On degenerations of graded maximal Cohen-Macaulay modules

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1. Introduction

The notion of degenerations of modules appears in geometric methods of representation theory of finite dimensional algebras. In [16], Yoshino gives a scheme-theoretical definition of degenerations, so that it can be considered for modules over a Noetherian algebra which is not necessary finite dimensional. Now, a theory of degenerations is considered for derived module categories [11] or stable module categories [17]. For the study of the degeneration problem of modules, several order relations for modules were introduced, and the connection among them has been studied [6, 13, 19] etc.

In this note we consider degenerations of graded maximal Cohen-Macaulay modules. First we define an order relation called the hom order on the category of graded maximal Cohen-Macaulay modules (Definition 2). Actually it is a partial order over a graded ring is Gorenstein with a graded isolated singularity.

Taking the scheme-theoretical definition of degenerations into account, we propose a notion of degenerations for graded modules and we state several properties on it. We show that if the graded ring is of graded finite representation type and representation directed, then the hom order, the degeneration order and the extension order are identical on the graded maximal Cohen-Macaulay modules.

We also consider a stable analogue of degenerations of graded maximal Cohen-Macaulay modules and study a relation between degenerations for derived module categories of a finite dimensional algebra and it.

2. Hom order on graded modules

Throughout the note let $R = \oplus_{i=0}^{\infty} R_i$ be a commutative Noetherian $\mathbb{N}$-graded Gorenstein ring with $R_0 = k$ an algebraically field of characteristic zero. We denote by $\mathrm{mod}^{\mathbb{Z}}(R)$ the category of finitely generated $\mathbb{Z}$-graded modules whose morphisms are homogenous morphisms that preserve degrees. For $i \in \mathbb{Z}$, $M(i) \in \mathrm{mod}^{\mathbb{Z}}(R)$ is defined by $M(i)_n = M_{n+i}$. Then $\mathrm{Hom}_R(M, N(i))$ consisting of homogenous morphisms of degree i, and we set

$$ {}^*\mathrm{Hom}_R(M, N) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(M, N(i)). $$

For a graded prime ideal $\mathfrak{p}$ of R, we denote by $R_{(\mathfrak{p})}$ a graded localization of R by $\mathfrak{p}$. For $M \in \mathrm{mod}^{\mathbb{Z}}(R)$, take a graded free resolution

$$ \cdots \to F_l \xrightarrow{d_l} F_{l-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to M \to 0. $$

This paper is an announcement of our result and the detailed version have been submitted to somewhere.

We define an lth syzygy module $\Omega^l M$ of M by $\mathrm{Im} d_l$. We say that a graded R-module M is said to be a graded maximal Cohen-Macaulay R-module if

$$^*\mathrm{Ext}^i_R(M, R(-a)) = 0 \quad \text{for any } i > 0,$$

where $a = a(R)$, that is an a-invarinat of R. We denote by $\mathrm{CM}^{\mathbb{Z}}(R)$ the full subcategory of $\mathrm{mod}^{\mathbb{Z}}(R)$ consisting of graded maximal Cohen-Macaulay R-modules. For $M \in \mathrm{mod}^{\mathbb{Z}}(R)$ we denote by $h(M)$ a sequence $(\dim_k M_n)_{n\in\mathbb{Z}}$ of non-negative integers.

Remark 1.
(1) R is *local, namely $\mathfrak{m} = \oplus_{i>0} R_i$ is a unique maximal graded ideal of R.
(2) By the definition, it is easy to see that $h(M) = h(N)$ if and only if they have the same Hilbert series. Moreover, we also have that $h(M) = h(N)$ if and only if $h(M^*) = h(N^*)$ where $(-)^* = {}^*\mathrm{Hom}_R(-, R(-a))$. See [7, Theorem 4.4.5.].
(3) $\mathrm{mod}^{\mathbb{Z}}(R)$ and $\mathrm{CM}^{\mathbb{Z}}(R)$ are Krull-Schmidt, namely each object can be decomposed into indecomposable objects up to isomorphism uniquely.

Our motivation of the note is to investigate the graded degenerations of graded maximal Cohen-Macaulay modules in terms of some order relations, so that we consider the following relation on $\mathrm{CM}^{\mathbb{Z}}(R)$ that is known as the hom order.

Definition 2. For M, $N \in \mathrm{CM}^{\mathbb{Z}}(R)$ we define $M \leq_{hom} N$ if $[M, X] \leq [N, X]$ for each $X \in \mathrm{CM}^{\mathbb{Z}}(R)$. Here $[M, X]$ is an abbreviation of $\dim_k \mathrm{Hom}_R(M, X)$.

Remark 3. As a consequence in [5], if R is of dimension 0, 1 or 2, $\leq_{hom}$ is a partial order on $\mathrm{CM}^{\mathbb{Z}}(R)$ since $\mathrm{CM}^{\mathbb{Z}}(R)$ is closed under kernels in such cases.

Lemma 4. *Let M and N be indecomposable graded maximal Cohen-Macaulay R-modules. Suppose that $h(M) = h(N)$. For $Y \in \mathrm{mod}^{\mathbb{Z}}(R)$ which is of finite projective dimension, we have $[M, Y] = [N, Y]$.*

Proof. For a graded R-module Y which is of finite projective dimension, $\Omega^i Y$ is also of finite projective dimension. Since R is Gorenstein, we have $\mathrm{Ext}^1_R(M, \Omega^i Y) = 0$ for all graded maximal Cohen-Macaulay R-modules M. Thus, take a graded free resolution of Y and apply $\mathrm{Hom}_R(M, -)$ to the resolution, we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, F_l) \to \mathrm{Hom}_R(M, F_{l-1}) \to \cdots \to \mathrm{Hom}_R(M, F_0) \to \mathrm{Hom}_R(M, Y) \to 0.$$

Hence

$$[M, Y] = \sum_{i=0}^{l} (-1)^i [M, F_i].$$

We also have

$$[N, Y] = \sum_{i=0}^{l} (-1)^i [N, F_i].$$

As mentioned in Remark 1 (2), we have an equality $[M, F] = [N, F]$ for each graded free module. Therefore we have

$$[M, Y] = \sum_{i=0}^{l} (-1)^i [M, F_i] = \sum_{i=0}^{l} (-1)^i [N, F_i] = [N, Y].$$

$\square$

In the note we use the theory of Auslander-Reiten (abbr. AR) sequences of graded maximal Cohen-Macaulay modules. For the detail, we recommend the reader to look at [10, 3, 4] and [14, Chapter 15.]. We say that $(R, \mathfrak{m})$ is a graded isolated singularity if each graded localization $R_{(\mathfrak{p})}$ is regular for each graded prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$.

Theorem 5. [3, 4, 14, 10] *Assume that $(R, \mathfrak{m})$ is a graded isolated singularity. Then $\mathrm{CM}^{\mathbb{Z}}(R)$ admits AR sequences.*

We denote by $\mu(M, Z)$ the multiplicity of Z as a direct summand of M.

Theorem 6. *Let M and N graded maximal Cohen-Macaulay R-modules. Assume that R is a graded isolated singularity. Then $[M, X] = [N, X]$ for each $X \in \mathrm{CM}^{\mathbb{Z}}(R)$ if and only if $M \cong N$. Particularly, $\leq_{hom}$ is a partial order on $\mathrm{CM}^{\mathbb{Z}}(R)$.*

Proof. We decompose M as $M = \oplus M_i^{\mu(M,M_i)}$ where M_i are indecomposable graded maximal Cohen-Macaulay R-modules. If M_i is not free, we can take the AR sequence ending in M_i

$$0 \to \tau M_i \to E_i \to M_i \to 0,$$

where τM_i is an AR translation of M_i. Apply $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(N, -)$ to the sequence, since k is an algebraically closed field, we have

$$0 \to \mathrm{Hom}_R(M, \tau M_i) \to \mathrm{Hom}_R(M, E_i) \to \mathrm{Hom}_R(M, M_i) \to k^{\mu(M,M_i)} \to 0$$

and

$$0 \to \mathrm{Hom}_R(N, \tau M_i) \to \mathrm{Hom}_R(N, E_i) \to \mathrm{Hom}_R(N, M_i) \to k^{\mu(N,M_i)} \to 0.$$

Counting the dimensions of terms, we conclude that $\mu(M, M_i) = \mu(N, M_i)$.

If M_i is free, we may assume that $M_i = R$. Let $\mathfrak{m}$ be a *maximal ideal of R. We consider the Cohen-Macaulay approximation (see [2]) of $\mathfrak{m}$

$$0 \to Y \to X \to \mathfrak{m} \to 0.$$

We note that X is a graded maximal Cohen-Macaulay R-module and Y is of finite injective dimension. We also note that Y is of finite projective dimension since R is Gorenstein. Let f be a composition map of the approximation $X \to \mathfrak{m}$ and a natural inclusion $\mathfrak{m} \to R$. Then we get the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & X \arrow[r, "f"] & R & \\
0 \arrow[r] & Y \arrow[r] \arrow[u, "g"] & X \arrow[r] \arrow[u, "="] & \mathfrak{m} \arrow[r] \arrow[u, "\subseteq"] & 0.
\end{tikzcd}$$

By a diagram chasing we see that g is surjective, so that K is of finite projective dimension. Now we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, K) \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, R) \to k^{\mu(M,R)} \to 0.$$

According to Lemma 4, $[M, K] = [N, K]$. Thus

$$[M, R] + [M, K] - [M, X] = [N, R] + [N, K] - [N, X].$$

Hence $\mu(M, R) = \mu(N, R)$. Consequently $M \cong N$. $\square$

3. Degenerations of graded maximal Cohen-Macaulay modules

Taking the scheme-theoretical definition of degenerations into account, we define a notion of degenerations for graded modules.

Definition 7. Let R be a Noetherian $\mathbb{N}$-graded ring where $R_0 = k$ is a field and let $V = k[[t]]$ with $\deg t = 1$ and K be the localization by t, namely $K = V_t = k(t)$. For R-modules M and $N \in \mathrm{mod}^{\mathbb{Z}}(R)$, we say that M gradually degenerates to N if there is a finitely generated graded $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module.
(2) $Q \otimes_V V/tV \cong N$ as a graded R-module.
(3) $Q \otimes_V K \cong M \otimes_k K$ as a graded $R \otimes_k K$-module.

In [16], Yoshino gives a necessary and sufficient condition for degenerations of (non-graded) modules. One can also show its graded version in a similar way.

Theorem 8. [16, Theorem 2.2] *The following conditions are equivalent for M and $N \in \mathrm{mod}^{\mathbb{Z}}(R)$.*

(1) *M gradually degenerates to N.*
(2) *There is a short exact sequence of finitely generated graded R-modules*

$$0 \longrightarrow Z \longrightarrow M \oplus Z \longrightarrow N \longrightarrow 0.$$

Remark 9. (1) As Yoshino has shown in [16], the endomorphism of Z in the sequence of Theorem 8 is nilpotent. However we do not need the nilpotency assumption here. Actually, since $\mathrm{End}_R(Z)$ is Artinian, by using Fitting theorem, we can describe the endomorphism as a direct sum of an isomorphism and a nilpotent morphism. See also [16, Remark 2.3.].

(2) Assume that M and N are graded maximal Cohen-Macaulay modules. Then Z is also a graded maximal Cohen-Macaulay R-module. See [16, Remark 4.3.].

(3) Assume that there is an exact sequence of finitely generated graded R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then M gradually degenerates to $L \oplus N$. See [16, Remark 2.5] for instance.

(4) Suppose that M gradually degenerates to N. Then $h(M) = h(N)$ and $M \leq_{hom} N$. In fact, M and N give the same class in the Grothendieck group.

(5) We can also prove in a similar way to the proof of [18, Theroem 2.1.] that, for L, M, $N \in \mathrm{mod}^{\mathbb{Z}}(R)$, if L gradually degenerates to M and if M gradually degenerates to N then L gradually degenerates to N.

Definition 10. For M, $N \in \mathrm{CM}^{\mathbb{Z}}(R)$, we define the relation $M \leq_{deg} N$, which is called the degeneration order, if M gradually degenerates to N. We also define the relation $M \leq_{ext} N$ if there are modules M_i, N'_i, N''_i and short exact sequences $0 \to N'_i \to M_i \to N''_i \to 0$ in $\mathrm{CM}^{\mathbb{Z}}(R)$ so that $M = M_1$, $M_{i+1} = N'_i \oplus N''_i$, $1 \leq i \leq s$ and $N = M_{s+1}$ for some s.

For $M \in \mathrm{CM}^{\mathbb{Z}}(R)$, we take a first syzygy module of M^*

$$0 \to \Omega^1 M^* \to F \to M^* \to 0.$$

Applying $(-)^*$ to the sequence, we have

$$0 \to M^{**} \cong M \to F^* \to (\Omega^1 M^*)^* \to 0.$$

Then we denote $(\Omega^1 M^*)^*$ by $\Omega^{-1}M$.

Now we state a key lemma of our result.

Lemma 11. *Let R be a graded isolated singularity and let M and N be graded maximal Cohen-Macaulay R-modules. Assume that $h(M) = h(N)$ and $M \leq_{hom} N$. Then, for each graded maximal Cohen-Macaulay R-module X, there exists an integer $l_X \gg 0$ such that $[M, X(\pm l_X)] = [N, X(\pm l_X)]$.*

Proof. For each $X \in \mathrm{CM}^{\mathbb{Z}}(R)$, we can take an exact sequence as above

$$0 \to X \to F^* \to \Omega^{-1}X \to 0.$$

Applying ${}^*\mathrm{Hom}_R(M, -)$ to the sequence, we have

$$0 \to {}^*\mathrm{Hom}_R(M, X) \to {}^*\mathrm{Hom}_R(M, F^*) \to {}^*\mathrm{Hom}_R(M, \Omega^{-1}X) \to {}^*\mathrm{Ext}^1_R(M, X) \to 0.$$

Since R is a graded isolated singularity, $\dim {}^*\mathrm{Ext}^1_R(M, X)$ is finite. Thus ${}^*\mathrm{Ext}^1_R(M, X)_{\pm l_1} = 0$ for sufficiently large $l_1 \gg 0$. Similarly there also exists an integer $l_2 \gg 0$ so that ${}^*\mathrm{Ext}^1_R(N, X)_{\pm l_2} = 0$.

Set $l = max\{l_1, l_2\}$. Then we have

$$0 \to {}^*\mathrm{Hom}_R(M, X)_{\pm l} \to {}^*\mathrm{Hom}_R(M, F^*)_{\pm l} \to {}^*\mathrm{Hom}_R(M, \Omega^{-1}X)_{\pm l} \to 0.$$

Therefore we obtain the equation

$$[M, X(\pm l)] = [M, F^*(\pm l)] - [M, \Omega^{-1}X(\pm l)].$$

We also have

$$[N, X(\pm l)] = [N, F^*(\pm l)] - [N, \Omega^{-1}X(\pm l)].$$

Suppose that $[M, X(\pm l)] < [N, X(\pm l)]$. Then the following inequality holds.

$$[M, F^*(\pm l)] - [M, \Omega^{-1}(X)(\pm l)] < [N, F^*(\pm l)] - [N, \Omega^{-1}X(\pm l)].$$

Since $h(M) = h(N)$, $[M, F^*(\pm l)] = [N, F^*(\pm l)]$. Hence we see that $[M, \Omega^{-1}X(\pm l)] > [N, \Omega^{-1}X(\pm l)]$. This is a contradiction since $M \leq_{hom} N$. Therefore we have some integer l such that $[M, X(\pm l)] = [N, X(\pm l)]$. $\square$

We say that the category $\mathrm{CM}^{\mathbb{Z}}(R)$ is of graded finite representation type if there are only a finite number of isomorphism classes of indecomposable graded maximal Cohen-Macaulay modules up to shift. We note that if $\mathrm{CM}^{\mathbb{Z}}(R)$ is of finite representation type, then R is a graded isolated singularity. See [14, Chapter 15.] for the detail.

For graded maximal Cohen-Macaulay R-modules M and N, we consider the following set of all the isomorphism classes of indecomposable graded maximal Cohen-Macaulay modules

$$\mathcal{F}_{M,N} = \{X \mid [N, X] - [M, X] > 0 \}/\cong .$$

As an immediate consequence of Lemma 11, we have the following.

Corollary 12. *Let R be of finite representation type and let M and N be graded maximal Cohen-Macaulay R-modules. Assume that $h(M) = h(N)$ and $M \leq_{hom} N$. Then $\mathcal{F}_{M,N}$ is a finite set.*

Proposition 13. [13] *Let R be of graded finite representation type and let M and N be graded maximal Cohen-Macaulay R-modules. Assume that $h(M) = h(N)$ and $M \leq_{hom} N$. Then there exists some graded maximal Cohen-Macaulay R-module L such that $M \oplus L$ degenerates to $N \oplus L$.*

Proof. Since R is a graded isolated singularity, $\mathrm{CM}^{\mathbb{Z}}(R)$ admits AR sequences. For each $X \in \mathcal{F}_{M,N}$, we can take an AR sequence starting from X.

$$\Sigma_X : 0 \to X \to E_X \to \tau^{-1}X \to 0.$$

Now we consider a sequence which is a direct sum of $[N.X]-[M,X]$ copies of Σ_X where X runs through all modules in $\mathcal{F}_{M,N}$. Namely

$$\bigoplus_{X \in \mathcal{F}_{M,N}} \Sigma_X^{[N.X]-[M,X]} = 0 \to U \to V \to W \to 0.$$

For any indecomposable $Z \in \mathrm{CM}^{\mathbb{Z}}(R)$, we obtain

$$0 \to \mathrm{Hom}_R(W, Z) \to \mathrm{Hom}_R(V, Z) \to \mathrm{Hom}_R(U, Z) \to k^{[N,Z]-[M,Z]} \to 0.$$

This implies that the equality

$$[U, Z] + [W, Z] + [M, Z] = [N, Z] + [V, Z]$$

for all $Z \in \mathrm{CM}^{\mathbb{Z}}(R)$. Hence this yields that

$$M \oplus U \oplus W \cong N \oplus V.$$

Since V degenerates to $U \oplus W$, therefore $M \oplus V$ degenerates to $M \oplus U \oplus W \cong N \oplus V$. $\square$

Now we focus on the case that a graded Gorenstein ring is of graded finite representation type and representation directed. We say that a graded maximal Cohen-Macaulay ring R is representation directed if the AR quiver of $\mathrm{CM}^{\mathbb{Z}}(R)$ has no oriented cyclic paths. Bongartz [6] has studied such a case over finite dimensional k-algebras. In our graded settings, the similar results hold.

Theorem 14. [8, Theorem 3.8.] *Let R be of graded finite representation type and representation directed. Then the following conditions are equivalent for M and $N \in \mathrm{CM}^{\mathbb{Z}}(R)$.*

(1) *$h(M) = h(N)$ and $M \leq_{hom} N$.*
(2) *$M \leq_{deg} N$.*
(3) *$M \leq_{ext} N$.*

To prove the theorem, we modify the arguments in [6, 19].

Lemma 15. [8, Lemma 3.9.] *Let M, N and X be finitely generated graded R-modules.*

(1) *Assume that $[X, M] = [X, N]$. If $M \oplus X$ gradually degenerates to $N \oplus X$, M gradually degenerates to N.*
(2) *Assume that R is Gorenstein and M and N graded maximal Cohen-Macaulay R-modules. If M gradually degenerates to $N \oplus F$ for some graded free R-module F then M/F gradually degenerates to N.*

For indecomposable graded maximal Cohen-Macaulay modules M and N, we write $X \preceq Y$ if $X \cong Y$ or if there exists a finite path from X to Y in the AR quiver of $\mathrm{CM}^{\mathbb{Z}}(R)$.

Lemma 16. [8, Lemma 3.10.][6] *Let R be a graded Gorenstein ring which is of graded finite representation type and representation directed and let M and $N \in \mathrm{CM}^{\mathbb{Z}}(R)$. Assume that $h(M) = h(N)$, $M \leq_{hom} N$ and M and N have no common direct summands. Let $X \in \mathrm{CM}^{\mathbb{Z}}(R)$ be an indecomposable module such that $\preceq$-minimal with the property $[N, X] - [M, X] > 0$ and E be a middle term of an AR sequence starting from X. Then $[E, N] = [E, M]$.*

Proof of Theorem 14. We give an outline of the proof of the implication (1) $\Rightarrow$ (2). Set $V = \oplus_{W \in \mathcal{F}_{M,N}} W$. Since $[N, X] - [M, X] = 0$ for any $X \notin \mathcal{F}_{M,N}$, we can show the implication by induction on $d = [N, V] - [M, V]$. If $d = 0$, $[N, Z] = [M, Z]$ for each $X \in \mathrm{CM}^{\mathbb{Z}}(R)$. Thus $M \cong N$. Hence assume that $d > 0$ and M and N have no summand in common in the inductive step. We take $X \in \mathrm{CM}^{\mathbb{Z}}(R)$ in Lemma 16 and let E be a middle term of the AR sequence starting from X. Then we can show $E \oplus M \leq_{deg} E \oplus N$. Hence, by virtue of Lemma 16 and Lemma 15 (1), we have $M \leq_{deg} N$. $\square$

4. An application

In the rest of the note we consider the stable analogue of degenerations of graded maximal Cohen-Macaulay modules.

Note that, since R is Gorenstein, $R \otimes_k V$ and $R \otimes_k K$ are also Gorenstein. Thus $\underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k V)$ and $\underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k K)$ are triangulated categories. We denote by $\mathcal{L} : \underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k V) \to \underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k K)$ (resp. $\mathcal{R} : \underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k V) \to \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$) the triangle functor defined by the localization by t (resp. taking $- \otimes_V V/tV$). See also [17, Definition 4.1].

Definition 17. Let $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. We say that $\underline{M}$ stably degenerates to $\underline{N}$ if there exists a graded maximal Cohen-Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}^{\mathbb{Z}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$.

One can show the following characterization of stable degenerations similarly to the proof of [17, Theorem 5.1].

Theorem 18. *The following conditions are equivalent for graded maximal Cohen-Macaulay R-modules M and N.*

1. *$F \oplus M$ degenerates to N for some graded free R-module F.*
2. *There is a triangle in $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$*
$$\underline{Z} \longrightarrow \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1].$$
3. *$\underline{M}$ stably degenerates to $\underline{N}$.*

Proof. We should note that $R \otimes_k V$ and $R \otimes_k K$ are *local. It makes difference from the non-graded case. $\square$

Remark 19. A theory of degenerations for derived categories has been studied in [11]. They have shown that, for complexes M, N in the bounded derived category of a finite dimensional algebra, M degenerates to N if and only if there exists a triangle of the form which appears in the above theorem. As shown in [1, 12, 10], suppose that R has a simple singularity then there exists a Dynkin quiver Q such that we have a triangle equivalence
$$\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \cong \mathrm{D}^b(kQ)$$

where $\mathrm{D}^b(kQ)$ is a bounded derived category of the category of finitely generated left modules over a path algebra kQ. By virtue of Theorem 18, we can describe the degenerations for $\mathrm{D}^b(kQ)$ in terms of the graded degenerations for $\mathrm{CM}^{\mathbb{Z}}(R)$.

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