On the limit of Frobenius in the Grothendieck group*

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A part is a joint work with Kosuke Ohta (Meiji University). Another part is a joint work with C-Y. Jean Chan (Central Michigan University) [1].

1 Introduction

We define the Cohen-Macaulay cone $C_{CM}(R)$, the strictly nef cone $SN(R)$, and the fundamental class $\overline{\mu_R}$ for a Noetherian local domain R. They satisfy

$$\begin{array}{ccccc} \overline{G_0(R)}_{\mathbb{R}} & \supset & SN(R) & \supset & C_{CM}(R) - \{0\} \\ \cup & & & & \\ \overline{G_0(R)}_{\mathbb{Q}} & \ni & \overline{\mu_R} & & \end{array}$$

where $G_0(R)$ is the Grothendick group of finitely generated R-modules, $\overline{G_0(R)}$ is the Grothendick group modulo numerical equivalence, and $\overline{G_0(R)}_K = \overline{G_0(R)} \otimes_{\mathbb{Z}} K$.

The fundamental class is deeply related to the homological conjectures as in Fact 8.

We are mainly interested in the problem whether $\overline{\mu_R}$ is in such cones or not. Theorem 11 is the main theorem, which states that if R is FFRT or F-rational, then $\overline{\mu_R}$ is in $C_{CM}(R)$. We shall give a corollary (Corollary 14).

2 Cohen-Macaulay cone

In this note, let R be a d-dimensional Noetherian local domain such that one of the following conditions are satisfied:

(a) R is a homomorphic image of an excellent regular local ring containing $\mathbb{Q}$.

(b) R is essentially of finite type over a field, $\mathbb{Z}$ or a complete DVR.

If either (a) or (b) is satisfied, there exists a regular alteration of $\operatorname{Spec} R$ by de Jong's theorem [4].

We always assume that modules are finitely generated.

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Let $G_0(R)$ be the Grothendieck group of finitely generated R-modules, that is,

$$G_0(R) := \frac{\bigoplus_{M\ :\ \text{f.g.}R\text{-module}} \mathbb{Z}[M]}{< [M]-[L]-[N] \mid 0 \to L \to M \to N \to 0 \text{ is exact} >}.$$

Let $C(R)$ be the category of bounded complexes of finitely generated R-free modules such that every homologies are of finite length. Let $C_d(R)$ be the subcategory of $C(R)$ consisting of complexes of length d with $H^0(\mathbb{F}.) \neq 0$. A complex $\mathbb{F}.$ in $C_d(R)$ is of the form

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

For example, the Koszul complex of a parameter ideal belongs to $C_d(R)$.

For $\mathbb{F}. \in C(R)$, we have a well-defined map

$$\chi_{\mathbb{F}.} : G_0(R) \longrightarrow \mathbb{Z}$$

by $\chi_{\mathbb{F}.}([M]) = \sum_i (-1)^i \ell_R(H_i(\mathbb{F}.\otimes_R M))$. We have the induced maps $\chi_{\mathbb{F}.} : G_0(R)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ and $\chi_{\mathbb{F}.} : G_0(R)_{\mathbb{R}} \longrightarrow \mathbb{R}$. We say that $\alpha \in G_0(R)$ ($\alpha \in G_0(R)_{\mathbb{Q}}$ or $\alpha \in G_0(R)_{\mathbb{R}}$) is numerically equivalent to 0 if $\chi_{\mathbb{F}.}(\alpha) = 0$ for any $\mathbb{F}. \in C(R)$. We define the Grothendieck group modulo numerical equivalence as follows:

$$\overline{G_0(R)} = G_0(R)/\{\alpha \in G_0(R) \mid \chi_{\mathbb{F}.}(\alpha) = 0 \text{ for any } \mathbb{F}. \in C(R)\}.$$

Then, by Theorem 3.1 and Remark 3.5 in [6], we know that $\overline{G_0(R)}$ is a non-zero finitely generated $\mathbb{Z}$-free module.[1]

Example 1 1) If $d \leq 2$, then $\overline{G_0(R)} = \mathbb{Z}$ (Proposition 3.7 in [6]).

2) Let X be a smooth projective variety with embedding $X \hookrightarrow \mathbb{P}^n$. Let R (resp. D) be the affine cone (resp. the very ample divisor) of this embedding. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \stackrel{\sim}{\longrightarrow} & A_*(R)_{\mathbb{Q}} & \stackrel{\sim}{\longleftarrow} & CH^{\cdot}(X)_{\mathbb{Q}}/D \cdot CH^{\cdot}(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{G_0(R)}_{\mathbb{Q}} & \stackrel{\sim}{\longrightarrow} & \overline{A_*(R)}_{\mathbb{Q}} & \stackrel{\phi}{\longleftarrow} & CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D \cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}} \end{array}$$

(a) By the commutativity of this diagram, ϕ is a surjection. Therefore, we have

$$\operatorname{rank} \overline{G_0(R)} \leq \dim_{\mathbb{Q}} CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D \cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}}. \tag{1}$$

There are examples that the equality does not hold in (1) (see [11]). If some conjectures are true, the equality holds in (1) for a smooth projective variety X defined over $\overline{\mathbb{Q}}$ (see [11]).

(b) If $CH^{\cdot}(X)_{\mathbb{Q}} \simeq CH^{\cdot}_{num}(X)_{\mathbb{Q}}$, then ϕ is an isomorphism ([6], [11]). In this case, the equality holds in (1). Using it, we can show the following.
If X is a blow-up at n points of $\mathbb{P}^k$ ($k \geq 2$), then $\operatorname{rank} \overline{G_0(R)} = n + 1$.
If $X = \mathbb{P}^m \times \mathbb{P}^n$, then $\operatorname{rank} \overline{G_0(R)} = min\{m, n\}$.

[1] We need the existence of a regular alteration in the proof of this result.

Consider groups $\overline{G_0(R)} \subset \overline{G_0(R)}_{\mathbb{Q}} \subset \overline{G_0(R)}_{\mathbb{R}}$. We shall define some cones in $\overline{G_0(R)}_{\mathbb{R}}$.

Definition 2 Let $C_{CM}(R)$ be the *Cohen-Macaulay cone*, i.e.,

$$C_{CM}(R) = \sum_{M:MCM} \mathbb{R}_{\geq 0}[M] \subset \overline{G_0(R)}_{\mathbb{R}}.$$

Let $C_{CM}(R)^-$ be the closure of $C_{CM}(R)$ with respect to the classical topology on $\overline{G_0(R)}_{\mathbb{R}}$.

We define the *strictly nef cone* by

$$SN(R) = \{\alpha \mid \ \chi_{\mathbb{F}.}(\alpha) > 0 \text{ for any } \mathbb{F}. \in C_d(R)\}.$$

By the depth sensitivity, $\chi_{\mathbb{F}.}([M]) = \ell_R(H_0(\mathbb{F}. \otimes M)) > 0$ for any maximal Cohen-Macaulay module M ($\neq 0$) and $\mathbb{F}. \in C_d(R)$. Therefore,

$$SN(R) \supset C_{CM}(R) - \{0\}.$$

Remark 3 Assume that R is a Cohen-Macaulay ring.

Let M be a torsion R-module. Taking a sufficiently high syzygies of M, we know

$$\pm[M] + n[R] \in C_{CM}(R) \quad \text{for } n \gg 0.$$

Therefore, we have $\operatorname{rank} \overline{G_0(R)} = \dim C_{CM}(R)$ and

$$C_{CM}(R)^- \supset C_{CM}(R) \supset Int(C_{CM}(R)^-) = Int(C_{CM}(R)) \ni [R].$$

Example 4 The following examples are given in [2]. Assume that k is an algebraically closed field of characteristic zero.

1) Put $R = k[x, y, z, w]_{(x,y,z,w)}/(xy - f_1 f_2 \cdots f_t)$. Here, we assume that $f_1, f_2, \ldots, f_t$ are pairwise coprime linear forms in $k[z, w]$. In this case, we have $\operatorname{rank} \overline{G_0(R)} = t$.

   We can prove that the Cohen-Macaulay cone is minimally spanned by the following $2^t - 2$ maximal Cohen-Macaulay modules of rank one:

   $$\{(x, f_{i_1} f_{i_2} \cdots f_{i_s}) \mid 1 \leq s < t, \quad 1 \leq i_1 < i_2 < \cdots < i_s \leq t\}$$

   Here, remark that this ring is of finite representation type if and only if $t \leq 3$.

2) The Cohen-Macaulay cone of $R = k[x_1, x_2, \ldots, x_6]_{(x_1,x_2,\ldots,x_6)}/(x_1x_2 + x_3x_4 + x_5x_6)$ is not spanned by maximal Cohen-Macaulay modules of rank one. It is of finite representation type since it has a simple singularity.

3 Fundamental class

Definition 5 We put

$$\mu_R = {\tau_R}^{-1}([\operatorname{Spec} R]) \in G_0(R)_{\mathbb{Q}},$$

where $\tau_R : G_0(R)_{\mathbb{Q}} \xrightarrow{\sim} A_*(R)_{\mathbb{Q}}$ is the singular Riemann-Roch map.

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \longrightarrow & \overline{G_0(R)_{\mathbb{Q}}} \\ \mu_R & \mapsto & \overline{\mu_R} \end{array}$$

We call the image $\overline{\mu_R}$ in $\overline{G_0(R)_{\mathbb{Q}}}$ the *fundamental class* of R.

Remark that $\overline{\mu_R} \neq 0$ since $\operatorname{rank}_R \mu_R = 1$.

Put $R = T/I$, where T is a regular local ring. The map τ_R is defined using not only R but also T. Therefore, $\mu_R \in G_0(R)_{\mathbb{Q}}$ may depend on the choice of T.[2] However, we can prove that $\overline{\mu_R} \in \overline{G_0(R)_{\mathbb{Q}}}$ is independent of T (Theorem 5.1 in [6]).

We shall explain why we call $\overline{\mu_R}$ the fundamental class of R.

Remark 6 1) If $X = \operatorname{Spec} R$ is a d-dimensional affine variety over $\mathbb{C}$, we have the cycle map cl

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} & \xrightarrow{cl} & H_*(X, \mathbb{Q}) \\ \mu_R & \mapsto & [\operatorname{Spec} R] & \mapsto & \mu_X \end{array}$$

such that $cl([\operatorname{Spec} R])$ is the fundamental class μ_X in $H_{2d}(X, \mathbb{Q})$ in the usual sense, where $H_*(X, \mathbb{Q})$ is the Borel-Moore homology. Here μ_X is the generator of $H_{2d}(X, \mathbb{Q}) \simeq \mathbb{Z}$.

Hence, we call $\overline{\mu_R}$ the fundamental class of R.

2) Let R have a subring S such that S is a regular local ring and R is a localization of a finite extension of S. Let L be a finite-dimensional normal extension of $Q(S)$ containing $Q(R)$. Let B be the integral closure of R in L. Then, we have

$$\mu_R = \frac{1}{\operatorname{rank}_R B}[B] \text{ in } G_0(R)_{\mathbb{Q}}.$$

In particular, $\overline{\mu_R} = \frac{\overline{[B]}}{\operatorname{rank}_R B}$ in $\overline{G_0(R)_{\mathbb{Q}}}$.

3) Assume that R is of characteristic $p > 0$ and F-finite. Assume that the residue class field is algebraically closed. By the singular Riemann-Roch theorem, we have

$$\overline{\mu_R} = \lim_{e \to \infty} \frac{\overline{[{}^e R]}}{p^{de}} \text{ in } \overline{G_0(R)_{\mathbb{R}}},$$

where ${}^e R$ is the eth Frobenius direct image.

Example 7 1) If R is a complete intersection, then μ_R is equal to $[R]$ in $G_0(R)_{\mathbb{Q}}$, therefore $\overline{\mu_R} = \overline{[R]}$ in $\overline{G_0(R)_{\mathbb{Q}}}$. There exists a Gorenstein ring such that $\overline{\mu_R} \neq \overline{[R]}$. However there exist many examples of rings satisfying $\overline{\mu_R} = \overline{[R]}$. Roberts ([8], [9]) proved the vanishing property of intersection multiplicity for rings satisfying $\overline{\mu_R} = \overline{[R]}$.

[2] There is no example that the map τ_R actually depend on the choice of T. For some excellent rings, it had been proved that τ_R is independent of the choice of T (Proposition 1.2 in [5]).

2) Let R be a normal domain. Then, we have

$$\begin{array}{ccl} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} = A_d(R)_{\mathbb{Q}} \oplus A_{d-1}(R)_{\mathbb{Q}} \oplus \cdots \\ [R] & \mapsto & [\operatorname{Spec} R] - \frac{K_R}{2} + \cdots \\ [\omega_R] & \mapsto & [\operatorname{Spec} R] + \frac{K_R}{2} + \cdots \end{array}$$

If $\tau_R^{-1}(K_R) \neq 0$ in $\overline{G_0(R)_{\mathbb{Q}}}$, then $[R] \neq \overline{\mu_R}$.

Sometimes $\overline{\mu_R} = \frac{1}{2}([R] + [\omega_R])$ is satisfied. But it is not true in general.

3) Let $R = k[x_{ij}]/I_2(x_{ij})$, where (x_{ij}) is the generic $(m+1) \times (n+1)$-matrix, and k is a field. Suppose $0 < m \leq n$.

Then, we have

$$\begin{array}{rcl} G_0(R)_{\mathbb{Q}} \simeq \overline{G_0(R)_{\mathbb{Q}}} & \simeq & \mathbb{Q}[a]/(a^{m+1}) \\ [R] & \mapsto & \left(\frac{a}{1-e^{-a}}\right)^m \left(\frac{-a}{1-e^{a}}\right)^n \\ & & = 1 + \frac{1}{2}(m-n)a + \frac{1}{24}(\cdots)a^2 + \cdots \\ [\omega_R] & \mapsto & \left(\frac{-a}{1-e^{a}}\right)^m \left(\frac{a}{1-e^{-a}}\right)^n \\ \overline{\mu_R} & \mapsto & 1 \\ \tau_R^{-1}(K_R) & \mapsto & (n-m)a \end{array}$$

Here, we shall explain the relationship between the fundamental class $\overline{\mu_R}$ and homological conjectures.

Fact 8 1) The small Mac conjecture is true if and only if $\overline{\mu_R} \in C_{CM}(R)$ for any R. We give an outline of the proof here.

"If" part is trivial. We shall show "only if" part. Suppose that S is a regular local ring such that R is a localization of a finite extension over S. Let L be a finite-dimensional normal extension of $Q(S)$ containing $Q(R)$. Let B be the integral closure of R in L. Here, *assume that there exists an maximal Cohen-Macaulay B-module M*. Put $\operatorname{Aut}_{Q(S)}(L) = \{g_1, \ldots, g_t\}$ and $N = \oplus_i ({}_{g_i}M)$, where ${}_{g_i}M$ denotes M with R-module structure given by $a \times m = g_i(a)m$. Then N is an maximal Cohen-Macaulay R-module such that $[N] = \operatorname{rank}_R N \cdot \mu_R$ in $G_0(R)_{\mathbb{Q}}$. Therefore, $\overline{\mu_R} = \frac{[N]}{\operatorname{rank}_R N} \in C_{CM}(R)$.

Even if R is an equi-characteristic Gorenstein ring, it is not known whether $\overline{\mu_R}$ is in $C_{CM}(R)$ or not. If R is a complete intersection, then $\overline{\mu_R} = [R] \in C_{CM}(R)$ as in 1) in Example 7.

2) If $\overline{\mu_R} = [R]$ in $\overline{G_0(R)_{\mathbb{Q}}}$, then the vanishing property of intersection multiplicity holds (Roberts [8], [9]).

3) Roberts [10] proved $\overline{\mu_R} \in SN(R)$ if $ch(R) = p > 0$. Using it, he proved the new intersection theorem in the mixed characteristic case.

4) $\overline{\mu_R} \in SN(R)$ if R contains a field (Kurano-Roberts [7]). *Even if R is a Gorenstein ring (of mixed characteristic), we do not know whether $\overline{\mu_R} \in SN(R)$ or not.*

5) If $\overline{\mu_R} \in SN(R)$ for any R, then Serre's positivity conjecture is true in the case where one of two modules is (not necessary maximal) Cohen-Macaulay.

If $C_{CM}(R) \neq \{0\}$ for any R, then the small Mac conjecture is true. Therefore, if $\overline{\mu_R} \in C_{CM}(R)$ for any R, then Serre's positivity conjecture is true.

Remark 9 1) If R is Cohen-Macaulay of characteristic $p > 0$, then ${}^eR$ is a maximal Cohen-Macaulay module. Since $\overline{\mu_R}$ is the limit of $[{}^eR]/p^{de}$ in $\overline{G_0(R)_{\mathbb{R}}}$, $\overline{\mu_R}$ is contained in $C_{CM}(R)^-$. *In the case where R is not of characteristic $p > 0$, we do not know whether $\overline{\mu_R}$ is contained in $C_{CM}(R)^-$ even if R is Gorenstein.*

2) As we have already seen, if R is Cohen-Macaulay, then $[R] \in Int(C_{CM}(R)) \subset C_{CM}(R)$.

There is an example of non-Cohen-Macaulay ring R such that $[R] \notin SN(R)$.[3] On the other hand, it is expected that $\overline{\mu_R} \in SN(R)$ for any R. Therefore, for the non-Cohen-Macaulay local ring R, $\overline{\mu_R}$ behaves better than $[R]$ in a sence.

4 Main theorem

The fundamental class $\overline{\mu_R}$ is deeply related to homological conjectures. Therefore, we propose the following question.

Question 10 Assume that R is a "good" Cohen-Macaulay local domain (for example, equi-characteristic, Gorenstein, etc). Is $\overline{\mu_R}$ in $C_{CM}(R)$?

The main theorem is the following:

Theorem 11 *Assume that R is an F-finite Cohen-Macaulay local domain of characteristic $p > 0$ with residue class field algebraically closed.*

1) If R is FFRT, then $\overline{\mu_R}$ is contained in $C_{CM}(R)$.

2) If R is F-rational, then $\overline{\mu_R}$ is contained in $Int(C_{CM}(R))$.

Here, we give an outline of the proof in the case where R is F-rational.

Proof. First, we shall prove that $[\omega_R] \in Int(C_{CM}(R))$ if R is Cohen-Macaulay. We have an isomorphism $\xi : G_0(R)_{\mathbb{R}} \to G_0(R)_{\mathbb{R}}$ given by $\xi([M]) = \sum_i (-1)^i [Ext^i_R(M, \omega_R)]$. Let $\mathbb{D}^{\cdot}$ be the dualizing complex of R. For $\mathbb{F}. \in C(R)$ and an R-module M, consider the double complex

$$Hom_R(\mathbb{F}. \otimes M, \mathbb{D}^{\cdot}),$$

we can show $(-1)^d \chi_{\mathbb{F}.} = \chi_{(\mathbb{F}.^*)}\xi$. Thus, we know that ξ preserves the numerical equivalence. Therefore we have the induced map

$$\overline{\xi} : \overline{G_0(R)_{\mathbb{R}}} \to \overline{G_0(R)_{\mathbb{R}}}.$$

The map $\overline{\xi}$ satisfies $\overline{\xi}([R]) = [\omega_R]$ and $\overline{\xi}(C_{CM}(R)) = C_{CM}(R)$. Since $[R] \in Int(C_{CM}(R))$, we obtain $[\omega_R] \in Int(C_{CM}(R))$.

[3] Peskine-Szpiro had conjectured $[R] \in SN(R)$.

Assume that M is a maximal Cohen-Macaulay module. For $e > 0$, consider the following exact sequence

$$0 \longrightarrow L_e \longrightarrow F_*^e(M) \longrightarrow M^{\oplus b_e} \longrightarrow 0$$

where $F_*^e(M)$ is the eth Frobenius direct image of M. Take b_e as large as possible. Recall that L_e is a maximal Cohen-Macaulay module. Put $r = \mathrm{rank}_R M$.

Here we define the dual F-signature following Sannai [12]

$$s(M) := \limsup_{e \to \infty} \frac{b_e}{rp^{de}}$$

Then, taking a subsequence of $\{\frac{b_e}{rp^{de}}\}_e$, we may assume that $s(M) = \lim_{e\to\infty} \frac{b_e}{rp^{de}}$. Then, $\frac{[L_e]}{rp^{de}}$ converges to some element in $\overline{G_0(R)}_{\mathbb{R}}$, say $\alpha(M)$.

$$\begin{array}{ccccccl} \frac{[F_*^e(M)]}{rp^{de}} & = & \frac{b_e[M]}{rp^{de}} & + & \frac{[L_e]}{rp^{de}} & \in \overline{G_0(R)}_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow & (e\to\infty) \\ \overline{\mu_R} & = & s(M)[M] & + & \alpha(M) & \end{array}$$

Since L_e is a maximal Cohen-Macaulay module, we know $\alpha(M) \in C_{CM}(R)^-$.

Here put $M = \omega_R$. Then

$$\overline{\mu_R} = s(\omega_R)[\omega_R] + \alpha(\omega_R) \in \overline{G_0(R)}_{\mathbb{R}}. \tag{2}$$

Here

$$\alpha(\omega_R) \in C_{CM}(R)^-. \tag{3}$$

and

$$[\omega_R] \in Int(C_{CM}(R)) = Int(C_{CM}(R)^-). \tag{4}$$

The most important point in this proof is the fact that

$$R \text{ is F-rational iff } s(\omega_R) > 0$$

due to Sannai [12].

Therefore, if R is F-rational, then $\overline{\mu_R} \in Int(C_{CM}(R)^-)$ by (2), (3), (4) and Remark 3.

q.e.d.

Remark 12 If the rank of $\overline{G_0(R)}$ is one for a Cohen-Macaulay local domain R, then $\overline{\mu_R} \in C_{CM}(R)$.

If R is a toric ring (a normal semi-group ring over a field k), then we can prove $\overline{\mu_R} \in C_{CM}(R)$ as in the case of FFRT without assuming that $ch(k)$ is positive.

Problem 13 1) As in the above proof, if there exists a maximal Cohen-Macaulay module in $Int(C_{CM}(R))$ such that its generalized F-signature or its dual F-signature is positive, then $\overline{\mu_R}$ is in $Int(C_{CM}(R)^-)$.

Without assuming that R is F-rational, do there exist such a maximal Cohen-Macaulay module?

2) How do we make mod p reduction? (the case of rational singularity)

3) If R is Cohen-Macaulay, is $\overline{\mu_R}$ in $C_{CM}(R)^-$? If R is a Cohen-Macaulay ring containing a field of positive characteristic, then $\overline{\mu_R}$ in $C_{CM}(R)^-$ as in 1) in Remark 9.

4) If R is of finite representation type, is $\overline{\mu_R}$ in $C_{CM}(R)$?

5) Find more examples of $C_{CM}(R)$ and $SN(R)$.

In order to prove the following corollary, we use a fact $\overline{\mu_R} \in Int(C_{CM}(R))$ for some F-rational ring R.

Corollary 14 ([1]) *Let d be a positive integer and p a prime number. Let ϵ_0, ϵ_1, ..., ϵ_d be integers such that*

$$\epsilon_i = \begin{cases} 1 & i = d, \\ -1,\ 0\ \textit{or}\ 1 & d/2 < i < d, \\ 0 & i \leq d/2. \end{cases}$$

Then, there exists a d-dimensional Cohen-Macaulay local ring R of characteristic p, a maximal primary ideal I of R of finite projective dimension, and positive rational numbers α, β_{d-1}, β_{d-2},..., β_0 such that

$$\ell_R(R/I^{[p^n]}) = \epsilon_d \alpha p^{dn} + \sum_{i=0}^{d-1} \epsilon_i \beta_i p^{in}$$

for any $n > 0$.

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