak{m}}(A)) & \text{if } 1 \leq i \leq d-1 \end{cases}$$

and one has the equality $\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_i(Q) \binom{n+d-i}{d-i}$ for all $n \geq 0$ ([9, Korollar 3.2]), so that $\{\mathrm{e}_i(Q)\}_{1 \leq i \leq d}$ are independent of the choice of the reduction Q of I and so, are invariants of A. Here $W = \mathrm{H}^0_{\mathfrak{m}}(A)$ denotes the 0-th local cohomology modules of A with respect to $\mathfrak{m}$ and $\mathrm{H}^i_M(G)$ the i-th local cohomology modules of G with respect to the graded maximal ideal $M = \mathfrak{m}T + T_+$ of T.

Theorem 1.1. *Suppose that conditions* (C_1) *and* (C_2) *are satisfied. Then the following two conditions are equivalent to each other.*

(1) $2e_0(I) - e_1(I) + e_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$.

(2) $I^3 \subseteq QI^2 + W$, $(Q+W) \cap (I^2 + W) = QI + W$, *and* $(a_1, a_2, \cdots, \check{a}_i, \cdots, a_d) :_A a_i \subseteq I^2 + Q$ *for all* $1 \le i \le d$.

When this is the case, we have $I^2 \supseteq W$ *and the following assertions also hold true.*

(*i*) *For all* $n \in \mathbb{Z}$,

$$[\mathrm{H}^0_M(G)]_n \cong \begin{cases} W/I^3 \cap W & \text{if } n = 2, \\ I^n \cap W/I^{n+1} \cap W & \text{if } n \ge 3, \\ (0) & \text{otherwise.} \end{cases}$$

Hence $[\mathrm{H}^0_M(G)]_2 \cong W/I^3 \cap W$, $[\mathrm{H}^0_M(G)]_3 \cong I^3 \cap W$, *and* $[\mathrm{H}^0_M(G)]_n = (0)$ *for all* $n \ne 2, 3$ *if* A *is a Buchsbaum local ring.*

(*ii*)

$$\mathrm{H}^i_M(G) = [\mathrm{H}^i_M(G)]_{2-i} \cong \mathrm{H}^i_{\mathfrak{m}}(A)$$

for all $1 \le i \le d-1$,

(*iii*) *the* a-*invariant*

$$\mathrm{a}(G) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^d_M(G)]_n \ne (0)\}$$

of G *is at most* $2-d$,

(*iv*) $e_2(I) = e_1(Q) + e_2(Q) - e_0(I) + e_1(I) + \ell_A(A/I)$,

(*v*) $e_i(I) = e_{i-2}(Q) + 2e_{i-1}(Q) + e_i(Q)$ *for all* $3 \le i \le d$, *and*

(*vi*) *the associated graded ring* G *is a Buchsbaum ring with the Buchsbaum invariant* $\mathbb{I}(G) = \mathbb{I}(A)$ *if* A *is a Buchsbaum local ring.*

The key of the proof of Theorem 1.1 is the use of the Sally module S of I with respect to Q.

We are now in a position to briefly explain how we organized this paper.

In Section 2 we will summarize some auxiliary results on Sally modules for the later use in this paper. We will give in Section 3 an outline of a proof of the implication (1) $\Rightarrow$ (2) and the last assertions of Theorem 1.1. In Section 3 we will also introduce some techniques of Sally modules which is the key for the proof of Theorem 1.1. In Section 4 we will give one example of an $\mathfrak{m}$-primary ideal I with $2e_0(I) - e_1(I) + e_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$ in a Buchsbaum local ring $(A, \mathfrak{m})$.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and put $Q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal of A which forms a reduction of I. We put

$$R = A[It], \quad T = A[Qt], \quad R' = A[It, t^{-1}], \quad G = R'/t^{-1}R', \quad \text{and} \quad F = T/IT.$$

We denote by $\mathrm{H}^i_{\mathfrak{m}}(*)$ ($i \in \mathbb{Z}$) the i-th local cohomology functor of A with respect to $\mathfrak{m}$. Let $\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ denotes the Buchsbaum invariant of A. Let $M = \mathfrak{m}T + T_+$ be the unique graded maximal ideal in T. We denote by $\mathrm{H}^i_M(*)$ ($i \in \mathbb{Z}$) the i-th local cohomology functor of T with respect to M and $\mathbb{I}(G) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_G(\mathrm{H}^i_M(G))$ the Buchsbaum invariant of G. Let L be a graded T-module. We denote by $L(\alpha)$, for each $\alpha \in \mathbb{Z}$, the graded T-module whose grading is given by $[L(\alpha)]_n = [L]_{\alpha+n}$ for all $n \in \mathbb{Z}$. Let $\mu_A(I)$ denotes the number of a minimal system of generators of I.

2. The structure of Sally modules

In our proof of Theorem 1.1 we need some structure theorems of Sally modules.

Following Vasconcelos [11], we define

$$S = \mathrm{S}_Q(I) := IR/IT \cong \bigoplus_{n \geq 1} I^{n+1}/Q^n I$$

and call it the Sally module of I with respect to Q.

The purpose of this section is to summarize some auxiliary results on Sally modules, which we need throughout this paper. See [5, 6, 11] for the detailed proofs.

Remark 2.1 (cf. [5, 6, 11]). We notice that S is a finitely generated graded T-module and $\mathfrak{m}^{\ell}{\cdot}S = (0)$ for some integer $\ell \gg 0$, since R is a module finite extension of the graded ring T and $\mathfrak{m} = \sqrt{Q}$, so that $\dim_T S \leq d$.

Lemma 2.2. *Suppose that conditions* (C_1) *and* (C_2) *are satisfied. Then*

$$F = T/IT \cong (A/I)[X_1, X_2, \cdots, X_d]$$

as graded A-algebras, where $(A/I)[X_1, X_2, \cdots, X_d]$ denotes the polynomial ring with d indeterminate over the Artinian local ring A/I. Hence F is a Cohen-Macaulay ring with $\dim F = d$.

Proof. See [6, Proposition 2.2]. □

Let us note the following lemma.

Lemma 2.3. *Suppose that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Then* $\mathrm{Ass}_T S \subseteq \{\mathfrak{m}T\}$, *whence* $\dim_T S = d$ *if* $S \neq (0)$.

Proof. See [6, Lemma 2.3]. □

Proposition 2.4. *Suppose that conditions* (C_1) *and* (C_2) *are satisfied. Then*

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I)\}\binom{n+d-1}{d-1} \\ &+ \sum_{i=2}^{d}(-1)^i\{\mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)\}\binom{n+d-i}{d-i} - \ell_A(S_n)\end{aligned}$$

for all $n \geq 0$.

Proof. See [6, Proposition 2.4]. $\square$

Put $s = \dim_T S \leq d$. Then we write

$$\ell_A(S_n) = e_0(S)\binom{n+s-1}{s-1} - e_1(S)\binom{n+s-2}{s-2} + \cdots + (-1)^{s-1}e_{s-1}(S)$$

for all $n \gg 0$ with integers $\{e_i(S)\}_{0 \leq i \leq s-1}$. Then by Proposition 2.4 we get the following result, which is also given in [8, Proposition 6.2].

Corollary 2.5. *Suppose that conditions* (C_1) *and* (C_2) *are satisfied. Then we have the following.*

(1) $e_1(I) = e_0(I) + e_1(Q) - \ell_A(A/I) + e_0(S)$ *and*
(2) $e_i(I) = e_{i-1}(Q) + e_i(Q) + e_{i-1}(S)$ *for all* $2 \leq i \leq d$.

3. Proof of our main theorem

In this section let us introduce some techniques, being inspired by [1, 2], which plays a crucial role throughout this paper.

Let us begin with the following.

Lemma 3.1. *Assume that* $I \supsetneq Q$ *and put* $\mu = \mu_A(I/Q)$. *Then there exists an exact sequence*

$$T(-1)^{\mu} \xrightarrow{\phi} R/T \to S(-1) \to 0$$

as graded T*-modules.*

Proof. Let us write $I = Q + (x_1, x_2, \cdots, x_\mu)$ and put

$$\phi : T(-1)^{\mu} \to R/T$$

denotes a homomorphism of graded T-modules with $\phi(\alpha_1, \alpha_2, \cdots, \alpha_\mu) = \sum_{i=1}^{\mu} \overline{\alpha_i x_i t} \in R/T$ for $\alpha_i \in T$ and $1 \leq i \leq \mu$, where $\overline{\alpha_i x_i t}$ denotes the image of $\alpha_i x_i t$ in R/T. Then we have

$$\mathrm{Coker}\phi = R/[It{\cdot}T + T] \cong R_+/It{\cdot}T$$

as graded T-modules. Then two isomorphisms

$$R_+ \overset{\widehat{t^{-1}}}{\to} IR(-1) \text{ and } It{\cdot}T \overset{\widehat{t^{-1}}}{\to} IT(-1)$$

of graded T-modules induce the isomorphism $R_+/It{\cdot}T \cong (IR/IT)(-1)$ of graded T-modules. Therefore $\mathrm{Coker}\phi \cong S(-1)$ as graded T-modules, whence we get a required exact sequence. $\square$

Tensoring the exact sequence of Lemma 3.1 with A/I, we get exact sequence

$$F(-1)^{\mu} \xrightarrow{\overline{\phi}} R/IR + T \to (S/IS)(-1) \to 0 \quad (*)$$

of graded T-modules, where $\overline{\phi} = A/I \otimes \phi$.

Then thanks to this exact sequence $(*)$ and Corollary 2.5, we get the following inequality which is originally proved by [8, Theorem 4.1] and [1, Theorem 3.1].

Proposition 3.2. *Suppose that $d > 0$. Then we have*

$$2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) \leq 2\ell_A(A/I) + \ell_A(I/I^2 + Q).$$

The following result plays a key role in our proof of Theorem 1.1.

Proposition 3.3. *Suppose that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Assume that $I \supsetneq Q$. Then the following two conditions are equivalent to each other.*

(1) $2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$,

(2) *there exist exact sequences*

$$0 \to ((I/I^2 + Q) \otimes F)(-1) \to R/IR + T \to S(-1) \to 0$$

and

$$0 \to F \to G \to R/IR + T \to 0$$

of graded T-modules.

We get the following corollary by Proposition 3.3.

Corollary 3.4. *Suppose that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Assume that* $2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$. *Then we have the following.*

(1) $I^{n+2} \subseteq Q^n I$ *for all* $n \geq 0$,

(2) $Q^n \cap I^{n+1} = Q^n I$ *for all* $n \geq 0$,

(3) $\mathrm{depth} G > 0$,

(4) $(a_1, a_2, \cdots, \check{a_i}, \cdots, a_d) : a_d \subseteq I^2 + Q$ *for all* $1 \leq i \leq d$.

In the rest of Section 3, let us introduce an outline of our proof of the implication (1) ⇒ (2) and the last assertions of Theorem 1.1.

Let us note the following lemma.

Lemma 3.5. *Put* $C = A/W$ *with* $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. *Then* $2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$ *if and only if* $2\mathrm{e}_0(IC) - \mathrm{e}_1(IC) + \mathrm{e}_1(QC) = 2\ell_A(C/IC) + \ell_A(IC/I^2C + QC)$ *and* $I^2 + Q \supseteq W$.

Suppose that condition (C_1) and (C_2) are satisfied and assume that

$$2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q).$$

Put $C = A/W$ then we have $2\mathrm{e}_0(IC) - \mathrm{e}_1(IC) + \mathrm{e}_1(QC) = 2\ell_A(C/IC) + \ell_A(IC/I^2C + QC)$ and $W \subseteq I^2 + Q$ by Lemma 3.5. Therefore, passing to the ring C, we get $(Q + W) \cap (I^2 + W) = QI + W$ and $[(a_1, a_2, \cdots, \check{a}_i, \cdots, a_d) : a_i] \subseteq I^2 + Q$ for all $1 \leq i \leq d$ by Corollary 3.4 (2) and (4).

Thus the implication (1) $\Rightarrow$ (2) in Theorem 1.1 has been proven modulo the following Theorem 3.6.

Theorem 3.6. *Suppose that conditions* (C_1) *and* (C_2) *are satisfied and assume that* $2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$. *Then we have* $I^3 \subseteq QI^2 + W$ *and* $I^2 \supseteq W$.

Taking the local cohomology functor $\mathrm{H}^i_M(*)$ to the exact sequences of Proposition 3.3 (2), we may calculate the local cohomology modules $\mathrm{H}^i_M(G)$ of the associated graded ring G. Thus we may prove that our assertions (*i*), (*ii*), and (*iii*) are satisfied. Our assertion (*iv*) and (*v*) are also satisfied by the exact sequences of Proposition 3.3 (2) and Corollary 2.5. To prove our assertion (*vi*) of Theorem 1.1, we need to compute the Koszul cohomology of the associated graded ring G of I.

4. An example

In this section we construct one example of $\mathfrak{m}$-primary ideal I which satisfies $2\mathrm{e}_1(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$ in a Buchsbaum local ring.

Our goal is the following.

Theorem 4.1. *Let* $\ell > 0$ *and* $d \geq 2$ *be integers. Then there exists an* $\mathfrak{m}$*-primary ideal* I *in a Buchsbaum local ring* $(A, \mathfrak{m})$ *such that*

$$d = \dim A, \quad \mathrm{H}^i_{\mathfrak{m}}(A) = (0) \ \ \text{for } i \neq 1, d, \quad \ell_A(\mathrm{H}^1_{\mathfrak{m}}(A)) = \ell, \quad \text{and}$$

$$2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ *of* I.

To construct necessary examples we need some techniques which due to [4, Section 6]. Let us begin with the following.

Let m, $\ell > 0$ and $d \geq 2$ be integers. Let

$$U = k[\{X_i\}_{1\leq i\leq m}, \{Y_j\}_{1\leq j\leq \ell}, \{V_{jk}\}_{1\leq j\leq \ell, 1\leq k\leq d}, \{Z_k\}_{1\leq k\leq d}]$$

be the polynomial ring with $m + \ell + \ell d + d$ indeterminates over an infinite field k and let

$$\begin{aligned} \mathfrak{a} \ = \ & (\{X_i\}_{1\leq i\leq m}, \{Y_j\}_{1\leq j\leq \ell}, \{V_{jk}\}_{1\leq j\leq \ell, 1\leq k\leq d})\cdot(\{X_i\}_{1\leq i\leq m}, \{Y_j\}_{1\leq j\leq \ell}) \\ & + (V_{jk}V_{j'k'} \mid 1 \leq j, j' \leq \ell, \ 1 \leq k, k' \leq d, j \neq j' \text{ or } k \neq k') \\ & + (V_{jk}^2 - Y_j Z_k \mid 1 \leq j \leq \ell, \ 1 \leq k \leq d). \end{aligned}$$

We put $C = U/\mathfrak{a}$ and denote the images of X_i, Y_j, V_{j_k}, and Z_k in C by x_i, y_j, v_{j_k}, and a_k, respectively. Then $\dim C = d$, since $\sqrt{\mathfrak{a}} = (X_i, Y_j, V_{j_k} \mid 1 \leq i \leq m,\ 1 \leq j \leq \ell,\ 1 \leq k \leq d)$. Let $\mathcal{M} = C_+ := (x_i, y_j, v_{j_k}, a_k \mid 1 \leq i \leq m, 1 \leq j \leq \ell, 1 \leq k \leq d)$ be the graded maximal ideal in C. Let Λ be a subset of $\{1, 2, \cdots, m\}$. We put

$$\mathfrak{q} = (a_i \mid 1 \leq i \leq d) \text{ and } J_\Lambda = \mathfrak{q} + (x_\alpha \mid \alpha \in \Lambda) + (v_{j_k} \mid 1 \leq j \leq \ell,\ 1 \leq k \leq d).$$

Then $\mathcal{M}^2 = \mathfrak{q}\mathcal{M}$, $J_\Lambda^2 = \mathfrak{q}J_\Lambda + \mathfrak{q}(y_1, y_2, \cdots, y_\ell)$, and $J_\Lambda^3 = \mathfrak{q}J_\Lambda^2$, whence $\mathfrak{q}$ is a reduction of both $\mathcal{M}$ and J_Λ, and $a_1, a_2, \cdots, a_d$ is a homogeneous system of parameters for the graded ring C.

Let $B = C_\mathcal{M}$ and put $\mathfrak{n} = \mathcal{M}B$ denotes the maximal ideal of B. We then have the following.

Theorem 4.2. *The following assertions hold true.*

(1) *B is a Cohen-Macaulay local ring with* $\dim B = d$.
(2) $\mathrm{e}_0(\mathfrak{q}B) = \mathrm{e}_0(J_\Lambda B) = m + \ell d + \ell + 1$.
(3) $\mathrm{e}_1(J_\Lambda B) = \mathrm{e}_0(J_\Lambda B) - \ell_B(B/J_\Lambda B) + \ell = \sharp\Lambda + \ell d + \ell$.
(4) $\mathrm{e}_i(J_\Lambda B) = 0$ *for all* $2 \leq i \leq d$.
(5) $\mathrm{G}(J_\Lambda B)$ *is a Buchsbaum ring with* $\operatorname{depth} \mathrm{G}(J_\Lambda B) = 0$ *and* $\mathbb{I}(\mathrm{G}(J_\Lambda B)) = \ell d$.

Let us now consider the following.

Put $J = J_{\{1,2,\cdots,m\}}B$ and $A = k + J$. Then A is a local k-subalgebra of B with maximal ideal $\mathfrak{m} = J$ and B is a module finite extension of A, because $\ell_A(B/A) = \ell_A(B/J) - 1 = \ell$. Hence A is a Noetherian local ring with $\dim A = \dim B = d$ by Eakin-Ngata's Theorem. We fix a subset Λ of $\{1, 2, \cdots, m\}$ and put

$$I = J_\Lambda B \text{ and } Q = (a_1, a_2, \cdots, a_d)A.$$

Then I is an $\mathfrak{m}$-primary ideal in A and Q is a parameter ideal in A and a reduction of I. We then have the following.

Theorem 4.3. *The following assertions hold true.*

(1) *A is a Buchsbaum local ring with* $\mathrm{H}^i_\mathfrak{m}(A) = (0)$ *for all* $i \neq 1, d$ *and* $\mathrm{H}^1_\mathfrak{m}(A) \cong B/A$, *whence* $h^1(A) = \ell$,
(2) $\mathrm{e}_0(I) = m + \ell d + \ell + 1$,
(3) $\mathrm{e}_1(I) = \sharp\Lambda + \ell d + \ell$,
(4) $\mathrm{e}_i(I) = 0$ *for* $2 \leq i \leq d-1$ *and* $\mathrm{e}_d(I) = (-1)^{d+1}\ell$,
(5) $2\mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_1(Q) = 2\ell_A(A/I) + \ell_A(I/I^2 + Q)$, *and*
(6) $\mathrm{G}(I)$ *is a Buchsbaum ring with* $\mathrm{H}^i_M(\mathrm{G}(I)) = (0)$ *for all* $i \neq 1, d$ *and* $\mathrm{H}^1_M(\mathrm{G}(I)) = [\mathrm{H}^1_M(\mathrm{G}(I))]_1 \cong \mathrm{H}^1_\mathfrak{m}(A) \cong B/A$.

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STABILITY OF QUASI-SOCLE IDEALS

JUN HORIUCHI

1. Introduction

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We study quasi-socle ideals, i.e., ideals of the form $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ where Q is a parameter ideal in A. We are interested in determining when $I^2 = QI$, in which case we call I is stable. To state the results, we need to first fix some notation and terminology.

For each $\mathfrak{m}$-primary ideal I in A we denote by $\{\mathrm{e}_I^i(A)\}_{0 \leq i \leq d}$ the Hilbert coefficients of A with respect to I. The Hilbert function of I is then given by the formula

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \ldots + (-1)^d \mathrm{e}_I^d(A)$$

for all $n \gg 0$, where $\ell_A(M)$ denotes the length of the A-module M.

Let Q be a parameter ideal in A. We set $\mathbb{I}(Q) = \ell_A(A/Q) - e_Q^0(A)$. Then A is a Cohen–Macaulay ring if and only if $\mathbb{I}(Q) = 0$ for some (and hence every) parameter ideal Q. We say that A is a *Buchsbaum* ring if $\mathbb{I}(Q)$ is constant and independent of the choice of parameter ideals Q in A.

We say that A is a *generalized Cohen–Macaulay* ring if $\sup_Q \mathbb{I}(Q) < \infty$, where Q runs through parameter ideals in A. This definition is equivalent to saying that all the local cohomology modules $\mathrm{H}_\mathfrak{m}^i(A)$ $(i \neq d)$ of A with respect to $\mathfrak{m}$ are finitely generated. When this is the case, one has the equality $\sup_Q \mathbb{I}(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_\mathfrak{m}^i(A))$. A good reference for generalized Cohen–Macaulay rings is [T].

Let $Q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal in a generalized Cohen–Macaulay ring A. Then we say that Q is *standard* if $\mathbb{I}(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_\mathfrak{m}^i(A))$. This condition is equivalent to saying that for all integers $n_i > 0$ the sequence $a_1^{n_1}, a_2^{n_2}, \ldots, a_d^{n_d}$ forms a d-sequence in any order ([T, Proposition 3.2]). It is known that for a given generalized Cohen–Macaulay ring A, one can find an integer $\ell \gg 0$ such that every parameter ideal Q contained in $\mathfrak{m}^\ell$ is standard ([T, Section 3]).

For each ideal I in A we set

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n, \quad \mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}, \quad \text{and} \quad \mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

and call them, respectively, the Rees algebra, the associated graded ring, and the fiber cone of I.

With this notation and terminology our purpose is to prove the following.

Theorem 1.1. *Let A be a generalized Cohen–Macaulay ring and suppose that* $\operatorname{depth} G(\mathfrak{m}) \geq 2$. *Let $\ell \geq 1$ be an integer such that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Then for each integer $q \geq 1$, one can find an integer $t = t(q) \geq q+\ell+1$ such that I is stable for every parameter ideal Q of A contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$. Moreover, for each positive integer q such that $1 \leq q \leq \ell$, the integer $t(q)$ is given by, $t(q) = q + \ell + 1$ if $q = \ell$ and $t(q) = 2\ell$ if $q < \ell$.*

Applying the results in [GN, Section 5] and [GO, Section 2] to our ideals $I = Q : \mathfrak{m}^q$, we readily get the following, which is the most important consequence of Theorem 1.1. Notice that in both Theorem 1.1 and Corollary 1.2 one can choose $\ell = 1$ when A is a Buchsbaum ring.

Corollary 1.2. *Let A be a generalized Cohen–Macaulay ring with* $\operatorname{depth} G(\mathfrak{m}) \geq 2$ *and choose an integer $\ell \geq 1$ so that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Then for each integer $q \geq \ell$, there exists an integer $t = t(q) \geq q + \ell + 1$ such that the following assertions hold true for every parameter ideal Q of A contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$.*

(1) $e_I^1(A) = e_I^0(A) + e_Q^1(A) - \ell_A(A/I)$.

(2) *The Hilbert function of I is given by*
$\ell_A(A/I^{n+1}) = e_I^0(A)\binom{n+d}{d} - e_I^1(A)\binom{n+d-1}{d-1} + \sum_{i=2}^{d}(-1)^i \left[e_Q^{i-1}(A) + e_Q^i(A)\right]\binom{n+d-i}{d-i}$
for all $n \geq 0$.

(3) $H_{\mathcal{M}}^i(G(I)) = [H_{\mathcal{M}}^i(G(I))]_{1-i} \cong H_{\mathfrak{m}}^i(A)$ *as an A-module for all $i < d$ and*

$$\max\left\{n \in \mathbb{Z} \mid [H_{\mathcal{M}}^d(G(I))]_n \neq (0)\right\} \leq 1 - d.$$

(4) *The associated graded ring $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ of I is a Buchsbaum ring whenever A is Buchsbaum.*

Here $\mathcal{M} = \mathfrak{m}G(I) + G(I)_+$ and $[H_{\mathcal{M}}^i(G(I))]_n$ $(i, n \in \mathbb{Z})$ denotes the homogeneous component with degree n in the i-th graded local cohomology module $H_{\mathcal{M}}^i(G(I))$ of $G(I)$ with respect to $\mathcal{M}$.

In [GHS] Goto, Sakurai and the author proved Theorem 1.1 and Corollary 1.2, assuming the extra condition on systems $a_1, a_2, \ldots, a_d$ of parameters that $a_d = ab$ for some $a \in \mathfrak{m}^q$ and $b \in \mathfrak{m}$. This is a technical but crucial condition in order to use the result of Goto and Sakurai [GSa3, Lemma2.3], and thanks to the condition, they were able to get the equality $I^2 = QI$ by induction on dimension d, where $I = Q : \mathfrak{m}^q$ and $Q \subseteq \mathfrak{m}^{q+\ell+1}$. The present proof of Theorem 1.1 and Corollary 1.2 is substantially different from the one in [GHS]. It is based on Proposition 2.3 and valid for every parameter ideal Q contained in $\mathfrak{m}^t$, choosing an integer t such that $t \geq q + \ell + 1$.

Our research dates back to the works of Corso, Polini, Huneke, Vasconcelos, and Goto, where they explored the socle ideals $Q : \mathfrak{m}$ for parameter ideals Q in Cohen–Macaulay rings and proved the following.

Theorem 1.3 ([CHV, CP1, CP2, CPV, G]). *Let Q be a parameter ideal in a Cohen–Macaulay ring A and let $I = Q : \mathfrak{m}$. Then the following conditions are equivalent.*

(1) $I^2 \neq QI$.
(2) *Q is integrally closed in A.*
(3) *A is a regular local ring and the A-module $\mathfrak{m}/Q$ is cyclic.*

Therefore, if A is a Cohen–Macaulay ring which is not regular, then $I^2 = QI$ for every parameter ideal Q in A, so that $\mathrm{G}(I)$ and $\mathrm{F}(I)$ are both Cohen–Macaulay rings, where $I = Q : \mathfrak{m}$. The Rees algebra $\mathcal{R}(I)$ is also Cohen–Macaulay if $\dim A \geq 2$.

Perhaps Wang has provided the greatest achievement so far by affirmatively answering a conjecture posed by Polini and Ulrich that is rooted in linkage theory. We state his result in the following way.

Theorem 1.4 ([Wan]). *Suppose that A is a Cohen–Macaulay ring and let $q \geq 1$ be an integer. Let Q be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^{q+1}$ and put $I = Q : \mathfrak{m}^q$. Then*

$$\mathfrak{m}^q I = \mathfrak{m}^q Q, \quad I \subseteq \mathfrak{m}^{q+1}, \quad \text{and} \quad I^2 = QI,$$

provided $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$.

It seems natural to ask what we can expect when the base local ring is not necessarily Cohen–Macaulay. Goto, Sakurai and the author [GHS, Theorem 1.1] gave an answer in the case where the base ring A is Buchsbaum, showing the assumption that $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$ is sufficient in order for Wang's methods to work. Generalizing the results in [GSa1, GSa2, GSa3, GHS] our theorem 1.1 answers the question with substantial generality in the case where A is a generalized Cohen–Macaulay ring.

2. Proof of Theorem 1.1

In what follows, we denote by A a Noetherian local ring with maximal ideal $\mathfrak{m}$ and dimension $d > 0$. Let $\mathrm{H}^i_{\mathfrak{m}}(*)$ $(i \in \mathbb{Z})$ be the local cohomology functors of A with respect to $\mathfrak{m}$. The purpose of this section is to prove Theorem 1.1.

Our proof is based on the following result of Cuong and Truong [CT, Theorem 3.3, Corollary 4.1]. They deal with the case when $q = 1$, but this can be generalized to when $q \geq 1$ in a straightforward manner.

Theorem 2.1 ([CT, Theorem 3.3, Corollary 4.1]). *Suppose that A is a generalized Cohen–Macaulay ring and let $q \geq 1$ be an integer. Then*

$$\sup_Q \ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_\mathfrak{m}(A)} \mathfrak{m}^q)$$

where Q runs through standard parameter ideals in A. Furthermore, one can find an integer $k = k(q) \geq 1$ such that every parameter ideal Q of A contained in $\mathfrak{m}^k$ is standard with

$$\ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_\mathfrak{m}(A)} \mathfrak{m}^q).$$

Following result is due to Cuong and Quy [CQ, Corollary 4.3]. They generalized the results of Cuong and Truong [CT, Theorem 3.3, Corollary 4.1] as one of applications of their splitting theorem [CQ, Theorem 1.1].

Theorem 2.2 ([CQ, Corollary 4.3]). *Suppose that A is a generalized Cohen–Macaulay ring. Let $\ell \geq 1$ be an integer such that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Then for each positive integer q such that $1 \leq q \leq \ell$ and all parameter ideal Q of A contained in $\mathfrak{m}^{2\ell}$, the length $\ell_A([Q : \mathfrak{m}^q]/Q)$ is independent of the choice of parameter ideal Q and given by*

$$\ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_\mathfrak{m}(A)} \mathfrak{m}^q).$$

We begin with the following.

Proposition 2.3. *Suppose that A is a generalized Cohen–Macaulay ring and let $q \geq 1$ be an integer. Let Q be a standard parameter ideal in A and assume that*

$$\ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_\mathfrak{m}(A)} \mathfrak{m}^q).$$

Then

$$[Q + W] : \mathfrak{m}^q = [Q : \mathfrak{m}^q] + W$$

where $W = \mathrm{H}^0_\mathfrak{m}(A)$.

Proof. We set $\bar{A} = A/W$. Then $Q \cap W = (0)$ ([T, Corollary 2.3]), we have the exact sequence

$$0 \to \mathrm{H}^0_\mathfrak{m}(A) \to A/Q \xrightarrow{\varepsilon} \bar{A}/Q\bar{A} \to 0.$$

Since $Q\overline{A}$ is also a standard parameter ideal of $\overline{A}$. By applying $\mathrm{Hom}_A(A/\mathfrak{m}^q, *)$ and using Theorem 2.1 we get

$$\begin{aligned}\ell_A([Q:\mathfrak{m}^q]/Q) &\le \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)}\mathfrak{m}^q) + \ell_A([Q\bar{A}:_{\overline{A}}\mathfrak{m}^q]/Q\bar{A})\\ &\le \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)}\mathfrak{m}^q) + \sum_{i=0}^{d}\binom{d}{i}\ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(\bar{A})}\mathfrak{m}^q)\\ &= \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)}\mathfrak{m}^q) + \sum_{i=1}^{d}\binom{d}{i}\ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)}\mathfrak{m}^q)\\ &\le \sum_{i=0}^{d}\binom{d}{i}\ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)}\mathfrak{m}^q),\end{aligned}$$

since $\mathrm{H}^0_{\mathfrak{m}}(\overline{A}) = (0)$ and $\mathrm{H}^i_{\mathfrak{m}}(\overline{A}) = \mathrm{H}^i_{\mathfrak{m}}(A)$ for all $i \ge 1$. Therefore, because

$$\ell_A([Q:\mathfrak{m}^q]/Q) = \sum_{i=0}^{d}\binom{d}{i}\ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)}\mathfrak{m}^q),$$

we have

$$\ell_A([Q:\mathfrak{m}^q]/Q) = \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)}\mathfrak{m}^q) + \ell_A([Q\bar{A}:_{\overline{A}}\mathfrak{m}^q]/Q\bar{A}).$$

This shows that homomorphism $A/Q \xrightarrow{\varepsilon} \bar{A}/Q\bar{A}$ gives rise to an epimorphism

$$\mathrm{Hom}_A(A/\mathfrak{m}^q, \varepsilon) : \mathrm{Hom}_A(A/\mathfrak{m}^q, A/Q) \to \mathrm{Hom}_A(A/\mathfrak{m}^q, \overline{A}/\overline{Q}).$$

Hence

$$[Q+W]:\mathfrak{m}^q = [Q:\mathfrak{m}^q] + W.$$

□

The following is the key for our proof of Theorem 1.1. This is a generalization of the result of Goto and Sakurai [GSa1, Theorem 3.9].

Theorem 2.4. *Suppose that A is a generalized Cohen–Macaulay ring and let $q \ge 1$ be an integer. Let Q be a standard parameter ideal in A and set $I = Q:\mathfrak{m}^q$. Assume that the following three conditions are satisfied.*

(1) $\ell_A(I/Q) = \sum_{i=0}^{d}\binom{d}{i}\ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)}\mathfrak{m}^q)$.

(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(3) $I^2 \subseteq Q$.

Then I is stable.

Proof. We have

$$[Q+W]:\mathfrak{m}^q = [Q:\mathfrak{m}^q] + W = I + W$$

by Proposition 2.3, where $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Let $Q = (a_1, a_2, \ldots, a_d)$.

Suppose that $d = 1$. We put $\overline{A} = A/W$, $\overline{\mathfrak{m}} = \mathfrak{m}/W$, $\overline{I} = I\overline{A}$, and $\overline{Q} = Q\overline{A}$. Then $\overline{\mathfrak{m}}^q{\cdot}\overline{I} = \overline{\mathfrak{m}}^q{\cdot}\overline{Q}$; hence, $\overline{\mathfrak{m}}^q{\cdot}\overline{I}^n = \overline{\mathfrak{m}}^q{\cdot}\overline{Q}^n$ for all $n \in \mathbb{Z}$. By the equality $[Q{+}W] : \mathfrak{m}^q = I{+}W$, we have $\overline{I} = \overline{Q} : \overline{\mathfrak{m}}^q$. Let $x \in \overline{I}^2$. Then, since $\overline{I}^2 \subseteq \overline{Q}$, we have $x = a_1y$ with $y \in \overline{A}$. Let $\alpha \in \overline{\mathfrak{m}}^q$. Then, $a_1(\alpha y) = \alpha x \in \overline{\mathfrak{m}}^q{\cdot}\overline{I}^2 = \overline{\mathfrak{m}}^q{\cdot}\overline{Q}^2$, we get $a_1(\alpha y) = a_1^2 z$ for some $z \in \overline{A}$. Hence $\alpha y \in \overline{Q}$ (notice that $\overline{A}$ is Cohen–Macaulay so that a_1 is $\overline{A}$-regular), hence, we have $x = a_1y \in \overline{Q}{\cdot}\overline{I}$, because $y \in \overline{Q} : \overline{\mathfrak{m}}^q = \overline{I}$. Thus we have $\overline{I}^2 = \overline{Q}{\cdot}\overline{I}$, so that $I^2 \subseteq QI + W$. Therefore, since $W \cap Q = (0)$ and $I^2 \subseteq Q$, we get $I^2 \subseteq (QI + W) \cap Q = QI$ as claimed.

Suppose that $d \geq 2$ and our assertion holds true for $d-1$. Let $B = A/(a_1)$. Then conditions (1), (2), and (3) are satisfied for the parameter ideal QB in B. This is clear for conditions (2) and (3). As for condition (1), for all $0 \leq i \leq d-2$ we have the short exact sequence

$$0 \to \mathrm{H}^i_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathfrak{m}}(B) \to \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \to 0$$

of local cohomology modules, since $a_1\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ $(0 \leq i \leq d-1)$ and $\ell_A((0) : a_1) = \ell_A(W) < \infty$ ([T, Theorem 2.5]). Hence, by Theorem 2.1, we get

$$\begin{aligned}
\ell_A(I/Q) &= \ell_A([QB :_B \mathfrak{m}^q]/QB) \\
&\leq \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(B)} \mathfrak{m}^q) \\
&\leq \sum_{i=0}^{d-1} \binom{d-1}{i} \left[\ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q) + \ell_A((0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(A)} \mathfrak{m}^q)\right] \\
&= \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q) \\
&= \ell_A(I/Q),
\end{aligned}$$

so that

$$\ell_A([QB :_B \mathfrak{m}^q]/QB) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(B)} \mathfrak{m}^q).$$

Therefore, condition (1) is satisfied also for QB. Thus we have $I^2 \subseteq QI + (a_1)$ by the hypothesis of induction on d. Let us now choose $x \in I^2$ and write $x = y + a_1z$ with $y \in QI$ and $z \in A$. Also, let $\alpha \in \mathfrak{m}^q$. We then have

$$\alpha x = \alpha y + a_1(\alpha z) \in Q^2,$$

because $x \in I^2$ and $\mathfrak{m}^qI = \mathfrak{m}^qQ$. Consequently $a_1(\alpha z) \in Q^2$ (notice that $\alpha y \in Q^2$), since $a_1, a_2, \ldots, a_d$ form a d-sequence in A ([T, Proposition 3.1]), we have $a_1(\alpha z) \in (a_1) \cap Q^2 = a_1Q$. Hence $\alpha z - v \in (0) : a_1 \subseteq W$ ([T, Theorem 2.5]) for some $v \in Q$, which guarantees $z \in (Q{+}W) : \mathfrak{m}^q = I{+}W$. Since $a_1W = (0)$, we get $x = y{+}a_1z \in QI$. Hence $I^2 = QI$. □

To prove Theorem 1.1 we need the following result of [GHS], in which we make use of the assumption that $\operatorname{depth} G(\mathfrak{m}) \geq 2$.

Proposition 2.5 ([GHS, Proposition 2.2]). *Let A be a generalized Cohen–Macaulay ring with* $\operatorname{depth} G(\mathfrak{m}) \geq 2$. *Choose an integer $\ell \geq 1$ so that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Let $q \geq 1$ be an integer and let Q be a parameter ideal of A such that $Q \subseteq \mathfrak{m}^{q+\ell+1}$. We then have*

$$\mathfrak{m}^q I = \mathfrak{m}^q Q, \quad I \subseteq \mathfrak{m}^{q+\ell+1}, \quad \text{and} \quad I^2 \subseteq Q,$$

where $I = Q : \mathfrak{m}^q$.

We are now ready to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Let $\ell \geq 1$ be an integer such that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Take $t(q) = \max\{k(q), q+\ell+1\}$, where $k(q)$ is the integer obtained by Theorem 2.1. Then by Theorem 2.4 and Proposition 2.5 we readily get I is stable for every parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ of A contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$. Moreover, for each positive integer q such that $1 \leq q \leq \ell$, the integer $t(q)$ is given by, $t(q) = \max\{2\ell, q+\ell+1\}$ by [CQ, Corollary 4.3] so that, $t(q) = q+\ell+1$ if $q = \ell$ and $t(q) = 2\ell$ if $q < \ell$. □

Before entering into the proof of Corollary 1.2, let us give the notion introduced by [GO]. Let I be an $\mathfrak{m}$–primary ideal of A and let $\underline{a} = a_1, a_2, \ldots, a_d$ be a system of parameters in A. We assume that $Q = (a_1, a_2, \ldots, a_d)$ is a reduction of I. Then we say that condition (C_2) is satisfied for $\underline{a}$ and I, if

$$(a_1, \ldots, \check{a}_i, \ldots, a_d) : a_i \subseteq I$$

for all $1 \leq i \leq d$.

Proof of Corollary 1.2. We have $I^2 = QI$ by Theorem 1.1. We notice that if $q \geq \ell$, condition (C_2) is satisfied for our system $\underline{a}$ of parameters and the ideal $I = Q : \mathfrak{m}^q$. In fact,

$$(a_1, \ldots, \check{a}_i, \ldots, a_d) : a_i = (a_1, \ldots, \check{a}_i, \ldots, a_d) : \mathfrak{m}^\ell$$

for each $1 \leq i \leq d$ ([T, Lemma 1.1]), because $Q \subseteq \mathfrak{m}^\ell$ and every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Therefore, since $q \geq \ell$, we get

$$(a_1, \ldots, \check{a}_i, \ldots, a_d) : a_i = (a_1, \ldots, \check{a}_i, \ldots, a_d) : \mathfrak{m}^\ell \subseteq (a_1, \ldots, \check{a}_i, \ldots, a_d) : \mathfrak{m}^q \subseteq I$$

as wanted. Hence the detailed description of the Hilbert function of our ideal $I = Q : \mathfrak{m}^q$ follows from [GO]. By [GN, Section 5] the associated graded ring $G(I)$ of I is Buchsbaum, if A is Buchsbaum. Assertions (1), (2) (resp. (3), (4)) of Corollary 1.2 readily follow from [GO, Propositions 2.4, 2.5] (resp. [GN, Theorem 1.3, Section 5]). □

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UPPER BOUND OF MULTIPLICITY OF F-RATIONAL RINGS AND F-PURE RINGS

KEI-ICHI WATANABE

1. Introduction

In the problem session of the workshop at AIM, August 2011, titled "Relating Test Ideals and Multiplier Ideals", Karl Schwede posed the following question.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ of dimension d and embedding dimension v. Assume that R is F-pure. Then is the multiplicity $e(R)$ of R always satisfy

$$e(R) \leq \binom{v}{d} \quad ?$$

Actually, this inequality is always true and follows from Briançon-Skoda type theorem, which was proved by C. Huneke.

This is a joint work with Craig Huneke.

2. Preliminaries

Let $(R, \mathfrak{m})$ be either a Noetherian local ring or $R = \oplus_{n \geq 0} R_n$ be a graded ring finitely generated over a field $R_0 = k$. We always assume that either R contains a field of characteristic $p > 0$ or R is essentially of finite type over a field of characteristic 0. **We always assume that our ring R is reduced.**

Definition 2.1. We denote by R° the set of elements of R that are not contained in any minimal prime ideal. The *tight closure* I^* of I is defined to be the ideal of R consisting of all elements $x \in R$ for which there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$.

Definition 2.2. We say that a local ring $(R, \mathfrak{m})$ is *F-rational* if it is a homomorphic image of a Cohen-Macaulay ring and for every parameter ideal J of R we have $J^* = J$. It is known that F-rational rings are normal and Cohen-Macaulay.

Definition 2.3. Assume that R contains a field of characteristic $p > 0$ and $q = p^e$ be a power of p.

(1) For a power $q = p^e$ and ideal I in R, we denote by $I^{[q]}$, the ideal generated by $\{a^q \mid a \in I\}$.
(2) We write $R^{1/q}$ then we say that R is *F-pure* if for every R module M, the natural map $M = M \otimes_R R \to M \otimes_R R^{1/p}$, sending $x \in M$ to $x \otimes 1$ is injective.

(3) Let I be an ideal of R and $x \in R$. If R is F-pure and if $x^q \in I^{[q]}$, then $x \in I$. This follows from (2) if we put $M = R/I$.

3. The main results

The following theorem is our main result in this article.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d$ and embedding dimension v. Then,*

(1) *If R is a rational singularity or F-rational, then $e(R) \leq \binom{v-1}{d-1}$.*

(2) *If R is F-pure, then $e(R) \leq \binom{v}{d}$.*

This theorem easily follows from the following theorem.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d$ and let $J \subset \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$.*

(1) *If R is a rational singularity of F-rational, then $\mathfrak{m}^d \subset J$.*

(2) *If R is F-pure, then $\mathfrak{m}^{d+1} \subset J$.*

Proof. The statement (1) is well known and follows from Bricançon-Skoda type theorem (cf. [HH], [LT]).

For the statement (2) we will prove the following statement.

Assume R is F-pure and I is an ideal generated by r elements, then $\overline{I^{r+1}} \subset I$. This is sufficient to prove 3.2 since $\mathfrak{m}^{d+1} \subset \overline{\mathfrak{m}^{d+1}} = \overline{J^{d+1}}$.

Now, take $x \in \overline{I^{r+1}}$. Then we can take $c \in R^{\circ}$ such that for sufficiently large $N, cx^N \in I^{(r+1)N}$. Then $cx^N \in c(I^{(r+1)N} : c)$. The latter is contained in $cR \cap I^{(r+1)N}$ and by Artin-Rees Lemma, there exists k such that $cR \cap I^{(r+1)N} \subset cI^{(r+1)N-k}$ for sufficiently large N. Now, we have shown that $cx^N \in cI^{(r+1)N-k}$. Note that $I^{rq} \subset I^{[q]}$. Taking sufficiently large $N = q = p^e$ and noting that c is a non zero divisor, we get $x^q \in I^{[q]}$. Since R is F-pure, we get $x \in I$. □

It is easy to prove 3.2 using 3.1.

Proof of 3.2 $\Longrightarrow$ 3.1. We have the following inequality and the equality holds if and only if R is Cohen-Macauly (cf. [BH], Corollary 4.7,11).

(3.1.1) $\quad e(R) \leq l_R(R/J)$

So, it suffices to show that $l_R(R/J)$ is bounded by the right-hand side of the inequalities in 3.1. Now, let $x_1, \ldots, x_d, y_1, \ldots, y_{v-d}$ be minimal generators of $\mathfrak{m}$ with $J = (x_1, \ldots, x_d)$. Then R/J is generated by the monomials of $y_1, \ldots, y_{v-d}$ of degree $\leq d-1$ (resp. degree $\leq d$) in case (1) (resp. case (2)) by 3.2. It is easy to see that the number of monomials of $y_1, \ldots, y_{v-d}$ of degree $\leq d-1$ (resp. degree $\leq d$) is $\binom{v-1}{d-1}$ (resp. $\binom{v}{d}$). □

Remark 3.3. Assume we have equality in 3.1 (1) or (2). Then R is Cohen-Macaulay since we must have equality in (3.1.1), too. Moreover, since the associated graded ring of R has the same embedding dimension and multiplicity with R, $\mathrm{gr}_{\mathfrak{m}}(R)$ is also Cohen-Macaulay in this case.

4. Actual Upper Bound

The upper bound in 3.1 (2) is taken by the following example.

Example 4.1. Let Δ be a simplicial complex on the vertex set $\{1, 2, \ldots, v\}$, whose maximal faces are all possible $d-1$ simplices. Then the Stanley-Reisner ring $R = k[\Delta]$ has dimension d and $e(R) = \binom{v}{d}$. Note that Stanley-Reisner rings are always F-pure.

Remark 4.2. (1) Are there other examples where we have equality in 3.1 (2) if $v \geq d+2$? It is shown in [GW] that in the case of $d = 1$, this is the only example if we assume $(R, \mathfrak{m})$ is complete local ring with algebraically closed residue field.

(2) Also, are there examples where we have equality in 3.1 (1) if $v \geq d+2$ and $d \geq 3$? If $d = 2$, we have always $e(R) = v - 1$ (cf. [Li]).

5. Case of Gorenstein Rings

If R is Gorenstein, the upper bound is largely reduced by the duality. If $(B, \mathfrak{n})$ is an Artinian Gorenstein ring with $\mathfrak{n}^s \neq 0$ and $\mathfrak{n}^{s+1} = 0$, then $l_B(\mathfrak{n}^t) \leq l_B([0 :_B \mathfrak{n}^{s-t+1}]) = l_B(B/\mathfrak{n}^{s-t+1})$. Hence we have the following inequalities by 3.2.

Theorem 5.1. *Let $(R, \mathfrak{m})$ be a Gorenstein Noetherian local ring with $\dim R = d$ and embedding dimension v.*

(1) *If R is a rational singularity or F-rational with $\dim R = 2r+1$, then* $e(R) \leq \binom{v-r-1}{r} + \binom{v-r-2}{r-1}$.

(2) *If R is a rational singularity or F-rational with $\dim R = 2r$, then* $e(R) \leq 2\binom{v-r-1}{r-1.}$

(3) *If R is F-pure with $\dim R = 2r+1$, then* $e(R) \leq 2\binom{v-r-1}{r}$.

(4) *If R is F-pure with $\dim R = 2r$, then* $e(R) \leq \binom{v-r}{r} + \binom{v-r-1}{r-1}$.

Remark 5.2. Again, the upper bound in (3), (4) is taken by the Stanley-Reisner ring of "Cyclic Polytopes" (cf. [St]).

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ALMOST GORENSTEIN RINGS

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ABSTRACT. The notion of almost Gorenstein ring given by Barucci and Fröberg [2] in the case where the local rings are analytically unramified is generalized, so that it works well also in the case where the rings are analytically ramified. As a sequel, the problem of when the endomorphism algebra $\mathfrak{m} : \mathfrak{m}$ of $\mathfrak{m}$ is a Gorenstein ring is solved in full generality, where $\mathfrak{m}$ denotes the maximal ideal in a given Cohen-Macaulay local ring of dimension one. Characterizations of almost Gorenstein rings are given in connection with the principle of idealization. Examples are explored.

1. INTRODUCTION

This paper studies a special class of one-dimensional Cohen-Macaulay local rings, which we call *almost Gorenstein* rings ([7]). Originally, almost Gorenstein rings were introduced by V. Barucci and R. Fröberg [2], in the case where the local rings are analytically unramified. They developed in [2] a very nice theory of almost Gorenstein rings and gave many interesting results, as well. Our paper aims at an alternative definition of almost Gorenstein ring which we can apply also to the rings that are not necessarily analytically unramified. One of the purposes of such an alternation is to go beyond a gap in the proof of [2, Proposition 25] and solve in full generality the problem of when the algebra $\mathfrak{m} : \mathfrak{m}$ is a Gorenstein ring, where $\mathfrak{m}$ denotes the maximal ideal in a given Cohen-Macaulay local ring of dimension one.

Before going into more details, let us fix our notation and terminology, which we maintain throughout this paper.

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. We denote by $\mathrm{Q}(R)$ the total quotient ring of R. Let K_R be the canonical module of R. Then we say that an ideal I of R is *canonical*, if $I \neq R$ and $I \cong \mathrm{K}_R$ as R–modules. As is known by [10, Satz 6.21], R possesses a canonical ideal if and only if $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring, where $\widehat{R}$ denotes the $\mathfrak{m}$–adic completion of R. Therefore, the ring R possesses a canonical ideal, once it is analytically unramified, that is the case where $\widehat{R}$ is a reduced ring.

Let I be a canonical ideal of R. Then because $\mathrm{Ann}_R I = (0)$, the ideal I is $\mathfrak{m}$–primary, and we have integers $\mathrm{e}_0(I) > 0$ and $\mathrm{e}_1(I)$ such that the Hilbert function of I is given by the polynomial

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I)$$

for all integers $n \gg 0$, where $\ell_R(M)$ denotes, for each R–module M, the length of M. Let $\mathrm{r}(R) = \ell_R(\mathrm{Ext}^1_R(R/\mathfrak{m}, R))$ be the Cohen-Macaulay type of R ([10, Definition 1.20]). Then our definition of almost Gorenstein ring is now stated as follows.

Definition (Definition 3.1). *We say that R is an almost Gorenstein ring, if R possesses a canonical ideal I such that* $\mathrm{e}_1(I) \leq \mathrm{r}(R)$.

If R is a Gorenstein ring, then we can choose any parameter ideal Q of R to be a canonical ideal and get $\mathrm{e}_1(Q) = 0 < \mathrm{r}(R) = 1$. Hence every Gorenstein local ring of dimension one is an almost Gorenstein ring.

Let us explain how this paper is organized. In Section 2 we would like to show some well-known results about the first Hilbert coeffficients $\mathrm{e}_1(I)$ of $\mathfrak{m}$–primary ideals I in R and the existence of canonical ideals in R. In Section 3 we shall give characterizations of Gorenstein rings and almost Gorenstein rings as well, according to Definition 3.1. In Section 4 we will study the problem of when the endomorphism algebra $\mathfrak{m} : \mathfrak{m}$ ($\cong \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$) of $\mathfrak{m}$ is a Gorenstein ring. This is the problem which Barucci and Fröberg wanted to solve in [2], but Barucci finally felt there was a gap in [2, Proof of Proposition 25]. However, we should note here that the counterexample [1, Example p. 995] given by Barucci to [2, Proposition 25] is wrong, and the proof stated in [2] still works with our modified definition of almost Gorenstein ring, which we shall closely discuss in Section 4. In the last section 5 we will give a series of characterizations of almost Gorenstein rings obtained by idealization (namely, trivial extension). Unless otherwise specified, in what follows, let $(R, \mathfrak{m})$ denote a Cohen-Macaulay local ring with $\dim R = 1$. Let $\mathrm{Q}(R)$ be the total quotient ring of R and $\overline{R}$ the integral closure of R in $\mathrm{Q}(R)$. For each finitely generated R–module M, let $\mu_R(M)$ denote the number of elements in a minimal system of generators for M. Let $v(R) = \mu_R(\mathfrak{m})$ and $e(R) = \mathrm{e}_0(\mathfrak{m})$, the multiplicity of R with respect to $\mathfrak{m}$. Let $\ell_R(*)$ stand for the length. For given fractional ideals F_1, F_2 of R, let $F_1 : F_2 = \{x \in \mathrm{Q}(R) \mid xF_2 \subseteq F_1\}$. When we consider the ideal colon $\{x \in R \mid xJ \subseteq I\}$ for integral ideals I, J of R, we denote it by $I :_R J$ in order to make sure of the meaning.

2. The first Hilbert coefficients and existence of canonical ideals

In this section we shall summarize preliminary results, which we need throughout this paper.

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let I be an $\mathfrak{m}$–primary ideal of R. Then there exist integers $\mathrm{e}_0(I) > 0$ and $\mathrm{e}_1(I)$ such that

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I)$$

for all integers $n \gg 0$. We assume that there exists an element $a \in I$ such that the ideal $Q = (a)$ is a reduction of I, i.e., $I^{r+1} = QI^r$ for some integer $r \geq 0$ (this condition is automatically satisfied, if the residue class field $R/\mathfrak{m}$ of R is infinite). We put

$$r = \mathrm{red}_Q(I) := \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}.$$

For each integer $n \geq 0$ let $\frac{I^n}{a^n} = \{\frac{x}{a^n} \mid x \in I^n\}$ and put $S = R[\frac{I}{a}]$ in $\mathrm{Q}(R)$. We then have $\frac{I^n}{a^n} \subseteq \frac{I^{n+1}}{a^{n+1}}$ for all $n \geq 0$. Therefore, since $S = \bigcup_{n \geq 0} \frac{I^n}{a^n}$ and $\frac{I^n}{a^n} = \frac{I^r}{a^r}$ for all $n \geq r$, we get $S = \frac{I^r}{a^r} \cong I^r$ as R–modules. Hence S is a finitely generated R–module, so that

$$R \subseteq S \subseteq \overline{R}.$$

Let $n \geq 0$ be an integer. Then, since $I^{n+1}/Q^{n+1} \cong [\frac{I^n}{a^n}]/R \subseteq S/R$, we have

$$\begin{aligned} \ell_R(R/I^{n+1}) &= \ell_R(R/Q^{n+1}) - \ell_R(I^{n+1}/Q^{n+1}) \\ &\geq \ell_R(R/Q^{n+1}) - \ell_R(S/R) \\ &= \ell_R(R/Q)\binom{n+1}{1} - \ell_R(S/R) \end{aligned}$$

and

$$\ell_R(R/I^{n+1}) = \ell_R(R/Q)\binom{n+1}{1} - \ell_R(S/R),$$

if $n \geq r-1$. Consequently we get the following.

Lemma 2.1. $\mathrm{e}_0(I) = \ell_R(R/Q)$ *and*

$$0 \leq \mathrm{e}_1(I) = \ell_R(I^r/Q^r) = \ell_R(S/R) \leq \ell_R(\overline{R}/R).$$

The following result is fairly well-known.

Proposition 2.2 (cf. [11]). $r \leq \mathrm{e}_1(I)$ *and*

$$\mu_R(I/Q) = \mu_R(I) - 1 \leq \ell_R(I/Q) = \mathrm{e}_0(I) - \ell_R(R/I) \leq \mathrm{e}_1(I).$$

We furthermore have the following.

(1) $\mu_R(I/Q) = \ell_R(I/Q)$ *if and only if* $\mathfrak{m}I \subseteq Q$, *i.e.*, $\mathfrak{m}I = \mathfrak{m}Q$.
(2) $\ell_R(I/Q) = \mathrm{e}_1(I)$ *if and only if* $I^2 = QI$.

Remark 2.3. Proposition 2.2 is a special case of the results which hold true for arbitrary Cohen-Macaulay local rings of positive dimension. The inequality $\ell_R(I/Q) \leq \mathrm{e}_1(I)$ is known as Northcott's inequality ([11]), and assertion (2) of Proposition 2.2 was proven by [9, 12] independently. The ideals I satisfying the condition that $\mathfrak{m}I \subseteq Q$ are called ideals of minimal multiplicity ([6]).

Corollary 2.4. *The following assertions hold true.*

(1) *Let I and J be $\mathfrak{m}$-primary ideals of R and suppose that I contains a reduction $Q = (a)$. If $I \subseteq J \subseteq \overline{I}$, then $\mathrm{e}_1(I) \leq \mathrm{e}_1(J)$.*
(2) *Suppose that R is not a discrete valuation ring. Then $\mathrm{e}_1(Q :_R \mathfrak{m}) = \mathrm{r}(R)$ for every parameter ideal $Q = (a)$ of R, where $\mathrm{r}(R)$ denotes the Cohen–Macaulay type of R.*

Let K_R denote the canonical module of R. The fundamental theory of canonical modules was developed by the monumental book [10] of E. Kunz and J. Herzog. In what follows, we shall freely consult [10] about basic results on canonical modules (see [4, Part I] also).

As is well-known, R possesses the canonical module K_R if and only if R is a homomorphic image of a Gorenstein ring ([13]). In the present research we are interested also in the condition for R to contain *canonical ideals.*

Let us begin with the following.

Definition 2.5. An ideal I of R is said to be a canonical ideal of R, if $I \neq R$ and $I \cong \mathrm{K}_R$ as R–modules.

Here we confirm that this definition implicitly assumes the existence of the canonical module K_R. Namely, the condition in Definition 2.5 that $I \cong \mathrm{K}_R$ as R–modules should be read to mean that R possesses the canonical module K_R and the ideal I of

R is isomorphic to K_R as an R–module. Notice that canonical ideals are $\mathfrak{m}$–primary, because they are faithful R–modules ([10, Bemerkung 2.5]).

We then have the following result [10, Satz 6.21].

Proposition 2.6 ([10]). *The following conditions are equivalent.*

(1) $\mathrm{Q}(\widehat{R})$ *is a Gorenstein ring.*
(2) R *contains a canonical ideal.*

Hence R contains a canonical ideal, if $\widehat{R}$ is a reduced ring.

Let $\overline{R}$ denote the integral closure of R in $\mathrm{Q}(R)$.

Corollary 2.7. *The following conditions are equivalent.*

(1) *There exists an R–submodule K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as R–modules.*
(2) *R contains a canonical ideal I and $a \in I$ such that (a) is a reduction of I.*

When this is the case, every canonical ideal I of R contains an element which generates a reduction of I and the first Hilbert coefficient $\mathrm{e}_1(I)$ is independent of the choice of canonical ideals I.

As an immediate consequence, we get the following.

Corollary 2.8. *Assume that $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring. If the residue class field $R/\mathfrak{m}$ of R is infinite, then there exists an R–submodule K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as R–modules.*

Remark 2.9. Corollary 2.8 is not true in general, unless the field $R/\mathfrak{m}$ is infinite. For example, we look at the local ring

$$R = k[[X,Y,Z]]/(X,Y) \cap (Y,Z) \cap (Z,X),$$

where $k[[X,Y,Z]]$ is the formal power series ring over a field k. Then R is reduced and $\dim R = 1$. We put $I = (x+y, y+z)$, where x, y, and z denote the images of X, Y, and Z in R, respectively. Then I is a canonical ideal of R. If $k = \mathbb{Z}/2\mathbb{Z}$, no element of I generates a reduction of I, so that no R–submodules K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ are isomorphic to K_R.

The R–submodules K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as R–modules play a very important role in our argument. The following result insures the existence of those *fractional* ideals K, after enlarging the residue class field $R/\mathfrak{m}$ of R until it will be infinite, or even algebraically closed.

Lemma 2.10 ([3, AC IX, p. 41, Corollaire]). *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $k = R/\mathfrak{m}$. Then for each extension k_1/k of fields, there exists a flat local homomorphism $(R,\mathfrak{m}) \to (R_1,\mathfrak{m}_1)$ of Noetherian local rings which satisfies the following conditions.*

(a) $\mathfrak{m}_1 = \mathfrak{m}R_1$.
(b) $R_1/\mathfrak{m}_1 \cong k_1$ *as k–algebras.*

We apply Lemma 2.10 to our context.

Proposition 2.11. *Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let $k = R/\mathfrak{m}$ and let k_1/k be an extension of fields. Suppose that $\varphi : (R,\mathfrak{m}) \to (R_1,\mathfrak{m}_1)$ is a flat local homomorphism of Noetherian local rings such that*

(a) $\mathfrak{m}_1 = \mathfrak{m}R_1$.
(b) $R_1/\mathfrak{m}_1 \cong k_1$ *as* k*–algebras.*

Then R_1 *is a Cohen–Macaulay ring with* $\dim R_1 = 1$. *We furthermore have the following.*

(1) $\mathrm{Q}(\widehat{R_1})$ *is a Gorenstein ring if and only if* $\mathrm{Q}(\widehat{R})$ *is a Gorenstein ring. When this is the case, for every canonical ideal* I *of* R *the ideal* IR_1 *of* R_1 *is a canonical ideal of* R_1 *and* $\mathrm{e}_1(IR_1) = \mathrm{e}_1(I)$.
(2) $\mathfrak{m}_1 : \mathfrak{m}_1$ *is a Gorenstein ring if and only if* $\mathfrak{m} : \mathfrak{m}$ *is a Gorenstein ring.*
(3) *Let* M, N *be finitely generated* R*–modules. Then* $M \cong N$ *as* R*–modules if and only if* $R_1 \otimes_R M \cong R_1 \otimes_R N$ *as* R_1*–modules.*

Corollary 2.12. *Suppose that* $\mathrm{Q}(\widehat{R})$ *is a Gorenstein ring. Then the first Hilbert coefficient* $\mathrm{e}_1(I)$ *of* I *is independent of the choice of canonical ideals* I *of* R.

Proposition 2.11 is sufficiently general for our purpose, since we need exactly the fact that the Gorenstein property of $\mathrm{Q}(\widehat{R})$ is preserved after enlarging the residue class field. We actually do not know whether the property in the ring R of being analytically unramified is preserved after enlarging the residue class field.

Let us note the following.

Question 2.13. Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let k_1/k be an extension of fields where $k = R/\mathfrak{m}$. Suppose that $\widehat{R}$ is a reduced ring. In this setting, can we always choose a flat local homomorphism $(R, \mathfrak{m}) \to (R_1, \mathfrak{m}_1)$ of Noetherian local rings so that the following three conditions are satisfied?

(a) $\mathfrak{m}_1 = \mathfrak{m}R_1$.
(b) $R_1/\mathfrak{m}_1 \cong k_1$ as k–algebras.
(c) $\widehat{R_1}$ is a reduced ring.

3. Almost Gorenstein rings

In this section we define almost Gorenstein rings and give characterizations.

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$.

Definition 3.1. We say that R is an almost Gorenstein ring, if R possesses a canonical ideal I such that $\mathrm{e}_1(I) \le \mathrm{r}(R)$.

This definition is well-defined, because by Corollary 2.12 the value $\mathrm{e}_1(I)$ is independent of the choice of canonical ideals I. If R is a Gorenstein ring, one can choose any parameter ideal Q of R to be a canonical ideal, so that $\mathrm{e}_1(Q) = 0 < 1 = \mathrm{r}(R)$. Hence every one–dimensional Gorenstein local ring is almost Gorenstein.

Before going ahead, let us note basic examples of almost Gorenstein rings which are not Gorenstein.

Example 3.2. Let k be a field.

(1) We look at the rings $R_1 = k[[t^3, t^4, t^5]]$, $R_2 = k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$, and $R_3 = k[[X, Y, Z, W]]/(Y^2, Z^2, W^2, YW, ZW, XW - YZ)$, where $k[[t]]$, $k[[X, Y, Z]]$, and $k[[X, Y, Z, W]]$ denote the formal power series rings over k. Then these rings R_1, R_2, and R_3 are almost Gorenstein rings with $\mathrm{r}(R_1) = \mathrm{r}(R_2) = 2$ and $\mathrm{r}(R_3) = 3$. The ring R_1 is an integral domain, R_2 is a reduced ring but not an integral domain, and R_3 is not a reduced ring.

(2) Let $a \geq 3$ be an integer and put $R = k[[t^a, t^{a+1}, t^{a^2-a-1}]]$. Then $\mathrm{e}_1(I) = \frac{a(a-1)}{2} - 1$ for canonical ideals I of R. Since $\mathrm{r}(R) = 2$, R is an almost Gorenstein ring if and only if $a = 3$. This example suggests that almost Gorenstein rings are rather rare.

We note the following.

Proposition 3.3. *Let $\varphi : (R, \mathfrak{m}) \to (R_1, \mathfrak{m}_1)$ be a flat local homomorphism of Noetherian local rings and assume that $\mathfrak{m}_1 = \mathfrak{m}R_1$. Then the following conditions are equivalent.*

(1) *R_1 is an almost Gorenstein ring.*

(2) *R is an almost Gorenstein ring.*

When this is the case, $\mathrm{r}(R_1) = \mathrm{r}(R)$ and for every canonical ideal I of R, IR_1 is a canonical ideal of R_1 with $\mathrm{e}_1(IR_1) = \mathrm{e}_1(I)$.

We now develop the theory of almost Gorenstein rings. For this purpose let us maintain the following setting throughout this section. Thanks to Lemma 2.10, Proposition 3.3, and Corollary 2.8, we may assume this setting, after enlarging the residue class field $R/\mathfrak{m}$ of R to be infinite.

Setting 3.4. Let K be an R–submodule of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as R–modules. Let $S = R[K]$ and $\mathfrak{c} = R : S$ the conductor of S. We choose a regular element $a \in \mathfrak{m}$ so that $aK \subsetneq R$ and put $I = aK$, $Q = (a)$.

Notice that Q is a reduction of the canonical ideal I of R and $S = R[\frac{I}{a}]$.
We begin with the following.

Lemma 3.5. (1) *Let T be a subring of $\mathrm{Q}(R)$ such that $K \subseteq T$ and T is a finitely generated R–module. Then $R : T = K : T$.*

(2) *$\mathfrak{c} = K : S$ and $\ell_R(R/\mathfrak{c}) = \ell_R(S/K)$.*

(3) *$\ell_R(I/Q) = \ell_R(K/R)$ and $\ell_R(S/R) = \ell_R(R/\mathfrak{c}) + \ell_R(I/Q)$.*

Proof. For each subring T of $\mathrm{Q}(R)$ such that $K \subseteq T$ and T is a finitely generated R–module, we have

$$K : T = K : KT = (K : K) : T = R : T,$$

since $R = K : K$ ([10, Bemerkung 2.5]). Therefore, taking $T = S$, we get $\ell_R(R/\mathfrak{c}) = \ell_R(R/(K : S))$, while

$$\ell_R(R/(K : S)) = \ell_R([K : (K : S)]/(K : R)) = \ell_R(S/K),$$

thanks to the canonical duality ([10, Bemerkung 2.5]). Thus $\ell_R(R/\mathfrak{c}) = \ell_R(S/K)$. Since $K = \frac{I}{a}$, we get $\ell_R(I/Q) = \ell_R(K/R)$, so that

$$\begin{aligned} \ell_R(S/R) &= \ell_R(S/K) + \ell_R(K/R) \\ &= \ell_R(K : K/K : S) + \ell_R(I/Q) \\ &= \ell_R(R/\mathfrak{c}) + \ell_R(I/Q). \end{aligned}$$

□

Since $\mu_R(I) = \mathrm{r}(R)$ ([10, Satz 6.10]), combining Proposition 2.2 with Lemma 3.5, we get the following, which is the key for our argument.

Proposition 3.6. $0 \leq \mathrm{r}(R) - 1 = \mu_R(I) - 1 \leq \ell_R(I/Q) \leq \mathrm{e}_1(I) = \ell_R(R/\mathfrak{c}) + \ell_R(I/Q)$.

First of all let us note a characterization of Gorenstein rings.

Theorem 3.7. *The following conditions are equivalent.*
(1) *R is a Gorenstein ring.*
(2) $K = R$.
(3) $S = K$.
(4) $S = R$.
(5) $\ell_R(S/R) = \ell_R(R/\mathfrak{c})$.
(6) $I^2 = QI$.
(7) $\mathrm{e}_1(I) = 0$.
(8) $\mathrm{e}_1(I) = \mathrm{r}(R) - 1$.

As a consequence of Theorem 3.7, we have the following.

Corollary 3.8. *The following assertions hold true.*
(1) *Suppose that R is not a Gorenstein ring. Then $K : \mathfrak{m} \subseteq S$.*
(2) *S is a Gorenstein ring if and only if $\mathfrak{c}^2 = a\mathfrak{c}$ for some $a \in \mathfrak{c}$.*

We now give a characterization of almost Gorenstein rings. The following result is exactly the same as the definition of almost Gorenstein ring that Barucci and Fröberg [2] gave in the case where the rings R are analytically unramified.

Theorem 3.9. *R is an almost Gorenstein ring if and only if $\mathfrak{m}K \subseteq R$, i.e., $\mathfrak{m}I = \mathfrak{m}Q$. When this is the case, $\mathfrak{m}S \subseteq R$.*

Proof. Suppose that R is an almost Gorenstein ring. If $\ell_R(I/Q) = \mathrm{e}_1(I)$, then $I^2 = QI$ by Proposition 2.2 (2), so that R is a Gorenstein ring by Theorem 3.7. If $\ell_R(I/Q) < \mathrm{e}_1(I)$, then we have $\mathrm{r}(R) - 1 = \ell_R(I/Q)$, because $\mathrm{r}(R) - 1 \leq \ell_R(I/Q) < \mathrm{e}_1(I) \leq \mathrm{r}(R)$. Therefore $\mathfrak{m}I = \mathfrak{m}Q$ by Proposition 2.2 (1). Hence $\mathfrak{m}I^n = \mathfrak{m}Q^n$ for all $n \in \mathbb{Z}$, so that $\mathfrak{m}S \subseteq R$, because $S = \frac{I^n}{a^n}$ for $n \gg 0$. Conversely, suppose that $\mathfrak{m}K \subseteq R$ and we will show R is an almost Gorenstein ring. We may assume that R is not a Gorenstein ring. Let $J = Q :_R \mathfrak{m}$. Then $J^2 = QJ$ by [5], since R is not a regular local ring. Therefore $I \subseteq J \subseteq \overline{I}$, so that $\mathrm{e}_1(I) \leq \mathrm{e}_1 J) = \mathrm{r}(R)$ by Corollary 2.4. Hence R is an almost Gorenstein ring. □

Since $\mathfrak{m}K \subseteq \mathfrak{m}\overline{R}$, we readily have the following.

Corollary 3.10. *If $\mathfrak{m}\overline{R} \subseteq R$, then R is an almost Gorenstein ring.*

We need the following.

Lemma 3.11. *The following assertions hold true.*
(1) $\ell_R(I^2/QI) = \ell_R(R/(R : K)) \leq \ell_R(S/K)$.
(2) $\ell_R((R : \mathfrak{m})/R) = \mathrm{r}(R)$.
(3) *R is a discrete valuation ring, if $\mathfrak{m} : \mathfrak{m} \subsetneq R : \mathfrak{m}$.*

As a consequence of Theorems 3.7 and 3.9, we get the following characterization of almost Gorenstein rings which are not Gorenstein. Condition (3) in Theorem 3.12 is called Sally's equality. $\mathfrak{m}$–primary ideals satisfying Sally's equality are known to enjoy very nice properties ([8, 15, 16]), where the ideals are not necessarily canonical ideals and the rings need not be of dimension one. For instance, the fact that condition (3) in Theorem 3.12 implies both the condition (5) and assertion (a) is due to [15].

Theorem 3.12. *The following conditions are equivalent.*

(1) *R is an almost Gorenstein ring but not a Gorenstein ring.*
(2) $\mathrm{e}_1(I) = \mathrm{r}(R)$.
(3) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_R(R/I) + 1$.
(4) $\ell_R(S/K) = 1$*, i.e.,* $S = K : \mathfrak{m}$.
(5) $\ell_R(I^2/QI) = 1$.
(6) $\mathfrak{m} : \mathfrak{m} = S$ *and R is not a discrete valuation ring.*

When this is the case, we have the following.

(a) $\mathrm{red}_Q(I) = 2$.
(b) $\ell_R(R/I^{n+1}) = (\mathrm{r}(R) + \ell_R(R/I) - 1)\binom{n+1}{1} - \mathrm{r}(R)$ *for all* $n \geq 1$.
(c) *Let* $G = \bigoplus_{n\geq 0} I^n/I^{n+1}$ *be the associated graded ring of I and* $M = \mathfrak{m}G + G_+$ *the graded maximal ideal of G. Then G is a Buchsbaum ring with* $\mathbb{I}(G) = 1$*, where* $\mathbb{I}(G)$ *stands for the Buchsbaum invariant of G.*

Proof. (1) $\Leftrightarrow$ (2) This follows from the fact that $\mathrm{r}(R) - 1 \leq \mathrm{e}_1(I)$ (Proposition 3.6). Remember that by Theorem 3.7 R is a Gorenstein ring if and only if $\mathrm{e}_1(I) = \mathrm{r}(R) - 1$ and that R is an almost Gorenstein ring if and only if $\mathrm{e}_1(I) \leq \mathrm{r}(R)$.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) We have by Lemma 3.5 and Proposition 3.6

$$\ell_R(S/K) = \ell_R(R/\mathfrak{c}) = \mathrm{e}_1(I) - \ell_R(I/Q) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_R(R/I).$$

Therefore, condition (3) is equivalent to saying that $\ell_R(S/K) = 1$, i.e., $\ell_R(R/\mathfrak{c}) = 1$. The last condition says that $\mathfrak{c} = \mathfrak{m}$, i.e., $\mathfrak{m}S \subseteq R$ but $S \neq R$, or equivalently, R is an almost Gorenstein ring but not a Gorenstein ring (Theorems 3.7 and 3.9). Remember that $\ell_R((K : \mathfrak{m})/K = 1$ ([10, Satz 3.3]) and that by Corollary 3.8 (1) $K : \mathfrak{m} \subseteq S$, if R is not a Gorenstein ring. Then, because R is not a Gorenstein ring if $S \neq K$ (Theorem 3.7), we get that $\ell_R(S/K) = 1$ if and only if $S = K : \mathfrak{m}$.

(4) $\Rightarrow$ (5) By Theorem 3.7 R is not a Gorenstein ring, so that $I^2 \neq QI$ and hence $\ell_R(I^2/QI) = 1$, because $\ell_R(I^2/QI) \leq \ell_R(S/K)$ by Lemma 3.11 (1).

(5) $\Rightarrow$ (1) By Lemma 3.11 (1) we have $R : K = \mathfrak{m}$. Therefore $\mathfrak{m}K \subseteq R$ and $K \neq R$, so that R is an almost Gorenstein ring but not a Gorenstein ring.

(1) $\Rightarrow$ (6) Suppose that R is an almost Gorenstein ring but not a Gorenstein ring. Then R is not a discrete valuation ring, $S \neq R$, and $\mathfrak{m}S \subseteq R$. Hence

$$R \subsetneq S \subseteq R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$$

by Lemma 3.11 (3). Since $\ell_R(S/R) = \mathrm{e}_1(I) = \mathrm{r}(R) = \ell_R((R : \mathfrak{m})/R)$ (thanks to Lemma 2.1, the equivalence of conditions (1) and (2), and Lemma 3.11 (2)), we get $S = \mathfrak{m} : \mathfrak{m}$.

(6) $\Rightarrow$ (2) By Lemma 3.11 (3) we have $S = \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$. Therefore $\mathrm{e}_1(I) = \ell_R(S/R) = \ell_R((R : \mathfrak{m})/R) = \mathrm{r}(R)$ by Lemma 2.1 and Lemma 3.11 (2).

Let us prove the last assertions. We put $J = Q :_R \mathfrak{m}$. Hence $I \subseteq J$, because R is an almost Gorenstein ring.

(a) We have $\ell_R(I/Q) = \mathrm{r}(R) - 1$ by Proposition 2.2 (1). Hence $\ell_R(J/I) = 1$, because $\ell_R(J/Q) = \mathrm{r}(R)$. Therefore $I^3 = QI^2$ by [8, Proposition 2.6], since $J^2 = QJ$ by [5]. Thus $\mathrm{red}_Q(I) = 2$ by Theorem 3.7, because R is not a Gorenstein ring.

(b) This is clear.

(c) Let $[\mathrm{H}^0_M(G)]_0$ denote the homogeneous component of the graded local cohomology module $\mathrm{H}^0_M(G)$ with degree 0. Then, thanks to the fact $I^3 = aI^2$, a direct computation shows

$$\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0 \cong (I^2 :_R a)/I.$$

We want to see that $J = I^2 :_R a$. Let $x \in I^2 :_R a$. Then $ax \in I^2 \subseteq J^2 = aJ$. Hence $x \in J$, so that we have

$$I \subseteq I^2 :_R a \subseteq J = (a) :_R \mathfrak{m}.$$

Claim 1. $I \neq I^2 :_R a$.

Proof of Claim 1. Suppose that $I = I^2 :_R a$. Then, since $I^2 \subseteq \mathfrak{m}I \subseteq (a)$, we have

$$I^2 = a(I^2 :_R a) = aI.$$

Hence R is a Gorenstein ring by Theorem 3.7, which is impossible. □

Thanks to Claim 1, we get $I \subsetneq I^2 :_R a$, so that $J = I^2 :_R a$, since $\ell_R(J/I) = 1$. Thus $\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0 \cong J/I \cong R/\mathfrak{m}$. Hence G is a Buchsbaum ring with $\mathbb{I}(G) = 1$. □

4. Gorensteiness in the algebra $\mathfrak{m} : \mathfrak{m}$

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. In this section we shall settle in full generality the problem of when the endomorphism algebra $\mathfrak{m} : \mathfrak{m}$ of $\mathfrak{m}$ is a Gorenstein ring. This is the question which V. Barucci and R. Fröberg [2, Proposition 25] tried to answer in the case where the rings R are analytically unramified and Barucci [1] eventually felt that there was a gap in their proof.

Let $v(R) = \mu_R(\mathfrak{m})$ denote the embedding dimension of R and $e(R) = \mathrm{e}_0(\mathfrak{m})$ the multiplicity of R with respect to $\mathfrak{m}$. We then have the following.

Theorem 4.1. *The following conditions are equivalent.*

(1) $\mathfrak{m} : \mathfrak{m}$ *is a Gorenstein ring.*

(2) R *is an almost Gorenstein ring and* $v(R) = e(R)$.

Proof. After enlarging the residue class field of R, by Proposition 2.11 we may assume that the field $R/\mathfrak{m}$ is algebraically closed and the ring $\mathrm{Q}(\widehat{R})$ is Gorenstein. We may also assume that R is not a discrete valuation ring. Therefore we have an R–submodule K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as R–modules (Corollary 2.8). Let us maintain the same notation as in Setting 3.4. Hence $S = R[K]$ and $\mathfrak{c} = R : S$. Let $A = \mathfrak{m} : \mathfrak{m}$.

(1) ⇒ (2) Since R is not a discrete valuation ring, $R \subsetneq R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m} = A$ by Lemma 3.11. Suppose that R is a Gorenstein ring. Then, since $\mathfrak{m} = R : A$ and A is a Gorenstein ring, the A–module $\mathfrak{m}$ is locally free of rank one ([10, Satz 5.12]), so that $\mathfrak{m} \cong A$ as A–modules. Hence $\mathfrak{m} = aA$ for some $a \in \mathfrak{m}$. Therefore $\mathfrak{m}^2 = a\mathfrak{m}$, i.e., $v(R) = e(R)$ (see [14]).

Suppose now that R is not a Gorenstein ring. Since $R : A = \mathfrak{m}$, we have

$$A \subseteq K : \mathfrak{m} \subseteq S$$

by Corollary 3.8 (1).

Claim 2. *Let X be a finitely generated A–submodule of* $\mathrm{Q}(R)$ *such that* $\mathrm{Q}(R)\cdot X = \mathrm{Q}(R)$. *Then X is a reflexive R–module, i.e.,* $X = R : (R : X)$.

Proof of Claim 2. Notice that $R : (R : A) = R : \mathfrak{m} = A$ and $A : (A : X) = X$ for every fractional ideal X of A ([10, Bemerkung 2.5]), since A is a Gorenstein ring. We write $A : X = \sum_{i=1}^{\ell} Ay_i$ where $y_i's$ are units of $\mathrm{Q}(R)$. Then

$$X = A : (A : X) = A : \sum_{i=1}^{\ell} Ay_i = \bigcap_{i=1}^{\ell} A\frac{1}{y_i}.$$

Therefore, because $A\frac{1}{y_i} \cong A$ and A is R–reflexive, we get

$$\begin{aligned} X &\subseteq R : (R : X) \\ &= R : (R : \bigcap_{i=1}^{\ell} A\frac{1}{y_i}) \\ &\subseteq R : \sum_{i=1}^{\ell} (R : A\frac{1}{y_i}) \\ &\subseteq \bigcap_{i=1}^{\ell} \left[R : (R : A\frac{1}{y_i})\right] \\ &= \bigcap_{i=1}^{\ell} A\frac{1}{y_i} \\ &= X, \end{aligned}$$

so that X is a reflexive R–module. □

Claim 3. $\ell_A(S/A) = \ell_A(\mathfrak{m}/\mathfrak{c})$.

Proof of Claim 3. Let $\ell = \ell_A(S/A)$ and take a composition series

$$A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_\ell = S$$

as A–modules. Then applying $[R : *]$, by Claim 2 we get

$$\mathfrak{m} = R : A = R : A_0 \supsetneq R : A_1 \supsetneq \cdots \supsetneq R : A_\ell = R : S = \mathfrak{c}.$$

Hence $\ell_A(\mathfrak{m}/\mathfrak{c}) \geq \ell$ and we get $\ell_A(S/A) = \ell_A(\mathfrak{m}/\mathfrak{c})$ by symmetry. □

We now notice that $\ell_A(X) = \ell_R(X)$ for every A–module X of finite length, because A is a module-finite extension of R and the field $R/\mathfrak{m}$ is algebraically closed. Consequently

$$\ell_R(S/A) = \ell_A(S/A) = \ell_A(\mathfrak{m}/\mathfrak{c}) = \ell_R(\mathfrak{m}/\mathfrak{c})$$

and therefore by Lemma 3.11 (2) we get

$$\begin{aligned} \ell_R(S/R) &= \ell_R(S/A) + \ell_R(A/R) \\ &= \ell_R(\mathfrak{m}/\mathfrak{c}) + \ell_R((R : \mathfrak{m})/R) \\ &= (\ell_R(R/\mathfrak{c}) - 1) + \mathrm{r}(R) \\ &= \ell_R(R/\mathfrak{c}) + (\mathrm{r}(R) - 1), \end{aligned}$$

so that $\ell_R(I/Q) = \mathrm{r}(R) - 1 = \mu_R(I/Q)$ by Lemma 3.5 (3). Thus R is an almost Gorenstein ring. Since $\mathfrak{m}S \subseteq R$, we have $S \subseteq R : \mathfrak{m} = A$. Hence $S = A$ and $\mathfrak{c}^2 = a\mathfrak{c}$ for some $a \in \mathfrak{c}$ by Corollary 3.8 (2). Thus $v(R) = e(R)$, because $\mathfrak{c} = \mathfrak{m}$.

(2) ⇒ (1) Suppose that R is a Gorenstein ring. Then $e(R) \leq 2$ and hence every finitely generated R–subalgebra of $\overline{R}$ is a Gorenstein ring. In particular, the ring

$A = \mathfrak{m} : \mathfrak{m}$ is Gorenstein. Suppose that R is not a Gorenstein ring. Then $S = \mathfrak{m} : \mathfrak{m}$ by Theorem 3.12 and S is a Gorenstein ring by Corollary 3.8 (2), because $\mathfrak{c} = \mathfrak{m}$ and $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$. □

Remark 4.2. In the proof of (1) ⇒ (2) of Theorem 4.1 the critical part is the fact that $\ell_R(S/A) = \ell_R(\mathfrak{m}/\mathfrak{c})$, which in our context we safely get by the assumption that $R/\mathfrak{m}$ is an algebraically closed filed. Except this part the above proof is essentially the same as was given by Barucci and Fröberg [2]. We nevertheless do not know whether we can still assume that $R/\mathfrak{m}$ is an algebraically closed field, even if we restrict the notion of almost Gorenstein ring within the rings which are analytically unramified. See Question 2.13.

5. Almost Gorenstein rings obtained by idealization

In this section we explore almost Gorenstein rings obtained by idealization. The purpose is to show how our modified notion of almost Gorenstein ring enriches examples and the theory as well.

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. For each R–module M we denote by $R \ltimes M$ the idealization of M over R. Hence $R \ltimes M = R \oplus M$ as additive groups and the multiplication in $R \ltimes M$ is given by

$$(a, x)(b, y) = (ab, ay + bx).$$

We then have $\mathfrak{a} := (0) \times M$ forms an ideal of $R \ltimes M$ and $\mathfrak{a}^2 = (0)$. Hence, because $R \cong (R \ltimes M)/\mathfrak{a}$, $R \ltimes M$ is a local ring with maximal ideal $\mathfrak{m} \times M$ and $\dim R \ltimes M = 1$. Remember that $M \cong \mathrm{K}_R$ as R–modules if and only if $R \ltimes M$ is a Gorenstein ring, provided M is a finitely generated R–module and $M \neq (0)$ ([13]).

Let us begin with the following.

Proposition 5.1. *Let I be an arbitrary $\mathfrak{m}$–primary ideal of R and suppose that there exists an element $a \in I$ such that $Q = (a)$ is a reduction of I. Assume that R possesses the canonical module K_R and put $I^\vee = \mathrm{Hom}_R(I, \mathrm{K}_R)$. Then the following conditions are equivalent.*

1. *$R \ltimes I^\vee$ is an almost Gorenstein ring.*
2. *$\mathfrak{m}I = \mathfrak{m}Q$ and $I^2 = QI$.*
3. *$\mu_R(I/Q) = \ell_R(I/Q) = \mathrm{e}_1(I)$.*

As an immediate consequence of Proposition 5.1 we get the following.

Corollary 5.2. *Suppose that R possesses the canonical module K_R. If R is not a discrete valuation ring, then $R \ltimes (Q :_R \mathfrak{m})^\vee$ is an almost Gorenstein ring for every parameter ideal Q in R, where $(Q :_R \mathfrak{m})^\vee = \mathrm{Hom}_R(Q :_R \mathfrak{m}, \mathrm{K}_R)$.*

Proof. Let $I = Q :_R \mathfrak{m}$. Then by [5] we have $I^2 = QI$, so that the ideal I satisfies condition (2) in Proposition 5.1. □

Theorem 5.3. *Suppose that R possesses the canonical module K_R and the residue class field $R/\mathfrak{m}$ of R is infinite. Let M be a Cohen-Macaulay R–module with $\mathrm{Ann}_R M = (0)$. Then the following conditions are equivalent.*

1. *$R \ltimes M$ is an almost Gorenstein ring.*
2. *There exists an $\mathfrak{m}$–primary ideal I of R and $a \in I$ such that $M \cong I^\vee$ as R–modules, $I^2 = QI$, and $\mathfrak{m}I = \mathfrak{m}Q$, where $Q = (a)$ and $I^\vee = \mathrm{Hom}_R(I, \mathrm{K}_R)$.*

When R is a Gorenstein ring, we can simplify Theorem 5.3 as follows.

Corollary 5.4. *Assume that R is a Gorenstein ring and let M be a Cohen-Macaulay R–module with $\operatorname{Ann}_R M = (0)$. Then the following conditions are equivalent.*

(1) *$R \ltimes M$ is an almost Gorenstein ring.*
(2) *Either $M \cong R$ or $M \cong \mathfrak{m}$.*

The following result shows the property of being an almost Gorenstein ring is preserved via idealization of the maximal ideal, and vice versa.

Theorem 5.5. *The following conditions are equivalent.*

(1) *$R \ltimes \mathfrak{m}$ is an almost Gorenstein ring.*
(2) *R is an almost Gorenstein ring.*

When this is the case, $v(R \ltimes \mathfrak{m}) = 2v(R)$.

Proof. We may assume that the residue class field $R/\mathfrak{m}$ of R is infinite and the ring $\mathrm{Q}(\widehat{R})$ is Gorenstein. Hence there exists an R–submodule K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as R–modules. We maintain the same notation as in Setting 3.4. Hence $S = R[K]$. We choose a non-zerodivisor $a \in \mathfrak{m}$ so that $aK \subsetneq R$. Let $I = aK$, $Q = (a)$, and $J = I :_R \mathfrak{m}$. Therefore I is a canonical ideal of R.

(1) $\Rightarrow$ (2) We may assume that R is not a Gorenstein ring. By Theorem 5.3 we may choose an $\mathfrak{m}$–primary ideal $\mathfrak{a}$ of R and $b \in \mathfrak{a}$ so that $\mathfrak{a}^2 = b\mathfrak{a}$, $\mathfrak{m}\mathfrak{a} = \mathfrak{m}b$, and $\mathfrak{m} \cong \mathfrak{a}^\vee$, where $*^\vee = \operatorname{Hom}_R(*, \mathrm{K}_R)$. Since $\ell_R((I : \mathfrak{m})/I) = 1$, we get $J = I : \mathfrak{m} = a(K : \mathfrak{m})$. On the other hand, since $K : \mathfrak{m} \subseteq S$ by Corollary 3.8 (1), we get

$$Q = (a) \subseteq I = aK \subseteq J = a(K : \mathfrak{m}) \subseteq aS \subseteq a\overline{R}.$$

Hence Q is also a reduction of J. Now notice that $\mathfrak{m} \cong J^\vee$, since $J = I : \mathfrak{m} \cong \mathfrak{m}^\vee$. Then, because $R \ltimes J^\vee$ is an almost Gorenstein ring, we get $\mathfrak{m}J \subseteq Q$ by Proposition 5.1, so that $\mathfrak{m}I \subseteq \mathfrak{m}J \subseteq Q$. Hence R is an almost Gorenstein ring.

(2) $\Rightarrow$ (1) By Corollary 5.4 we may assume that R is not a Gorenstein ring. Choose a regular element $b \in \mathfrak{m}$ so that $bS \subsetneq R$ and put $\mathfrak{a} = bS$. Then $b \in \mathfrak{a}$, $\mathfrak{a}^2 = b\mathfrak{a}$, and $\mathfrak{m}\mathfrak{a} \subseteq (b)$, since R is an almost Gorenstein ring. Now notice that $S = K : \mathfrak{m} \cong \mathfrak{m}^\vee$ (Theorem 3.12) and we have $\mathfrak{m} \cong S^\vee \cong \mathfrak{a}^\vee$. Hence $R \ltimes \mathfrak{m}$ is an almost Gorenstein ring by Proposition 5.1.

Since the maximal ideal of $R \ltimes \mathfrak{m}$ is $\mathfrak{m} \times \mathfrak{m}$, we get

$$\begin{aligned}
v(R \ltimes \mathfrak{m}) &= \ell_{R\ltimes\mathfrak{m}}((\mathfrak{m} \times \mathfrak{m})/(\mathfrak{m} \times \mathfrak{m})^2) \\
&= \ell_R((\mathfrak{m} \oplus \mathfrak{m})/(\mathfrak{m}^2 \oplus \mathfrak{m}^2)) \\
&= 2\ell_R(\mathfrak{m}/\mathfrak{m}^2) \\
&= 2v(R),
\end{aligned}$$

which proves the last equality. □

We need the following.

Lemma 5.6. *The following conditions are equivalent.*

(1) *$R \ltimes \mathfrak{m}$ is a Gorenstein ring.*
(2) *R is a discrete valuation ring.*

Let us note examples of almost Gorenstein rings obtained by idealization.

Example 5.7. Let $n \geq 0$ be an integer. We put

$$R_n = \begin{cases} R & (n = 0), \\ R \ltimes \mathfrak{m} & (n = 1), \\ (R_{n-1})_1 & (n > 1). \end{cases}$$

Then the following assertions hold true.

(1) If R is a Gorenstein ring, then R_n is an almost Gorenstein ring for all $n \geq 0$.

(2) R_n is not a discrete valuation ring for every $n \geq 1$. Therefore by Lemma 5.6 R_{n+1} is not a Gorenstein ring for all $n \geq 1$.

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A Remark on the Finiteness Dimension [1]

PHAM HUNG QUY

Dedicated to Professor Nguyen Tu Cuong on the occasion of his sixtieth birthday

Abstract

This note is the main part of my report at the 33rd symposium on Commutative Algebra in Japan. The rest of my report can be seen in [14]. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M a finitely generated R-module. The finiteness dimension of M relative to $\mathfrak{a}$ is defined by

$$f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 \,:\, H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\},$$

where $H^i_{\mathfrak{a}}(M)$ is the i-th local cohomology with respect to $\mathfrak{a}$. The aim of this paper is to show that if $x_1, ..., x_t$ is an $\mathfrak{a}$-filter regular sequence of M with $t \leq f_{\mathfrak{a}}(M)$, then the set

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M$$

is finite.

1 Introduction

Throughout this paper, let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M a finitely generated R-module. For basic facts about local cohomology refer to [2]. We use $\mathbb{N}_0$ (resp. $\mathbb{N}$) to denote the set of non-negative (resp. positive) integers.

Local cohomology was introduced by A. Grothendieck. In general, the i-th local cohomology of M with respect to $\mathfrak{a}$, $H^i_{\mathfrak{a}}(M)$, may not be finitely generated. An important problem in Commutative Algebra is to find certain finiteness properties of local cohomology. In [4], C. Huneke raised the following conjecture: Is the number of associated prime ideals of a local cohomology module $H^i_{\mathfrak{a}}(M)$ always finite? This question has received much attention in the case when $M = R$ is a regular ring (cf. [5], [10], [16]). Although A.K. Singh in [15] gave the first counterexample to Huneke's conjecture, it has positive answer in many cases. For a given positive integer t, $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is finite if either $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$ (cf. [1], [8]) or $\operatorname{Supp} H^i_{\mathfrak{a}}(M)$ is finite for all $i < t$ (cf. [8]). Combining these results, the author in [14] showed that $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is finite if for each $i < t$ either $H^i_{\mathfrak{a}}(M)$ is finitely generated or $\operatorname{Supp} H^i_{\mathfrak{a}}(M)$ is a finite set.

As mentioned above, if t is the least integer such that $H^t_{\mathfrak{a}}(M)$ is not finitely generated, then $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is finite. Such integer is called the *finiteness dimension, denoted by $f_{\mathfrak{a}}(M)$, of M relative to $\mathfrak{a}$* (see, [2, Chapter 9]). The purpose of this paper is to show that the finiteness dimension provides a stronger result about the finiteness of certain sets of associated primes. Namely, let $x_1, ..., x_t$ be an $\mathfrak{a}$-filter regular sequence of M with $t \leq f_{\mathfrak{a}}(M)$, i.e. $\operatorname{Supp}((x_1, ..., x_{i-1})M : x_i)/(x_1, ..., x_{i-1})M \subseteq V(\mathfrak{a})$ for all $i = 1, ..., t$, where $V(\mathfrak{a})$ denotes the set of prime ideals containing $\mathfrak{a}$. Then the set

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M$$

is finite.

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2 The main result

Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R, and M a finitely generated R-module. We begin by recalling some facts about the finiteness dimension of M relative to $\mathfrak{a}$.

Definition 2.1. (i) The *finiteness dimension of M relative to $\mathfrak{a}$* is defined by

$$f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 \, : \, H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\},$$

with the usual convention that the infimum of the empty set of integers is ∞.

(ii) The *$\mathfrak{a}$-minimum $\mathfrak{a}$-adjusted depth of M* is defined by

$$\lambda_{\mathfrak{a}}(M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \, : \, \mathfrak{p} \in \operatorname{Supp}(M) \setminus V(\mathfrak{a})\},$$

with the convention that $\operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} = \infty$ if $\mathfrak{a}+\mathfrak{p} = R$.

Remark 2.2. (i) $f_{\mathfrak{a}}(M) \in \mathbb{N}_0$ provided $\mathfrak{a}M \neq M$ and M is not $\mathfrak{a}$-torsion.

(ii) $f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N} \, : \, \mathfrak{a}^n H^i_{\mathfrak{a}}(M) \neq 0 \text{ for all } n \in \mathbb{N}\}$. Therefore there exists a positive integer n_0 such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < f_{\mathfrak{a}}(M)$.

(iii) $f_{\mathfrak{a}}(M) \leq \lambda_{\mathfrak{a}}(M)$ and the equality holds when R is universally catenary and all the formal fibres of all its localizations are Cohen-Macaulay rings (see, [2, 9.6.7]).

We next recall the notion of $\mathfrak{a}$-filter regular sequence of M and its relation with local cohomology.

Definition 2.3. We say a sequence $x_1, ..., x_t$ of elements contained in $\mathfrak{a}$ is an *$\mathfrak{a}$-filter regular sequence of M* if

$$\operatorname{Supp}\left((x_1, ..., x_{i-1})M : x_i)/(x_1, ..., x_{i-1})M \subseteq V(\mathfrak{a})\right.$$

for all $i = 1, ..., t$, where $V(\mathfrak{a})$ denotes the set of prime ideals containing $\mathfrak{a}$.

Remark 2.4. Let $x_1, ..., x_t$ be an $\mathfrak{a}$-filter regular sequence of M. Then

(i) For all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\mathfrak{a})$, $\frac{x_1}{1}, \cdots, \frac{x_t}{1}$ is a poor $M_{\mathfrak{p}}$-sequence i.e. for each $i = 2, ..., t$, the element x_i is a non-zerodivisor on $M/(x_1, ..., x_{t-1})M$ (cf. [12, Proposition 2.2]).

(ii) $x_1^{n_1}, ..., x_t^{n_t}$ is an $\mathfrak{a}$-filter regular sequence of M for all $n_1, ..., n_t \in \mathbb{N}$, moreover

$$\operatorname{Ass}(M/(x_1^{n_1}, ..., x_t^{n_t})M) \setminus V(\mathfrak{a}) = \operatorname{Ass}(M/(x_1, ..., x_t)M) \setminus V(\mathfrak{a}).$$

(iii) By [12, Proposition 3.4] we have $H^t_{\mathfrak{a}}(M) \cong H^0_{\mathfrak{a}}(H^t_{(x_1,...,x_t)}(M))$. Combining with the well-known fact that $H^t_{(x_1,...,x_t)}(M) \cong \lim_{\rightarrow} M/(x_1^{n_1}, ..., x_t^{n_t})M$, it follows that

$$\operatorname{Ass} H^t_{\mathfrak{a}}(M) \subseteq \bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M.$$

Proof of (ii). By [12, Proposition 2.2] we have $x_1^{n_1}, ..., x_t^{n_t}$ is an $\mathfrak{a}$-filter regular sequence of M for all $n_1, ..., n_t \in \mathbb{N}$. Let $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, ..., x_t)M) \setminus V(\mathfrak{a})$. By localization at $\mathfrak{p}$ we have $\mathfrak{p}R_{\mathfrak{p}}\operatorname{Ass}(M_{\mathfrak{p}}/(\frac{x_1}{1}, \cdots, \frac{x_t}{1})M_{\mathfrak{p}})$ and $\frac{x_1}{1}, \cdots, \frac{x_t}{1}$ is an $M_{\mathfrak{p}}$-sequence. The assertion now follows from the fact that

$$\operatorname{Ass}(M/(x_1^{n_1}, ..., x_t^{n_t})M) = \operatorname{Ass}(M/(x_1, ..., x_t)M)$$

for all $n_1, ..., n_t \in \mathbb{N}$ provided $x_1, ..., x_t$ is an M-sequence. □

Recently, N.T. Cuong and the author proved the following splitting theorem (cf. [3]) whose consequence plays a key role in this paper.

Theorem 2.5 ([3], Theorem 1.1). *Let M be a finitely generated module over a Noetherian ring R and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then, for all $\mathfrak{a}$-filter regular element $x \in \mathfrak{a}^{2n_0}$ of M, it holds that*

$$H^i_{\mathfrak{a}}(M/xM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(M),$$

for all $i < t-1$, and

$$0 :_{H^{t-1}_{\mathfrak{a}}(M/xM)} \mathfrak{a}^{n_0} \cong H^{t-1}_{\mathfrak{a}}(M) \oplus 0 :_{H^t_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0}.$$

Corollary 2.6 ([3], Corollary 4.4). *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then for every $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M contained in $\mathfrak{a}^{2n_0}$, we have*

$$\bigcup_{i=0}^{j} \operatorname{Ass} H^i_{\mathfrak{a}}(M) = \operatorname{Ass}(M/(x_1, ..., x_j)M) \bigcap V(\mathfrak{a}),$$

for all $j = 1, \ldots, t$. In particular, $H^t_{\mathfrak{a}}(M)$ has only finitely many associated primes.

Corollary 2.6 implies that $\cup_{n\in\mathbb{N}}\operatorname{Ass} M/(x_1^n, ..., x_t^n)M$ is finite for every $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M with $t \leq f_{\mathfrak{a}}(M)$. In order to prove the main result we need some preliminary lemmas. The author is grateful to K. Khashyarmanesh for information that the following is a sharp of [7, Lemma 2.1].

Lemma 2.7. *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then for every $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M, we have $\mathfrak{a}^{2^j n_0}H^i_{\mathfrak{a}}(M/(x_1, ..., x_j)M) = 0$ for all $0 \leq j \leq t-1$ and $i < t-j$.*

Proof. The case $j = 0$ is trivial and by induction it is sufficient to show the assertion in the case $j = 1 < t$. The short exact sequence

$$0 \longrightarrow M/(0 :_M x_1) \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

induces the exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/x_1M) \longrightarrow H^{i+1}_{\mathfrak{a}}(M/(0 :_M x_1)) \longrightarrow \cdots.$$

Notice that $0 :_M x_1$ is $\mathfrak{a}$-torsion, hence $H^{i+1}_{\mathfrak{a}}(M/(0 :_M x_1)) \cong H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$. Thus $\mathfrak{a}^{n_0}H^{i+1}_{\mathfrak{a}}(M/(0 :_M x_1)) = 0$ for all $i < t-1$. The assertion is now clear. □

Proposition 2.8. *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Let $x_1, ..., x_t$ be an $\mathfrak{a}$-filter regular sequence of M and $j < t$ a non-negative integer. For all $n_1, ..., n_t \in \mathbb{N}$ such that $n_i \geq 2^t n_0$ for all $j+1 \leq i \leq t$, we have*

$$\operatorname{Ass}M/(x_1^{n_1}, ..., x_t^{n_t})M = \operatorname{Ass}M/(x_1^{n_1}, ..., x_j^{n_j}, x_{j+1}^{2^t n_0}, ..., x_t^{2^t n_0})M.$$

Proof. By Remark 2.4 (ii) we have

$$\operatorname{Ass}(M/(x_1^{n_1}, ..., x_t^{n_t})M) \setminus V(\mathfrak{a}) = \operatorname{Ass}(M/(x_1^{n_1}, ..., x_j^{n_j}, x_{j+1}^{2^t n_0}, ..., x_t^{2^t n_0})M) \setminus V(\mathfrak{a}).$$

On the other hand $\mathfrak{a}^{2^j n_0} H^i_{\mathfrak{a}}(M/(x_1^{n_1},...,x_j^{n_j})M) = 0$ for all $i < t-j$ by Lemma 2.7, and Corollary 2.6 implies that

$$\begin{aligned}\operatorname{Ass}(M/(x_1^{n_1},...,x_t^{n_t})M)\bigcap V(\mathfrak{a}) &= \bigcup_{i=0}^{t-j} \operatorname{Ass} H^i_{\mathfrak{a}}(M/(x_1^{n_1},...,x_j^{n_j})M)\\ &= \operatorname{Ass}(M/(x_1^{n_1},...,x_j^{n_j},x_{j+1}^{2^t n_0},...,x_t^{2^t n_0})M)\bigcap V(\mathfrak{a}).\end{aligned}$$

The proof is complete. □

Lemma 2.9. *Let $(R,\mathfrak{m})$ be a local ring. Let $x_1,...,x_t$ be an $\mathfrak{a}$-filter regular sequence of M such that $t \le \lambda_{\mathfrak{a}}(M)$, the $\mathfrak{a}$-minimum $\mathfrak{a}$-adjusted depth of M. Then $x_1,...,x_t$ is an $\mathfrak{a}$-filter regular sequence of M in any order.*

Proof. It is sufficient to show the assertion in the case $t = 2 \le \lambda_{\mathfrak{a}}(M)$. Moreover we only need to prove that x_2 is an $\mathfrak{a}$-filter regular element of M (see [6, Theorem 117]). Indeed, let $\mathfrak{p} \in \operatorname{Ass} M\backslash V(\mathfrak{a})$. Then $\operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \ge 2$ by the definition of $\lambda_{\mathfrak{a}}(M)$. Thus there exists $\mathfrak{q} \in \operatorname{Spec}(R) \setminus V(\mathfrak{a})$ such that $\mathfrak{q}$ is a minimal prime ideal of $(x_1)+\mathfrak{p}$. By localization at $\mathfrak{q}$ we have $\frac{x_1}{1}$ is a $M_{\mathfrak{q}}$-regular element. Hence $\mathfrak{q}R_{\mathfrak{q}} \in \operatorname{Ass}(M/x_1M)_{\mathfrak{q}}$ since $\operatorname{ht}(\mathfrak{q}R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) = 1$ and $\mathfrak{p}R_{\mathfrak{q}} \in \operatorname{Ass} M_{\mathfrak{q}}$. Thus $\mathfrak{q} \in \operatorname{Ass} M/x_1M$. Hence $x_2 \notin \mathfrak{q}$ because x_2 is an $\mathfrak{a}$-filter regular element of M/x_1M. Therefore $x_2 \notin \mathfrak{p}$ and so x_2 is an $\mathfrak{a}$-filter regular element of M. □

We now give the main result of this paper.

Theorem 2.10. *Let M be a finitely generated R-module, and $\mathfrak{a}$ an ideal of R. Let t be a positive integer such that $t \le f_{\mathfrak{a}}(M)$, the finiteness dimension of M relative to $\mathfrak{a}$, and $x_1,...,x_t$ an $\mathfrak{a}$-filter regular sequence of M. Then the set*

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M$$

is finite.

Proof. Let n_0 be a positive integer such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < f_{\mathfrak{a}}(M)$. For each $(n_1,...,n_t) \in \mathbb{N}^t$ we consider a t-tuple of positive integers $(m_1,...,m_t) \in \mathbb{N}^t$ such that $m_i = n_i$ if $n_i < 2^t n_0$, and $m_i = 2^t n_0$ if $n_i \ge 2^t n_0$. We have that $\mathfrak{p} \in \operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M$ iff $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} M_{\mathfrak{p}}/(x_1^{n_1},...,x_t^{n_t})M_{\mathfrak{p}}$. By Lemma 2.9 and a change of the order of the x_i, if necessary, we can assume that $n_i < 2^t n_0$ for all $i \le j$, and $n_i \ge 2^t n_0$ for all $j+1 \le i \le t$, for some $j \le t$. Now, Proposition 2.8 implies that $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} M_{\mathfrak{p}}/(x_1^{n_1},...,x_t^{n_t})M_{\mathfrak{p}}$ iff $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} M_{\mathfrak{p}}/(x_1^{m_1},...,x_t^{m_t})M_{\mathfrak{p}}$. Therefore

$$\operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M = \operatorname{Ass} M/(x_1^{m_1},...,x_t^{m_t})M.$$

Hence

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M = \bigcup_{1 \le m_1,...,m_t \le 2^t n_0} \operatorname{Ass} M/(x_1^{m_1},...,x_t^{m_t})M$$

is a finite set. □

It should be noted that L.T. Nhan in [13, Theorem 3.1] proved a similar result for generalized regular sequences of M. We recall that in a local ring $(R,\mathfrak{m})$ a sequence $x_1,...,x_t$ of elements is said to be a *generalized regular sequence of M* if $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M/(x_1,...,x_{i-1})M$ satisfying $\dim R/\mathfrak{p} > 1$, for all $i = 1,...,t$.

Question 2.11. Notice that $H^i_{\mathfrak{a}}(M) = \lim_{\to} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$, by virtue of Theorem 2.10 it raises the following natural questions.

(i) Is $\cup_n \operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ finite for all $i \leq f_{\mathfrak{a}}(M)$?

(ii) Is

$$\bigcup_{n_1,\dots,n_t \in \mathbb{N}} \operatorname{Ass}\operatorname{Ext}^i_R(R/(x_1^{n_1}, \dots, x_t^{n_t}), M)$$

finite for all $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_t$ of M and $i \leq t \leq f_{\mathfrak{a}}(M)$?

If M is an $\mathfrak{a}$-torsion module, then $f_{\mathfrak{a}}(M) = \infty$. The following is a special case of Question 2.11(i).

Question 2.12. Is $\cup_n \operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ finite for all i provided M is $\mathfrak{a}$-torsion?

In [11], L. Melkersson and Schenzel asked whether the sets $\operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ become stable for sufficiently large n. This question is not true in general since $\cup_n \operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ may be infinite. However, Khashyarmanesh and Salarian have proved that $\operatorname{Ass}\operatorname{Ext}^1_R(R/\mathfrak{a}^n, M)$ become stable for sufficiently large n (cf. [9, Corollary 2.3]). Thus, Melkersson-Schenzel's question and Question 2.11 (i) has an affirmative answer in the cases $f_{\mathfrak{a}}(M) \leq 1$. We may modify Melkersson-Schenzel's question as follows.

Question 2.13. *whether the sets* $\operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ *become stable for sufficiently large* n *and for all* $i \leq f_{\mathfrak{a}}(M)$*?*

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A New Definition of the Transcendence Degree over a Ring

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Abstract

This note reports on results from the recent paper [5]. A new definition of the transcendence degree of an algebra over a ring is given. This has the property that for a finitely generated algebra over a Noetherian Jacobson ring, the transcendence degree is equal to the Krull dimension. This generalizes a well-known result in commutative algebra. As a consequence, the transcendence degree of a finitely generated algebra over a Noetherian Jacobson ring cannot increase when passing to a subalgebra.

The starting point of the investigations is the following result.

Theorem 1 (Coquand and Lombardi [2]). *Let R be a commutative ring with unity and $n \in \mathbb{N}$ a positive integer. Then the following statements are equivalent:*

(a) $\dim(R) < n$.

(b) For every $a_1, \ldots, a_n \in R$ there exist $m_1, \ldots, m_n \in \mathbb{N}_0$ such that

$$\prod_{i=1}^{n} a_i^{m_i} \in \Big(a_j \cdot \prod_{i=1}^{j} a_i^{m_i} \,\Big|\, j = 1, \ldots, n\Big)_R, \tag{1}$$

where $(S)_R$ denotes the ideal in R generated by a set $S \subseteq R$.

A proof will be presented later in this note. Notice that (1) tells us that $\prod_{i=1}^{n} a_i^{m_i}$ is an R-linear combination of monomials in $a_1, \ldots, a_n$ that are lexicographically larger than $\prod_{i=1}^{n} a_i^{m_i}$. This may be paraphrased by saying that $(a_1, \ldots, a_n)$ is a zero of a polynomial over R whose trailing coefficient, with respect to the lexicographic monomial ordering, is 1. This observation motivates several questions: Can the lexicographic monomial ordering be replaced by other monomial orderings? Does there exist a relative version of Theorem 1, similar to the statement that the dimension of a finitely generated algebra over a field equals its Krull dimension?

Before proceeding, let us recall the concept of a monomial ordering. Throughout this note, R will stand for a commutative ring with unity.

Definition 2. *Let $R[x_1, x_2, \ldots]$ be the polynomial ring with infinitely many indeterminates and M the set of monomials (i.e., finite products of powers of the x_i). A* **monomial ordering** *is a total ordering "$\preceq$" on M such that:*

(a) if $t \in M$, then $1 \preceq t$;

(b) if $s, t_1, t_2 \in M$ with $t_1 \preceq t_2$, then $st_1 \preceq st_2$.

The most important example of a monomial ordering is the **lexicographic** monomial ordering, defined by:

$$\underbrace{\prod_{i=1}^{n} x_i^{e_i}}_{=:t} \preceq \underbrace{\prod_{i=1}^{n} x_i^{e'_i}}_{=:t'} \;:\Leftrightarrow\; t = t' \text{ or } e_i < e'_i \text{ for the smallest } i \text{ with } e_i \neq e'_i.$$

Given a monomial ordering "$\preceq$" and a nonzero polynomial $f \in R[x_1, x_2, \ldots]$, we can speak of the **leading coefficient** and the **trailing coefficient** of f, i.e., the coefficient of the largest and smallest monomial, respectively, appearing in f with nonzero coefficient.

The following definition generalizes the notion of the transcendence degree of an algebra over a field.

Definition 3. *(a) A nonzero polynomial $f \in R[x_1, x_2, \ldots]$ is called* **submonic** *if there exists a monomial ordering "$\preceq$" such that the trailing coefficient of f is* 1.

(b) Let A be an R-algebra (i.e., a commutative ring A with unity together with a ring homomorphism $R \to A$). Elements $a_1, \ldots, a_n \in A$ are called **algebraically dependent** *over R if there exists a submonic polynomial $f \in R[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$. (Of course the homomorphism $R \to A$ is applied to the coefficients of f before evaluating at $a_1, \ldots, a_n$.) Otherwise, $a_1, \ldots, a_n$ are called* **algebraically independent** *over R.*

(c) For a nonzero R-algebra A, the **transcendence degree** *of A over R is defined as*

$$\operatorname{trdeg}(A : R) := \sup\big\{n \in \mathbb{N} \mid \text{there exist } a_1, \ldots, a_n \in A \text{ that are algebraically independent over } R\big\}.$$

If $A = \{0\}$ is the zero ring, we define $\operatorname{trdeg}(A) = \dim(A) = -1$.

Let us consider some examples.

Example 4. (1) If R is an integral domain, then an element $a \in R$ is algebraically dependent over R if and only if it is zero or a unit. In fact, algebraic dependence means that there exist $n \in \mathbb{N}$ and $b \in R$ such that $a^n = ba^{n+1}$.

(2) If R is a nonzero finite ring, then $\operatorname{trdeg}(R : R) = 0$ since for each $a \in R$ there exist nonnegative integers $m < n$ such that $a^m = a^n$. So a satisfies the submonic polynomial $x^m - x^n$.

(3) Let us consider $R = \mathbb{Z}$. Since $\mathbb{Z}$ has nonzero elements that are not units, (1) shows that $\operatorname{trdeg}(\mathbb{Z} : \mathbb{Z}) \geq 1$. We claim that all pairs of integers $a, b \in \mathbb{Z}$ are algebraically dependent over $\mathbb{Z}$. We may assume a and b to be nonzero and write

$$a = \pm \prod_{i=1}^{r} p_i^{d_i} \quad \text{and} \quad b = \pm \prod_{i=1}^{r} p_i^{e_i},$$

where the p_i are pairwise distinct prime numbers and $d_i, e_i \in \mathbb{N}_0$. Choose $n \in \mathbb{N}_0$ such that $n \geq d_i/e_i$ for all i with $e_i > 0$. Then

$$\gcd(a, b^{n+1}) = \prod_{i=1}^{r} p_i^{\min\{d_i,(n+1)e_i\}} \quad \text{divides} \quad \prod_{i=1}^{r} p_i^{ne_i} = b^n,$$

so there exist $c, d \in \mathbb{Z}$ such that

$$b^n = ca + db^{n+1}. \tag{2}$$

Hence (a, b) satisfies the polynomial $x_2^n - cx_1 - dx_2^{n+1}$, which is submonic (with respect to the lexicographic ordering with $x_1 > x_2$). This proves our claim, so

$$\operatorname{trdeg}(\mathbb{Z} : \mathbb{Z}) = 1.$$

Clearly this argument shows that every principal ideal domain that is not a field has transcendence degree 1 over itself. It is remarkable that although the transcendence degree is an algebraic invariant, the above calculation has a distinctly arithmetic flavor. ◁

Of course by specifying a particular monomial ordering "$\preceq$" in Definition 3, one gets the notions of submonicity, algebraic (in-)dependence and transcendence degree with respect to "$\preceq$". The latter will be written as $\operatorname{trdeg}_{\preceq}(A : R)$. Using this notation, Theorem 1 may be expressed by the equation

$$\operatorname{trdeg}_{\text{lex}}(R : R) = \dim(R), \tag{3}$$

which holds for every commutative ring with unity.

We are now ready to state the main result. Recall that a commutative ring with unity is called a *Jacobson ring* if every prime ideal is an intersection of maximal ideals. (Some authors use the term *Hilbert ring.*)

Theorem 5 (Kemper [5]). *Let R be a Noetherian Jacobson ring and A a finitely generated R-algebra. Then*

$$\operatorname{trdeg}(A : R) = \dim(A).$$

Since every field is a Jacobson ring, this generalizes the well-known classical result that the dimension of a finitely generated algebra over a field equals its transcendence degree. Comparing Theorem 5 to (3) raises the question whether the hypothesis that R is a Jacobson ring is really necessary. The following example shows that it is.

Example 6. Let $p \in \mathbb{Z}$ be a prime and let

$$R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b \right\}$$

be the localization of $\mathbb{Z}$ at $(p)_{\mathbb{Z}}$. Then

$$\mathbb{Q} = R\,[1/p]$$

is a finitely generated R-algebra. But we have

$$\dim(\mathbb{Q}) = 0 < \operatorname{trdeg}(R : R) \leq \operatorname{trdeg}(\mathbb{Q} : R),$$

where the first inequality follows from Example 4(1) and the second from $R \subseteq \mathbb{Q}$. So the statement of Theorem 5 fails in this example. ◁

In fact, more can be said: For every Noetherian ring that is not Jacobson, there exists an example of the above type (see [5, Remark 2.7]. So the validity of the statement of Theorem 5 characterizes Jacobson rings.

To give the reader an idea of the proof, we present a proof of Theorem 1.

Proof of Theorem 1. We prove (3), which is equivalent to Theorem 1. To this end, we claim that the equivalence

$$\operatorname{trdeg}_{\text{lex}}(R : R) \geq n \quad \Longleftrightarrow \quad \dim(R) \geq n$$

holds for all $n \in \mathbb{N}_0$. We use induction on n. There is nothing to show for $n = 0$, so we may assume $n > 0$.

First assume that $\operatorname{trdeg}_{\text{lex}}(R : R) \geq n$, so we have $a_1, \ldots, a_n \in R$ that are algebraically independent over R with respect to lex. The set

$$U := \{ f(a_n) \mid f \in R[x_n] \text{ is submonic} \} \subseteq R$$

is multiplicative. It follows from the choice of the lexicographic monomial ordering that $a_1, \ldots, a_{n-1}$ are, as elements of the localization $U^{-1}R$, algebraically independent over $U^{-1}R$ with respect to lex. By induction, there exists a strictly increasing sequence $Q_0 \subsetneqq \cdots \subsetneqq Q_{n-1}$ with $Q_i \in \operatorname{Spec}(U^{-1}R)$. This yields a strictly increasing sequence of prime ideals $P_i \in \operatorname{Spec}(R)$ with $U \cap P_i = \emptyset$. The last equation means that the class of a_n in R/P_i is algebraically independent, so Example 4(1) tells us that R/P_i is not a field. Therefore P_{n-1} is not maximal, and we conclude that $\dim(R) \geq n$.

Conversely, assume that $\dim(R) \geq n$, so we have a strictly increasing sequence $P_0 \subsetneqq \cdots \subsetneqq P_n$ with $P_i \in \operatorname{Spec}(R)$. Choose $a_n \in P_n \setminus P_{n-1}$. By Example 4(1), the class of a_n in R/P_{n-1} is algebraically independent, so $U \cap P_{n-1} = \emptyset$, with U defined as above. It follows that $\dim(U^{-1}R) \geq n-1$, so by induction there exist $a_1, \dots, a_{n-1} \in R$ that are, as elements of $U^{-1}R$, algebraically independent with respect to lex. To show that $a_1, \dots, a_n \in R$ are algebraically independent with respect to lex, let $f \in R[x_1, \dots, x_n]$ be submonic with respect to lex. Viewing f as a polynomial in the indeterminates $x_1, \dots, x_{n-1}$ and extracting its trailing coefficient $c_0 \in R[x_n]$, we conclude that c_0 is submonic, so $c_0(a_n) \in U$. But $c_0(a_n)^{-1} f(x_1, \dots, x_{n-1}, a_n) \in U^{-1}R[x_1, \dots, x_{n-1}]$ is submonic with respect to lex, so it follows from the algebraic independence of $a_1, \dots, a_{n-1}$ as elements of $U^{-1}R$ that $f(a_1, \dots, a_n) \neq 0$. This completes the proof. □

The proof of Theorem 5 is much more involved. If A is an R-algebra, it follows directly from Definition 3 that

$$\operatorname{trdeg}(A : A) \leq \operatorname{trdeg}(A : R) \leq \operatorname{trdeg}_{\mathsf{lex}}(A : R). \tag{4}$$

The proof of Theorem 5 proceeds by establishing the inequalities

$$\dim(A) \leq \operatorname{trdeg}(A : A) \tag{5}$$

for every Noetherian ring A and

$$\operatorname{trdeg}_{\mathsf{lex}}(A : R) \leq \dim(A) \tag{6}$$

for every finitely generated algebra A over a Noetherian Jacobson ring R. Together with (4), the inequalities (5) and (6) imply the theorem. The proof of (5) uses the convex cone of a monomial ordering and the Hilbert–Samuel polynomial. The proof of (6) uses an induction argument similar to the first part of the above proof of Theorem 1. For this induction, the following lemma, which may be of interest in itself, is required.

Lemma 7 ([5]). *Let a be an element of a Noetherian ring R and set*

$$U_a := \{a^n(1 + ax) \mid n \in \mathbb{N}_0,\ x \in R\}.$$

Then the localization $U_a^{-1}R$ is a Jacobson ring.

In fact, the proof of (6) extends to the more general case that A is a subalgebra of a finitely generated R-algebra. Let us call such algebras **subfinite**. So a refined version of Theorem 5 can be stated as follows.

Theorem 8 ([2, 5]). *Let A be an algebra over a Noetherian ring R. Then*

$$\operatorname{trdeg}_{\mathsf{lex}}(A : A) = \dim(A) \overset{\substack{\text{if } A \text{ is Noetherian} \\ \downarrow}}{=} \operatorname{trdeg}(A : A) \leq \\ \leq \operatorname{trdeg}(A : R) \leq \operatorname{trdeg}_{\mathsf{lex}}(A : R) \underset{\substack{\uparrow \\ \text{if } R \text{ is Jacobson, } A \text{ subfinite}}}{=} \dim(A).$$

Let us remark that the equality of dimension and transcendence degree for subfinite algebras over a field is known by Giral [3] (see also Kemper [4, Exercise 5.3]).

Example 9. Let a and b be two nonzero algebraic numbers (i.e., elements of an algebraic closure of $\mathbb{Q}$). There exists $d \in \mathbb{Z} \setminus \{0\}$ such that a and b are integral over $\mathbb{Z}[d^{-1}]$, so $A := \mathbb{Z}[a, b, d^{-1}]$ has Krull dimension 1. By Theorem 8, $\operatorname{trdeg}_{\text{lex}}(A : \mathbb{Z}) = 1$, so a, b satisfy a polynomial $f \in \mathbb{Z}[x_1, x_2]$ that is submonic with respect to lex. If $x_1^m x_2^n$ is the trailing monomial of f, then all monomials of f are divisible by x_1^m, so we may assume $m = 0$. We obtain

$$b^n = g(a, b) \cdot a + h(a, b) \cdot b^{n+1}$$

with $g, h \in \mathbb{Z}[x_1, x_2]$ polynomials. This generalizes (2). It is not so clear how the existence of such a relation follows directly from the properties of algebraic numbers. ◁

Theorem 8 has the following corollary which, to the best of the author's knowledge, is new.

Corollary 10. *Let R be a Noetherian Jacobson ring, B a subfinite R-algebra, and $A \subseteq B$ a subalgebra. Then*

$$\dim(A) \leq \dim(B).$$

Example 11. Let R be a Noetherian Jacobson ring and A a finitely generated R-algebra. Furthermore, let G be a group of automorphisms of A (as an R-algebra) and $H \subseteq G$ a subgroup. Then it follows from Corollary 10 that

$$\dim\left(A^G\right) \leq \dim\left(A^H\right),$$

even though the invariant rings need not be finitely generated (see Nagata [6]). ◁

Example 6 tells us that Corollary 10 fails if the hypothesis that R be Jacobson is dropped.

This work is still in progress. Let us point to some open questions. In Theorem 8, the lexicographic monomial ordering still plays a special role. This seems annoying. The question is whether lex can be substituted by any other monomial ordering. This is certainly the case if R is a field or, more generally, if R contains a field over which A is subfinite. We also have the following result.

Theorem 12 ([5]). *Let A be a Noetherian algebra over a Noetherian ring R with $0 \leq \dim(A) \leq 1$. Then Theorem 8 holds with* lex *replaced by any other monomial ordering "$\preceq$".*

In view of Theorem 12, a candidate that comes to mind for a ring A such that $\operatorname{trdeg}_{\preceq}(A) > \dim(A)$ for some monomial ordering "$\preceq$" is the polynomial ring $\mathbb{Z}[x]$. Using a short program written in MAGMA [1], the author tested millions of randomly selected triples of polynomials from $\mathbb{Z}[x]$ and verified that they were all algebraically dependent with respect to the graded reverse lexicographic ordering, even over the subring $\mathbb{Z}$. This prompts the following conjecture.

Conjecture 13. *Let A be a Noetherian algebra over a Noetherian ring R. Then Theorem 8 holds with* lex *replaced by any other monomial ordering "$\preceq$".*

A further, more general question is to what extent the definition of submonic polynomials given here is natural. More precisely, how far can the class of submonic polynomials be extended such that

$$\dim(A) \leq \operatorname{trdeg}(A : R) \tag{7}$$

still holds for every finitely generated algebra A over a Noetherian ring R? How far can it be shrunk such that

$$\operatorname{trdeg}(A : R) \leq \dim(A) \tag{8}$$

remains true for every finitely generated algebra A over a Noetherian Jacobson ring R? For example, (7) clearly still holds if all divisors of submonic polynomials are included, and (8) holds when restricting to polynomials that are submonic with respect to lex.

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An algorithm for computing the value-semigroup of an irreducible algebroid curve

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1 Introduction

Let K be an algebraic closed field of arbitrary characteristic. A Noetherian complete local ring A of dimension one with a coefficient field K is called an *algebroid curve* over K. If A is domain, we say that A is *irreducible*. By Cohen's theorem, A is isomorphic to $K[\![\boldsymbol{x}]\!]/I$ for some $K[\![\boldsymbol{x}]\!] = K[\![x_1, \ldots, x_r]\!]$ and $I \subset K[\![\boldsymbol{x}]\!]$. Since the integral closure of A in its fraction field is isomorphic to $K[\![t]\!]$, we may regard A as a subring of $K[\![t]\!]$. We define

$$\mathrm{S}(A) := \{\dim_K(A/\eta) \mid 0 \neq \eta \in A\} = \{\mathrm{ord}_t(\eta) \mid 0 \neq \eta \in A\},$$

where ord_t is the normalized valuation on $K[\![t]\!]$, and call it the *value-semigroup* (or semigroup of values) of A. Since $\dim_K(A/\eta_1\eta_2) = \dim_K(A/\eta_1) + \dim_K(A/\eta_2)$, $\mathrm{S}(A)$ is actually a semigroup. It is know that $\mathrm{S}(A)$ is deeply related to the singularity of A. Kunz [2] has shown that A is Gorenstein if and only if $\mathrm{S}(A)$ is symmetric. Here, we say that a numerical semigroup $H \subset \mathbb{N}$, $\gcd(H) = 1$, is symmetric if for any $n \in \mathbb{Z}$, $n \in H \Leftrightarrow m - n \notin H$ where $m = \max\{n \in \mathbb{N} \mid n \notin \mathbb{N}\}$.

We will give an algorithm for computing $\mathrm{S}(A)$. For plane curves $\mathbb{C}[\![x, y]\!]/\langle F(x, y)\rangle$, $\mathrm{S}(A)$ can be computed using resolution of singularity, or Puiseux expansion of F. For algebroid curves of arbitrary codimension, Hefez–Hernandes [1] has given an algorithm for computing $\mathrm{S}(A)$ if A is expressed as a subring of $K[\![t]\!]$, and then they mentioned a way of computing $\mathrm{S}(A)$ if A is expressed as a residue ring $K[\![\boldsymbol{x}]\!]/\mathfrak{p}$ of $K[\![\boldsymbol{x}]\!]$. In this note, we will give a more effective way to compute $\mathrm{S}(A)$ in the case where $A = K[\![\boldsymbol{x}]\!]/\mathfrak{p}$.

2 Value-semigroup

We denote by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}_+$, and $\mathbb{R}_+$, the set of non-negative integers, positive integers, and positive real numbers, respectively. For $\boldsymbol{x} = (x_1, \dots, x_r)$ and $\boldsymbol{a} = (a_1, \dots, a_r)$, we write $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \cdots x_r^{a_r}$.

Definition 2.1. For $\boldsymbol{w} = (w_1, \dots, w_r) \in \mathbb{N}_+^r$ and $0 \neq f = \sum_{\boldsymbol{a} \in \mathbb{N}^r} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}} \in K[\![\boldsymbol{x}]\!]$, $c_{\boldsymbol{a}} \in K$, we define $\mathrm{ord}_{\boldsymbol{w}}(f) = \min\{\boldsymbol{w} \cdot \boldsymbol{a} \mid c_{\boldsymbol{a}} \neq 0\} \in \mathbb{N}$, and $\mathrm{in}_{\boldsymbol{w}}(f) = \sum_{\boldsymbol{w} \cdot \boldsymbol{a} = \mathrm{ord}_{\boldsymbol{w}}(f)} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}} \in K[\boldsymbol{x}]$. We set $\mathrm{ord}_{\boldsymbol{w}}(0) = \infty$ and $\mathrm{in}_{\boldsymbol{w}}(0) = 0$. For an ideal $I \subset K[\![\boldsymbol{x}]\!]$, we define $\mathrm{in}_{\boldsymbol{w}}(I) = \langle \mathrm{in}_{\boldsymbol{w}}(f) \mid f \in I \rangle \subset K[\boldsymbol{x}]$.

For $K[\![t]\!]$, we write $\mathrm{ord}_t(-) = \mathrm{ord}_1$, and $\mathrm{in}_t(-) = \mathrm{in}_1(-)$.

Let $\mathfrak{p} \subset K[\![\boldsymbol{x}]\!]$ be a one-dimensional prime ideal such that $x_i \notin \mathfrak{p}$ for all i, and $A = K[\![\boldsymbol{x}]\!]/\mathfrak{p}$. For $f \in K[\![\boldsymbol{x}]\!]$, we define $\mathrm{int}(f; \mathfrak{p}) = \dim_K K[\![\boldsymbol{x}]\!]/\langle \mathfrak{p}, f \rangle$. Then

$$\mathrm{S}(A) = \{\mathrm{int}(f, \mathfrak{p}) \mid f \in K[\![\boldsymbol{x}]\!], f \notin \mathfrak{p}\}.$$

Let $K[\![t]\!]$ be the integral closure of A in its fraction field, and write $A = K[\![\boldsymbol{\xi}]\!]$ for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$, $\xi_i \in K[\![t]\!]$. We set

$$\begin{aligned} \phi_{\boldsymbol{\xi}} &: K[\![\boldsymbol{x}]\!] \to K[\![t]\!], & f(\boldsymbol{x}) &\mapsto f(\boldsymbol{\xi}), \\ \phi_{\mathrm{in}_t(\boldsymbol{\xi})} &: K[\boldsymbol{x}] \to K[t], & g(\boldsymbol{x}) &\mapsto g(\mathrm{in}_t(\boldsymbol{\xi})). \end{aligned}$$

From now on, we write

$$\boldsymbol{w} = (w_1, \dots, w_r), w_i = \mathrm{int}(x_i; \mathfrak{p}).$$

Note that $\boldsymbol{w} = (\mathrm{ord}_t(\xi_1), \dots, \mathrm{ord}_T(\xi_r))$. $\mathrm{S}(A)$ contains the numerical semigroup $\langle \boldsymbol{w} \rangle$ generated by $w_1, \dots, w_r$. However, the equality $\mathrm{S}(A) = \langle \boldsymbol{w} \rangle$ does not hold in general. We will give a sufficient and necessary condition for this equality.

Definition 2.2. We call $K[\mathrm{in}_t(A)] := K[\mathrm{in}_t(\eta) \mid \eta \in A] = K[t^i \mid i \in \mathrm{S}(A)]$ the *initial algebra* of A. We say that $\boldsymbol{\xi}$ is a *local SAGBI basis* of A if $K[\mathrm{in}_t(A)] = K[\mathrm{in}_t(\boldsymbol{\xi})]$, in other words, $\mathrm{S}(A) = \langle \boldsymbol{w} \rangle$.

An algorithm for computing local SAGBI bases is given in [1] Algorithm 3.2.

Proposition-Definition 2.3 (local reduction, see [1] Section 2). Let $\eta \in K[\![t]\!]$. Then there exist $q \in K[\![\boldsymbol{x}]\!]$ and $\zeta \in K[\![\boldsymbol{\xi}]\!]$ satisfying the following:

(1) $\eta = q(\boldsymbol{\xi}) + \zeta$.

(2) $\mathrm{in}_t(\zeta) \notin K[\mathrm{in}_t(\boldsymbol{\xi})]$ if $\zeta \neq 0$.

(3) $\mathrm{ord}_t(\eta) = \mathrm{ord}_{\boldsymbol{w}}(q)$ if $q \neq 0$.

We call q a quotient, and ζ a remainder of η on local reduction by $\boldsymbol{\xi}$.

Proof. We define $\eta_i \in K[\![\boldsymbol{\xi}]\!]$, $\alpha_i \in K$, and $\boldsymbol{c}_i \in \mathbb{N}^r$ inductively on i in the following manner: Set $\eta_0 = \eta$. If $\mathrm{in}_t(\eta_i) \in K[\mathrm{in}_t(\boldsymbol{\xi})]$, take $\boldsymbol{c}_i \in \mathbb{N}^r$ and $\beta_i \in K$ such that $\mathrm{ord}_t(\eta_i) = \mathrm{ord}_t(\boldsymbol{\xi}^{\boldsymbol{c}_i})$ and $\mathrm{in}_t(\eta_i) = \beta_i \,\mathrm{in}_t(\boldsymbol{\xi}^{\boldsymbol{c}_i})$. We set $\eta_{i+1} = \eta_i - \beta_i \boldsymbol{\xi}^{\boldsymbol{c}_i}$.

If $\eta_m \notin K[\mathrm{in}_t(\boldsymbol{\xi})]$ or $\eta_m = 0$ for some $m \in \mathbb{N}$, then $q := \sum_{i=0}^{m-1} \beta_i \boldsymbol{x}^{\boldsymbol{c}_i}$ and $\zeta := \eta_m$ satisfy the desired conditions. If $0 \neq \eta_i \in K[\mathrm{in}_t(\boldsymbol{\xi})]$ for all i, then $q := \sum_{i=0}^{\infty} \beta_i \boldsymbol{x}^{\boldsymbol{c}_i}$ and $\zeta := 0$ satisfy the desired conditions. □

Lemma 2.4. *Then* $\mathrm{in}_{\boldsymbol{w}}(\mathrm{Ker}\,\phi_{\boldsymbol{\xi}}) \subset \mathrm{Ker}\,\phi_{\mathrm{in}_t(\boldsymbol{\xi})}$.

Proof. Let $f \in \mathrm{Ker}\,\phi_{\boldsymbol{\xi}}$, and $f_0 = \mathrm{in}_{\boldsymbol{w}}(f) \in K[\boldsymbol{x}]$. Since the lowest order terms appearing in the expansion of $f(\boldsymbol{\xi})$ is $f_0(\mathrm{in}_t(\boldsymbol{\xi}))$ which should be also zero. Hence $f_0 \in \mathrm{Ker}\,\phi_{\mathrm{in}_t(\boldsymbol{\xi})}$. Thus $\mathrm{in}_{\boldsymbol{w}}(\mathrm{Ker}\,\phi_{\boldsymbol{\xi}}) \subset \mathrm{Ker}\,\phi_{\mathrm{in}_t(\boldsymbol{\xi})}$. □

Furthermore, the following holds.

Theorem 2.5 ([3]). *Then* $\sqrt{\mathrm{in}_{\boldsymbol{w}}(\mathrm{Ker}\,\phi_{\boldsymbol{\xi}})} = \mathrm{Ker}\,\phi_{\mathrm{in}_t(\boldsymbol{\xi})}$.

Similarly to SAGBI bases ([4] Theorem 11.4), the following holds.

Theorem 2.6 ([3]). *The following are equivalent:*

(1) $\boldsymbol{\xi}$ *is a local SAGBI basis of* A.

(2) *Any remainder of* $f(\boldsymbol{\xi})$ *on local reduction by* $\boldsymbol{\xi}$ *is zero for all* $f \in K[\![\boldsymbol{x}]\!]$.

(3) *One can take zero as a remainder of* $f(\boldsymbol{\xi})$ *on local reduction by* $\boldsymbol{\xi}$ *for all* $f \in \mathrm{Ker}(\phi_{\mathrm{in}_t(\boldsymbol{\xi})})$.

(4) $\mathrm{in}_{\boldsymbol{w}}(\mathrm{Ker}\,\phi_{\boldsymbol{\xi}}) = \mathrm{Ker}(\phi_{\mathrm{in}_t(\boldsymbol{\xi})})$.

By Theorem 2.6 (4) and Theorem 2.5, we can decide whether or not $\mathrm{S}(A) = \langle \boldsymbol{w} \rangle$ holds without computing $\boldsymbol{\xi}$.

Corollary 2.7. $\mathrm{S}(A) = \langle \boldsymbol{w} \rangle$ *if and only if* $\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p}) = \sqrt{\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p})}$.

Lemma 2.8. *Assume that* $\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p}) \neq \sqrt{\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p})}$. *Take* $f_0 \in \sqrt{\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p})} \setminus \mathrm{in}_{\boldsymbol{w}}(\mathfrak{p})$ *a* $\boldsymbol{w}$*-homogeneous element. Then there exist* $\ell \in \mathbb{N}$ *and* $\boldsymbol{c}_i \in \mathbb{N}^r$, $\alpha_i \in K^\times$ *for* $1 \leq i \leq \ell$ *such that* $\mathrm{int}(f_i; \mathfrak{p}) = \boldsymbol{c}_i \cdot \boldsymbol{w} < \mathrm{int}(f_{i+1}; \mathfrak{p})$, *and* $\mathrm{int}(f_\ell; \mathfrak{p}) \notin \sum_{i=1}^{r} \mathbb{N} w_i$ *where* $f_i = f_{i-1} - \alpha_i \boldsymbol{x}^{\boldsymbol{c}_i}$ *for* $1 \leq i \leq \ell$.

Algorithm 2.9 ([3]). Let $\mathfrak{p} \subset R = K[\![x_1, \dots, x_r]\!]$ be a one-dimensional prime ideal such that $x_i \notin \mathfrak{p}$ for all i, and set $A = R/\mathfrak{p}$. Then $\mathrm{S}(A)$ is computed as follows.

(1) Let $\boldsymbol{w} = (w_1, \dots, w_r)$, $w_i = \mathrm{int}(x_i; \mathfrak{p})$.
(2) Continue the following procedure while $\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p}) \neq \sqrt{\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p})}$.
   - Take $g \in R$ such that $\mathrm{int}(g; \mathfrak{p}) \notin \sum_{i=1}^{r} \mathbb{N} w_i$ as in Lemma 2.8.
   - Replace $\mathfrak{p}$ by $\langle \mathfrak{p}, x_{r+1} - g \rangle \subset K[\![x_1, \dots, x_{r+1}]\!]$.
   - $R := K[\![x_1, \dots, x_{r+1}]\!]$, $w_{r+1} := \mathrm{int}(g; \mathfrak{p})$, $\boldsymbol{w} = (w_1, \dots, w_r, w_{r+1}) \in \mathbb{N}^{r+1}$, $r := r + 1$,
(3) Eventually it holds that $\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p}) = \sqrt{\mathrm{in}_{\boldsymbol{w}}(\mathfrak{p})}$, and $\mathrm{S}(A) = \sum_{i=1}^{r} \mathbb{N} w_i$

For the output $\mathfrak{p}$ of Algorithm 2.9, $\text{in}_w(\mathfrak{p})$ is a prime ideal, and it is not hard to show that an ideal $I \subset K[\![\boldsymbol{x}]\!]$ is prime if $\text{in}_v(I) \subset K[\boldsymbol{x}]$ is prime for some $v \in \mathbb{N}_+^r$. Thus one can use Algorithm 2.9 to obtain an evidence of the primeness of $\mathfrak{p}$.

Example 2.10. Let $\mathfrak{p} = \langle x^3 - z^2 + x^4yz, (y^2 - xz)^2 - x^2y^5z^3\rangle \subset K[\![x, y, z]\!]$. Then $w = (8, 10, 12)$, and $\text{in}_w(\mathfrak{p}) = \langle x^3 - z^2, (y^2 - xz)^2\rangle$. Take $y^2 - xz \in \sqrt{\text{in}_w(\mathfrak{p})} \setminus \text{in}_w(\mathfrak{p})$. Then $\text{int}(y^2 - xz; \mathfrak{p}) = 51 \notin \langle 8, 10, 12\rangle$. Let $\mathfrak{p}' = \langle \mathfrak{p}, u - (y^2 - xz)\rangle \subset K[\![x, y, z, u]\!]$, and $w' = (8, 10, 12, 51)$. Then $\text{in}_{w'}(\mathfrak{p}') = \langle x^3 - z^2, u^2 - x^2y^5z^2, y^2 - xz\rangle$. As $\text{in}_{w'}(\mathfrak{p}') = \sqrt{\text{in}_{w'}(\mathfrak{p}')}$, $\text{S}(K[\![x, y, z]\!]/\mathfrak{p}) = \text{S}(K[\![x, y, z, u]\!]/\mathfrak{p}') = \langle 8, 10, 12, 51\rangle$.

Example 2.11. Let $\mathfrak{p} = \langle y^2 - xz, x^5 + z^3 - 2x^2yz - y^4\rangle \subset K[\![x, y, z]\!]$. Then $w = (6, 8, 10)$, $\text{in}_w(\mathfrak{p}) = \langle y^2 - xz, x^5 + z^3 - 2x^2yz\rangle$, and $\sqrt{\text{in}_w(\mathfrak{p})} = \langle x^3 - yz, y^2 - xz, z^2 - x^2y\rangle$. Let $g_1 = x^3 - yz$ and $g_2 = x^2y - z^2$. Then $\text{int}(g_1; \mathfrak{p}) = 19$, and $\text{int}(g_2; \mathfrak{p}) = 21$. Let $\mathfrak{p}' = \langle \mathfrak{p}, u_1 - g_1, u_2 - g_2\rangle \subset K[\![x, y, z, u_1, u_2]\!]$. and $w' = (6, 8, 10, 19, 21)$. Then $\text{in}_{w'}(\mathfrak{p}')$ is prime. Thus $\mathfrak{p}$ is prime, and $\text{S}(K[\![x, y, z]\!]/\mathfrak{p}) = \langle 6, 8, 10, 19, 21\rangle$.

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ROOTS OF EHRHART POLYNOMIALS AND SYMMETRIC δ-VECTORS

AKIHIRO HIGASHITANI

ABSTRACT. The conjecture on roots of Ehrhart polynomials, stated by Matsui, the author, Nagazawa, Ohsugi and Hibi, says that all the roots α of the Ehrhart polynomial of a Gorenstein Fano polytope of dimension d satisfy $-\frac{d}{2} \leq \mathrm{Re}(\alpha) \leq \frac{d}{2} - 1$. In this article, we observe the behaviors of roots of the generalized Ehrhart polynomials of Gorenstein Fano polytopes. As a result, we prove that this conjecture is true when the roots are real numbers or when $d \leq 5$.

INTRODUCTION

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\partial\mathcal{P}$ its boundary. Given a positive integer n, we write $i(\mathcal{P}, n) = \sharp(n\mathcal{P} \cap \mathbb{Z}^N)$, where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$. In 1950's, Ehrhart [7] succeeded in proving some fundamental properties on $i(\mathcal{P}, n)$. The numerical function $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$ which satisfies $\sharp(n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N) = (-1)^d i(\mathcal{P}, -n)$, which is called Ehrhart's "loi de réciprocité". We call $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of $\mathcal{P}$.

We define the sequence $\delta_0, \delta_1, \ldots, \delta_d$ of integers by the formula

$$(1-\lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n)\lambda^n = \sum_{j=0}^{d} \delta_j \lambda^j.$$

The sequence $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ is called the *δ-vector* of $\mathcal{P}$. Thus, $\delta_0 = 1$ and $\delta_1 = \sharp(\mathcal{P} \cap \mathbb{Z}^N) - (d+1)$. Each δ_j is nonnegative ([14]). It follows from the reciprocity law that $\delta_d = \sharp((\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N)$. In particular, one has $\delta_1 \geq \delta_d$. We refer the reader to [3] or [9] for further information on Ehrhart polynomials and δ-vectors.

Note that the Ehrhart polynomial can be expressed with the δ-vector by using the binomial coefficients as follows:

$$i(\mathcal{P}, n) = \sum_{j=0}^{d} \delta_j \binom{n+d-j}{d}.$$

On roots of Ehrhart polynomials, the following conjecture is proposed:

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Conjecture 0.1 ([2, Conjecture 1.4]). *All roots α of the Ehrhart polynomials of integral convex polytopes of dimension d satisfy*

$$-d \le \mathrm{Re}(\alpha) \le d-1,$$

where $\mathrm{Re}(\alpha)$ *denotes the real part of* $\alpha \in \mathbb{C}$.

It is proved in [2] and [6] that the conjecture is true when the roots are real numbers or when $d \le 5$. However, this has been disproved in [11] and [13]. There exists a certain counterexample of dimension 15.

A *Gorenstein Fano polytope*, which is also said to be a *reflexive polytope*, is one of the most interesting objects to study from viewpoints of both algebraic geometry on toric Fano varieties and combinatorics on Fano polytopes. Moreover, the Ehrhart polynomials and δ-vectors of Gorenstein Fano polytopes have remarkable properties. For a Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ with its δ-vector $\delta(\mathcal{P})$, it follows from [1] and [10] that the following conditions are equivalent:

- $\mathcal{P}$ is Gorenstein;
- $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ is symmetric, i.e., $\delta_i = \delta_{d-i}$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$;
- $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$.

When $\mathcal{P} \subset \mathbb{R}^d$ is a Gorenstein Fano polytope, since $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$, the roots of $i(\mathcal{P}, n)$ are distributed symmetrically in the complex plane with respect to the line $\mathrm{Re}(z) = -\frac{1}{2}$. Thus, in particular, if d is odd, then $-\frac{1}{2}$ is a root of $i(\mathcal{P}, n)$. It is known [4, Proposition 1.8] that, if all roots $\alpha \in \mathbb{C}$ of $i(\mathcal{P}, n)$ of an integral convex polytope $\mathcal{P}$ of dimension d satisfy $\mathrm{Re}(\alpha) = -\frac{1}{2}$, then $\mathcal{P}$ is unimodularly equivalent to a Gorenstein Fano polytope whose volume is at most 2^d. In [8], the roots of the Ehrhart polynomials of smooth Fano polytopes with small dimensions are completely determined.

It seems to be meaningful to investigate root distributions of Ehrhart polynomials of Gorenstein Fano polytopes. In [12], the following conjecture is also proposed:

Conjecture 0.2 ([12, Conjecture 3.10]). *All roots α of the Ehrhart polynomials of Gorenstein Fano polytopes of dimension d satisfy*

$$-\frac{d}{2} \le \mathrm{Re}(\alpha) \le \frac{d}{2} - 1.$$

This conjecture says that if we restrict the objects to Gorenstein Fano polytopes in Conjecture 0.1, then the range becomes half. We note that there also exists a certain counterexample of dimension 34. (See [13].)

On many results on roots of the Ehrhart polynomials, Stanley's nonnegativity of δ-vectors [14] plays an important role. (For example, see [4, 5] and [6].) Hence, it is natural to define the following polynomial, which is derived from [6, Definition 1.2].

Definition 0.3. Given a sequence of nonnegative real numbers $(\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{R}_{\ge 0}^{d+1}$ satisfying that these numbers are symmetric, i.e., $\delta_i = \delta_{d-i}$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$, we define the polynomial

$$f(n) = \sum_{i=0}^{d} \delta_i \binom{n+d-i}{d}$$

in n of degree d. We call $f(n)$ a *symmetric Stanley's nonnegative* or *SSNN* polynomial.

Thus, we consider the following question as a generalized form of Conjecture 0.2.

Question 0.4. *Do all roots α of an SSNN polynomial of degree d satisfy*

$$-\frac{d}{2} \le \mathrm{Re}(\alpha) \le \frac{d}{2} - 1\,?$$

In this article, we consider this question and it turns out that this is true when the roots are real numbers or when $d \le 5$. In fact,

Theorem 0.5. *Let $f(n)$ be an SSNN polynomial of degree d and $\alpha \in \mathbb{C}$ an arbitrary root of $f(n)$.*
(a) *If $\alpha \in \mathbb{R}$, then α satisfies $-\frac{d}{2} \le \alpha \le \frac{d}{2} - 1$, more strictly, $-\left\lfloor\frac{d}{2}\right\rfloor \le \alpha \le \left\lfloor\frac{d}{2}\right\rfloor - 1$.*
(b) *If $d \le 5$, then α satisfies $-\frac{d}{2} \le \mathrm{Re}(\alpha) \le \frac{d}{2} - 1$, more strictly, $-\left\lfloor\frac{d}{2}\right\rfloor \le \alpha \le \left\lfloor\frac{d}{2}\right\rfloor - 1$.*

We prove Theorem 0.5 in Section 1. Moreover, in Section 2, we make computational experiments for observing that Question 0.4 seems to be also affirmative for $d = 6$ and 7. However, this is no longer true when $d = 8$. (See Remark 2.1.)

1. A proof of Theorem 0.5

In this section, we give a proof of Theorem 0.5.

Let $f(n) = \sum_{i=0}^{d} \delta_i \binom{n+d-i}{d}$ be an SSNN polynomial of degree d. First of all, we verify that $f(n)$ satisfies

$$f(n) = (-1)^d f(-n-1). \tag{1}$$

Let

$$N_i(n) = \prod_{j=0}^{d-1}(n+d-i-j) + \prod_{j=0}^{d-1}(n+i-j)$$

for $0 \le i \le \lfloor\frac{d}{2}\rfloor - 1$ and

$$N_{\lfloor\frac{d}{2}\rfloor}(n) = \begin{cases} \prod_{j=0}^{d-1}(n+\frac{d}{2}-j), & \text{if } d \text{ is even,} \\ \prod_{j=0}^{d-1}(n+\frac{d+1}{2}-j) + \prod_{j=1}^{d}(n+\frac{d-1}{2}-j), & \text{if } d \text{ is odd.} \end{cases}$$

It then follows that

$$f(n) = \sum_{i=0}^{\lfloor\frac{d}{2}\rfloor} \frac{\delta_i N_i(n)}{d!}.$$

Since one has

$$\begin{aligned}(-1)^d N_i(-n-1) &= (-1)^d \prod_{j=0}^{d-1}(-n-1+d-i-j) + (-1)^d \prod_{j=0}^{d-1}(-n-1+i-j) \\ &= \prod_{j=0}^{d-1}(n+1-d+i+j) + \prod_{j=0}^{d-1}(n+1-i+j) \\ &= \prod_{j=0}^{d-1}(n+i-j) + \prod_{j=0}^{d-1}(n+d-i-j) = N_i(n)\end{aligned}$$

for $0 \le i \le \lfloor \frac{d}{2} \rfloor - 1$ and $(-1)^d N_{\lfloor \frac{d}{2} \rfloor}(-n-1) = N_{\lfloor \frac{d}{2} \rfloor}(n)$, we obtain $f(n) = (-1)^d f(-n-1)$.

We prove Theorem 0.5 (a) by using the above notations.

Proof of Theorem 0.5 (a). Let

$$g(n) = d! f\left(n - \frac{1}{2}\right) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \delta_i N_i \left(n - \frac{1}{2}\right).$$

Then, it is sufficient to prove that all the real roots of $g(n)$ are contained in the closed interval $[-\lfloor \frac{d}{2} \rfloor + \frac{1}{2}, \lfloor \frac{d}{2} \rfloor - \frac{1}{2}]$. Notice that $g(n)$ satisfies

$$g(n) = (-1)^d g(-n). \tag{2}$$

For $N_i\left(n - \frac{1}{2}\right), 0 \le i \le \lfloor \frac{d}{2} \rfloor$, we have the following:

$$\begin{aligned}N_i\left(n - \frac{1}{2}\right) &= \prod_{j=0}^{d-1}\left(n + d - \frac{1}{2} - i - j\right) + \prod_{j=0}^{d-1}\left(n - \frac{1}{2} + i - j\right) \\ &= \prod_{l=0}^{2i-1}\left(n - \frac{1}{2} + i - l\right) \times \\ &\qquad \left(\prod_{j=0}^{d-2i-1}\left(n - \frac{1}{2} + d - i - j\right) + \prod_{j=0}^{d-2i-1}\left(n - \frac{1}{2} - i - j\right)\right) \\ &= \prod_{l=0}^{2i-1}\left(n - \frac{1}{2} + i - l\right) M_i(n),\end{aligned}$$

where

$$M_i(n) = \prod_{j=0}^{d-2i-1}\left(n + \frac{1}{2} + i + j\right) + \prod_{j=0}^{d-2i-1}\left(n - \frac{1}{2} - i - j\right),$$

and

$$N_{\lfloor \frac{d}{2} \rfloor}\left(n - \frac{1}{2}\right) = \prod_{j=0}^{d-1}\left(n + \frac{d}{2} - \frac{1}{2} - j\right)$$

when d is even.

Let α be a real number with $\alpha > \lfloor \frac{d}{2} \rfloor - \frac{1}{2}$. On the coefficients of $M_i(n)$, it is obvious that its coefficients are nonnegative rational numbers. Thus, we have $M_i(\alpha) > 0$ since $\alpha > 0$. In addition, one has $\prod_{l=0}^{2i-1} \left(\alpha - \left(\frac{1}{2} - i + l\right)\right) > 0$ since $0 \leq l \leq 2i - 1$ and $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Hence, α cannot be a root of $g(n)$ from the nonnegativity of $\delta_0, \delta_1, \ldots, \delta_{\lfloor \frac{d}{2} \rfloor}$. Moreover, by virtue of (2), for a real number β with $\beta < -\lfloor \frac{d}{2} \rfloor + \frac{1}{2}$, β cannot be a root of $g(n)$, as desired. □

The rest part of this section is devoted to proving Theorem 0.5 (b).

1.1. **The case where $d = 2$ and 3.**

- An SSNN polynomial of degree 2 has two roots. If both of them are real numbers, then the assertion holds from Theorem 0.5 (a). If both of them are non-real numbers, then it is easy from (1) to see that each of their real parts is $-\frac{1}{2}$.
- An SSNN polynomial of degree 3 has three roots and one of them is $-\frac{1}{2}$. Thus, we consider the rest two roots. However, the same discussion as above can be done.

1.2. **The case where $d = 4$.** Let $f(n) = \frac{a}{4!}N_0(n) + \frac{b}{4!}N_1(n) + \frac{c}{4!}N_2(n)$, where a, b, c are nonnegative real numbers. Then $f(n)$ has four roots and the possible cases are as follows:

(i) those four roots are all real numbers;
(ii) two of them are real numbers and the others are non-real numbers;
(iii) those four roots are all non-real numbers.

We need not to discuss the cases (i) and (ii) by virtue of Theorem 0.5 (a) and (1). Thus, we consider the case (iii), i.e., we assume that $f(n)$ has four non-real roots. Moreover, when $a = 0$, $f(n)$ cannot have four non-real roots since both 0 and -1 are the roots of $f(n)$. Hence, we may also assume that $a \neq 0$. In addition, we may set $a = 1$ since the roots of $f(n)$ exactly coincide with those of $\frac{f(n)}{a}$.

Let

$$g(n) = 4!f\left(n - \frac{1}{2}\right) = (2 + 2b + c)n^4 + \left(43 + 7b - \frac{5}{2}c\right)n^2 + \frac{105}{8} - \frac{15}{8}b + \frac{9}{16}c.$$

Our work is to show that if the roots α of $g(n)$ are all non-real numbers, then α satisfies $-\frac{3}{2} \leq \mathrm{Re}(\alpha) \leq \frac{3}{2}$. We set

$$G(X) = (2 + 2b + c)X^2 + \left(43 + 7b - \frac{5}{2}c\right)X + \frac{105}{8} - \frac{15}{8}b + \frac{9}{16}c.$$

We consider the roots of $G(X)$. Let α and β (resp. $D(G(X))$) denote the roots of $G(X)$ (resp. the discriminant of $G(X)$). Then we may assume that $D(G(X)) < 0$. In fact, when $D(G(X)) \geq 0$, i.e., both α and β are real numbers, then the roots of

$g(n)$ are $\pm\sqrt{\alpha}, \pm\sqrt{\beta}$. Even if α (resp. β) is positive or negative, $\pm\sqrt{\alpha}$ (resp. $\pm\sqrt{\beta}$) are either real numbers or pure imaginary numbers.

Let, say, $\alpha = re^{\theta\sqrt{-1}}$ with $r > 0$ and $0 < \theta < \pi$. Then one has $\beta = \bar{\alpha} = re^{-\theta\sqrt{-1}}$. Thus, the roots of $g(n)$ are $\sqrt{r}e^{\frac{\theta}{2}\sqrt{-1}}, \sqrt{r}e^{(\pi-\frac{\theta}{2})\sqrt{-1}}, \sqrt{r}e^{-\frac{\theta}{2}\sqrt{-1}}$ and $\sqrt{r}e^{-(\pi-\frac{\theta}{2})\sqrt{-1}}$. Hence, it is enough to show that

$$0 < \mathrm{Re}(\sqrt{r}e^{\frac{\theta}{2}\sqrt{-1}}) = \sqrt{r}\cos\frac{\theta}{2} = \sqrt{r}\sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{r + r\cos\theta}{2}} \leq \frac{3}{2}.$$

Since $G(X) = (2+2b+c)(X-\alpha)(X-\beta)$, we have

$$r = \sqrt{\alpha\beta} = \sqrt{\frac{\frac{105}{8} - \frac{15}{8}b + \frac{9}{16}c}{2+2b+c}} = \frac{1}{4}\sqrt{\frac{210 - 30b + 9c}{2+2b+c}}$$

and

$$r\cos\theta = \frac{\alpha+\beta}{2} = \frac{-43 - 7b + \frac{5}{2}c}{2(2+2b+c)} = -\frac{1}{4}\cdot\frac{86 + 14b - 5c}{2+2b+c}.$$

By the way, one has

$$\begin{aligned} D(G(X)) &= \left(43 + 7b - \frac{5}{2}c\right)^2 - 4(2+2b+c)\left(\frac{105}{8} - \frac{15}{8}b + \frac{9}{16}c\right) \\ &= 4(c^2 - 4(2b+17)c + 4(4b^2 + 32b + 109)). \end{aligned}$$

Let $h(c) = \frac{D(G(X))}{4}$. Then one has $h(c) < 0$ and the range of c satisfying $h(c) < 0$ is

$$2(2b+17) - 12\sqrt{b+5} < c < 2(2b+17) + 12\sqrt{b+5}.$$

When b and c satisfy this, we have the following:

$$\begin{aligned} 4(r + r\cos\theta) &= \sqrt{\frac{210-30b+9c}{2+2b+c}} - \frac{86+14b-5c}{2+2b+c} \\ &= \sqrt{9 - 48\cdot\frac{b-4}{2+2b+c}} - 24\cdot\frac{b+4}{2+2b+c} + 5 \\ &< \sqrt{9 - 48\cdot\frac{b-4}{2+2b+2(2b+17)+12\sqrt{b+5}}} \\ &\qquad -24\cdot\frac{b+4}{2+2b+2(2b+17)+12\sqrt{b+5}} + 5 \\ &= \sqrt{9 - 8\cdot\frac{b-4}{b+6+2\sqrt{b+5}}} - 4\cdot\frac{b+4}{b+6+2\sqrt{b+5}} + 5 \quad (=: H(b)) \\ &\leq \sqrt{9 - 8\cdot\frac{-4}{6+2\sqrt{5}}} - 4\cdot\frac{4}{6+2\sqrt{5}} + 5, \quad \left(\text{since } \frac{dH(b)}{db} < 0 \text{ when } b \geq 0,\right) \\ &= 4\sqrt{5} - 2. \end{aligned}$$

Therefore, one has

$$\sqrt{\frac{r+r\cos\theta}{2}} < \sqrt{\frac{4\sqrt{5}-2}{8}} = \frac{\sqrt{2\sqrt{5}-1}}{2} < \frac{3}{2},$$

as required.

1.3. **The case where** $d=5$. Let $f(n)=\frac{a}{5!}N_0(n)+\frac{b}{5!}N_1(n)+\frac{c}{5!}N_2(n)$. Then $f(n)$ has five roots and one of them is $-\frac{1}{2}$. Thus we can deduce the case where $d=4$ and a proof can be given in the similar way, so we omit a precise proof.

2. The case where $d \geq 6$

In this section, in order to observe that Theorem 0.5 seems to be also true when $d=6$ and 7, we make computational experiments for the case where $d=6$ and 7. Moreover, we present an example which shows that Theorem 0.5 is no longer true when $d=8$. In addition, we propose a possible counterexample of Conjecture 0.2 with $d=10$, while such example is already known in [13].

Our method how to make experiments, say, the case where $d=6$, is as follows. We produce 4 nonnegative real numbers a,b,c,d at random, construct the polynomial

$$a\left(\binom{n+6}{6}+\binom{n}{6}\right)+b\left(\binom{n+5}{6}+\binom{n+1}{6}\right)+ c\left(\binom{n+4}{6}+\binom{n+2}{6}\right)+d\binom{n+3}{6},$$

compute its roots and plot them on the complex plane. In Figure 1 drawn below, we show the root distributions of a large sample (approximately 20,000) of SSNN polynomials of degree 6. Similarly, in Figure 2, we show the root distributions of a large sample (approximately 20,000) of SSNN polynomials

$$a\left(\binom{n+7}{7}+\binom{n}{7}\right)+b\left(\binom{n+6}{7}+\binom{n+1}{7}\right)+ c\left(\binom{n+5}{7}+\binom{n+2}{7}\right)+d\left(\binom{n+4}{7}+\binom{n+3}{7}\right)$$

with random nonnegative real numbers a,b,c,d. (Those are computed by `Maple`.)

Remark 2.1. (a) There exists an SSNN polynomial of degree 8 such that there is a root α which does not satisfy $-4 \leq \mathrm{Re}(\alpha) \leq 3$. In fact, if we set $(\delta_0,\delta_1,\ldots,\delta_8)=(1,0,0,0,14,0,0,0,1)$ and $f(n)=\sum_{i=0}^{8}\delta_i\binom{n+8-i}{8}$, then the roots of $f(n)$ are approximately

$$-0.5 \pm 0.44480014\sqrt{-1},\quad -0.5 \pm 1.78738687\sqrt{-1},$$
$$3.00099518 \pm 5.29723208\sqrt{-1} \text{ and } -4.00099518 \pm 5.29723208\sqrt{-1}.$$

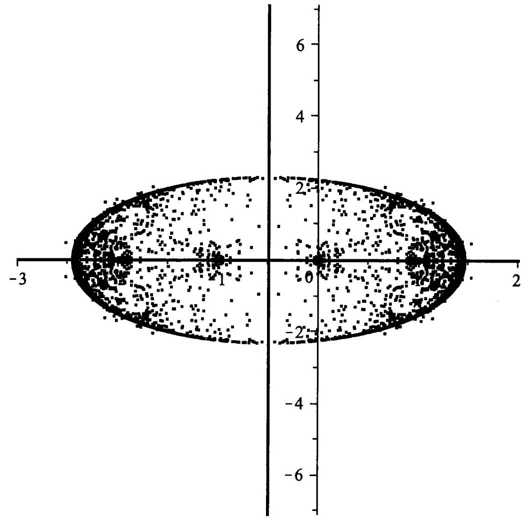

FIGURE 1. $d = 6$

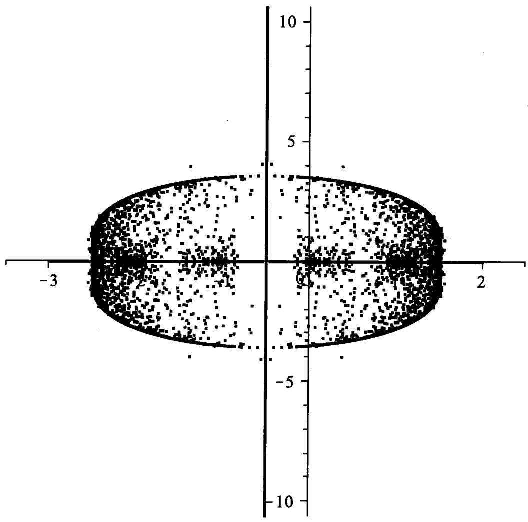

FIGURE 2. $d = 7$

On the other hand, $f(n)$ cannot be the Ehrhart polynomial of any Gorenstein Fano polytope of dimension 8 since $\delta_1 < \delta_8$.
(b) When $d = 10$, some possible candidates which are counterexamples of Conjecture 0.2 appear. For example, let $(\delta_0, \delta_1, \ldots, \delta_{10}) = (1, 1, 1, 1, 1, 23, 1, 1, 1, 1, 1)$ and $f(n) = \sum_{i=0}^{10} \delta_i \binom{n+10-i}{10}$. Then one of approximate roots of $f(n)$ is

$$4.02470021 + 8.22732653\sqrt{-1}.$$

However, in a recent paper [13], a counterexample of Conjecture 0.2 is provided. There exists a Gorenstein Fano polytope of dimension 34 whose Ehrhart polynomial has a root α which violates $-17 \leq \mathrm{Re}(\alpha) \leq 16$.

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On the Auslander-Bridger type approximation of modules

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Throughout this article, let R be a commutative noetherian complete local ring with maximal ideal $\mathfrak{m}$ and residue field k. All modules considered in this article are assumed to be finitely generated. An R-module C is said to be *semidualizing* if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. Various homological dimensions with respect to a fixed semidualizing module C such as C-projective dimension are invented and investigated. Here the *C-projective dimension* of a nonzero R-module M, denoted by $C\text{-proj.dim}_R M$, is defined as the infimum of integers n such that there is an exact sequence of the form

$$0 \to C^{b_n} \to C^{b_{n-1}} \to \cdots \to C^{b_1} \to C^{b_0} \to M \to 0,$$

where each b_i is a positive integer. We denote by $\mathrm{mod}(R)$ the category of finitely generated R-modules, by $\mathcal{T}_n^C$ the full subcategory of $\mathrm{mod}(R)$ consisting of all modules X such that $\mathrm{Tor}_i^R(X, C) = 0$ for any $1 \leq i \leq n$, by $\mathcal{C}_n^C$ the full subcategory of $\mathrm{mod}(R)$ consisting of all modules X such that $X \cong Y \otimes_R C$ for some module $Y \in \mathcal{T}_n^C$. A free module of rank one is a typical example of a semidualizing module.

Let M be an R-module. Let

$$\cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

be a minimal free resolution of M. We define the *C-transpose* of M to be the cokernel of the map $\mathrm{Hom}(\partial_1, C) : \mathrm{Hom}(F_0, C) \to \mathrm{Hom}(F_1, C)$, and denote it by $\mathrm{Tr}_C M$. We say that M is *n-C-torsionfree* if the R-modules $\mathrm{Ext}^i_R(\mathrm{Tr}_C M, C)$ equal to zero for all $1 \leq i \leq n$.

If an R-module M belongs to $\mathcal{C}_n^C$, then there exists an exact sequence

$$C^{b_{n+1}} \xrightarrow{\partial_{n+1}} C^{b_n} \xrightarrow{\partial_n} C^{b_{n-1}} \to \cdots \to C^{b_1} \xrightarrow{\partial_1} C^{b_0} \xrightarrow{\partial_0} M \to 0.$$

We define the *C-transpose prime* of M to be the cokernel of the map $\mathrm{Hom}(\partial_1, C)$: $\mathrm{Hom}(C^{b_0}, C) \to \mathrm{Hom}(C^{b_1}, C)$, and denote it by $\mathrm{Tr}'_{(C,\partial_1)} M$. And we define the *$i$-$C$-syzygy* of M to be the image of the map $\partial_i : C^{b_i} \to C^{b_{i-1}}$, and denote it by $\Omega^i_{(C,\partial.)} M$ for each $0 \leq i \leq n+1$.

The following two theorems is well-known as Auslander-Bridger approximation theorem and Cohen-Macaulay approximation theorem.

Theorem 1 ([1]) The following are equivalent for a finitely generated R-module M:

(1) $\Omega_R^n M$ is *n-torsionfree*.

(2) There exists an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules such that $\mathrm{proj.dim}_R Y < n$, $\mathrm{Ext}_R^i(X, R) = 0$ $(1 \le i \le n)$ and X is isomorphic to $\mathrm{Tr}_R \Omega_R^n \mathrm{Tr}_R \Omega_R^n M$.

Theorem 2 ([2]) Let R be a Cohen-Macaulay local ring with the canonical module. Then, for every finitely generated R-module M, there exists an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules such that $\mathrm{inj.dim}_R Y < n$ and X is a maximal Cohen-Macaulay R-module.

Takahashi unifies above two approximation theorems using a semidualizing R-module. The approximation theorem of Takahashi is stated as follows.

Theorem 3 ([5]) Let M and C be finitely generated R-modules. Assume that C is semidualizing. Then the following conditions on M are equivalent:

(1) $\Omega_R^n M$ is *n-C-torsionfree*.

(2) There exists an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules such that $C\text{-proj.dim}_R Y < n$ and $\mathrm{Ext}_R^i(X, C) = 0$ $(1 \le i \le n)$.

Our purpose is to give a middle term of the sequence in Theorem 3 explicitly. To prove our theorem, we establish the following lemma.

Lemma 4 Let M and C be finitely generated R-modules. Assume that C is semidualizing, M belongs to $\mathcal{C}_0^C$ and $\mathrm{Ext}_R^i(\mathrm{Tr}'_C M, C) = 0$ for $1 \le i \le n$, then M is *n-C-torsionfree*.

The main result in this article is the following theorem.

Theorem 5 Let M be an R-module and let $n \ge 0$.

If M belongs to $\mathcal{C}_n^C$ and $\mathrm{Ext}_R^i(\mathrm{Tr}'_C \Omega_C^n M, C) = 0$ for $1 \le i \le n$, then there exists an exact sequence

$$0 \to T \to \mathrm{Tr}_C \Omega_R^n \mathrm{Tr}'_C \Omega_C^n M \to M \to 0$$

of R-modules with $C\text{-proj.dim}_R T < n$.

We obtain another approximation theorem is the following.

Theorem 6 Let M be an R-module and let $n \ge 0$.

If $\mathrm{Tr}_R \Omega_R^n M$ belongs to $\mathcal{T}_n^C$ and $\Omega_R^n M$ is n-C-torsionfree, then there exists an exact sequence

$$0 \to T \to \mathrm{Tr}'_C \Omega_C^n \mathrm{Tr}_C \Omega_R^n M \to M \to 0$$

of R-modules with $\mathrm{proj.dim}_R T < n$.

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SUBCATEGORIES OF EXTENSION MODULES BY SERRE SUBCATEGORIES

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ABSTRACT. We consider subcategories consisting of the extensions of modules in two given Serre subcategories to nd a method of constructing Serre subcategories of the module category. We shall give a criterion for this subcategory to be a Serre subcategory.

INTRODUCTION

Through this paper, R is a commutative noetherian ring and all modules are unitary. We denote by R-Mod the category of R-modules and by R-mod the full subcategory consisting of finitely generated R-modules.

In [2], P. Gabriel showed that one has lattice isomorphisms between the set of Serre subcategories of R-mod, the set of Serre subcategories of R-Mod which are closed under arbitrary direct sums and the set of specialization closed subsets of Spec(R). By this result, Serre subcategories of R-mod are classified. However, it has not yet classified Serre subcategories of R-Mod.

The main purpose in this paper is to give a way of constructing Serre subcategories of R-Mod. To do this, we shall consider subcategory consisting of extension modules given by two Serre subcategories. In particular, we give a necessary and sufficient condition for this subcategory to be a Serre subcategory.

1. THE DEFINITION AND BASIC PROPERTIES OF A SUBCATEGORY OF EXTENSION MODULES

We assume that all full subcategories of R-Mod are closed under isomorphisms. We recall that a subcategory $\mathcal{S}$ of R-Mod is said to be Serre subcategory if the following condition is satisfied: For any short exact sequence

$$0 \to L \to M \to N \to 0$$

of R-modules, it holds that M is in $\mathcal{S}$ if and only if L and N are in $\mathcal{S}$. In other words, $\mathcal{S}$ is called a Serre subcategory if it is closed under submodules, quotient modules and extensions.

We give the definition of a subcategory of extension modules by Serre subcategories.

Definition 1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We denote by $(\mathcal{S}_1, \mathcal{S}_2)$ a subcategory consisting of R-modules M with a short exact sequence

$$0 \to X \to M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$, that is

$$(\mathcal{S}_1, \mathcal{S}_2) = \left\{ M \in R\text{-Mod} \quad \begin{array}{c} \text{there are } X \in \mathcal{S}_1 \text{ and } Y \in \mathcal{S}_2 \text{ such that} \\ 0 \to X \to M \to Y \to 0 \\ \text{is a short exact sequence.} \end{array} \right\}.$$

Remark 2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod.

(1) Since the zero module belongs to any Serre subcategory, one has $\mathcal{S}_1 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ and $\mathcal{S}_2 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

(2) It holds $\mathcal{S}_1 \supseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_1$.

(3) It holds $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_2$.

(4) A subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under finite direct sums.

Example 3. We denote by $\mathcal{S}_{f.g.}$ the subcategory consisting of finitely generated R-modules and by $\mathcal{S}_{Artin}$ the subcategory consisting of Artinian R-modules. If R is a complete local ring, then a subcategory $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is known as the subcategory consisting of Matlis re exive R-modules. Therefore, $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is a Serre subcategory of R-Mod.

Proposition 4. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. Then a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under submodules and quotient modules.*

Proof. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. We assume that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ and shall show that L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$.

It follows from the definition of $(\mathcal{S}_1, \mathcal{S}_2)$ that there exists a short exact sequence

$$0 \to X \xrightarrow{\varphi} M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Then we can construct the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \cap L & \longrightarrow & X & \longrightarrow & \dfrac{X}{X \cap L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \varphi & & \downarrow \overline{\varphi} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{L}{X \cap L} & \longrightarrow & Y & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of R-modules with exact rows and columns where $\overline{\varphi}$ is a natural map induced by φ and $N' = \mathrm{Coker}(\overline{\varphi})$. Since each subcategory $\mathcal{S}_i$ is closed under submodules and quotient modules, we can see that $X \cap L$, $X/(X \cap L)$ are in $\mathcal{S}_1$ and $L/(X \cap L)$, N' are in $\mathcal{S}_2$. Consequently, L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$. □

Here, a natural question arises.

Question. Is a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$?

The following example shows that a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ needs not be a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$.

Example 5. We shall see that the subcategory $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ needs not be closed under extensions.

Let R be a one dimensional Gorenstein local ring with a maximal ideal $\mathfrak{m}$. Then one has a minimal injective resolution

$$0 \to R \to \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}\mathfrak{p} = 0}} E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{m}) \to 0$$

of R. ($E_R(M)$ denotes the injective hull of an R-module M.) We note that R and $E_R(R/\mathfrak{m})$ are in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$.

Now, we assume that a subcategory $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is closed under extensions. Then $E_R(R) = \oplus_{\mathrm{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p})$ is in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$. It follows from the definition of $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ that there exists an Artinian R-submodule X of $E_R(R)$ such that $E_R(R)/X$ is a finitely generated R-module.

If $X = 0$, then $E_R(R)$ is a finitely generated injective R-module. It follows from the Bass formula that one has $\dim R = \operatorname{depth} R = \operatorname{inj\,dim} E_R(R) = 0$. However, this equality contradicts $\dim R = 1$. On the other hand, if $X \neq 0$, then X is a non-zero Artinian R-module. Therefore, one has $\mathrm{Ass}_R(X) = \{\mathfrak{m}\}$. Since X is an R-submodule of $E_R(R)$, one has

$$\mathrm{Ass}_R(X) \subseteq \mathrm{Ass}_R(E_R(R)) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}\,\mathfrak{p} = 0\}.$$

This is contradiction as well.

2. The main result

In this section, we shall give a criterion for a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ to be a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$.

We start to show that the following assertion holds.

Lemma 6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-*Mod*. We suppose that a sequence $0 \to L \to M \to N \to 0$ of R-modules is exact. Then the following assertions hold.*

(1) *If $L \in \mathcal{S}_1$ and $N \in (\mathcal{S}_1, \mathcal{S}_2)$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

(2) *If $L \in (\mathcal{S}_1, \mathcal{S}_2)$ and $N \in \mathcal{S}_2$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) We assume that L is in $\mathcal{S}_1$ and N is in $(\mathcal{S}_1, \mathcal{S}_2)$. Since N belongs to $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a short exact sequence

$$0 \to X \to N \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Then we consider the following pull buck diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y & = & Y & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of R-modules with exact rows and columns. Since $\mathcal{S}_1$ is a Serre subcategory, it follows from the first row in the diagram that X' belongs to $\mathcal{S}_1$. Consequently, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by the middle column in the diagram.
(2) We can show that the assertion holds by the similar argument in (1). □

Now, we can show the main purpose of this paper.

Theorem 7. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-*Mod*. Then the following conditions are equivalent:*

(1) *A subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory;*

(2) *One has $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) $\Rightarrow$ (2) We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$. By the definition of a subcategory $(\mathcal{S}_2, \mathcal{S}_1)$, there exists a short exact sequence

$$0 \to Y \to M \to X \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. We note that X and Y are also in $(\mathcal{S}_1, \mathcal{S}_2)$. Since a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extensions by the assumption (1), we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$.
(2) $\Rightarrow$ (1) We only have to prove that a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extensions by Proposition 4. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules such that L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$. We shall show that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$.

Since L is in $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a short exact sequence

$$0 \to S \to L \to L/S \to 0$$

of R-modules where S is in $\mathcal{S}_1$ such that L/S is in $\mathcal{S}_2$. We consider the following push out diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L/S & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

of R-modules with exact rows and columns. Next, since N is in $(\mathcal{S}_1, \mathcal{S}_2)$, we have a short exact sequence

$$0 \to T \to N \to N/T \to 0$$

of R-modules where T is in S_1 such that N/T is in $\mathcal{S}_2$. We consider the following pull back diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/S & \longrightarrow & P' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/S & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N/T & = & N/T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

of R-modules with exact rows and columns.

In the first row of the second diagram, since L/S is in $\mathcal{S}_2$ and T is in $\mathcal{S}_1$, P' is in $(\mathcal{S}_2, \mathcal{S}_1)$. Now here, it follows from the assumption (2) that P' is in $(\mathcal{S}_1, \mathcal{S}_2)$. Next, in the middle column of the second diagram, we have the short exact sequence such that P' is in $(\mathcal{S}_1, \mathcal{S}_2)$ and N/T is in $\mathcal{S}_2$. Therefore, it follows from Lemma 6 that P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Finally, in the middle column of the first diagram, there exists the short exact sequence such that S is in $\mathcal{S}_1$ and P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Consequently, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by Lemma 6.

The proof is completed. □

Corollary 8. *A subcategory $(\mathcal{S}_{f.g.}, \mathcal{S})$ is a Serre subcategory for a Serre subcategory $\mathcal{S}$ of* R-Mod.

Proof. Let $\mathcal{S}$ be a Serre subcategory of R-Mod. To prove our assertion, it is enough to show that one has $(\mathcal{S}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S})$ by Theorem 7. Let M be in $(\mathcal{S}, \mathcal{S}_{f.g.})$. Then there exists a short exact sequence $0 \to Y \to M \to M/Y \to 0$ of R-modules where Y is in $\mathcal{S}$ such that M/Y is in $\mathcal{S}_{f.g.}$. It is easy to see that there exists a finitely generated R-submodule X of M such that $M = X + Y$. Since $X \oplus Y$ is in $(\mathcal{S}_{f.g.}, \mathcal{S})$ and M is a homomorphic image of $X \oplus Y$, M is in $(\mathcal{S}_{f.g.}, \mathcal{S})$ by Proposition 4. □

Remark 9. A Serre subcategory is closed under arbitrary direct sums if it is obtained from Gabriel's correspondence. Therefore, if $(\mathcal{S}_{\text{f. g.}}, \mathcal{S}) \neq R$-Mod, then $(\mathcal{S}_{\text{f. g.}}, \mathcal{S})$ is a Serre subcategory which is not obtained from Gabriel's correspondence.

We note that a subcategory $\mathcal{S}_{Artin}$ consisting of Artinian R-modules is a Serre subcategory which is closed under injective hulls. (Also see [1, Example 2.4].) Therefore we can see that a subcategory $(\mathcal{S}, \mathcal{S}_{Artin})$ is also Serre subcategory for a Serre subcategory of R-Mod by the following assertion.

Corollary 10. *Let $\mathcal{S}_2$ be a Serre subcategory of* R-Mod *which is closed under injective hulls. Then a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory for a Serre subcategory $\mathcal{S}_1$ of* R-Mod.

Proof. By Theorem 7, it is enough to show that one has $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$ and shall show that M is in $(\mathcal{S}_1, \mathcal{S}_2)$. Then there exists a short exact sequence

$$0 \to Y \to M \to X \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Since $\mathcal{S}_2$ is closed under injective hulls, we note that the injective hull $E_R(Y)$ of Y is also in $\mathcal{S}_2$. We consider a push out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E_R(Y) & \longrightarrow & T & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows and injective vertical maps. The second exact sequence splits, and we have an injective homomorphism $M \to X \oplus E_R(Y)$. Since there is a short exact sequence

$$0 \to X \to X \oplus E_R(Y) \to E_R(Y) \to 0$$

of R-modules, the R-module $X \oplus E_R(Y)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$. Consequently, we see that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$ by Proposition 4.

The proof is completed. □

Example 11. Let R be a domain but not a field and let Q be a field of fractions of R. We denote by $\mathcal{S}_{Tor}$ a subcategory consisting of torsion R-modules, that is

$$\mathcal{S}_{Tor} = \{M \in R\text{-Mod} \mid M \otimes_R Q = 0\}.$$

Then we shall see that one has

$$(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor}) = \{M \in R\text{-Mod} \mid \dim_Q M \otimes_R Q < \infty\}.$$

Therefore, a subcategory $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ is a Serre subcategory by Corollary 8, but a subcategory $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$ is not closed under extensions by Theorem 7.

First of all, we shall show that the above equality holds. We suppose that M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Then there exists a short exact sequence

$$0 \to X \to M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_{f.g.}$ and Y is in $\mathcal{S}_{Tor}$. We apply an exact functor $- \otimes_R Q$ to this sequence. Then we see that one has $M \otimes_R Q \cong X \otimes_R Q$ and this module is a finite dimensional Q-vector space.

Conversely, let M be an R-module with $\dim_Q M \otimes_R Q < \infty$. Then we can denote $M \otimes_R Q = \sum_{i=1}^{n} Q(m_i \otimes 1_Q)$ with $m_i \in M$ and the unit element 1_Q of Q. We consider a short exact sequence

$$0 \to \sum_{i=1}^{n} Rm_i \to M \to M/\sum_{i=1}^{n} Rm_i \to 0$$

of R-modules. It is clear that $\sum_{i=1}^{n} Rm_i$ is in $\mathcal{S}_{f.g.}$ and $M/\sum_{i=1}^{n} Rm_i$ is in $\mathcal{S}_{Tor}$. So M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Consequently, the above equality holds.

Next, it is clear that $M \otimes_R Q$ has finite dimension as Q-vector space for an R-module M of $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Thus, one has $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$.

Finally, we shall see that a field of fractions Q of R is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ but not in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$, so one has $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Indeed, it follows from $\dim_Q Q \otimes_R Q = 1$ that Q is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. On the other hand, we assume that Q is in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Since R is a domain, a torsion R-submodule of Q is only the zero module. It means that Q must be a finitely generated R-module. But, this is a contradiction.

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A note on Cohen-Macaulay algebras with straightening law

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1 Introduction

Let A be a graded algebra with straightening law (ASL for short) over a field K on a poset P and A_{dis} the discrete counterpart of A, that is, the discrete ASL over K on P. It is known that $\dim A_{\mathrm{dis}} = \dim A = \mathrm{rank} P + 1$. On the other hand, $\mathrm{depth} A_{\mathrm{dis}} \leq \mathrm{depth} A$ but little is known except for this fact. The present author showed that if A_{dis} is Buchsbaum, then the equality holds.

If α is a minimal element of $\mathrm{ind}(A)$, the indiscrete part of A, then the associated graded ring $G = \mathrm{Gr}_{\alpha A}(A)$ of the filtration $\{\alpha^n A\}$ is an ASL on P with $\mathrm{ind}(G) = \mathrm{ind}(A) \setminus \{\alpha\}$. And it is well known that there is a flat family whose general fiber is A and G is a special fiber. In particular, there is a sequence $A = A_0, A_1, \ldots, A_s = A_{\mathrm{dis}}$ of ASL's over K on P such that for each i, there is a flat family whose general fiber is A_{i-1} and A_i is a special fiber. Therefore, if there is a Cohen-Macaulay ASL on P whose discrete counterpart is not Cohen-Macaulay, then there is a Cohen-Macaulay ASL B and $\alpha \in \min(\mathrm{ind}(B))$ such that $\mathrm{Gr}_{\alpha B}(B)$ is not Cohen-Macaulay.

In this note, we show that if there is a Cohen-Macaulay ASL A whose discrete counterpart is not Cohen-Macaulay, then if we take A to be of minimal dimension among such ASL's, there is a minimal element α of P such that $\mathrm{Gr}_{\alpha A}(A)$ is not Cohen-Macaulay.

2 Preliminaries

All rings and algebras in this note are commutative with identity element. Let K be a field and P a finite partially ordered set (poset for short). We call a totally ordered subset of P a chain in P. The length of a chain X in

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P is $\#X - 1$. The maximal length of the chains in P is called the rank of P and is denoted as rank$P$. If every maximal chain in P has the same length, we say that P is pure. We denote by $K[P]$ the Stanley-Reisner ring of the chain complex $\Delta(P)$ of P over K.

A monomial on P is a formal power product $\xi_1^{e_1}\xi_2^{e_2}\cdots\xi_s^{e_s}$ of elements in P. The support $\mathrm{supp}(\xi_1^{e_1}\xi_2^{e_2}\cdots\xi_s^{e_s})$ of $\xi_1^{e_1}\xi_2^{e_2}\cdots\xi_s^{e_s}$ is $\{\xi_i \mid e_i > 0\}$. A monomial μ is called a standard monomial if $\mathrm{supp}(\mu)$ is a totally ordered subset of P or $\emptyset$. A poset ideal of P is a subset I of P such that $x \in I$ and $y < x$ imply $y \in I$. Note $\emptyset$ is a poset ideal of P.

Definition 2.1 Let A be a graded K-algebra and $\varphi\colon P \to A$ is an injective map. A is an algebra with straightening law (ASL for short) over K on P with structure map φ if we embed P in A by φ, the following conditions are satisfied.

(ASL-0) $A = \bigoplus_{n\geq 0} A_n$ is a graded ring with $A_0 = K$ and every element of P is a homogeneous element of positive degree.

(ASL-1) The standard monomials on P form a K-vector space basis of A.

(ASL-2) If α, $\beta \in P$ and $\alpha \not\sim \beta$, then

$$\alpha\beta = \sum_{\mu:\text{standard}} b_{\alpha\beta\mu}\mu$$

where $0 \neq b_{\alpha\beta\mu} \in K$ and $\min(\mathrm{supp}(\mu)) < \alpha$, β. (This is called the straightening relation.)

The right hand side of the straightening relation may be the empty sum, i.e., $\alpha\beta = 0$. If $\alpha\beta = 0$ for any α, $\beta \in P$ with $\alpha \not\sim \beta$, we say that A is the discrete ASL over K on P. If A is an ASL over K on P, we say the discrete ASL over K on P the discrete counterpart of A and denote A_{dis}. We set

$$\mathrm{ind}A := \bigcup_{b_{\alpha\beta\mu}\neq 0} \mathrm{supp}(\mu)$$

and call the indiscrete part of A. Note that if α is an element of P such that there is no $x \in \mathrm{ind}A$ such that $x < \alpha$, then $\alpha\beta = 0$ for any $\beta \in P$ with $\alpha \not\sim \beta$.

Fact 2.2 $\dim A = \mathrm{rank}P + 1$.

Fact 2.3 *There is a flat family whose general fiber is A and A_{dis} is a special fiber. In particular,*

$$\mathrm{depth}A_{\mathrm{dis}} \leq \mathrm{depth}A.$$

Fact 2.4 ([Miy1, Theorem 4.4] see also [Miy2, Theorem 1.2]) *If* A_{dis} *is Buchsbaum, then* $\text{depth}A_{\text{dis}} = \text{depth}A$.

Fact 2.5 ([Miy1, Corollary 3.4]) *If there is a Cohen-Macaulay ASL on* P*, then* P *is pure.*

Definition 2.6 Let Ω be a subset of P. If ΩA is a K-vector space with basis $\{\mu \mid \mu$ is a standard monomial and $\text{supp}(\mu) \cap \Omega \neq \emptyset\}$, we say that Ω is a standard subset of P.

Remark 2.7 If Ω is a standard subset of P, then $A/\Omega A$ is an ASL on $P \setminus \Omega$.

Fact 2.8 ([DEP, Proposition 1.2]) *Let* Ω *be a subset of* P*. If* $x \in \Omega$*,* $y < x$ *and* $y \in \text{ind}A$ *imply* $y \in \Omega$*, then* Ω *is a standard subset. In particular, a poset ideal is a standard subset.*

Let

$$\mathcal{F}: \ A = I_0 \supset I_1 \supset I_2 \supset \cdots$$

be a filtration of homogeneous ideals of A such that $\bigcap_{n \geq 1} I_n = (0)$. For $a \in A$ with $a \neq 0$ set $\text{ord}(a) := \max\{n \mid a \in I_n\}$. For a monomial $\xi_1^{e_1} \cdots \xi_s^{e_s}$ on P, we set $\text{eord}(\xi_1^{e_1} \cdots \xi_s^{e_s}) := \sum_{i=1}^{s} e_i \text{ord}(\xi_i)$.

Definition 2.9 $\mathcal{F}$ is called a standard filtration if I_n is a K-vector space with basis $\{\mu \mid \mu$ is a standard monomial and $\text{eord}(\mu) \geq n\}$ for any n.

Let $\mathcal{R}_{\mathcal{F}}(A)$ be the Rees algebra with respect to $\mathcal{F}$, i.e., $\mathcal{R}_{\mathcal{F}}(A) = A[I_n T^n \mid n \geq 0]$, where $A[T]$ the polynomial ring with variable T and $\text{Gr}_{\mathcal{F}}(A)$ the associated graded ring. Then the following fact is known.

Fact 2.10 ([DEP, Corollary 2.2]) *If* $\mathcal{F}$ *is a standard filtration, then* $\text{Gr}_{\mathcal{F}}(A)$ *is an ASL over* K *on* P *with structure map* $\xi \mapsto \xi T^{\text{ord}(\xi)} + I_{\text{ord}(\xi)+1} \in I_{\text{ord}(\xi)}/I_{\text{ord}(\xi)+1}$*. And there is a flat family whose general fiber is* A *and* $\text{Gr}_{\mathcal{F}}(A)$ *is a special fiber. In particular,* $\text{depth}\text{Gr}_{\mathcal{F}}(A) \leq \text{depth}A$.

Let α be an element in P such that if $\beta < \alpha$ then $\beta \notin \text{ind}(A)$. Then it is easily verified that the filtration

$$A \supset \alpha A \supset \alpha^2 a \supset \cdots$$

is a standard filtration with $\text{ord}(\alpha) = 1$ and $\text{ord}(\beta) = 0$ for $\beta \in P \setminus \{\alpha\}$ and $\text{Gr}_{\alpha A}(A)$ is an ASL on P with $\text{ind}(\text{Gr}_{\alpha A}(A)) = \text{ind}(A) \setminus \{\alpha\}$ ([DEP, Theorem 3.1]).

Note that $\mathcal{R}_{\mathcal{F}}(A)$ and $\text{Gr}_{\mathcal{F}}(A)$ are bigraded rings. We denote the degree inherited form A as the first entry and the degree defined by the Rees algebra structure as the second entry.

3 Counter example of minimal dimension

In this section, we show that if there is a Cohen-Macaulay graded ASL A on a poset P such that A_{dis} is not Cohen-Macaulay and A is of minimal dimension among such examples, there is a minimal element α of P such that $\mathrm{Gr}_{\alpha A}(A)$ is not Cohen-Macaulay.

First let us recall the following result which is contained in the proof of [Miy1, Theorem 4.4].

Lemma 3.1 *Let α be a minimal element of $\mathrm{ind}(A)$ and set $G = \mathrm{Gr}_{\alpha A}(A)$. If $e = \mathrm{depth} G < \mathrm{depth} A$, then $[H_M^e(G)]_{(u,0)} = 0$ for any $u \in \boldsymbol{Z}$, where M is the unique bigraded maximal ideal of the Rees algebra $\mathcal{R}_{\alpha A}(A)$.*

Now let α be an element of P such that $\{x \in P \mid x < \alpha\} \cap \mathrm{ind}(A) = \emptyset$ and set $\Omega = \{x \in P \mid x \not> \alpha\}$, $G = \mathrm{Gr}_{\alpha A}(A)$, $\mathfrak{m}$ the irrelevant maximal ideal of A and M the unique bigraded maximal ideal of the Rees algebra $\mathcal{R}_{\alpha A}(A)$. Then, by Fact 2.8, Ω is a standard subset of P and $A/\Omega A \simeq \alpha A$ as A-modules, since $(0 :_A \alpha) = \Omega A$.

Proposition 3.2 *In the situation above, if A is Cohen-Macaulay, then*

$$\mathrm{depth} G = \mathrm{depth} \alpha A = \mathrm{depth} A/\Omega A.$$

Proof Set $d = \dim A$. Let $\xi_1, \xi_2, \ldots, \xi_d$ be a homogeneous system of parameters of A. We compute $H_M^i(G)$ by the Čech complex with respect to $\xi_1, \xi_2, \ldots, \xi_d, \alpha T$. Denote the Čech complex with respect to $\xi_1, \xi_2, \ldots, \xi_d$ by $C(-)^\bullet$. Then the Čech complex with respect to $\xi_1, \xi_2, \ldots, \xi_d, \alpha T$ is the double complex

$$\begin{array}{c} C(-)^\bullet \\ \downarrow \\ C(-_{\alpha T})^\bullet, \end{array} .$$

Note that

$$\bigoplus_{n \in \boldsymbol{Z}} (G_\alpha)_{(n,i)} \simeq \alpha A/\alpha^2 A$$

for any $i \in \boldsymbol{Z}$ and

$$\bigoplus_{n \in \boldsymbol{Z}} G_{(n,i)} \simeq \begin{cases} \alpha A/\alpha^2 A & i > 0, \\ A/\alpha A & i = 0, \\ 0 & i < 0. \end{cases}$$

Now first suppose that $\mathrm{depth} G = d$. Then $[H^i(C(G_{\alpha T})^\bullet)]_{(n,j)} = 0$ for any $i, j, n \in \boldsymbol{Z}$ with $j < 0$ and $i < d-1$, since $G_{(n,j)} = 0$, $H_M^{i+1}(G) = 0$ and ξ_1,

$\xi_2, \ldots, \xi_d$ are contained in $\bigoplus_{n \in \boldsymbol{Z}} \mathcal{R}_{\alpha A}(A)_{(n,0)}$. Therefore, $H^i_{\mathfrak{m}}(\alpha A/\alpha^2 A) = 0$ for any i with $i < d-1$ and $H^i_{\mathfrak{m}}(\alpha A) = 0$ for any i with $i < d$. It follows that

$$\mathrm{depth} A/\Omega A = \mathrm{depth}(\alpha A) \geq d.$$

Since $\dim A/\Omega A = d$, we see that

$$\mathrm{depth} A/\Omega A = \mathrm{depth}(\alpha A) = d = \mathrm{depth} G.$$

Next suppose that $\mathrm{depth} G = e < d$. Then by Lemma 3.1, we see that $[H^e_M(G)]_{(n,0)} = 0$ for any $n \in \boldsymbol{Z}$. And since $H^e_M(G)$ is a subquotient of a local cohomology module of a Stanley-Reisner ring $K[P]$, we see that

$$[H^e_M(G)]_{(n,i)} = 0 \quad \text{if } n > 0 \text{ or } i > 0.$$

Therefore, there is $j < 0$ and $n \in \boldsymbol{Z}$ such that $[H^e_M(G)]_{(n,j)} \neq 0$. So we see that

$$H^{e-1}(C(G_{\alpha T})^\bullet) \neq 0 \quad \text{and} \quad H^i(C(G_{\alpha T})^\bullet) = 0 \quad \text{for } i < e-1.$$

Therefore, since $\bigoplus_{n \in \boldsymbol{Z}} (G_{\alpha T})_{(n,j)} \simeq \alpha A/\alpha^2 A$ for any $j \in \boldsymbol{Z}$, we see that

$$\mathrm{depth}(\alpha A/\alpha^2 A) = e - 1.$$

It follows that

$$\mathrm{depth} A/\Omega A = \mathrm{depth} \alpha A = e = \mathrm{depth} G.$$

▮

Now we prove the following

Theorem 3.3 *Suppose that there is a Cohen-Macaulay ASL whose discrete counterpart is not Cohen-Macaulay. Set* $d = \min\{\dim A \mid A$ *is a Cohen-Macaulay ASL whose discrete counterpart is not Cohen-Macaulay*$\}$ *and let* A *be a Cohen-Macaulay ASL on a poset* P *of dimension* d *whose discrete counterpart is not Cohen-Macaulay. Then there is a minimal element* α *of* P *such that* $\mathrm{Gr}_{\alpha A}(A)$ *is not Cohen-Macaulay.*

This theorem follows directly from the following

Lemma 3.4 *Let* A *be a Cohen-Macaulay ASL of dimension* d*, where* d *is as in Theorem 3.3, and let* α *be a minimal element of* $\mathrm{ind}(A)$*. If* α *is not a minimal element of* P*, then* $\mathrm{Gr}_{\alpha A}(A)$ *is Cohen-Macaulay.*

Proof Assume the contrary. Since α is not a minimal element of P by assumption, there is a minimal element β of P such that $\beta < \alpha$.

Set $\Pi = \{x \in P \mid x \not> \beta\}$. Then by Proposition 3.2, we see that

$$\mathrm{depth} A/\Pi A = \mathrm{depth}\mathrm{Gr}_{\beta A}(A) = \mathrm{depth} A = d$$

since $\beta \notin \mathrm{ind}(A)$. Therefore, $A/\Pi A$ is a Cohen-Macaulay ASL on $P \setminus \Pi$. Since β is the unique minimal element of $P \setminus \Pi$, β is a NZD for $A/\Pi A$. So $A/(\Pi A + \beta A)$ is a Cohen-Macaulay ASL on $P \setminus (\Pi \cup \{\beta\})$ of dimension $d-1$. By the minimality of d, we see that $P \setminus (\Pi \cup \{\beta\})$ and therefore $P \setminus \Pi$ is a Cohen-Macaulay poset.

Now set $P_1 = \{x \in P \mid x \geq \alpha\}$, $P_2 = \{x \in P \mid x < \alpha\}$ and $\Omega = \{x \in P \mid x \not> \alpha\}$ and take a saturated chain $\beta < \gamma_1 < \gamma_2 < \cdots < \gamma_u < \alpha$ in P. Then $P_1 = \mathrm{link}_{P \setminus \Pi}(\{\beta, \gamma_1, \gamma_2, \ldots, \gamma_u\})$. In particular, P_1 is a Cohen-Macaulay poset.

Since Ω is a standard subset by Fact 2.8, $A/\Omega A$ is an ASL on $P \setminus \Omega$. And by Proposition 3.2, we see that

$$\mathrm{depth} A/\Omega A = \mathrm{depth}\mathrm{Gr}_{\alpha A}(A) < d.$$

On the other hand, by Fact 2.5, P is pure and so $\dim A/\Omega A = d$. Therefore, $A/\Omega A$ is not Cohen-Macaulay. So the poset $P \setminus \Omega$ is not Cohen-Macaulay. But $P \setminus \Omega = P_2 \cup P_1$ and so $K[P \setminus \Omega] = K[P_2] \otimes K[P_1]$. Since P_1 is a Cohen-Macaulay poset, we see that P_2 is not a Cohen-Macaulay poset.

Set $B = A/(\Omega A + P_2 A)$. Then $A/\Omega A = B[P_2]$, the Stanley-Reisner ring of poset P_2 over the base ring B, since $P_2 \cap \mathrm{ind}(A) = \emptyset$. Now since $\alpha(\Omega A) = 0$, we see that

$$A[\alpha^{-1}] = (A/\Omega A)[\alpha^{-1}] = (B[\alpha^{-1}])[P_2].$$

(Neither $A[\alpha^{-1}]$ nor $B[\alpha^{-1}]$ is an ASL. But we do not use the ASL property from now on.)

$A[\alpha^{-1}]$ is a Cohen-Macaulay ring because it is a localization of a Cohen-Macaulay ring. On the other hand, $(B[\alpha^{-1}])[P_2]$ is not a Cohen-Macaulay ring since P_2 is not a Cohen-Macaulay poset. This is a contradiction. ∎

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ALTERNATIVE POLARIZATIONS OF BOREL FIXED IDEALS AND ELIAHOU-KERVAIRE TYPE RESOLUTION

RYOTA OKAZAKI AND KOHJI YANAGAWA

1. Introduction

Let $S := \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\Bbbk$. For a monomial ideal $I \subset S$, $G(I)$ denotes the set of minimal (monomial) generators of I. We say a monomial ideal $I \subset S$ is *Borel fixed* (or *strongly stable*), if $\mathsf{m} \in G(I)$, $x_i|\mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Borel fixed ideals are important, since they appear as the *generic initial ideals* of homogeneous ideals (if $\mathrm{char}(\Bbbk) = 0$).

Recall that a squarefree monomial ideal I is said to be *squarefree strongly stable*, if $\mathsf{m} \in G(I)$, $x_i|\mathsf{m}$, $x_j \not|\mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Any monomial $\mathsf{m} \in S$ with $\deg(\mathsf{m}) = e$ has a unique expression

$$\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with} \quad 1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n. \tag{1.1}$$

Now we can consider the squarefree monomial

$$\mathsf{m}^{\mathsf{sq}} = \prod_{i=1}^{e} x_{\alpha_i + i - 1}$$

in the "larger" polynomial ring $T = \Bbbk[x_1, \dots, x_N]$ with $N \gg 0$. If $I \subset S$ is Borel fixed, then $I^{\mathsf{sq}} := (\, \mathsf{m}^{\mathsf{sq}} \mid \mathsf{m} \in G(I)\,) \subset T$ is squarefree strongly stable. Moreover, for a Borel fixed ideal I and all i, j, we have $\beta^S_{i,j}(I) = \beta^T_{i,j}(I^{\mathsf{sq}})$. Recall that this operation plays a role in the *shifting theory* for simplicial complexes (see [1]).

A minimal free resolution of a Borel fixed ideal I has been constructed by Eliahou and Kervaire [4]. While the minimal free resolution is unique up to isomorphism, its "description" depends on the choice of a free basis, and further analysis of the minimal free resolution is still an interesting problem. See, for example, [2, 6, 7, 8, 10]. In this paper, we will give a new approach which is applicable to both I and I^{sq}. Our main tool is the "alternative" polarization $\mathsf{b}\text{-}\mathsf{pol}(I)$ of I.

Let $\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \le i \le n,\ 1 \le j \le d\,]$ be the polynomial ring, and set

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \le i \le n,\ 2 \le j \le d\} \subset \widetilde{S}.$$

Then there is an isomorphism $\widetilde{S}/(\Theta) \cong S$ induced by $\widetilde{S} \ni x_{i,j} \longmapsto x_i \in S$. Throughout this paper, $\widetilde{S}$ and Θ are used in this meaning.

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Assume that $\mathsf{m} \in G(I)$ has the expression (1.1). If $\deg(\mathsf{m})\ (= e) \le d$, we set

$$\text{(1.2)} \qquad \mathsf{b\text{-}pol}(\mathsf{m}) = \prod_{i=1}^{e} x_{\alpha_i, i} \in \widetilde{S}.$$

Note that $\mathsf{b\text{-}pol}(\mathsf{m})$ is a squarefree monomial. If there is no danger of confusion, $\mathsf{b\text{-}pol}(\mathsf{m})$ is denoted by $\widetilde{\mathsf{m}}$. If $\mathsf{m} = \prod_{i=1}^{n} x_i^{a_i}$, then we have

$$\widetilde{\mathsf{m}}\ (= \mathsf{b\text{-}pol}(\mathsf{m})) = \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j} \in \widetilde{S}, \quad \text{where} \quad b_i := \sum_{l=1}^{i} a_l.$$

If $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$, we set

$$\mathsf{b\text{-}pol}(I) := (\mathsf{b\text{-}pol}(\mathsf{m}) \mid \mathsf{m} \in G(I)) \subset \widetilde{S}.$$

The second author ([12]) showed that if I is Borel fixed, then $\widetilde{I} := \mathsf{b\text{-}pol}(I)$ is a polarization of I, that is, Θ forms an $\widetilde{S}/\widetilde{I}$-regular sequence with the natural isomorphism $\widetilde{S}/(\widetilde{I} + (\Theta)) \cong S/I$. Note that $\mathsf{b\text{-}pol}(-)$ does not give a polarization for a general monomial ideal, and $\mathsf{b\text{-}pol}(I)$ is essentially different from the standard polarization even for a Borel fixed ideal I. Moreover,

$$\Theta' = \{\, x_{i,j} - x_{i+1,j-1} \mid 1 \le i < n, 1 < j \le d \,\} \subset \widetilde{S}$$

forms an $\widetilde{S}/\widetilde{I}$-regular sequence too, and we have $\widetilde{S}/(\widetilde{I} + (\Theta')) \cong T/I^{\mathsf{sq}}$ through $\widetilde{S} \ni x_{i,j} \longmapsto x_{i+j-1} \in T$ (if we adjust the value of $N = \dim T$). The equation $\beta^S_{i,j}(I) = \beta^T_{i,j}(I^{\mathsf{sq}})$ mentioned above easily follows from this observation.

In this paper, we will construct a minimal $\widetilde{S}$-free resolution $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$, which is analogous to the Eliahou-Kervaire resolution of S/I. However, their description can *not* be lifted to $\widetilde{I}$, and we need modification. Clearly, $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ and $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta')$ give the minimal free resolutions of S/I and T/I^{sq} respectively.

Under the assumption that a Borel fixed ideal I is generated in one degree (i.e., all elements of $G(I)$ have the same degree), Nagel and Reiner [10] constructed the alternative polarization $\widetilde{I} = \mathsf{b\text{-}pol}(I)$ of I, and described a minimal $\widetilde{S}$-free resolution of $\widetilde{I}$ explicitly (and induced minimal free resolutions of I itself and I^{sq}). Their resolution is equivalent to our description. In this sense, our results are generalizations of those for [10].

Batzies and Welker ([2]) developed a strong theory which tries to construct minimal free resolutions of monomial ideals using Forman's *discrete Morse theory* ([5]). If a monomial ideal J is *shellable* in the sense of [2] (i.e., has *linear quotients*, in the sense of [6]), their method is applicable to J, and we can get a *Batzies-Welker type* minimal free resolution. However, in many cases, it is almost impossible to compute the differential map of their resolution explicitly.

A Borel fixed ideal I and its polarization $\widetilde{I} = \mathsf{b\text{-}pol}(I)$ are shellable. We will show that our resolution $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$ and the induced resolutions of S/I and T/I^{sq} are Batzies-Welker type. In particular, these resolutions are cellular. As far as the authors know, an *explicit* description of a Batzies-Welker type resolution of a general Borel fixed ideal has never been obtained before.

If I is generated in one degree, the CW complex supporting $\widetilde{P}_\bullet$ is regular by [10]. We believe that it is also true in general, but we can not prove it now.

2. The Eliahou-Kervaire type resolution of $\widetilde{S}/\,\mathsf{b}\text{-}\mathsf{pol}(I)$

Throughout the rest of the paper, I is a Borel fixed monomial ideal with $\deg \mathsf{m} \leq d$ for all $\mathsf{m} \in G(I)$. For the definitions of the alternative polarization $\mathsf{b}\text{-}\mathsf{pol}(I)$ of I and related concepts, consult the previous section. For a monomial $\mathsf{m} = \prod_{i=1}^n x_i^{a_i} \in S$, set $\mu(\mathsf{m}) := \min\{\, i \mid a_i > 0\,\}$ and $\nu(\mathsf{m}) := \max\{\, i \mid a_i > 0\,\}$. In [4], it is shown that any monomial $\mathsf{m} \in I$ has a unique expression $\mathsf{m} = \mathsf{m}_1 \cdot \mathsf{m}_2$ with $\nu(\mathsf{m}_1) \leq \mu(\mathsf{m}_2)$ and $\mathsf{m}_1 \in G(I)$. Following [4], we set $g(\mathsf{m}) := \mathsf{m}_1$. For i with $i < \nu(\mathsf{m})$, let

$$\mathfrak{b}_i(\mathsf{m}) = (x_i/x_k) \cdot \mathsf{m}, \text{ where } k := \min\{\, j \mid a_j > 0,\ j > i\}.$$

Since I is Borel fixed, $\mathsf{m} \in I$ implies $\mathfrak{b}_i(\mathsf{m}) \in I$.

Definition 2.1. For a finite subset $\widetilde{F} = \{\,(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)\,\}$ of $\mathbb{N} \times \mathbb{N}$ and a monomial $\mathsf{m} = \prod_{i=1}^e x_{\alpha_i} = \prod_{i=1}^n x_i^{a_i} \in G(I)$ with $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_e \leq n$, we say the pair $(\widetilde{F}, \widetilde{\mathsf{m}})$ is *admissible* (for $\mathsf{b}\text{-}\mathsf{pol}(I)$), if the following are satisfied:

(a) $1 \leq i_1 < i_2 < \cdots < i_q < \nu(\mathsf{m})$,

(b) $j_r = \max\{\, l \mid \alpha_l \leq i_r\,\} + 1$ (equivalently, $j_r = 1 + \sum_{l=1}^{i_r} a_l$) for all r.

For $\mathsf{m} \in G(I)$, the pair $(\emptyset, \widetilde{\mathsf{m}})$ is also admissible.

The following are fundamental properties of admissible pairs.

Lemma 2.2. *Let $(\widetilde{F}, \widetilde{\mathsf{m}})$ be an admissible pair with $\widetilde{F} = \{\,(i_1, j_1), \ldots, (i_q, j_q)\,\}$ and $\mathsf{m} = \prod x_i^{a_i} \in G(I)$. Then we have the following.*

(i) $j_1 \leq j_2 \leq \cdots \leq j_q$.

(ii) $x_{k,j_r} \cdot \mathsf{b}\text{-}\mathsf{pol}(\mathfrak{b}_{i_r}(\mathsf{m})) = x_{i_r,j_r} \cdot \mathsf{b}\text{-}\mathsf{pol}(\mathsf{m})$, *where* $k = \min\{\, l \mid l > i_r, a_l > 0\,\}$.

For $\mathsf{m} \in G(I)$ and an integer i with $1 \leq i < \nu(\mathsf{m})$, set $\mathsf{m}_{\langle i\rangle} := g(\mathfrak{b}_i(\mathsf{m}))$ and $\widetilde{\mathsf{m}}_{\langle i\rangle} := \mathsf{b}\text{-}\mathsf{pol}(\mathsf{m}_{\langle i\rangle})$. If $i \geq \nu(\mathsf{m})$, we set $\mathsf{m}_{\langle i\rangle} := \mathsf{m}$ for the convenience. In the situation of Lemma 2.2, $\widetilde{\mathsf{m}}_{\langle i_r\rangle}$ divides $x_{i_r,j_r} \cdot \widetilde{\mathsf{m}}$ for all $1 \leq r \leq q$.

For $\widetilde{F} = \{\,(i_1, j_1), \ldots, (i_q, j_q)\,\}$ and r with $1 \leq r \leq q$, set $\widetilde{F}_r := \widetilde{F} \setminus \{\,(i_r, j_r)\,\}$, and for an admissible pair $(\widetilde{F}, \widetilde{\mathsf{m}})$ for $\mathsf{b}\text{-}\mathsf{pol}(I)$,

$$B(\widetilde{F}, \widetilde{\mathsf{m}}) := \{\, r \mid (\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle}) \text{ is admissible}\,\}.$$

Lemma 2.3. *Let $(\widetilde{F}, \widetilde{\mathsf{m}})$ be as in Lemma 2.2.*

(i) *For all r with $1 \leq r \leq q$, $(\widetilde{F}_r, \widetilde{\mathsf{m}})$ is admissible.*

(ii) *We always have $q \in B(\widetilde{F}, \widetilde{\mathsf{m}})$.*

(iii) *Assume that $(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle})$ satisfies the condition (a) of Definition 2.1. Then $r \in B(\widetilde{F}, \widetilde{\mathsf{m}})$ if and only if either $j_r < j_{r+1}$ or $r = q$.*

(iv) *For r, s with $1 \leq r < s \leq q$ and $j_r < j_s$, we have $\mathfrak{b}_{i_r}(\mathfrak{b}_{i_s}(\mathsf{m})) = \mathfrak{b}_{i_s}(\mathfrak{b}_{i_r}(\mathsf{m}))$ and hence $(\widetilde{\mathsf{m}}_{\langle i_r\rangle})_{\langle i_s\rangle} = (\widetilde{\mathsf{m}}_{\langle i_s\rangle})_{\langle i_r\rangle}$.*

(v) *For r, s with $1 \leq r < s \leq q$ and $j_r = j_s$, we have $\mathfrak{b}_{i_r}(\mathsf{m}) = \mathfrak{b}_{i_r}(\mathfrak{b}_{i_s}(\mathsf{m}))$ and hence $\widetilde{\mathsf{m}}_{\langle i_r\rangle} = (\widetilde{\mathsf{m}}_{\langle i_s\rangle})_{\langle i_r\rangle}$.*

Example 2.4. Let $I \subset S = \Bbbk[x_1, x_2, x_3, x_4]$ be the smallest Borel fixed ideal containing $\mathsf{m} = (x_1)^2 x_3 x_4$. In this case, $\mathsf{m}'_{\langle i \rangle} = g(\mathfrak{b}_i(\mathsf{m}'))$ for all $\mathsf{m}' \in G(I)$. Hence, we have $\mathsf{m}_{\langle 1 \rangle} = (x_1)^3 x_4$, $\mathsf{m}_{\langle 2 \rangle} = (x_1)^2 x_2 x_4$ and $\mathsf{m}_{\langle 3 \rangle} = (x_1)^2 (x_3)^2$. The following 3 pairs are all admissible.

- $(\widetilde{F}, \widetilde{\mathsf{m}}) = (\{ (1,3), (2,3), (3,4) \}, x_{1,1}\, x_{1,2}\, x_{3,3}\, x_{4,4})$
- $(\widetilde{F}_2, \widetilde{\mathsf{m}}_{\langle 2 \rangle}) = (\{ (1,3), (3,4) \}, x_{1,1}\, x_{1,2}\, x_{2,3}\, x_{4,4})$
- $(\widetilde{F}_3, \widetilde{\mathsf{m}}_{\langle 3 \rangle}) = (\{ (1,3), (2,3) \}, x_{1,1}\, x_{1,2}\, x_{3,3}\, x_{3,4})$

(For this $\widetilde{F}$, $i_r = r$ holds and the reader should be careful). However, $(\widetilde{F}_1, \widetilde{\mathsf{m}}_{\langle 1 \rangle}) = (\{ (2,3), (3,4) \}, x_{1,1}\, x_{1,2}\, x_{1,3}\, x_{4,4})$ does not satisfy the condition (b) of Definition 2.1. Hence $B(\widetilde{F}, \widetilde{\mathsf{m}}) = \{2, 3\}$.

Next let I' be the smallest Borel fixed ideal containing $\mathsf{m} = (x_1)^2 x_3 x_4$ and $(x_1)^2 x_2$. For $\widetilde{F} = \{ (1,3), (2,3), (3,4) \}$, $(\widetilde{F}, \widetilde{\mathsf{m}})$ is admissible again. However $\widetilde{\mathsf{m}}_{\langle 2 \rangle} = (x_1)^2 x_2$ in this time, and $(\widetilde{F}_2, \widetilde{\mathsf{m}}_{\langle 2 \rangle}) = (\{ (1,3), (3,4) \}, x_{1,1}\, x_{1,2}\, x_{2,3})$ is no longer admissible. In fact, it does not satisfy (a) of Definition 2.1. Hence $B(\widetilde{F}, \widetilde{\mathsf{m}}) = \{3\}$ for $\mathsf{b\text{-}pol}(I')$.

For $F = \{i_1, \ldots, i_q\} \subset \mathbb{N}$ with $i_1 < \cdots < i_q$ and $\mathsf{m} \in G(I)$, Eliahou-Kervaire call the pair (F, m) admissible for I, if $i_q < \nu(\mathsf{m})$. In this case, there is a unique sequence $j_1, \ldots, j_q$ such that $(\widetilde{F}, \widetilde{\mathsf{m}})$ is admissible for $\widetilde{I}$, where $\widetilde{F} = \{ (i_1, j_1), \ldots, (i_q, j_q) \}$. In this way, there is a one-to-one correspondence between the admissible pairs for I and those of $\widetilde{I}$. As the free summands of the Eliahou-Kervaire resolution of I are indexed by the admissible pairs for I, our resolution of $\widetilde{I}$ are indexed by the admissible pairs for I.

We will define a $\mathbb{Z}^{n \times d}$-graded chain complex $\widetilde{P}_\bullet$ of free $\widetilde{S}$-modules as follows. First, set $\widetilde{P}_0 := \widetilde{S}$. For each $q \geq 1$, we set

$$A_q := \text{the set of admissible pairs } (\widetilde{F}, \widetilde{\mathsf{m}}) \text{ for } \mathsf{b\text{-}pol}(I) \text{ with } \#\widetilde{F} = q,$$

and

$$\widetilde{P}_q := \bigoplus_{(\widetilde{F}, \widetilde{\mathsf{m}}) \in A_{q-1}} \widetilde{S}\, e(\widetilde{F}, \widetilde{\mathsf{m}}),$$

where $e(\widetilde{F}, \widetilde{\mathsf{m}})$ is a basis element with

$$\deg \left(e(\widetilde{F}, \widetilde{\mathsf{m}}) \right) = \deg \left(\widetilde{\mathsf{m}} \times \prod_{(i_r, j_r) \in \widetilde{F}} x_{i_r, j_r} \right) \in \mathbb{Z}^{n \times d}.$$

We define the $\widetilde{S}$-homomorphism $\partial : \widetilde{P}_q \to \widetilde{P}_{q-1}$ for $q \geq 2$ so that $e(\widetilde{F}, \widetilde{\mathsf{m}})$ with $\widetilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\}$ is sent to

$$\sum_{1 \leq r \leq q} (-1)^r \cdot x_{i_r, j_r} \cdot e(\widetilde{F}_r, \widetilde{\mathsf{m}}) - \sum_{r \in B(\widetilde{F}, \widetilde{\mathsf{m}})} (-1)^r \cdot \frac{x_{i_r, j_r} \cdot \widetilde{\mathsf{m}}}{\widetilde{\mathsf{m}}_{\langle i_r \rangle}} \cdot e(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r \rangle}),$$

and $\partial : \widetilde{P}_1 \to \widetilde{P}_0$ by $e(\emptyset, \widetilde{\mathsf{m}}) \longmapsto \widetilde{\mathsf{m}} \in \widetilde{S} = \widetilde{P}_0$. Clearly, ∂ is a $\mathbb{Z}^{n \times d}$-graded homomorphism.

Set

$$\widetilde{P}_\bullet : \cdots \xrightarrow{\partial} \widetilde{P}_i \xrightarrow{\partial} \cdots \xrightarrow{\partial} \widetilde{P}_1 \xrightarrow{\partial} \widetilde{P}_0 \longrightarrow 0.$$

Let $\succ$ be the lexicographic order on the monomials of S with $x_1 \succ x_2 \succ \cdots \succ x_n$.

Theorem 2.5. *The complex $\widetilde{P}_\bullet$ is a $\mathbb{Z}^{n\times d}$-graded minimal $\widetilde{S}$-free resolution for $\widetilde{S}/\operatorname{b-pol}(I)$.*

Sketch of Proof. Calculation using Lemma 2.3 shows that $\partial \circ \partial(e(\widetilde{F}, \widetilde{\mathsf{m}})) = 0$ for each admissible pair $(\widetilde{F}, \widetilde{\mathsf{m}})$. That is, $\widetilde{P}_\bullet$ is a chain complex.

Now let $I = (\mathsf{m}_1, \dots, \mathsf{m}_t)$ with $\mathsf{m}_1 \succ \cdots \succ \mathsf{m}_t$, and set $I_r := (\mathsf{m}_1, \dots, \mathsf{m}_r)$. It is easy to show that the I_r are also Borel fixed. The acyclicity of the complex $\widetilde{P}$ can be shown inductively by means of mapping cones. □

Remark 2.6. In their paper [6], Herzog and Takayama explicitly gave a minimal free resolution of a monomial ideal with *linear quotients* admitting a *regular decomposition function.* A Borel fixed ideal I is a typical example with this property. However, while our $\widetilde{I}$ has linear quotients, the decomposition function can not be regular in general. Hence the method of [6] is not applicable to our case.

3. Applications and Remarks

Let $I \subset S$ be a Borel fixed ideal, and $\Theta \subset \widetilde{S}$ the sequence defined in Introduction. As remarked after Example 2.4, there is a one-to-one correspondence between the admissible pairs for $\widetilde{I}$ and those for I, and if $(\widetilde{F}, \widetilde{\mathsf{m}})$ corresponds to (F, m) then $\#\widetilde{F} = \#F$. Hence we have

$$\beta_{i,j}^{\widetilde{S}}(\widetilde{I}) = \beta_{i,j}^{S}(I) \tag{3.1}$$

for all i, j, where S and $\widetilde{S}$ are considered to be $\mathbb{Z}$-graded. Of course, this equation is clear, if we know the fact that $\widetilde{I}$ is a polarization of I ([12, Theorem 3.4]). Conversely, we can show this fact by the equation (3.1) and [10, Lemma 6.9].

Corollary 3.1 ([12, Theorem 3.4]). *The ideal $\widetilde{I}$ is a polarization of I.*

The next result also follows from [10, Lemma 6.9].

Corollary 3.2. *$\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ is a minimal S-free resolution of S/I.*

Remark 3.3. (1) The correspondence between the admissible pairs for I and those for $\widetilde{I}$, does *not* give a chain map between the Eliahou-Kervaire resolution and our $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$. In this sense, two resolutions are not the same. See Example 4.9 below.

(2) Eliahou and Kervaire ([4]) constructed minimal free resolutions of *stable monomial ideals*, which form a wider class than Borel fixed ideals. As shown in [12, Example 2.3 (2)], $\operatorname{b-pol}(J)$ is not a polarization for a stable monomial ideal J in general, and the construction of $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ does not work for J.

(3) Even if I is Borel fixed, the *lcm lattice* of I and that of $\widetilde{I}$ are not isomorphic in general. Recall that the lcm-lattice of a monomial ideal J is the set $\mathrm{LCM}(J) := \{\operatorname{lcm}\{\mathsf{m} \mid \mathsf{m} \in \sigma\} \mid \sigma \subset G(J)\}$ with the order given by divisibility. Clearly, $\mathrm{LCM}(J)$ forms a lattice. For the Borel fixed ideal $I = (x^2, xy, xz, y^2, yz)$, we have $xy \vee xz = xy \vee yz = xz \vee yz = xyz$ in $\mathrm{LCM}(I)$. On the other hand, $\widetilde{xy} \vee \widetilde{xz} = x_1 y_2 z_2$, $\widetilde{xy} \vee \widetilde{yz} = x_1 y_1 y_2 z_2$ and $\widetilde{xz} \vee \widetilde{yz} = x_1 y_1 z_2$ are all distinct in $\mathrm{LCM}(\widetilde{I})$.

Let $a = \{a_0, a_1, a_2, \dots\}$ be a non-decreasing sequence of non-negative integers with $a_0 = 0$, and $T = \Bbbk[x_1, \dots, x_N]$ a polynomial ring with $N \gg 0$. In his paper [9], Murai defined an operator $(-)^{\gamma(a)}$ acting on monomials and monomial ideals of S. For a monomial $\mathsf{m} \in S$ with the expression $\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i}$ as (1.1), set

$$\mathsf{m}^{\gamma(a)} := \prod_{i=1}^{e} x_{\alpha_i + a_{i-1}} \in T,$$

and for a monomial ideal $I \subset S$,

$$I^{\gamma(a)} := (\mathsf{m}^{\gamma(a)} \mid \mathsf{m} \in G(I)) \subset T.$$

If $a_{i+1} > a_i$ for all i, then $I^{\gamma(a)}$ is a squarefree monomial ideal. Particularly in the case $a_i = i$ for all i, $(-)^{\gamma(a)}$ is just $(-)^{\mathsf{sq}}$ mentioned in Introduction.

The operator $(-)^{\gamma(a)}$ also can be described by $\mathsf{b}\text{-}\mathsf{pol}(-)$ as is shown in [12]. Let L_a be the $\Bbbk$-subspace of $\widetilde{S}$ spanned by $\{x_{i,j} - x_{i',j'} \mid i + a_{j-1} = i' + a_{j'-1}\}$, and Θ_a a basis of L_a. For example, we can take

$$\{\, x_{i,j} - x_{i+1,j-1} \mid 1 \le i < n,\ 1 < j \le d \,\}$$

as Θ_a in the case $a_i = i$ for all i. With a suitable choice of the number N, the ring homomorphism $\widetilde{S} \to T$ with $x_{i,j} \mapsto x_{i+a_{j-1}}$ induces the isomorphism $\widetilde{S}/(\Theta_a) \cong T$.

Proposition 3.4 ([12, Proposition 4,1]). *With the above notation, Θ_a forms an $\widetilde{S}/\widetilde{I}$-regular sequence, and we have $(\widetilde{S}/(\Theta_a)) \otimes_{\widetilde{S}} (\widetilde{S}/\widetilde{I}) \cong T/I^{\gamma(a)}$ through the isomorphism $\widetilde{S}/(\Theta_a) \cong T$.*

Applying Proposition 3.4 and [3, Proposition 1.1.5], we have the following.

Corollary 3.5. *The complex $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_a)$ is a minimal T-free resolution of $T/I^{\gamma(a)}$. In particular, a minimal free resolution of T/I^{sq} is given in this way.*

For a Borel fixed ideal I generated in one degree, Nagel and Reiner [10] constructed a CW complex, more precisely a polytopal complex, which supports a minimal free resolution of $\widetilde{I}$ (or I, I^{sq}). See [10, Theorem 3.13].

Proposition 3.6. *Let I be a Borel fixed ideal generated in one degree. Then Nagel-Reiner description of a minimal free resolution of $\widetilde{I}$ coincides with our $\widetilde{P}_\bullet$.*

We do not give a proof of the above theorem here. We just remark that if I is generated in one degree then $\mathsf{m}_{\langle i \rangle} = \mathfrak{b}_i(\mathsf{m})$ for all $\mathsf{m} \in G(I)$ and $\widetilde{P}_\bullet$ becomes simpler than the general case.

4. Relation to Batzies-Welker theory

In [2], Batzies and Welker connected the theory of *cellular resolutions* of monomial ideals with Forman's discrete Morse theory ([5]).

Definition 4.1. A monomial ideal J is called *shellable* if there is a total order $\sqsubset$ on $G(J)$ satisfying the following condition.

(∗) For any $\mathsf{m}, \mathsf{m}' \in G(J)$ with $\mathsf{m} \sqsupset \mathsf{m}'$, there is an $\mathsf{m}'' \in G(J)$ such that $\mathsf{m} \sqsupseteq \mathsf{m}''$, $\deg\left(\frac{\operatorname{lcm}(\mathsf{m},\mathsf{m}'')}{\mathsf{m}}\right) = 1$ and $\operatorname{lcm}(\mathsf{m}, \mathsf{m}'')$ divides $\operatorname{lcm}(\mathsf{m}, \mathsf{m}')$.

Let $\sqsubset$ be the total order on $G(\widetilde{I}) = \{\, \widetilde{\mathsf{m}} \mid \mathsf{m} \in G(I)\,\}$ such that $\widetilde{\mathsf{m}}' \sqsubset \widetilde{\mathsf{m}}$ if and only if $\mathsf{m}' \succ \mathsf{m}$ in the lexicographic order on S with $x_1 \succ x_2 \succ \cdots \succ x_n$. In the rest of this section, $\sqsubset$ means this order.

Lemma 4.2. *The order $\sqsubset$ makes $\widetilde{I}$ shellable.*

The following construction is taken from [2, Theorems 3.2 and 4.3]. For the background of their theory, the reader is recommended to consult the original paper.

For $\emptyset \neq \sigma \subset G(\widetilde{I})$, let $\widetilde{\mathsf{m}}_\sigma$ denote the largest element of σ with respect to the order $\sqsubset$, and set $\mathrm{lcm}(\sigma) := \mathrm{lcm}\{\, \widetilde{\mathsf{m}} \mid \widetilde{\mathsf{m}} \in \sigma\,\}$.

Definition 4.3. We define a total order $\prec_\sigma$ on $G(\widetilde{I})$ as follows. Set

$$N_\sigma := \{\, (\widetilde{\mathsf{m}}_\sigma)_{\langle i\rangle} \mid 1 \le i < \nu(\mathsf{m}_\sigma),\ (\widetilde{\mathsf{m}}_\sigma)_{\langle i\rangle} \text{ divides } \mathrm{lcm}(\sigma)\,\}.$$

For all $\widetilde{\mathsf{m}} \in N_\sigma$ and $\widetilde{\mathsf{m}}' \in G(\widetilde{I}) \setminus N_\sigma$, define $\widetilde{\mathsf{m}} \prec_\sigma \widetilde{\mathsf{m}}'$. The restriction of $\prec_\sigma$ to N_σ is set to be $\sqsubset$, and the same is true for the restriction to $G(\widetilde{I}) \setminus N_\sigma$.

Let X be the $(\#G(\widetilde{I}) - 1)$-simplex associated with $2^{G(\widetilde{I})}$ (more precisely, $2^{G(\widetilde{I})} \setminus \{\emptyset\}$). Hence we freely identify $\sigma \subset G(\widetilde{I})$ with the corresponding cell of the simplex X. Let G_X be the directed graph defined as follows. The vertex set of G_X is $2^{G(\widetilde{I})} \setminus \{\emptyset\}$. For $\emptyset \neq \sigma, \sigma' \subset G(\widetilde{I})$, there is an arrow $\sigma \to \sigma'$ if and only if $\sigma \supset \sigma'$ and $\#\sigma = \#\sigma' + 1$. For $\sigma = \{\, \widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2, \dots, \widetilde{\mathsf{m}}_k\,\}$ with $\widetilde{\mathsf{m}}_1 \prec_\sigma \widetilde{\mathsf{m}}_2 \prec_\sigma \cdots \prec_\sigma \widetilde{\mathsf{m}}_k \,(= \widetilde{\mathsf{m}}_\sigma)$ and $l \in \mathbb{N}$ with $1 \le l < k$, set $\sigma_l := \{\, \widetilde{\mathsf{m}}_{k-l}, \widetilde{\mathsf{m}}_{k-l+1}, \dots, \widetilde{\mathsf{m}}_k\,\}$ and

$$u(\sigma) := \sup\{\, l \mid \exists \widetilde{\mathsf{m}} \in G(\widetilde{I}) \text{ s.t. } \widetilde{\mathsf{m}} \prec_\sigma \widetilde{\mathsf{m}}_{k-l} \text{ and } \widetilde{\mathsf{m}} \,|\, \mathrm{lcm}(\sigma_l)\ \}.$$

If $u := u(\sigma) \neq -\infty$, we can define $\widetilde{\mathsf{n}}_\sigma := \min_{\prec_\sigma}\{\, \widetilde{\mathsf{m}} \mid \widetilde{\mathsf{m}} \text{ divides } \mathrm{lcm}(\sigma_u)\,\}$. Let E_X be the set of edges of G_X. We define a subset A of E_X by

$$A := \{\, \sigma \cup \{\widetilde{\mathsf{n}}_\sigma\} \to \sigma \mid u(\sigma) \neq -\infty, \widetilde{\mathsf{n}}_\sigma \notin \sigma\,\}.$$

It is easy to see that A is a *matching*, that is, every σ occurs in at most one edges of A. We say $\emptyset \neq \sigma \subset G(\widetilde{I})$ is *critical*, if it does not occurs in any edge of A.

We have the directed graph G_X^A with the vertex set $2^{G(\widetilde{I})} \setminus \{\emptyset\}$ (i.e., same as G_X) and the set of edges $(E_X \setminus A) \cup \{\, \sigma \to \tau \mid (\tau \to \sigma) \in A\,\}$. By the proof of [2, Theorem 3.2], we see that the matching A is *acyclic*, that is, G_X^A has no directed cycle. A directed path in G_X^A is called a *gradient path.*

Forman's discrete Morse theory [5] guarantees the existence of a CW complex X_A with the following conditions.

- There is a one-to-one correspondence between the i-cells of X_A and the *critical* i-cells of X (equivalently, the critical subsets of $G(\widetilde{I})$ consisting of $i+1$ elements).
- X_A is contractible, that is, homotopy equivalent to X.

The cell of X_A corresponding to a critical cell σ of X is denoted by σ_A. By [2, Proposition 7.3], the closure of σ_A contains τ_A if and only if there is a gradient path from σ to τ. See also Proposition 4.6 below and the argument before it.

Assume that $\emptyset \neq \sigma \subset G(\widetilde{I})$ is critical. Recall that $\widetilde{\mathsf{m}}_\sigma$ denotes the largest element of σ with respect to $\sqsubset$. Take $\mathsf{m}_\sigma = \prod_{l=1}^n x_l^{a_l} \in G(I)$ with $\widetilde{\mathsf{m}}_\sigma = \text{b-pol}(\mathsf{m}_\sigma)$, and set $q := \#\sigma - 1$. Then there are integers $i_1, \dots, i_q$ with $1 \le i_1 < \ldots < i_q < \nu(\mathsf{m}_\sigma)$ and

$$\sigma = \{\, (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r\rangle} \mid 1 \le r \le q\,\} \cup \{\widetilde{\mathsf{m}}_\sigma\} \tag{4.1}$$

(see the proof of [2, Proposition 4.3]). Equivalently, we have $\sigma = N_\sigma \cup \{\widetilde{\mathsf{m}}_\sigma\}$. Set $j_r := 1 + \sum_{l=1}^{i_r} a_l$ for each $1 \leq r \leq q$, and $\widetilde{F}_\sigma := \{(i_1, j_1), \ldots, (i_q, j_q)\}$. Then $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ is an admissible pair for $\widetilde{I}$. Conversely, any admissible pair comes from a critical cell $\sigma \subset G(\widetilde{I})$ in this way. Hence there is a one-to-one correspondence between critical cells and admissible pairs.

Let X_A^i denote the set of all the critical subset $\sigma \subset G(\widetilde{I})$ with $\#\sigma = i+1$, and for (not necessarily critical) subsets σ, τ of $G(\widetilde{I})$, let $\mathscr{P}_{\sigma,\tau}$ denote the set of all the gradient paths from σ to τ. For $\sigma \in X_A^q$ of the form (4.1), $e(\sigma)$ denotes a basis element with degree $\deg(\mathrm{lcm}(\sigma)) \in \mathbb{Z}^{n \times d}$. Set

$$\widetilde{Q}_q = \bigoplus_{\sigma \in X_A^q} \widetilde{S}\, e(\sigma) \qquad (q \geq 0).$$

The differential map $\widetilde{Q}_q \to \widetilde{Q}_{q-1}$ sends $e(\sigma)$ to

$$\sum_{r=1}^{q} (-1)^r x_{i_r, j_r} \cdot e(\sigma \setminus \{(\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}\}) - (-1)^q \sum_{\substack{\tau \in X_A^{q-1} \\ \mathcal{P} \in \mathscr{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau}}} m(\mathcal{P}) \cdot \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\tau)} \cdot e(\tau), \tag{4.2}$$

where $m(\mathcal{P}) = \pm 1$ is the one defined in [2, p.166].

The following is a direct consequence of [2, Theorem 4.3] (and [2, Remark 4.4]).

Proposition 4.4 (Batzies-Welker, [2]). *$\widetilde{Q}_\bullet$ is a minimal free resolution of $\widetilde{I}$, and has a cellular structure supported by X_A.*

Theorem 4.5. *Our description of the resolution $\widetilde{P}_\bullet$ coincides the Batzies-Welker resolution $\widetilde{Q}_\bullet$ (more precisely, the truncation $\widetilde{P}_{\geq 1}$ of $\widetilde{P}_\bullet$ coincides with $\widetilde{Q}_\bullet$).*

First, note that the following hold.

(1) Let σ and τ be (not necessarily critical) cells with $\mathscr{P}_{\sigma,\tau} \neq \varnothing$. Then $\mathrm{lcm}(\tau)$ divides $\mathrm{lcm}(\sigma)$.

(2) Let $\sigma \in X_A^q$, $\tau \in X_A^{q-1}$ and assume that there is a gradient path $\sigma \to \sigma \setminus \{\widetilde{\mathsf{m}}\} = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_l = \tau$. Then $\#\sigma_{l-1} = \#\tau + 1 = q+1$, $\#\sigma_i = q$ or $q+1$ for each i, and σ_i is not critical for all $0 \leq i < l$.

(3) If σ is critical, so is $\sigma \setminus \{(\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}\}$ for $1 \leq r \leq q$.

Next, we will show the following.

Proposition 4.6. *Let σ, τ be critical cells with $\#\sigma = \#\tau + 1$, and $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ and $(\widetilde{F}_\tau, \widetilde{\mathsf{m}}_\tau)$ the admissible pairs corresponding to σ and τ respectively. Set $\widetilde{F}_\sigma = \{(i_1, j_1), \ldots, (i_q, j_q)\}$ with $i_1 < \cdots < i_q$. Then $\mathscr{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau} \neq \varnothing$ if and only if there is some $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ with $(\widetilde{F}_\tau, \widetilde{\mathsf{m}}_\tau) = ((\widetilde{F}_\sigma)_r, (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle})$. If this is the case, the gradient path is the unique one from $(\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\})$ to τ (hence $\#\mathscr{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau} = 1$).*

Sketch of Proof. Only if part follows from the above remark. Note that the second index j of each $x_{i,j} \in \widetilde{S}$ restricts the choice of paths and it makes the proof easier.

Next, assuming $\widetilde{F}_\tau = (\widetilde{F}_\sigma)_r$ and $\widetilde{\mathsf{m}}_\tau = (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}$ for some $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$, we will construct a gradient path from $\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}$ to τ. For short notation, set $\widetilde{\mathsf{m}}_{[s]} := (\widetilde{\mathsf{m}}_\sigma)_{\langle i_s \rangle}$ and $\widetilde{\mathsf{m}}_{[s,t]} := ((\widetilde{\mathsf{m}}_\sigma)_{\langle i_s \rangle})_{\langle i_t \rangle}$. By (4.1), we have $\sigma_0 := (\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}) = \{\widetilde{\mathsf{m}}_{[s]} \mid 1 \leq s \leq q\}$ and $\tau = \{\widetilde{\mathsf{m}}_{[r,s]} \mid 1 \leq s \leq q, s \neq r\} \cup \{\widetilde{\mathsf{m}}_{[r]}\}$. We can inductively construct a gradient path $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_t \to \cdots \sigma_{2(q-r+1)r-2}$ as follows. Write $t = 2pr + \lambda$ with $t \neq 0$, $0 \leq p \leq q-r$, and $0 \leq \lambda < 2r$. For $0 < t \leq 2(q-r)$, we set

$$\sigma_t = \begin{cases} \sigma_{t-1} \cup \{ \widetilde{\mathsf{m}}_{[q-p,s]} \} & \text{if } \lambda = 2s-1 \text{ for some } 1 \leq s \leq r; \\ \sigma_{t-1} \setminus \{ \widetilde{\mathsf{m}}_{[q-p+1,s]} \} & \text{if } \lambda = 2s \text{ for some } 0 < s < r; \\ \sigma_t \setminus \{ \widetilde{\mathsf{m}}_{[q-p+1]} \} & \text{if } \lambda = 0, \end{cases}$$

where we set $\widetilde{\mathsf{m}}_{[q+1,s]} = \widetilde{\mathsf{m}}_{[s]}$ for all s. In the case $\widetilde{\mathsf{m}}_{[s,t]} = \widetilde{\mathsf{m}}_{[s+1,t]}$, it seems to cause a problem, but skipping the corresponding part of path, we can avoid the problem. Since $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$, we have $\widetilde{\mathsf{m}}_{[s,r]} = \widetilde{\mathsf{m}}_{[r,s]}$ for all $s > r$ by Lemma 2.3 (iv). Hence

$$\sigma_{2(q-r)} = \{ \widetilde{\mathsf{m}}_{[r+1,s]} \mid 1 \leq s < r \} \cup \{ \widetilde{\mathsf{m}}_{[r]} \} \cup \{ \widetilde{\mathsf{m}}_{[r,s]} \mid r < s \leq q \}.$$

Now for s with $0 < s \leq r-1$, set σ_t with $2(q-r)r < t \leq 2(q-r+1)r-2$ to be $\sigma_{t-1} \cup \{ \widetilde{\mathsf{m}}_{[r,s]} \}$ if s is odd and otherwise $\sigma_{t-1} \setminus \{ \widetilde{\mathsf{m}}_{[r+1,s]} \}$. Then we have $\sigma_{2(q-r+1)r-2} = \tau$, and the gradient path $\sigma \rightsquigarrow \tau$.

The uniqueness of the path follows from elementally (but lengthy) argument. □

Sketch of Proof of Theorem 4.5. Recall that there is the one-to-one correspondence between the critical cells $\sigma \subset G(\widetilde{I})$ and the admissible pairs $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$. Hence, for each q, we have the isomorphism $\widetilde{Q}_q \to \widetilde{P}_q$ induced by $e(\sigma) \longmapsto e(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$.

By Proposition 4.6, if we forget "coefficients", the differential map of $\widetilde{Q}_\bullet$ and that of $\widetilde{P}_\bullet$ are compatible with the maps $e(\sigma) \longmapsto e(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$. So it is enough to check the equality of the coefficients. But it follows from direct computation. □

Corollary 4.7. *The free resolution $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ (resp. $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_a)$) of S/I (resp. $T/I^{\gamma(a)}$) is also a cellular resolution supported by X_A. In particular, these resolutions are Batzies-Welker type.*

Remark 4.8. Recently, Mermin [8] showed that the Eliahou-Kervaire resolution of a Borel fixed ideal is cellular and supported by a *regular* CW complex. In the previous section, we showed that our resolution $\widetilde{P}_\bullet$ is cellular. However, the regularity of the complex X_A supporting $\widetilde{P}_\bullet$ is not clear, while we have the following.

(1) If the closure of an $(i+1)$-cell σ_A contains an $(i-1)$-cell τ_A, there are exactly two cells between them.

(2) The incidence number $[\sigma_A : \sigma'_A]$ is $1, -1$ or 0 for all σ_A, σ'_A.

If X_A is regular, the conditions of the above proposition hold obviously.

When I is generated in one degree, $\widetilde{P}$ is equivalent to Nagel-Reiner's, and hence a polytopal (hence regular) CW complex can be taken as the supporting complex X_A of our resolution $\widetilde{P}_\bullet$.

Example 4.9. Consider the Borel fixed ideal $I = (x^2, xy^2, xyz, xyw, xz^2, xzw)$. Then $\mathsf{b\text{-}pol}(I) = (x_1x_2, x_1y_2y_3, x_1y_2z_3, x_1y_2w_3, x_1z_2z_3, x_1z_3w_3)$, and easy computation shows that the CW complex X_A, which supports our resolutions $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$ and $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ of S/I, is the one illustrated in Figure 1. The complex consists of a square pyramid and a tetrahedron glued along trigonal faces of each. For a Borel fixed ideal generated in one degree, any face of the Nagel-Reiner CW complex is a product of several simplices. Hence a square pyramid can not appear in the case of Nagel and Reiner.

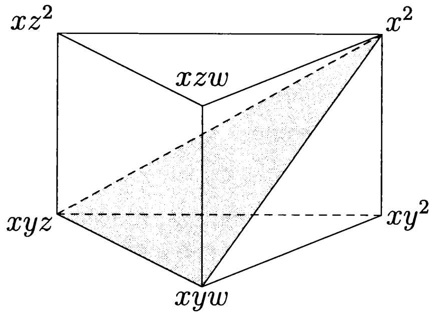

FIGURE 1.

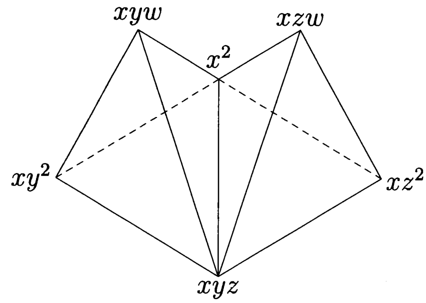

FIGURE 2.

We remark that the Eliahou-Kervaire resolution of I is supported by the CW complex illustrated in Figure 2. This complex consists of two tetrahedrons glued along edges of each. These figures show visually that the description of the Eliahou-Kervaire resolution and that of ours are really different.

Question 4.10. Is the CW complex X_A is regular for a general Borel fixed ideal?

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F-purity of isolated log canonical singularities

Shunsuke Takagi

In this article, we explain the correspondence of log canonical singularities and F-pure singularities in the case of isolated singularities. This article is based on the joint work [4] with Osamu Fujino, and the reader is referred to [4] for the proofs.

1 Preliminaries

First we recall the definition of log canonical singularities.

Definition 1.1. Let $x \in X$ be a point of a normal $\mathbb{Q}$-Gorenstein complex algebraic variety. Let $f : Y \to X$ be a resolution of singularities such that the exceptional locus $\mathrm{Exc}(f)$ is a simple normal crossing divisor. Then we can write

$$K_Y = f^*K_X + \sum_i a_i E_i,$$

where the a_i are rational numbers and the E_i are f-exceptional prime divisors on Y. We say that $x \in X$ is a *log canonical singularity* (resp. a *log terminal singularity*) if $a_i \geq -1$ (resp. $a_i > -1$) for all i such that $x \in f(E_i)$. This definition is independent of the choice of the resolution f.

Example 1.2. Let $X = \mathrm{Spec}\,\mathbb{C}[X,Y,Z]/(X^a + Y^b + Z^c)$. Then X has only log canonical singularities if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1$.

Fujino [3] introduced the invariant $\mu(x \in X)$ of an isolated log canonical singularity $x \in X$.

Definition 1.3. Let $x \in X$ be an isolated log canonical singularity which is not log terminal. First we assume that $x \in X$ is quasi-Gorenstein. Take a projective birational morphism $f : Y \to X$ from a smooth variety $\widetilde{X}$ such that $\mathrm{Exc}(f)$ and $\mathrm{Supp}\, f^{-1}(x)$ are simple normal crossing divisors. Then we can write

$$K_Y = f^*K_X + F - E,$$

where E and F are effective divisors on Y and have no common irreducible components. By assumption, E is a reduced simple normal crossing divisor on Y. We define $\mu(x \in X)$ by

$$\mu(x \in X) = \min\{\dim W \mid W \text{ is a stratum of } E\}.$$

This definition is independent of the choice of the resolution f.

In general, we take an index one cover $\rho : X' \to X$ with $x' = \rho^{-1}(x)$ to define $\mu(x \in X)$ by

$$\mu(x \in X) = \mu(x' \in X').$$

Since the index one cover is unique up to étale isomorphisms, the above definition of $\mu(x \in X)$ is well-defined. By definition,

$$0 \leq \mu(x \in X) \leq \dim X - 1.$$

Remark 1.4. A Gorenstein isolated log canonical singularity $x \in X$ with $\mu = \mu(x \in X)$ is called in [9] as a purely elliptic singularity (X, x) of type $(0, \mu)$.

Next we recall the definition of F-pure singularities.

Definition 1.5. Let $x \in X$ be a (closed) point of an F-finite integral scheme X of characteristic $p > 0$.

(i) $x \in X$ is said to be *F-pure* if the Frobenius map

$$F : \mathcal{O}_{X,x} \to F_*\mathcal{O}_{X,x} \quad a \mapsto a^p$$

splits as an $\mathcal{O}_{X,x}$-module homomorphism.

(ii) $x \in X$ is said to be *strongly F-regular* if for every nonzero $c \in \mathcal{O}_{X,x}$, there exist an integer $e \geq 1$ such that

$$cF^e : \mathcal{O}_{X,x} \to F^e_*\mathcal{O}_{X,x} \quad a \mapsto ca^{p^e}$$

splits as an $\mathcal{O}_{X,x}$-module homomorphism.

Example 1.6. Let $X = \operatorname{Spec} \mathbb{F}_p[X, Y, Z]/(X^3 + Y^3 + Z^3)$. Then X has only F-pure singularities if and only if $p \equiv 1 \bmod 3$.

Using reduction from characteristic zero to positive characteristic, we can define the notion of F-purity in characteristic zero.

Definition 1.7. Let $x \in X$ be a point of a complex algebraic variety X. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra $A \subseteq \mathbb{C}$, we can construct a point x_A of a scheme X_A of finite type over A such that $(X_A, x_A) \times_{\operatorname{Spec} A} \mathbb{C} \cong (X, x)$. By the generic freeness, we may assume that (X_A, x_A) is at over Spec A. We refer to $x_A \in X_A$ as a *model* of $x \in X$ over A. Given a closed point $s \in \operatorname{Spec} A$, we denote by $x_s \in X_s$ the fiber of $x \in X$ over s.

We say that $x \in X$ is of *strongly F-regular type* (resp. *dense F-pure type*) if for a model of $x \in X$ over a finitely generated $\mathbb{Z}$-subalgebra $A \subseteq \mathbb{C}$, there exists a dense open subset (resp. a dense subset) of closed points $S \subseteq \operatorname{Spec} A$ such that $x_s \in X_s$ is strongly F-regular (resp. F-pure) for all $s \in S$.

Example 1.8. Let $X = \operatorname{Spec} \mathbb{C}[X, Y, Z]/(X^3 + Y^3 + Z^3)$. By Lemma 1.6, X is of dense F-pure type.

Hara proved the equivalence of log terminal singularities and strongly F-regular singularities.

Theorem 1.9 ([5]). *Let $x \in X$ be a point of a normal $\mathbb{Q}$-Gorenstein complex algebraic variety X. Then $x \in X$ is log terminal if and only if it is of strongly F-regular type.*

In this article, we will discuss an analogous statement for isolated log canonical singularities.

2 Main Theorem

In order to state our main result, we need the following conjecture.

Conjecture A_n. *Let V be an n-dimensional projective variety over an algebraically closed field k of characteristic zero with only rational singularities. Assume that $K_V \sim 0$. Given a model of V over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the natural Frobenius action on $H^n(V_s, \mathcal{O}_{V_s})$ is bijective for every $s \in S$.*

Lemma 2.1. *Conjecture A_n is true if $n \leq 2$.*

Proof. By an argument similar to the proof of [8, Proposition 5.3], we may assume that $k = \overline{\mathbb{Q}}$ without loss of generality. Conjecture A_0 is trivial. Conjecture A_1 follows from a result of Serre [11].

So we consider the case when $n = 2$. Let $\;: \widetilde{V} \to V$ be a minimal resolution. $\widetilde{V}$ is an abelian surface or a K3 surface. Suppose given a model of $\;$ over a finitely generated $\mathbb{Z}$-subalgebra A of k. Then there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on $H^2(\widetilde{V}_s, \mathcal{O}_{\widetilde{V}_s})$ is bijective for every $s \in S$ (the abelian surface case follows from a result of Ogus [10] and the K3 surface case follows from a result of Bogomolov–Zarhin [2] or that of Joshi and Rajan [7]). Since X has only rational singularities, we may assume that $H^2(V_s, \mathcal{O}_{V_s}) \cong H^2(\widetilde{V}_s, \mathcal{O}_{\widetilde{V}_s})$ as $\kappa(s)[F]$-modules for all $s \in S$. Thus, we obtain the assertion. □

The recent progress in the minimal model program [1] allows us to prove the following theorem.

Theorem 2.2 ([4, Theorem 3.3]). *Let $x \in X$ be an isolated log canonical singularity. If Conjecture A_μ holds where $\mu = \mu(x \in X)$, then $x \in X$ is of dense F-pure type. In particular, if $\mu(x \in X) \leq 2$, then $x \in X$ is of dense F-pure type.*

Sketch of Proof. Let $d = \dim X$. After passing through an index one cover, we may assume that $x \in X$ is quasi-Gorenstein. We take a dlt blow-up $g : Z \to X$ of $x \in X$. That is, g is a projective birational morphism satisfying the following properties:

(i) $K_Z + D = f^*K_X$, where D is a reduced divisor on X,

(ii) (Z, D) is a $\mathbb{Q}$-factorial dlt pair,

(iii) g is an isomorphism outside x,

(iv) Z has only canonical singularities.

Then we can take a minimal log canonical center V of (Z, D) such that

(v) V is a projective variety with only rational singularities,

(vi) $\dim V = \mu = \mu(x \in X)$,

(vii) $K_V \sim 0$,

(viii) $H^{\mu}(V, \mathcal{O}_V) \cong H^{d-1}(D, \mathcal{O}_D)$.

Applying Conjecture A_{μ} to this V and running a K_Z-minimal model program with scaling over X, we can obtain the assertion. □

Corollary 2.3. *Let $x \in X$ be an isolated singularity of a normal $\mathbb{Q}$-Gorenstein complex algebraic variety of dimension ≤ 3. Then $x \in X$ is log canonical if and only if it is of dense F-pure type.*

Proof. The if part follows from [6, Theorem 3.9]. The only if part follows from Theorem 2.2, because $\mu(x \in X) \leq \dim X - 1 = 2$. □

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