Finite geometry and the Lefschetz property

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Introduction

We discuss the Lefschetz property for Artinian Gorenstein algebras constructed from combinatorial data. As a consequence, we obtain the Sperner property for the geometric modular lattice. This article is a résumé of the results from [6], [7], [8].

The Lefschetz property for a commutative graded algebra is a ring-theoretic abstraction of the Hard Lefschetz Theorem for compact Kähler manifolds, which is defined as follows.

Definition 1 Let $A = \bigoplus_{i=0}^{c} A_i$, $A_c \neq 0$, be a commutative graded algebra.
(1) If there exists an element $L \in A_1$ such that the maps $\times L : A_i \to A_{i+1}$ are of full rank for $0 \leq i \leq c-1$, then A is said to have the weak Lefschetz property. The element L satisfying the above condition is called a weak Lefschetz element.
(2) If there exists an element $L \in A_1$ such that the maps $\times L^j : A_i \to A_{i+j}$ are of full rank for $0 \leq i \leq c-1$ and $1 \leq j < c-i$, then A is said to have the weak Lefschetz property. In this case, the element L is called a strong Lefschetz element.
(3) If there exists an element $L \in A_1$ such that the maps $\times L^{c-2i} : A_i \to A_{c-i}$ are bijective for $0 \leq i \leq [c/2]$, then A is said to have the strong Lefschetz property in the narrow sense.

In the present article, our main interest is the strong Lefschetz property for Gorenstein algebras over a field k. For a graded Gorenstein k-algebra $A = \bigoplus_{i=0}^{c} A_i$, we have $\dim_k A_i = \dim_k A_{c-i}$, so the strong Lefschetz property is equivalent to the one in the narrow sense. In the following, we just say that A has the Lefschetz property when A has the strong Lefschetz property in the narrow sense. The strong Lefschetz element will be called the Lefschetz element for short.

The Lefschetz property for Artinian algebras has interesting combinatorial implications. As an example, we discuss the Sperner property for finite ranked posets. The maximal cardinality of antichains of a poset P is called the Dilworth number of P, which is denoted by $d(P)$. For a finite ranked poset $P = \bigcup_i P_i$ with level sets P_i, each level set P_i is an antichain of P. So we have the inequality $d(P) \geq \max_i(\#P_i)$. If the equality holds, the poset P is said to have the Sperner property. Sperner [10] proved that the

Dilworth number of the Boolean lattice $2^{\{1,\dots,n\}}$ is equal to $\binom{n}{[n/2]}$. The Sperner property was named after his work. See e.g. [4], [11] for applications of the Lefschetz property to the Sperner property.

1 Lefschetz property for Artinian Gorenstein algebras

Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian Gorenstein commutative algebra over a field k of characteristic 0. We assume that $A_0 = k$ and $A_c \neq 0$. Let $P = P_n = k[x_1, \dots, x_n]$ and $Q = Q_n = k[X_1, \dots, X_n]$ be polynomial rings over k. We consider P as a Q-module by identifying each variable X_i with the operator $\partial/\partial x_i$ on P. For a polynomial $f \in P$, define the ideal $\operatorname{Ann} f$ of Q by $\operatorname{Ann} f := \{\alpha \in Q \mid \alpha(X)f(x) = 0\}$.

Proposition 1 ([2]) Let $A = \oplus_{i=0}^{c} A_i$ be a finite-dimensional k-algebra with $\dim_k A_1 = n$. The algebra A is Gorenstein if and only if there exists a polynomial $f \in P_n$ such that $A \cong Q_n / \operatorname{Ann} f$.

Now we give a characterization of a Lefschetz element of a Gorenstein algebra $Q/\operatorname{Ann} f$ in terms of a polynomial $f \in P$. For a polynomial $g \in P$, fix a family of polynomials $\mathbf{B}_i = \{\beta_\mu^{(i)}(X)\}_{\mu=1}^{\dim_k A_i} \subset Q$ affording a linear basis of degree i part of $Q/\operatorname{Ann} g$. We define the i-th Hessian of g as follows:

$$\operatorname{Hess}_{\mathbf{B}_i}^{(i)}(g) := \det\left(\beta_\mu^{(i)}(X)\beta_\nu^{(i)}(X)g(x)\right)_{\mu,\nu=1}^{\dim_k A_i}.$$

Proposition 2 ([9], [12]) An element $L = a_1X_1 + \cdots + a_nX_n \in Q$ gives a Lefschetz element of $Q/\operatorname{Ann} f$ if and only if

$$f(a_1, \dots, a_n) \neq 0 \quad \text{and} \quad \operatorname{Hess}_{\mathbf{B}_i}^{(i)}(f)(a_1, \dots, a_n) \neq 0, \qquad 1 \leq i \leq [\deg f/2].$$

Corollary 1 For a polynomial $f \in P$, the algebra $Q/\operatorname{Ann} f$ has the strong Lefschetz property if and only if $\operatorname{Hess}_{\mathbf{B}_i}^{(i)}(f) \neq 0$ as a function on degree one part of $Q/\operatorname{Ann} f$ for $1 \leq i \leq [\deg f/2]$.

2 Gorenstein algebras associated with matroids

The matroid $M = (E, \mathcal{F})$ is a pair of a finite set E and a family $\mathcal{F}$ of subsets of E satisfying the following conditions (i) - (iii):
(i) $\emptyset \in \mathcal{F}$,
(ii) $X \in \mathcal{F}$, $Y \subset X \Rightarrow Y \in \mathcal{F}$,
(iii) $X, Y \in \mathcal{F}$, $\#X < \#Y \Rightarrow \exists y \in Y \setminus X$, $X \cup \{y\} \in \mathcal{F}$.

For a matroid $M = (E, \mathcal{F})$, a maximal element of $\mathcal{F}$ with respect to the inclusion is called a basis of M. Denote by $\mathcal{B}_M$ the set of the bases of M. For $S \subset E$, define $r(S) := \max\{\#F \mid F \subset S,\ F \in \mathcal{F}\}$. The closure $\sigma(S)$ of a subset $S \subset E$ is defined as $\sigma(S) := \{x \in E \mid r(S \cup \{x\}) = r(S)\}$. A subset $S \subset E$ is called a flat of M if $\sigma(S) = S$.

We construct a Gorenstein algebra A_M for a given matroid $M = (E, \mathcal{F})$. Consider the polynomial rings $P_M = k[x_e \mid e \in E]$ and $Q_M = k[X_e \mid e \in E]$. We regard P_M as a Q_M-module as in the previous section. For a subset S of E, we set $x_S := \prod_{e \in S} x_e$. For a matroid M, take a polynomial $\Phi_M := \sum_{S \in \mathcal{B}_M} x_S \in P_M$ and define the Gorenstein algebra A_M by $A_M := Q_M / \operatorname{Ann} \Phi_M$. It is easy to see that the set

$$\Lambda_M = \{X_e^2 \mid e \in E\} \cup \{X_S \mid S \subset E,\ S \notin \mathcal{F}\} \cup \{X_A - X_{A'} \mid A, A' \subset E,\ \sigma(A) = \sigma(A')\}$$

is contained in the ideal $\operatorname{Ann} \Phi_M$. Denote by J_M the ideal of Q_M generated by Λ_M. It can be shown that the set Λ_M is a universal Gröbner basis of J_M. For a general matroid M, the ideal J_M does not necessarily coincide with $\operatorname{Ann} \Phi_M$. The equality $J_M = \operatorname{Ann} \Phi_M$ can be shown for a special class of matroids. The projective space $\mathbb{P}^{n-1}(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ forms a matroid under the linear independence in usual sense, which we denote by $M(q, n)$.

Theorem 1 ([7]) For the matroid $M = M(q, n)$, we have $A_M = Q_M / J_M$. Moreover, the algebra A_M has the strong Lefschetz property.

The idea of the proof is based on Corollary 1. We can show that $\mathrm{Hess}^{(i)}_{\mathbf{B}_i}(\Phi_{M(q,n)})$ is a nonzero polynomial for $1 \le i \le [n/2]$.

Corollary 2 The lattice of the linear subspaces of $\mathbb{P}^{n-1}(\mathbb{F}_q)$ has the Sperner property.

The Sperner property of the vector space lattice was proved by Kantor [5]. We have given another proof based on the Lefschetz property of $A_{M(q,n)}$.

3 Geometric modular lattice

For a matroid M, the set of the flats of M forms a lattice $L(M)$. A characterization of a matroid M such that $J_M = \operatorname{Ann} \Phi_M$ is given as follows.

Theorem 2 ([7]) The algebra Q_M / J_M is Gorenstein if and only if $L(M)$ is a geometric modular lattice. Moreover, for a geometric modular lattice M, the algebra A_M has the strong Lefschetz property.

The above theorem is proved by using Greene's theorem [3] on the characterization of the geometric modular lattice.

Corollary 3 The geometric modular lattice has the Sperner property.

The result in the above Corollary was proved by Baker [1]. It can be obtained also as a consequence of the above theorem.

4 Block design and Gorenstein algebra

It is known that the geometric modular lattice decomposes into a product of Boolean lattices, lattices of rank 2, finite projective spaces and (non-Desarguesian) finite projective planes. The finite projective planes play an important role in the proof of Theorem 2. In this section, we introduce Gorenstein algebras for block designs as an analogue of the construction of the previous section.

The pair $\mathbb{D} = (E, \mathbb{B})$ of a finite set E and a family $\mathbb{B}$ of subsets of E is called a t-(v, k, λ) design if the following conditions are satisfied:
(i) $\# E = v$,
(ii) each element of $\in \mathbb{B}$ is a k-element subset of E,
(iii) For any t distinct elements $a_1, \ldots, a_t \in E$, there exist exactly λ elements of $\mathbb{B}$ containing $a_1, \ldots, a_t$.

An element of $\mathbb{B}$ is called a block of $\mathbb{D}$. We construct an algebra $A_{\mathbb{D}}$ for a t-(v, k, λ) design $\mathbb{D}$. Define a set Δ of subsets of E as follows:

$$\Delta := \{ S \in 2^E \mid \# S = t+1,\ S \not\subset B, \forall B \in \mathbb{B} \}.$$

Consider the polynomial rings $P_{\mathbb{D}} = k[x_e \mid e \in E]$ and $Q_{\mathbb{D}} = k[X_e \mid e \in E]$. We take a polynomial $\Psi_{\mathbb{D}} := \sum_{S \in \Delta} x_S$ to define the algebra $A_{\mathbb{D}} := Q_{\mathbb{D}} / \operatorname{Ann} \Psi_{\mathbb{D}}$. We can show the following in a similar manner to the case of $A_{M(q,n)}$.

Theorem 3 ([6]) Let $\mathbb{D} = (E, \mathbb{B})$ be a t-(v, k, λ) design. For a t-element subset $T \subset E$, define $Z(T) := \bigcup_{B \in \mathbb{B},\, T \subset B} B$. If $\# Z(T)$ is independent of the choice of T, then $A_{\mathbb{D}}$ has the strong Lefschetz property.

The assumption $t \leq 5$ in the statement of [6, Theorem 4] is not necessary.

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