Normality of edge rings and Minkowski sums

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In this article, the results on the normality and the integer decomposition property of the Minkowski sum of edge polytopes will be presented.

1 Introduction

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope, which is a convex polytope each of whose vertices belongs to $\mathbb{Z}^N$, and let

$$\widetilde{\mathcal{P}} = \{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\}, \ \ \mathcal{C}(\mathcal{P}) = \mathbb{R}_{\geq 0}\widetilde{\mathcal{P}} \ \text{ and } \ \mathcal{A}_\mathcal{P} = \widetilde{\mathcal{P}} \cap \mathbb{Z}^{N+1}.$$

Then it is well known that the semigroups $\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}$ and $\mathcal{C}(\mathcal{P}) \cap \mathbb{Z}\mathcal{A}_\mathcal{P}$ are finitely generated.

Let k be a field. Given an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$, we define two affine semigroup K-algebras $k[\mathcal{P}]$ and $\mathcal{E}_k(\mathcal{P})$ by setting

$$k[\mathcal{P}] := k[\mathbf{x}^\alpha t^n : (\alpha, n) \in \mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}] \ \text{ and } \ \mathcal{E}_k(\mathcal{P}) := k[\mathbf{x}^\alpha t^n : (\alpha, n) \in \mathcal{C}(\mathcal{P}) \cap \mathbb{Z}\mathcal{A}_\mathcal{P}],$$

respectively, where $\mathbf{x}^\alpha$ denotes the Laurent monomial $x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N$. We call $k[\mathcal{P}]$ the *toric ring* of $\mathcal{P}$ and $\mathcal{E}_k(\mathcal{P})$ the *Ehrhart ring* of $\mathcal{P}$.

We say that $k[\mathcal{P}]$ (or $\mathcal{P}$) is *normal* if $k[\mathcal{P}] = \mathcal{E}_k(\mathcal{P})$ holds, equivalently, $\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P} = \mathcal{C}(\mathcal{P}) \cap \mathbb{Z}\mathcal{A}_\mathcal{P}$ holds. On the other hand, we say that $\mathcal{P}$ possesses the *integer decomposition property* (IDP, for short) if for any positive integer m and $\alpha \in m\mathcal{P} \cap \mathbb{Z}^N$, where $m\mathcal{P} = \{m\alpha : \alpha \in \mathcal{P}\}$, there exist $\alpha_1, \ldots, \alpha_m \in \mathcal{P} \cap \mathbb{Z}^N$ such that $\alpha = \alpha_1 + \cdots + \alpha_m$. Note that $\mathcal{P}$ has IDP if and only if $\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P} = \mathcal{C}(\mathcal{P}) \cap \mathbb{Z}^{N+1}$ holds. Thus, in particular, $\mathcal{P}$ is normal if $\mathcal{P}$ has IDP.

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For convex polytopes $\mathcal{P}$ and $\mathcal{P}'$, let

$$\mathcal{P} + \mathcal{P}' = \{\alpha + \alpha' : \alpha \in \mathcal{P}, \alpha' \in \mathcal{P}'\}.$$

This is called the *Minkowski sum* of $\mathcal{P}$ and $\mathcal{P}'$. Note that the Minkowski sum of m copies of $\mathcal{P}$ coincides with the dilated polytope $m\mathcal{P}$. Hence, to take Minkowski sum of some convex polytopes can be understood as a generalization of to dilate a convex polytope.

In general, the normality of Minkowski sum of integral convex polytopes is rather difficult. For example, let $\mathcal{P}_1 = \text{conv}(\{(0,0,0),(1,0,0),(0,1,0)\})$ and $\mathcal{P}_2 = \text{conv}(\{(0,0,0),(1,1,3)\})$. Then each of $\mathcal{P}_1$ and $\mathcal{P}_2$ has IDP (in particular, is normal). However, one sees that $\mathcal{P}_1 + \mathcal{P}_2$ is not normal (does not have IDP). This example appears in [2, page 2315].

In this article, we discuss the normality of Minkowski sum of edge polytopes, which are integral convex polytopes arising from graphs (see Definition 1). In Section 2, we first consider the dimension of Minkowski sum of edge polytopes in terms of graphs. Next, in Section 3, the equivalence of the normality and IDP for Minkowski sum of edge polytopes is claimed. In Section 4, some sufficient condition for Minkowski sum of edge poyltopes to be normal is given. Finally, in Section 5, we present a result on Minkowski sum of dilated integral convex polytopes.

2 Minkowski sum of edge polytopes

In this section, we consider the dimension of Minkowski sum of edge polytopes. Before stating a proposition, we recall a graph theory terminology.

- We say that G is *bipartite* if the vertex set of G can be deomposed into two disjoint non-empty subsets U and V and every edge in G belongs to $U \times V$. We call such partition $U \sqcup V$ the *partition* of the bipartite graph G.

For $1 \leq i \leq d$, let $\mathbf{e}_i$ be the ith coordinate vectors of $\mathbb{R}^d$.

Definition 1 Let G be a graph on the vertex set $[d] := \{1, \ldots, d\}$. Given an edge $\{i, j\} \in E(G)$ in G, let $\rho(e) \in \mathbb{R}^d$ denote the vector $\mathbf{e}_i + \mathbf{e}_j$. We write $\mathcal{P}_G$ for the convex hull of the set of integer points $\{\rho(e) : e \in E(G)\}$. This polytope $\mathcal{P}_G$ is called the *edge polytope* of G.

Let $G_1, \ldots, G_m$ be graphs on the vertex set $[d]$ with the edge set $E(G_1), \ldots, E(G_m)$, respectively. We define $G_1 + \cdots + G_m$ by setting the graph on the vertex set $[d]$ with the edge set $\bigcup_{i=1}^m E(G_i)$.

Proposition 2 *Let $G_1, \ldots, G_m$ be connected graphs on the same vertex set $[d]$. Then*

$$\dim(\mathcal{P}_{G_1} + \cdots + \mathcal{P}_{G_m}) = \begin{cases} d-2, & \text{if } G_1 + \cdots + G_m \text{ is bipartite,} \\ d-1, & \text{if } G_1 + \cdots + G_m \text{ is non-bipartite.} \end{cases}$$

Note that this proposition is a generalization of [4, Proposition 1.3].

3 The equivalence of normality and IDP

Next, in this section, we consider the equivalence of the normality and IDP for Minkowski sums of edge polytopes. More precisely, the following holds:

Theorem 3 *Let $G_1, \ldots, G_m$ be connected graphs on the same vertex set $[d]$. Then $\mathcal{P}_{G_1} + \cdots + \mathcal{P}_{G_m}$ is normal if and only if $\mathcal{P}_{G_1} + \cdots + \mathcal{P}_{G_m}$ has IDP.*

Strategy of Proof

Let $\mathcal{P} = \mathcal{P}_{G_1} + \cdots + \mathcal{P}_{G_m}$. We may show that if $\mathcal{P}$ is normal, then $\mathcal{P}$ has IDP. In other words, it suffices to show that for every $k \in \mathbb{Z}_{>0}$ and $\alpha \in k\mathcal{P} \cap \mathbb{Z}^d$, one has $\alpha \in \mathbb{Z}\mathcal{A}_{\mathcal{P}}$.

In the case where $G_1 + \cdots + G_m$ is bipartite, let $U \cup V$ be the partition of $G_1 + \cdots + G_m$. Then it turns out that $\mathbb{Z}\mathcal{A}_{\mathcal{P}}$ coincides with $\mathbb{Z}\mathcal{A}$, where

$$\mathcal{A} = \{\mathbf{e}_{p_1} + \cdots + \mathbf{e}_{p_m} + \mathbf{e}_{q_1} + \cdots + \mathbf{e}_{q_m} : p_i \in U, q_i \in V\}.$$

It is clear that $k\mathcal{P} \cap \mathbb{Z}^d \subset \mathbb{Z}\mathcal{A}$. Hence, we are done.

In the case where $G_1 + \cdots + G_m$ is non-bipartite, it turns out that $\mathbb{Z}\mathcal{A}_{\mathcal{P}}$ coincides with $\mathbb{Z}\mathcal{B}$, where

$$\mathcal{B} = \{\mathbf{e}_{p_1} + \cdots + \mathbf{e}_{p_m} + \mathbf{e}_{q_1} + \cdots + \mathbf{e}_{q_m} : 1 \leq p_i, q_i \leq d\}.$$

Since $k\mathcal{P} \cap \mathbb{Z}^d \subset \mathbb{Z}\mathcal{B}$, we are done. ■

We also note that in the proof of [4, Theorem 2.2], the equivalence of normality and IDP for edge polytopes of connected graphs is essentially proved. Theorem 3 says that this equivalence is also true for Minkowski sums of edge polytopes. However, if we drop the connectedness, then this equivalence does not hold.

Example 4 Consider the graphs G_1 and G_2 in Figure 1. Then G_2 is not connected. Now one has

$$\left(\frac{1}{2}\rho(\{1,2\}) + \frac{1}{2}\rho(\{2,3\}) + \frac{1}{2}\rho(\{3,4\}) + \frac{1}{2}\rho(\{5,6\})\right) + \left(\frac{1}{2}\rho(\{1,5\}) + \frac{3}{2}\rho(\{4,6\})\right)$$
$$= (1,1,1,2,1,2) \in 2(\mathcal{P}_{G_1} + \mathcal{P}_{G_2}) \cap \mathbb{Z}^6.$$

One can see that $(1, 1, 1, 2, 1, 2)$ cannot be written as a sum of any two integer points in $(\mathcal{P}_{G_1} + \mathcal{P}_{G_2}) \cap \mathbb{Z}^6$. Hence, $\mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ does not have IDP. On the other hand, since $(1, 1, 1, 2, 1, 2)$ does not belong to $\mathbb{Z}\mathcal{A}_{\mathcal{P}_{G_1}+\mathcal{P}_{G_2}}$, we need not consider this integer point for the normality. Actually, one can check that $\mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ is normal.

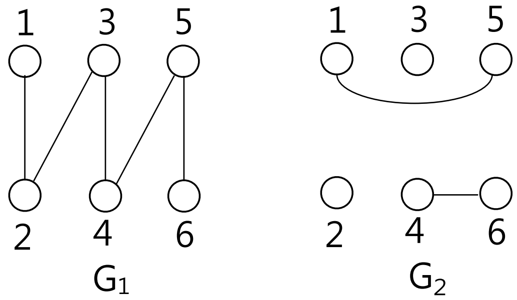

Figure 1: An example of graphs G_1 and G_2 such that $\mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ is normal but does not have IDP

4 When does the Minkowski sum of edge polytopes have IDP?

In this section, we give a sufficient condition for Minkowski of edge polytopes to have IDP (be normal) as follows.

Theorem 5 *Let G_1 and G_2 be graphs on the same vertex set $[d]$. We assume that G_1 is connected and arbitrary two odd cycles in G_1 always have a common vertex. We also assume that G_2 is a subgraph of G_1. Then $\mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ has IDP, and thus, this is normal.*

Example 6 (a) Theorem 5 is no longer true for the case of three graphs. Let G_1, G_2 and G_3 be graphs in Figure 2.

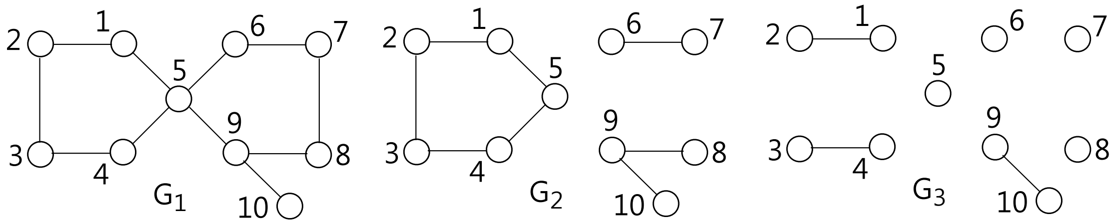

Figure 2: A counterexample for Theorem 5 in the case of three graphs

Then

$$\left(\frac{3}{5}\rho(\{5,6\})+\frac{3}{5}\rho(\{7,8\})+\frac{3}{5}\rho(\{5,9\})+\frac{1}{5}\rho(\{9,10\})\right)+$$
$$\left(\frac{2}{5}\rho(\{1,5\})+\frac{2}{5}\rho(\{2,3\})+\frac{2}{5}\rho(\{4,5\})+\frac{2}{5}\rho(\{6,7\})+\frac{2}{5}\rho(\{8,9\})\right)+$$
$$\left(\frac{3}{5}\rho(\{1,2\})+\frac{3}{5}\rho(\{3,4\})+\frac{4}{5}\rho(\{9,10\})\right)$$
$$=(1,1,1,1,2,1,1,1,2,1)\in 2(\mathcal{P}_{G_1}+\mathcal{P}_{G_2}+\mathcal{P}_{G_3})\cap\mathbb{Z}^{10}.$$

One can check that $(1,1,1,1,2,1,1,1,2,1)$ cannot be written as any sum of two integer points in $(\mathcal{P}_{G_1}+\mathcal{P}_{G_2}+\mathcal{P}_{G_3})\cap\mathbb{Z}^{10}$. Thus this does not have IDP. Moreover, by Theorem 3, this is not normal, either.

(b) Furthermore, the following example shows that the assumption "two odd cycles alywas have a common vertex" is necessary. The graphs G_1 and G_2 in Figure 3 are an example such that $\mathcal{P}_{G_1}$ is normal but $\mathcal{P}_{G_1}+\mathcal{P}_{G_2}$ is not, where G_1 does not satisfy the condition "two odd cycles alywas have a common vertex".

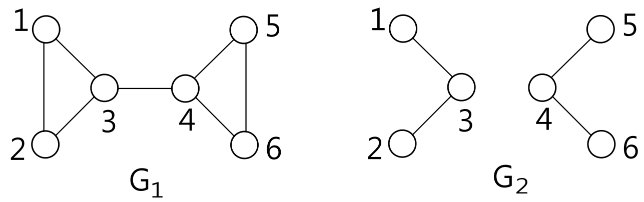

Figure 3: An example showing that our assumption is necessary

(c) In addition, the following graphs in Figure 4 show that the assumption "G_2 is a subgraph of G_1" is also necessary. Namely, each of $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ satisfies that two odd cycles alywas have a common vertex but $\mathcal{P}_{G_1}+\mathcal{P}_{G_2}$ is not normal.

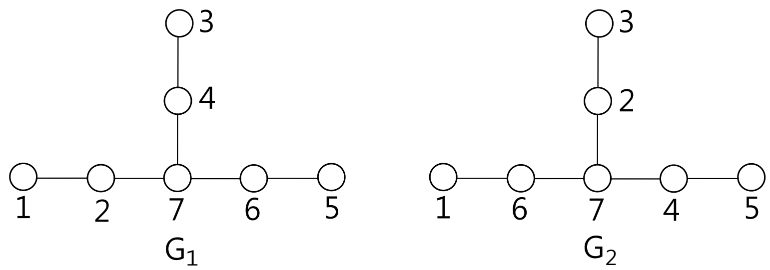

Figure 4: Another example showing that our assumption is necessary

5 Toric ring associated to Minkowski sum of dilated polytopes

Finally, we present a result on toric rings associated to Minkowski sum of dilated polytopes.

Theorem 7 *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^N$ be m integral convex polytopes and let $n_1, \ldots, n_m$ positive integers. Then*

(a) *$n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m$ has IDP if $n_i \geq \dim \mathcal{P}_i$ for every $1 \leq i \leq m$;*

(b) *$k[n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m]$ is level with a-invariant -1 if $n_i \geq \dim \mathcal{P}_i + 1$ for every $1 \leq i \leq m$.*

This theorem is an analogy of [1, Theorem 1.3.3 (a)]. For an integral convex polytope $\mathcal{P}$ of dimension d, it is proved in [1, Theorem 1.3.3 (a)] that $n\mathcal{P}$ is normal if $n \geq d - 1$, $k[n\mathcal{P}]$ is Koszul if $n \geq d$ and $k[n\mathcal{P}]$ is level with a-invariant -1 if $n \geq d + 1$.

We remain the following question:

Question 8 *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^N$ be m integral convex polytopes and let $n_1, \ldots, n_m$ positive integers. Is $k[n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m]$ Koszul if $n_i \geq \dim \mathcal{P}_i$ for every $1 \leq i \leq m$?*

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