Strong Koszulness of toric rings associated with stable set polytopes of trivially perfect graphs [1]

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1. Introduction

In this note, we study the notion of strongly Koszul algebra. This notion was introduced by Herzog, Hibi and Restuccia[HeHiR].

Let K be a field, R be a graded K-algebra, and $\mathfrak{m} = R_+$ be the homogeneous maximal ideal of R. The definition of strongly Koszul algebra is as follows.

Definition 1.1 ([HeHiR])**.** A graded K-algebra R is said to be *strongly Koszul* if $\mathfrak{m}$ admits a minimal system of generators $\{u_1, \ldots, u_t\}$ which satisfies the following condition:

> For all subsequences $u_{i_1}, \ldots, u_{i_r}$ of $\{u_1, \ldots, u_t\}$ $(i_1 \leq \cdots \leq i_r)$ and for all $j = 1, \ldots, r-1$, $(u_{i_1}, \ldots, u_{i_{j-1}}) : u_{i_j}$ is generated by a subset of elements of $\{u_1, \ldots, u_t\}$.

Example 1.2. (1) Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over K. It is clear that R is strongly Koszul since $(x_{i_1}, \ldots, x_{i_{j-1}}) : x_{i_j} = (x_{i_1}, \ldots, x_{i_{j-1}})$.

(2) Let $R = K[x_1, x_2, x_3]/(x_1^2 - x_2x_3)$. Clearly, $(0) : x_i = (0)$ for all $i = 1, 2, 3$ since R is domain. Moreover,

$$(x_1) : x_2 = (x_1, x_3),\ (x_1) : x_3 = (x_1, x_2),\ (x_2) : x_3 = (x_2),$$
$$(x_1, x_2) : x_3 = (x_1, x_2).$$

Hence we have R is strongly Koszul.

A graded K-algebra R is called Koszul if $K = R/\mathfrak{m}$ has a linear resolution[P]. By the following propsition, we have that a strongly Koszul algebra is Koszul.

Proposition 1.3 ([HeHiR, Theorem 1.2])**.** If R is strongly Koszul with respect to the minimal homogeneous generators $\{u_1, \ldots, u_t\}$ of $\mathfrak{m} = R_+$, then for all subsequences $\{u_{i_1}, \ldots, u_{i_r}\}$ of $\{u_1, \ldots, u_t\}$, $R/(u_{i_1}, \ldots, u_{i_r})$ has a linear resolution. In particular, strongly Koszul algebras are Koszul.

Remark 1.4. (1) In my talk, I stated that strongly Koszul semigroup rings are normal, but my proof of normality of strongly Koszul semigroup rings was wrong. Please accept my apology. However, strongly Koszul squarefree semigroup rings are normal. See [MO] for the details.

[1] This paper is a résumé of our result. See [M] for the details.

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(2) In general, Koszul algebras are not normal even in the case of squarefree semigroup ring. In [OH], Ohsugi and Hibi showed that
$$R = K[x_1x_2, x_1x_5, x_1x_6, x_2x_3, x_2x_6, x_3x_4, x_3x_6, x_4x_5, x_4x_6, x_5x_6]$$
is non-normal but Koszul.

Herzog, Hibi and Restuccia gave a useful criterion for strongly Koszul semigroup rings.

Proposition 1.5 (HeHiR, Proposition 1.4)**.** Let $S = \langle s_i = (s_{i1}, \ldots, s_{in}) \mid 1 \leq i \leq r\rangle$ be a semigroup of $\mathbb{N}^n$ and $K[S] = K[X^{s_1}, \ldots, X^{s_r}]$ be the semigroup ring generated by S, subring of $K[X_1, \ldots, X_n]$, where $X^{s_i} = X_1^{s_{i1}} \cdots X_n^{s_{in}}$. Then the following assertions are equivalent:

(1) $K[S]$ is strongly Koszul.

(2) The intersection ideal $(X^{s_i}) \cap (X^{s_j})$ is generated in degree 2 for all $i \neq j$.

Example 1.6. (1) Let $R = K[X^2, XY, Y^2]$ (this is isomorphic to the ring in Example 1.2(2)). We can check easily that
$$(X^2, XY) = (X^3Y, X^2Y^2), (X^2, Y^2) = (X^2Y^2), (XY, Y^2) = (X^2Y^2, XY^3).$$
Hence R is strongly Koszul by Proposition 1.5.

(2) Let $R = K[X^4, X^3Y, X^2Y^2, Y^4]$. Then R is Koszul since its defining ideal has quadratic Gröbner basis. However, $(X^4, X^3Y) = (X^7Y, X^8Y^4)$ since $X^8Y^4 = X^4 \cdot X^4 \cdot Y^4 = X^3Y \cdot X^3Y \cdot X^2Y^2$. Hence we have R is not strongly Koszul.

The aim of this note is to give necessary and sufficient conditions for strong Koszulness of *toric rings associated with stable set polytopes of graphs.*

Let G be a simple graph on the vertex set $V(G) = [n]$ with the edge set $E(G)$. $S \subset V(G)$ is said to be *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in S$. Note that $\emptyset$ is stable. For each stable set S of G, we define $\rho(S) = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^n$, where $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^n$.

The convex hull of $\{\rho(S) \mid S$ is a stable set of $G\}$ is called the *stable set polytope* of G (see [C]) , denoted by $\mathcal{Q}_G$. $\mathcal{Q}_G$ is a kind of $(0, 1)$-polytope. For this polytope, we define the subring of $K[T, X_1, \ldots, X_n]$ as follows:
$$K[\mathcal{Q}_G] := K[T \cdot X_1^{a_1} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n) \text{ is a vertex of } \mathcal{Q}_G].$$
$K[\mathcal{Q}_G]$ is called the *toric ring associated with the stable set polytope of* G. We can regard $K[\mathcal{Q}_G]$ as a graded k-algebra by setting $\deg T \cdot X_1^{a_1} \cdots X_n^{a_n} = 1$.

In the theory of graded algebras, the notion of Koszulness plays an important role and is closely related to the Gröbner basis theory.

Let $\mathcal{P}$ be an integral convex polytope (i.e., a convex polytope such that each of whose vertices has integer coordinates) and
$$k[\mathcal{P}] := k[T \cdot X_1^{a_1} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n) \text{ is a vertex of } \mathcal{P}]$$
be the toric ring associated with $\mathcal{P}$. In general, it is known that

$$\text{The defining ideal of } k[\mathcal{P}] \text{ possesses a quadratic Gröbner basis}$$
$$\Downarrow$$
$$k[\mathcal{P}] \text{ is Koszul}$$
$$\Downarrow$$
$$\text{The defining ideal of } k[\mathcal{P}] \text{ is generated by quadratic binomials}$$

follows from general theory (for example, see [BHeV]).

In [HeHiR], Herzog, Hibi, and Restuccia proposed the conjecture that the strong Koszulness of R is at the top of the above hierarchy, that is,

Conjecture 1.7 (see [HeHiR]). The defining ideal of a strongly Koszul algebra $k[\mathcal{P}]$ possesses a quadratic Gröbner basis.

A ring R is *trivial* if R can be constructed by starting from polynomial rings and repeatedly applying tensor and Segre products. In this note, we propose the following conjecture.

Conjecture 1.8. Let $\mathcal{P}$ be a $(0,1)$-polytope and $k[\mathcal{P}]$ be the toric ring generated by $\mathcal{P}$. If $k[\mathcal{P}]$ is strongly Koszul, then $k[\mathcal{P}]$ is trivial.

In the case of a $(0,1)$-polytope, Conjecture 1.8 implies Conjecture 1.7. If $\mathcal{P}$ is an order polytope or an edge polytope of bipartite graphs, then Conjecture 1.8 holds true [HeHiR].

In this note, we prove Conjecture 1.8 for stable set polytopes. The main theorem of this note is the following:

Theorem 1.9. Let G be a graph. Then the following assertions are equivalent:
(1) $k[\mathcal{Q}_G]$ is strongly Koszul.
(2) G is a trivially perfect graph.

In particular, if $k[\mathcal{Q}_G]$ is strongly Koszul, then $k[\mathcal{Q}_G]$ is trivial.

Remark 1.10. We note that Conjecture 1.8 is false. In fact, the edge ring $K[K_4]$ is strongly Koszul but not trivial, where K_4 is the complete graph with 4 vertices. See [HMO] for the details.

Throughout this note, we will use the standard terminologies of graph theory in [Diest].

2. Hibi ring and comparability graph

In this section, we introduce the concepts of a Hibi ring and a comparability graph. Both are defined with respect to a partially ordered set.

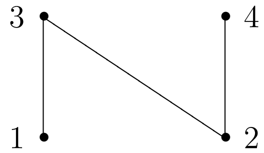
Let $P = \{p_1, \ldots, p_n\}$ be a finite partially ordered set consisting of n elements, which is referred to as a *poset*. Let $J(P)$ be the set of all poset ideals of P, where a poset ideal of P is a subset I of P such that if $x \in I$, $y \in P$, and $y \leq x$, then $y \in I$. Note that $\emptyset \in J(P)$.

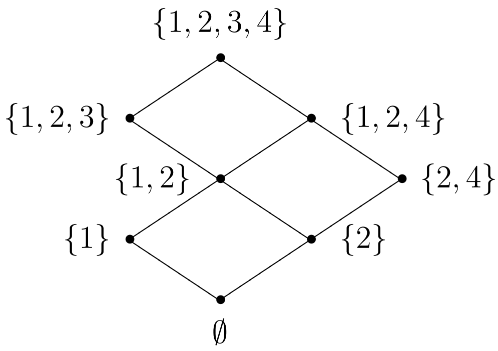
First, we give the definition of the Hibi ring introduced by Hibi.

Definition 2.1 ([Hib])**.** For a poset $P = \{p_1, \ldots, p_n\}$, the *Hibi ring* $\mathcal{R}_k[P]$ is defined as follows:

$$\mathcal{R}_k[P] := k[T \cdot \prod_{i \in I} X_i \mid I \in J(P)] \subset k[T, X_1, \ldots, X_n]$$

Example 2.2. Consider the following poset $P = (1 \leq 3, 2 \leq 3 \text{ and } 2 \leq 4)$.

$P =$ 

$J(P) =$ 

Then we have

$$\mathcal{R}_k[P] = k[T, TX_1, TX_2, TX_1X_2, TX_2X_4, TX_1X_2X_3, TX_1X_2X_4, TX_1X_2X_3X_4].$$

Hibi showed that a Hibi ring is always normal. Moreover, a Hibi ring can be represented as a factor ring of a polynomial ring: if we let

$$I_P := (X_I X_J - X_{I \cap J} X_{I \cup J} \mid I, J \in J(P), I \not\subseteq J \text{ and } J \not\subseteq I)$$

be the binomial ideal in the polynomial ring $k[X_I \mid I \in J(P)]$ defined by a poset P, then $\mathcal{R}_k[P] \cong k[X_I \mid I \in J(P)]/I_P$. Hibi also showed that I_P has a quadratic Gröbner basis for any term order which satisfies the following condition: the initial term of $X_I X_J - X_{I \cap J} X_{I \cup J}$ is $X_I X_J$. Hence a Hibi ring is always Koszul from general theory.

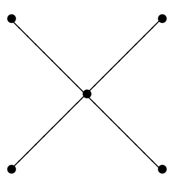
Next, we introduce the concept of a comparability graph.

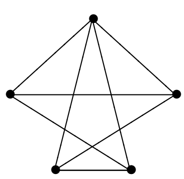
Definition 2.3. A graph G is called a *comparability graph* if there exists a poset P which satisfies the following condition:

$$\{i, j\} \in E(G) \iff i \geq j \quad \text{or} \quad i \leq j \quad \text{in} \quad P.$$

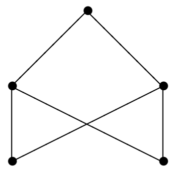
We denote the comparability graph of P by $G(P)$.

Example 2.4. The lower-left poset P defines the comparability graph $G(P)$.

$P =$ 

$G(P) =$ 

Remark 2.5. It is possible that $P \neq P'$ but $G(P) = G(P')$. Indeed, for the following poset P', $G(P')$ is identical to $G(P)$ in the above example.

$P' =$ 

Complete graphs are comparability graphs of totally ordered sets. Bipartite graphs and trivially perfect graphs (see the next section) are also comparability graphs. Moreover, if G is a comparability graph, then the suspension (e.g., see [HiNOS, p.4]) of G is also a comparability graph.

Recall the following definitions of two types of polytope which are defined by a poset.

Definition 2.6 (see [St1])**.** Let $P = \{p_1, \ldots, p_n\}$ be a finite poset.

(1) The *order polytope* $\mathcal{O}(P)$ of P is the convex polytope which consists of $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_i \leq 1$ with $a_i \geq a_j$ if $p_i \leq p_j$ in P.
(2) The *chain polytope* $\mathcal{C}(P)$ of P is the convex polytope which consists of $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_i \leq 1$ with $a_{i_1} + \cdots + a_{i_k} \leq 1$ for all maximal chain $p_{i_1} < \cdots < p_{i_k}$ of P.

Let $\mathcal{C}(P)$ and $\mathcal{O}(P)$ be the chain polytope and order polytope of a finite poset P, respectively. In [St1], Stanley proved that

$$\{\text{The vertices of } \mathcal{O}(P)\} = \{\rho(I) \mid I \text{ is a poset ideal of } P\},$$
$$\{\text{The vertices of } \mathcal{C}(P)\} = \{\rho(A) \mid A \text{ is an anti-chain of } P\},$$

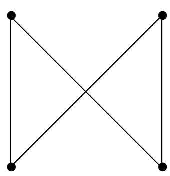
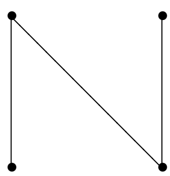
where $A = \{p_{i_1}, \ldots, p_{i_k}\}$ is an anti-chain of P if $p_{i_s} \not\leq p_{i_t}$ and $p_{i_s} \not\geq p_{i_t}$ for all $s \neq t$. Hence we have $\mathcal{Q}_{G(P)} = \mathcal{C}(P)$.

In [HiL], Hibi and Li answered the question of when $\mathcal{C}(P)$ and $\mathcal{O}(P)$ are unimodularly equivalent. From their study, we have the following theorem.

Theorem 2.7 ([HiL, Theorem 2.1])**.** Let P be a poset and $G(P)$ be the comparability graph of P. Then the following are equivalent:

(1) The X-poset in Example 2.4 does not appear as a subposet (refer to [St2, Chapter 3]) of P.
(2) $\mathcal{R}_k[P] \cong k[\mathcal{Q}_{G(P)}]$.

Example 2.8. The cycle of length 4 C_4 and the path of length 3 P_4 are comparability graphs of Q_1 and Q_2, respectively.

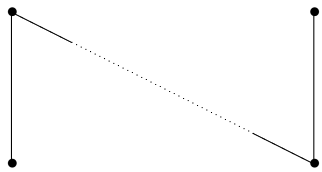
$Q_1 =$  $Q_2 =$ 

Hence $k[\mathcal{Q}_{C_4}] \cong \mathcal{R}_k[Q_1]$ and $k[\mathcal{Q}_{P_4}] \cong \mathcal{R}_k[Q_2]$.

A ring R is *trivial* if R can be constructed by starting from polynomial rings and repeatedly applying tensor and Segre products. Herzog, Hibi and Restuccia gave an answer for the question of when is a Hibi ring strongly Koszul.

Theorem 2.9 (see [HeHiR, Theorem 3.2])**.** Let P be a poset and $R = \mathcal{R}_k[P]$ be the Hibi ring constructed from P. Then the following assertions are equivalent:

(1) R is strongly Koszul.
(2) R is trivial.
(3) The N-poset as described below does not appear as a subposet of P.



Note that if $k[\mathcal{Q}_G]$ is strongly Koszul, then $k[\mathcal{Q}_{G_W}]$ is strongly Koszul for all induced subgraphs G_W of G (see [OHeHi]). By this fact, Example 2.8 and Theorem 2.9, we have

Corollary 2.10. If G contains C_4 or P_4 as an induced subgraph, then $k[\mathcal{Q}_G]$ is not strongly Koszul.

3. TRIVIALLY PERFECT GRAPH

In this section, we introduce the concept of a trivially perfect graph. As its name suggests, a trivially perfect graph is a kind of perfect graph; it is also a kind of comparability graph, as described below.

Definition 3.1. For a graph G, we set

$$\alpha(G) := \max\{\#S \mid S \text{ is a stable set of } G\},$$
$$m(G) := \#\{\text{the set of maximal cliques of } G\}.$$

We call $\alpha(G)$ the *stability number* (or *independence number*) of G.

In general, $\alpha(G) \leq m(G)$. Moreover, if G is chordal, then $m(G) \leq n$ by Dirac's theorem [Dir]. In [G], Golumbic introduced the concept of a trivially perfect graph.

Definition 3.2 ([G])**.** We say that a graph G is *trivially perfect* if $\alpha(G_W) = m(G_W)$ for any induced subgraph G_W of G.

For example, complete graphs and star graphs (i.e., the complete bipartite graph $K_{1,r}$) are trivially perfect.

We define some additional concepts related to perfect graphs. Let C_G be the set of all cliques of G. Then we define

$$\omega(G) := \max\{\#C \mid C \in C_G\},$$
$$\theta(G) := \min\{s \mid C_1 \coprod \cdots \coprod C_s = V(G), C_i \in C_G\},$$
$$\chi(G) := \theta(\overline{G}),$$

where $\overline{G}$ is the complement of G. These invariants are called the *clique number*, *clique covering number*, and *chromatic number* of G, respectively.

In general, $\alpha(G) = \omega(\overline{G})$, $\theta(G) \leq m(G)$ and $\omega(G) \leq \chi(G)$. The definition of a perfect graph is as follows.

Definition 3.3. We say that a graph G is *perfect* if $\omega(G_W) = \chi(G_W)$ for any induced subgraph G_W of G.

Lovász proved that G is perfect if and only if $\overline{G}$ is perfect [Lo]. The theorem is now called the weak perfect graph theorem. With it, it is easy to show that a trivially perfect graph is perfect.

Proposition 3.4. Trivially perfect graphs are perfect.

Proof. Assume that G is trivially perfect. By [Lo], it is enough to show that $\overline{G}$ is perfect. For all induced subgraphs $\overline{G}_W$ of $\overline{G}$, we have

$$m(G_W) = \alpha(G_W) = \omega(\overline{G_W}) \leq \chi(\overline{G_W}) = \theta(G_W) \leq m(G_W)$$

by general theory (note that $\overline{G}_W = \overline{G_W}$). $\square$

Golumbic gave a characterization of trivially perfect graphs.

Theorem 3.5 ([G, Theorem 2])**.** The following assertions are equivalent:

(1) G is trivially perfect.
(2) G is C_4, P_4-*free*, that is, G contains neither C_4 nor P_4 as an induced subgraph.

Proof. (1) $\Rightarrow$ (2): It is clear since $\alpha(C_4) = 2$, $m(C_4) = 4$, and $\alpha(P_4) = 2$, $m(P_4) = 3$.

(2) $\Rightarrow$ (1): Assume that G contains neither C_4 nor P_4 as an induced subgraph. If G is not trivially perfect, then there exists an induced subgraph G_W of G such that $\alpha(G_W) < m(G_W)$. For this G_W, there exists a maximal stable set S_W of G_W which satisfies the following:

There exists $s \in S_W$ such that $s \in C_1 \cap C_2$ for some distinct pair of cliques C_1, $C_2 \in C_{G_W}$.

Note that $\#S_W > 1$ since G_W is not complete. Then there exist $x \in C_1$ and $y \in C_2$ such that $\{x, s\}$, $\{y, s\} \in E(G_W)$ and $\{x, y\} \notin E(G_W)$.

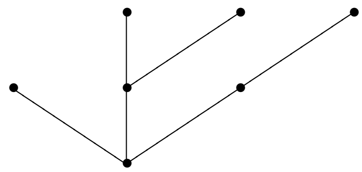
Let $u \in S_W \setminus \{s\}$. If $\{x, u\} \in E(G_W)$ or $\{y, u\} \in E(G_W)$, then the induced graph $G_{\{x,y,s,u\}}$ is C_4 or P_4, a contradiction. Hence $\{x, u\} \notin E(G_W)$ and $\{y, u\} \notin E(G_W)$. Then $\{x, y\} \cup \{S \setminus \{s\}\}$ is a stable set of G_W, which contradicts that S is maximal. Therefore, G is trivially perfect. $\square$

Next, we show that a trivially perfect graph is a kind of comparability graph. First, we define the notion of a tree poset.

Definition 3.6 (see [W]). A poset P is a *tree* if it satisfies the following conditions:

(1) Each of the connected components of P has a minimal element.
(2) For all p, $p' \in P$, the following assertion holds: if there exists $q \in P$ such that $p, p' \leq q$, then $p \leq p'$ or $p \geq p'$.

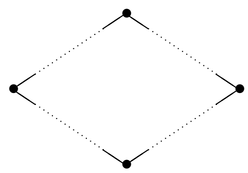
Example 3.7. The following poset is a tree:



Tree posets can be characterized as follows.

Proposition 3.8. Let P be a poset. Then the following assertions are equivalent:

(1) P is a tree.
(2) Neither the X-poset in Example 2.4, the N-poset in Theorem 2.9, nor the diamond poset as described below appears as a subposet of P.



In [W], Wolk discussed the properties of the comparability graphs of a tree poset and showed that such graphs are exactly the graphs that satisfy the "diagonal condition". This condition is equivalent to being C_4, P_4-free, and hence we have

Corollary 3.9. Let G be a graph. Then the following assertions are equivalent:

(1) G is trivially perfect.
(2) G is a comparability graph of a tree poset.
(3) G is C_4, P_4-free.

Remark 3.10. A graph G is a *threshold graph* if it can be constructed from a one-vertex graph by repeated applications of the following two operations:

(1) Add a single isolated vertex to the graph.
(2) Take a suspension of the graph.

The concept of a threshold graph was introduced by Chvátal and Hammer [CHam]. They proved that G is a threshold graph if and only if G is C_4, P_4, $2K_2$-free. Hence a trivially perfect graph is also called a *quasi-threshold graph*.

4. Proof of Main theorem

In this section, we prove the main theorem.

Theorem 4.1. Let G be a graph. Then the following assertions are equivalent:

(1) $k[\mathcal{Q}_G]$ is strongly Koszul.
(2) G is trivially perfect.

Proof. We assume that G is trivially perfect. Then there exists a tree poset P such that $G = G(P)$ from Corollary 3.9. This implies that neither the X-poset in Example 2.4 nor the N-poset in Theorem 2.9 appears as a subposet of P by Proposition 3.8, and hence $k[\mathcal{Q}_{G(P)}] \cong \mathcal{R}_k[P]$ is strongly Koszul by Theorems 2.7 and 2.9.

Conversely, if G is not trivially perfect, G contains C_4 or P_4 as an induced subgraph by Corollary 3.9. Therefore, we have that $k[\mathcal{Q}_G]$ is not strongly Koszul by Corollary 2.10. $\square$

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